JACOBI ELLIPTIC MONOPOLE-ANTIMONOPOLE PAIR OF THE SU(2) YANG-MILLS-HIGGS THEORY

by

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LIST OF SYMBOLS

m	θ-winding number
n	ϕ-winding number
s	integer parameter in axially symmetric solutions
μ	Higgs field mass
λ	strength of Higgs potential
$T_{\mu\nu}$	energy-momentum tensor
$F^a_{\mu\nu}$	gauge field strength tensor
A^a_μ	gauge field
Φ^a	Higgs field
$\hat{\Phi}^a$	unit Higgs field
$\mathscr{L}$	Lagrangian density/Lagrangian
G	Lie group
σ^a	Pauli matrices
E	Energy
r_i	spatial orthornormal unit vector
θ_i	spatial orthornormal unit vector
ϕ_i	spatial orthornormal unit vector
u^a_r	isospin orthornormal unit vector
u^a_θ	isospin orthornormal unit vector

u^a_ϕ	isospin orthornormal unit vector
$F_{\mu\nu}$	electromagnetic tensor
E_i	Abelian electric field
B_i	Abelian magnetic field
k_μ	topological/magnetic current
M	topological/magnetic charge
ψ_1	profile function dependent on r and θ
ψ_2	profile function dependent on r and θ
R_1	profile function dependent on r and θ
R_2	profile function dependent on r and θ
Φ_1	profile function dependent on r and θ
Φ_2	profile function dependent on r and θ
$\bar{x}$	compactified coordinate
ϕ	scalar field
ε_{ijk}	Levi-Civita symbol
ξ	vacuum expectation energy
g	gauge coupling constant
E_1, E_2	Jacobi elliptic functions
k	Jacobi elliptic parameter
∂_μ	partial derivative
D_μ	covariant derivative

PASANGAN MONOKUTUB-ANTIMONOKUTUB ELIPTIK JACOBI KEPADA TEORI YANG-MILLS-HIGGS SU(2)

ABSTRAK

Monokutub magnet dan multikutub magnet adalah penyelesaian soliton topologi dalam ruang tiga dimensi bagi model Georgi-Glashow SU(2) tak Abelian. Ianya merupakan hasil sampingan akibat pemecahan simetri secara spontan daripada kumpulan SU(2) kepada kumpulan U(1) dan seterusnya memperolehi cas magnet.

Dalam tesis ini, model Georgi-Glashow SU(2) ataupun dikenali sebagai teori Yang-Mills-Higgs SU(2) dikaji untuk mencari lebih banyak konfigurasi klasikal monokutub magnet berserta dengan ciri-cirinya. Dalam kajian konfigurasi dalam model tersebut, gantian ansatz yang bersesuaian diperlukan dalam persamaan gerakan pembezaan tertib kedua dan seterusnya mencari penyelesaian analitik ataupun berangka.

Konfigurasi monokutub eliptik Jacobi bersimetri paksi (Teh et al. 2010) diperolehi dengan mengitlakkan penyelesaian asimptot jarak besar kepada fungsi eliptik Jacobi dan kemudiannya menyelesaikan persamaan pembezaan medan tertib kedua secara berangka. Kami mengkaji penyelesaian ini secara berangka dengan mengubah nombor magnetnya dan menganalisis sifat-sifatnya apabila keupayaan Higgs tidak sifar. Semua penyelesaian ini adalah tak-BPS, sekata dan memiliki jumlah tenaga yang sama dengan monokutub umum ’t Hooft-Polyakov. Sesetengah monokutub ini didapati terherot dan memperolehi momen dwikutub magnet.

Penyelesaian baharu satu pasangan monokutub-antimonokutub (1-MAP) dan cincin vorteks eliptik Jacobi bersimetri paksi juga dikaji. Dengan cara yang sama, 1-MAP eliptik Jacobi ini turut diperolehi dengan kaedah mengitlakkan penyelesaian asimptot jarak besar kepada fungsi eliptik Jacobi dan menyelesaikan persamaan permbezaan medan tertib kedua secara berangka apabila keupayaan Higgs sifar dan tidak sifar. Sifat-sifat penyelesaian baru ini dibandingkan dengan 1-MAP piawai dan 1-MAP yang terhasil daripada bilangan belitan-θ $m = 2$. Secara konklusinya, walaupun ciri-ciri 1-MAP yang berbilangan belitan $m = 1$ adalah setara dengan 1-MAP berbilangan belitan $m = 2$, jumlah tenaga 1-MAP dengan $m = 1$ adalah jauh lebih rendah daripada 1-MAP dengan $m = 2$.

JACOBI ELLIPTIC MONOPOLE-ANTIMONOPOLE PAIR OF THE SU(2) YANG-MILLS-HIGGS THEORY

ABSTRACT

Magnetic monopoles and multimonopole are well known three dimensional topological soliton solutions of the non-Abelian SU(2) Georgi-Glashow model. They are remnants of the spontaneous symmetry breaking of the gauge group SU(2) into the group U(1) with net magnetic charge.

In this thesis, the SU(2) Georgi-Glashow model or synonymously SU(2) Yang-Mills-Higgs theory is studied to seek for more magnetic monopole configurations along with their properties at the classical level. To find such configurations in the model, one need to substitute a suitable ansatz into the second order equations of motions and look for an analytical or numerical solutions.

The axially symmetric Jacobi elliptic one-monopole (Teh et al. 2010) configurations were obtained by generalizing the large distance asymptotic solutions to the Jacobi elliptic functions and solving the second order field equations numerically. We study them numerically by varying its magnetic number and analyze its properties when the Higgs potential is non-vanishing. These are non-BPS, regular solutions which possess the same total energy as the generalized 't Hooft-Polyakov monopole. Some of these monopoles are distorted and possess magnetic dipole moment.

The new axially symmetric Jacobi elliptic one monopole-antimonopole pair (1-MAP) and vortex rings are studied as well. Similarly, these Jacobi elliptic 1-MAP are obtained by using large distance asymptotic solutions generalization to Jacobi elliptic

functions and solving the second order field equations numerically when the Higgs potential is vanishing and non-vanishing. The properties of these new solutions are compared with the standard 1-MAP and 1-MAP obtained from θ-winding number $m = 2$. It can be concluded that while the properties of the 1-MAP of winding number $m = 1$ are comparable to the 1-MAP of winding number $m = 2$, the total energy of the former is significantly lower than the latter.

CHAPTER 1

INTRODUCTION

1.1 Particle Physics and Gauge Theory

The subject of physics would most probably be well remembered with the lessons of classical mechanics. This is the 'mechanics' that emerged during the 17th century and lasted until early 20th century. Since then, it was realized that classical mechanics is not sufficient to explain everything and it was eventually superceded by relativistic mechanics. From that moment onwards, relativistic mechanics (or sometimes known as special relativity) forms another branch or domain of physics. At the same time, another equally bizarre but as well tested as the relativistic mechanics was forming another domain in physics. It was constructed by renown figures such as Planck, Bohr, Einstein, and many others and the name of this new domain is called the quantum mechanics. For particle physicists, these two new domains are essential for the study of subatomic matter, forces and interactions between them. Nevertheless, particle physicists would require another well established domain that could incorporate both the quantum effect as well as the relativistic effect. This is the point where quantum field theory comes in.

A culmination of quantum mechanics and relativistic mechanics, quantum field theory is widely considered as the correct method to study the elementary particles. It has a good reason to be a valid theory since it is accountable for theoretical predictions with accuracies up to one part in a billion. The major driving force behind the success of quantum field theory is the idea of gauge theory. By definition, gauge theory is a field theory where its Lagrangian remains invariant under a continuous group of local transformations. Field with such property is sometimes referred as gauge invariance or gauge symmetry. We will discuss about gauge theory in more detail in later chapter.

Hermann Weyl (1918) was the first person who invoked the idea of gauge theory in his attempt to unify general relativity and electromagnetism, the two fundamental forces known that time. While admiring Weyl's work, Einstein did not believe that it truly reflects the Nature and this leads to some intense exchanges of letters between the two. Einstein's hunch proved to be right and Weyl's initial attempt was a failed physical theory. However, with the development of quantum theory, Weyl succesfully showed that electrodynamics was invariant under the gauge transformation of the gauge field and wave function of a charged particle (suggested earlier by London and Fock). Very importantly, Weyl enunciated the role of gauge invariance as a symmetry principle to rederive the electromagnetism (O'Raifeartaigh and Straumann, 2000). This was the moment when gauge theory was born. The similar gauge theory is also applicable to the quantum version of the electrodynamics. Dubbed as Quantum Electrodynamics (QED), it has the gauge group, $G =$U(1) and is an Abelian group. It describes how the light and matter interact and is fully compatible with quantum mechanics and special relativity. The prominent physicist, Richard Feynman called it 'the jewel of physics' for its extraordinary predictions of physical quantities. In fact, QED predictions are so good that it is considered as the most accurate theoretical prediction in the history of science.

Following the immense success of QED, physicists are curious whether the non-Abelian gauge theory could be applied on other fundamental forces as well, notably the weak interaction and strong interaction? In 1954, Chen Ning Yang and Robert Mills invented the non-Abelian gauge theory in an attempt to study isospin doublet of proton and neutron (Yang and Mills, 1954). This brings the generalization of the gauge group, G from U(1) to SU(2), which is the simplest non-Abelian group. However, this generalization of Yang-Mills SU(2) suffers a serious drawback simply because the gauge fields are predicted to be massless. At that time, the only known massless gauge field is the photon. Therefore, this generalization is incompatible with the known characteristic of weak interactions which is short-range and being mediated by a massive boson. Receiving heavy criticism from Pauli and lack of experimental support, Yang-Mills theory was soon abandoned until the late 1960s except by a very few theorists.It seems that there are only two logical solutions to why experimentalists could not see any massless particles except for photon (Zee, 2003). First is the Yang-Mills particles somehow through some mechanism did acquire mass. Second is the Yang-Mills particles are in fact massless but somehow are not observed. Later, it turns out that both are right in the sense that the former was realized in electroweak theory and the latter in strong interaction. We will briefly discuss the weak interaction before moving on into strong interaction.

Schwinger (1957) was the first person to try to unify electromagnetic and weak forces into a larger gauge group. Soon, Glashow realized that a renormalizable theory of weak force necessarily involves this unification and in 1961, proposed a model of SU(2)$\times$ U(1) but lacked the important 'Higgs fields'. The major breakthrough was made by Weinberg (1967) and Salam (1968) when they introduce the Higgs field to break the gauge symmetry, which will render the gauge bosons to acquire mass. The resulting field theory, known as Glashow-Weinberg-Salam model (famously known as Weinberg-Salam electroweak theory) was confirmed sixteen years later with the discovery of $W^{\pm}, Z^0$ bosons at European Center for Nuclear Research (CERN). Glashow, Weinberg and Salam shared the 1979 Physics Nobel Prize for their contributions to the theory of the unified weak and electromagnetic interaction as well as the prediction of the weak neutral current.

On the other hand, the application of Yang-Mills theory on strong interaction (or strong force/color force) was equally formidable. Around late 1960th period, the particle physics was in deep mess with hundreds of 'elementary particles' keep pouring out from the accelerator. Frustrated, J. Robert Oppenheimer once said that the Nobel Prize should be awarded to physicist who did not discover a new particle. In 1964, Gell-Mann and his student George Zweig, proposed a quark model to explain the variety of hadrons (Fritzsch, 2012). In quark model, hadrons are not elementary but instead comprise of smaller constituents called 'quarks'. Each quarks carry a fractional charge of the hadrons. Although this model can classify any hadron by its constituents, it has a huge problem for violating the Pauli exclusion principle. To get around this problem, Greenberg at the same year and subsequently, Han and Nambu suggested that quark neccessarily carry an extra quantum number called color charge. The color charge as we know today is taken to be the three primary color: red, blue and green. These 'colors' are quantum property and are not related to the visual perception of color. Fritzsch, Leutwyler and Gell-Mann then proposed the quark's triplet color as the fundamental representation of the gauge group $SU(3)_C$ (index C for color) or just simply SU(3) (Fritzsch et al., 1973). This gauge theory that describes the strong interaction using color symmetry is named the Quantum Chromodynamics (QCD).

There are eight gauge bosons in QCD called the gluons and they intrinsically carry color charge too. To our best knowledge of SU(3) symmetry, it is unbroken in Nature, the reason that gluons do not have mass. Gluons interact with quarks as well as with themselves. At very high energy, gluon-gluon interaction actually reduces the strength of coupling constant, causing the quarks and gluons to behave like a free particle (Fritzsch, 2012). This behaviour prediction known as asymptotic freedom was discovered in 1972 by Gerard 't Hooft (unpublished) and in 1973 by David Gross, David Politzer and Frank Wilczek (Gross and Wilczek, 1973; Politzer, 1973). The Nobel Prize in Physics 2004 were awarded jointly to Gross, Politzer, and Wilczek for their contribution to the theory of strong interaction.

1.2 Standard Model of Particle Physics

Putting the Glashow-Weinberg-Salam model and QCD together is the famous Standard Model of Particle Physics or simply 'Standard Model' with gauge group $SU(3)_C \times SU(2)_L \times U(1)$. Standard Model accurately describes three of the four fundamental forces, namely electomagnetism, weak interaction and strong interaction. Standard Model is constructed with the combined effort of the best theorists and the finest experimentalists, involving global scale collaboration between the largest group of intellects. It was finalized in the mid 1970s and has correctly predicted elementary particles ahead of its time. Comprised of 61 elementary particles, all of them are experimentally verified except for the one very important Higgs boson, which is the key building block of the Standard Model. For many years, Higgs boson has eluded experimentalists and some might suggest that Standard Model has been in the wrong path. It was until very recently on July 4th 2012, strong hints of Higgs boson finally emerges from the Large Hadron Collider (LHC) in CERN. In the following year on March 14th, the ATLAS (A Toroidal LHC Apparatus) and CMS (Compact Muon Solenoid) collaborations at LHC finally confirmed the existence of Higgs boson to complete up the final piece of the Standard Model.

Powerful as it may be, Standard Model does have its weaknesses as well. In particular, there are several experimental observations that Standard Model could not explain adequately. First and foremost, Standard Model does not incorporate the final fundamental force, which we know now as the general relativity. Of course, physicists have attempted this before since 1970s but the effort of bringing these two 'recipes' together seems to be disastrous. The mathematics between them are inconsistent with each other under certain conditions and will only yield illogical result. Moreover, the recent discovery of dark matter and dark energy which make up the 96% constituents of the universe has nothing accountable to Standard Model. There are simply no known interaction between ordinary matter with dark matter nor the origin of the dark energy.

Effort to answer some of these shortcomings lead to the development of Physics beyond Standard Model (BSM). One of the main research field in BSM is the Grand Unified Theories or GUT in short. GUT is a model to merge the three interactions: electromagnetism, weak interaction, and strong interaction into a single interaction categorized by a larger gauge group and with just one coupling constant. The motivation behind GUT is that while the Standard Model has three gauge groups, the coupling constant of each groups actually varies with the energy. Based on experiments, physicists predicts that the strength of the three coupling constants almost converge at energy approaching 10^{16} GeV. The first and simplest model of GUT was proposed by

Georgi and Glashow which based on the gauge group SU(5) :

$$\mathrm{SU(5)} \supset \mathrm{SU(3)} \times \mathrm{SU(2)} \times \mathrm{U(1)}$$

Perhaps the most compelling GUT is the Georgi-Glashow SU(5) model with the inclusion of the theory of supersymmetry called the Minimal Supersymmetric Standard Model (MSSM). In supersymmetry, it is theorized that for every particle of spin one (force particle), there is neccessarily a 'superpartner' that differs by half a unit spin (matter particle). These superpartners or superparticles are believed to be a hundred to a thousand times heavier than a proton and will effectively double the number of particles in Standard Model. Apparently MSSM was put forward to explain the hierarchy problem, a long standing question in physics (and still is). Secondly, supersymmetry offers a possibility that the unseen, dormant dark matter which makes up 24% of the universe is somehow a manifest of superparticles. Finally and probably the most appealing feature of MSSM is that supersymmetry allows the three coupling constants to match and converge perfectly at energy around 10^{16} GeV called the grand unification energy. This feature is almost but cannot be attained without supersymmetry and many physicists believe that this perfect matching is no coincidence. In the experimental point of view, it is speculated that the LHC in CERN might be powerful enough to find the tell tale signs of these exotic superparticles.

Besides MSSM, there are many candidates of GUT such as the $SO(10)$, Pati-Salam model, E_6 to name a few. When discussing the topic of GUT, there is one very important fact that we need to bear in mind. As for today, we live in a time when theoretical physics has developed way far beyond where the experiments can go. The scenario nowadays is the complete opposite of the 1960th era where a theory is desperately needed to explain the bulk of elementary particles spewing out from accelerator. Nowadays, advances in theory allows theorists to construct various plausible models to explain the Nature but then the question lingers on which model is the most accurate? Only through experimental guidance can physicists rule out the inconsistent model and re-focus their attention on the other model. At least this is the way the physics works since Galileo's time. Unfortunetely, the new physics behind GUT lies within the energy range of 10^{16} GeV. This fantastically huge figure is so much higher than what the accelerator can achieve today and would be unlikely to be attained in future. In short, we have absolutely no way to determine which model is the best. The best that theorists can do is to investigate each and every possibility, which translates into a slow and pain staking progress in physics for the pass several decades. Of course, the discovery of Higgs boson in LHC or potentially superparticles would serves as a major motivation for physicists to strive on.

As readers should realize, even the GUT is still fall short of including gravity. Combining these two together into a single Theory of Everything (TOE) is the ultimate dream of every theorists. So far, there is no mutual agreement on which theory is the best candidate of TOE but one of the hottest subject pursued is the superstring theory (or M-theory). Likewise, superstring theory severely lacks of any experimental guidances to make any progress in physical theories. However, one thing for certain is that combining GUT and gravity would be one notoriously difficult task to accomplish.

1.3 Magnetic Monopole

Like its name implies, the term magnetic monopole is reserved for a hypothetical particles with only one magnetic pole. Specifically, a magnetic monopole possesses a net 'magnetic charge'. This particle is totally different from the ordinary magnet or to be more precise, a 'magnetic dipole' that has two poles (one south and one north). A magnet cannot be broken into two single pole magnets by any means (cutting, splitting) possible and the outcome would only yield two magnetic dipole instead.

The scientific development of magnetism came hand in hand with electricity during the late 19th century and it was Maxwell who unify them together in one swift stroke using his four famous equations. In vacuum state, the Maxwell equations display some degree of symmetry between electricity and magnetism. However in reality, this symmetry is ruined by the fact that we could only find electric monopole (charge) and not magnetic monopole. It is said that the absence of magnetic monopole leads to the broken symmetry in classical electrodynamics. Using quantum approach, Dirac

(1931) showed that the existence of magnetic monopole is compatible with Maxwell's equations and at the same time deduce some important result. His approach in one way or another, symmetrized the Maxwell's equations thus making it mathematically more appealing. But one question remains, is magnetic monopole a necessity?

Despite the absence of magnetic monopole, the research to investigate one is still very much alive. The next major development made after Dirac was the 't Hooft-Polyakov monopole, found independently by Gerard 't Hooft and Alexander Polyakov (1974). They discovered that the magnetic monopole exists as a 'soliton solution' in gauge theory with spontaneous symmetry breaking. Solitons are defined as stable solutions with well defined energy to the non-linear classical gauge field theories. Since gauge theories have become a very important tool in physics, it is imperative to take solitons seriously as well. We will explore more about 't Hooft-Polyakov monopole later on.

Even though solitons are classical solutions, they are not deem to be unimportant. Time and again, classical approach proves to be an extremely useful method for constructing the quantum theory. For example, development of classical mechanics led to the principle of conservation of energy and ideas of Hamiltonian mechanics. These ideas became crucial ingredients for the development of quantum mechanics, even as classical mechanics itself was superceded. If one knew everything about classical field configurations, then in principle all questions concerning the quantum theory could be answered (Actor, 1979). Even partial information about classical fields provide some insight into the quantum theory.

On top of that, magnetic monopoles inevitably reappear in the context of GUT. Despite the various possible models of GUT, many of them commonly predict the existence of magnetic monopoles even though its properties are model dependent. This came as no suprise as was stressed by 't Hooft and Polyakov, any 'grand unified' theory of particle physics necessarily contains magnetic monopoles. Even the more ambitious TOE model also predicts their existence.

1.4 The Search for Magnetic Monopoles

Given the ample arguement that monopoles should exist, physicists are still riddled by the same old mystery: where are they? Of course, there have been many attempts to detect the monopoles experimentally. These attempts include producing monopoles in the particle accelerator and secondly by finding them through cosmic rays activities.

1.4.1 Accelerator Searches

In principle, particles or monopoles are always reproducible in the particle accelerator provided that the collision energy is high enough. Another fact is that GUT monopoles require energy at least a trillion times more powerful than LHC and this seems to be unrealistic to achieve in foreseeable future. However, our best accelerators did help to determine the law of physics up to electroweak scale, which is at around 100 GeV and there are monopoles much lighter than GUT's being predicted. For example, there is Cho-Maison monopole (1997) resides in the electroweak model. These so called intermediate-mass monopoles might be light enough to be seen in LHC. In 2009, CERN Research Board approved the LHC's seventh experiment : the Monopole and Exotics Detector At the LHC (MoEDAL) with its primary objective to detect monopoles (Pinfold, 2010). The detector of MoEDAL comprises of an array of approximately 400 plastic nuclear track detectors (NTDs). Conceptually, MoEDAL detectors act like a giant camera to capture the activities of highly ionizing particles with the plastic NTDs as its 'photographic film'. Monopoles with its high ionizing power ($>$ 4000 times of e) will microscopically damage the polymeric bonds in NTD to register a detection. These plastic detectors can always be removed and analyzed during the short shutdown of LHC.

1.4.2 Cosmic Rays Searches

Probably our best hope to observe a monopole is through the detection from cosmic rays. It has been postulated that monopoles are created during the early universe and they should be around even to this day. Monopoles in cosmic rays are measured in flux and generally higher flux means higher possibility of hitting the detector (thus registrating detection). One type of detector uses the concept of electromagnetic induction when a monopole passes by a superconducting loop. The attractive feature of these inductive detectors is that they respond specifically to a magnetically-charged particle, not any electric charges or magnetic dipoles (Caplin et al., 1986). Monopole that traverses in the superconducting loop induces a current by Faraday's law of induction. There are seven groups that did this experiment using the electromagnetic induction technique (Groom, 1986) and some early experiments do show promising evidence for them. One notable evidence was reported by Blas Cabrera (1982) where his superconducting loop recorded a jump in current by exactly the same amount that a monopole would generate; a perfect signature of monopole. Another group by Caplin et al. (1986) reported a similar event three years later in Imperial College. However, because later experiments have not been able to reproduce them, no conclusive statement was made. Even Cabrera himself never claimed that the event were due to a monopole's passage.

From past experience, perhaps it would be wise to be more critical on the subject of monopole. For example, the announcement of the detection of moving magnetic monopole in cosmic rays by Price et al. (1975) was retracted after some error was found. A possible explaination for the detection was offered by Alvarez (1975) as the consequence of the decay of platinum nucleus. Recent prediction on how the GUT monopoles could have catalyze the proton decay still yield no result after prolonged research.

Nevertheless, monopole theory still remains a great interest to theorist as it has been closely connected with many actual directions of theoretical physics such as the problem of confinement in QCD, proton decay, and evolution of universe (Shnir, 2005). The distinguised string theorist Joseph Polchinski (2004), famously quoted that the existence of monopoles as 'one of the safest bets that one can make about physics not yet seen'. In any way, theoretical study of monopole proves to be highly beneficial to physics and mathematics while we are still waiting for some empirical evidence.

1.5 Objectives

Since the idea of gauge theory has been deeply enrooted in our physical world, it is not suprising that magnetic monopole too has gained much enthusiasm. In the next chapter, we will journey into the important concepts and ideas of gauge theory. Besides, some basic and vital mathematical expressions will be highlighted. In the chapter after that, we will give some literature review on the magnetic monopoles in the SU(2) Yang-Mills-Higgs field theory and discussed some of the more recent research on multimonopoles. In Chapter 4 and Chapter 5, research works on the Jacobi elliptic monopole systems will be constructed and discussed. The monopole solutions are obtained by using suitable substitution on the SU(2) Yang-Mills-Higgs second order field equations and solving these equations numerically. Through these solutions, we could retrieved informations on their energies, magnetic field, magnetic dipole moment and others. From here, it is hopeful that these informations will give us insights into the more realistic GUT monopoles model.

CHAPTER 2

GAUGE FIELD THEORIES

2.1 What is Gauge Theory?

As was defined earlier, gauge theory is a type of field theory where its Lagrangian is invariant under continuous group of local transformations. These transformations are called the gauge transformations and they form the Lie group or synonymously, gauge group of the theory. For every Lie group, there are generators that form the Lie algebra. As an analogy, one can think of Lie algebra which resembles elements that spanned the vector space. For each generators, there exists corresponding gauge field that responsible for restoring the Lagrangian and ensure its invariance. In quantum field theory, act of quantizing the gauge fields will produce the gauge bosons. Gauge groups can be categorized to be commutative (Abelian) or non-commutative (non-Abelian).

2.2 Generators and Lie Algebras

The idea of symmetry often becomes the driving force for physicists in constructing a physical theory. The mathematical machinery that is closely related to symmetry is called the group theory and its branch known as the Lie groups. The circle group, orthogonal groups O(n), unitary groups U(n) and special unitary groups SU(n) are a few examples of the compact Lie groups. Most gauge theory uses compact Lie groups to ensure that the Hamiltonian (energy) of the system is bounded from below. In a 'loose' definition, Lie groups belong to a continuous groups that can be parameterized by one or more continuous variables. For example, every point on a unit circle can always be specified by an angle or variable θ measured from the positive z-axis. Associated with every variable or parameter is a generator, which is akin to the 'basis' of the vector space. Every particular element in the group can be expressed in term of the generator

$$e^{i\alpha_j X_j}, \tag{2.2.1}$$

where α_j is the parameter and X_j is a Hermitian generator. (Index j runs from 1,2,...,n where n is dimension of the group)

The product of two elements, one with parameter α_j and another with β_k is necessarily an element in the group (attributed by the closure property)

$$e^{i\alpha_j X_j} e^{i\beta_k X_k} = e^{i\delta_l X_l}. \tag{2.2.2}$$

The fact that the generators are matrices implies that their product do not commute in general. However, after some mathematical manipulation, expression (2.2.2) can be written as

$$e^{i\alpha_j X_j} e^{i\beta_k X_k} = e^{i(\alpha_j X_j + \beta_k X_k) - \frac{1}{2}[\alpha_j X_j, \beta_k X_k]}. \tag{2.2.3}$$

Eq.(2.2.3) is called the Baker-Campbell-Hausdorff formula and it is a generalization of exponential multiplication rule. The commutator $[X_j, X_k]$ is proportional to a linear combination of generators of the group (again due to closure property).

$$[X_j, X_k] = i f_{jkl} X_l, \tag{2.2.4}$$

where f_{jkl} is called the structure constants of the group.

The generators that satisfy the commutation relationship (2.2.4) defined by the structure constants form the Lie algebra of the group. In other words, the Lie algebra is the vector space spanned by the generators X_j under the rule of (2.2.4). The groups SU(n) are often used in the construction of particle physics model. Here, SU(n) stands for Special Unitary $n \times n$ matrices with unit determinant and generally has $(n^2 - 1)$

number of generators. Common SU(n) groups encountered are the SU(2) which is associated with isotopic-spin vector and SU(3) which relates to color quarks.

2.3 Abelian Gauge Theory

Gauge field theory first appeared in Maxwell's formulation of electrodynamics in 1864 and in addition, was the first field theory to appear in physics. Classical electrodynamics can be described by using the gauge group, $G =$U(1) or Unitary matrix of dimension 1. Mathematically, this group belongs to the circle group, and is the simplest gauge group possible. It consists of only one generator and is always commutative (Abelian). Therefore, classical electrodynamics is synonymously known as the Abelian gauge theory.

2.3.1 Global Gauge Transformations

Consider the Lagrangian of a complex scalar field ϕ with two real components (Rubakov, 2002), $\phi = \phi_1 + i\phi_2$

$$\mathscr{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \ \phi^*, \tag{2.3.1}$$

where μ, ν is the space-time indices that runs from 0,1,2 and 3. Applying Euler-Lagrange equations on Lagrangian (2.3.1) gives two Klein-Gordon equations

$$\begin{aligned} \partial_\mu \partial^\mu \phi + m^2 \phi &= 0 \, , \\ \partial_\mu \partial^\mu \phi^* + m^2 \phi^* &= 0 \, . \end{aligned} \tag{2.3.2}$$

Let us now consider the scalar field that transforms under the gauge group, G according to the rule

$$\phi(x) \to \phi'(x) \ = \ \omega\phi(x), \tag{2.3.3}$$

where $\omega \in G$. In particular, for $G =$U(1), $\omega = e^{i\alpha}$ where α is independent of space-time or 'global'.

Obviously, Lagrangian (2.3.1) is invariant under the transformation of (2.3.3). This kind of transformation is called the gauge transformation of the first kind or global gauge transformation (Ryder, 1996). Noether theorem then gives the conserved current (Gross, 1992)

$$
\begin{aligned}
j^{\mu} &= \sum_{\phi_i} \frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi_i)}\delta\phi_i \\
&= -i(\phi^{*}\partial^{\mu}\phi - \phi\partial^{\mu}\phi^{*}) \ . \qquad (2.3.4)
\end{aligned}
$$

Using equations (2.3.2), the current has a vanishing 4-divergence, as it should

$$
\partial_{\mu} j^{\mu} = 0 \ . \qquad (2.3.5)
$$

This implies that due to the invariance of action under the gauge transformation (2.3.3), there is a conserved quantity. Since α is a constant (global), the gauge transformation is the same for all points in space-time. Physically, it means that suppose a rotation is performed on one point by angle α, then all the other points will also be rotated by an angle α instantaneously. This is not a realistic idea since it violates the law of special relativity. In the next section, we will consider the case where α being space-time dependent and see how it leads to a much more interesting result.

2.3.2 Local Gauge Transformations

Suppose we let α to be space-time dependent, $\alpha(x)$ instead of being a constant. Clearly, Lagrangian (2.3.1) would not be invariant anymore due to the derivative term

$$
\begin{aligned}
\partial_{\mu}\phi(x) \to \partial_{\mu}\phi'(x) &= \partial_{\mu}(e^{i\alpha(x)}\phi(x)) \\
&= e^{i\alpha(x)}[i\partial_{\mu}\alpha(x)\phi(x) + \partial_{\mu}\phi(x)] \ , \qquad (2.3.6)
\end{aligned}
$$

which contains an extra term of $i\partial_{\mu}\alpha(x)\phi(x)$ and this effectively ruins the invariance. To allow the field ϕ to transforms covariantly again, the normal derivative, ∂_{μ} has to be replaced with a covariant derivative, D_{μ} which is defined as

$$D_\mu = \partial_\mu - ieA_\mu \ , \tag{2.3.7}$$

and postulates that A_μ transforms according to $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$. Here, A_μ is the gauge field and it's introduced to make the Lagrangian invariant once more. To illustrate this, it can be seen that the idea of gauge field enables the term $(D_\mu \phi)$ to transform covariantly as itself again

$$\begin{aligned}
(D_\mu \phi) \to (D_\mu \phi)' &= \partial_\mu \phi' - ieA'_\mu \phi' \\
&= e^{i\alpha}\partial_\mu \phi + e^{i\alpha} i\phi(\partial_\mu \alpha) - ieA_\mu e^{i\alpha}\phi - i(\partial_\mu \alpha)\phi e^{i\alpha} \\
&= e^{i\alpha}[\partial_\mu \phi - ieA_\mu \phi] \\
&= e^{i\alpha}(D_\mu \phi) \ .
\end{aligned} \tag{2.3.8}$$

This is called the gauge transformation of the second kind or local gauge transformation (Ryder, 1996). Now, we need to add in the kinetic term for the A_μ to make it dynamical and a suitable term would be the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.3.9}$$

Of course, the field strength tensor is invariant under the local transformation as can be verified by reader. After some modification, the final Lagrangian can be written as

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi \ \phi^* \ . \tag{2.3.10}$$

Expression (2.3.10) is actually the Lagrangian of the Maxwell classical electrodynamics. As an illustration, we use the Euler-Lagrange equation and take the variation of A_μ on (2.3.10)

$$\begin{aligned}
\partial_\nu F^{\mu\nu} &= -ie(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) - 2eA_\mu \phi^* \phi \\
&= -i[\phi^* D^\mu \phi - \phi D_\mu \phi^*] \\
&= J^\mu \ ,
\end{aligned} \tag{2.3.11}$$

where

$$J^{\mu} = -i[\phi^{*} D^{\mu} \phi - \phi D_{\mu} \phi^{*}] \; . \tag{2.3.12}$$

Eq.(2.3.11) is exactly the inhomogeneous Maxwell equations. The electric and magnetic components of the field strength tensor are given by

$$E_i = F_{i0} \; , \tag{2.3.13}$$

$$B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk} \; . \tag{2.3.14}$$

The current here, J^{μ} which is the covariant version of (2.3.4) is conserved by the antisymmetric properties of field strength tensor, $\partial_{\mu} J^{\mu}$ =0.

To summarize the ideas, it can be shown that classical electrodynamics can be derived solely by gauge theory. We see that the electromagnetic field arises naturally just by demanding the invariance of action under the local gauge transformation. By doing so, gauge field A_{μ} is introduced and for G =U(1), this field is identified as the electromagnetic field. The constant e in (2.3.7) is the coupling constant. Besides serving as a conserved quantity, it also measures the interaction between scalar field with electromagnetic field and particles with electromagnetic field.

2.4 Non-Abelian Gauge Theory

After the Abelian group, the next simplest group is the non-Abelian SU(2). This brings the generalization of the gauge group, G from U(1) to SU(2). Let us construct the gauge invariance model for the group SU(2) with two complex scalar fields, forming the column, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathscr{L} = (\partial_{\mu} \phi^{\dagger})(\partial^{\mu} \phi) - m^2 (\phi^{\dagger} \phi) - \lambda (\phi^{\dagger} \phi)^2 \; . \tag{2.4.1}$$

As has been clarified earlier, Lagrangian (2.4.1) is invariant under global gauge transformation. However, we would like to generalize (2.4.1) so that it becomes invariant under the local transformation of group SU(2),

$$\phi(x) \to \phi(x)' = \omega(x)\phi(x) \;\; ; \;\; \omega(x) \in SU(2) \; . \tag{2.4.2}$$

Adapting similar idea as in Abelian case, a covariant derivative is introduced to replace the normal derivative so that under transformation (2.4.2),

$$(D_\mu \phi)' \to \omega(D_\mu \phi) \; . \tag{2.4.3}$$

This is done by introducing a gauge field, A_μ and defining the covariant derivative as

$$D_\mu \phi = \partial_\mu \phi + A_\mu \phi \; . \tag{2.4.4}$$

At such, the transformation rule for A_μ is

$$A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} - (\partial_\mu \omega)\omega^{-1} \; . \tag{2.4.5}$$

The gauge potential, A_μ which is known as Yang-Mills fields take the values of the Lie algebra in group SU(2). Next is to construct a kinetic term for the gauge field by using field strength tensor, $F_{\mu\nu}$. Since the A_μ transforms according to the adjoint representation : $A'_\mu \to \omega A_\mu \omega^{-1}$, the field strength tensor, $F_{\mu\nu}$ should transforms in a similar manner as well, that is

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu}(x) \omega^{-1}(x) \; . \tag{2.4.6}$$

A suitable field strength tensor that transforms according to the adjoint representation is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.4.7}$$

$$= [D_\mu, D_\nu] \; . \tag{2.4.8}$$

The kinetic term invariant Lagrangian for the gauge field can be chosen to be

$$\mathscr{L}_{kin} = \frac{1}{2g^2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \ , \tag{2.4.9}$$

where g^2 is some positive constant.

2.4.1 Gauge Group SU(2) Representation

The gauge field and strength tensor terms which take the values of the SU(2) algebra can be expressed in terms of three real fields (Rubakov, 2002), which equal the total number of generators in gauge group $SU(2)$,

$$\begin{aligned} A_\mu(x) &= -ig\frac{\sigma^a}{2}A^a_\mu(x) \ , & (2.4.10)\\ F_{\mu\nu}(x) &= -ig\frac{\sigma^a}{2}F^a_{\mu\nu}(x) \ , & (2.4.11)\end{aligned}$$

whereby a,b,c are SU(2) internal indices which run from 1 to 3; $A^a_\mu(x)$ are real fields, $\frac{\sigma^a}{2}$ are Hermitian generators of the SU(2) algebra and g is gauge coupling constant. To be precise, σ_a here refers to the famous Pauli matrices which are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.4.12}$$

Pauli matrices have the following algebraic properties

$$\sigma_a\sigma_b = \delta_{ab} + i\varepsilon_{abc}\sigma_c, \quad \mathrm{Tr}(\sigma_a) = 0, \quad \mathrm{Tr}(\sigma_a\sigma_b) = 2\delta_{ab} \ , \tag{2.4.13}$$

and they obey the commutation (square bracket) and anticommutation (braces) relation

$$[\sigma_a,\sigma_b] = 2i\varepsilon_{abc}\sigma_c, \quad \{\sigma_a,\sigma_b\} = 2\delta_{ab}I \ , \tag{2.4.14}$$

where ε_{abc} is the Levi-Civita symbol, δ_{ab} as the Kronecker delta and I is the identity matrix. The field strength tensor can be written in term of real fields by using definition

(2.4.10)

$$\begin{aligned} F_{\mu\nu} &= -ig\frac{\sigma^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g^2 A^a_\mu A^b_\nu\,[\frac{\sigma^a}{2},\frac{\sigma^b}{2}] \\ &= -ig\frac{\sigma^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g^2 A^a_\mu A^b_\nu i\varepsilon^{abc}\frac{\sigma^c}{2} \\ &= -ig\frac{\sigma^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu)\ . \end{aligned} \tag{2.4.15}$$

Comparing with expression (2.4.11), the real components of the strength tensor, $F_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu\ . \tag{2.4.16}$$

Similarly, the kinetic term of (2.4.9) can be expressed as

$$\mathscr{L}_{kin} = \frac{1}{2g^2}F^{a\mu\nu}F^b_{\mu\nu}(-ig)^2\mathrm{Tr}(\frac{\sigma^a}{2}\frac{\sigma^b}{2}) = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\ . \tag{2.4.17}$$

The local SU(2) gauge transformation are usually written in 2×2 matrix form

$$\begin{aligned} \omega(x) &= \exp[\frac{i}{2}\sigma_a\theta_a(x)] \\ &= \cos\frac{1}{2}\theta(x) + i\hat{n}_a(x)\sigma_a\sin\frac{1}{2}\theta(x)\ , \end{aligned} \tag{2.4.18}$$

where the first line is the usual representation of unitary and hermitian matrices and second line is derived using Pauli matrices properties. Here, θ_a is parameter and $\hat{n}_a$ is unit vector defined by

$$\theta_a(x) \equiv \hat{n}_a(x)\theta(x)\ . \tag{2.4.19}$$

It is possible to determine the gauge transformation formulas for components of the gauge potentials. From (2.4.5) and (2.4.10), the pure gauge is given by

$$\begin{aligned} gA^a_\mu &= -i\mathrm{Tr}[\sigma_a(\partial_\mu\omega)\omega^{-1}] \\ &= \frac{1}{2}\mathrm{Tr}\big(\sigma_a\hat{n}_b\sigma_b(\partial_\mu\theta) + \sigma_a(\partial_\mu\hat{n}_b)\sigma_b\sin\theta + 2\varepsilon_{bck}(\partial_\mu\hat{n}_b)\hat{n}_c\sigma_k\sigma_a\sin^2\theta/2\big) \\ &= \hat{n}_a(\partial_\mu\theta) + \sin\theta(\partial_\mu\hat{n}_a) + 2\varepsilon_{abc}(\partial_\mu\hat{n}_b)\hat{n}_c\sin^2(\theta/2)\ . \end{aligned} \tag{2.4.20}$$

Meanwhile, the first term in (2.4.5) is given by the formula (Goddard and Olive, 1978)

$$\omega A_\mu \omega^{-1} = \cos\theta\ \sigma_a A^a_\mu + \sin\theta \varepsilon_{abc} A^a_\mu \hat{n}_b \sigma_c + 2\hat{n}_a A^a_\mu (\hat{n}\cdot\sigma)\sin^2(\theta/2)\ . \quad (2.4.21)$$

Finally, expression (2.4.5) can be written as

$$\begin{aligned} A'^a_\mu &= \cos\theta A^a_\mu + \sin\theta \varepsilon_{abc} A^b_\mu \hat{n}_c + 2\sin^2(\theta/2)\hat{n}_a(\hat{n}_b A^b_\mu) \\ &+ \frac{1}{g}\Big(\hat{n}_a(\partial_\mu\theta) + \sin\theta(\partial_\mu \hat{n}_a) + 2\varepsilon_{abc}(\partial_\mu \hat{n}_b)\hat{n}_c \sin^2(\theta/2)\Big)\ . \end{aligned} \quad (2.4.22)$$

2.4.2 Gauge Field Masses

Taking the kinetic term in (2.4.17) and Lagrangian (2.4.1), we could write the SU(2) gauge-invariant Lagrangian

$$\mathscr{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{1}{2}(D_\mu\phi^a)(D^\mu\phi^a) - m^2(\phi^a\phi^a) - \lambda(\phi^a\phi^a)^2\ . \quad (2.4.23)$$

A quick glance on above Lagrangian shows a serious defect since it hinders the introduction of mass terms for the gauge fields. If mass term such as

$$m^2 A_\mu A^\mu\ , \quad (2.4.24)$$

are put in by hand, it will destroy the gauge invariance since this term is not invariant (Ryder, 1996). Early works were plagued by this problem because the only observable massless gauge field is photon. Meanwhile, experimental results show the presence of massive gauge fields. How then can masses be introduced without destroying the gauge invariance of the Lagrangian? The answer is provided by spontaneous symmetry breaking or in physics jargon, through Higgs mechanism which will be discussed shortly.

2.5 SU(2) Yang-Mills-Higgs Theory

The model that will be used extensively in this thesis is the non-Abelian SU(2) Yang-Mills-Higgs (YMH) model with Yang-Mills fields, A^a_μ coupled with Higgs triplet, Φ^a in adjoint representation in 3+1 dimensions. It is also known as the SU(2) Georgi-Glashow model and was once competitor of the electroweak model before the discovery of 'neutral currents' (Georgi and Glashow, 1972a). A^a_μ and Φ^a are vector and scalar fields respectively under the Lorentz transformation. The Lagrangian for the SU(2) YMH model is given by

$$\mathscr{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}D^\mu\Phi^a D_\mu\Phi^a - V(\Phi)\,, \tag{2.5.1}$$

$$V(\Phi) = \frac{1}{4}\lambda(\Phi^a\Phi^a - \frac{\mu^2}{\lambda})^2\,. \tag{2.5.2}$$

Here, λ is the strength of the Higgs potential and μ is Higgs field mass in which both are constants. The vacuum expectation value of the Higgs field is given by ξ=$\mu/\sqrt{\lambda}$. Lagrangian (2.5.1) is gauge invariant under the set of independent local SU(2) transformations at each space-time point. The covariant derivative of the Higgs field and the gauge field strength tensor are given respectively by

$$D_\mu\Phi^a = \partial_\mu\Phi^a + g\varepsilon^{abc}A^b_\mu\Phi^c\,, \tag{2.5.3}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu\,, \tag{2.5.4}$$

where g is the gauge field coupling constant and the metric used is $g_{\mu\nu} = (-+++)$. By taking the variation with respect to gauge field and Higgs field, the equations of motion emerged from Lagrangian (2.5.1) are

$$D^\mu F^a_{\mu\nu} = \partial^\mu F^a_{\mu\nu} + g\varepsilon^{abc}A^{b\mu}F^c_{\mu\nu} = g\varepsilon^{abc}\Phi^b D_\nu\Phi^c, \tag{2.5.5}$$

$$D^\mu D_\mu\Phi^a = \lambda\Phi^a(\Phi^b\Phi^b - \xi^2)\,. \tag{2.5.6}$$

The symmetric energy-momentum tensor, $T_{\mu\nu}$ which follows from Lagrangian density (2.5.1) is given by (Prasad and Sommerfield, 1975)

$$
\begin{aligned}
T_{\mu\nu} &= F^a_{\mu\lambda}F^{a\lambda}_{\nu} + D_\mu\Phi^a D_\nu\Phi^a + g_{\mu\nu}\mathscr{L} \\
&= F^a_{\mu\alpha}F^{a\alpha}_{\nu} + D_\mu\Phi^a D_\nu\Phi^a - \frac{1}{4}g_{\mu\nu}F^a_{\alpha\beta}F^{a\alpha\beta} \\
&- \frac{1}{2}g_{\mu\nu}D_\alpha\Phi^a D_\alpha\Phi^a - \frac{1}{4}g_{\mu\nu}\lambda(\Phi^a\Phi^a - \xi^2)^2 \,, \qquad (2.5.7)
\end{aligned}
$$

and it is conserved by the virtue of field equation

$$
\partial_\mu T^{\mu\nu} = 0 \,. \qquad (2.5.8)
$$

The static energy or Hamiltonian can be expressed explicitly as

$$
\begin{aligned}
E &= \int T_{00}d^3x = \int (F^a_{0\alpha}F^{a\alpha}_0 + D_0\Phi^a D_0\Phi^a + g_{00}\mathscr{L})\, d^3x \\
&= \int \{E^a_iE^a_i + D_0\Phi^a D_0\Phi^a + \frac{1}{4}(-E^a_iE^a_i - E^a_iE^a_i + 2B^a_iB^a_i) \\
&+ \frac{1}{2}(-D_0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a) + \frac{1}{4}\lambda(\Phi^a\Phi^a - \xi^2)^2\}\, d^3x \\
&= \int \left\{\frac{1}{2}(E^a_iE^a_i + B^a_iB^a_i + D_0\Phi^a D_0\Phi^a + D_i\Phi^a D_i\Phi^a) + V\right\} d^3x \,. \qquad (2.5.9)
\end{aligned}
$$

where the electric and magnetic field are given respectively by

$$
E^a_i = F^a_{i0} \quad \text{and} \quad B^a_i = -\frac{1}{2}\varepsilon_{ijk}F^a_{jk} \,. \qquad (2.5.10)
$$

2.6 Spontaneous Symmetry Breaking

The YMH theory differs from the SU(2) Yang-Mills theory in the sense that it borrows the idea of Higgs-like mechanism to 'spontaneously' breaks the local gauge invariance (Actor, 1979). Even though the Lagrangian (2.5.1) appears to be symmetrical, its ground state is not. The introduction of Higgs fields forces the vacuum to take up a similar direction as the field itself and this effectively breaks the symmetry. Specifically, the SU(2) YMH will be spontaneously broken into subgroup, U(1). As a result, it has one massless boson, corresponds to the unbroken U(1) and with the remaining Yang-Mills bosons being massive.

By looking at the energy functional (2.5.9), we know that the first two terms are minimum when the electric and magnetic fields are equal to zero, $F^a_{\mu\nu} = 0$ or when A_μ is a pure gauge. Subsequent third and fourth terms are minimal when $D_\mu \Phi^a = 0$, meaning that the field, Φ^a is covariantly constant. Finally, minimization of the potential energy tells us that the minima is at

$$\Phi^a_0 = \xi = \frac{\mu}{\sqrt{\lambda}} \; . \tag{2.6.1}$$

For simplicity, we choose the ground state or vacuum field configuration, Φ^a_0 such that

$$\Phi^a_0 = (\Phi^1_0, \Phi^2_0, \Phi^3_0) = (0, 0, \xi) \; . \tag{2.6.2}$$

In practice, to observe a field in the ground state involves some changes of physical quantities in space-time. Therefore, it is crucial to perturb the field around the ground state in order to create some small excitations around it. In the context of field theory, these excitations corresponds to the elementary particles. In other words, perturbation around a ground state will break the symmetry and produce a spectrum of particles.

Consider a perturbation term, $\chi(x)$ around the ground state

$$\Phi^a_0 = (\Phi^1_0, \Phi^2_0, \Phi^3_0) = (0, 0, \xi + \chi(x)) \; . \tag{2.6.3}$$

Substituting Higgs configuration (2.6.3) into the Lagrangian (2.5.1) and the potential term gives (keeping the term up to quadratic order and neglecting constant)

$$\begin{aligned} V(\Phi) &= \frac{1}{4}\lambda(\Phi^a\Phi^a - \xi^2)^2 \\ &= \frac{1}{4}\lambda((\xi + \chi(x))^2 - \xi^2)^2 \approx \frac{1}{4}\lambda(4\xi^2\chi^2) \\ &= \lambda\xi^2\chi^2 \; . \end{aligned} \tag{2.6.4}$$

Assuming a small field A^a_μ, the first and second term in the Lagrangian reduces to quadratic order to just