Possibilistic Functional Dependencies and Their Relationship to Possibility Theory

Sebastian Link and Henri Prade

Abstract—**This paper introduces possibilistic functional dependencies. These dependencies are associated with a particular possibility distribution over possible worlds of a classical database. The possibility distribution reflects a layered view of the database. The highest layer of the (classical) database consists of those tuples that certainly belong to it, while the other layers add tuples that only possibly belong to the database, with different levels of possibility. The relation between the confidence levels associated with the tuples and the possibility distribution over possible database worlds is discussed in detail in the setting of possibility theory. A possibilistic functional dependency is a classical functional dependency associated with a certainty level that reflects the highest confidence level where the functional dependency no longer holds in the layered database. Moreover, the relationship between possibilistic functional dependencies and possibilistic logic formulas is established. Related work is reviewed, and the intended use of possibilistic functional dependencies is discussed in the conclusion.**

Index Terms—**Functional dependency (FD), possibilistic logic, possibility theory, uncertain data.**

I. Introduction

Functional dependencies (FDs) constitute a core notion in database theory [1], for database decomposition, safe updating, redundancy elimination, and query optimization. This fact has led to a great number of works on *fuzzy* functional dependencies (FFDs) in the fuzzy set literature, especially in the 1990s and in the first half of the next decade (see [2], [3], and Section V of this paper for some overview discussions). This is due to the existence of different views of fuzzy databases, as well as different proposals for FFDs. FFDs may be stronger or weaker than classical FDs. They may extend classical FDs to fuzzy databases, or may already differ from classical FDs on classical databases.

The view we investigate in this short note remains close to the one of a classical database where classical FDs hold. We only depart from it by admitting that some tuples may be uncertain, in the sense that we are not sure if some tuple, as it is, belongs or not to the database. This uncertainty may be due to several reasons, for example, when the database gathers tuples from different sources with different confidence levels. The uncertainty of some of the tuples will result in levels of certainty associated with classical FDs. The proposal presented here has some similarity in its basic features with an old one, published more than 20 years ago by Kiss [4], and which has had a limited impact in the literature until now. However, Kiss' proposal was cast in the setting of multiple-valued logic, while the approach in this short note relies on a possibility theory [5], [6] view. Moreover, the possibilistic view makes more precise the meaning of the weights associated with the tuples and the FDs, respectively. It provides a richer semantic characterization of the weighted FDs. We would like to stress that the simple model we propose may be useful for managing databases with uncertain tuples.

This paper is structured as follows. We start with a motivating example in Section II. We then discuss the relation between a possibility distribution over possible database worlds and the confidence in the tuples of a database. We make clear that these confidence levels are degrees of possibility. However, the highest one is also associated with a full certainty degree. The uncertain database is then viewed as a layered database. Section III introduces possibilistic FDs in this setting and establishes properties for them. It is shown that we can reason with the weighted FDs that hold in an uncertain database using possibilistic logic. We establish soundness and completeness theorems for inference from the weighted FDs with respect to the FDs that hold in the level cuts of the uncertain database, or, in other words, with respect to the possibility distribution over possible database worlds and the FDs that hold in each world. Section IV reviews related work that deals either with FFDs, or with classical FDs in fuzzy tuple databases; it also discusses classical FDs in possibilistic uncertain databases. Section V concludes by outlining some potential uses and future developments.

II. Motivating Example

There has been an increase in recognition over recent years that a database may contain uncertain pieces of information, although it has been a concern for a long time [7]–[10]. This uncertainty may take different forms. Attribute values may be imprecise or pervaded with uncertainty, or one may just be uncertain about the fact that a tuple, as it is, should be considered or not as belonging to a database. In the following, we take the latter view. The tuples are standard tuples (without null values), but we do not have full confidence in some of them.

To illustrate the idea, let us consider the database in Table I. It consists of a unique relation r with attributes C (*Course*), T (*Time*), L (*Lecturer*), and R (*Room*). As can be seen, each tuple is associated with a weight α_i. These weights α_i belong

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S. Link is with the Department of Computer Science, The University of Auckland, Auckland 1010, New Zealand (e-mail: s.link@auckland.ac.nz).

H. Prade was with the Department of Computer Science, The University of Auckland, Auckland 1010, New Zealand. He is now with the Institute of Research in Informatics of Toulouse, University of Toulouse, 31062, Toulouse Cedex 9, France, and with the , Centre for Quantum Computation and Intelligent Systems, University of Technology, Sydney, Ultimo, N.S.W. 2007, Australia (e-mail: prade@irit.fr).

TABLE I
EXAMPLE OF AN UNCERTAIN DATABASE

Course	*Time*	*Lecturer*	*Room*	$Poss(t)$
DB	Monday, 9 A.M.	Ann	Aqua	α_1
IS	Monday, 1 P.M.	Ann	Aqua	α_1
CS	Monday, 1 P.M.	Pete	Buff	α_1
CS	Tuesday, 2 P.M.	Pete	Buff	α_1
AI	Tuesday, 4 P.M.	Gill	Buff	α_1
AI	Wednesday, 3 P.M.	Gill	Cyan	α_1
Math	Thursday, 4 P.M.	Mary	Lava	α_3
Logic	Thursday, 4 P.M.	Mary	Pink	α_3
HCI	Friday, 9 A.M.	Bob	Tan	α_4
OR	Friday, 9 A.M.	Bob	Tan	α_4
OR	Friday, 9 A.M.	Jack	Tan	α_4

to a linearly ordered scale $S = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$ with $\alpha_1 > \cdots > \alpha_n > \alpha_{n+1}$. They may be encoded numerically, e.g., $\alpha_1 = 1, \alpha_2 = 0.8, \ldots, \alpha_n = 0.2, \alpha_{n+1} = 0$, but this is not compulsory. Indeed, a numerical encoding will have no particular meaning beyond the ordering of the numbers. These levels may also receive a linguistic reading. We shall come back to that in the next section.

Clearly, this encoding suggests a layer-based view of the relation r: We have first the tuples with the highest confidence level α_1, followed by those with a smaller confidence level (in the example α_3), and so on (in the example, we have a third layer with level α_4). It also implicitly suggests a possibility distribution over possible database worlds. How this distribution can be related to the weights α_i is discussed in the following.

III. Relating Possible Database Worlds and Confidence in Tuples

The problem we are facing is how to relate a possibility distribution over a *power set* of tuples to a distribution over a set of tuples. Although this kind of problem has not been considered very often, it already received an answer many years ago in [11]. We first recall these results using the motivating example used at that time [12], namely the representation of an imprecise and uncertain information about a multiple-valued attribute, here, the set of languages spoken by a person.

A. Possibility Distribution on a Power Set and Its Upper and Lower Approximations

For instance, we have the partial information that "John speaks either English and French, or English and German, and no other languages." In that case, it can be described by a two-valued possibility distribution π defined over the power set $2^{\mathcal{L}}$ of the set of languages $\mathcal{L}$, namely let $A_1 = \{English, French\}$, and let $A_2 = \{English, German\}$; then, we have $\pi(A_1) = \pi(A_2) = 1$ and $\pi(A_k) = 0$ for any $k \neq 1, 2$. Clearly, this information has an upper approximation by the set of languages *possibly* spoken by John, here $A^+ = \{English, French, German\}$, and the set of languages *certainly* spoken by John, here $A^- = \{English\}$ is a lower approximation. Note that this is only an approximation of the information conveyed by the original distribution π over $2^{\mathcal{L}}$, since we have lost the information that John speaks (only) two languages. However, the two approximations are now distributions over $\mathcal{L}$. This is simpler, namely, $\mu_{A^+}(l) = 1$ if $l \in \{English, French, German\}$ and $\mu_{A^+}(l) = 0$ otherwise, while $\mu_{A^-}(l) = 1$ if $l = English$ and $\mu_{A^-}(l) = 0$ otherwise.

This can be generalized to multiple-valued possibility distributions [12]. Let π be a mapping from a power set $2^{\mathcal{L}}$ (we keep the same notation, but $\mathcal{L}$ now denotes any set) to a linearly ordered scale S, where 1 and 0 continue to denote the top and the bottom element, respectively. We assume that π is normalized, i.e., $\sup_{i \in I} \pi(A_i) = 1$ (where I is an index set for the subsets in $2^{\mathcal{L}}$). The upper and lower approximations of the ill-known set described by π are defined, respectively, by

$$\mu_{A^+}(l) = \sup_{i: l \in A_i} \pi(A_i) \quad (1)$$

$$\mu_{A^-}(l) = 1 - \sup_{i: l \notin A_i} \pi(A_i) = \inf_{i: l \notin A_i} (1 - \pi(A_i)) \quad (2)$$

where the complementation $1 - (\cdot)$ denotes a mapping from $S = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$ with $\alpha_1 = 1 > \cdots > \alpha_n > \alpha_{n+1} = 0$ into scale $S' = \{\beta_1, \ldots, \beta_n, \beta_{n+1}\}$ with $\beta_1 = 1 > \cdots > \beta_n > \beta_{n+1} = 0$, such that $\beta_1 = 1 - (\alpha_{n+1}), \ldots, \beta_i = 1 - (\alpha_{n+2-i}), \ldots, \beta_{n+1} = 1 - (\alpha_1)$. When S is a subset of $[0, 1]$, $1 - (\cdot)$ is just the complementation to 1; otherwise, it is the order-reversing map of the scale S (for S finite). Since S is a *possibility* scale, S' is a *certainty* scale (the distinction between S and S' is important since the duality between possibility and certainty (necessity) is essential in possibility theory).

Equation (2) means that we are all the more certain that $l \in \mathcal{L}$ belongs to the ill-known set A described by π, i.e., $\mu_{A^-}(l)$ is all the higher, as it is impossible to find an A_i such that $l \notin A_i$. Similarly, it is all the more possible that $l \in \mathcal{L}$ belongs to the ill-known set A, i.e., $\mu_{A^+}(l)$ is all the higher, as there exists an A_i such that $l \in A_i$ having a high possibility level. The quantity $1 - \mu_{A^+}(l)$ is called by Yager [13] "rebuff measure," since it expresses to what extent l is impossible to be an element of A.

B. Some Linkage With Evidence Theory

The construction made here is reminiscent of Shafer's [14] setting for his evidence theory, where he starts with a mass function m, called "basic probability assignment" defined over the subsets A_i of some referential, say $\mathcal{L}$, which is such that $\sum_i m(A_i) = 1$. Then, m is nothing but the representation of a random subset A of $\mathcal{L}$. Then, a so-called *contour function* can be defined as $c(l) = \sum_{i: l \in A_i} m(A_i)$, which represents the plausibility that l belongs to A. Due to the probabilistic normalization of m, note that we also have $c(l) = 1 - \sum_{i: l \notin A_i} m(A_i)$. Here, the construct is similar, except that m is replaced by a *possibilistic mass function* π, and $\sum$ is replaced by sup to agree with the idea of possibility. Such a qualitative counterpart of Shafer evidence theory was first suggested in [15] (see [16] for recent developments). Then, the contour function splits into upper and lower approximation functions, i.e., μ_{A^+} and μ_{A^-}, respectively, which no longer coincide. Still, the following strong inclusion of the fuzzy set A^- in A^+ can be checked:

$$\forall l \in \mathcal{L}, \mu_{A^-}(l) > 0 \Rightarrow \mu_{A^+}(l) = 1.$$

It can also be observed that if $\mu_{A^-}(l)$ is interpreted as the certainty that l belongs to A (the ill-known set represented by π), namely $\mu_{A^-}(l) = cert(l \in A)$, the expected duality between possibility and certainty holds, namely, $\mu_{A^+}(l) = 1 - cert(l \in \overline{A})$. Indeed, if the ill-known set A is represented by the possibility distribution $\{(A_i, \pi(A_i))|i \in I\}$ (where I is an index set) over $2^{\mathcal{L}}$, then its complement $\overline{A}$ should be represented by $\{(B_i, \overline{\pi}(B_i))|i \in I\}$, where the possibility distribution $\overline{\pi}$ is defined by $\forall i \in I, \overline{\pi}(\overline{A_i}) = \pi(A_i)$, i.e., the possibility degrees are now allocated to the complement subsets. Then, $1 - cert(l \in \overline{A}) = 1 - \mu_{\overline{A}^-}(l) = 1 - (1 - \sup_{i:l \notin \overline{A_i}} \pi(A_i)) = \sup_{i:l \in A_i} \pi(A_i) = \mu_{A^+}(l)$.

C. Recovering the Possibility Distribution on the Power Set

We have shown how a normalized possibility distribution π over $2^{\mathcal{L}}$ induces upper and lower approximation functions over $\mathcal{L}$ for the information conveyed by π. Conversely, since (A^-, A^+) is only an approximation of the information contained in $\{(A_i, \pi(A_i))|i \in I\}$, there are several possibility distributions over $2^{\mathcal{L}}$ in general that agree with (A^-, A^+) in the sense of (1) and (2). This can be easily seen using the example already considered at the beginning of Section III-A. Take again $A^- = \{English\}$ and $A^+ = \{English, French, German\}$; other examples of possibility distributions over $2^{\mathcal{L}}$, distinct from π, the one already given, are $\pi'(\{English\}) = \pi'(\{English, French\}) = \pi'(\{English, German\}) = \pi'(\{English, French, German\}) = 1$, while $\pi'(B) = 0$ for any other subset B of $\mathcal{L}$, or $\pi''(\{English\}) = \pi''(\{English, French, German\}) = 1$, while $\pi''(B) = 0$ for any other $B \subseteq \mathcal{L}$. Note that π'' fully differs from π given in Section III-A. However, it can be shown that there exists a *unique* possibility distribution, which is the largest one in the sense of the fuzzy set inclusion defined on $2^{\mathcal{L}}$ ($\pi \subseteq \pi' \iff \forall i \in I, \pi(A_i) \leq \pi'(A_i)$). This is the least committed one (since it does not arbitrarily weaken the possibility level of any subset). This possibility distribution is defined by

$$\pi^*(B) = \min(\inf_{l \in B} \mu_{A^+}(l), \inf_{l \notin B}(1 - \mu_{A^-}(l))). \qquad (3)$$

This equation is easy to understand, a subset B is all the more possible, as both all its elements are possible, and no elements outside B are certain. Entering π^* in (1) and (2), we recover μ_{A^+} and μ_{A^-}. In the previous example, it can be checked that π^* is nothing but the possibility distribution π' given above.

D. Application to Layered Databases

We can now apply these results to our original problem. Here, we consider subsets of tuples $t \in \mathcal{T}$; therefore, these subsets are in $2^{\mathcal{T}}$, which plays the role of $2^{\mathcal{L}}$ in the previous sections. The possibility distribution π_r associated with the relation r is now defined as (denoting B a subset of tuples)

$$\pi_r(B) = \alpha_i \text{ if } \exists i, B = r_{\alpha_i}; \pi_r(B) = 0 \text{ otherwise} \qquad (4)$$

where $r_{\alpha_i} = \{t \in r | c(t) \geq \alpha_i\}$ is the cut of level α_i of the relation r, and $c(t)$ is the confidence level associated with tuple t. Thus, the different possible database worlds are precisely the level cuts of the fuzzy relation induced by the confidence weights. Any other possible database world that would not coincide with such level cuts has a possibility level equal to $\alpha_{n+1} = 0$. Note that the level cuts are nested, i.e., $r_{\alpha_i} \subseteq r_{\alpha_{i+1}}$, and thus, r_{α_1} is included in any possible database world that has a nonzero possibility level.

Applying (1) and (2) to the distribution defined by (4), we get

$$c^+(t) = \sup_{t \in r_{\alpha_i}} \alpha_i = \sup_{B:t \in B} \pi_r(B) \ (= \alpha_i \text{ if } t \in r_{\alpha_i} \text{ but } t \notin r_{\alpha_{i-1}}) \qquad (5)$$

$$c^-(t) = \begin{cases} \inf_{B:t \notin B}(1 - \pi_r(B)) = \alpha_1 = 1, \text{ if } t \in r_{\alpha_1} \\ \alpha_{n+1} = 0, \quad \text{otherwise.} \end{cases} \qquad (6)$$

This means that with the exception of the tuples that are in r_{α_1}, which are certainly in the database, the other tuples are only*possibly* in the database r; the possibility levels $c^+(t)$ then correspond exactly to the confidence levels, i.e., $c^+(t) = c(t)$.

Now, applying (3), we get

$$\pi^*(B) = \min(\inf_{t \in B} c^+(t), \inf_{t \notin B}(1 - c^-(t))). \qquad (7)$$

The distribution π^* coincides with the original distribution π_r for the subsets corresponding to the level cuts of r, i.e., $\forall B = r_\alpha, \pi^*(B) = \pi_r(B)$. Indeed, $\inf_{t \in r_\alpha} c^+(t) = \alpha$ and $\inf_{t \notin B}(1 - c^-(t)) = 0$ for any B that fails to include some t in r_{α_1}; otherwise, $\inf_{t \notin B}(1 - c^-(t)) = 1$. However, as in the spoken language example, π^* is larger than the possibility distribution we start with, namely here $\pi^* > \pi_r$. Indeed, for any B that contains r_{α_1} and that is a *strict* subpart of some level cut r_{α_k}, which is not itself a level cut of higher level (i.e., $B \neq r_{\alpha_j}$ for any $1 \leq j \leq k$), we have $\pi^*(B) = \alpha_k$, while $\pi_r(B) = 0$. Still, we have $\bigcup_{B:\pi^*(B)=\alpha} B = r_\alpha$.

Thus, the distribution π_r over $2^{\mathcal{T}}$ can be recovered from the pair (c^+, c^-) of upper and lower contour functions defined on $\mathcal{T}$, although π_r is smaller than the least committed distribution π^* on $2^{\mathcal{T}}$ associated with this pair. In the perspective of studying FDs in an uncertain database, it is natural to work with π_r, and thus with the level cuts r_α, since one should consider the tuples having a level of possibility at least equal to α *altogether* (for each α), which corresponds to the layer-based view of the relation r introduced at the beginning. Viewed in terms of the pair (c^+, c^-), the relation r has a fully certain subpart, namely r_1, which gathers all tuples t such that $c^+(t) = c^-(t) = 1$, while the rest of the relation is partitioned into the subsets of tuples t such that $c^+(t) = \alpha$ and $c^-(t) = 0$, for $\alpha_2 \leq \alpha \leq \alpha_n$.

The α_i's may now receive a proper linguistic counterpart. Since they are possibility levels, one may interpret them on a linguistic scale such that (taking, e.g., $n = 4$) α_1 = "*fully possible*," α_2 = "*quite possible*," α_3 = "*medium possible*," α_4 = "*somewhat possible*," α_5 = "*not at all possible*."

Since a database whose tuples are associated with confidence levels has now received a clear interpretation in the setting of possibility theory, we are in a position to study what the concept of an FD means in this setting. This approach promotes the idea to keep confidence levels fully qualitative in practice.

IV. Possibilistic Functional Dependencies

An FD $X \to Y$, where X and Y are sets of attributes, is a constraint of the form $\forall t, t' \in r,\ t.X = t'.X \Rightarrow t.Y = t'.Y$. It is obvious that if an FD holds in a database, it also holds in any subpart of the original database. Here, our layered set of tuples results in a nested sequence of possible database worlds. Therefore, if an FD holds in $r_{\alpha_{i+1}}$, the FD also holds in r_{α_i}. Conversely, if an FD does not hold in r_{α_i}, then the FD does not hold in $r_{\alpha_{i+1}}$.

Thus, if we examine the example of Table I, we can check that $CT \to R$ holds everywhere, namely in r_{α_4}, $C \to L$ and $RT \to C$ holds in r_{α_3}, and $LT \to C$ in $r_{\alpha_1} = r_{\alpha_2}$ only. This suggests to attach a certainty level to an FD, such that the FD is all the more certain as it holds in a larger database world provided that it is possible to some extent.

A. Defining Possibilistic Functional Dependencies

The above discussion leads to the following definition for the certainty level of an FD

$$Cert_r(fd) = 1 - \sup\{\pi(r_{\alpha_i}) | fd \text{ does not hold in } r_{\alpha_i}\} \quad (8)$$

where fd denotes an FD, and we have $\pi(r_{\alpha_i}) = \alpha_i$. Equation (8) is nothing but the *necessity* of the event "fd holds in r" with respect to the possibility distribution π_r, since by definition, the necessity $N(p)$ of a statement p is equal to $1 - \Pi(\neg p)$, which corresponds to 1 minus the possibility, i.e., to the impossibility of the opposite event "fd does not hold in r." Thus, if fd fails to hold in $r_{\alpha_{i+1}}$, but holds in r_{α_i}, $Cert_r(fd) = 1 - \pi(r_{\alpha_{i+1}}) = 1 - \alpha_{i+1} = \beta_{n+1-i}$ (indeed, the possibility that fd fails is the *greatest* possibility to be in a database world where fd fails, since possibility is maxitive [5], [6]). Thus, $Cert_r(fd) = \beta_1 = 1 - \alpha_{n+1} = 1$ if fd holds for any level cut of r. Note also that in particular, we get $Cert_r(fd) = 0$ if fd fails to hold in r_{α_1}; in fact, since the tuples in r_{α_1} are not only fully possible, but also fully certain, there is no possibility at all that fd holds in a database world having a nonzero possibility level, and thus, it is fully certain that fd fails to hold. Besides, in case the scales S and S' are included in $[0, 1]$, we just have $Cert_r(fd) = 1 - \alpha_{i+1}$ where now $1 - (\cdot)$ is the usual complementation to 1, as soon as fd fails to hold in $r_{\alpha_{i+1}}$, but holds in r_{α_i}. We call a classical FD associated with a certainty level a *possibilistic FD*.

Coming back to our example, it can be checked that the set Σ made up of the previously mentioned FDs associated with their certainty weights is

$$\Sigma = \{(CT \to R, \beta_1); (C \to L, \beta_2); (RT \to C, \beta_2); (LT \to C, \beta_3)\}.$$

Then, the following proposition can be stated:

$$\models_r (X \to Y, c) \Leftrightarrow \models_{r_{\underline{1-c}}} X \to Y \quad (9)$$

where $\models_r (X \to Y, c)$ means that $Cert_r(X \to Y) \geq c$ and where $r_{\underline{\alpha}}$ denotes the *strict* α-level cut of r, namely $r_{\underline{\alpha}} = \{t | c(t) > \alpha\}$. This proposition is easy to prove. First observe that $\models_r (X \to Y, c)$ entails $\models_r (X \to Y, c')$ as soon as $c \geq c'$. If $Cert_r(X \to Y) \geq c$, it follows from Definition 8 that $X \to Y$ may be violated at most in r_{1-c}, but certainly not in $r_{\underline{1-c}}$. Conversely, if $X \to Y$ holds for any level cut of r of level strictly greater than $1 - c$, $Cert_r(X \to Y)$ cannot be less than $1 - (1 - c)) = c$.

The careful definition of the concept of a possibilistic FD which is fully justifiable in terms of possibility theory is also of great potential in database practice. In particular, it allows us to take full advantage of previous results on classical dependencies, which we will explore in future work. For example, if a relation satisfies a classical FD, then that relation can be decomposed into two of its projections without loss of information [17], [18]. More generally, if a possibilistic relation satisfies a possibilistic FD with certainty c, then the strict level cut of the possibilistic relation with level $1 - c$ can be decomposed into two of their projections without loss of information.

B. Relation With Possibilistic Propositional Logic

It is well known [19]–[21] that FDs in classical databases have a simple propositional logic counterpart in terms of Horn clauses. In fact, the following holds:

$$\models_r \ \{A_1, \ldots, A_k\} \to B \Leftrightarrow \forall t, t' \in r, \models_{\omega_{\{t,t'\}}} \neg A'_1 \vee \cdots \vee \neg A'_k \vee B' \quad (10)$$

where $A'_1, \ldots, A'_k, B'$ are propositional variables associated with attributes $A_1, \ldots, A_k, B$, respectively, and $\omega_{\{t,t'\}}(A') =$ `True` if $\forall i, t.A_i = t'.A_i$ and $\omega_{\{t,t'\}}(A') =$ `False` otherwise. Equation (10) expresses that a given relation satisfies a given FD if and only if for all pairs of tuples in the relation, the special truth assignment derived from that pair is a Boolean model for the propositional Horn clause associated with the FD. Indeed, (10) can be seen as a semantic justification for the definition of the special truth assignment $\omega_{t,t'}$ that assigns to each propositional variable A' the value `True` iff tuples t, t' have the same instantiation on attribute A. This semantically relates the identity of tuples to propositional variable formulas expressing the counterparts of FDs. Moreover, (10) can be used to prove that a dependency statement is a consequence of a set of dependency statements if and only if the corresponding implicational statement is a consequence of the corresponding set of implicational statements [19].

This result extends to our setting, just as propositional logic extends to possibilistic logic [22]. Let us first have a brief refresher on possibilistic logic. A (standard) propositional possibilistic logic formula is a pair (p, β), where p is proposition and β is a certainty level. At the semantic level, it corresponds to the semantic constraint $N(p) \geq \beta$, where N is a necessity measure, associated with a possibility distribution π on the set of interpretations Ω in the following way $N(p) = \inf_{\omega \not\models p} 1 - \pi(\omega)$. The lower the possibility of an interpretation that makes p `False`, the higher the necessity degree of p. Therefore, given a formula (p, β), an interpretation ω that makes p `True` is possible at the maximal level in the scale S, say 1, while an interpretation ω that makes p `False` is at most possible at level $1 - \beta$. A possibilistic logic knowledge base K is a collection of possibilistic logic formulas, namely $K = \{(p_i, \beta_i) | i = 1, \ldots, n\}$, whose semantic counterpart is $\pi_K(\omega) = \min_{i=1,\ldots,n} \max(1 - \beta_i, [p_i](\omega))$, where $[p](\omega) = 1$ if $\omega \models p$ and where $[p](\omega) = 0$ otherwise.

Then, in possibilistic logic, the following soundness and completeness theorem holds:

$$\vDash_K (p, \beta) \Leftrightarrow \vdash_K (p, \beta) \Leftrightarrow \vdash_{K_\beta} p \Leftrightarrow \vDash_{K_\beta} p$$

where $\vDash_K (p, \beta)$ means $\forall \omega, \pi_K(\omega) \leq \pi_{\{(p,\beta)\}}(\omega)$, and $K_\beta = \{p_i | (p_i, \beta_i) \in K \text{ and } \beta_i \geq \beta\}$. Therefore, the last half of the above expression reduces to the soundness and completeness theorem of propositional logic, applied to each level cut of K, which is an ordinary propositional logic knowledge base. Finally, $\vdash_K (p, \beta)$ refers to the syntactic part of possibilistic logic, which relies on the repeated use of the resolution rule $(\neg p \vee q, \beta), (p \vee r, \gamma) \vdash (q \vee r, \min(\beta, \gamma))$. It is also interesting to notice that, due to the characteristic property of necessity measures, i.e., $N(p \wedge q) = \min(N(p), N(q))$, a possibilistic logic base can be easily put in clausal form.

Thus, we have seen that the semantics for the possibilistic logic formula (p, β) amounts to rank-order interpretations according to the possibility distribution $\pi_{\{(p,\beta)\}}$, where $\pi_{\{(p,\beta)\}}(\omega) = 1$ if $\omega \vDash p$ (i.e., ω makes p `True`) and $\pi_{\{(p,\beta)\}}(\omega) = 1 - \beta$ if ω is an interpretation that makes p `False` (i.e., $\omega \nvDash p$). Going back to possibilistic FDs, interpretations now refer to pairs of tuples, but one may have a similar construct. The counterpart of equivalence (10) can be stated in the following way:

$$\vDash_r (\{A_1, \ldots, A_k\} \rightarrow B, \beta) \Leftrightarrow \forall t, t' \in r^*, \vDash_{\pi_{\{t,t'\}}} (\neg A'_1 \vee \cdots \vee \neg A'_k \vee B', \beta) \quad (11)$$

where r denotes a possibilistic database (in the sense of this paper), and r^* is the same database without the levels. The notation $\vDash_{\pi_{\{t,t'\}}}$ in (11) reminds us that the semantics of a possibilistic propositional logic base is no longer in terms of truth assignment as in propositional logic, but in terms of a possibility distribution induced by the possible failure of the certainty-qualified propositions in the base, as recalled above; the index $\{t, t'\}$ points out that the semantics of propositional variables pertains to pairs of tuples here. Thus, the possibility distribution $\pi_{\{t,t'\}}$ over logical interpretations accounts for the possible failure of the FD in the possibilistic database. Indeed, the distribution $\pi_{\{t,t'\}}$ is defined in the following way:

1) $\pi_{\{t,t'\}}(\omega^*_{\{t,t'\}}) = \min(\alpha, \alpha')$, with $c(t) = \alpha$, $c(t') = \alpha'$, if (t, t') violates $\{A_1, \ldots, A_k\} \rightarrow B$ in $r_{\min(\alpha,\alpha')}$.
2) $\pi_{\{t,t'\}}(\omega^*_{\{t,t'\}}) = 0$, if (t, t') satisfies $\{A_1, \ldots, A_k\} \rightarrow B$ in r^*.
3) $\pi_{\{t,t'\}}(\omega) = 1$ for all $\omega \neq \omega^*_{\{t,t'\}}$.

Here, the interpretations ω are the ones induced by the literals $A'_1, \ldots, A'_k, B'$ (where A'_i is `True` iff $t.A_i = t'.A_i$, and B' is `True` iff $t.B = t'.B$), and $\omega^*_{\{t,t'\}}$ is the particular interpretation $A'_1 \ldots A'_k \neg B'$ (where $A'_1 \cdots A'_k$ are `True` and B' is `False`) that falsifies $\neg A'_1 \vee \cdots \vee \neg A'_k \vee B'$.

Proof of (11): Let $\varphi = \{A_1, \ldots, A_k\} \rightarrow B$, and $\varphi' = \neg A'_1 \vee \cdots \vee \neg A'_k \vee B'$. When (t, t') violates φ it means that $\min(\alpha, \alpha') \leq 1 - \beta$ assuming $\vDash_r (\varphi, \beta)$. Since $\pi_{\{(\varphi',\beta)\}}(\omega_{\{t,t'\}}) = 1 - \beta$ and $\pi_{\{(\varphi',\beta)\}}(\omega) = 1$ for all $\omega \neq \omega_{\{t,t'\}}$, it is clear that we have $\forall \omega, \pi_{\{t,t'\}}(\omega) \leq \pi_{\{(\varphi',\beta)\}}(\omega)$. Conversely, if this later inequality holds, there cannot exist t, t' such that $\min(\alpha, \alpha') > 1 - \beta$, and thus, $Cert_r(\varphi) \geq \beta$, i.e., $\vDash_r (\varphi, \beta)$. Q.E.D.

The above result indicates that Horn clauses in possibilistic propositional logic are the counterparts of possibilistic FDs, just as Horn clauses in Boolean propositional logic are the counterparts of FDs.

V. Related Work

The literature on FFDs is quite abundant. It is not the place here to survey it in detail, and some overview papers exist [2], [3], [23] for the first decade of literature on the topic. We first briefly mention the main existing types of FFDs and then compare in detail the proposed approach to a somewhat similar proposal, which originates from a different perspective. In the second part of this section, we discuss FDs in the context of the possible world semantics of another type of possibilistic databases.

A. Fuzzy Functional Dependencies

FFDs may refer to a quite large variety of situations. First, we may consider classical databases (where one mines FDs with satisfaction degrees [24], or fuzzy approximate dependencies [25]), or databases with precise attribute values but weighted tuples, or databases with fuzzy attribute values, or still fuzzy similarity-based relational databases (moreover, the database may have null values [26]). Then, we may either study classical FDs on weighted tuple databases or on fuzzy attribute value databases [27] or even fuzzy values with imprecise membership functions [28], or we may consider FFDs on classical databases [29] as well as on more general databases allowing for weighted tuples, fuzzy attribute values, or fuzzy values defined by means of fuzzy similarity relations [30]–[41]. For instance, the authors in [36] use fuzzy closeness relations between ill-known attribute values represented by possibility distributions and relate closeness degrees in the condition part of the FD's to closeness degrees in their conclusion part by means of Gödel implication (i.e., $a \rightarrow_G b = 1$ if $a \leq b$, and $a \rightarrow_G b = b$ otherwise). Such a generalized view of an FD $X \rightarrow Y$ may express not only that equal Y-values follow from equal X-values, but also that close Y-values follow from close X-values, for different closeness levels. Such a concern, discussed in [2], has nothing to do with the possibilistic FD's discussed here.

FFDs have been also considered in relation with a fuzzy Entity-Relationship model [42]. FFDs may be stronger or weaker than classical FDs depending on whether they are adding further constraints to the one conveyed by a classical FD (such as ordered FDs that agree with orderings existing in attribute domains [43], or gradual FDs [44]), or whether they weaken the constraint associated with a classical FD. Clearly, all these different options may serve different goals [2], which may depart from the role of classical FDs for database design in classical databases (such as data summarization [45], building of linguistic summaries [46], or a Bayesian network [47]).

However, in this short note, we are not dealing with any FFDs of any kind. The proposal made here is motivated by the idea that FDs may fail to hold in the presence of some tuples in which we have not full confidence. This might be related to the idea of partial FDs [48], where FDs hold up to exceptions whose number may be quantified. However, here, we take advantage of the confidence levels of the tuples for accommodating the exceptions. There has been another proposal made more than two decades ago, by Kiss [4] for dealing with classical FDs in a weighted tuple database, viewed as a fuzzy relation r. The author computes the degree of truth with which an FD $X \to Y$ holds, in the following way (where μ denotes membership functions):

$$\mathtt{Truth}(X \to Y) = \min_{\{t,t'\}}(\min(\mu_r(t), \mu_r(t'), \mu_=(t.X, t'.X)) \Rightarrow_L \mu_=(t.Y, t'.Y)),$$

where $\mu_=$ denotes the exact equality relation, and $\Rightarrow_L$ is Łukasiewicz implication. An easy computation leads to

$$\mathtt{Truth}(X \to Y) = 1 - \sup_{t,t':t.X=t.X \text{ and } t.Y\neq t'.Y} \min(\mu_r(t), \mu_r(t')).$$

Reorganizing the weighted tuples into layers of decreasing degrees, we see that the above formula coincides with our definition of $Cert_r(X \to Y)$, and indeed, $X \to Y$ holds in any level cut r_α of r such that $\alpha > 1 - \mathtt{Truth}(X \to Y)$. However, this simple multiple-valued logic view has no clear interpretation from an uncertainty modeling point of view, while a possible database world perspective also enables us to get a possibilistic logic counterpart. Moreover, interestingly enough, the author wrote about his proposal some years after: "The so-defined fuzzy relations can be handled mathematically well, but they have less practical importance" [49]. On the point of usefulness, we disagree with this view. Indeed, just as possibilistic logic is a valuable extension of propositional logic, one may expect that certainty-based FDs with a layer-based view of databases can help to control the normalization of the decomposition process of uncertain relations.

B. Functional Dependencies in Possibilistic Databases. Discussing the Meaning of the Levels

In this short note, we have emphasized the relationship between the levels attached to the tuples and the associated possibility distribution over possible database worlds. Several authors have pointed out the interest of seeing a possibilistic database as a set of classical databases associated with possibility degrees. When the possibilistic database is a database where attribute values are fuzzy (i.e., for each tuple and each attribute, we have a possibility distribution restricting the possible values), the possibility degrees associated with database worlds can be computed from the possibility degrees attached to the possible attribute values chosen for building each classical database compatible with the possibilistic database. One may then precisely define the possibility degree and the necessity degree with which a particular FD holds in the possibilistic database [50].

As can be seen, we have not used here this view of a possibilistic database. However, let us consider the particular case where all the attribute values of each tuple t would be precise but uncertain, with the same certainty level β_t, which would correspond to particular possibility distributions equal to 1 for the precise value, and equal to $1 - \beta_t$ everywhere else. Then, the database would contain only certainty-qualified values in the sense studied in [51] and [52]. Since here the certainty of all the attribute values is the same for a given tuple, one can associate this certainty level to the whole tuple (without losing any information), in agreement with the min-decomposability of necessity measures. Thus, what is obtained looks a bit like the possibilistic database considered in this note, except that tuples are now associated with *certainty levels* rather with *possibility levels*. Therefore, one may wonder, if an approach similar to the one presented here, but with certainty levels, would not be interesting as well. The answer is negative. This is because as soon as an FD is violated in r^* (the database without the certainty levels here), there would be a fully possible world where the FD is violated, and then, the FD would have no certainty, and one cannot reason in a possibilistic logic manner with FDs that are just possible to some extent. Besides, if we only consider relations r where the FDs are not violated in $r*$, we would be in a position to associate a certainty level with the FDs, but it would always be the same, namely the minimal value of all the certainty values attached to tuples in r, which is not very interesting. This confirms that the approach taken here with *possibility levels* is the right one if one does not want to trivialize the approach.

VI. Concluding Remarks

This short note has introduced the notion of possibilistic FDs based on the idea of a classical database, layered according to possibility levels attached to tuples, and where the first layer is the only certain one. We have shown that in such a case, the associated possibility distribution over possible database worlds is uniquely determined by the possibility levels attached to tuples, and *vice versa*. This has led us to associate certainty levels with FDs in a natural way. Furthermore, this definition allows us to extend the well-known propositional logic counterpart of FDs in the setting of possibilistic logic.

The notion of possibilistic FDs proposed here seems particularly appealing for use in database practice. Indeed, the layered view of the database together with the different levels of certainty of the FDs suggest their use in the control of the decomposition process of relations in third normal forms, or in Boyce–Codd normal forms, which can then be layered. The full investigation of these issues, with the study of the weighted counterpart of Armstrong's system of axioms, is the topic of a companion paper [53] and patent application [54]. Moreover, possibilistic keys [55] have been investigated as an important special case of possibilistic FDs and correspond to goal Horn clauses via (11). Besides, rather than starting with a layered database, and computing the certainty levels associated with FDs, one may also think of doing the converse, namely starting with a set of more or less certain FDs that should hold in a classical database, and looking for a stratification of the database which agrees with the certainty levels of the FDs.

References

[1] W. W. Armstrong, "Dependency structures of data base relationships," in *Proc. Proc. IFIP Congr.*, 1974, pp. 580–583.

[2] P. Bosc, D. Dubois, and H. Prade, "Fuzzy functional dependencies and redundancy elimination," *J. Amer. Soc. Inf. Syst.*, vol. 49, no. 3, pp. 217–235, 1998.

[3] S. B. Yahia, H. Ounalli, and A. Jaoua, “An extension of classical functional dependency: Dynamic fuzzy functional dependency,” *Inf. Sci.*, vol. 119, nos. 3/4, pp. 219–234, 1999.

[4] A. Kiss, “λ-decomposition of fuzzy relational databases,” *Ann. Univ. Budapest, Sec. Comput.*, vol. 12, pp. 133–142, 1991.

[5] D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. New York, NY, USA: Plenum, 1988.

[6] L. A. Zadeh, “Fuzzy sets as a basis for a theory of possibility,” *Fuzzy Sets Syst.*, vol. 1, pp. 3–28, 1978.

[7] E. Wong, “A statistical approach to incomplete information in database systems,” *ACM Trans. Database Syst.*, vol. 7, no. 3, pp. 470–488, 1982.

[8] A. Motro and P. Smets, Eds.,*Uncertainty Management in Information Systems: From Needs to Solutions*. Norwell, MA, USA: Kluwer, 1996.

[9] S. Abiteboul, R. Agrawal, P. A. Bernstein, M. J. Carey, S. Ceri, W. B. Croft, D. J. DeWitt, M. J. Franklin, H. Garcia-Molina, D. Gawlick, J. Gray, L. M. Haas, A. Y. Halevy, J. M. Hellerstein, Y. E. Ioannidis, M. L. Kersten, M. J. Pazzani, M. Lesk, D. Maier, J. F. Naughton, H.-J. Schek, T. K. Sellis, A. Silberschatz, M. Stonebraker, R. T. Snodgrass, J. D. Ullman, G. Weikum, J. Widom, and S. B. Zdonik, “The Lowell database research self-assessment,” *Commun. ACM*, vol. 48, no. 5, pp. 111–118, 2005.

[10] D. Suciu, D. Olteanu, C. Ré, and C. Koch, *Probabilistic Databases*. San Rafael, CA, USA: Morgan & Claypool, 2011.

[11] D. Dubois and H. Prade, “Incomplete conjunctive information,” *Comput. Math. Appl.*, vol. 15, pp. 797–810, 1988.

[12] H. Prade and C. Testemale, “Representation of soft constraints and fuzzy attribute values by means of possibility distributions in databases,” in *Analysis of Fuzzy Information. Vol. II: Artificial Intelligence and Decision Systems*, J. C. Bezdek, Ed. Boca Raton, FL, USA: CRC Press, 1987, pp. 213–229.

[13] R. R. Yager, “On the knowledge structure of multi-solution variables, including quantified statements,” *Int. J. Approx. Reason.*, vol. 1, pp. 23–70, 1987.

[14] G. Shafer, *A Mathematical Theory of Evidence*. Princeton, NJ, USA: Princeton Univ. Press, 1976.

[15] D. Dubois and H. Prade, “Evidence measures based on fuzzy information,” *Automatica*, vol. 21, pp. 547–562, 1985.

[16] H. Prade and A. Rico, “Possibilistic evidence,” in *Proc. 11th Eur. Conf. Symbolic Quantitative Approaches Reason. Uncertainty*, 2011, pp. 713–724.

[17] J. D. Ullman, *Principles of Database and Knowledge-Base Systems, Volume II*. New York, NY, USA: Comput. Sci. Press, 1989.

[18] S. Abiteboul, R. Hull, and V. Vianu, *Foundations of Databases*. Reading, MA, USA: Addison-Wesley, 1995.

[19] R. Fagin, “Functional dependencies in a relational data base and propositional logic,” *IBM J. Res. Develop.*, vol. 21, no. 6, pp. 543–544, 1977.

[20] S. Hartmann and S. Link, “Characterising nested database dependencies by fragments of propositional logic,” *Ann. Pure Appl. Logic*, vol. 152, nos. 1/3, pp. 84–106, 2008.

[21] S. Hartmann and S. Link, “The implication problem of data dependencies over SQL table definitions: Axiomatic, algorithmic and logical characterizations,” *ACM Trans. Database Syst.*, vol. 37, no. 2, pp. 17:1–17:40, 2012.

[22] D. Dubois, J. Lang, and H. Prade, “Automated reasoning using possibilistic logic: Semantics, belief revision, and variable certainty weights,” *IEEE Trans. Knowl. Data Eng.*, vol. 6, no. 1, pp. 64–71, Feb. 1994.

[23] P. Bosc, D. Dubois, and H. Prade, “Fuzzy functional dependencies—An overview and a critical discussion,” in *Proc. 3rd IEEE Int. Conf. Fuzzy Syst.*, Orlando, FL, USA, Jun. 26–29, 1994, pp. 325–330.

[24] Q. A.Wei, G. Q. Chen, and X. C. Zhou, “Properties and pre-processing strategies to enhance the discovery of functional dependency with degree of satisfaction,” *Control Cybern.*, vol. 38, no. 2, pp. 367–394, 2009.

[25] F. Berzal, I. Blanco, D. Sánchez, J. M. Serrano, and M. A. Vila, “A definition for fuzzy approximate dependencies,” *Fuzzy Sets Syst.*, vol. 149, no. 1, pp. 105–129, 2005.

[26] S. Liao, H. Wang, and W. Liu, “Functional dependencies with null values, fuzzy values, and crisp values,” *IEEE Trans. Fuzzy Syst.*, vol. 7, no. 1, pp. 97–103, Feb. 1999.

[27] P. Bosc and O. Pivert, “On the impact of regular functional dependencies when moving to a possibilistic database framework,” *Fuzzy Sets Syst.*, vol. 140, no. 1, pp. 207–227, 2003.

[28] A. Lu and W. Ng, “Maintaining consistency of vague databases using data dependencies,” *Data Knowl. Eng.*, vol. 68, no. 7, pp. 622–641, 2009.

[29] S. Al-Hamouz and R. Biswas, “Fuzzy functional dependencies in relational databases,” *Int. J. Comput. Cognition*, vol. 4, no. 1, pp. 39–43, 2006.

[30] K. V. S. V. N. Raju and A. K. Majumdar, “Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems,” *ACM Trans. Database Syst.*, vol. 13, no. 2, pp. 129–166, 1988.

[31] B. Bhuniya and P. Niyogi, “Lossless join property in fuzzy relational databases,” *Data Knowl. Eng.*, vol. 11, no. 2, pp. 109–124, 1993.

[32] J. C. Cubero, J. M. Medina, O. Pons, and M. A. Vila, “Weak and strong resemblance in fuzzy functional dependencies,” in *Proc. 3rd IEEE Conf. Fuzzy Syst.*, 1994, vol. 1, pp. 162–166.

[33] J. C. Cubero, J. M. Medina, O. Pons, and M. A. Vila, “Fuzzy loss less decompositions in databases,” *Fuzzy Sets Syst.*, vol. 97, pp. 145–167, 1998.

[34] J. C. Cubero, J. M. Medina, O. Pons, and M. A. Vila, “Non-transitive fuzzy dependencies (i),” *Fuzzy Sets Syst.*, vol. 106, no. 3, pp. 401–431, Sep. 1999.

[35] G. Q. Chen, E. Kerre, and J. Vandenbulcke, “The dependency-preserving decomposition and a testing algorithm in a fuzzy relational data model,” *Fuzzy Sets Syst.*, vol. 72, no. 1, pp. 27–37, 1995.

[36] G. Q. Chen, E. E. Kerre, and J. Vandenbulcke, “Normalization based on fuzzy functional dependency in a fuzzy relational data model,” *Inf. Syst.*, vol. 21, no. 3, pp. 299–310, 1996.

[37] L. Wei-Yi, “Constraints on fuzzy values and fuzzy functional dependencies,” *Inf. Sci.*, vol. 78, nos. 3/4, pp. 303–309, 1994.

[38] W.-Y. Liu, “Fuzzy data dependencies and implication of fuzzy data dependencies,” *Fuzzy Sets Syst.*, vol. 92, no. 3, pp. 341–348, 1997.

[39] M. I. Sözat and A. Yazici, “A complete axiomatization for fuzzy functional and multivalued dependencies in fuzzy database relations,” *Fuzzy Sets Syst.*, vol. 117, no. 2, pp. 161–181, 2001.

[40] B. K. Tyagi, A. Sharfuddin, R. N. Dutta, and D. K. Tayal, “A complete axiomatization of fuzzy functional dependencies using fuzzy function,” *Fuzzy Sets Syst.*, vol. 151, no. 2, pp. 363–379, 2005.

[41] R. Belohlávek and V. Vychodil, “Data tables with similarity relations: Functional dependencies, complete rules and non-redundant bases,” in *Proc. 11th Int. Conf. Database Syst. Adv. Appl.*, Singapore, Apr. 12–15, 2006, pp. 644–658.

[42] N. A. Chaudhry, J. R. Moyne, and E. A. Rundensteiner, “An extended database design methodology for uncertain data management,” *Inf. Sci.*, vol. 121, nos. 1/2, pp. 83–112, 1999.

[43] W. Ng, “Ordered functional dependencies in relational databases,” *Inf. Syst.*, vol. 24, no. 7, pp. 535–554, 1999.

[44] D. Rasmussen and R. R. Yager, “Finding fuzzy and gradual functional dependencies with SummarySQL,” *Fuzzy Sets Syst.*, vol. 106, no. 2, pp. 131–142, Sep. 1999.

[45] J. C. Cubero, J. M. Medina, O. Pons, and M. A. V. Miranda, “Data summarization in relational databases through fuzzy dependencies,” *Inf. Sci.*, vol. 121, nos. 3/4, pp. 233–270, 1999.

[46] P. Bosc, L. Liétard, and O. Pivert, “Extended functional dependencies as a basis for linguistic summaries,” in *Proc. 2nd Eur. Symp. Principles Data Mining Knowl. Discovery*, Nantes, France, Sep. 23–26, 1998, pp. 255–263.

[47] W. Y. Liu and N. Song, “Fuzzy functional dependencies and Bayesian networks,” *J. Comput. Sci. Technol.*, vol. 18, no. 1, pp. 56–66, 2003.

[48] F. B. Galiano, J. C. Cubero, F. Cuenca, and J. M. Medina, “Relational decomposition through partial functional dependencies,” *Data Knowl. Eng.*, vol. 43, no. 2, pp. 207–234, 2002.

[49] T. Nikovits, A. Kiss, and D. Chretien, “Representation and query languages of fuzzy relational databases,” *Annal. Univ. Budapest, Sec. Comput.*, vol. 17, pp. 293–306, 1998.

[50] P. Bosc and O. Pivert, “Functional dependencies over possibilistic databases: An interpretation based on the possible worlds semantics,” in *Proc. 3rd VLDB Workshop Manag. Uncertain Data*, Lyon, France, Aug. 28, 2009, pp. 1–16.

[51] P. Bosc, O. Pivert, and H. Prade, “A model based on possibilistic certainty levels for incomplete databases,” in *Proc. 3rd Int. Conf. Scalable Uncertainty Manag.*, Washington, DC, USA, Sep. 28–30, 2009, pp. 80–94.

[52] O. Pivert and H. Prade, “A certainty-based model for uncertain databases,” *IEEE Trans. Fuzzy Syst.*, vol. 23, no. 4, pp. 1181–1196, Aug. 2015.

[53] S. Link and H. Prade, “Relational database schema design for uncertain data,” Univ. Auckland, Auckland, New Zealand, Tech. Rep. CDMTCS-469, 2014.

[54] S. Link and H. Prade, “Database schema design generation system and method,” Patent 1 457 862, Aug. 18, 2014.

[55] H. Köhler, U. Leck, S. Link, and H. Prade, “Logical foundations of possibilistic keys,” in *Proc. 14th Eur. Conf. Logics Artif. Intell.*, Madeira, Portugal, Sep. 24–26, 2014, pp. 181–195.