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A capacity-based framework encompassing Belnap–Dunn logic for reasoning about multisource information [☆,☆☆]

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ABSTRACT

Belnap–Dunn four-valued logic is one of the best known logics for handling elementary information items coming from several sources. More recently, a conceptually simple framework, namely a two-tiered propositional logic augmented with classical modal axioms (here called BC logic), was suggested by the second author and colleagues, for the handling of multisource information. It is a fragment of the non-normal modal logic EMN, whose semantics is expressed in terms of two-valued monotonic set functions called Boolean capacities. We show BC logic is more expressive than Belnap–Dunn logic by proposing a consequence-preserving translation of Belnap–Dunn logic in this setting. As special cases, we can recover already studied translations of three-valued logics such as Kleene and Priest logics. Moreover, BC logic is compared with the source-processor logic of Avron, Ben Naim and Konikowska. Our translation bridges the gap between Belnap–Dunn logic, epistemic logic, and theories of uncertainty like possibility theory or belief functions, and paves the way to a unified approach to various methods for handling inconsistency due to several conflicting sources of information.

Keywords:
Belnap–Dunn logic
Inconsistency handling
Kleene and Priest logics
Capacities
Possibility theory
Modal logic

1. Introduction

Reasoning with inconsistent information is a topic that has emerged in the last thirty years as a key issue in logic-based Artificial Intelligence, because the usual approach in mathematics according to which from a contradiction everything follows is not useful in practice. Due to the large amount of data available today, inconsistencies are unavoidable, and people often cope with inconsistent information in daily life. Many works have been published proposing approaches to deal with inconsistent knowledge bases in such a way as to extract useful information from it in a non-explosive way. See [52,12,15] for surveys.

Belnap–Dunn four-valued logic [32,10,9] is one of the earliest approaches to this problem. It is based on a very natural set-up whereby several sources of information tentatively declare some elementary propositions to be true or false. The set of truth-values thus collected for each proposition from all sources is summarized by a so-called *epistemic truth-value* referring to whether sources are in conflict or not, informed or not about this elementary proposition. There are four such

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epistemic truth-values, two of which refer to ignorance and conflict. Truth tables for conjunction, disjunction, and negation are used to compute the epistemic status of other more complex formulas. This logic underlies both Kleene three-valued logic [40] (recovered when no conflict between sources is observed) and the three-valued Priest Logic of Paradox [46,47] (when sources are never ignorant and always assign truth-values to each elementary proposition). In the former logic, the truth-value referring to the idea of "contradiction" is eliminated, while in the latter, the truth-value referring to the idea of "ignorance" is eliminated. Belnap–Dunn logic uses all four values forming a bilattice structure, in which one partial ordering expresses relative strength of information (from no information to too much information), and the other partial ordering represents relative truth (from false to true).

In recent works [20,21], we have focused attention on three-valued logics of uncertainty or inconsistency (paraconsistent logics). We were able to express them in a two-tiered propositional logic called MEL [8], which adopts the syntax of a fragment of epistemic logic, and borrows axioms of the KD logic. MEL can be viewed as the simplest logical framework for reasoning about incomplete information and at the semantic level it expresses the all-or-nothing version of possibility theory [28]. This kind of two-tiered logic dates backs to works by Hájek and colleagues [38] who tried to capture probability and other uncertainty logics, using a two-tiered logic, with propositional logic at the lower level and a multivalued logic at the higher level. The idea is that the degree of truth of an atomic statement *p is probable* is the degree of probability of p. This scheme has been systematized in [34,18]. The logic MEL is in the same vein, albeit using propositional logic at both levels, and a modal language.

In the MEL logic, it is possible to express that an atomic proposition, or its complement, is unknown (when translating Kleene logic, for instance), or that it is contradictory (when translating Priest Logic of Paradox, for instance). The main merit of such translations in [20,21] is to lay bare the meaning of the various three-valued connectives usually defined by means of truth tables, and especially to exhibit the limited expressive power of such three-valued logics. Namely, modalities can only appear in front of literals in the translations. These results facilitate the practical use of three-valued logics and the MEL logic seems to offer a unified logical framework for all of them, where only usual binary truth-values are needed. In this paper we focus on the case of Belnap–Dunn logic, for which we propose a similar translation into a two-tiered propositional setting that borrows axioms from a non normal modal logic and supports both ideas of conflict and ignorance. Indeed, the setting of KD logics is not general enough to capture Belnap–Dunn approach.

A more general modal framework for multiple source information, here called BC logic, first outlined by the second author [24], and proved sound and complete in [30], is recalled. It is based on a fragment of the non-normal modal logic EMN [17]. At the semantic level this logic accounts for the notion of Boolean capacity, i.e., set functions valued on $\{0, 1\}$ that are monotonic under inclusion. This logic brings us closer to uncertainty theories and can encompass variants of probabilistic and belief function logics, for instance the logic of risky knowledge [43], where the adjunction rule is not valid. Our framework seems to be tailored for reasoning from multiple source information.

Then, we show that our two-tiered propositional setting related to the EMN logic can encode Belnap–Dunn four-valued logic, namely that the four truth-values in this logic are naturally represented by means of capacities taking values on $\{0, 1\}$. We provide a consequence-preserving translation of this four-valued logic into BC. Thus, we construct a bridge between Belnap–Dunn logic and uncertainty theories. As special cases, we recover our previous translations of Kleene logic for incomplete information [20] and Priest Logic of Paradox [21].

This translation is not a gratuitous exercise. BC logic has potential to support a number of inconsistency handling approaches and to act as a unifying framework for several logics of uncertainty and conflicting information; so, it is important to show that the best known logic for inconsistency handling is encompassed. The framework of BC logic is more expressive than the one of Belnap–Dunn logic, because it does away with the assumption that sources provide information about propositional variables only, and it does not require truth-functionality assumptions. The source-processor logic of Avron et al. [6] also does away with the first restriction but it still requires a weak form of truth functionality. We discuss the connections between that setting and ours. The BC logic has thus potential to encompass various approaches to inconsistency-tolerant formalisms, and is easier to relate to information fusion methods that are proposed in uncertainty theories [26], due to the use of monotonic set functions. For instance, a possible extension of Belnap set-ups with belief functions was already suggested in [23].

This paper is organized as follows. Section 2 presents the propositional logic of Boolean capacities BC and shows its capability to capture the notion of information coming from several sources. Section 3 recalls Belnap–Dunn four-valued logic from the point of view of its motivation, its syntactical inference and its semantics, as well as some of its three-valued extensions, i.e., Kleene and Priest logics. Section 4 contains the main results pertaining to the translation of Belnap–Dunn logic into BC. Results in this section were more succinctly announced in a former conference paper [22]. Section 5 shows how Kleene and Priest logic translations in [20,21] can be retrieved by translating additional axioms or inference rules into BC. Section 6 discusses some related approaches that deal with conflicting information, including the general source processor logic of [6]. We provide some proofs about Belnap–Dunn logic in Appendix A, for the sake of self-containedness.

2. The logic of Boolean capacities and multisource information management

In this section, we consider a general approach to the handling of pieces of incomplete and conflicting information coming from several sources. We show, following an intuition already suggested in [5,24,30], that if we represent each body of information items supplied by one source by means of a set of possible states of affairs, the collective information supplied by the sources can be modeled in a lossless way by a monotonic set function called a capacity, that takes value on $\{0, 1\}$. These set functions can serve as natural multi-source semantics for a simple non-normal modal logic in which we shall capture Belnap–Dunn four truth-values as already suggested in [24]. This logic, formally studied in [30] is driven by what we believe is the minimal set of axioms to express multisource reasoning. It can naturally serve as a target language to express Belnap–Dunn, Kleene and Priest logics, as well as the source processor logic of Avron et al. [6], and other approaches as well.

2.1. Boolean capacities and multisource information

Consider a set of states of affairs Ω which may be the set of interpretations of a propositional language. The complement of a subset $A \subseteq \Omega$ is denoted by A^c.

Definition 1. A capacity (or fuzzy measure) is a mapping $\gamma : 2^\Omega \to [0, 1]$ such that $\gamma(\emptyset) = 0$; $\gamma(\Omega) = 1$; and if $A \subseteq B$ then $\gamma(A) \leq \gamma(B)$.

This capacity is supposed to inform us about what the state of the world, say s, is. The value $\gamma(A)$ can be interpreted as the degree of support of a proposition of the form $s \in A$. For a textbook on (numerical) capacities, see Grabisch [37]. A Boolean capacity (B-capacity, for short) is a capacity with values in $\{0, 1\}$ [30]. It can be defined from a usual capacity and any threshold $\lambda > 0$ as $\beta(A) = 1$ if $\gamma(A) \geq \lambda$ and 0 otherwise.

The useful information in a B-capacity consists of its focal sets. A focal set E is such that $\beta(E) = 1$ (hence not empty) and $\beta(E \setminus \{w\}) = 0, \forall w \in E$. Let $\mathcal{F}_\beta$ be the set of focal sets of β. $\mathcal{F}_\beta$ collects all minimal sets for inclusion such that $\beta(E) = 1$. $\mathcal{F}_\beta$ is thus an antichain (no inclusion between sets). We can check that $\beta(A) = 1$ if and only if there is a subset E of A in $\mathcal{F}_\beta$ with $\beta(E) = 1$. So, the focal sets of a Boolean capacity are the necessary and sufficient information to recover the capacity.

We can interpret a Boolean capacity as the information provided by a set of sources. Consider n sources providing information in the form of non-empty sets $E_i \subseteq \Omega$ that are supposed to contain the actual world s after each source. In other words, E_i can be viewed as the epistemic state of the source i: it is only known from source i that the real state of affairs s should lie in E_i. A capacity β can be built from these pieces of information letting $\beta(A) = 1$ when $\exists i$ such that $E_i \subseteq A$ and 0 otherwise. Then $\beta(A) = 1$ really means that there is at least one source i that claims that $s \in A$ is true (that is, $s \in A$ is true in all states of affairs in E_i). This way of synthesizing information is not destructive, since it preserves every non-redundant initial piece of information: the set of focal sets $\mathcal{F}_\beta$ is clearly included in $\{E_1, E_2, \ldots, E_n\}$ as it contains minimal sets E_i for inclusion.

Given a proposition $s \in A$, there are four epistemic statuses based on the information from sources, that the capacity can express:

- *Support*: $\beta(A) = 1$ and $\beta(A^c) = 0$. Then $s \in A$ is asserted by at least one source and negated by no other one.
- *Rejection*: $\beta(A^c) = 1$ and $\beta(A) = 0$. Then $s \in A$ is negated by at least one source and asserted by no other one.
- *Ignorance*: $\beta(A) = \beta(A^c) = 0$. No source asserts nor negates $s \in A$.
- *Conflict*: $\beta(A) = \beta(A^c) = 1$. Some sources assert $s \in A$, some negate it.

Important special cases of Boolean capacities are [30]:

- when there is a single focal set: $\mathcal{F}_\beta = \{E\}$. This is equivalent to having β *minitive*, i.e., $\beta(A \cap B) = \min(\beta(A), \beta(B))$. It is then a *necessity measure* [27]. There is only one source and its information is incomplete, but there is no conflict.
- when the focal sets are singletons $\{w_1\}, \{w_2\}, \ldots, \{w_n\}$. This is equivalent to having β is *maxitive*, i.e., $\beta(A \cup B) = \max(\beta(A), \beta(B))$. All sources have complete information, so there are conflicts, but no ignorance. Letting $E = \{w_1, w_2, \ldots, w_n\}$, then $\beta(A) = 1$ if and only if $A \cap E \neq \emptyset$; so, formally β is *a possibility measure* [27]. But here E is a conjunction of non-mutually exclusive elements (fully informed sources), not a possibility distribution.
- β is both maxitive and minitive if and only if its unique focal set is a singleton. This is usually called a *Dirac function*, representing complete (deterministic) information.

2.2. The multisource set-up in the propositional setting

This approach can be described in more syntactic terms. Consider a standard propositional language $\mathcal{L}$ with variables $V = \{a, b, c, \ldots\}$ and connectives $\wedge, \vee, \neg$, for conjunction, disjunction and negation, respectively. We denote the propositional formulas of $\mathcal{L}$ by letters p, q, r. The truth set is $\{0, 1\}$ and we denote by $v : V \to \{0, 1\}$ a truth-assignment. The set of interpretations of $\mathcal{L}$ is Ω. The set of models of p is denoted by $[p] \subseteq \Omega$. The information provided by each source can then equivalently take the form of a consistent set K_i of propositional formulas claimed to be true. If we assume each source is logically sophisticated,[1] then K_i is equivalently modeled by the set E_i of its models. Moreover, for any formula p, source

[1] That is, is capable to reason according to propositional logic.

i declares that p is true (1) whenever $K_i \vdash p$, false (0) whenever $K_i \vdash \neg p$, and is silent otherwise. Then, we can again compute the status of all propositions with respect to the sources, using propositional inference, i.e., decide whether any proposition p is

- *supported* if for some source i, $K_i \vdash p$ and $K_i \vdash \neg p$ for no other source;
- *rejected* if for some source i, $K_i \vdash \neg p$ and $K_i \vdash p$ for no other source;
- *unknown* if p is neither supported nor rejected, i.e., for all sources j, $K_j \nvdash p$ and $K_j \nvdash \neg p$;
- *conflicting* if p is both supported and rejected, i.e., for some source i, $K_i \vdash p$ and for another one j, $K_j \vdash \neg p$.

This setting is somewhat similar to the one of Konieczny and Pino-Pérez [41,42] who consider the problem of syntax-independent merging of sets of propositional formulas that may contradict each other. Their aim is to compute a unique consistent closed set of formulas as the result of the merging process. They provide rationality axioms to do so, and examples of merging operations that follow these principles. However thus doing, the method is destructive, namely, it does away with some pieces of information provided by sources, in order to restore consistency. In contrast, the capacity-based information processing set-up proposed here, while it is also syntax-independent, is non-destructive. Indeed, all information items provided by the sources remain unaltered (unless redundant): we do not solve inconsistency, we only confine it to some conclusions.

Example 1. This example is inspired by one proposed by Revesz [50] in the literature on knowledge base merging. There are three variables $\{a, b, c\}$. There are three sources which declare the following formulas and their consequences as being true:

- $K_1 = \{a \vee c, \neg b\}$ with models $E_1 = \{100, 001, 101\}$
- $K_2 = \{\neg a, (b \vee \neg c) \wedge (c \vee \neg b)\}$ with models $E_2 = \{000, 011\}$
- $K_3 = \{a, b\}$ with models $E_3 = \{110, 111\}$

where models are denoted by triples of truth-values for a, b, c in this order. The reader can check that, following our approach, c is unknown (implied by none of the bases), $a \wedge b$ is conflicting (implied by K_3 but $\neg a \vee \neg b$ is implied by K_2), $a \wedge b \wedge c$ is rejected (implied by none of the bases but its negation is implied by K_1 and K_2) as well as $a \wedge \neg b$ (implied by none of the bases but its negation is implied by K_2 and K_3), $a \vee b$ and $a \vee c$ are supported (implied by K_3, but rejected by none).

2.3. The logic BC

In this section, we recall the logical framework for multiple source information, proposed in [24,30], which accounts for the above capacity-based framework. Consider a two-tiered propositional language $\mathcal{L}_\Box$ based on another propositional language $\mathcal{L}$. The atomic propositions of $\mathcal{L}_\Box$ form the set $\mathcal{V}_\Box = \{\Box p : p \in \mathcal{L}\}$. The language $\mathcal{L}_\Box$, whose formulas are denoted by Greek letters $\phi, \psi, \ldots$, is then of the form:

- If $p \in \mathcal{L}$ then $\Box p \in \mathcal{L}_\Box$;
- if $\phi, \psi \in \mathcal{L}_\Box$ then $\neg\phi \in \mathcal{L}_\Box$, $\phi \wedge \psi \in \mathcal{L}_\Box$.

Note that the language $\mathcal{L}$ is embedded inside (but disjoint from) $\mathcal{L}_\Box$. As usual $\Diamond p$ stands for $\neg\Box\neg p$. It defines a very elementary fragment, proposed by Banerjee and Dubois [7,8], of a modal logic language.

A minimal epistemic logic with conflicts using the language $\mathcal{L}_\Box$ defined above has been proposed [24]. It is a two-tiered propositional logic augmented with some modal axioms:

(PL) All axioms of propositional logic for $\mathcal{L}_\Box$-formulas;
(RM) $\Box p \to \Box q$ if $\vdash p \to q$ in PL;
(N) $\Box p$, whenever p is a propositional tautology;
(P) $\Diamond p$, whenever p is a propositional tautology.

The only inference rule is modus ponens: If ψ and $\psi \to \phi$ then ϕ.

This is a fragment of the non-normal logic EMN [17]. In particular the axiom (RE): $\Box p \equiv \Box q$ if and only if $\vdash p \equiv q$ is valid. Note that the two dual modalities $\Box$ and $\Diamond$ play the same role. Namely the above axioms remain valid if we exchange $\Box$ and $\Diamond$. So these modalities are not distinguishable.

Semantics of non-normal logics are usually expressed in terms of neighborhoods, which attach a family of subsets of interpretations to each possible world [17]. Here we do not need this complex semantics since modalities are not nested. Capacities on the set of interpretations Ω of the language $\mathcal{L}$ encode a set of subsets A such that $\beta(A) = 1$, which is a special case of world-independent neighborhood containing Ω and closed under inclusion (after axiom RM).

We call this formalism *BC logic*, where BC stands for Boolean capacities.[2]

Models in this logic are Boolean capacities. A BC-model of an atomic formula $\Box p$ for this modal logic is then a Boolean capacity β such that $\beta([p]) = 1$. The satisfaction of BC-formulas $\phi \in \mathcal{L}_\Box$ is then defined recursively as usual:

- $\beta \models \Box p$, if and only if $\beta([p]) = 1$;
- $\beta \models \neg\phi$, $\beta \models \phi \wedge \psi$ are defined in the standard way.

Satisfiability can be equivalently expressed in terms of the focal sets of β, as follows:

$$\beta \models \Box p \quad \text{iff} \quad \exists E \in \mathcal{F}_\beta, E \subseteq [p].$$

So one might as well define a model in this logic as any antichain of subsets that form a family of focal sets of a capacity, thus laying bare the multiple-source nature of this logic. The antichain condition can be relaxed to any n-tuple of non-empty subsets $(E_1, \ldots, E_n), \forall n > 0$. However, several such tuples may generate the same Boolean capacity.

Semantic entailment is defined classically, and syntactic entailment is classical propositional entailment taking RM, N, P as axiom schemata: $\Gamma \vdash_{BC} \phi$ if and only if $\Gamma \cup \{\text{all instances of } RM, N, P\} \vdash \phi$ (classically defined). It has been proved that BC logic is sound and complete wrt Boolean capacity models [30]. In fact, axiom RM clearly expresses the monotonicity of capacities, and it is easy to realize that a classical propositional interpretation of $\mathcal{L}_\Box$ that respects the axioms of BC can be precisely viewed as a Boolean capacity. BC is a special case of modal logic of uncertainty with a two-layered syntax in the sense of [34,18].

2.4. Special case: incomplete information

A first important special case of BC is when there is only one source with epistemic state E. Then the corresponding capacity N, called necessity measure in possibility theory [27], is minitive and defined by $N(A) = 1$ if $E \subseteq A$ and 0 otherwise. The logic BC particularized to necessity measures is the logic MEL [8]. The following KD axioms and inference rule are valid in MEL under this restriction to a single source:

- All axioms of propositional logic for $\mathcal{L}_\Box$-formulas.
- (K): $\Box(p \to q) \to (\Box p \to \Box q)$.
- (N): $\Box p$, whenever p is a propositional tautology.
- (D): $\Box p \to \Diamond p$.
- Modus Ponens: If $\phi, \phi \to \psi$ then ψ.

Axiom (C): $\Box(p \wedge q) \equiv (\Box p \wedge \Box q)$, which is the adjunction axiom (the Boolean form of the minitivity axiom) is valid in this system. Satisfiability reduces to $E \models \Box p$ if and only if $E \subseteq [p]$. This logic is sound and complete for this standard system of modal axioms of the logic KD. Note that the conflicting situation cannot be expressed using necessity measures since it never holds that $N([p]) = N(\neg[p]) = 1$. In previous publications, we have shown that Kleene logic can be encoded in MEL [20]. Classical logic can be encoded in MEL as well, if we restrict the language to conjunctions of atomic propositions $\Box p \in \mathcal{L}_\Box$: it has been shown that $\{\Box p_1, \ldots, \Box p_k\} \vdash \Box q$ in MEL if and only if $\{p_1, \ldots, p_k\} \vdash q$ in classical logic [8].

Given a Boolean capacity β, let N_i denote the Boolean capacity induced by the single focal set E_i. Then, it is easy to check that the Boolean capacity induced by the tuple $(E_1, \ldots, E_n)$ is of the form $\beta(A) = \max_{i=1}^n N_i(A)$ [30]. Denote by $\Box_i$ the KD modality associated to the necessity measure N_i. As a consequence, for any capacity of BC-model, there are KD models $\{E_1, \ldots, E_n\}$ and KD modalities $\Box_1, \ldots \Box_n$ such that $\beta \models_{BC} \Box p$ if and only if $\exists i \in \{1, \ldots, n\}$ such that $E_i \models_{KD} \Box_i p$.

We can also consider dropping axiom K in MEL. In that case, only Axiom D remains, which still entails that $\Box p \wedge \Box\neg p$ is always a contradiction, while $\Diamond p \vee \Diamond\neg p$ is a tautology. In terms of Boolean capacities, axiom D corresponds to requiring that a Boolean capacity β is dominated by its conjugate $\beta^c(A) = 1 - \beta(A^c)$, i.e., $\beta(A) \leq \beta^c(A)$, for all $A \subseteq \Omega$. We call such a Boolean capacity *pessimistic* while its conjugate β^c is then said to be *optimistic* [29]. For pessimistic (resp. optimistic) capacities, it is equivalent to have $\min(\beta(A), \beta(A^c)) = 0$ (resp. $\max(\beta(A), \beta(A^c)) = 1$) for all $A \subseteq \Omega$.

Pessimistic Boolean capacities were characterized by means of their focal sets in [29]:

Proposition 1. *A Boolean capacity obeys* $\min(\beta(A), \beta(A^c)) = 0$ *for all* $A \subseteq \Omega$ *if and only if its focal sets intersect.*

Viewing the focal sets as information coming from various sources, it is clear that the sources leading to pessimistic Boolean capacities are globally coherent since the information items do not contradict each other. If there were two non

[2] In [30], it is called QC logic, where QC stands for qualitative capacities, i.e., mappings $\gamma : 2^\Omega \to L$, where L is a finite bounded chain with bottom 0 and top 1. A logic for qualitative capacities would extend the one for Boolean ones using atomic formulas of the form $\Box_\lambda p$ standing for $\gamma([p]) \geq \lambda$, for $\lambda \in L \setminus \{0\}$. QC logic is a natural name for this multimodal setting, while here we use $\Box_\lambda p$ with a fixed λ, which comes down to using a Boolean capacity. Such a QC logic would be to BC logic what generalized possibilistic logic GPL [31] is to the logic of necessity measures MEL [8], mentioned in the introduction.

overlapping focal sets E and F there would be some set A containing E and disjoint from F, and then $\min(\beta(A), \beta(A^c)) = 1$. So, no conflict situation ($E_i \subseteq A, E_j \subseteq A^c$ for some $i \neq j$) can be expressed by pessimistic capacities. However, information items from the sources can be incomplete.

Remark 1. In BC, we can define a new modality $\blacksquare$ such that $\blacksquare p \equiv \Box p \wedge \Diamond p$ whose dual is $\blacklozenge p \equiv \Box p \vee \Diamond p$. It is clear that $(\blacksquare, \blacklozenge)$ satisfies the axioms of BC and also axiom D. It reflects the fact that given any B-capacity β, the set function $\beta_*(A) = \min(\beta(A), \beta^c(A)), \forall A \subset \Omega$ is a pessimistic B-capacity [29].

2.5. Special case: conflicting information

An opposite interesting special case is when the focal sets are singletons $\{w_1\}, \ldots, \{w_n\}$. This situation corresponds to n totally informed but conflicting sources, each of which is capable of asserting that any proposition p is true or false, while they are in total disagreement. Then, as seen earlier, the corresponding Boolean capacity Π is a possibility measure defined by $\Pi(A) = 1$ if and only if $\exists w_i \in A$, which is equivalent to $\Pi(A) = 1$ if and only if $A \cap E \neq \emptyset$, with $E = \{w_1, \ldots, w_n\}$.

In previous publications, we have shown that Priest logic can be encoded in MEL [19–21], provided that $\Pi([p]) = 1$ is understood as "at least one source in E supports p" and is expressed as $\Diamond p$, contrary to the convention used here for BC logic. The fact that in Priest logic $p \vee \neg p$ is a tautology is then expressed by the fact that $\Diamond p \vee \Diamond \neg p$ is a tautology in MEL. As we use possibility measures for modeling conflict, the situation of ignorance cannot be expressed using possibility measures under this convention since it never holds that $\Pi([p]) = \Pi(\neg[p]) = 0$.

Consider dropping axiom K, and adding only D to BC axioms from the MEL setting devoted to Priest Logic. Then, keeping the convention of $\beta([p]) = 1$ as $\Box p$ when β is pessimistic, and $\Diamond p$ if it is optimistic, we note that axiom D now reflects the property $\max(\beta(A), \beta(A^c)) = 1$ as characteristic of optimistic capacities. We can specify this notion using focal sets (a result not highlighted in [29]):

Proposition 2. *A Boolean capacity obeys* $\max(\beta(A), \beta(A^c)) = 1$ *for all* $A \subseteq \Omega$ *if and only if it possesses at least one focal set that is a singleton disjoint from the other focal sets.*

Proof. If no focal set E_i is a singleton, then one can choose A such that neither A nor its complement contain any E_i (splitting each focal set into two non-empty parts). Then $\beta(A) = \beta(A^c) = 0$. Suppose β has one singleton focal set. Then, it can only be disjoint from the other ones since $\mathcal{F}_\beta$ is an antichain. Thus, for any set A, either A contains w_1 and $\beta(A) = 1$ or A does not, hence it contains another focal set, and $\beta(A^c) = 1$. □

In terms of information supplied by sources, Proposition 2 tells that an optimistic capacity corresponds to a set of sources, at least one of which is fully informed and contradicts all other sources, which ensures that optimistic Boolean capacities account for conflicting sources and forbid ignorance (at least one source always informs about any proposition p). We can notice that the role of axiom K is not more essential in the modeling of conflicting information than for incomplete information.

In the following we try to show that the framework of Boolean capacities and its logic are capable of accounting for some previous logical approaches to the handling of inconsistent and incomplete information due to conflicting sources. We start with one of the oldest approaches, the four-valued logic of Dunn and Belnap.

3. Belnap–Dunn four-valued logic

Belnap–Dunn logic [32,10,9] handles both incomplete and inconsistent information coming from several sources. Forty years later, it remains one of the most influential approaches to this problem. This logic is also mathematically important as it has focused attention on a specific algebraic structure called a bilattice, that has been extensively studied and used for non-monotonic reasoning and logic programming [36,33,13]. This section recalls the intuition behind this logic, and presents both its four-valued semantics and its syntax, as well as its connections with a three-valued logic of incomplete information (Kleene logic) and another three valued paraconsistent logic (Priest Logic of Paradox). Indeed, Belnap–Dunn four-valued logic can be viewed, from the point of view of its truth tables, as an augmented Kleene logic: on top of the truth-value supposed to represent ignorance, one adds another truth-value representing conflict. However, this four-valued logic can also be considered as more basic than Kleene logic as it has less inference rules from a syntactic point of view [51]. Likewise, adding the excluded middle law to Belnap–Dunn logic yields Priest Logic of Paradox [48]. The material in this section is scattered in various publications [32,10,9,48,35], and it is useful to present a synthesis in a single place.

3.1. The Belnap multisource set-up

Belnap [10,9] considers an artificial information processor, fed from a variety of sources, and capable of answering queries on propositions of interest. In this context, inconsistency threatens, all the more so as the information processor is supposed never to subtract information. The basic assumption is that the computer receives information about *atomic propositions* in

Table 1
Belnap–Dunn negation, disjunction and conjunction.

	$\neg$
F	**T**
U	**U**
C	**C**
T	**F**

$\vee$	**F**	**U**	**C**	**T**
F	**F**	**U**	**C**	**T**
U	**U**	**U**	**T**	**T**
C	**C**	**T**	**C**	**T**
T	**T**	**T**	**T**	**T**

$\wedge$	**F**	**U**	**C**	**T**
F	**F**	**F**	**F**	**F**
U	**F**	**U**	**F**	**U**
C	**F**	**F**	**C**	**C**
T	**F**	**U**	**C**	**T**

a cumulative way from outside sources, each asserting for each atomic proposition whether it is true, false, or being silent (hence ignorant) about it. The notion of *epistemic set-up* is defined as an assignment, of one of four so-called *epistemic truth-values*, here denoted by **T**, **F**, **C**, **U**, to each atomic proposition $a, b, \ldots$:

1. Assigning **T** to a means the computer has only been told that a is true (1) by at least one source, and false (0) by none.
2. Assigning **F** to a means the computer has only been told that a is false by at least one source, and true by none.
3. Assigning **C** to a means the computer has been told at least that a is true by one source and false by another.
4. Assigning **U** to a means the computer has been told nothing about a.

In Belnap's epistemic setting, the role of the computer is not to interpret the information provided by the sources, but just to store it. In particular, any piece of information supplied is viewed as a logical atom to which an epistemic qualifier is attached as per the above rules. Having attached an epistemic truth-value to all atoms of the language $\mathcal{L}$, this assignment can be extended to all formulas in $\mathcal{L}$ using the truth tables in Table 1.

If $\{0, 1\}$ is the set of usual truth-values (as assigned by the information sources to atoms), then the set $\mathbb{V}_4 = \{\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathbf{U}\}$ of epistemic truth-values coincides with the power set of $\{0, 1\}$, letting $\mathbf{T} = \{1\}$, $\mathbf{F} = \{0\}$, $\mathbf{C} = \{0, 1\}$, $\mathbf{U} = \emptyset$, according to the convention initiated by Dunn [32]. Then, the empty set corresponds to no information received, while $\{0, 1\}$ represents the presence of conflicting sources, some claiming the truth of a some its falsity. **C** expresses an excess of truth-values, the set $\{0, 1\}$ being viewed as expressing 0 *and* 1 at the same time.

Belnap's approach relies on two orderings in $\mathbb{V}_4 = \{\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathbf{U}\}$, equipping it with two lattice structures:

- *The information ordering*, $\sqsubset$ whose meaning is "less informative than", such that $\mathbf{U} \sqsubset \mathbf{T} \sqsubset \mathbf{C}$; $\mathbf{U} \sqsubset \mathbf{F} \sqsubset \mathbf{C}$. This ordering reflects the inclusion relation of the sets $\emptyset$, $\{0\}$, $\{1\}$, and $\{0, 1\}$. It is in fact also in agreement with the specificity ordering of possibility theory [24]. It intends to reflect the amount of (possibly conflicting) data provided by the sources. **U** is at the bottom because (to quote) "it gives no information at all". **C** is at the top because (following Belnap) it gives too much information. It yields *the information lattice*, a Scott approximation lattice with join and meet defined by union and intersection of sets of truth-values (in this lattice, the maximum of **T** and **F** is **C**).
- *The truth ordering*, $<_t$, representing "more true than" according to which $\mathbf{F} <_t \mathbf{C} <_t \mathbf{T}$ and $\mathbf{F} <_t \mathbf{U} <_t \mathbf{T}$, each chain reflecting the truth-set of Kleene's logic. In other words, ignorance and conflict play the same role with respect to **F** and **T** according to this ordering. It yields *the logical lattice*, based on the truth ordering, and the interval extension of $\wedge$, $\vee$ and $\neg$ from $\{0, 1\}$ to $2^{\{0,1\}} \setminus \{\emptyset\}$, as appears in Table 1. In this lattice, the maximum of **U** and **C** is **T** and their minimum is **F**.

Remark 2. The convention for representing epistemic stances in the previous section, in agreement with possibility theory [27], is opposite to Dunn's, as the set $\{0, 1\}$ represents the hesitation between *true* and *false* and means *0 or 1*, just like an epistemic state E is a disjunction of possible worlds. Then, conflict is expressed by the empty set of truth-values (reflecting the fact that a conflicting formula has no model). Under this convention, subsets of $\{0, 1\}$ represent constraints on mutually exclusive truth-values. Denoting by π a Boolean possibility distribution, **U** can be encoded by $\pi(0) = \pi(1) = 1$ (representing ignorance), **C** by $\pi(0) = \pi(1) = 0$ representing the contradiction (corresponding to no possible truth-value left). The information ordering between epistemic truth-values induced by this convention is the same as in the Belnap–Dunn setting, using the opposite inclusion ordering between subsets of $\{0, 1\}$.

3.2. The syntax

Consider the previously introduced propositional language $\mathcal{L}$ with variables $V = \{a, b, c, \ldots\}$ with connectives $\wedge, \vee, \neg$, for conjunction, disjunction and negation, respectively. Formulas p in $\mathcal{L}$ are generated as usual. However, as explained above, connectives of negation, conjunction and disjunction are defined truth-functionally in Belnap–Dunn four-valued logic, following the rules of the bilattice (see Table 1).

Belnap–Dunn four-valued logic has no tautologies, hence no axioms. It can be defined only via a set of inference rules, as those that can be found in [48,35]:

Definition 2. Let $a, b, c \in V$. The *Belnap–Dunn four-valued logic* is defined by no axiom and the following set of rules

$$(R1) : \frac{a \wedge b}{a} \qquad (R2) : \frac{a \wedge b}{b} \qquad (R3) : \frac{a \quad b}{a \wedge b} \qquad (R4) : \frac{a}{a \vee b}$$

$$(R5): \frac{a \vee b}{b \vee a} \quad (R6): \frac{a \vee a}{a} \quad (R7): \frac{a \vee (b \vee c)}{(a \vee b) \vee c}$$

$$(R8): \frac{a \vee (b \wedge c)}{(a \vee b) \wedge (a \vee c)} \quad (R9): \frac{(a \vee b) \wedge (a \vee c)}{a \vee (b \wedge c)} \quad (R10): \frac{a \vee c}{\neg\neg a \vee c}$$

$$(R11): \frac{\neg(a \vee b) \vee c}{(\neg a \wedge \neg b) \vee c} \quad (R12): \frac{\neg(a \wedge b) \vee c}{(\neg a \vee \neg b) \vee c} \quad (R13): \frac{\neg\neg a \vee c}{a \vee c}$$

$$(R14): \frac{(\neg a \wedge \neg b) \vee c}{\neg(a \vee b) \vee c} \quad (R15): \frac{(\neg a \vee \neg b) \vee c}{\neg(a \wedge b) \vee c}$$

The syntactic inference $\Gamma \vdash_B p$, where $\Gamma \subseteq \mathcal{L}$ is a set of formulas, means that p can be derived from Γ using the above inference rules.

These rules express that conjunction is idempotent, and it yields a more specific proposition than each of the conjuncts (R1, R2, R3); disjunction is idempotent and it yields a less specific proposition than its disjuncts (R4, R5, R6). Disjunction is associative (R7), conjunction is distributive on disjunction (R8) and conversely (R9). Negation is involutive (R10, R13) and De Morgan Laws are satisfied (R11, R12, R14, R15). It makes it clear that the underlying algebra is a De Morgan algebra.

Formulas in this logic can be put in normal form, by applying the above rules. Namely, distributivity, idempotence and De Morgan laws imply that a formula can be expressed as a conjunction of clauses: $p = p_1 \wedge \ldots \wedge p_n$, where the p_i's are disjunctions of literals $l_{ij} = a$ or $\neg a$, for $a \in V$.

3.3. The semantics

Consider again the four epistemic truth-values $\{\mathbf{F}, \mathbf{U}, \mathbf{C}, \mathbf{T}\}$ forming the bilattice $\mathbb{V}_4$. A Belnap–Dunn valuation is a mapping $v_4 : \mathcal{L} \mapsto \mathbb{V}_4$ constructed from an assignment of epistemic truth-values $V \to \mathbb{V}_4$ to atoms and the truth tables of Table 1. Let $\Gamma \subseteq \mathcal{L}$ and $p \in \mathcal{L}$, then we define the semantic consequence relation by means of the truth ordering $\leq_t$ as follows:

$$\Gamma \vDash_B p \qquad \text{iff} \qquad \exists p_1, \ldots, p_n \in \Gamma,\ \forall v_4,\ v_4(p_1) \wedge \ldots \wedge v_4(p_n) \leq_t v_4(p)$$

Usually, semantic inference is defined via the preservation of truth, and more generally of designated truth-values. In Belnap–Dunn setting natural designated truth-values are $\{\mathbf{C}, \mathbf{T}\}$, since in their set-theoretic encoding under Dunn's conventions both include 1. An alternative consequence relation can thus be defined by:

$$\Gamma \vDash_C p \qquad \text{iff} \qquad \forall v_4, \quad \text{if } \forall p_i \in \Gamma,\ v_4(p_i) \in \{\mathbf{C}, \mathbf{T}\} \text{ then } \ v_4(p) \in \{\mathbf{C}, \mathbf{T}\}$$

Due to the fact that $\mathbf{U}$ and $\mathbf{C}$ play the same role in the bilattice, Font [35] proves that the consequence relation $\Gamma \vDash_C p$ is equivalent to $\Gamma \vDash_U p$, defined by:

$$\Gamma \vDash_U p \qquad \text{iff} \qquad \forall v_4, \quad \text{if } \forall p_i \in \Gamma,\ \ v_4(p_i) \in \{\mathbf{U}, \mathbf{T}\}, \text{then } \ v_4(p) \in \{\mathbf{U}, \mathbf{T}\}$$

using $\{\mathbf{U}, \mathbf{T}\}$ as designated truth-values. Font [35] also independently proves that $\Gamma \vDash_B p$ iff both $\Gamma \vDash_U p$ and $\Gamma \vDash_C p$. Thus, each of the consequence relations $\vDash_C$ and $\vDash_U$ is equivalent to $\vDash_B$. We give explicit proofs in Appendix A, in the case of two premises, for the sake of self-containedness.

It should be clear that Belnap–Dunn logic has neither tautologies nor contradictions, e.g., $a \vee \neg a$ is not a tautology, nor is $a \wedge \neg a$ a contradiction.

Example 2. $a \vee b \nvDash_B a \vee \neg a$ since for instance it fails if $v_4(b) = \mathbf{T}$ and $v_4(a) = v_4(\neg a) = \mathbf{U}$. Also $a \wedge \neg a \nvDash_B b \vee \neg b$, since for instance it fails if $v_4(a) = \mathbf{C}$, $v_4(b) = \mathbf{U}$.

The agreement between the Hilbert-style axiomatization of Belnap–Dunn logic and the above semantics is proved by Pynko [48] and Font [35]:

Theorem 1. *Belnap–Dunn logic is sound and complete with respect to the bilattice semantics with designated truth-values* $\{\mathbf{C}, \mathbf{T}\}$*, that is* $\Gamma \vdash_B p$ *iff* $\Gamma \vDash_B p$.

3.4. Special cases

Two special cases of Belnap–Dunn logic, namely Priest Logic of Paradox [46,47] and Kleene logic [40], that use the same language, can be obtained by adding an axiom and an inference rule respectively.

In Priest Logic of Paradox [46,47], the truth-set is restricted to the three linearly ordered truth-values $\mathbb{P}_3 = \{\mathbf{F}, \mathbf{C}, \mathbf{T}\}$, and the designated set of truth-values is $\{\mathbf{C}, \mathbf{T}\}$; it comes down to merging $\mathbf{U}$ and $\mathbf{F}$ in Belnap $\mathbb{V}_4$. The truth tables for conjunction, disjunction and negation are obtained from the restriction of Belnap–Dunn truth tables to $\mathbb{P}_3$. It is then clear that, letting v_3 denote a three-valued truth assignment ranging on $\mathbb{P}_3$, we have $\forall a \in V, v_3(a \vee \neg a) \in \{\mathbf{C}, \mathbf{T}\}$, which gives intuition for the following result by Pynko [48]:

Proposition 3. *A Hilbert system for* Priest Logic of Paradox *is obtained by Belnap–Dunn logic inference rules plus axiom $a \vee \neg a$.*

Semantic inference in the Logic of Paradox is then defined by:

$$\Gamma \vDash_P p \text{ iff } \forall v_3, \text{if } v_3(p_i) \in \{\mathbf{C}, \mathbf{T}\}, \forall p_i \in \Gamma \text{ then } v_3(p) \in \{\mathbf{C}, \mathbf{T}\}$$

Any sentence of the form $p \vee \neg p$ is then clearly a tautology, since $v_3(p \vee \neg p) \in \{\mathbf{C}, \mathbf{T}\}$ for any proposition p in the Logic of Paradox. In fact all tautologies of propositional calculus can be recovered, but modus ponens does not hold in this logic.

On the other hand, the strong Kleene logic is another extension of Belnap–Dunn logic that uses as truth-set a Kleene lattice $\mathbb{K}_3 = \{\mathbf{F}, \mathbf{U}, \mathbf{T}\}$ (merging **C** and **T**). This logic has the same truth tables as the Logic of Paradox, if we identify **U** and **C**. However there is one designated truth-value **T**,[3] so that semantic inference $\vDash_K$ is of the form

$$\Gamma \vDash_K p \quad \text{iff } \forall v_3, \text{if } \forall p_i \in \Gamma,\ v_3(p_i) = \{\mathbf{T}\}, \text{ then } v_3(p) = \{\mathbf{T}\}$$

By adding one rule to Belnap–Dunn logic, we get a Hilbert system for Kleene logic, as recently explained by Albuquerque et al. [1], correcting a claim in [35].

Proposition 4. *A Hilbert system for strong* Kleene logic *is obtained by Belnap–Dunn logic inference rules plus the rule of suppression of contradictions:*

$$SC\text{:}\ \frac{a \vee (b \wedge \neg b)}{a}.$$

To get an intuition of why this is so, we borrow some remarks from [1]. First, it is easy to realize the duality between Priest and Kleene logic as $p \vdash_K q$ if and only if $\neg q \vdash_P \neg p$, since

$$v_3(p) = \mathbf{T} \text{ implies } v_3(q) = \mathbf{T} \iff v_3(\neg q) \in \{\mathbf{F}, \mathbf{U}\} \text{ implies } v_3(\neg p) \in \{\mathbf{F}, \mathbf{U}\}.$$

This is in contrast with Belnap–Dunn logic where $p \vDash_B q$ if and only if $\neg q \vDash_B \neg p$. It is easy to see that rule (SC) is semantically valid in strong Kleene logic. Conversely, suppose $p \vdash_K q$, that is, $\neg q \vdash_P \neg p$, using completeness of the Logic of Paradox. It means that there are some atomic formulas $a_i, i = 1, \ldots, n$ such that $\neg q \wedge \bigwedge_{i=1}^n (a_i \vee \neg a_i) \vdash_B \neg p$ in Belnap–Dunn logic. It also reads $p \vdash_B q \vee \bigvee_{i=1}^n (a_i \wedge \neg a_i)$, and, using (SC) $q \vee \bigvee_{i=1}^n (a_i \wedge \neg a_i)$ implies q. So, inference relation $\vdash_K$ is captured by Belnap–Dunn logic rules (R1–15) plus (SC).

It is interesting to notice, as done in [1], that rule (SC) is equivalent to the resolution rule

$$\text{RR: } \frac{a \vee b, c \vee \neg b}{a \vee c}.$$

This is because on the one hand $(a \vee b) \wedge (c \vee \neg b) \vdash_B (a \wedge c) \vee (a \wedge \neg b) \vee (b \wedge c) \vee (b \wedge \neg b)$ (distributivity) using (R1–15), which implies $(a \wedge c) \vee (a \wedge \neg b) \vee (b \wedge c)$, using (SC), which in turn implies $a \vee c$ using (R1–15). So (R1–15) + (SC) implies (RR). Conversely, the premisse of (SC) also reads $(a \vee b) \wedge (a \vee \neg b)$ using (R1–15). Then (SC) is (RR) where $a = c$. The inference system made of the 15 rules of Belnap–Dunn logic plus resolution is thus sound and complete with respect to semantic entailment in strong Kleene logic.

Remark 3 *(Nothing but the truth).* As a consequence, modus ponens is valid in Kleene logic, noticing that from $\{b, \neg b \vee c\}$, one can deduce $\{b \vee c, \neg b \vee c\}$ from (R4), and then c using (RR). However, adding modus ponens to the 15 rules of Belnap–Dunn logic does not yield Kleene logic. It corresponds to using **T** as the single designated value for semantic inference in Belnap–Dunn setting, while keeping the four epistemic truth-values. This four-valued logic, preserving "nothing but the truth" (and called NBT), has been studied in [51,45].

Remark 4 *(Kleene logic of order).* The following inference rule holds in strong Kleene logic:

$$\text{R16: } \frac{a \wedge \neg a}{b \vee \neg b}$$

since $a \wedge \neg a \vdash_B (a \vee b) \wedge (\neg a \vee \neg b)$ which entails $b \vee \neg b$, using (RR). It also holds in NBT logic. Although R16 reflects a known property of Kleene algebras, it is insufficient to recover Kleene logic when added to Belnap–Dunn logic rules as

[3] http://plato.stanford.edu/entries/truth-values/.

explained in [1], contrary to what is claimed in [35]. In fact, (R16) is valid in both Priest and Kleene logics as well as the rule:

$$\text{R17: } \frac{(a \wedge \neg a) \vee c}{(b \vee \neg b) \vee c}$$

Adding R17 to Belnap–Dunn logic rules (R1–15) provides a Hilbert style system for the so-called Kleene logic of order $\mathcal{K}^{\leq}$ which considers consequences common to both the Logic of Paradox and the strong Kleene logic (see Th. 3.4 in [1]). It has the same three-valued truth set as strong Kleene and Priest logics (say $\mathbb{V}_3$, as $\mathbf{C}$ and $\mathbf{U}$ are equal in $\mathcal{K}^{\leq}$), and semantic entailment is of the form

$$\begin{aligned}\Gamma \vDash_{\leq} p &\text{ iff } v_3(\wedge_{p_i \in \Gamma} p_i) \leq v_3(p)\\ &\text{iff } \forall v_3, \text{ if } \ v_3(p_i) \in \{\mathbf{C}, \mathbf{T}\}\ \forall p_i \in \Gamma, \text{then } \ v_3(p) \in \{\mathbf{C}, \mathbf{T}\}\\ &\quad \text{and } \forall v, \text{if } \ v_3(p_i) = \mathbf{T}\ \forall p_i \in \Gamma, \text{then } \ v_3(p) = \mathbf{T}.\end{aligned}$$

Other extensions of Belnap–Dunn logic are studied in [51,45,1].

4. A Translation of Belnap–Dunn logic into BC

In previous papers [20,21], we have proposed a consequence-preserving translation of Kleene and Priest logics in the two-tiered logic MEL [8] with semantics in terms of necessity and possibility measures. These results strongly suggest that Belnap–Dunn logic can be in turn expressed in BC. Indeed, Belnap–Dunn truth-values described in the previous section can be expressed by means of pairs $(\beta(A), \beta(A^c))$ introduced in Section 2, restricting A to sets of classical models $[a]$ of atomic formulas a, namely:

- $v_4(a) = \mathbf{T}$ is interpreted as $\beta([a]) = 1$ and $\beta([\neg a]) = 0$
- $v_4(a) = \mathbf{F}$ is interpreted as $\beta([a]) = 0$ and $\beta([\neg a]) = 1$
- $v_4(a) = \mathbf{U}$ is interpreted as $\beta([a]) = 0$ and $\beta([\neg a]) = 0$
- $v_4(a) = \mathbf{C}$ is interpreted as $\beta([a]) = 1$ and $\beta([\neg a]) = 1$

Based on this intuition, the translation of Belnap–Dunn logic into BC naturally follows. Results of this section were already announced in the conference paper [22].

4.1. *Principle of the translation*

As mentioned in Section 2, $\Box p$ is encoding $\beta([p]) = 1$ for a Boolean capacity, which clearly means that at least one source (corresponding to a focal set in $\mathcal{F}_\beta$) asserts p. The case where $\beta([\neg p]) = 0$ thus corresponds to $\Diamond p = \neg\Box\neg p$, which clearly means that no source is asserting $\neg p$. Hence, formulas in BC can be related to Belnap–Dunn truth-values [24]:

- $\Box p \wedge \Diamond p$ holds when $\beta([p]) = 1$ and $\beta([\neg p]) = 0$ (related to the epistemic truth-value $\mathbf{T}$).
- $\Box\neg p \wedge \Diamond\neg p$ holds when $\beta([\neg p]) = 1$ and $\beta([p]) = 0$ (related to the epistemic truth-value $\mathbf{F}$).
- $\Diamond p \wedge \Diamond\neg p$ holds when $\beta([p]) = 0$ and $\beta([\neg p]) = 0$ (related to the epistemic truth-value $\mathbf{U}$).
- $\Box p \wedge \Box\neg p$ holds when $\beta([p]) = 1$ and $\beta([\neg p]) = 1$ (related to the epistemic truth-value $\mathbf{C}$).

Belnap–Dunn approach first assigns epistemic truth-values to atomic propositions only. We can encode these truth-qualified atoms into the modal language of BC as follows. Let $\mathcal{T}$ denote the translation operation that takes a partial Belnap–Dunn truth-value assignment of the form $v_4(a) \in \Theta \subseteq \mathbb{V}_4$ to atomic propositional formulas, (indicating their epistemic status w.r.t a set of sources), and turns it into a modal formula in the logic BC. For instance:

$$\mathcal{T}(v_4(a) = \mathbf{T}) = \Box a \wedge \Diamond a \qquad \mathcal{T}(v_4(a) = \mathbf{F}) = \Box\neg a \wedge \Diamond\neg a \tag{1a}$$

$$\mathcal{T}(v_4(a) = \mathbf{U}) = \Diamond a \wedge \Diamond\neg a \qquad \mathcal{T}(v_4(a) = \mathbf{C}) = \Box a \wedge \Box\neg a \tag{1b}$$

$$\mathcal{T}(v_4(a) \geq_t \mathbf{C}) = \Box a \qquad \mathcal{T}(v_4(a) \leq_t \mathbf{C}) = \Box\neg a \tag{1c}$$

$$\mathcal{T}(v_4(a) \geq_t \mathbf{U}) = \Diamond a \qquad \mathcal{T}(v_4(a) \leq_t \mathbf{U}) = \Diamond\neg a \tag{1d}$$

(where $\geq_t$ refers to the truth ordering). In Belnap–Dunn logic, since valuations of propositions other than elementary ones are obtained via truth tables, the translation of $\mathbb{V}_4$-truth-qualified formulas will be carried out by respecting the truth tables of the logic. However, not all formulas of $\mathcal{L}_\Box$ can be reached via the translation: as we shall see, only literals can appear in the scope of modalities.

Table 2
Translation table for negation, conjunction and disjunction.

$\mathcal{T}$	$v_4(\neg a)$	$v_4(a \wedge b)$	$v_4(a \vee b)$
$= \mathbf{T}$	$\Box\neg a \wedge \Diamond\neg a$	$\Box a \wedge \Diamond a \wedge \Box b \wedge \Diamond b$	$\Box a \vee \Box b \vee (\Diamond a \wedge Ub) \vee (\Diamond b \wedge Ua)$
$= \mathbf{F}$	$\Box a \wedge \Diamond a$	$\Box\neg a \vee \Box\neg b \vee (\Diamond\neg a \wedge Ub) \vee (\Diamond\neg b \wedge Ua)$	$\Box\neg a \wedge \Diamond\neg a \wedge \Box\neg b \wedge \Diamond\neg b$
$\geq_t \mathbf{U}$	$\Diamond a$	$\Diamond a \wedge \Diamond b$	$\Diamond a \vee \Diamond b$
$\geq_t \mathbf{C}$	$\Box a$	$\Box a \wedge \Box b$	$\Box a \vee \Box b$
$\leq_t \mathbf{U}$	$\Diamond\neg a$	$\Diamond\neg a \vee \Diamond\neg a$	$\Diamond\neg a \wedge \Diamond\neg a$
$\leq_t \mathbf{C}$	$\Box\neg a$	$\Box\neg a \vee \Box\neg a$	$\Box\neg a \wedge \Box\neg a$

4.2. *Translating formulas from Belnap–Dunn logic to BC*

We can proceed to the translation of Belnap–Dunn truth tables into BC. First consider negation. It is easy to check that

$$\mathcal{T}(v_4(\neg a) = \mathbf{T}) = \mathcal{T}(v_4(a) = \mathbf{F})$$
$$\mathcal{T}(v_4(\neg a) = \mathbf{F}) = \mathcal{T}(v_4(a) = \mathbf{T})$$
$$\mathcal{T}(v_4(\neg a) = \mathbf{x}) = \mathcal{T}(v_4(a) = \mathbf{x}), \mathbf{x} \in \{\mathbf{U}, \mathbf{C}\}$$
$$\mathcal{T}(v_4(\neg a) \geq_t \mathbf{x}) = \mathcal{T}(v_4(a) \leq_t \mathbf{x}), \mathbf{x} \in \{\mathbf{U}, \mathbf{C}\}$$

For compound formulas built with conjunction and disjunction, we get:

$$\mathcal{T}(v_4(p \wedge q) = \mathbf{T}) = \mathcal{T}(v_4(p) = \mathbf{T}) \wedge \mathcal{T}(v_4(q) = \mathbf{T})$$
$$\mathcal{T}(v_4(p \vee q) = \mathbf{T}) = \mathcal{T}(v_4(p) = \mathbf{T}) \vee \mathcal{T}(v_4(q) = \mathbf{T})$$
$$\vee(\mathcal{T}(v_4(p) = \mathbf{U}) \wedge \mathcal{T}(v_4(q) = \mathbf{C})) \vee (\mathcal{T}(v_4(p) = \mathbf{C}) \wedge \mathcal{T}(v_4(q) = \mathbf{U}))$$
$$\mathcal{T}(v_4(p \wedge q) \geq_t \mathbf{x}) = \mathcal{T}(v_4(p) \geq_t \mathbf{x}) \wedge \mathcal{T}(v_4(q) \geq_t \mathbf{x})$$
$$\mathcal{T}(v_4(p \vee q) \geq_t \mathbf{x}) = \mathcal{T}(v_4(p) \geq_t \mathbf{x}) \vee \mathcal{T}(v_4(q) \geq_t \mathbf{x})$$
$$\mathcal{T}(v_4(p \wedge q) \leq_t \mathbf{x}) = \mathcal{T}(v_4(p) \leq_t \mathbf{x}) \vee \mathcal{T}(v_4(q) \leq_t \mathbf{x})$$
$$\mathcal{T}(v_4(p \vee q) \leq_t \mathbf{x}) = \mathcal{T}(v_4(p) \leq_t \mathbf{x}) \wedge \mathcal{T}(v_4(q) \leq_t \mathbf{x})$$

with $\mathbf{x} \in \{\mathbf{U}, \mathbf{C}\}$. For elementary formulas $\neg a$, $a \vee b$, $a \wedge b$, we give explicit translations using the truth tables of Belnap–Dunn logic in Table 2. For simplicity we shorten $\Diamond a \wedge \Diamond\neg a$ as Ua (a is unknown). Then, some translations can be simplified as for the expressions of $\mathcal{T}(v_4(a \vee b) = \mathbf{T})$ and $\mathcal{T}(v_4(a \wedge b) = \mathbf{F})$. The first one is of the form (as per the above results):

$$(\Box a \wedge \Diamond a) \vee (\Box b \wedge \Diamond b) \vee (\Box a \wedge \Box\neg a \wedge Ub) \vee (Ua \wedge \Box b \wedge \Box\neg b).$$

Developing $(\Box a \wedge \Diamond a) \vee (\Box a \wedge \Box\neg a \wedge Ub)$ simplifies into $\Box a \vee (\Diamond a \wedge Ub)$ and likewise exchanging a and b. The result appears on Table 2 line 2, last column. The result for $v_4(a \wedge b)$ obtains similarly on line 3, column 3 of this table.

Beyond their technicalities, these translations lay bare the price paid by assuming a truth-functional approach to multiple source reasoning, namely we cannot really assign an epistemic status to genuine compound Boolean formulas. For instance, consider the translation $\mathcal{T}(v_4(a \wedge b) = \mathbf{T}) = \Box a \wedge \Diamond a \wedge \Box b \wedge \Diamond b$. It says that attaching the epistemic truth-value $\mathbf{T}$ to a conjunction of atoms in Belnap–Dunn logic just comes down to having the Boolean atom a confirmed by one source ($\Box a$) and not disconfirmed by other ones ($\Diamond a$), and likewise for the Boolean atom b, which is unsurprising. But it tells nothing about the epistemic status of the *Boolean* formula $a \wedge b$. In some sense, the truth-functional conjunction in Belnap–Dunn logic becomes trivial in BC, since in BC, the truth-functionality statement $v_4(a \wedge b) = v_4(a) \wedge v_4(b)$ (Table 1) just tells us that the conjunction of $\Box a \wedge \Diamond a$ and $\Box b \wedge \Diamond b$ is $\Box a \wedge \Diamond a \wedge \Box b \wedge \Diamond b$; it does not inform about the truth of $\Box(a \wedge b) \wedge \Diamond(a \wedge b)$. In fact, in BC, we do not have that $\Box a \wedge \Diamond a \wedge \Box b \wedge \Diamond b \vdash \Box(a \wedge b) \wedge \Diamond(a \wedge b)$ (by lack of adjunction axiom). The same comment applies to the other translations.

4.3. *Belnap–Dunn logic inference in BC*

We have two designated values: $\mathbf{T}$ and $\mathbf{C}$. So, for inference purposes, we only have to consider the translation of expressions of the form $\mathcal{T}(v_4(p) \geq_t \mathbf{C})$. Since $\mathcal{T}(v_4(a) \geq_t \mathbf{C}) = \Box a$, and $\mathcal{T}(v_4(\neg a) \geq_t \mathbf{C}) = \Box\neg a$ for literals, the other formulas being translated via Belnap–Dunn truth tables, this translation sends Belnap–Dunn logic into the following fragment of the language of BC:

$$\mathcal{L}^B_\Box = \Box a | \Box\neg a | \phi \wedge \psi | \phi \vee \psi$$

where no negation appears in front of $\Box$.

Conversely, from the fragment $\mathcal{L}^B_\Box$ we can go back to Belnap–Dunn logic. Namely any formula ϕ in $\mathcal{L}^B_\Box$ can be translated into a formula $\theta(\phi)$ of the propositional logic language by means of the following translation rules:

- $\theta(\Box a) = a$
- $\theta(\Box\neg a) = \neg a$
- $\theta(\psi \wedge \phi) = \theta(\psi) \wedge \theta(\phi)$
- $\theta(\psi \vee \phi) = \theta(\psi) \vee \theta(\phi)$

We remark, *only using the properties of classical logic*, that $\Box a \vee \Box\neg a$ is not a tautology in BC. More generally, no tautologies can be expressed in the above fragment. This is coherent with the fact that Belnap–Dunn logic has no theorems. Likewise, $\Box a \wedge \Box\neg a$ is not a contradiction in BC. Now we show that the inference rules of Belnap–Dunn logic are valid in BC via the proposed translation.

Theorem 2. *Let* $\dfrac{\phi}{\psi}$ *be any of the rules of Belnap–Dunn logic. Then, the following inference rule*

$$\frac{\mathcal{T}(v_4(\psi) \geq_t \mathbf{C})}{\mathcal{T}(v_4(\phi) \geq_t \mathbf{C})}$$

holds in BC.

Proof. The result easily follows by the definition of the translation. In order to give an example, let us show why the translation of rule (R11) is valid in BC. At first, we translate $\mathcal{T}(v_4(\neg(a \vee b) \vee c) \geq_t \mathbf{C})$. By the translation rules for disjunction and negation given previously, we get $\mathcal{T}(v_4(\neg(a \vee b)) \geq_t \mathbf{C}) \vee \mathcal{T}(v_4(c) \geq_t \mathbf{C})$ and then $[\mathcal{T}(v_4(a) \leq_t \mathbf{C}) \wedge \mathcal{T}(v_4(b) \leq_t \mathbf{C})] \vee \mathcal{T}(v_4(c) \geq_t \mathbf{C})$. Then, by the rules on atomic propositions given in Table 2, we get $(\Box\neg a \wedge \Box\neg b) \vee \Box c$. Now, the consequent $\mathcal{T}(v_4((\neg a \wedge \neg b) \vee c) \geq_t \mathbf{C})$ gives exactly the same expression, that is, once translated, antecedent and consequent are the same. $\Box$

As a consequence we can mimick syntactic inference of Belnap–Dunn logic in BC, provided that we restrict to formulas in $\mathcal{L}^B_\Box$.

4.4. Capacity semantics of the Belnap–Dunn fragment of BC

The restriction of the scope of modalities to literals required by the translation of Belnap–Dunn logic into BC also affects the set of B-capacities that can act as a semantic counterpart of the logic. We can check that semantic inference in Belnap–Dunn logic can be expressed in the modal setting of BC by constraining the capacities that can be used as models of $\mathcal{L}^B_\Box$ formulas. Namely, consider a Belnap set-up where each source i provides a set T_i of atoms considered true, a set F_i of atoms considered false, where $T_i \cap F_i = \emptyset$. Should the source behave like a propositional reasoner, this information would correspond to a *partial model*, which is a special kind of epistemic state of rectangular shape, namely:

$$E_i = [(\bigwedge_{a \in T_i} a) \wedge (\bigwedge_{b \in F_i} \neg b)].$$

As there are n sources, we could consider n-tuples of partial models $(E_1, \ldots, E_n)$, and restrict to capacities with such rectangular focal sets.

In fact, Belnap sources are not propositional reasoners, and pieces of atomic information they provide cannot be expressed by their propositional conjunctions. This is as if there were, behind each source i, as many primitive sources as the number of atoms in $T_i \cup F_i$. This is expressed by the use of atoms in $\mathcal{L}^B_\Box$, of the form $\Box\ell$ where ℓ is a literal. As we cannot put $\Box$ in front of conjunctions nor disjunctions, we should use capacities whose focal sets are the form $[a], a \in \cup_{i=1}^n T_i$ and $[\neg b], b \in \cup_{i=1}^n F_i$ to interpret formulas in $\mathcal{L}^B_\Box$. We call such capacities *atomic*.

Considering the Belnap–Dunn valuation v_4 associated to the information supplied by n sources, there is a one-to-one correspondence between Belnap–Dunn valuations and atomic B-capacities α induced by this information:

Proposition 5. *For each Belnap–Dunn valuation* v_4*, there exists a unique atomic B-capacity* α_{v_4} *such that* $v_4 \models p$ *if and only if* $\alpha_{v_4} \models \mathcal{T}(v_4(p) \geq_t \mathbf{C})$.

Proof. Given Belnap–Dunn valuation v_4, define $T = \{a : v_4(a) = \mathbf{T} \text{ or } \mathbf{C}\}$, $F = \{a : v_4(a) = \mathbf{F} \text{ or } \mathbf{C}\}$, and let $\alpha([a]) = 1$ if $a \in T$, $\alpha([\neg a]) = 1$ if $a \in F$. Clearly, if $a \in T$, then $v_4 \models a$ and $\alpha \models \mathcal{T}(v_4(a) \geq_t \mathbf{C}) = \Box a$. If $a \in F$, then $v_4 \models \neg a$ and $\alpha \models \mathcal{T}(v_4(\neg a) \geq_t \mathbf{C}) = \Box\neg a$. The remainder follows using truth tables of Belnap–Dunn logic translated into BC. $\Box$

In the other way around,

Proposition 6. *For any B-capacity* β*, there is a single Belnap–Dunn valuation* v_4^β *such that* $\beta \models \phi \in \mathcal{L}^B_\Box$ *if and only if* $v_4^\beta(\theta(\phi)) \in \{\mathbf{C}, \mathbf{T}\}$.

Proof. Given a B-capacity β, define a Belnap–Dunn valuation $v_4^\beta(a) = \mathbf{T}$ if $\beta([a]) = 1$ and $\beta([\neg a]) = 0$, $v_4^\beta(a) = \mathbf{F}$ if $\beta([a]) = 0$ and $\beta([\neg a]) = 1$, $v_4^\beta(a) = \mathbf{C}$ if $\beta([a]) = 1 = \beta([\neg a])$, $v_4^\beta(a) = \mathbf{U}$ if $\beta([a]) = \beta([\neg a]) = 0$. The reader can check the equivalence for $\phi = \Box a, \Box\neg a, \Box a \vee \Box b, \Box a \wedge \Box b$ etc. $\Box$

However there are several Belnap set-ups inducing a given Belnap–Dunn valuation v_4: for instance only two sources are enough to model the four values [44]. We thus introduce an equivalence relation on the set of B-capacities, whereby two of them are equivalent if they correspond to the same four-valued truth assignment: $\beta \sim_B \beta'$ if and only if $v_4^\beta = v_4^{\beta'}$.

Proposition 7. *For any B-capacity β, there exists an atomic B-capacity α such that $\beta \sim_B \alpha$.*

Proof. Indeed, consider β with focal sets $E_1, \ldots E_n$. Let $T_i = \{a \in V : E_i \subseteq [a]\}$ and $F_i = \{b \in V : E_i \subseteq [\neg b]\}$. The focal sets of α are based on such literals and form the family $\mathcal{F}_\alpha = \{[a] : a \in \cup_{i=1}^n T_i\} \cup \{[\neg b] : b \in \cup_{i=1}^n F_i\}$, an antichain. Then, it is obvious that $v_4^\beta(a) = \mathbf{T}$ if and only if there is a source i such that $E_i \subseteq [a]$, if and only if $a \in \mathcal{F}_\alpha$, if and only if $v_4^\alpha(a) = \mathbf{T}$, and so on for the other three epistemic truth-values. $\Box$

From Proposition 7 we can conclude that for any B-capacity β, there exists an atomic B-capacity $\alpha \sim_B \beta$ such that $\beta \models \phi \in \mathcal{L}_\Box^B$ if and only if $\alpha \models \phi$. It is precisely the unique atomic capacity encoding the Belnap valuation v_4^β. The atomic capacity α uses only a small part of the information conveyed by β.

4.5. Main result

We then can establish the fact that our translation of Belnap–Dunn logic into BC is consequence-preserving:

Theorem 3. *Let Γ be a set (conjunction) of formulas in propositional logic interpreted in Belnap–Dunn logic, and p another such formula. Then $\Gamma \vdash_B p$ if and only if $\{\mathcal{T}(v_4(q) \geq_t \mathbf{C}) : q \in \Gamma\} \vdash_{BC} \mathcal{T}(v_4(p) \geq_t \mathbf{C})$.*

Proof. Suppose $\Gamma \vdash_B p$. Then from Theorem 2, all inference rules of Belnap–Dunn logic become valid inferences in BC using the translations of their premises and conclusions. So the inference can be made in BC. Conversely, by completeness of BC, suppose $\forall\beta$, if $\beta \models \mathcal{T}(v_4(q) \geq_t \mathbf{C}), \forall q \in \Gamma$ then $\beta \models \mathcal{T}(v_4(p) \geq_t \mathbf{C})$. Using Proposition 7, for all B-capacities β, $\exists\alpha \sim_B \beta$, where α is atomic, such that $\forall q \in \Gamma$, $\alpha \models \mathcal{T}(v_4(q) \geq_t \mathbf{C})$ if and only if $\beta \models \mathcal{T}(v_4(q) \geq_t \mathbf{C})$ and $\alpha \models \mathcal{T}(v_4(p) \geq_t \mathbf{C})$ if and only if $\beta \models \mathcal{T}(v_4(p) \geq_t \mathbf{C})$. Then, we have that if $v_4(q) \geq_t \mathbf{C}, \forall q \in \Gamma$ then $v_4(p) \geq_t \mathbf{C}$ for the Belnap–Dunn valuation v_4 associated to α. So $\Gamma \models_B p$. By completeness of Belnap–Dunn logic, $\Gamma \vdash_B p$ follows. $\Box$

It is important to notice that Belnap logic actually corresponds to a very small fragment of the BC logic. Indeed, due to the restriction of the language to literals in the scope of the $\Box$ modality, no modal axiom of BC can be expressed in $\mathcal{L}_\Box^B$. Moreover since no formula in this language contains a negation in the front of $\Box$, Belnap logic is thus mapped to the positive fragment of propositional logic where only conjunction and disjunction appears (atoms of the form $\Box\neg a$ can be considered positive since negation is internalized, and $\Box\neg a$ is logically independent from $\Box a$ in BC).

Example 1 *(continued).* The sources in the case of Belnap logic can only inform about a, b, c. It is clear that $K_1 \vdash \neg b$, $K_2 \vdash \neg a$, $K_3 \vdash a$ and $K_3 \vdash b$. So it is clear that based on this information, a and b both have epistemic truth value $\mathbf{C}$, while c has truth value $\mathbf{U}$. Using the truth tables, it can be seen that $a \wedge b$ is conflicting ($\mathbf{C}$), $a \wedge b \wedge c$ is rejected ($\mathbf{F}$), $a \vee b$ as well as $a \wedge \neg b$ are conflicting (because a and b are $\mathbf{C}$) and $a \vee c$ is supported ($\mathbf{T}$). These results are similar to those of the BC approach, with some differences due to the fact that Belnap sources are less expressive and the truth tables are less discriminant. For instance, we have shown that the BC approach supports $a \vee b$ and rejects $a \wedge \neg b$, but the two logics give the same conclusions on other formulas cited above.

5. Recovering Kleene and Priest fragments of MEL

In this section, we show how to recover our previous translations of three-valued Kleene logic and the Logic of Paradox into MEL [20,21], from our translation of Belnap–Dunn logic into BC, by translating into BC the properties that must be added to Belnap–Dunn logic in order to recover these logics (as explained in subsection 3.4). We also consider the translation of Kleene logic of order.

5.1. Strong Kleene logic as the logic of incomplete atomic information

Since $\Gamma \vdash_K p$ in strong Kleene logic stands for $\Gamma \vdash_{B+SC} p$, we must enforce the translation of the (SC) rule as an inference rule in the $\mathcal{L}_\Box^B$-fragment of BC logic. It is rather clear that the translation of (SC) into BC reads

(SCK): $(\Box a \wedge \Box\neg a) \vee \Box b \vdash \Box b$

The above inference rule is equivalent to $\Box a \wedge \Box\neg a \vdash \Box b \wedge \neg\Box b = \bot$ in BC. In other words, $\Box a \wedge \Box\neg a$ is a contradiction, and we can simplify logical expressions in disjunctive normal form in the language $\mathcal{L}^B_\Box$, deleting the products containing $\Box a \wedge \Box\neg a$.

In terms of capacity semantics, the validity of $(\Box a \wedge \Box\neg a) \vdash \bot$ enforces the constraints

$$\min(\beta(A), \beta(A^c)) = 0, \forall A = [a], a \in V,$$

i.e., A is a set of models of a propositional atom. The capacities serving as models for BC + (SCK) are pessimistic, excluding conflicting knowledge, which comes down to banning the epistemic value **C**.

Alternatively to (SC) we can add the resolution rule (RR) to the rules of Belnap–Dunn logic, whose translation into $\mathcal{L}^B_\Box$ is

(RK): $\{\Box a \vee \Box b, \Box c \vee \Box\neg a\} \vdash \Box b \vee \Box c$.

One way of recovering (SCK) in the logic BC is to enforce $\Box p \wedge \Box\neg p$ to be a contradiction for any $p \in \mathcal{L}$, which comes down to adding axiom D to BC. In that case, $\Box a \wedge \Diamond a = \Box a$ in BC + D, and the basic translation schemes for truth-assigned Belnap–Dunn logic atoms reduce to the ones in [20] for Kleene logic:

$$\mathcal{T}(v_4(a) = \mathbf{T}) = \Box a; \qquad \mathcal{T}(v_4(a) = \mathbf{F}) = \Box\neg a; \tag{3a}$$

$$\mathcal{T}(v_4(a) = \mathbf{U}) = \Diamond a \wedge \Diamond\neg a; \tag{3b}$$

$$\mathcal{T}(v_4(a) \geq_t \mathbf{U}) = \Diamond a; \qquad \mathcal{T}(v_4(a) \leq_t \mathbf{U}) = \Diamond\neg a. \tag{3c}$$

It becomes clear that reasoning in Kleene logic with designated truth-value **T** is equivalent to reasoning in BC + D since (SCK) and (RK) hold in BC + D and can be used in $\mathcal{L}^B_\Box$. This is summarized by the following specialization of Theorem 3, saying that there is a consequence-preserving translation into BC + D of Kleene logic with designated truth-value **T** (note that $\mathcal{T}(v_4(q) \geq \mathbf{C}) = \mathcal{T}(v_3(q) = \mathbf{T})$ under axiom D).

Theorem 4. *Let Γ be a set (conjunction) of formulas in propositional logic interpreted in Kleene logic, and p another such formula. Then $\Gamma \vdash_K p$ if and only if $\{\mathcal{T}(v_4(q) \geq \mathbf{C}) : q \in \Gamma\} \vdash_{BC+D} \mathcal{T}(v_4(p) \geq \mathbf{C})$.*

In [20], Kleene logic was translated into the same fragment $\mathcal{L}^B_\Box$ of $\mathcal{L}_\Box$, albeit into MEL, namely BC + $\{K, D\}$. Theorem 4 shows we can drop axiom K, which is not surprising as the latter cannot be expressed in the language $\mathcal{L}^B_\Box$. In fact, axiom K uses disjunction in the scope of modalities and is instrumental in deducing $\Box(p \wedge q)$ from $\Box p$ and $\Box q$. But such formulas have no counterpart in Kleene logic. Actually, $\Box(a \wedge b) \notin \mathcal{L}^B_\Box$. Axiom D cannot be expressed inside $\mathcal{L}^B_\Box$ either, but it can be added to BC in order to ensure the validity of inference rules (SCK) and (RK) in $\mathcal{L}^B_\Box$. It is then clear that MEL was more than what was needed to express Kleene logic and inference $\vdash_K$ while BC + D is sufficient as it coincides with MEL for language $\mathcal{L}^B_\Box$ and allows for resolution in this fragment.

Nevertheless, the translation into MEL [20] can be semantically described in terms of necessity measures, i.e., having unique rectangular focal sets representing incomplete information coming from a single source, which is the simplest semantics for incomplete non-contradictory information, while BC + D has pessimistic B-capacities for models. So the translation into MEL [20] is the simplest from a semantic point of view. However, in the latter translation, information from sources is interpreted in terms of knowledge or belief, due to axiom K, while, if we use BC + D, this is not the case as it is no longer possible to infer $\Box(p \wedge q)$ from $\Box p$ and $\Box q$, thus remaining more in the spirit of Belnap set-ups, where pieces of information are just collected (see Belnap and Wansing [54] response to the paper [23], where they argue against the interpretation of information items in terms of beliefs). In terms of capacities, adding axiom K comes down to restricting pessimistic capacities to necessity measures.

Remark 5. Modus ponens is a consequence of the resolution rule (RR) and its translation into BC according to Belnap and Kleene logics is

$$\{\Box a, \Box\neg a \vee \Box b\} \vdash_{BC+D} \Box b.$$

However, if we add modus ponens to the 15 rules of Belnap–Dunn logic, we do not get Kleene logic, but only the four-valued logic NBT already mentioned in Remark 3 [45]. The translation of a formula p from NBT into BC is $\mathcal{T}(v_4(p) = \mathbf{T})$. In particular, asserting an atomic formula a in NBT translates into $\Box a \wedge \Diamond a$. See Table 2 first line for conjunction and disjunction. The target modal language allows negation in front of boxes (as $\Diamond a$ stands for $\neg\Box\neg a$). As explained in Remark 1 the modalities $\blacksquare a = \Box a \wedge \Diamond a$ and $\blacklozenge a = \Box a \vee \Diamond a$ are dual and obey axiom D. In particular, it can be checked that $\blacksquare a \wedge \blacksquare\neg a \vdash \bot$ in BC. The translation of the modus ponens rule of NBT is of the form $\blacksquare a \wedge (\blacksquare\neg a \vee \blacksquare b) \vdash \blacksquare b$, which is valid in BC. This is because $\mathcal{T}(v_4(a \wedge (\neg a \vee b)) = \mathbf{T})$ is provably equivalent to $\blacksquare a \wedge (\blacksquare\neg a \vee \blacksquare b)$ (see Proposition 10 in Appendix A). The logic NBT can thus be translated into BC as well. But the translation of the resolution rule (RR) using modality $\blacksquare$ is not valid in BC, just as (RR) is not valid in NBT.

5.2. Complete conflicting information

To recover the translation of Priest Logic of Paradox into MEL [21] from the translation of Belnap's, we must add to BC an axiom that is the translation of the new axiom $a \vee \neg a$ added to Belnap–Dunn logic. Based on our translation principles for Belnap–Dunn logic, it follows that $\mathcal{T}(v_4(a \vee \neg a) \geq_t \mathbf{C}) = \Box a \vee \Box\neg a$, which must be added as an (unusual) axiom to the Belnap fragment of BC.

More generally, we can strengthen BC by adding the axiom D^d: $\Diamond p \to \Box p$, the converse of D. Capacities that are models of this axiom are optimistic, i.e., such that $\max(\beta(A), \beta(A^c)) = 1$. It is known from Proposition 2 that the focal sets of such β include a singleton, which comes down to letting the truth-value $\mathbf{U}$ disappear in Belnap–Dunn logic, since the case of a proposition being ignored by the set of sources ($\beta(A) = \beta(A^c) = 0$) cannot happen: it is ruled out by the presence of a focal singleton.

Translating formulas of Priest logic (Belnap–Dunn's plus axiom $a \vee \neg a$) to BC + D^d yields:

$$\mathcal{T}(v_4(a) = \mathbf{T}) = \Diamond a; \qquad \mathcal{T}(v_4(a) = \mathbf{F}) = \Diamond\neg a; \tag{4a}$$

$$\mathcal{T}(v_4(a) = \mathbf{C}) = \Box a \wedge \Box\neg a; \tag{4b}$$

$$\mathcal{T}(v_4(a) \geq_t \mathbf{C}) = \Box a; \qquad \mathcal{T}(v_4(a) \leq_t \mathbf{C}) = \Box\neg a. \tag{4c}$$

while $\mathcal{T}(v_4(a) = \mathbf{U}) = \Diamond a \wedge \Diamond\neg a = \bot$ in BC + D^d. Stating elementary formula a in Priest logic is translated into $\Box a$ in BC. But since $\Box a \wedge \Box\neg a$ is no longer a contradiction, we cannot use modus ponens in BC + D^d in order to infer $\Box b$ from $\{\Box a, \Box\neg a \vee \Box b\}$.

We thus get for the translation of Priest logic, the logic BC restricted to language $\mathcal{L}^B_\Box$ with the supplementary axiom $\Box a \vee \Box\neg a$. But this axiom is not in the spirit of modal logics. Due to the symmetric roles of $\mathbf{C}$ and $\mathbf{U}$ in the bilattice, and the corresponding symmetry of modalities $\Box$ and $\Diamond$ in BC, we can modify our translation principles by switching the modalities $\Box$ and $\Diamond$ and express Priest logic inference inside BC + D. It comes down, as pointed out in [21], to replacing $\mathbf{C}$ by $\mathbf{U}$, assuming the same conventions as for the translation of Kleene logic, albeit now considering both $\mathbf{U}$ and $\mathbf{T}$ as designated truth-values. It forces to changing the target sublanguage $\mathcal{L}^B_\Box$ of $\mathcal{L}_\Box$ into $\mathcal{L}^B_\Diamond$ containing only atoms $\Diamond a, \Diamond\neg a$ and their combinations via $\wedge$ and $\vee$. Note that $\mathcal{L}^B_\Box \cap \mathcal{L}^B_\Diamond = \emptyset$. It leads to translating the assertion of a proposition p in Priest logic as $\mathcal{T}(v_3(p) \in \{\mathbf{T}, \mathbf{U}\})$ and adding axiom $\Diamond a \vee \Diamond\neg a$ (an avatar of axiom D) to be used in $\mathcal{L}^B_\Diamond$.

This is summarized by the following theorem, saying that there is a consequence-preserving translation into BC + D of Priest logic, viewed as Kleene logic with designated truth-values $\mathbf{T}, \mathbf{U}$. It is another specialization of Theorem 3 [21]:

Theorem 5. *Let Γ be a set (conjunction) of formulas in propositional logic interpreted in Priest logic (rules R1–R15, plus axiom $a \vee \neg a$), and p another such formula. Then $\Gamma \vdash_P p$ if and only if $\mathcal{T}(v_3(\Gamma) \in \{\mathbf{T}, \mathbf{U}\}) \vdash_{BC+D} \mathcal{T}(v_3(p) \in \{\mathbf{T}, \mathbf{U}\})$.*

A similar comment as in the previous subsection, regarding the impossibility to write axiom K in the target propositional language for Priest logic, applies. The above results show that this axiom is not required. Using it in BC enforces a semantics in terms of Boolean possibility measures whose focal sets are singletons corresponding to an arbitrary number of completely informed hence inconsistent sources. The above results show that pure paraconsistency already prevails when only one of the sources is completely informed and disagree with the other ones.

5.3. Kleene logic of order

Kleene logic of order (discussed in Remark 4) is recovered from Belnap's adding the Kleene inference rule (R17) [51]. Translated into BC, it yields $\mathcal{T}(v_4((a \wedge \neg a) \vee c) \geq_t \mathbf{C}) \vdash \mathcal{T}(v_4((b \vee \neg b) \vee c) \geq_t \mathbf{C})$, that is, $(\Box a \wedge \Box\neg a) \vee \Box c \vdash_{BC} (\Box b \vee \Box\neg b) \vee \Box c$. The proper logic to capture this logic is BC plus a generalization of the translation of (R16), i.e., $\Box a \wedge \Box\neg a \vdash \Box b \vee \Box\neg b$:

$$(KLO): \quad \Box p \wedge \Box\neg p \vdash \Box q \vee \Box\neg q.$$

Note that in this form (KLO) is equivalent to $(\Box p \wedge \Box\neg p) \vee \Box r \vdash_{BC} (\Box q \vee \Box\neg q) \vee \Box r$, since the former is retrieved if $r = \bot$. They are not equivalent when p, q, r are literals. Adding (KLO) to BC, and applying the deduction theorem (remember BC is a propositional logic), we get the equivalent statement $(\Box a \to \Diamond a) \vee (\Diamond b \to \Box b)$. It means that either axiom D holds or its converse (D^d: $\Diamond p \to \Box p$) hold, or both.

In terms of Boolean capacities, (KLO) reads

$$\forall A, B \subseteq \Omega, \min(\beta(A), \beta(A^c)) \leq \max(\beta(B), \beta(B^c)) \tag{5}$$

We show that in this case, β is either optimistic or pessimistic.

Proposition 8. *The inequality $\forall A, B \subseteq \Omega, \min(\beta(A), \beta(A^c)) \leq \max(\beta(B), \beta(B^c))$ holds if and only if either $\forall A \subseteq \Omega, \beta(A) \leq \beta^c(A)$ or $\forall A \subseteq \Omega, \beta(A) \geq \beta^c(A)$ or yet β is a Dirac function.*

Proof. Suppose property (5) holds and there exists B such that $\beta(B)=\beta(B^c)=0$; then $\beta(A)=\beta(A^c)=0$ holds for all $A\subseteq\Omega$, and β is pessimistic. Alternatively, suppose there exists A such that $\beta(A)=\beta(A^c)=1$. Then $\beta(B)=\beta(B^c)=1$ holds for all $B\subseteq\Omega$, and β is optimistic. Otherwise, for all $A\subseteq\Omega$, $A\subseteq\Omega, \beta(A)=1$ if and only if $\beta(A^c)=0$. It means that its unique focal set is a singleton, i.e., it is a Dirac function (equivalently, a standard truth-assignment), and $\beta=\beta^c$. $\square$

The semantic entailment of Kleene logic of order clearly indicates that $p\vdash_{\leq} q$ if and only if both $p\vdash_K q$ and $p\vdash_P q$. In BC, it means that the inference of a BC formula from another one should be valid for both optimistic and pessimistic Boolean capacity models. So one may conjecture the following result:

Claim Let Γ be a set (conjunction) of formulas in Belnap–Dunn logic, and p another such formula. Then $\Gamma\vdash_{\leq} p$ if and only if $\mathcal{T}(v_3(\Gamma)\in\{\mathbf{T},\mathbf{U}\})\vdash_{BC+D}\mathcal{T}(v_3(p)\in\{\mathbf{T},\mathbf{U}\})$ and $\{\mathcal{T}(v_3(q)=\mathbf{T}):q\in\Gamma\}\vdash_{BC+RK}\mathcal{T}(v_3(q)=\mathbf{T})$.

Suppose $\Gamma=\{p_1,\dots p_n\}$ where $p_i=\vee_{j=1}^{k_i}\ell_{ij}$, and ℓ_{ij} are literals. Likewise $p=\vee_{j=1}^{k}\ell_j$. Then one may claim that $\Gamma\vdash_{\leq} p$ if and only if $\wedge_{i=1}^{n}\vee_{j=1}^{k_i}\Box\ell_{ij}\vdash_{BC+D}\vee_{j=1}^{k}\Box\ell_j$ and $\wedge_{i=1}^{n}\vee_{j=1}^{k_i}\Diamond\ell_{ij}\vdash_{BC+D}\vee_{j=1}^{k}\Diamond\ell_j$.

6. Some related inconsistency management approaches

The logic of Boolean capacities presupposes an information set-up where sources provide information about formulas, not just atoms of the language. Each source has an epistemic state that can be of the form of a set of propositions, or a set of possible worlds. Namely, each source is a propositional reasoner that provides classically consistent information. A source can directly compute, for any proposition, whether it supports it, rejects it, or can neither reject nor support it. Information collected from the sources via an existential strategy, like in Belnap set-ups, is encoded by a Boolean capacity instead of truth-values, and can be computed by the inference system of the logic BC. We have shown that Belnap–Dunn logic can be expressed in BC. In this section we show that it is also true to a large extent for some other inconsistency management approaches.

6.1. *The source-processor logic*

Avron et al. [6] are also concerned with extending the Belnap set-ups by dispensing with the constraint that sources provide information about atomic formulas only. In agreement with Dunn's convention, they assume that truth-values assigned by sources to each formula p are collected in a subset $d(p)\subseteq\{0,1\}$. Namely:

- $d(p)=\emptyset$ if no source declared p true or false (**U**);
- $d(p)=\{1\}$ if some source declared p true and none declared it false (**T**);
- $d(p)=\{0\}$ if some source declared p false and none declared it true (**F**);
- $d(p)=\{0,1\}$ if some source declared p false and another one declared it true (**C**).

Function d is called a valuation processor, and it follows a number of conditions:

- (d1): $0\in d(\neg p)$ if and only if $1\in d(p)$
- (d2): $1\in d(\neg p)$ if and only if $0\in d(p)$
- (d3): $1\in d(p\vee q)$ if $1\in d(p)$ or $1\in d(q)$
- (d4): $0\in d(p\vee q)$ if and only if $0\in d(p)$ and $0\in d(q)$
- (d5): $1\in d(p\wedge q)$ if and only if $1\in d(p)$ and $1\in d(q)$
- (d6): $0\in d(p\wedge q)$ if $0\in d(p)$ or $0\in d(q)$

It is clear that a valuation processor can be expressed in terms of Boolean capacities (we write $\beta(p)$ for $\beta([p])$, for simplicity). We can translate the statement $1\in d(p)$ by $\beta(p)=1$ and $0\in d(p)$ by $\beta(\neg p)=1$ in our capacity-based setting; also $1\notin d(p)$ by $\beta(p)=0$, and $0\notin d(p)$ by $\beta(\neg p)=0$. Then:

- $d(p)=\emptyset$ reads $\beta(p)=\beta(\neg p)=0$;
- $d(p)=\{1\}$ reads $\beta(p)=1$, $\beta(\neg p)=0$;
- $d(p)=\{0\}$ reads $\beta(p)=0$, $\beta(\neg p)=1$;
- $d(p)=\{0,1\}$ reads $\beta(p)=\beta(\neg p)=1$.

Let us check if the requirements (d1–d6) are sanctioned by the approach based on capacities.

- (d1) and (d2) trivially hold in the language of capacities. Indeed, $0\in d(\neg p)$ reads $\beta(\neg\neg p)=\beta(p)=1$, and likewise, $1\in d(\neg p)$ reads $\beta(\neg p)=1$.
- (d3) and (d6) respectively read: $\beta(p\vee q)=1$ if $\beta(p)=1$ or $\beta(q)=1$, and $\beta(\neg p\vee\neg q)=1$ if $\beta(\neg p)=1$ or $\beta(\neg q)=1$, which holds due to the monotonicity of β. The two conditions are equivalent due to (d1–d2).

Table 3
Disjunction and conjunction for general source-processor logic.

$\tilde{\vee}$	**F**	**U**	**C**	**T**	$\tilde{\wedge}$	**F**	**U**	**C**	**T**
F	{**F**, **C**}*	{**T**, **U**}	{**C**}*	{**T**}	**F**	{**F**}	{**F**}	{**F**}	{**F**}
U	{**T**, **U**}	{**T**, **U**}	{**T**}	{**T**}	**U**	{**F**}	{**F**, **U**}	{**F**}	{**F**, **U**}
C	{**C**}*	{**T**}	{**C**}*	{**T**}	**C**	{**F**}	{**F**}	{**C**}*	{**C**}*
T	{**T**}	{**T**}	{**T**}	{**T**}	**T**	{**F**}	{**F**, **U**}	{**C**}*	{**T**, **C**}*

Table 4
Non-deterministic disjunction and conjunction in the capacity-based approach.

$\tilde{\vee}$	**F**	**U**	**C**	**T**	$\tilde{\wedge}$	**F**	**U**	**C**	**T**
F	$\mathcal{V}$	{**T**, **U**}	{**T**, **C**}	{**T**}	**F**	{**F**}	{**F**}	{**F**}	{**F**}
U	{**T**, **U**}	{**T**, **U**}	{**T**}	{**T**}	**U**	{**F**}	{**F**, **U**}	{**F**}	{**F**, **U**}
C	{**T**, **C**}	{**T**}	{**T**, **C**}	{**T**}	**C**	{**F**}	{**F**}	{**F**, **C**}	{**F**, **C**}
T	{**T**}	{**T**}	{**T**}	{**T**}	**T**	{**F**}	{**F**, **U**}	{**F**, **C**}	$\mathcal{V}$

- (d4) and (d5) respectively read: $\beta(\neg p \wedge \neg q) = 1$ if and only if $\beta(\neg p) = 1$ and $\beta(\neg q) = 1$, and $\beta(p \wedge q) = 1$ if and only if $\beta(p) = 1$ and $\beta(q) = 1$, again two equivalent conditions due to (d1–d2). *They generally do not hold for capacities*. In fact we only have, due to the monotonicity condition, that $\beta(p \vee q) = 0$ implies $\beta(p) = 0$ and $\beta(q) = 0$ and $\beta(p \wedge q) = 1$ implies $\beta(p) = 1$ and $\beta(q) = 1$, i.e., the two conditions
  - (d4') $1 \notin d(p \vee q)$ implies $1 \notin d(p)$ and $1 \notin d(q)$
  - (d5') $1 \in d(p \wedge q)$ implies $1 \in d(p)$ and $1 \in d(q)$

  This is the major difference between our setting and the source-processor logic setting.

These authors justify the equivalences (d4–d5) by assuming that after collecting truth-values in $d(p)$ the origin of these truth-values is forgotten. As a consequence, if for instance $1 \in d(p)$ and $1 \in d(q)$, then since under these truth-values $p \wedge q$ should have truth-value 1, it follows that $1 \in d(p \wedge q)$. However, $\beta(p) = 1$ and $\beta(q) = 1$ implies $\beta(p \wedge q) = 1$ only for necessity measures.

Besides, in the capacity-based approach, there are limit conditions $\beta(\bot) = 0$ and $\beta(\top) = 1$. However, after (d4–d5), one may have that $1 \in d(\bot)$, i.e., $\beta(\bot) = 1$, which, by monotonicity, would yield $\beta(p) = 1$ for all propositional formulas p. This situation cannot occur with capacities. In terms of truth value collector, the limit condition of capacities reads:

$$(\mathrm{d}^{lim}) : \mathrm{d}(\bot) = \{0\}; \mathrm{d}(\top) = \{1\}.$$

Likewise we could think of a condition dual to (d5) in the form

$$(\mathrm{d5}^{\neg}) : 1 \notin d(p \vee q) \text{ if and only if } 1 \notin d(p) \text{ and } 1 \notin d(q),$$

that is, $\beta(p) = 0$ and $\beta(q) = 0$ if and only if $\beta(p \vee q) = 0$, but this condition only holds for possibility measures. Requiring both (d5) and ($\mathrm{d5}^{\neg}$) turns β into a mere Boolean truth-assignment function.

Avron et al. [6] show that their framework leads to what they call the "most general source processor logic", in the form of a four-valued matrix $(\mathcal{V}, \mathbb{D}, \mathcal{O})$ with $\mathcal{V} = \mathbb{V}_4$, the Belnap–Dunn four-valued truth set, $\mathbb{D} = \{\mathbf{T}, \mathbf{C}\}$ the designated truth-values, and $\mathcal{O}$ containing three extended connectives denoted by $\tilde{\neg}, \tilde{\vee}, \tilde{\wedge}$, with non-deterministic truth tables for the latter two connectives given on Table 3. In contrast with the general source-processor logic, the non-deterministic truth tables computed from our approach (replacing (d4–d5) by (d4'–d5')) are more imprecise than the truth tables in Table 3.

There are four (starred) entries in each truth table of Table 3 that are not in agreement with the capacity-based approach, approximated by the truth tables reported in Table 4:

- $\mathbf{T}\tilde{\wedge}\mathbf{T}$ is totally indeterminate. Indeed, assigning **T** to p and q reads $\beta(p) = 1 = \beta(q)$, and $\beta(\neg p) = 0 = \beta(\neg q)$. It does not induce any constraint on $\beta(p \wedge q)$ nor on $\beta(\neg p \vee \neg q)$ except if $p = \neg q$, and then it yields **F** (lost by the truth table). Likewise, by De Morgan duality, $\mathbf{F}\tilde{\vee}\mathbf{F}$ is indeterminate.
- $\mathbf{C}\tilde{\vee}\mathbf{C} = \{\mathbf{T}, \mathbf{C}\}$: in terms of capacities, it is $\beta(p) = \beta(q) = \beta(\neg p) = \beta(\neg q) = 1$. Hence $\beta(p \vee q) = 1$ but there is no constraint on $\beta(\neg p \wedge \neg q)$. So only two cases remain for $p \vee q$: **C** if $\beta(\neg p \wedge \neg q) = 1$ and **T** if $\beta(\neg p \wedge \neg q) = 0$. Likewise, by De Morgan duality, $\mathbf{C}\tilde{\wedge}\mathbf{C} = \{\mathbf{F}, \mathbf{C}\}$.
- $\mathbf{C}\tilde{\wedge}\mathbf{T} = \{\mathbf{F}, \mathbf{C}\}$: in terms of capacities, it is $\beta(p) = \beta(\neg p) = 1$ and $\beta(q) = 1, \beta(\neg q) = 0$. Hence $\beta(\neg p \vee \neg q) = 1$ but there is no constraint on $\beta(p \wedge q)$. So only two cases remain for $p \wedge q$: **C** if $\beta(p \wedge q) = 1$ and **F** if $\beta(p \wedge q) = 0$. Likewise, by De Morgan duality, $\mathbf{C}\tilde{\vee}\mathbf{F} = \{\mathbf{T}, \mathbf{C}\}$.
- In contrast, $\mathbf{U}\tilde{\wedge}\mathbf{C} = \{\mathbf{F}\}$ like in Belnap–Dunn logic. Indeed it reads $\beta(p) = \beta(\neg p) = 0$ and $\beta(q) = \beta(\neg q) = 1$. It implies by monotonicity $\beta(p \wedge q) = 0$ and $\beta(\neg p \vee \neg q) = 1$, so {**F**}.

It can be argued that (up to the limit condition (d^{lim})) our approach is the least constrained one for reasoning about sources, in the sense that, given n sources providing n pieces of information in the form of sets of possible worlds

$\{E_1,\ldots,E_n\}$ the set of derived propositions corresponding to $\{A:\beta(A)=1\}$ is precisely the neighborhood $\cup_{i=1}^n\{A:E_i\subseteq A\}$. In contrast, the general source-processor logic also derives conjunctions of the $E_i's$ (due to (d4–d5)). In particular, Avron et al. [6] consider the case of a general source processor with complete information, understood as the situation where each atom of the language is informed by at least one source. It means that $1\in d(\ell_i)$ for one literal pertaining to each propositional variable p_i ($\ell_i=p_i$ or $\ell_i=\neg p_i$). As a consequence of (d5), $1\in d(\wedge_{i=1}^k\ell_i)$ so that by (d5) and (d6) again for any formula p, either $1\in d(p)$ or $0\in d(p)$, and epistemic truth-value **U** is never assigned. In our capacity logic, this situation is not enough to rule out truth-value **U**. Complete information in the capacity-based approach requires that *one* source can assign truth-value 1 or 0 to *all* formulas, i.e., has complete information (see Proposition 2).

Another difference with our approach is their use of a sequent calculus for the logic. The motivation for going beyond classical logic is the same as ours when using a modal framework, namely, the fact that in the multisource problem, the truth of p (in the sense of **T**) is not equivalent to the falsity of $\neg p$ (in the sense of **F**), and the difficulty to express disjunctive knowledge. So, these authors use sequents of the form $p_1,\ldots,p_n\Rightarrow q_1,\ldots,q_m$, understood as

"either $1\notin d(p_1)$ or ... or $1\notin d(p_n)$ or $1\in d(q_1)$ or ... or $1\in d(q_m)$".

They propose logical rules that express properties (d1–d6) using sequents, and show that the obtained logic exactly accounts for the partial truth tables in Table 3. In the BC logic, the above sequent can be expressed as claiming the truth of:

$$\Diamond\neg p_1\vee\cdots\vee\Diamond\neg p_n\vee\Box q_1\vee\cdots\vee\Box q_m$$

or equivalently, $\Box p_1\wedge\cdots\wedge\Box p_n\vdash\Box q_1\vee\cdots\vee\Box q_m$. However, there is one major difference between the logic BC and the source-processor logic using sequents: the capacity-based approach is not truth-functional, in the sense that applying the partial truth tables in Table 4 using an appropriate sequent system will produce results that are sometimes weaker than the BC logic. For instance, even if $\beta(p)=\beta(\neg p)=1$ still we have $\beta(\perp)=0$, because $\Box\perp$ and $\Diamond\perp$ are contradictions in BC. In the source processor logic, due to (d5), $1\in d(p)$ and $1\in d(\neg p)$ imply $1\in d(\perp)$ which prevents $p\wedge\neg p$ from being considered false. Using the non-deterministic truth table induced by capacities in Table 4, we also get a completely indeterminate result for $p\wedge\neg p$ since $\mathbf{T}\tilde{\wedge}\mathbf{T}=\mathcal{V}$.

6.2. *Propositional logic approaches*

Other approaches to inconsistency handling seem to be representable inside the logic BC. For instance, a very standard approach is the use of all maximal consistent subsets $K_i, i=1,\ldots,n$ of formulas in an inconsistent propositional base K, in order to circumvent the problem of explosive classical consequences of K (Rescher and Manor [49]). The set of consequences of K is defined by the union or the intersection of the sets of consequences of the bases K_i. Viewing each maximal consistent base K_i as a source, a proposition p is deduced in the existential approach if $K_i\vdash p$ for some source i, and, in the universal approach, if $K_i\vdash p$ for all $i=1,\ldots,n$.

This method can be captured by Boolean capacities with disjoint focal sets $E_i=[K_i], i=1,\ldots,n$. First encode the propositional logic base $K=\{p_1,\ldots,p_m\}$ in BC, letting $K^\Box=\{\Box p_1,\ldots,\Box p_m\}$. Then, apply to $K^\Box$ the inference rule

MC: $\Box p,\Box q\vdash\Box p\wedge q$ whenever $p\wedge q\neq\perp$.

Let $Cl^{MC}(K^\Box)$ be the deductive closure of $K^\Box$ using inference rule MC. Consider the capacity β such that $\beta(A)=1$ if $[p]\subseteq A$ for some $\Box p\in Cl^{MC}(K^\Box)$. It is clear that this capacity has disjoint focal sets E_i, one for each maximal consistent subset K_i of K. It is then possible to apply the inference rules of BC to $Cl^{MC}(K^\Box)$. The set $\{p:Cl^{MC}(K^\Box)\vdash_{BC}\Box p\}$ is the set of consequences of K in the sense of the existential closure in the maximal consistent subset approach. Besides if K is consistent, β has just one focal set $[K]$, and we recover the classical deductive closure of K.

Note that, in contrast, the deductive closure of $K^\Box$, obtained directly using the inference rules of BC, is more cautious than in classical logic as the atoms it contains are $\{\Box p:\exists i, p_i\vdash p\}$, even if K is consistent. The universal approach to the maximum consistent subset method cannot be directly expressed in BC, because the necessity modality in the latter logic is used to express the epistemic truth-value **T** of Belnap, which has an existential flavor.

Another inconsistency handling method that can be recovered by the BC logic is the one proposed by Benferhat et al. [11], that is halfways between the existential approach to maximal consistent subset entailment and the universal approach. The idea is to define a so-called *argumentative inference* from a possibly inconsistent propositional base K defining $K\vdash_{arg}p$ whenever there is a consistent subset $C\subseteq K$ such that $C\vdash p$ and there is no consistent subset $C\subseteq K$ such that $C\vdash\neg p$, which corresponds to assigning epistemic truth-value **T** to p when $K\vdash_{arg}p$ [23]. Clearly, it can be easily seen that we have $K\vdash_{arg}p$ if and only if $K^\Box\vdash_{BC}\Box p\wedge\Diamond p$, which suggests a possible connection with the NBT logic of Rivieccio [51,45], i.e., using Belnap–Dunn bilattice with only designated value **T**.

7. Conclusion

In this paper, we have pursued our work towards a better understanding of a class of many-valued logics dealing with inconsistent or incomplete information processing. We have shown that just like Kleene logic and Priest's Logic of Paradox, we can capture Belnap–Dunn four-valued logics in a simple higher-order propositional logic BC couched in the language of modal logic involving only depth-1 formulas. The natural semantics for this propositional logic is in terms of all-or-nothing set functions that can capture both incomplete and inconsistent pieces of information. The set function semantics is akin to neighborhood semantics [17].

The logic BC can capture general information collection set-ups not confined to atomic propositions. In that sense the logic BC is a general (seemingly the weakest possible) setting for modeling incomplete and inconsistent information having as particular instance Belnap–Dunn four-valued logic. The use of set functions beyond possibility and necessity measures to interpret the BC logic, thus dropping axiom K, is in agreement with the fact that propositions in this four-valued logic cannot be viewed as S5-like beliefs. We have indicated that our framework may be used to capture some other approaches to inconsistent and incomplete information handling, basically those starting from an inconsistent set of Boolean propositions (for instance, using maximal consistent subsets), or yet the generalized Belnap set-ups considered by Avron et al. [6]. In previous works, we showed that several three-valued logics of incomplete information (including Łukasewicz and Nelson logics [20]) and several paraconsistent three-valued logics (including Jaśkowski and Sobociński's logics [21]) can be captured in the setting of MEL. It could be of interest to pursue this effort, using BC as a setting, towards more general paraconsistent settings (like the Logics of Formal Inconsistency [16] or ideal paraconsistent logics of Arieli et al. [4]), but also four-valued logics richer than Belnap's.

Besides, the use of set functions clarifies the connection between Belnap–Dunn four-valued logic and uncertainty modeling. In this sense, our paper paves the way to graded or numerical extensions of Belnap-like inconsistency handling methods (for instance modeling the reliability of the sources) in agreement with uncertainty theories (as suggested in [23] and more recently in [25]). It may also bridge the gap with modal versions of probabilistic logic (Hamblin [39], Burgess [14], see also the pioneering survey by Walley and Fine [53], and more recently, the logic of risky knowledge [43]) where axiom K is not valid.

Along the same line, recent works in argumentation theory try to find a ranking of arguments in terms of degrees of acceptability, computed from an attack relation between arguments extracted from an inconsistent propositional base [3]. On this basis, the propositional base is ranked in terms of relative plausibility, according to which a formula is more plausible than another if supported by an argument that is more acceptable than any argument that supports the other formula [2]. This plausibility ranking is consistent with logical inference in the sense that if p implies q then p is not more plausible than q. Expressed in our terminology, these new argumentation-based ranking logics essentially compute a capacity, which is in agreement with the spirit of our BC logic, albeit more expressive.

From the logical standpoint, an infinite set of logics extending the Belnap–Dunn one has been defined in [51]. Some of them, that use the 4 truth values seem to be captured as well by our setting, as suggested in the previous sections. Other extensions consist in adding the generalized ex-falso quodlibet rules: $\frac{(p_1 \wedge \neg p_1) \vee \ldots (p_n \wedge \neg p_n)}{q}$ with $n = 1, \ldots, \infty$ to Belnap–Dunn logic inference rules. A infinite chain of intermediate logics between Belnap–Dunn and strong Kleene logics is obtained. However, most of these logics cannot be interpreted in terms of multisource settings, and require a semantics with many more than four truth-values, thus escaping our Boolean capacity setting.

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Appendix A. Proofs

Equivalence between inference rules in Belnap–Dunn logic. We give some results concerning the equivalence $\vDash_B = \vDash_C = \vDash_U$ [35], thus completing the discussion in Subsection 3.3.

Lemma 1. *$p \vDash_U q$ iff $p \vDash_C q$.*

Proof. Suppose $p \vDash_U q$. It means that $\forall v_4$, if $v_4(p) \in \{\mathbf{U}, \mathbf{T}\}$ then $v_4(q) \in \{\mathbf{U}, \mathbf{T}\}$. Let us show that $p \vDash_C q$ as well. For all v such that $v_4(p) = \mathbf{U}$, the condition for $p \vDash_C q$ is not violated as it cannot be applied. If $v_4(p) = \mathbf{T}$ and $v_4(q) = \mathbf{T}$, $p \vDash_C q$ is not violated either. Suppose $v_4(p) = \mathbf{T}$ and $v_4(q) = \mathbf{U}$. Then v_4 violates the condition for $p \vDash_C q$. However if such a valuation exists, exchanging **C** and **U** in the assignment of atoms by v_4, preserving assignments **T** and **F**, there exists another valuation v'_4 such that $v'_4(p) = \mathbf{T}$ and $v'_4(q) = \mathbf{C}$. But v'_4 then violates $p \vDash_U q$ which thus does not hold. It contradicts our assumption. So assignments such that $v_4(p) = \mathbf{T}$ and $v_4(q) \neq \mathbf{T}$ are in agreement neither with $p \vDash_U q$ nor $p \vDash_C q$. The only possibility for v_4 to be in agreement with $p \vDash_U q$ is that $v_4(p) = \mathbf{T} = v_4(q) = \mathbf{T}$, or $v_4(p) = \mathbf{U}$ and $v_4(q) \geq_t \mathbf{U}$, or $v_4(p) = \mathbf{C}$ and $v_4(q) \geq_t \mathbf{C}$. So, $\vDash_U$ and $\vDash_C$ correspond to the same pairs (p, q). □

Lemma 2. *$\Gamma \vDash_B p$ iff both $\Gamma \vDash_U p$ and $\Gamma \vDash_C p$.*

Proof. We show it for two premises, i.e., consider the case when $v_4(p_1) \wedge v_4(p_2) \leq_t v_4(p)$. The interesting cases are when $(v_4(p_1), v_4(p_2), v_4(p))$ are of the form $(\mathbf{T}, \mathbf{C}, \mathbf{C})$, $(\mathbf{T}, \mathbf{U}, \mathbf{U})$, $(\mathbf{C}, \mathbf{C}, \mathbf{C})$, $(\mathbf{U}, \mathbf{U}, \mathbf{U})$, and $(\mathbf{T}, \mathbf{T}, \mathbf{T})$ (otherwise $v_4(p_1) \wedge v_4(p_2) = \mathbf{F}$).

Then, it is obvious that $p_1 \wedge p_2 \vDash_U p$ and $p_1 \wedge p_2 \vDash_C p$. For the converse, suppose that $p_1 \wedge p_2 \vDash_U p$ and $p_1 \wedge p_2 \vDash_C p$, but $v_4(p_1) \wedge v_4(p_2) \nleq_t v_4(p)$. The latter condition holds in the following cases only:

- if $v_4(p) = \mathbf{C}$ and $v_4(p_1) = \mathbf{T}$, $v_4(p_2) = \mathbf{U}$, which violates $p_1 \wedge p_2 \vDash_U p$,
- if $v_4(p) = \mathbf{U}$ and $v_4(p_1) = \mathbf{T}$, $v_4(p_2) = \mathbf{C}$, which violates $p_1 \wedge p_2 \vDash_C p$,
- if $v_4(p) = \mathbf{F}$ and $v_4(p_1) > \mathbf{F}$, $v_4(p_2) > \mathbf{F}$, which violates both. $\square$

Proposition 9. *Let $p_1 = \vee_i l_i$, $p_2 = \vee_j l_j$. Then, $p_1 \vDash_B p_2$ iff $p_1 \vDash_U p_2$.*

Proof. One direction is trivial. Let us suppose that $p_1 \vDash_U p_2$ and at the same time that $p_1 \nvDash_B p_2$. It means that there exists v such that $v_4(p_1) \nleq_t v_4(p_2)$. Either $v_4(p_1) >_t v_4(p_2)$, or $v_4(p_1) = \mathbf{C}$ and $v_4(p_2) = \mathbf{U}$. In the first case, due to the fact that $p_1 \vDash_U p_2$ this is only possible if $v_4(p_1) = \mathbf{T}$ and $v_4(p_2) = \mathbf{U}$. Thus,

- either there exists a literal l^* in p_1 such that $v_4(l^*) = \mathbf{T}$ and $l^* \notin p_2$. But if this is the case, we can define $v_4^*(l^*) = \mathbf{T}$, $v_4^*(l) = \mathbf{C}$ for $l \neq l^*$, and contradict our hypothesis $p_1 \vDash_U p_2$
- or there exist literals $l, l' \in p_1$ and v_4^* with $v_4^*(l) = \mathbf{C}$, $v_4^*(l') = \mathbf{U}$ (so that $v_4^*(l \vee l') = \mathbf{T}$). The two literals cannot be both in p_2. Then we can define $v_4^*(l) = \mathbf{C}$ for $l \neq l^*$, and contradict our hypothesis $p_1 \vDash_U p_2$ because $v_4^*(p_2) = \mathbf{C}$ (if $l' \notin p_2$ or if $v_4^*(l') = \mathbf{C}$) or $v_4^*(p_2) = \mathbf{F}$ if $l' \in p_2$, $v_4^*(l') = \mathbf{U}$.

In the other case, there exists v_4^*, such that $v_4^*(p_1) = \mathbf{C}$ and $v_4^*(p_2) = \mathbf{U}$, which means that there exists $l^* \in p_1$ with $v_4^*(l^*) = \mathbf{C}$ while $v_4^*(l) \in \{\mathbf{F}, \mathbf{C}\}$ for other literals in p_1, and $l' \in p_2$ with $v_4^*(l') = \mathbf{U}$ while $v_4^*(l) \in \{\mathbf{F}, \mathbf{U}\}$ for other literals in p_2. $\square$

On modus ponens in NBT. (See Remark 5 of Subsection 3.4.)

Proposition 10. *$\mathcal{T}(v_4(a \wedge (\neg a \vee b))) = \mathbf{T})$ is equivalent to $\blacksquare a \wedge (\blacksquare \neg a \vee \blacksquare b)$ in BC.*

Proof. $\mathcal{T}(v_4(a \wedge (\neg a \vee b)) = \mathbf{T}) = \mathcal{T}(v_4(a) = \mathbf{T}) \wedge \mathcal{T}(v_4(\neg a \vee b) = \mathbf{T}) = (\Box a \wedge \Diamond a) \wedge [(\Box \neg a \wedge \Diamond \neg a) \vee (\Box b \wedge \Diamond b) \vee (\Box a \wedge \Box \neg a \wedge \Diamond b \wedge \Diamond \neg b) \vee (\Diamond a \wedge \Diamond \neg a \wedge \Box b \wedge \Box \neg b)] = \blacksquare a \wedge (\blacksquare \neg a \vee \blacksquare b)$, since $(\Box a \wedge \Diamond a)$ is inconsistent with $\Box \neg a = \neg \Diamond a$, and with $\Diamond \neg a = \neg \Box a$. $\square$

As $\blacksquare a \wedge \blacksquare \neg a \vdash \perp$, $\blacksquare a \wedge (\blacksquare \neg a \vee \blacksquare b)$ can be further simplified as $\blacksquare a \wedge \blacksquare b$ that implies $\blacksquare b$. So the modus ponens of the NBT logic can be justified in BC.

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