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Közgazdaság- és Regionális Tudományi Kutatóközpont
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Popular Matchings in Complete Graphs

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CERS-IE WP – 2020/4

January 2020

https://www.mtakti.hu/wp-content/uploads/2020/01/CERSIEWP202004.pdf

INSTITUTE OF ECONOMICS, CENTRE FOR ECONOMIC AND REGIONAL STUDIES,
BUDAPEST, 2020

ABSTRACT

Our input is a complete graph G on n vertices where each vertex has a strict ranking of all other vertices in G. The goal is to construct a matching in G that is "globally stable" or popular. A matching M is popular if M does not lose a head-to-head election against any matching M': here each vertex casts a vote for the matching in {M,M'} in which it gets a better assignment. Popular matchings need not exist in the given instance G and the popular matching problem is to decide whether one exists or not. The popular matching problem in G is easy to solve for odd n. Surprisingly, the problem becomes NP-hard for even n, as we show here. This seems to be the first graph theoretic problem that is efficiently solvable when n has one parity and NP-hard when n has the other parity.

JEL codes: C63, C78
Keywords: popular matching, NP-completeness, polynomial algorithm, stable matching

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Népszerű párosítások teljes gráfokban

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ÖSSZEFOGLALÓ

Adott egy teljes gráf, ahol minden csúcs szigorú listában rangsorolja a szomszédjait. Egy M párosítást akkor nevezünk népszerűnek, ha nincsen olyan M' párosítás, hogy több csúcs részesíti előnyben M'-t M-mel szemben, mint fordítva. Nem minden inputban létezik népszerű párosítás. Ha a csúcsok száma a gráfban páratlan, akkor könnyű eldönteni, hogy az inputon van-e népszerű párosítás. Meglepő módon ugyanez a probléma NP-teljes olyan gráfokon, amik páros sok csúcsot tartalmaznak. Ez az első olyan gráfelméleti probléma, ahol a csúcshalmaz paritása ilyen bonyolultságelméleti különbséget indukál.

JEL: C63, C78
Kulcsszavak: népszerű párosítás, NP-teljesség, polinomiális algoritmus, stabil párosítás

Popular Matchings in Complete Graphs

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Abstract. Our input is a complete graph G on n vertices where each vertex has a strict ranking of all other vertices in G. The goal is to construct a matching in G that is "globally stable" or *popular*. A matching M is popular if M does not lose a head-to-head election against any matching M': here each vertex casts a vote for the matching in $\{M, M'\}$ in which it gets a better assignment. Popular matchings need not exist in the given instance G and the popular matching problem is to decide whether one exists or not. The popular matching problem in G is easy to solve for odd n. Surprisingly, the problem becomes NP-hard for even n, as we show here. This seems to be the first graph theoretic problem that is efficiently solvable when n has one parity and NP-hard when n has the other parity.

1 Introduction

Consider a complete graph $G = (V, E)$ on n vertices where each vertex ranks all other vertices in a strict order of preference. Such a graph is called a *roommates* instance with complete preferences. The problem of computing a stable matching in G is a classical and well-studied problem. Recall that a matching M is stable if there is no *blocking pair* with respect to M, i.e., a pair (u, v) where both u and v prefer each other to their respective assignments in M.

Stable matchings need not always exist in a roommates instance. For example, the instance given in Fig. 1 on 4 vertices d_0, d_1, d_2, d_3 has no stable matching. (Here d_0's top choice is d_1, second choice is d_2, and last choice is d_3, and similarly for the vertices.)

d_0 : $d_1 > d_2 > d_3$
d_1 : $d_2 > d_3 > d_0$
d_2 : $d_3 > d_1 > d_0$
d_3 : $d_1 > d_2 > d_0$

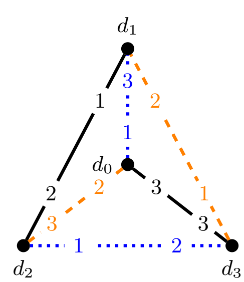

Fig. 1. An instance that admits two popular matchings—marked by dotted blue and dashed orange edges—but no stable matching. The preference list of each vertex is illustrated by the numbers on its edges: a lower number indicates a more preferred neighbor.

Irving [17] gave an efficient algorithm to decide if G admits a stable matching or not. In this paper we consider a notion that is more relaxed than stability: this is the notion of *popularity*. For any vertex u, a ranking over neighbors can be extended naturally to a ranking over matchings as follows: u prefers matching M to matching M' if (i) u is matched in M and unmatched in M' or

⋆ Supported by the Cooperation of Excellences Grant (KEP-6/2017), by the Ministry of Human Resources under its New National Excellence Programme (ÚNKP-18-4-BME-331), the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3/2016), its János Bolyai Research Fellowship, and OTKA grant K128611.

⋆⋆ This work was done while visiting the Hungarian Academy of Sciences, Budapest.

(ii) u is matched in both and u prefers $M(u)$ to $M'(u)$. For any two matchings M and M', let $\phi(M, M')$ be the number of vertices that prefer M to M'.

Definition 1. *Let M be any matching in G. M is* popular *if $\phi(M, M') \geq \phi(M', M)$ for every matching M' in G.*

Suppose an election is held between M and M' where each vertex casts a vote for the matching that it prefers. So $\phi(M, M')$ (similarly, $\phi(M', M)$) is the number of votes for M (resp., M'). A popular matching M never loses an election to another matching M' since $\phi(M, M') \geq \phi(M', M)$: thus it is a weak *Condorcet winner* [5, 6] in the corresponding voting instance. This is why popularity can be regarded as a natural notion of "global stability".

The notion of popularity was first introduced in bipartite graphs in 1975 by Gärdenfors—popular matchings always exist in bipartite graphs since stable matchings always exist here [10] and every stable matching is popular [11]. The proof that every stable matching is popular holds in non-bipartite graphs as well [4]; in fact, it is easy to show that every stable matching is a min-size popular matching [14]. Relaxing the constraint of stability to popularity allows us to find globally stable matchings that may exist in instances that do not admit stable matchings; moreover, even when stable matchings exist, there may be popular matchings that achieve more "social good" (such as larger size), which might be relevant in many applications.

Observe that the instance in Fig. 1 has two popular matchings: $M_1 = \{(d_0, d_1), (d_2, d_3)\}$ and $M_2 = \{(d_0, d_2), (d_1, d_3)\}$. However as was the case with stable matchings, popular matchings also need not always exist in the given instance G. Just take, for example, the same instance as in Fig. 1, but without vertex d_0. The *popular roommates problem* is to decide if G admits a popular matching or not. When the graph is not complete, it is known that the popular roommates problem is NP-hard [9, 12]. Here we are interested in the complexity of the popular matching problem when the input instance is complete.

Interestingly, several popular matching problems that are intractable in bipartite graphs become tractable in *complete bipartite* graphs. The min-cost popular matching problem in bipartite graphs is such a problem—this is NP-hard in a bipartite graph with incomplete lists [9], however it can be solved in polynomial time in a bipartite graph with complete lists [8]. The difference is due to the fact that while there is no efficient description of the convex hull of all popular matchings in a general bipartite graph, this polytope has a compact extended formulation in a complete bipartite graph.

It is a simple observation (see Section 2) that when n is *odd*, a matching in a complete graph G on n vertices is popular only if it is stable. Since there is an efficient algorithm to decide if G admits a stable matching or not, the popular roommates problem in a complete graph G can be efficiently solved when n is odd. We show the following result here.

Theorem 1. *Let G be a complete graph on n vertices, where n is even. The problem of deciding whether G admits a popular matching or not is* NP*-hard.*

So the popular roommates problem with complete preference lists is NP-hard for even n while it is easy to solve for odd n. Note that the popular roommates problem is non-trivial for every $n \geq 5$, i.e., there are both "yes instances" and "no instances" of size n. It is rare and unusual for a natural decision problem in combinatorial optimization to be efficiently solvable when n has one parity and become NP-hard when n has the other parity. We are not aware of any natural optimization problem on graphs that is non-trivially tractable when the cardinality of the vertex set has one parity, which becomes intractable for the other parity.

1.1 Background and related work

The first polynomial time algorithm for the stable roommates problem was given by Irving [17] in 1985. Roommates instances that admit stable matchings were characterized in [25]. New polynomial time algorithms for the stable roommates problem were given in [24, 26].

Algorithmic questions for popular matchings in bipartite graphs have been well-studied in the last decade [1, 8, 14, 16, 18–20]. Not much was known on popular matchings in non-bipartite graphs.

Biró et al. [1] proved that validating whether a given matching is popular can be done in polynomial time, even when ties are present in the preference lists. It was shown in [15] that every roommates instance $G = (V, E)$ admits a matching with *unpopularity factor* $O(\log |V|)$ and that it is NP-hard to compute a least unpopularity factor matching. It was shown in [16] that computing a max-weight popular matching in a roommates instance with edge weights is NP-hard, and more recently, that computing a max-size popular matching in a roommates instance is NP-hard [3].

The complexity of the popular roommates problem was open for several years [1, 7, 15, 16, 22] and two independent NP-hardness proofs [9, 12] of this problem were announced very recently. Interestingly, both these hardness proofs need "incomplete preference lists", i.e., the underlying graph is *not* complete. The reduction in [12] is from a variant of the vertex cover problem called the *partitioned vertex cover* problem and we discuss the reduction in [9] in Section 1.2 below. So the complexity status of the popular roommates problem in a complete graph was an open problem and we resolve it here.

Computational hardness for instances with complete lists has been investigated in various matching problems under preferences. An example is the three-sided stable matching problem with cyclic preferences: this involves three groups of participants, say, men, women, and dogs, where dogs have weakly ordered preferences over men only, men have preferences over women only, and finally, women only list the dogs. If these preferences are allowed to be incomplete, the problem of finding a *weakly* stable matching is known to be NP-complete [2]. It is one of the most intriguing open questions in stable matchings [22, 27] as to whether the same problem becomes tractable when lists are complete.

1.2 Techniques

The 1-in-3 SAT problem is a well-known NP-hard problem [23]: it consists of a 3-SAT formula ϕ with no negated literals and the problem is to find a truth assignment to the variables in ϕ such that every clause has exactly one variable set to $\mathtt{true}$. We show a polynomial time reduction from 1-in-3 SAT to the popular roommates problem with complete lists.

Our construction is based on the reduction in [9] that proved the NP-hardness of the popular roommates problem. However there are several differences between our reduction and the reduction in [9]. The reduction in [9] considered a popular matching problem in bipartite graphs called the "exclusive popular set" problem and showed it to be NP-hard—when preference lists are complete, this problem can be easily solved. Thus the reduction in [9] needs incomplete preference lists.

The exclusive popular set problem asks if there is a popular matching in the given bipartite graph where the set of matched vertices is S, for a given even-sized subset S. A key step in the reduction in [9] from this problem in bipartite graphs to the popular matching problem in non-bipartite graphs merges all vertices outside S into a single node. Thus the total number of vertices in the non-bipartite graph used in [9] is *odd*. Moreover, the fact that popular matchings always exist in bipartite graphs is crucially used in this reduction. However in our setting, the whole problem is to decide if *any* popular matching exists in the given graph—thus there are no popular matchings that "always exist" here.

The reduction in [9] primarily uses the LP framework of popular matchings in bipartite graphs from [18, 19, 21] to analyze the structure of popular matchings in their instance. The LP framework characterizing popular matchings in non-bipartite graphs is more complex [21], so we use the combinatorial characterization of popular matchings [14] in terms of forbidden alternating paths/cycles to show that any popular matching in our instance will yield a 1-in-3 satisfying assignment for ϕ. To show the converse, we use a dual certificate similar to the one used in [9] to prove the popularity of the matching that we construct using a 1-in-3 satisfying assignment for ϕ.

Organization of the paper. We discuss preliminaries in Section 2. Section 3 describes the construction of our complete graph G corresponding to a given a 1-in-3 SAT formula ϕ. Section 4 studies the structure of the graph G and Section 5 shows that any popular matching in G yields a 1-in-3 satisfying assignment for ϕ. Section 6 completes the reduction by showing how to obtain a popular matching in G from any 1-in-3 satisfying assignment for ϕ.

2 Preliminaries

This section contains a characterization of popular matchings from [14]. We also include a simple proof of the claim stated in Section 1 that when n is odd, every popular matching in G has to be stable.

Let M be any matching in $G = (V, E)$. For any pair $(u, v) \notin M$, define $\mathsf{vote}_u(v, M)$ as follows: (here $M(u)$ is u's partner in M and $M(u) = \mathsf{null}$ if u is unmatched in M)

$$\mathsf{vote}_u(v, M) = \begin{cases} + & \text{if } u \text{ prefers } v \text{ to } M(u); \\ - & \text{if } u \text{ prefers } M(u) \text{ to } v. \end{cases}$$

Label every edge (u, v) that does not belong to M by the pair $(\mathsf{vote}_u(v, M), \mathsf{vote}_v(u, M))$. Thus every non-matching edge has a label in $\{(\pm, \pm)\}$. For example, if consider the matching marked by the dashed orange edges in Fig. 1, then (d_1, d_2) is labeled $(+, +)$, (d_2, d_3) is labeled $(+, -)$, (d_0, d_1) is labeled $(+, -)$, and (d_0, d_3) is labeled $(-, -)$. Note that an edge is labeled $(+, +)$ if and only if it is a blocking edge to M. Let G_M be the subgraph of G obtained by deleting edges labeled $(-, -)$ from G. The following theorem characterizes popular matchings in G.

Theorem 2 ([14]). *M is popular in G if and only if G_M does not contain any of the following with respect to M:*

(1) an alternating cycle with a $(+, +)$ edge;
(2) an alternating path with two distinct $(+, +)$ edges;
(3) an alternating path with a $(+, +)$ edge and an unmatched vertex as an endpoint.

Using the above characterization, it can be easily checked whether a given matching is popular or not [14]. Thus our NP-hardness result implies that the popular roommates problem is NP-complete.

When n is odd. Recall the claim made in Section 1 that when n is odd, every popular matching in G has to be stable. A simple proof of this statement is included below.

Observation 1 ([13]) *Let G be a complete graph on n vertices, where n is odd. Any popular matching in G has to be stable.*

Proof. Since n is odd and G is complete, any popular matching leaves exactly one vertex unmatched. Let M be a popular matching and let v be the vertex left unmatched in M. Consider a vertex u adjacent to v. We know that $(u, w) \in M$ for some $w \in V \setminus \{v\}$, and due to Part (3) in Theorem 2, no $(+, +)$ edge is incident to w. Since v is adjacent not only to u, but to all vertices in the graph, this holds for all $w \in V$. Thus M is stable. ◻

3 The graph G

Recall that ϕ is the input formula to 1-in-3 SAT. The graph G that we construct here consists of gadgets in 4 levels along with 2 special gadgets that we will call the D-gadget and Z-gadget. Gadgets in level 1 correspond to variables in the formula ϕ while gadgets in levels 0, 2, and 3 correspond to clauses in ϕ. Variants of the gadgets in levels 0-3 and the D-gadget were used in [9] while the Z-gadget is new.

We will now describe these gadgets: along with a figure, we provide the preference lists of vertices in this gadget. The tail of each list consists of all vertices not listed yet, in an arbitrary order. Even though the preference lists are complete, the structure of the gadgets and the preference lists will ensure that inter-gadget edges will not belong to any popular matching, as we will show in Section 4.

The D-gadget. The D-gadget is on 4 vertices d_0, d_1, d_2, d_3 and the preference lists of these vertices are as given in Fig. 1 with all vertices outside the D-gadget at the tail of each list (in an arbitrary order). Recall that this gadget admits no stable matching.

We describe gadgets from level 1 first, then levels 0, 2, 3, and finally, the Z-gadget. The stable matchings within the gadgets are highlighted by colors in the figures. The gray elements in the preference lists denote vertices that are outside this gadget. We will assume that D in a preference list stands for $d_0 > d_1 > d_2 > d_3$.

Level 1. For each variable X_i in the formula ϕ, we construct a gadget on four vertices as shown in Fig. 2. The bottom vertices x'_i and y'_i will be preferred by some vertices in level 0 to vertices in their own gadget, while the top vertices x_i and y_i will be preferred by some vertices in level 2 to vertices in their own gadget. All four vertices in a level 1 gadget prefer to be matched among themselves, along the four edges drawn than be matched to any other vertex in the graph. This gadget has a unique stable matching $\{(x_i, y_i), (x'_i, y'_i)\}$.

$x_i \ : y_i > y'_i > D > \dots$
$x'_i \ : y_i > y'_i > D > \dots$
$y_i \ : x_i > x'_i > D > \dots$
$y'_i \ : x_i > x'_i > D > \dots$

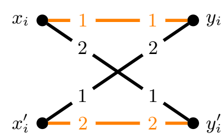

Fig. 2. The variable gadget in level 1.

Level 0. To each clause $c = X_i \vee X_j \vee X_k$ in the formula ϕ, we create *6 gadgets* in level 0. One of these can be seen in Fig. 3. The top two vertices, i.e. a^c_1 and b^c_1, rank y'_j and x'_k in level 1, as their respective second choices. Recall that indices j and k are well-defined in the clause $c = X_i \vee X_j \vee X_k$. Within this level 0 gadget on $a^c_1, b^c_1, a^c_2, b^c_2$, both $\{(a^c_1, b^c_1), (a^c_2, b^c_2)\}$ and $\{(a^c_1, b^c_2), (a^c_2, b^c_1)\}$ are stable matchings. In the preference lists below (and also for gadgets in levels 2 and 3), we have omitted the superscript c in their lists for the sake of readability.

$a_1 \ : b_1 > y'_j > b_2 > D > \dots$
$a_2 \ : b_2 > b_1 > D > \dots$
$b_1 \ : a_2 > x'_k > a_1 > D > \dots$
$b_2 \ : a_1 > a_2 > D > \dots$

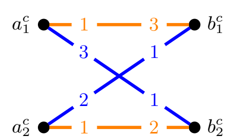

Fig. 3. A clause gadget in level 0.

The gadget on vertices $\{a^c_3, a^c_4, b^c_3, b^c_4\}$ is built analogously: the vertex a^c_3 ranks y'_k as its second choice, while b^c_3 ranks x'_i second. In the third gadget, the vertex a^c_5 ranks y'_i second, while b^c_5 ranks x'_j second. Observe the shift in i, j, k indices as second choices for vertices a^c_1, a^c_3, a^c_5 (and similarly, for b^c_1, b^c_3, b^c_5).

The fourth, fifth and sixth gadgets are analogous to the first, second, and third gadgets, respectively, but there is a slight twist. More precisely, the preferences of $a'^c_1, a'^c_2, b'^c_1, b'^c_2$ in the fourth gadget are analogous to the preferences in Fig. 3, except that a'^c_1 ranks y'_k second, while b'^c_1 ranks x'_j second. Similarly, the second choice of a'^c_3 is y'_i, the second choice of b'^c_3 is x'_k, and finally, a'^c_5 ranks y'_j second, while b'^c_5 ranks x'_i second. Observe the change in *orientation* of the indices i, j, k as second choice neighbors when comparing the first three level 0 gadgets of c with its last three level 0 gadgets. This will be important to us later.

Level 2. To each clause $c = X_i \vee X_j \vee X_k$ in the formula ϕ, we create *6 gadgets* in level 2. The first gadget in level 2 is on vertices $p^c_0, p^c_1, p^c_2, q^c_0, q^c_1, q^c_2$ and their preference lists are described in Fig. 4. Note that p^c_2 ranks y_j from level 1 as its second choice, while q^c_2 ranks x_k from level 1 second.

The second gadget in level 2 is on vertices $p^c_3, p^c_4, p^c_5, q^c_3, q^c_4, q^c_5$ and it is built analogously. That is, p^c_3 and q^c_3 are each other's top choices and similarly, p^c_4 and q^c_4 are each other's top choices, and so on. The preference list of p^c_5 is $q^c_3 > y_k > q^c_4 > q^c_5 > D > \dots$ and the preference list of q^c_5 is $p^c_4 > x_i > p^c_3 > p^c_5 > D > \dots$

$$
\begin{aligned}
p_0 &: q_0 > q_2 > D > \dots\\
p_1 &: q_1 > q_2 > D > \dots\\
p_2 &: q_0 > y_j > q_1 > q_2 > D > \dots\\
q_0 &: p_0 > p_2 > D > \dots\\
q_1 &: p_1 > p_2 > D > \dots\\
q_2 &: p_1 > x_k > p_0 > p_2 > D > \dots
\end{aligned}
$$

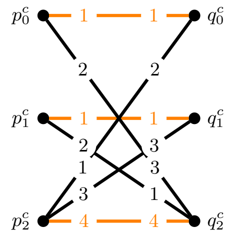

Fig. 4. A clause gadget in level 2.

The third gadget in level 2 is on vertices $p_6^c, p_7^c, p_8^c, q_6^c, q_7^c, q_8^c$ and it is built analogously. In particular, the preference list of p_8^c is $q_6^c > y_i > q_7^c > q_8^c > D > \dots$ and the preference list of q_8^c is $p_7^c > x_j > p_6^c > p_8^c > D > \dots$

The fourth gadget in level 2 is on vertices $p_0'^c, p_1'^c, p_2'^c, q_0'^c, q_1'^c, q_2'^c$ and it is totally analogous to the first gadget in level 2. That is, $p_0'^c$ and $q_0'^c$ are each other's top choices and similarly, $p_1'^c$ and $q_1'^c$ are each other's top choices, and so on. In particular, the preference list of $p_2'^c$ is $q_0'^c > y_j > q_1'^c > q_2'^c > D > \dots$ and the preference list of $q_2'^c$ is $p_1'^c > x_k > p_0'^c > p_2'^c > D > \dots$

Similarly, the fifth gadget in level 2 is on vertices $p_3'^c, p_4'^c, p_5'^c, q_3'^c, q_4'^c, q_5'^c$ and it is totally analogous to the second gadget in level 2. Also, the sixth gadget in level 2 is on vertices $p_6'^c, p_7'^c, p_8'^c, q_6'^c, q_7'^c, q_8'^c$ and it is totally analogous to the third gadget in level 2.

Level 3. To each clause $c = X_i \vee X_j \vee X_k$ in the formula ϕ, we create *2 gadgets* in level 3. The first gadget is on vertices $s_0^c, s_1^c, s_2^c, s_3^c, t_0^c, t_1^c, t_2^c, t_3^c$ and the preference lists of these vertices are described in Fig. 5.

$$
\begin{aligned}
s_0 &: t_1 > q_0 > t_2 > q_3 > t_3 > D > t_0 > \dots\\
t_0 &: s_3 > p_7 > s_2 > p_4 > s_1 > D > s_0 > \dots\\
s_1 &: t_1 > t_0 > D > \dots\\
t_1 &: s_1 > s_0 > D > \dots\\
s_2 &: t_2 > t_0 > D > \dots\\
t_2 &: s_2 > s_0 > D > \dots\\
s_3 &: t_3 > t_0 > D > \dots\\
t_3 &: s_3 > s_0 > D > \dots
\end{aligned}
$$

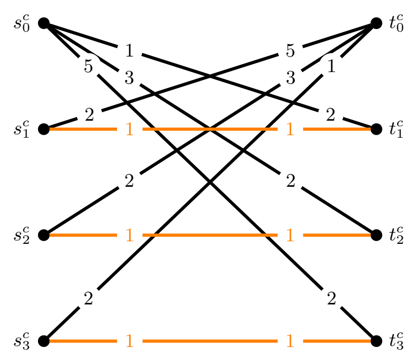

Fig. 5. A clause gadget in level 3.

The second gadget in level 3 is on $s_0'^c, s_1'^c, s_2'^c, s_3'^c, t_0'^c, t_1'^c, t_2'^c, t_3'^c$ and their preference lists are absolutely analogous to the preference lists of the first gadget in level 3.

The Z-gadget. The Z-gadget is on 6 vertices $z_0, z_1, z_2, z_3, z_4, z_5$ and the preference lists of these vertices are given in Fig. 6. The vertices in a set stand for all these vertices in an arbitrary order. For example, $\cup_i\{x_i, y_i\}$ denotes all the "top" vertices belonging to variable gadgets in an arbitrary order.

Note that G is a complete graph on an even number of vertices and so every popular matching in G has to be a perfect matching.

4 Popular edges in G

Call an edge e in G *popular* if there is a popular matching M in G such that $e \in M$. In this section we identify edges that are not popular and show that every popular edge is an *intra-gadget* edge, connecting two vertices of the same gadget.

$$
\begin{aligned}
z_0 &: z_4 > z_5 > \cup_i \{x_i, y_i\} > \cup_{c,i} \{p^c_{3i+1}, q^c_{3i}, p'^c_{3i+1}, q'^c_{3i}\} > \\
&\quad \cup_{c,i} \{a^c_i, b^c_i, a'^c_i, b'^c_i\} > z_1 > z_2 > z_3 > D > \ldots \\
z_1 &: z_5 > z_4 > \cup_i \{x_i, y_i\} > \cup_{c,i} \{p^c_{3i+1}, q^c_{3i}, p'^c_{3i+1}, q'^c_{3i}\} > \\
&\quad \cup_{c,i} \{a^c_i, b^c_i, a'^c_i, b'^c_i\} > z_0 > z_3 > z_2 > D > \ldots \\
z_2 &: z_0 > z_1 > z_3 > z_4 > z_5 > D > \ldots \\
z_3 &: z_1 > z_0 > z_2 > z_5 > z_4 > D > \ldots \\
z_4 &: z_2 > z_3 > z_5 > z_0 > z_1 > D > \ldots \\
z_5 &: z_3 > z_2 > z_4 > z_1 > z_0 > D > \ldots
\end{aligned}
$$

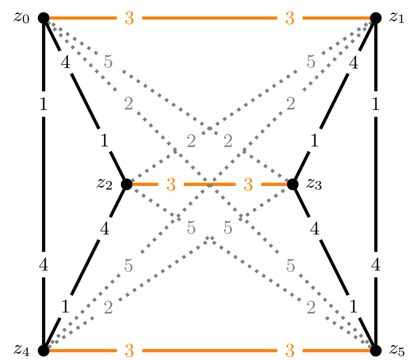

Fig. 6. The Z-gadget.

The following observation, which is straightforward, will be used repeatedly in our proofs.

Observation 2 *Let v be u's top choice neighbor. If v is matched in M to a neighbor worse than u then (u, v) is a blocking edge to M.*

Lemma 1. *For any clause c, no popular matching in G can match s^c_0 (similarly, t^c_0) to a neighbor worse than t^c_0 (resp., s^c_0). An analogous statement holds for s'^c_0 and t'^c_0.*

Proof. Let M be a popular matching such that $(s^c_0, v) \in M$ for some vertex v such that $t^c_0 > v$ in s^c_0's list, i.e., s^c_0 prefers t^c_0 to v. We claim this implies:

- a $(+,+)$ edge reachable from v via an alternating path in G_M that begins with a non-matching edge incident to v *and*
- a $(+,+)$ edge reachable from s^c_0 via an alternating path in G_M that begins with a non-matching edge incident to s^c_0.

If this is the same $(+,+)$ edge then we have an alternating cycle in G_M with a $(+,+)$ edge, a contradiction to M's popularity (by Theorem 2). If these are two different $(+,+)$ edges then there is an alternating path in G_M with two $(+,+)$ edges, again a contradiction to M's popularity (by Theorem 2).

(1) If v is a top choice neighbor for some vertex (such as $z_k, x_j, y_j, d_1, d_2, d_3, a^c_i, b^c_i, p^c_0, p^c_1$, and so on) then there is a $(+,+)$ edge incident to v (by Observation 2).
(2) Suppose v is one of $s^r_0, t^r_0, s'^r_0, t'^r_0$ for some r. Assume without loss of generality that $v = s^r_0$. Then either (s^r_0, t^r_0) is a $(+,+)$ edge or t^r_0 is matched in M to a neighbor better than s^r_0.
Recall t^r_0's preference list: every vertex that t^r_0 prefers to s^r_0 is either a top choice neighbor or it is d_0. In the former case, there is a $(+,+)$ edge incident to t^r_0's partner (by Observation 2) and in the latter case also there is a $(+,+)$ edge incident to d_0 since one of d_1, d_2, d_3 is matched to a neighbor worse than d_0 and so there is a $(+,+)$ edge between this d_i and d_0. Since the edge (s^r_0, t^r_0) is a $(+,-)$ edge, there is a $(+,+)$ edge reachable from s^r_0 via an alternating path of length 2.
(3) The only case left is when v is neither a top choice neighbor of some vertex nor one of $s^r_0, t^r_0, s'^r_0, t'^r_0$ for some r. So u is a vertex such as d_0 or x'_i, y'_i or $p^c_{3j+2}, q^c_{3j+2}, p'^c_{3j+2}, q'^c_{3j+2}$ (for $j = 0, 1, 2$ and some c). It is easy to see that there is a $(+,+)$ edge reachable from v via an alternating path of length at most 2. For instance, either (x'_i, y'_i) is a $(+,+)$ edge or $(x_i, y'_i) \in M$ which creates the alternating path (s^c_0, x'_i)-(y'_i, x_i)-$(y_i, *)$, where (x_i, y_i) is a $(+,+)$ edge.

Similarly, we can argue that there is a $(+,+)$ edge reachable from s^c_0 via an alternating path in G_M. If t^c_0 is matched to a neighbor worse than s^c_0 then the edge (s^c_0, t^c_0) is a $(+,+)$ edge. Else t^c_0 is matched to a neighbor u better than s^c_0 and this means there is a $(+,+)$ edge incident to u, as we argued above in case (2). Hence there is a $(+,+)$ edge reachable from s^c_0 via an alternating path of length at most 2 in G_M. □

Lemma 2. *Every popular matching matches the vertices in the D-gadget among themselves.*

Proof. Let M be a matching that matches d_i for some $i \in \{0,1,2,3\}$ to a vertex v outside the D-gadget. This means at least 2 vertices d_i and d_j in the D-gadget are matched to vertices outside the D-gadget. So (d_i, d_j) is a $(+,+)$ edge. We now claim there is a forbidden alternating path or cycle (as given in Theorem 2) to M's popularity.

If v is a top choice neighbor or a vertex such as x'_i, y'_i or $p^c_{3j+2}, q^c_{3j+2}, p'^c_{3j+2}, q'^c_{3j+2}$ (for $j = 0,1,2$ and some c) then there is a $(+,+)$ edge e reachable from v via an alternating path of length at most 2 as seen in the proof of Lemma 1. This creates an alternating path in G_M with 2 $(+,+)$ edges: (d_j, d_i) and e.

The other possibility is that v is $s^c_0, t^c_0, s'^c_0, t'^c_0$ for some clause c. Assume without loss of generality that $v = s^c_0$. Consider the vertex t^c_0. We know from Lemma 1 that t^c_0 has to be matched to a neighbor at least as good as s^c_0. So we have the following cases:

(1) t^c_0 is matched to a vertex $d_{i'}$ in the D-gadget: this means there is either an alternating path with 2 $(+,+)$ edges or an alternating cycle with a $(+,+)$ edge:

$$(s^c_0, d_i) \stackrel{(+,+)}{-} (d_{i'}, t^c_0) \stackrel{(+,+)}{-} (s^c_1, *) \qquad \text{or} \qquad (s^c_0, d_i) \stackrel{(+,+)}{-} (d_{i'}, t^c_0) \stackrel{(+,-)}{-} (s^c_1, t^c_1) \stackrel{(-,+)}{-} (s^c_0, d_i).$$

If s^c_1 is matched to a neighbor worse than t^c_0 then the former is an alternating path with two $(+,+)$ edges: these are $(d_i, d_{i'})$ and (t^c_0, s^c_1). Else s^c_1 is matched to t^c_1 and the latter is an alternating cycle with a $(+,+)$ edge, which is $(d_i, d_{i'})$.

(2) t^c_0 is matched to s^c_i for some $i \in \{1,2,3\}$: this means t^c_i is matched to a neighbor worse than s^c_0 and so (s^c_0, t^c_i) is a $(+,+)$ edge and thus we have the following alternating path with two $(+,+)$ edges (t^c_1, s^c_0) and (d_i, d_j):

$$(*, t^c_1) \stackrel{(+,+)}{-} (s^c_0, d_i) \stackrel{(+,+)}{-} (d_j, *).$$

(3) t^c_0 is matched to either p^c_4 or p^c_7: we will show that this results in an alternating path with two $(+,+)$ edges. Assume without loss of generality that t^c_0 is matched to p^c_4. Consider the following alternating path:

$$(*, s^c_3) \stackrel{(+,+)}{-} (t^c_0, p^c_4) \stackrel{(+,+)}{-} (q^c_5, *) \qquad \text{or} \qquad (*, d_j) \stackrel{(+,+)}{-} (d_i, s^c_0) \stackrel{(+,-)}{-} (t^c_3, s^c_3) \stackrel{(-,+)}{-} (t^c_0, p^c_4) \stackrel{(+,+)}{-} (q^c_5, *).$$

Recall that s^c_3 is the top choice neighbor of t^c_0 and the vertex p^c_4 is the top choice neighbor of q^c_5. If the vertex s^c_3 is matched to a neighbor worse than t^c_0 then the former path is an alternating path in G_M with two $(+,+)$ edges in it: these are (s^c_3, t^c_0) and (p^c_4, q^c_5). Else $(s^c_3, t^c_3) \in M$ and recall that the edge (s^c_0, t^c_3) is a $(+,-)$ edge as s^c_0 prefers t^c_3 to d_i. This creates the latter path which is an alternating path in G_M with 2 $(+,+)$ edges in it: these are (d_i, d_j) and (p^c_4, q^c_5). □

The gadget D admits 2 popular matchings: $\{(d_0,d_1),(d_2,d_3)\}$ and $\{(d_0,d_2),(d_1,d_3)\}$. So if M is a popular matching then either $\{(d_0,d_1),(d_2,d_3)\} \subset M$ or $\{(d_0,d_2),(d_1,d_3)\} \subset M$.

Lemma 3. *Let (u,v) be an edge in G where both u and v prefer d_0 to each other. Then (u,v) cannot be a popular edge.*

Proof. Let M be a popular matching in G that contains such an edge (u,v). We know from Lemma 2 that either $\{(d_0,d_1),(d_2,d_3)\} \subset M$ or $\{(d_0,d_2),(d_1,d_3)\} \subset M$. So there is always a blocking edge $(d_i,d_j) \in \{(d_1,d_3),(d_1,d_2)\}$ to M.

Observe that both u and v cannot belong to the D-gadget as there is no such pair within D. If exactly one of u, v belongs to the D-gadget then (u,v) is not a popular edge (by Lemma 2). So neither u nor v belongs to the D-gadget and this implies that u prefers d_0, d_1, d_2, d_3 to v and symmetrically, v prefers d_0, d_1, d_2, d_3 to u.

Consider the following alternating cycle C with respect to M:

$$(u,v) \stackrel{(+,-)}{-} (d_{i'}, d_i) \stackrel{(+,+)}{-} (d_j, d_{j'}) \stackrel{(-,+)}{-} (u,v),$$

where $(d_{i'}, d_i)$ and $(d_j, d_{j'})$ are edges from the D-gadget in M and (d_i, d_j) is a blocking edge. Thus C is an alternating cycle in G_M with a $(+,+)$ edge. This contradicts the popularity of M (by Theorem 2). □

Corollary 1. *The edges* (s_0^c, t_0^c) *and* $(s_0'^c, t_0'^c)$ *are not popular edges for any clause* c.

Corollary 1 follows from Lemma 3 by setting u and v to s_0^c and t_0^c (similarly, $s_0'^c$ and $t_0'^c$), respectively. Let us call u a level i vertex if u belongs to a level i gadget.

Lemma 4. *No edge between a level* i *vertex and a level* $i+1$ *vertex is popular, for* $0 \le i \le 2$.

The proof of Lemma 4 follows from Claims 1-3 proved below.

Claim 1 *There is no popular edge between a level 0 vertex and a level 1 vertex.*

Proof. Let M be a popular matching in G with such an edge, say (a_1^c, y_j'). We claim this would create an alternating path in G_M with two $(+,+)$ edges in it. Theorem 2 forbids such an alternating path. Consider the vertex b_1^c. There are 3 possibilities for b_1^c's partner in M.

(1) b_1^c is matched to a_2^c
So (a_2^c, b_2^c) is labeled $(+,+)$. Recall that b_2^c is a_2^c's top choice and the only neighbor that b_2^c prefers to a_2^c is a_1^c (matched to y_j'). Consider the following alternating path in G_M:

$$(*, b_2^c) \overset{(+,+)}{-} (a_2^c, b_1^c) \overset{(-,+)}{-} (a_1^c, y_j') \overset{(+,+)}{-} (x_j', *).$$

If x_j' is matched to a neighbor worse than y_j' then the above is an alternating path in G_M with two $(+,+)$ edges: these are (b_2^c, a_2^c) and (y_j', x_j'). Else replace $(x_j', *)$ in the above path with $(x_j', y_j) - (x_j, *)$: the $(+,+)$ edges here are (b_2^c, a_2^c) and (y_j, x_j).

(2) b_1^c is matched to x_k'
Either the edge (x_k', y_k') or the edge (x_k, y_k) will block M. Suppose y_k' is matched to a neighbor worse than x_k' in M. Consider the following alternating path in G_M:

$$(*, y_k') \overset{(+,+)}{-} (x_k', b_1^c) \overset{(-,+)}{-} (a_1^c, y_j') \overset{(+,+)}{-} (x_j', *).$$

Either the above is an alternating path in G_M with two $(+,+)$ edges or by replacing $(x_j', *)$ with $(x_j', y_j) - (x_j, *)$ (as done in case (1)), we get an alternating path in G_M with two $(+,+)$ edges. If y_k' is matched to a neighbor better than x_k' in M, i.e., if $(x_k, y_k') \in M$ then prefix both these alternating paths with $(*, y_k)$. This will yield an alternating path in G_M with (x_k, y_k) as a blocking edge and either (x_j', y_j') or (x_j, y_j) as a blocking edge.

(3) b_1^c is matched to a neighbor worse than a_1^c
The edge (a_1^c, b_1^c) is labeled $(+,+)$. Consider the following alternating path in G_M:

$$(*, b_1^c) \overset{(+,+)}{-} (a_1^c, y_j') \overset{(+,+)}{-} (x_j', *) \qquad \text{or} \qquad (*, b_1^c) \overset{(+,+)}{-} (a_1^c, y_j') \overset{(+,-)}{-} (x_j', y_j) \overset{(+,+)}{-} (x_j, *).$$

That is, if x_j' is matched to a neighbor worse than y_j' then consider the first alternating path above: this is an alternating path in G_M with both (b_1^c, a_1^c) and (y_j', x_j') as $(+,+)$ edges. Else $(x_j', y_j) \in M$ and the second alternating path is an alternating path in G_M with (b_1^c, a_1^c) and (y_j, x_j) as $(+,+)$ edges.

□

Claim 2 *There is no popular edge between a level 1 vertex and a level 2 vertex.*

Proof. Let M be a popular matching in G that contains such an edge, say (p_2^c, y_j). Consider the following alternating path with respect to M:

$$(p_2^c, y_j) \overset{(+,+)}{-} (x_j', y_j') \overset{(+,+)}{-} (x_j, *).$$

Since M is a perfect matching, x_j' is matched in M. We know that no edge between x_j' and a level 0 vertex belongs to M (by Claim 1). Also, M cannot match x_j' to a neighbor that it regards worse than d_0 (by Lemma 3). Thus x_j' has to be matched to y_j' in M and so the above alternating path has two $(+,+)$ edges: (x_j', y_j) and (x_j, y_j'). This is a contradiction to M's popularity (by Theorem 2).

Claim 3 *There is no popular edge between a level 2 vertex and a level 3 vertex.*

Proof. Let M be a popular matching in G that contains such an edge, say (s_0^c, q_0^c). Consider the following alternating path with respect to M:

$$(s_0^c, q_0^c) \overset{(+,+)}{-} (p_2^c, q_2^c) \overset{(+,+)}{-} (p_0^c, *) \qquad \text{or} \qquad (s_0^c, q_0^c) \overset{(+,+)}{-} (p_2^c, q_1^c) \overset{(+,+)}{-} (p_1^c, *).$$

The vertex p_2^c is matched in M and its partner cannot be a level 1 vertex (by Claim 2) or a neighbor worse than d_0 (by Lemma 3). So either (p_2^c, q_2^c) or (p_2^c, q_1^c) is in M. This means either the first alternating path given above or the second one is an alternating path in G_M with two $(+,+)$ edges: (p_2^c, q_0^c) and (p_0^c, q_2^c) in the former and (p_2^c, q_0^c) and (p_1^c, q_1^c) in the latter. This is a contradiction to M's popularity (by Theorem 2). ☐

Lemma 5. *All popular matchings match the 6 vertices of the Z-gadget among themselves.*

Proof. Let M be any popular matching in G. It follows from Lemma 3 that M has to pair each of z_2, z_3, z_4, and z_5 to a vertex in the Z-gadget. Let us now show that z_0 also has to be matched within the Z-gadget. Then it immediately follows that z_1 also has to be matched within the Z-gadget. We have the following 3 cases:

(1) Suppose z_0 is matched in M to a level 0 neighbor, say b_1^c. Then (a_1^c, b_1^c) is a blocking edge to M. Lemmas 2, 3, and 4 ensure that a_1^c is either matched to z_1 or to b_2^c. We investigate these two cases below.
  - $(a_1^c, z_1) \in M$: Here both z_0 and z_1 are matched to vertices they prefer to all their neighbors inside the Z-gadget, except for z_4 and z_5. We know that z_4 and z_5 must be matched inside the Z-gadget. There are 3 subcases and in each case there is an alternating cycle in G_M with a blocking edge (a_1^c, b_1^c): a contradiction to M's popularity (by Theorem 2).
    * $(z_4, z_2) \in M$: the alternating cycle is $(b_1^c, z_0) \overset{(+,-)}{-} (z_4, z_2) \overset{(+,-)}{-} (z_1, a_1^c) \overset{(+,+)}{-} (b_1^c, z_0)$.
    * $(z_4, z_3) \in M$: the alternating cycle is $(b_1^c, z_0) \overset{(+,-)}{-} (z_4, z_3) \overset{(+,-)}{-} (z_1, a_1^c) \overset{(+,+)}{-} (b_1^c, z_0)$.
    * $(z_4, z_5) \in M$: the alternating cycle is $(b_1^c, z_0) \overset{(+,-)}{-} (z_4, z_5) \overset{(-,+)}{-} (z_1, a_1^c) \overset{(+,+)}{-} (b_1^c, z_0)$.
  - $(a_1^c, b_2^c) \in M$: Lemmas 2, 3, and 4 ensure that a_2^c is matched to z_1 (recall that M is perfect). This leads to the same 3 subcases as above, except that instead of the edge (z_1, a_1^c), there is the path $(z_1, a_2^c) - (b_2^c, a_1^c)$ in G_M: here (a_2^c, b_2^c) is labeled $(+,-)$.

(2) Suppose z_0 is matched in M to a level 1 neighbor, say y_i.
This case is similar to the previous case. Here the edge (x_i, y_i) becomes the blocking edge to M. It follows from Lemmas 2, 3, and 4 that x_i is either matched to z_1 or to y_i'. The latter case leaves x_i' unmatched and the subcases that arise in the former case are analogous to the ones in case (1).

(3) Suppose z_0 is matched in M to a level 2 neighbor, say q_0^c.
It follows from Lemmas 2, 3, and 4 that $(p_0^c, q_2^c), (p_2^c, q_1^c)$, and (p_1^c, z_1) are in M. Consider the alternating path $(z_0, q_0^c) - (p_2^c, q_1^c) - (p_1^c, z_1)$: it has two blocking edges (p_2^c, q_0^c) and (p_1^c, q_1^c). This is again a contradiction to M's popularity.

Recall that Lemma 2 showed that all vertices of D must be matched within the gadget. Thus z_0 cannot be matched to a vertex in the D-gadget. The case where z_0 is matched in M to a level 3 neighbor does not arise as such an edge would violate Lemma 3. This finishes our proof that any popular matching M matches the 6 vertices of the Z-gadget among themselves. ☐

Lemma 6. *The only popular matching inside the Z-gadget is $\{(z_0, z_1), (z_2, z_3), (z_4, z_5)\}$.*

Proof. The matching $\{(z_0, z_1), (z_2, z_3), (z_4, z_5)\}$ is stable in the Z-gadget, thus this is a popular matching. Note that this gadget has no other stable matching.

Let M be any matching that matches the 6 vertices of the Z-gadget among themselves. Suppose M contains one or more of the edges (z_i, z_j) where $i = j \bmod 2$ (colored black in Fig. 6). Without loss of generality, let $(z_0, z_2) \in M$. There are three candidate matchings that we need to check for popularity: note that none is popular (by Theorem 2).

- $\{(z_0, z_2), (z_1, z_3), (z_4, z_5)\}$: this has the alternating cycle $(z_2, z_0) \overset{(+,+)}{-} (z_1, z_3) \overset{(-,+)}{-} (z_5, z_4) \overset{(+,-)}{-} (z_2, z_0)$ with the blocking edge (z_0, z_1).
- $\{(z_0, z_2), (z_1, z_4), (z_3, z_5)\}$: this has the alternating cycle $(z_0, z_2) \overset{(-,+)}{-} (z_3, z_5) \overset{(-,+)}{-} (z_1, z_4) \overset{(+,+)}{-} (z_0, z_2)$ with the blocking edge (z_0, z_4).
- $\{(z_0, z_2), (z_1, z_5), (z_3, z_4)\}$: this has the alternating cycle $(z_2, z_0) \overset{(+,-)}{-} (z_1, z_5) \overset{(+,+)}{-} (z_3, z_4) \overset{(+,-)}{-} (z_2, z_0)$ with the blocking edge (z_3, z_5).

Thus we can conclude that $M \subset \{z_0, z_2, z_4\} \times \{z_1, z_3, z_5\}$. Suppose M contains an unstable edge here (dotted and gray in Fig. 6), say (z_0, z_3): among the vertices in the Z-gadget, z_3 is the last choice of z_0 and the edge (z_0, z_2) blocks M. Since z_2 has to be matched in M, there are two cases.

- $(z_1, z_2) \in M$: the 4 vertices z_0, z_1, z_2, z_3 prefer $\{(z_0, z_2), (z_1, z_3)\}$ to $\{(z_0, z_3), (z_1, z_2)\} \subset M$.
- $(z_2, z_5) \in M$: the 4 vertices z_0, z_2, z_3, z_5 prefer $\{(z_0, z_2), (z_3, z_5)\}$ to $\{(z_0, z_3), (z_2, z_5)\} \subset M$.

Thus in both cases we have a contradiction to M's popularity. Analogous proofs hold for other unstable edges chosen from $\{z_0, z_2, z_4\} \times \{z_1, z_3, z_5\}$. Thus the only popular matching inside the Z-gadget is $\{(z_0, z_1), (z_2, z_3), (z_4, z_5)\}$. ◻

5 Stable states versus unstable states

In this section we will show how to obtain a 1-in-3 satisfying assignment for the input ϕ from any popular matching in G. The following definition will be useful to us.

Definition 2. *A gadget A in $G = (V, E)$ is said to be in* unstable state *with respect to matching M if there is a blocking edge $(u, v) \in V(A) \times V(A)$ with respect to M. If there is no such blocking edge to M then we say A is in* stable state *with respect to M.*

In Figures 2-6 depicting our gadgets, corresponding to matchings that consist of colored edges within the gadget, the relevant gadget is in *stable* state. A level 1 gadget in unstable state will encode the corresponding variable being set to true while a level 1 gadget in stable state will encode the corresponding variable being set to false. We will now analyze what gadgets are in stable/unstable state with respect to any popular matching M in G. This will lead to the proof that for any clause c, exactly one of the level 1 gadgets corresponding to the 3 variables in c is in unstable state.

Lemma 7. *For any clause c, the following statements hold:*

- *all its 6 level 0 gadgets are in stable state with respect to M;*
- *both its level 3 gadgets in G are in unstable state with respect to M.*

Proof. Consider a level 0 gadget corresponding to clause c, say the one on vertices $a_1^c, b_1^c, a_2^c, b_2^c$. Lemmas 2, 3, 4, and 5 imply that either $\{(a_1^c, b_1^c), (a_2^c, b_2^c)\} \subset M$ or $\{(a_1^c, b_2^c), (a_2^c, b_1^c)\} \subset M$. Thus there is no blocking edge within this gadget. As this holds for every level 0 gadget corresponding to c and for every clause c, the first part of the lemma follows.

We will now prove the second part of the lemma. Since M is a perfect matching, the vertices s_0^c, t_0^c (also $s_0'^c, t_0'^c$) have to be matched in M, for all clauses c. It follows from Lemmas 2 and 3 that both s_0^c and t_0^c (similarly, $s_0'^c$ and $t_0'^c$) have to be matched to neighbors that are better than d_0. Lemma 4 showed that there is no popular edge between a level 3 vertex and a level 2 vertex. Thus s_0^c is matched to t_i^c for some $i \in \{1, 2, 3\}$.

If s_0^c is matched to t_i^c then s_i^c has to be matched to t_0^c—otherwise Lemma 3 would be violated by s_i^c and its partner. So (s_i^c, t_i^c) blocks M and this holds for every clause c. Similarly, there is a blocking edge $(s_i'^c, t_i'^c)$ for some $i \in \{1, 2, 3\}$ for every clause c. ◻

Lemma 8. *For any clause c, at least* one *of the following two conditions has to hold:*

- *two or more of its first three level 2 gadgets are in unstable state with respect to M;*
- *two or more of its last three level 2 gadgets are in unstable state with respect to M.*

Proof. Suppose both statements are false. Let M be a popular matching such that corresponding to clause c, at least two among its *first* three level 2 gadgets are in stable state with respect to M *and* at least two among its *last* three level 2 gadgets are in stable state with respect to M.

Consider the two level 3 gadgets corresponding to c. We know that $(s^c_0, t^c_i), (s^c_i, t^c_0)$ are in M for some $i \in \{1, 2, 3\}$ and similarly, $(s'^c_0, t'^c_j), (s'^c_j, t'^c_0)$ are in M for some $j \in \{1, 2, 3\}$ (see the proof of Lemma 7). We will now show the existence of an alternating path ρ that will contradict M's popularity.

For this, we claim it suffices to show in *stable state* the following:

- one among the *first* three level 2 gadgets with a vertex that either s^c_0 or t^c_0 prefers to its partner in M, and
- one among the *last* three level 2 gadgets with a vertex that either s'^c_0 or t'^c_0 prefers to its partner in M.

For instance, suppose $i = 1$ and $j = 2$. So t^c_0 prefers p^c_4 and p^c_7 to its partner s^c_1 in M and s'^c_0 prefers q'^c_0 to its partner t'^c_2 in M and t'^c_0 prefers p'^c_7 to its partner s'^c_2 in M. Consider the level 2 gadgets containing p^c_4, p^c_7, q'^c_0, and p'^c_7.

Observe that by our assumption in the first paragraph above, either the gadget of p^c_4 or the gadget of p^c_7 is in stable state, similarly either the gadget of q'^c_0 or the gadget of p'^c_7 is in stable state. In all 4 cases, we will show the existence of an alternating path ρ in G_M with *two* blocking edges (s^c_1, t^c_1) and (s'^c_2, t'^c_2), which is a contradiction to M's popularity (by Theorem 2).

1. Suppose the gadgets of p^c_4 and q'^c_0 are in stable state. So the edges $(p^c_i, q^c_i) \in M$ for $i = 3, 4, 5$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 0, 1, 2$. Consider the following alternating path ρ with respect to M:
$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_4, q^c_4) \overset{(-,+)}{-} (p^c_5, q^c_5) \overset{(+,-)}{-} (p^c_3, q^c_3) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (p'^c_1, q'^c_1) \overset{(-,+)}{-} (p'^c_2, q'^c_2) \overset{(+,-)}{-} (p'^c_0, q'^c_0) \overset{(-,+)}{-} (s'^c_0, t'^c_2) \overset{(+,+)}{-} (s'^c_2, t'^c_0).$$
We know that $(z_0, z_1) \in M$ (by Lemma 6). Note that ρ is the desired alternating path in G_M with two blocking edges (s^c_1, t^c_1) and (s'^c_2, t'^c_2).
2. Suppose the gadgets of p^c_4 and p'^c_7 are in stable state. So the edges $(p^c_i, q^c_i) \in M$ for $i = 3, 4, 5$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 6, 7, 8$. Consider the following alternating path ρ with respect to M:
$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_4, q^c_4) \overset{(-,+)}{-} (p^c_5, q^c_5) \overset{(+,-)}{-} (p^c_3, q^c_3) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (q'^c_6, p'^c_6) \overset{(-,+)}{-} (q'^c_8, p'^c_8) \overset{(+,-)}{-} (q'^c_7, p'^c_7) \overset{(-,+)}{-} (t'^c_0, s'^c_2) \overset{(+,+)}{-} (t'^c_2, s'^c_0).$$
Observe that the labels on edges of $\rho \setminus M$ are absolutely identical to the first case and thus ρ is the desired alternating path in G_M with two blocking edges (s^c_1, t^c_1) and (s'^c_2, t'^c_2).
3. Suppose the gadgets of p^c_7 and q'^c_0 are in stable state. So the edges $(p^c_i, q^c_i) \in M$ for $i = 6, 7, 8$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 0, 1, 2$. Consider the following alternating path ρ with respect to M:
$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_7, q^c_7) \overset{(-,+)}{-} (p^c_8, q^c_8) \overset{(+,-)}{-} (p^c_6, q^c_6) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (p'^c_1, q'^c_1) \overset{(-,+)}{-} (p'^c_2, q'^c_2) \overset{(+,-)}{-} (p'^c_0, q'^c_0) \overset{(-,+)}{-} (s'^c_0, t'^c_2) \overset{(+,+)}{-} (s'^c_2, t'^c_0).$$
Again, observe that the labels on edges of $\rho \setminus M$ are absolutely identical to the first two cases and ρ is the desired alternating path with two blocking edges (s^c_1, t^c_1) and (s'^c_2, t'^c_2).
4. Suppose the gadgets of p^c_7 and p'^c_7 are in stable state. So the edges $(p^c_i, q^c_i) \in M$ for $i = 6, 7, 8$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 6, 7, 8$. Consider the following alternating path ρ with respect to M:
$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_7, q^c_7) \overset{(-,+)}{-} (p^c_8, q^c_8) \overset{(+,-)}{-} (p^c_6, q^c_6) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (q'^c_6, p'^c_6) \overset{(-,+)}{-} (q'^c_8, p'^c_8) \overset{(+,-)}{-} (q'^c_7, p'^c_7) \overset{(-,+)}{-} (t'^c_0, s'^c_2) \overset{(+,+)}{-} (t'^c_2, s'^c_0).$$

As before, the labels on edges of $\rho \setminus M$ are absolutely identical to the above three cases and ρ is the desired alternating path with two blocking edges (s^c_1, t^c_1) and (s'^c_2, t'^c_2).

For any $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$, an analogous construction can be shown.

- Let $i = j = 1$. So t^c_0 prefers p^c_4 and p^c_7 to its partner s^c_1 in M and t'^c_0 prefers p'^c_4 and p'^c_7 to its partner s'^c_1 in M. We know that either the gadget of p^c_4 or the gadget of p^c_7 is in stable state, and similarly, either the gadget of p'^c_4 or the gadget of p'^c_7 is in stable state.
  Suppose the gadgets of p^c_4 and p'^c_4 are in stable state. So the edges $(p^c_i, q^c_i) \in M$ for $i = 3, 4, 5$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 3, 4, 5$. Consider the following alternating path ρ with respect to M:

$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_4, q^c_4) \overset{(-,+)}{-} (p^c_5, q^c_5) \overset{(+,-)}{-} (p^c_3, q^c_3) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (q'^c_3, p'^c_3) \overset{(-,+)}{-} (q'^c_5, p'^c_5) \overset{(+,-)}{-} (q'^c_4, p'^c_4) \overset{(-,+)}{-} (t'^c_0, s'^c_1) \overset{(+,+)}{-} (t'^c_1, s'^c_0).$$

  Observe again that the labels on edges of $\rho \setminus M$ are absolutely identical to the labels obtained for the desired alternating paths when $i = 1$ and $j = 2$. The path ρ is the desired alternating path in G_M with two blocking edges (s^c_1, t^c_1) and (s'^c_1, t'^c_1).
  The case when the gadgets of p^c_7 and p'^c_7 are in stable state was already seen in Case 4 of $i = 1$ and $j = 2$. The only difference between the path that we will construct now with the path ρ seen there is in the last two edges: now we will have (t'^c_0, s'^c_1) and (t'^c_1, s'^c_0) in M; thus the blocking edges to our path will be (s^c_1, t^c_1) and (s'^c_1, t'^c_1).
  The proofs for the remaining two cases: (i) when the gadgets of p^c_4 and p'^c_7 are in stable state and (ii) when the gadgets of p^c_7 and p'^c_4 are in stable state are absolutely analogous to the above case. Thus in all 4 cases, we can show the existence of an alternating path ρ in G_M with *two* blocking edges (s^c_1, t^c_1) and (s'^c_1, t'^c_1): a contradiction to M's popularity.
- Let $i = 1$ and $j = 3$. So t^c_0 prefers p^c_4 and p^c_7 to its partner s^c_1 in M and s'^c_0 prefers q'^c_0 and q'^c_3 to its partner t'^c_3 in M. We know that either the gadget of p^c_4 or the gadget of p^c_7 is in stable state, and similarly, either the gadget of q'^c_0 or the gadget of q'^c_3 is in stable state.
  The cases when the gadgets of p^c_4 and q'^c_0 are in stable state and when the gadgets of p^c_7 and q'^c_0 are in stable state were already seen in Case 1 and Case 3 of $i = 1$ and $j = 2$: thus we can construct analogous alternating paths in these cases. So let us consider the case when the gadgets of p^c_7 and q'^c_3 are in stable state.
  So the edges $(p^c_i, q^c_i) \in M$ for $i = 6, 7, 8$ and the edges $(p'^c_j, q'^c_j) \in M$ for $j = 3, 4, 5$. Consider the following alternating path ρ with respect to M:

$$(s^c_0, t^c_1) \overset{(+,+)}{-} (s^c_1, t^c_0) \overset{(+,-)}{-} (p^c_7, q^c_7) \overset{(-,+)}{-} (p^c_8, q^c_8) \overset{(+,-)}{-} (p^c_6, q^c_6) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (p'^c_4, q'^c_4) \overset{(-,+)}{-} (p'^c_5, q'^c_5) \overset{(+,-)}{-} (p'^c_3, q'^c_3) \overset{(-,+)}{-} (s'^c_0, t'^c_3) \overset{(+,+)}{-} (s'^c_3, t'^c_0).$$

  The above path ρ is the desired alternating path in G_M with two blocking edges (s^c_1, t^c_1) and (s'^c_3, t'^c_3).
  The remaining case, i.e., when the gadgets of p^c_4 and q'^c_3 are in stable state, is absolutely analogous to the above case and we can again show an alternating path in G_M with two blocking edges (s^c_1, t^c_1) and (s'^c_3, t'^c_3).
- Let $i = 2$ and $j = 1$. This is a "mirror image" of the very first case considered: when $i = 1$ and $j = 2$. The only difference is that we swap *primed variables* and *unprimed variables* in ρ. For example, when the gadgets of q^c_0 and p'^c_4 are in stable state, the desired alternating path is exactly the same as ρ in Case 1 there, except for this swapping of roles. Thus ρ, with blocking edges (s^c_2, t^c_2) and (s'^c_1, t'^c_1), would be:

$$(t^c_0, s^c_2) \overset{(+,+)}{-} (t^c_2, s^c_0) \overset{(+,-)}{-} (q^c_0, p^c_0) \overset{(-,+)}{-} (q^c_2, p^c_2) \overset{(+,-)}{-} (q^c_1, p^c_1) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (q'^c_3, p'^c_3) \overset{(-,+)}{-} (q'^c_5, p'^c_5) \overset{(+,-)}{-} (q'^c_4, p'^c_4) \overset{(-,+)}{-} (t'^c_0, s'^c_1) \overset{(+,+)}{-} (t'^c_1, s'^c_0).$$

– Let $i = j = 2$. So s_0^c prefers q_0^c to its partner t_2^c in M and and t_0^c prefers p_7^c to its partner s_2^c in M and $s_0'^c$ prefers $q_0'^c$ to its partner $t_2'^c$ in M and $t_0'^c$ prefers $p_7'^c$ to its partner $s_1'^c$ in M. We know that either the gadget of q_0^c or the gadget of p_7^c is in stable state, and similarly, either the gadget of $q_0'^c$ or the gadget of $p_7'^c$ is in stable state.

The cases when the gadgets of p_7^c and $q_0'^c$ are in stable state and when the gadgets of p_7^c and $p_7'^c$ are in stable state were already seen in Cases 3 and 4 of $i = 1$ and $j = 2$. Let us consider the case when the gadgets of q_0^c and $q_0'^c$ are in stable state.

So the edges $(p_i^c, q_i^c) \in M$ for $i = 0, 1, 2$ and the edges $(p_j'^c, q_j'^c) \in M$ for $j = 0, 1, 2$. Consider the following alternating path ρ with respect to M:

$$(t_0^c, s_2^c) \overset{(+,+)}{-} (t_2^c, s_0^c) \overset{(+,-)}{-} (q_0^c, p_0^c) \overset{(-,+)}{-} (q_2^c, p_2^c) \overset{(+,-)}{-} (q_1^c, p_1^c) \overset{(-,+)}{-}$$
$$(z_0, z_1) \overset{(+,-)}{-} (p_1'^c, q_1'^c) \overset{(-,+)}{-} (p_2'^c, q_2'^c) \overset{(+,-)}{-} (p_0'^c, q_0'^c) \overset{(-,+)}{-} (s_0'^c, t_2'^c) \overset{(+,+)}{-} (s_2'^c, t_0'^c).$$

The path ρ is the desired alternating path in G_M with two blocking edges (s_2^c, t_2^c) and $(s_2'^c, t_2'^c)$. The proof for the remaining case is absolutely analogous. Thus in all 4 cases, we can show the existence of an alternating path ρ in G_M with *two* blocking edges (s_2^c, t_2^c) and $(s_2'^c, t_2'^c)$: a contradiction to M's popularity.

It is easy to see that the remaining cases of (i, j) are absolutely analogous to the ones listed above and this finishes the proof of the lemma. ◻

Recall that there are three level 1 gadgets associated with any clause c: these gadgets correspond to the three variables in c.

Lemma 9. *Let $c = X_i \vee X_j \vee X_k$. At least one of the level 1 gadgets corresponding to X_i, X_j, X_k is in unstable state with respect to M.*

Proof. Suppose not. That is, assume that for some clause c, all three of its level 1 gadgets are in stable state. Let $c = X_i \vee X_j \vee X_k$. So (x_r, y_r) and (x_r', y_r') are in M for all $r \in \{i, j, k\}$.

We know from Lemma 8 that either two or more of the *first* three level 2 gadgets corresponding to c are in unstable state with respect to M or two or more of the *last* three level 2 gadgets corresponding to c are in unstable state with respect to M. Assume without loss of generality that the first and second gadgets, i.e., those on p_i^c, q_i^c, for $0 \le i \le 5$, are in unstable state with respect to M.

We know from our lemmas in Section 4 that there is no popular edge across gadgets. Thus M matches the 6 vertices of a level 2 gadget with each other. In particular, it follows from Lemma 3 that for the level 2 gadget on p_i^c, q_i^c for $i = 0, 1, 2$, we have (i) $(p_0^c, q_0^c), (p_1^c, q_1^c), (p_2^c, q_2^c)$ in M or (ii) $(p_0^c, q_2^c), (p_1^c, q_1^c), (p_2^c, q_0^c)$ in M or (iii) $(p_0^c, q_0^c), (p_1^c, q_2^c), (p_2^c, q_1^c)$ in M.

There are two unstable states for each level 2 gadget, i.e., either (ii) or (iii) above for the gadget on p_i^c, q_i^c for $i = 0, 1, 2$. A level 2 gadget can be in either of these two unstable states in M—without loss of generality assume that M contains $(p_0^c, q_0^c), (p_1^c, q_2^c), (p_2^c, q_1^c)$ and $(p_3^c, q_5^c), (p_4^c, q_4^c), (p_5^c, q_3^c)$. Observe that p_2^c likes y_j more than q_1^c and similarly, q_5^c likes x_i more than p_3^c. Consider the following alternating path ρ with respect to M:

$$(q_2^c, p_1^c) \overset{(+,+)}{-} (q_1^c, p_2^c) \overset{(+,-)}{-} (y_j, x_j) \overset{(-,+)}{-} (z_0, z_1) \overset{(+,-)}{-} (y_i, x_i) \overset{(-,+)}{-} (q_5^c, p_3^c) \overset{(+,+)}{-} (q_3^c, p_5^c).$$

Note that M has to contain (z_0, z_1) (by Lemma 6). Observe that ρ is an alternating path in G_M with *two* blocking edges (p_1^c, q_1^c) and (p_3^c, q_3^c). This is a contradiction to M's popularity (by Theorem 2) and the lemma follows. ◻

Lemma 10. *Let $c = X_i \vee X_j \vee X_k$. At most one of the level 1 gadgets corresponding to X_i, X_j, X_k is in unstable state with respect to M.*

Proof. Suppose not. So at least two of the three level 1 gadgets corresponding to X_i, X_j, X_k are in unstable state with respect to M. Assume without loss of generality that the gadgets corresponding to variables X_i and X_j are in unstable state. So the edges $(x_i, y_i'), (x_i', y_i)$ are in M, similarly the edges $(x_j, y_j'), (x_j', y_j)$ are in M.

Recall that a_5^c regards y_i' as its second choice neighbor and b_5^c regards x_j' as its second choice neighbor. Similarly, $b_5'^c$ regards x_i' as its second choice neighbor and $a_5'^c$ regards y_j' as its second choice neighbor.

In the popular matching M, level 0 vertices are matched within their own gadget. Therefore, either $\{(a_5^c, b_5^c), (a_6^c, b_6^c)\} \subset M$ or $\{(a_5^c, b_6^c), (a_6^c, b_5^c)\} \subset M$; similarly $\{(a_5'^c, b_5'^c), (a_6'^c, b_6'^c)\} \subset M$ or $\{(a_5'^c, b_6'^c), (a_6'^c, b_5'^c)\} \subset M$. Thus the following two observations clearly hold:

- either a_5^c or b_5^c is matched to its third choice neighbor;
- either $a_5'^c$ or $b_5'^c$ is matched to its third choice neighbor.

Based on which of these vertices are matched to their third choice neighbors, we have four cases as shown below. Each of these 4 cases results in a forbidden alternating path/cycle (as given in Theorem 2), thus proving the lemma.

Case 1. The vertices a_5^c and $a_5'^c$ are matched to their third choice neighbors. So $(a_5^c, b_6^c), (a_6^c, b_5^c)$ and $(a_5'^c, b_6'^c), (a_6'^c, b_5'^c)$ are in M. Consider the following alternating path ρ with respect to M:

$$(x_i', y_i) \overset{(+,+)}{-} (x_i, y_i') \overset{(-,+)}{-} (a_5^c, b_6^c) \overset{(-,+)}{-} (z_0, z_1) \overset{(+,-)}{-} (b_6'^c, a_5'^c) \overset{(+,-)}{-} (y_j', x_j) \overset{(+,+)}{-} (y_j, x_j').$$

Observe that ρ is an alternating path in G_M with two blocking edges (x_i, y_i) and (x_j, y_j), a contradiction to M's popularity.

Case 2. The vertices a_5^c and $b_5'^c$ are matched to their third choice neighbors. So $(a_5^c, b_6^c), (a_6^c, b_5^c)$ and $(a_5'^c, b_5'^c), (a_6'^c, b_6'^c)$ are in M. Consider the following alternating cycle C with respect to M:

$$(b_6^c, a_5^c) \overset{(+,-)}{-} (y_i', x_i) \overset{(+,+)}{-} (y_i, x_i') \overset{(-,+)}{-} (b_5'^c, a_5'^c) \overset{(-,+)}{-} (z_1, z_0) \overset{(+,-)}{-} (b_6^c, a_5^c).$$

Observe that C is an alternating cycle in G_M with a blocking edge (x_i, y_i), a contradiction to M's popularity.

Case 3. The vertices b_5^c and $a_5'^c$ are matched to their third choice neighbors. So $(a_5^c, b_5^c), (a_6^c, b_6^c)$ and $(a_5'^c, b_6'^c), (a_6'^c, b_5'^c)$ are in M. Consider the following alternating cycle C' with respect to M:

$$(a_5^c, b_5^c) \overset{(+,-)}{-} (x_j', y_j) \overset{(+,+)}{-} (x_j, y_j') \overset{(-,+)}{-} (a_5'^c, b_6'^c) \overset{(-,+)}{-} (z_0, z_1) \overset{(+,-)}{-} (a_5^c, b_5^c).$$

Observe that C' is an alternating cycle in G_M with a blocking edge (x_j, y_j), a contradiction to M's popularity.

Case 4. The vertices b_5^c and $b_5'^c$ are matched to their third choice neighbors. So $(a_5^c, b_5^c), (a_6^c, b_6^c)$ and $(a_5'^c, b_5'^c), (a_6'^c, b_6'^c)$ are in M. Consider the following alternating path ρ' with respect to M:

$$(y_i', x_i) \overset{(+,+)}{-} (y_i, x_i') \overset{(-,+)}{-} (b_5'^c, a_5'^c) \overset{(-,+)}{-} (z_0, z_1) \overset{(+,-)}{-} (a_5^c, b_5^c) \overset{(+,-)}{-} (x_j', y_j) \overset{(+,+)}{-} (x_j, y_j').$$

Observe that ρ' is an alternating path in G_M with two blocking edges (x_i, y_i) and (x_j, y_j), a contradiction to M's popularity. ◻

We have shown that *at most one* of the level 1 gadgets corresponding to X_i, X_j, X_k is in unstable state with respect to M. So *exactly one* of the level 1 gadgets corresponding to X_i, X_j, X_k is in unstable state with respect to M. This allows us to set a 1-in-3 satisfying assignment to instance ϕ. For each variable X_i in ϕ do:

— if the gadget corresponding to X_i is in *unstable* state then set $X_i = \mathsf{true}$ else set $X_i = \mathsf{false}$.

It follows from our above discussion that this is a 1-in-3 satisfying assignment for ϕ. We have thus shown the following result.

Theorem 3. *If G admits a popular matching then ϕ has a 1-in-3 satisfying assignment.*

6 The converse

We will now show the converse of Theorem 3, i.e., if ϕ has a 1-in-3 satisfying assignment S then G admits a popular matching. We will use S to construct a popular matching M in G as follows. To begin with, $M = \emptyset$.

Level 1. For each variable X_i do:

- if X_i is set to true in S then add (x_i, y_i') and (x_i', y_i) to M;
- else add (x_i, y_i) and (x_i', y_i') to M.

For each clause $c = X_i \vee X_j \vee X_k$, we know that exactly one of X_i, X_j, X_k is set to true in S. Assume without loss of generality that $X_k = \mathsf{true}$ in S. For the level 0, 2, and 3 gadgets corresponding to c, we do as follows:

Level 0. Recall that there are *six* level 0 gadgets that correspond to c. For the first 3 gadgets (these are on vertices a_i^c, b_i^c for $i = 1, \ldots, 6$) do:

- include $(a_1^c, b_2^c), (a_2^c, b_1^c)$ from the first gadget;
- include $(a_3^c, b_3^c), (a_4^c, b_4^c)$ from the second gadget;
- choose either $(a_5^c, b_5^c), (a_6^c, b_6^c)$ or $(a_5^c, b_6^c), (a_6^c, b_5^c)$ from the third gadget.

Observe that since the third variable X_k of c was set to be true, cross edges are fixed in the first gadget (see Fig. 3), while the other stable matching (horizontal edges) is chosen in the second gadget.

For the fourth and fifth gadgets, we will do exactly the opposite. Also, it will not matter which stable pair of edges is chosen from the third and sixth gadgets. So for the last 3 level 0 gadgets corresponding to c (these are on vertices $a_i'^c, b_i'^c$ for $i = 1, \ldots, 6$) do:

- include $(a_1'^c, b_1'^c), (a_2'^c, b_2'^c)$ from the fourth gadget;
- include $(a_3'^c, b_4'^c), (a_4'^c, b_3'^c)$ from the fifth gadget.
- choose either $(a_5'^c, b_5'^c), (a_6'^c, b_6'^c)$ or $(a_5'^c, b_6'^c), (a_6'^c, b_5'^c)$ from the sixth gadget.

Level 2. Recall that there are *six* level 2 gadgets that correspond to c. For the first 3 gadgets (these are on vertices p_i^c, q_i^c for $i = 0, \ldots, 8$) do:

- include $(p_0^c, q_2^c), (p_1^c, q_1^c), (p_2^c, q_0^c)$ from the first gadget
- include $(p_3^c, q_3^c), (p_4^c, q_5^c), (p_5^c, q_4^c)$ from the second gadget
- include $(p_6^c, q_6^c), (p_7^c, q_7^c), (p_8^c, q_8^c)$ from the third gadget

In the first three gadgets, because $X_k = \mathsf{true}$, the third one is set to parallel edges, reaching the stable state, while the first one is blocked by the top horizontal edge and the second one is blocked by the middle horizontal edge. Include isomorphic edges (to the above ones) from the last three level 2 gadgets corresponding to c, i.e., include $(p_0'^c, q_2'^c), (p_1'^c, q_1'^c), (p_2'^c, q_0'^c)$ from the fourth gadget, and so on. On this level, the last three gadgets mimic the matching edges from the first three gadgets, unlike in level 0.

Level 3. For the first level 3 gadget corresponding to c do:

- include $(s_0^c, t_3^c), (s_1^c, t_1^c), (s_2^c, t_2^c), (s_3^c, t_0^c)$ in M.

Since the third variable in c was set to be true, the vertices s_0^c and t_0^c are matched to t_3^c and s_3^c, respectively—thus the bottom horizontal edge (s_3^c, t_3^c) blocks M. Include isomorphic edges (to the above ones) for the second level 3 gadget corresponding to c, i.e., include $(s_0'^c, t_3'^c), (s_1'^c, t_1'^c), (s_2'^c, t_2'^c), (s_3'^c, t_0'^c)$ in M. Once again, the second gadget mimics the matching edges on the first gadget.

Z-gadget and D-gadget. Finally include the edges $(z_0, z_1), (z_2, z_3), (z_4, z_5)$ from the Z-gadget in M. By Lemma 6, every popular matching in G has to include these edges. Also include $(d_0, d_1), (d_2, d_3)$ from the D-gadget in M.

6.1 The popularity of M

We will now prove the popularity of the above matching M via the LP framework of popular matchings initiated in [18] for bipartite graphs. This framework generalizes to provide a sufficient condition for popularity in non-bipartite graphs [9]. This involves showing a witness $\boldsymbol{\alpha} \in \{0, \pm 1\}^{|V|}$ such that $\boldsymbol{\alpha}$ is a *certificate* of M's popularity. In order to define the constraints that $\boldsymbol{\alpha}$ has to satisfy so as to certify M's popularity, let us define an edge weight function w_M as follows.

For any edge (u, v) in G do:

- if (u, v) is labeled $(-, -)$ then set $w_M(u, v) = -2$;
- if (u, v) is labeled $(+, +)$ then set $w_M(u, v) = 2$;
- else set $w_M(u, v) = 0$. (So $w_M(e) = 0$ for all $e \in M$.)

Let N be any perfect matching in G. It is easy to see from the definition of the edge weight function w_M that $w_M(N) = \phi(N, M) - \phi(M, N)$.

Let the max-weight perfect fractional matching LP in the graph G with edge weight function w_M be our primal LP. This is LP1 defined below.

$$\text{maximize} \sum_{e \in E} w_M(e) x_e \qquad \text{(LP1)}$$

subject to

$$\textstyle\sum_{e \in \delta(u)} x_e \;=\; 1 \;\; \forall\, u \in V \qquad \text{and} \qquad x_e \geq 0 \;\; \forall\, e \in E.$$

If the primal optimal value is at most 0 then $w_M(N) \leq 0$ for all perfect matchings N in G, i.e., $\phi(N, M) \leq \phi(M, N)$. This means $\phi(M', M) \leq \phi(M, M')$ for *all* matchings M' in G, since G is a complete graph on an even number of vertices (so $M' \subseteq$ some perfect matching). That is, M is a popular matching in G.

Consider the LP that is dual to LP1. This is LP2 given below in variables α_u, where $u \in V$.

$$\text{minimize} \sum_{u \in V} \alpha_u \qquad \text{(LP2)}$$

subject to

$$\alpha_u + \alpha_v \;\; \geq \;\; w_M(u, v) \;\; \forall\, (u, v) \in E.$$

If we show a dual feasible solution $\boldsymbol{\alpha}$ such that $\sum_{u \in V} \alpha_u = 0$ then the primal optimal value is at most 0, i.e., M is a popular matching.

In order to prove the popularity of M, we define $\boldsymbol{\alpha}$ as follows. For each variable X_r do:

- if X_r was set to true then set $\alpha_{x_r} = \alpha_{y_r} = 1$ and $\alpha_{x'_r} = \alpha_{y'_r} = -1$;
- else set $\alpha_{x_r} = \alpha_{y_r} = \alpha_{x'_r} = \alpha_{y'_r} = 0$.

Let clause $c = X_i \vee X_j \vee X_k$. Recall that we assumed that $X_i = X_j = \mathsf{false}$ and $X_k = \mathsf{true}$. For the vertices in clauses corresponding to c, we will set α-values as follows.

- For every level 0 vertex v do: set $\alpha_v = 0$.
- For the first three level 2 gadgets corresponding to c do:
  - set $\alpha_{p_0^c} = \alpha_{q_0^c} = 1$, $\alpha_{p_1^c} = 1, \alpha_{q_1^c} = -1$, and $\alpha_{p_2^c} = \alpha_{q_2^c} = -1$;
  - set $\alpha_{p_3^c} = -1, \alpha_{q_3^c} = 1$, $\alpha_{p_4^c} = \alpha_{q_4^c} = 1$, and $\alpha_{p_5^c} = \alpha_{q_5^c} = -1$;
  - set $\alpha_{p_6^c} = \alpha_{q_6^c} = \alpha_{p_7^c} = \alpha_{q_7^c} = \alpha_{p_8^c} = \alpha_{q_8^c} = 0$.

The setting of α-values is analogous for vertices in the last three level 2 gadgets corresponding to c. For the first level 3 gadget corresponding to c do:

- set $\alpha_{s_0^c} = \alpha_{t_0^c} = -1$, $\alpha_{s_1^c} = -1, \alpha_{t_1^c} = 1$, $\alpha_{s_2^c} = -1$, $\alpha_{t_2^c} = 1$, and $\alpha_{s_3^c} = \alpha_{t_3^c} = 1$.

The setting of α-values is analogous for vertices in the other level 3 gadget corresponding to c. For the z-vertices do: set $\alpha_u = 0$ for all $u \in \{z_0, \ldots, z_5\}$. For the d-vertices do:

– set $\alpha_{d_0} = \alpha_{d_2} = -1$ and $\alpha_{d_1} = \alpha_{d_3} = 1$.

Properties of $\boldsymbol{\alpha}$. For every $(u, v) \in M$, either $\alpha_u = \alpha_v = 0$ or $\{\alpha_u, \alpha_v\} = \{-1, 1\}$; so $\alpha_u + \alpha_v = 0$. Since M is a perfect matching, we have $\sum_{u \in V} \alpha_u = 0$. The claims stated below show that $\boldsymbol{\alpha}$ is a feasible solution to LP2. This will prove the popularity of M.

We need to show that every edge (u, v) is *covered*, i.e., $\alpha_u + \alpha_v \geq w_M(u, v)$. We have already observed that for any $(u, v) \in M$, $\alpha_u + \alpha_v = 0 = w_M(u, v)$.

Claim 4 *Let (u, v) be a blocking edge to M. Then $\alpha_u + \alpha_v = 2 = w_M(u, v)$.*

Proof. Level 1 gadgets that correspond to variables set to true have blocking edges. More precisely, for every variable X_k set to true, (x_k, y_k) is a blocking edge to M and we have $\alpha_{x_k} = \alpha_{y_k} = 1$. Thus $\alpha_{x_k} + \alpha_{y_k} = 2 = w_M(x_k, y_k)$. Similarly, consider any level 2 or level 3 gadget that is in unstable state: such a gadget has a blocking edge within it, say (p^c_0, q^c_0) or (p^c_4, q^c_4) or (s^c_3, t^c_3), and both endpoints of such an edge have their α-values set to 1. For the D-gadget, (d_1, d_3) is a blocking edge and we have $\alpha_{d_1} = \alpha_{d_3} = 1$. There are no blocking edges to M in the Z-gadget or in a level 0 gadget. Thus all blocking edges are covered. ◻

Claim 5 *Let (u, v) be an intra-gadget edge that is non-blocking. Then $\alpha_u + \alpha_v \geq w_M(u, v)$.*

Proof. For any edge (z_i, z_j) where $i, j \in \{0, 1, \ldots, 5\}$, we have $\alpha_{z_i} + \alpha_{z_j} = 0 = w_M(z_i, z_j)$. Similarly, all edges within the D-gadget are covered.

For any (a^c_i, b^c_i), we have $\alpha_{a^c_i} + \alpha_{b^c_i} = 0 = w_M(a^c_i, b^c_i)$. Similarly, $\alpha_{a^c_{2i-1}} + \alpha_{b^c_{2i}} = 0 = w_M(a^c_{2i-1}, b^c_{2i})$, also $\alpha_{a^c_{2i}} + \alpha_{b^c_{2i-1}} = 0 = w_M(a^c_{2i}, b^c_{2i-1})$ for all i and c.

We also have for all c: $\alpha_{p^c_1} + \alpha_{q^c_2} = 0 = w_M(p^c_1, q^c_2)$ while $\alpha_{p^c_2} + \alpha_{q^c_1} = -2 = w_M(p^c_2, q^c_1)$ and $\alpha_{p^c_2} + \alpha_{q^c_2} = -2 = w_M(p^c_2, q^c_2)$. It is similar for all other edges within level 2 gadgets and also for edges within level 3 gadgets. Thus it is easy to see that for all intra-gadget non-blocking edges (u, v), we have $\alpha_u + \alpha_v \geq w_M(u, v)$. ◻

Claim 6 *Let (u, v) be any inter-gadget edge. Then $\alpha_u + \alpha_v \geq w_M(u, v)$.*

Proof. No inter-gadget edge blocks M. The vertices z_0 and z_1 prefer some neighbors in levels 0, 1, 2 to each other and the α-value of each of these neighbors is either 0 or 1. In particular, $\alpha_{x_i} \geq 0$ and $\alpha_{y_i} \geq 0$, $\alpha_{p^c_1} \geq 0$ and $\alpha_{q^c_0} \geq 0$, and so on while $\alpha_{a^c_i} = \alpha_{b^c_i} = 0$ for all i and c. Since $\alpha_{z_i} = 0$ for all i, the edges incident to z_i are covered for all i.

Consider edges between a level 0 vertex and a level 1 vertex, such as (a^c_1, y'_j) or (b^c_1, x'_k): regarding the former edge, we have $w_M(a^c_1, y'_j) = 0 = \alpha_{a^c_1} + \alpha_{y'_j}$ and for the latter edge, we have $w_M(b^c_1, x'_k) = -2 < -1 = \alpha_{b^c_1} + \alpha_{x'_k}$. It can similarly be verified that every edge between a level 0 vertex and a level 1 vertex is covered.

Consider edges between a level 1 vertex and a level 2 vertex, such as (p^c_2, y_j) or (x_k, q^c_2): recall that $(p^c_0, q^c_2) \in M$ and so $w_M(x_k, q^c_2) = 0$; we set $\alpha_{q^c_2} = -1$ and $\alpha_{x_k} = 1$, thus $\alpha_{x_k} + \alpha_{q^c_2} = w_M(x_k, q^c_2)$. We have $(p^c_2, y_j) = -2$ since $(p^c_2, q^c_0) \in M$ and so this edge is covered. It can similarly be verified that every edge between a level 1 vertex and a level 2 vertex is covered.

Consider edges between a level 2 vertex and a level 3 vertex, such as those incident to s^c_0 or t^c_0: we have $w_M(s^c_0, q^c_0) = w_M(s^c_0, q^c_3) = 0$ and $\alpha_{s^c_0} = -1$ while $\alpha_{q^c_0} = \alpha_{q^c_3} = 1$. Similarly, $w_M(p^c_7, t^c_0) = w_M(p^c_4, t^c_0) = -2$ and so these edges are covered. It is analogous with edges incident to s'^c_0 or t'^c_0.

Consider any edge e whose one endpoint is in the D-gadget and the other endpoint is outside the D-gadget. It is easy to see that $w_M(e) = -2$, hence this edge is covered. ◻

Thus we have shown the following theorem.

Theorem 4. *If ϕ has a 1-in-3 satisfying assignment then G admits a popular matching.*

Theorem 1 stated in Section 1 follows from Theorems 3 and 4. Thus the popular matching problem in a roommates instance on n vertices with complete preference lists is NP-hard for even n.

Acknowledgment. Thanks to Chien-Chung Huang for asking us about the complexity of the popular roommates problem with complete preference lists.

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