(I Can't Get No) Antisatisfaction.

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Substructural approaches to paradoxes have attracted much attention from the philosophical community in the last decade. In this paper we focus on two substructural logics, named ST and TS, along with two structural cousins, LP and K3. It is well known that LP and K3 are duals in the sense that an inference is valid in one logic just in case the contrapositive is valid in the other logic. As a consequence of this duality, theories based on either logic are tightly connected since many of the arguments for and objections against one theory reappear in the other theory in dual form. The target of the paper is making explicit in exactly what way, if any, ST and TS are dual to one another. The connection will allow us to gain a more fine-grained understanding of these logics and of the theories based on them. In particular, we will obtain new insights on two questions concerning ST which are being intensively discussed in the current literature: whether ST preserves classical logic and whether it is LP in sheep's clothing. Explaining in what way ST and TS are duals requires comparing these logics at a metainferential level. We provide to this end a uniform proof theory to decide on valid metainferences for each of the four logics. This proof procedure allows us to show in a very simple way how different properties of inferences (unsatisfiability, supersatisfiability and antivalidity) that behave in very different ways for each logic can be captured in terms of the validity of a metainference.

Substructural approaches to paradox – Non-transitive logic – Non-reflexive logic – Strong Kleene

1 Substructural Logics and Paradoxes

In a sequent calculus a structural rule (as opposed to an operational rule) is a rule that does not mention any particular piece of logical vocabulary. We can think of it as expressing some structural property of the consequence relation itself, more than giving meaning to a particular item in the logical vocabulary. Consider, for example, the rules of Identity and Cut:

$$\frac{}{A \Rightarrow A}\ \text{Id} \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \text{Cut}$$

We can think of Identity as expressing the property of reflexivity and of Cut as expressing the property of transitivity of a consequence relation.

A good number of papers in the recent literature about paradoxes focus on the so-called "substructural logics", that is, non-classical logics obtained by restricting or completely abandoning one or more structural rules. There are good reasons indeed for going substructural – at least if you want to hold on to some intuitively correct features about truth, like transparency. Paradoxes can arise in different forms: the Liar (involving negation), the Curry (involving the conditional), the Validity Curry (involving a Validity predicate) and some more. While the structuralist needs to solve these paradoxes one by one, playing with the rules for each connective, the substructuralist makes the promise of getting rid of paradoxes in a uniform way while keeping the meaning of classical connectives – or introducing minimal disturbance. After a persuasive argument in this direction, David Ripley concludes,

> Rather than rushing from paradox to paradox making ad hoc modifications, these substructural approaches grapple with the paradoxes where they live: in the basic features of argumentation. This way, they can avoid having to worry about rules governing particular pieces of vocabulary; in a single fell swoop they address liars, curries, validity curries, Hinnion-liberts, and so on.(Ripley, 2015, 310)

Each possible substructural approach to paradoxes has been defended by some philosopher in the recent literature.[1] In this paper we focus on two substructural approaches named ST and TS, and on two structural cousins, LP and $\mathsf{K3}$. In subsection 2.1 we introduce these logics as based on Strong Kleene valuations. In subsection 2.2 we describe three properties of inferences, in addition to the standard property of validity. Although the relationship between these properties are straightforward in the classical setting, in the context of Strong Kleene logics the picture is more complex. Subsection 2.3 concludes by briefly discussing the notion of duality for consequence relations. In section 3 we introduce a proof procedure for metainferential validity for any of the four logics discussed. This procedure will allow us to show in section 4 that the properties discussed in section 2.2, and par-

[1] See Petersen (2000), Shapiro (2010) Zardini (2011), Beall and Murzi (2013) and Rosenblatt (2019) for non-contractive approaches; Weir (2005), Cobreros et al. (2013) and Ripley (2013) for non-transitive approaches; French (2016) and Nicolai and Rossi (2018) for non-reflexive approaches and Da Ré (2020) for non-monotonic.

ticularly that of xy-antivalidity, can be captured by the (xy-)validity of a metainference. Section 5 concludes.

2 Strong Kleene Logics

2.1 Four Three-valued Logics

Let $\mathcal{L}$ be a propositional language with the usual connectives ($\vee, \wedge, \supset, \neg, \top, \bot$)[2] and let an interpretation v be a function from propositional letters to the set of truth values $\{1, \frac{1}{2}, 0\}$. Interpretations extend to formulas according to the *Strong Kleene* scheme:

$$v(A \vee B) = max(v(A), v(B))$$

$$v(A \wedge B) = min(v(A), v(B))$$

$$v(A \supset B) = max(1 - v(A), v(B))$$

$$v(\neg A) = 1 - v(A)$$

$$v(\top) = 1$$

$$v(\bot) = 0$$

The semantics make room for two notions of formula satisfaction. We say that a formula A is *strictly satisfied* by an interpretation v, written $v \Vdash_s A$, when $v(A) = 1$ and that it is *tolerantly satisfied*, written $v \Vdash_t A$, when $v(A) > 0$. Strict and tolerant satisfaction are *duals* in the sense that $v \nVdash_t A$ if and only if $v \Vdash_s \neg A$ and $v \nVdash_s A$ if and only if $v \Vdash_t \neg A$ (most of what comes later hangs on this fact).

If we understand validity as a form of preservation of satisfaction from premises to conclusions it is natural to consider two notions of validity out of our two notions of satisfaction: one preserving strict satisfaction and one preserving tolerant satisfaction. It is possible, however, to consider *mixed* forms of validity, where both strict and tolerant satisfaction appear in the definition.[3] Therefore, given these two notions of formula satisfaction, we can define four different notions of *inference satisfaction*, substituting the

[2]Although we introduce here $\top$ and $\bot$ as logical constants their only role in this paper will be marking an empty position in a sequent.

[3]The idea that, in addition to preservation of a designated value, logical consequence can mix different notions of satisfaction appears, as far as we know, in the works of: Malinowski (1990), Nait-Abdallah (1995), Bennett (1998), Frankowski (2004), Zardini (2008), Cobreros et al. (2012), Cobreros et al. (2013) and Ripley (2013).

x's and y's by t's and s' in the following schematic definition,[4]

$v \vDash_{xy} \Gamma \Rightarrow \Delta$ iff

if $v \Vdash_x A$ (for every A in Γ) then $v \Vdash_y B$ (for some B in Δ).

We say that an inference is xy-valid, written $\vDash_{xy} \Gamma \Rightarrow \Delta$, iff it is xy-satisfied by every valuation. For example, $\vDash_{ss} \Gamma \Rightarrow \Delta$ just in case for every valuation, if all premises are strictly satisfied, then some conclusion is strictly satisfied (we read premises conjunctively and conclusions disjunctively). The resulting logics[5] are ordered (valid inferences in the logic below are included into the valid inferences of the logic above) according to diagram 1.

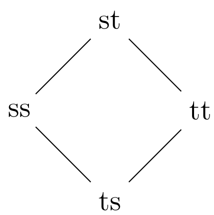


Figure 1: Four 3-valued logics

tt corresponds to Priest's Logic of Paradox LP, ss to Strong Kleene K3 while st and ts correspond to the logics ST and TS. In the following we will therefore use ss (respectively, tt, st and ts) and K3 (respectively LP, ST and TS) interchangeably. If we consider just inferences, TS is the weakest logic, since no sequent is TS-valid (apart from sequents involving $\top$ or $\bot$). Regarding valid inferences again, ST is the strongest logic (any TS, K3 or LP-valid inference is also ST-valid); actually, the set of ST-valid inferences coincides with the set of classically valid inferences (these facts about TS and ST will be made evident after the discussion in section 3).

Although ST coincides with classical logic in the set of valid sequents, these logics differ in an important respect: it is possible to consistently extend ST with a *transparent* truth predicate ('A' and '$T\langle A \rangle$' are intersubstitutable in every extensional context without a change in validity, see Cobreros et al.

[4]Where '$\Rightarrow$' stands for the sequent arrow. We take, as usual, a sequent as expressing an inference.

[5]Under the assumption that the notion of satisfaction applies equally to sequents at any metainferential level, see Scambler (2019) section 3.3 for a clarification of this point. Following Barrio et al. (2019), Chris Scambler shows that there are uncountably many logics differing perhaps only at some metainferential level. A discussion of these results is beyond the scope of this paper.

(2013), Ripley (2013)). The key feature that makes this extension possible is that ST admits failures of transitivity. That is, ST admits cases where B follows from A and C from B while C does not follow from A. This transitivity property can be characterised as a *metainference*, in the sense of an inference relating some inferences to other inferences. It is the failure of this metainference that makes ST "substructural". Similarly, TS is substructural since it admits failures of reflexivity, which in turn can be characterised as the failure of a metainference: that one allowing us to draw the inference that A follows from A out of any inference whatsoever (again, these facts should be evident after our discussion in section 3).

2.2 Inference properties

There are different properties of inferences that might be of interest in addition to validity. In this section we focus on four such properties, including validity itself.

In a classical setting, we will say that a valuation v *unsatisfies* a sequent $A \Rightarrow B$, written $v \mathrel{\underline{\perp}} A \Rightarrow B$,[6] just in case if it satisfies A then it does not satisfy B. A valuation v *supersatisfies* $A \Rightarrow B$, written $v \mathrel{\overline{\top}} A \Rightarrow B$ when if it does not satisfy A, it does satisfy B. Finally, a valuation v *antisatisfies* $A \Rightarrow B$, written $v \mathrel{\dashv\!|} A \Rightarrow B$, when if it does not satisfy A, it does not satisfy B either. We can represent these notions in a visually more appealing way:

- $v \Vdash A \Rightarrow B$ iff $v \Vdash A \leadsto v \Vdash B$
- $v \mathrel{\underline{\perp}} A \Rightarrow B$ iff $v \Vdash A \leadsto v \nVdash B$
- $v \mathrel{\overline{\top}} A \Rightarrow B$ iff $v \nVdash A \leadsto v \Vdash B$
- $v \mathrel{\dashv\!|} A \Rightarrow B$ iff $v \nVdash A \leadsto v \nVdash B$

Quantifying over valuations we can define the corresponding notions of validity ($\vDash$), unsatisfiability ($\perp\!\!\!\perp$), supersatisfiability ($\top\!\!\!\top$) and antivalidity ($=\!\mid$).

Since classical satisfaction is self-dual (in the sense that $v \nVdash A$ just in case $v \Vdash \neg A$) all the three new properties above (uns-, super- and anti-) can be defined in terms of satisfaction via negation. Thus, that a sequent $A \Rightarrow B$ is classically unsatisfied by v, written $v \mathrel{\underline{\perp}} A \Rightarrow B$, means that $v \Vdash A \Rightarrow \neg B$;

[6] For simplicity we will talk about sequents involving only one premise and only one conclusion. In the four logics, the comma in the premises works as a conjunction and the comma in the conclusions as a disjunction; this is the reason why we find this simplification unproblematic.

that $v \mathrel{\overline{\top}} A \Rightarrow B$ means that $v \Vdash \neg A \Rightarrow B$, and that $v \mathrel{\dashv\!\mid} A \Rightarrow B$ means that $v \Vdash \neg A \Rightarrow \neg B$ (similarly, that $A \Rightarrow B$ is classically unsatisfiable means that $A \Rightarrow \neg B$ is classically valid etc.)

In the setting of Strong Kleene logics, analogous notions can be defined using tolerant and strict satisfaction for formulas. For instance, a sequent $A \Rightarrow B$ is xy-unsatisfied by a valuation v, written $v \mathrel{\underline{\perp}}_{xy} A \Rightarrow B$, iff if A is x-satisfied by v, B is not y-satisfied by v. Perhaps visually more appealing,

- $v \Vdash_{xy} A \Rightarrow B$ iff $\quad v \Vdash_x A \rightsquigarrow v \Vdash_y B$
- $v \mathrel{\underline{\perp}}_{xy} A \Rightarrow B$ iff $\quad v \Vdash_x A \rightsquigarrow v \nVdash_y B$
- $v \mathrel{\overline{\top}}_{xy} A \Rightarrow B$ iff $\quad v \nVdash_x A \rightsquigarrow v \Vdash_y B$
- $v \mathrel{\dashv\!\mid}_{xy} A \Rightarrow B$ iff $\quad v \nVdash_x A \rightsquigarrow v \nVdash_y B$

Since tolerant and strict satisfaction are not self-duals but, rather, each other's dual, the situation is now quite different than in the classical case. The un-, super-, and anti-satisfaction of a sequent can be expressed in terms of the satisfaction of a sequent in which either the antecedent, or the consequent (or both) are negated. However, the notion of satisfaction used will be that of *another* Strong Kleene logic. For example, that a sequent is ST-unsatisfied by v, written $v \mathrel{\underline{\perp}}_{\mathsf{ST}} A \Rightarrow B$ means that the sequent $A \Rightarrow \neg B$ is K3-satisfied by v, that is, $v \Vdash_{\mathsf{K3}} A \Rightarrow \neg B$. More generally, using negation we can map the un-, super- and anti-satisfaction of a sequent into the satisfaction of another sequent by jumping from one logic to another as shown in diagram 2.

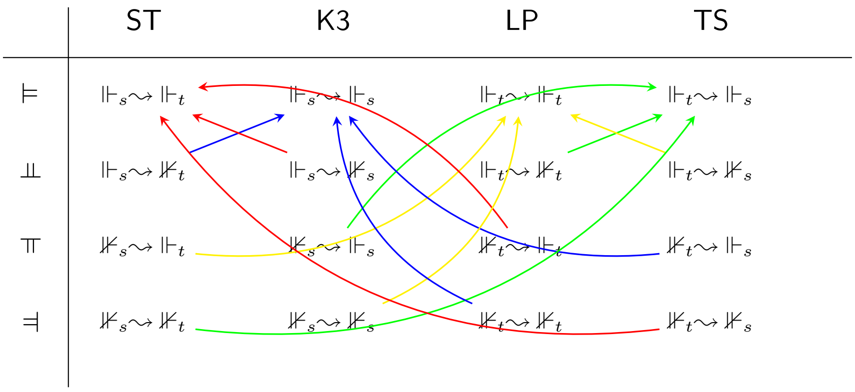


Figure 2: Inference properties in Strong Kleene logics

2.3 Duality about consequence

Duality about connectives usually refers to a certain kind of De Morgan relation between them. Thus, we say that '$\wedge$' and '$\vee$' are duals in a given logic when $\neg(A \wedge B)$ is equivalent to $\neg A \vee \neg B$. Strict and tolerant satisfaction are duals in very much this sense since $\nVdash_s A$ just in case $\Vdash_t \neg A$ (although this second duality involves two different negations: one in the metalanguage and one in the object language).

The duality in the case of a consequence relation does not fit this mould, as it is not generally the case that $\nvDash A \Rightarrow B$ just in case $\vDash \neg A \Rightarrow \neg B$. So when we say that classical logic is self-dual we must mean something different. Plausibly, duality about a consequence relation refers to a direct consequence of the underlying notion of satisfaction: the connection between validity and antivalidity.

We defined validity in terms of a (universally closed) metalinguistic conditional connecting satisfaction in the premises to satisfaction in the conclusions. Being it a conditional we can contrapose it: swap premises and conclusions and negate both. There are, however, two ways of negating premises and conclusions: one in the metalanguage and one in the object language. All these come to the same thing in classical logic, since classical satisfaction is self-dual (see figure 3).

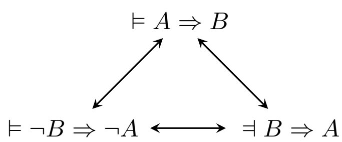


Figure 3: Classical duality

The connections in figure 3 motivate the adoption of the following terminology. We will say that two consequence relations $\vdash^*$ and $\vdash^\dagger$ are **operationally duals** when

$$\vDash^* A \Rightarrow B \quad \text{iff} \quad \vDash^\dagger \neg B \Rightarrow \neg A$$

and that they are **structurally duals** when

$$\vdash^* A \Rightarrow B \quad \text{iff} \quad {}^\dagger\!\dashv B \Rightarrow A.$$

Thus, according to this terminology, classical logic is both operationally and structurally self-dual. The situation is different in the case of Strong Kleene Logics, as the notions of satisfaction involved are not self-dual. We already know that LP and K3 are operationally duals. Now it can be seen, by inspecting the definition of anti-validity above, that LP is structurally self-dual. Figure 4 depicts the situation for LP about operational and structural duality.

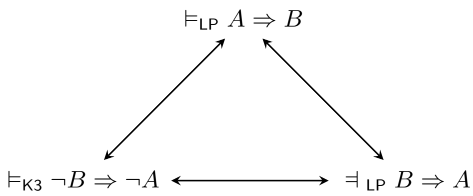


Figure 4: LP-K3 duality

The situation is somehow reversed for the logic ST. Again, by looking into the definitions of validity and anti-validity, it can be seen that ST is operationally self-dual but structurally it is the dual of TS, amounting to the picture in figure 5.

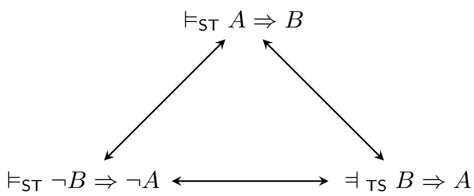


Figure 5: ST-TS duality

One of the arguments for ST lies in its alleged preservation of classical logic.[7] But the diagram in figure 2 seems to show that ST is K3-ish about unsatisfiability, LP-ish about supersatisfiability and TS-ish about antivalidity. Further, ST is self-dual only in the operational sense, but dual of TS in the structural sense. Do these non-classical features threaten in any way the alleged classicality of the logic ST?

At first, the answer to this question seems to be negative. The defender of ST might claim that she is committed just to validity and no other property of inferences. Whether other ST-properties of inferences (such as, say, ST-antivalidity) are classical or not, is none of her business. She might insist that her theory is committed to validity alone and this property behaves in the classical way.

In section 4 we will argue that this line of response is not available to the defender of ST. Although none among ST-antivalidity, ST-unsatisfiability and ST-supersatisfiability can be captured in terms of the ST-validity of an

[7] See, in particular, (Cobreros et al., 2013, 853)

inference, they can all be captured in terms of the ST-validity of a *metainference*. In order to show this, we introduce first a method to decide on metainferences.

3 Trees: inferences and metainferences

3.1 Trees for inferences

It is possible to adapt Smullyan's (1995) trees to decide the validity of inferences for all the logics described above.[8] The underlying idea is the following. In order for an inference to be xy-valid, there must not be an interpretation x-satisfying the premises and not y-satisfying the conclusions. Our trees provide a systematic search for a valuation of the relevant kind for any given inference (if there is no such valuation the trees will tell us).

Since there are two notions of satisfaction, our trees need to keep track of which is the relevant notion of satisfaction for which formulas. To this end, formulas in a tree go with a tag, 's' for 'strict' and 't' for 'tolerant'. Tree rules are exactly like the classical rules with the exception of the tag. For example:

$$\frac{A \supset B, x}{\neg A, x \qquad B, x} \qquad\qquad \frac{\neg(A \supset B), x}{\begin{array}{c} A, x \\ \neg B, x \end{array}}$$

The justification is the following. (For the rule on the left taking x to be s): $A \supset B$ is strictly true exactly when either A is strictly false or B strictly true. (For the rule on the right taking x to be s): $A \supset B$ is strictly false exactly when A is strictly true and B strictly false. A similar justification applies when we take x to be t: $A \supset B$ is tolerantly true exactly when either A is tolerantly false or B tolerantly true; $A \supset B$ is tolerantly false exactly when A is tolerantly true and B tolerantly false. Tree rules are summarised in figure 6.

A branch *closes* when it contains any of the following pairs of formulas '$A, s/\neg A, s$' or '$A, t(s)/\neg A, s(t)$', or it contains the formula '$\bot, t(s)$' but, crucially, the pair '$A, t/\neg A, t$' is not enough to close a branch (since both are tolerantly satisfied when A takes the middle value).

[8] See also Priest (2008) specially sections 8.4.8 and 8.4.11, although the trees there for LP and K3 are presented in a slightly different way. The exact kind of trees used here are described in Cobreros et al. (2012).

$$
\begin{array}{c|c|c|c|c|c}
A \wedge B, x & \neg(A \vee B), x & \neg(A \supset B), x & \neg\neg A, x & \neg\top, x & \neg\bot, x \\
\hline
A, x & \neg A, x & A, x & A, x & \bot, x & \top, x \\
B, x & \neg B, x & \neg B, x & & &
\end{array}
$$

$$
\begin{array}{cc|cc|cc}
\multicolumn{2}{c|}{\neg(A \wedge B), x} & \multicolumn{2}{c|}{A \vee B, x} & \multicolumn{2}{c}{A \supset B, x} \\
\hline
\neg A, x & \neg B, x & A, x & B, x & \neg A, x & B, x
\end{array}
$$

Figure 6: Tree rules

The difference between trees for either logic is just the tags in the initial list of formulas. Let $d()$ be a function swapping t's to s' and s' to t's. We say $\vdash_{xy} A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ iff there is a closed tree with initial list,[9]

$$
\begin{array}{c}
A_1, x \\
\vdots \\
A_n, x \\
\neg B_1, d(y) \\
\vdots \\
\neg B_m, d(y)
\end{array}
$$

In order to see the rationale for the initial list of formulas, recall that "B is not y-satisfied" is equivalent to "$\neg B$ is $d(y)$-satisfied".

Example 1. $\nvdash_{\mathsf{LP}} (A \supset B) \wedge (B \supset C) \Rightarrow A \supset C$

$$
\begin{array}{c}
(A \supset B) \wedge (B \supset C), t \\
\neg(A \supset C), s \\
A \supset B, t \\
B \supset C, t \\
A, s \\
\neg C, s \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
\begin{array}{c} \neg A, t \\ \times \end{array} &
\begin{array}{c} B, t \\ \swarrow \quad \searrow \\ \begin{array}{cc} \neg B, t & \begin{array}{c} C, t \\ \times \end{array} \end{array} \end{array}
\end{array}
\end{array}
$$

The tree shows that the conditional in LP is not transitive. Observe that

[9] Soundness and completeness proofs are adaptations of the corresponding proofs in Priest (2008).

we can rearrange tags to find the tree corresponding to a different logic. For example, if we change t's by s' and s' by t's in the tree above, we get the tree corresponding to K3 (which shows, by the way, that the conditional is transitive in K3).

Notice also that the trees for ST contain only s' in their initial list of formulas. Since tagged trees and classical trees are identical, except for the tags, if a branch closes in a classical tree (with a pair $A/\neg A$) it will close in the corresponding ST-tree (with a pair $A, s/\neg A, s$). This proves the claim above that any classically valid sequent is also valid in ST. Observe finally, that trees for TS contain only t's and therefore they never close (so no sequent is valid according to TS, as claimed before).

Notice that the method can be applied to decide xy-unsatisfiability, xy-supersatisfiability or xy-antivalidity of an inference by selecting the appropriate initial list of formulas (as shown un figure 7).

$\perp_{xy} A \Rightarrow B$ just in case there is a closed tree	$\top_{xy} A \Rightarrow B$ just in case there is a closed tree	$\dashv_{xy} A \Rightarrow B$ just in case there is a closed tree
A, x B, y	$\neg A, d(x)$ $\neg B, d(y)$	$\neg A, d(x)$ B, y

Figure 7: Unsatisfiability, supersatisfiability, antivalidity

3.2 Trees for metainferences

Given our propositional language $\mathcal{L}$, an inference is an expression of the form '$\Gamma \Rightarrow \Delta$' where $\Gamma, \Delta \subseteq \mathcal{L}$. Call $Inf(\mathcal{L})$ to the set of all inferences. A metainference is an expression of the form '$\Gamma \Rrightarrow \Delta$' where $\Gamma, \Delta \subseteq Inf(\mathcal{L})$. Thus, for example,

(i) $A \supset B, B \supset C \Rightarrow A \supset C$ is an inference, and

(ii) $A \Rightarrow B, B \Rightarrow C \Rrightarrow A \Rightarrow C$ is a metainference.

An inference can be considered a metainference without premises. Thus, for example, the inference '$A \Rightarrow A$' can also be read as the metainference '$\Rrightarrow$

$A \Rightarrow A$'. For readability we will often mark the empty side of a metainference with either '$\top$' (for an empty left-hand side) and '$\bot$' (for an empty right-hand side). Notice further that, '$\top$' can be read as an abbreviation of '$\bot \Rightarrow \top$' and '$\bot$' can be read as an abbreviation of '$\top \Rightarrow \bot$', so that we will write a metainference like '$\Rrightarrow A \Rightarrow A$' simply as '$\top \Rrightarrow A \Rightarrow A$' and '$A \Rightarrow A \Rrightarrow$' as '$A \Rightarrow A \Rrightarrow \bot$'.

A metainference $\Gamma \Rrightarrow \Delta$ is xy-satisfied by a valuation v, written $v \Vdash_{xy} \Gamma \Rrightarrow \Delta$, iff either $v \nVdash_{xy} \Gamma$ or $v \Vdash_{xy} \Delta$. A metainference is xy-valid, when it is xy-satisfied by every valuation.[10]

The conditional and the arrow sequent have different meanings in each of the four logics we are considering. For example, we know that the conditional in LP is not transitive, and so inference (i) is not LP-valid. However, the LP consequence relation is transitive, which means that the metainference (ii) above is LP-valid. This motivates extending trees to cover metainferences, and we add rules for the arrow sequent to that end (relative to each logic):

$$\frac{A \Rightarrow B, xy}{\neg A, d(x) \qquad B, y} \qquad\qquad \frac{A \Rightarrow B, \overline{xy}}{\begin{array}{c} A, x \\ \neg B, d(y) \end{array}}$$

The expression "$A \Rightarrow B, \overline{xy}$" means that the sequent $A \Rightarrow B$ is not xy-satisfied. The justification of the rules can be easily seen keeping in mind that t and s are duals. Thus, for example, the sequent '$A \Rightarrow B$' is LP-satisfied just in case either A is not tolerantly satisfied or B is tolerantly satisfied. But recall that 'A is not tolerantly satisfied' means the same as '$\neg A$ is strictly satisfied'.

We write $\vdash_{xy} A_1 \Rightarrow A_{1'}, \ldots\ A_n \Rightarrow A_{n'} \Rrightarrow B_1 \Rightarrow B_{1'}, \ldots\ B_m \Rightarrow B_{m'}$ iff there is a closed tree with initial list,

$$\begin{array}{c} A_1 \Rightarrow A_{1'}, xy \\ \vdots \end{array}$$

[10]These definitions come from Dicher and Paoli (2019) and are also used in Barrio et al. (2019). For a matter of uniformity we define metainferences with possibly multiple inferences as conclusions, although the examples we will consider below involve all a single inference in the conclusion side. The idea of validity for a metainference is also present in Barrio et al. (2015) although in a slightly different ("global-substitutional") sense. In the paper Cobreros et al. (2013) the validity of a metainference is used in a third different way ("simply global"). Which of these notions is the appropriate one, if any, is an interesting question beyond the scope of this paper. The notion of a metainference can be generalised to cover metainferences of any order, where a metainference of order n is an arrow with metainferences of order $n-1$ at each side. Barrio et al. (2019) use this generalisation to prove some intriguing results. See Scambler (2019) for a rejoinder.

$$A_n \Rightarrow A_{n'}, xy$$

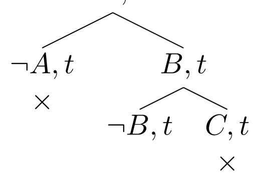
$$B_1 \Rightarrow B_{1'}, \overline{xy}$$

$$\vdots$$

$$B_m \Rightarrow B_{m'}\overline{xy}$$

Example 2. $\nvdash_{\mathsf{ST}} A \Rightarrow B, B \Rightarrow C \Rrightarrow A \Rightarrow C$

$$A \Rightarrow B, st$$
$$B \Rightarrow C, st$$
$$A \Rightarrow C, \overline{st}$$
$$A, s$$
$$\neg C, s$$



The tree shows that the metainference expressing the transitivity of the arrow is not valid in ST. Observe that, except for the initial list of formulas, the tree is identical to that of Example 1 showing the non-transitivity of the conditional in LP. Comparing trees for LP-inferences and trees for ST-metainferences, we can see that there's a correspondence between the two logics. A metainference is ST-valid just in case the result of "lowering" it to an inference (that is, substituting '$\Rightarrow$' by '$\supset$' and '$\Rrightarrow$' by '$\Rightarrow$') is LP-valid (this connection was noticed and proved in Barrio et al. (2015)). An analogous correspondence connects TS metainferences to K3 inferences (see Cobreros et al. (2019) for more on this).[11]

[11]In a companion paper in preparation we show how the trees can be "turned upside down" to obtain a sequent calculus that covers the four 3-valued logic under consideration as well the hierarchy of meta-inferential logics considered by Barrio et al. (2019). As it is implicit in the construction of the trees (in particular, from the fact that to test the xy-validity of a sequent we start a tree by simply taking the formulas in the antecedent labelled by x together with the negation of those in the antecedent marked by the dual of y), the corresponding sequent calculus is two-sided and labelled, where the labels are used to reflect syntactically the key features of the semantics. The calculus (like the trees presented here) therefore fulfils the elegance conditions introduced in Fjellstad (2017) (in particular, if one forgets the labels, the rules are just those for classical logic; and one has a straightforward connection between derivability of sequents and validity), and it stands to the st semantics as the dual-two-sided calculus of Fjellstad stands to his dual valuation semantics. The duality issues we investigate in the present paper could in principle be reformulated in terms of Fjellstad's dual valuation semantics and hence also in his sequent calculus, but they can be more straightforwardly formulated using the strict-tolerant semantic settings, which is the reason why we preferred introducing these

4 Expressing xy-antivalidity

As announced at the end of subsection 2.3, in this section we explain how xy-antivalidity can be expressed in terms of xy-validity. Consider again the trees for xy-validity and xy-antivalidity (figure 8),

$\vdash_{xy} A \Rightarrow B$	$\dashv_{xy} A \Rightarrow B$
just in case there is a closed tree	just in case there is a closed tree,
A, x $\neg B, d(y)$	$\neg A, d(x)$ B, y

Figure 8: Validity and antivalidity trees

Notice that in the case of ST (and similarly for TS), the ST-antivalidity of an inference is not given by the ST-validity of any other inference, since ST-validity trees contain only s' while ST-antivalidity trees only t's. There is a general way, however, to express the xy-antivalidity of an *inference* in terms of the xy-validity of a *metainference*.

Proposition. ${}_{xy}\dashv A \Rightarrow B \quad$ iff $\quad \vdash_{xy} A \Rightarrow \bot, \top \Rightarrow B \Rrightarrow \bot$.

Proof. Write the tree for the xy-validity of the metainference at the right-hand side of the biconditional,

$$A \Rightarrow \bot, xy$$
$$\top \Rightarrow B, xy$$
$$\neg\bot, xy$$
$$\neg A, d(x) \qquad \bot, y \; \times$$
$$\neg\top, d(x) \; \times \qquad B, y$$

and notice that the only open branch of this tree contains exactly the formulas in the initial tree for xy-antivalidity of the inference at the left-hand side (as shown in figure 8). □

Corollary.

trees rather than working with Fjellstad's calculus.

(i) $\bot_{xy} A \Rightarrow B$ iff $\vdash_{xy} \top \Rightarrow B \Rrightarrow A \Rightarrow \bot$ and
(ii) $\top_{xy} A \Rightarrow B$ iff $\vdash_{xy} A \Rightarrow \bot \Rrightarrow \top \Rightarrow B$.

By inspecting the corresponding trees, we can verify the equivalences in figures 4 and 5,

- $\vdash_{\mathsf{LP}} A \Rightarrow B$ iff ${}_{\mathsf{LP}}\dashv B \Rightarrow A$
- $\vdash_{\mathsf{ST}} A \Rightarrow B$ iff ${}_{\mathsf{TS}}\dashv B \Rightarrow A$

In words: an inference is LP-valid just in case the result of swapping premises and conclusions gives you an LP-antivalid inference. In contrast, an inference is ST-valid just in case the result of swapping premises and conclusions gives you a TS-antivalid inference. Or adopting the terminology in subsection 2.3, LP is structurally self-dual while ST and TS are structurally duals.

We end up with a final observation. If we consider the sequent $A \Rightarrow A$, the structural duality of ST and TS unpacks into,

$$\vDash_{\mathsf{ST}} \top \Rightarrow A,\ A \Rightarrow \bot \Rrightarrow \bot$$

iff

$$\vDash_{\mathsf{TS}} \top \Rrightarrow A \Rightarrow A$$

Which are metainferences representing instances of the rule of Cut and Identity, respectively. Given the close relations between ST and LP, on the one hand, and TS and K3, on the other, the failure of Cut can be read as a form of structural paraconsistency and the failure of Identity as a form of structural paracompleteness.

5 Final remarks

The goal of this paper was that of explaining in what sense the logics ST and TS are duals, in analogy with the already well-known duality between LP and K3. As in the case of LP and K3, the duality spelled out in this paper (if successful), should allow us to deepen the understanding of these logics and of the theories based on them. In this line, the discussion above supports at least two claims in the debate about ST.

In the first place, it supports the idea that the logic ST is more akin to other Strong Kleene logics than to classical logic and that it is, therefore, on the non-classical side of the scene. Particularly, the fact that we can record all (non-classically behaving) ST-like properties of inferences in terms of the ST-validity of a metainference shows that the supporter of a theory based on the logic ST is not in a position to avoid the commitment to those properties. In a similar way in which the supporter of LP cannot avoid the commitment to $\dashv_{\mathsf{LP}} A \Rightarrow A$, since she is already committed to $\vdash_{\mathsf{LP}} \top \Rightarrow A, A \Rightarrow \bot \Rrightarrow \bot$, the supporter of ST cannot avoid the commitment to $\not\dashv_{\mathsf{ST}} A \Rightarrow A$ since she's already committed to $\nvdash_{\mathsf{ST}} \top \Rightarrow A, A \Rightarrow \bot \Rrightarrow \bot$. We don't think this fact should be something terribly worrying for the supporters of ST, since Strong Kleene logics like LP and K3 have many interesting properties and have been very fruitful in their application to different phenomena. Arguments for ST based on it's alleged classicality, however, seem to loose their bite under the above considerations.

The second point concerns the status of a theory based on ST as a distinctive approach to paradoxes. The connections shown by Barrio et al. (2015) and Dicher and Paoli (2019) between ST and LP ground their claim that ST is nothing more than LP in sheep's clothing. Although a proper assessment of their arguments is outside the scope of this paper, the fact that we can represent *xy*-unsatisfiability, *xy*-supersatisfiability and *xy*-antivalidity of an inference in terms of the *xy*-validity of a metainference shows that all these logics are connected by different symmetries to one another. This fact can be used to argue that either all logics are different or all come to the same thing, no middle position seems to be tenable. We grant that a proper assessment of this last claim would require a more detailed explanation of the connections between the properties discussed above. We intend to examine these connections in future work.

In addition to these two points of discussion, the connection between ST and TS raises some general questions in the area of Philosophical Logic. First, whether it is possible to extract from any given non-transitive consequence relation a *dual* non-reflexive consequence relation. Second, whether the failure of transitivity can be generally understood as a form of paraconsistency and failure of reflexivity as a form of paracompleteness. Third, whether there are interesting logical theories that are structurally paraconsistent and structurally paracomplete beyond ST and TS.

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