Modified gravitational backgrounds: Horndeski's theory and non-geometrical backgrounds in string theory

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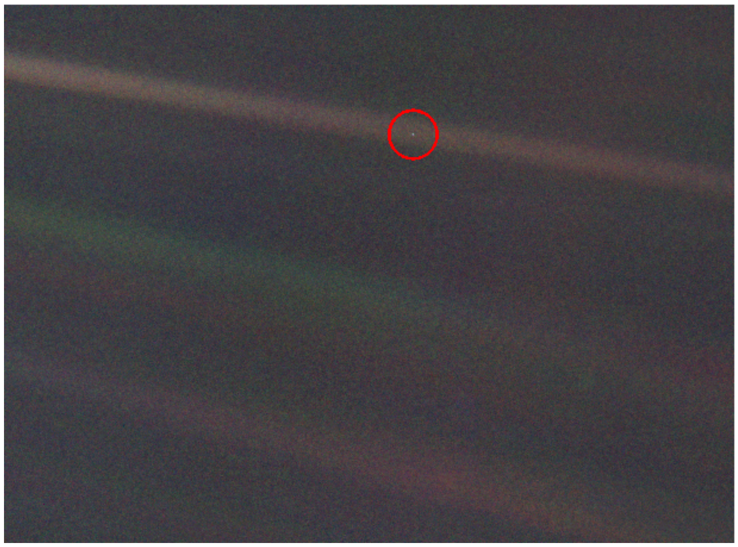
To my family and friends,
who bear with me in every step I take.



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"We succeeded in taking that picture [from deep space], and, if you look at it, you see a dot. That's here. That's home. That's us. On it, everyone you ever heard of, every human being who ever lived, lived out their lives. The aggregate of all our joys and sufferings, thousands of confident religions, ideologies and economic doctrines, every hunter and forager, every hero and coward, every creator and destroyer of civilizations, every king and peasant, every young couple in love, every hopeful child, every mother and father, every inventor and explorer, every teacher of morals, every corrupt politician, every superstar, every supreme leader, every saint and sinner in the history of our species, lived there on a mote of dust, suspended in a sunbeam."

– Carl Sagan
extract from speech at Cornell University
October 13, 1994

Contents

Zusammenfassung

Diese These befasst sich mit einem zweiteiligen Studium modifizierter Hintergründe in Theorien, die die Gravitation verallgemeinern. Im ersten Teil erkunden wir Horndeskis Lagrange-Dichte im Rahmen des Cartan Formalismus erster Ordnung, und wir studieren Gravitationswellen in Anbetracht einer nicht-verschwindenden Torsion. Im zweiten Teil gehen wir zur Stringtheorie über, und studieren T-Dualitäts-Transformationen zu einem konkreten nichtlinearen Sigma-Modell, das offene Strings enthält. Dies motiviert das Studium der Hintergründe, die sich aus solchen Transformationen ergeben.

Im ersten Teil analysieren wir Horndeskis Lagrange-Dichte in Cartan Formalismus erster Ordnung. Dieser Formalismus erlaubt der Torsion, ungleich Null zu sein, während sie in der allgemeinen Relativitätstheorie eine verschwindende Quantität ist. Horndeskis Lagrange-Dichte ist die allgemeinste Lagrange-Dichte in vier Dimensionen, die jede mögliche Wechselwirkung zwischen einem Skalarfeld ϕ und der Gravitation zulässt. Ihre Feldgleichungen sind bis zu zweiter Ordung partielle Differentialgleichungen. Diese Besonderheit jener Feldgleichungen verhindert die Existenz von Geistfeldern. Da diese Lagrange-Dichte bekannte modifizierte Theorien der Gravitation als Sonderfälle enthält, konzentrieren wir uns auf die Rolle der Torsion und ihrer Wirkung auf die linearen Störungen der Felder. Um unsere Untersuchung handhabbar zu machen, formulieren wir Horndeskis Lagrange-Diche in die Sprache der Differentialformen um. Dem Cartan Formalismus folgend nehmen wir an, dass der Spin-Zusammenhang ω^{ab} und das Vierbein e^a voneinander unabhängig sind. Wir nehmen die gesamte Horndeskis Lagrange-Dichte und leiten daraus die Feldgleichungen des Skalarfeldes, des Spin-Zusammenhanges und des Vierbeines ab. Um den torsionslosen Fall und die gewöhnliche allgemeine Relativitätstheorie zu erreichen, müssen wir eine Nebenbedingung durch Lagrange-Mutiplikatoren einführen.

Als Vorbereitung zur Analyse linearer Störungen unserer Felder, definieren wir mehrere Differentialoperatoren, um die Zeitraumstorsion unterscheiden zu können. Diese Operatoren können auf Lorentz-Indizes tragende $p-$Formen kovariant angewendet werden. Insbesondere beweisen wir eine Verallgemeinerung der Weitzenböck-Identität, die Torsion enthält.

Später erforschen wir Horndeskis Lagrange-Dichte unter linearen Störungen des Skalarfeldes, des Spin-Zusammenhanges und des Vierbeines. Wir entdecken, dass nicht-minimale Kopplungen und zweite Ableitungen des Skalarfeldes generische

Quellen der Torsion sind. Das steht in Kontrast zu dem, was bekannt in Einstein-Cartan-Sciama-Kibble-Schema ist, bei dem sich die Torsion ausschließlich auf Fermionen bezieht. Tatsächlich entdecken wir, dass die Hintergrundtorsion mit den sich ausbreitenden metrischen Freiheitsgraden koppelt. Das stellt eine Möglichkeit dar, Torsion durch Gravitationswellen zu falsifizieren.

Im zweiten Teil behandeln wir T-Dualitäts-Transformationen durch Buschers Verfahren zum offenen String im Rahmen der Stringtheorie. Solche Transformationen führen zu interessanten Geometrien, die eine wichtige Rolle in der Stringtheorie spielen, wie beispielsweise in der Modulistabilisierung oder beim Aufbau von Inflationspotentialen. Diese Geometrien werden als *nicht-geometrische Hintergründe* bezeichnet. In diesem zweiten Teil arbeiten wir technische Details aus, um Lücken in der Literatur zu schließen. Diese Details betreffen die Anwesenheit von D-Branen und die Wirkung von T-Dualitäts-Transformationen auf diese.

Zu Beginn studieren wir hierfür ein nichtlineares Sigma-Modell zum offenen String mit Feldern, die auf der Weltfläche Σ des offenen Strings und ihrer Grenze $\partial\Sigma$ definiert sind. Wir ziehen nicht-triviale Topologien für diese Weltfläche in Betracht und präsentieren die entsprechenden Grenzbedingungen für den offenen String. Buschers Verfahren zufolge nehmen wir gewisse Bedingungen für die Hintergrundskonfiguration an, und wir definieren zusätzliche Felder auf Σ und $\partial\Sigma$, um die Anwesenheit von D-Branen ebenfalls zu betrachten. Wir folgen Buschers Verfahren und führen T-Dualitäts-Transformationen durch, indem wir eine Symmetrie der Weltfläche eichen und Weltfläche-Eichfelder entlang der Isometrierichtungen der Zielraummetrik integrieren. Wir erreichen somit die duale Konfiguration für den offenen und geschlossenen Stringsektor. Insbesondere finden wir heraus, dass das duale Kalb-Ramond-Feld B einen verbleibenden Teil enthält, der eine wichtige Rolle spielt, wenn wir über die duale Konfiguration für den offenen String diskutieren.

Um unseren Formalismus zu illustrieren, betrachten wir die Standardkonfiguration des 3-Torus mit H-Fluss und führen eins, zwei und drei aufeinanderfolgende T-Dualitäts-Transformationen für verschiedene D-Branen Konfigurationen durch. Beim Nachlesen der dualen Hintergründe für den geschlossenen und den offenen Stringsektor finden wir die in der Literatur bekannten nicht-geometrischen Hintergründe. Allerdings merken wir an, dass solche Hintergründe Beiträge enthalten können, welche von dem dualen offenen Stringsektor herrühen. Im Bezug auf den dualen offenen Stringsektor studieren wir die Grenzbedingungen der offenen Strings der dualen Konfiguration. Dabei wurde entdeckt, dass sie mit bekannten Ergebnissen in CFT übereinstimmen. Zuletzt studieren wir die globale Wohldefiniertheit der D-Branen in solchen dualen Hintergründen und illustrieren die Anwendung der Freed-Witten-Anomalie-Auslöschungs-Bedingung für einige der hier präsentierten Beispiele.

Abstract

This thesis addresses a two-part study of modified backgrounds in theories that generalize gravity. In the first part of this work we explore Horndeski's theory within Cartan's first order formalism, and study gravitational waves considering torsion to be non-vanishing. In part two we move on to string theory and study T-duality transformations for a particular non-linear sigma model containing open strings. This prompts the study of the backgrounds arising from such transformations.

In part one we analyze Horndeski's Lagrangian in Cartan's first-order formalism. This formalism allows torsion to be non-zero, whereas in standard general relativity it is a vanishing quantity. Horndeski's Lagrangian is the most general Lagrangian in four dimensions featuring all possible interactions between a scalar field ϕ and gravity whose equations of motion are partial differential equations up to second order. This feature of such equations of motion prevents the existence of ghosts. Since this Lagrangian contains well-known modified theories of gravity as particular cases, we focus on the role of torsion and its impact at the linear perturbation regime. In order to make our analysis manageable, we cast Horndeski's Lagrangian in differential form language and we take the spin connection ω^{ab} and the vierbein e^a to be independent of each other, following Cartan's formalism. We take the full Horndeski Lagrangian and compute the equations of motion for the scalar field, the spin connection and the vierbein. We argue that in order to recover the torsionless case and make contact with standard General Relativity, we have to impose a constraint via Lagrange multipliers.

As a preparation for the analysis of the linear perturbation regime, we define several differential operators capable to discern spacetime torsion. These operators are capable to act covariantly on p-forms carrying Lorentz indices. In particular, we provide with a generalization of the Weitzenböck identity that includes torsion.

Later on, we consider linear perturbations around a generic background for the vielbein, spin connection and scalar field and study Horndeski's Lagrangian under such perturbations. What we find is that non-minimal couplings and second derivatives of the scalar field are generic sources of torsion. This makes a contrast to what was known from the Einstein-Cartan-Sciama-Kibble framework, where torsion can be sourced only from fermions. In fact, we find that background torsion couples with the propagating metric degrees of freedom. This provides with a potential way to falsify torsion via gravitational waves.

In part two we work inside the framework of string theory and we set to study T-duality transformations via Buscher's procedure for the open string. Such transformations lead to the study of interesting geometries which play an important role in string theory, as in moduli stabilization or in the construction of inflationary potentials. Such spaces are called *non-geometric backgrounds*. In this second part, we work out technical details which have been missing in the literature. These details regard the presence of D-branes and the effect of T-duality transformations on them.

We study a non-linear sigma model for the open string with fields defined on the worldsheet of such open string Σ and on its boundary $\partial\Sigma$. We take into account non-trivial topologies for this worldsheet and we present the appropriate boundary conditions for the open string. According to Buscher's procedure, we assume certain conditions for the background configuration and define additional fields on Σ and $\partial\Sigma$ taking into account the presence of D-branes. We follow Buscher's procedure and perform T-duality transformations by gauging a worldsheet symmetry and intregrating-out worldsheet gauge fields. We reach in this way the dual configuration for the open and closed string sector. In particular, we find that the dual Kalb-Ramond field B features a residual part which will play a major role when we discuss the dual configuration for the open string.

To illustrate our formalism, we consider the standard configuration of the three-torus with H-flux and perform one, two and three collective T-duality transformations for different D-brane configurations. We read off the dual backgrounds for the closed and open string sector. We find the standard non-geometric backgrounds found in the literature, noting that such backgrounds can receive contributions coming from the dual open string sector. Regarding the dual open string sector, we study the boundary conditions of the open strings in the dual configuration and we find that they comply with the usual results in CFT. We study the global well-definedness of these D-branes on such dual backgrounds and we illustrate the application of the Freed-Witten anomaly cancelation condition for some of the examples presented.

List of Figures

List of Tables

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General introduction

Motivation

During the 20th century, physics underwent a series of major changes. Our understanding of gravity was challenged in 1915 when Einstein published his series of papers on general relativity. What had started as an attempt to extend special relativity to incorporate gravity, ended up becoming a foundational framework which elegantly brought geometry and matter together. Not short after – around the 1920s Bohr, Schrödinger, Heisenberg, Pauli and many others laid the foundations of quantum mechanics. These basic pillars of modern physics have led generations of scientists to push further and lead to outstanding developments in physics: just in 2012 the discovery of the Higgs boson was announced, on February 2016 the first direct detection of gravitational waves was confirmed by the LIGO and VIRGO scientific collaboration and just in April 2019, the first direct photo of the shadow of a black hole was taken by the Event Horizon Telescope collaboration.

These breakthroughs in the history of physics were driven by our own curiosity. Our search for the fundamental laws of Nature has relied to a great extent on the construction of general and unifying frameworks.

The unification of seemingly unrelated phenomena has lead to discoveries in physics. A prime example of this is classical electromagnetism [1]: in the beginning, electricity and magnetism were considered to be independent phenomena. The research of Cavendish and Coulomb between 1771 and 1785 lead to the theory of electrostatics. Later on, the study of magnetism by Biot, Savart and Ampère suggested a connection between magnetic fields and electric currents. In 1831 Michael Faraday showed that changing magnetic fields generate electric currents. It wasn't until 1873 [2] when James Clerk-Maxwell provided a set of equations that addressed such phenomena as a whole. Such equations included an extra term fixing some remaining inconsistencies between the equations describing electricity and magnetism known at that time. This unification of electricity and magnetism led to the discovery of electromagnetic waves.

The speed of propagation of electromagnetic waves posed a problem within classical mechanics [2]. By the end of the 19th century, it was thought that light propagated through a medium called aether at such a speed, and the ad-hoc theory of aether was proposed. Thanks to the collective effort of many theoreticians and experimentalists like Hendrik Lorentz, Henri Poincaré, Albert Michelson and Edward Morley, Einstein published the theory of special relativity in 1905. The idea of aether was dismissed, and we no longer thought of space and time as separate entities: Such concepts were unified under the idea of *spacetime*. With the advent of General Relativity, spacetime became a dynamical entity and we could understand gravity not simply as a force, but as the curvature of spacetime itself.

Another example of this tendency is the Standard Model. This model describes three of the four fundamental forces of Nature (Electromagnetism, strong nuclear force and weak nuclear force) and all known elementary particles. This theory comprises electroweak theory together with quantum chromodynamics (QCD) and each of them were developed following unification as a central motivation. For instance, electroweak theory was established in the late 1960s by joining electromagnetism and the weak force as one Yang-Mills field whose gauge group is $SU(2) \times U(1)_Y$ [1]. QCD on the other hand binds quarks in atomic nuclei by means of the color charge, whose gauge group is $SU(3)$.

One crucial test of the Standard Model took place in the year 2012. CERN announced the discovery of the Higgs boson, thus explaining the mass for the gauge bosons. In physics, unification has allowed us to have a better understanding of the underlying mechanisms of Nature.

However, there are still caveats surrounding the Standard Model and general relativity. For instance, the existence of dark matter cannot be explained from the framework of SM. In cosmology, the accelerated expansion of the universe is thought to be driven by dark energy, whose origin is still unclear. In the standard cosmological model (the ΛCDM model) the cosmological constant Λ contained within Einstein's field equations accounts for the existence of dark energy.

These problems drive the development of new models and frameworks. When we consider the problem of dark energy, one approach to explain it involves the incorporation of scalar fields in gravity. The study of scalar fields in the gravitational theory was treated in a beginning by Pascual Jordan between in 1948 and 1959 and by Carl Brans and Robert Dicke in 1961 [3]. Since then, scalar fields have been used to explore the problem of inflation and dark energy. In 1974 Gregory Horndeski proposed a four-dimensional theory exhibiting all possible interactions between a scalar field and gravity. What makes this theory interesting is that the equations of motion are of second order, preventing the existence of ghosts.

Following unification as a guiding principle, string theory is a framework that encapsulates all interactions and all particles in the same scheme. It is a leading candidate for a theory of quantum gravity. Basically, the idea of the point particle is replaced by a one-dimensional extended string that can oscillate, and can be either open or closed. Since the fundamental object is now a string, it means that it can probe the underlying geometry in other ways. The study of the motion of strings on certain backgrounds has lead to the discovery of geometries that cannot be described in terms of Riemannian geometry. Such spaces are called non-geometric backgrounds.

This dissertation is concerned with the study of the two aforementioned frameworks. First, we will explore Horndeski's theory of gravitation, and later on, we will work within the framework of string theory and explore non-geometrical backgrounds from the open string point of view. In both cases we will explore modifications of the background: For the Horndeski part, we will allow spacetime torsion to be non-vanishing. This is a consequence of relaxing the geometrical assumptions of our manifold. On the other hand, the exploration of non-geometrical backgrounds involves the modification of the background on which a string propagates, via T-duality transformations.

Thesis' structure

In order to make exposition easier, this thesis is separated in two parts: Part one deals with Horndeski's theory of gravity within the first order formalism of gravity. Part two, on the other hand, works within the framework of string theory and studies non-geometric backgrounds from the open-string point of view. Each part contains its own introductory sections to the subject and its own conclusions. Furthermore, each part contains a chapter that serves as an introduction for the topic as well.

The final part of this doctoral work contains the final conclusion, an appendix and the corresponding bibliography.

Part I

Horndeski's theory with non-vanishing torsion

Chapter 1

Introduction of Part I

Einstein gravity and its extensions

More than 100 years ago Einstein published in a series of papers his theory of general relativity (GR). GR is still passing each observational test in the weak and strong field regime [4], proving to be the one of the most successful theories of physics. Five years ago, on February 11th, 2016 the LIGO and VIRGO collaboration announced the first direct observation of gravitational waves (GWs) from coalescing black holes [5] – a major prediction of GR – launching a new era of observational astronomy. In April 2019, the Event Horizon Telescope Collaboration announced through a series of six papers [6–11] the first image ever of a black hole and its surroundings located at the center of the galaxy Messier 87. The ring asymmetry, the "shadow"corresponding to its event horizon and brightness excess from the southern side were expected features predicted by Einstein's theory. The field equations of GR can be derived from the action S_{GR}

$$\begin{aligned} S_{\mathrm{GR}} &= S_{\mathrm{EH}} + S_{\mathrm{M}} \\ &= \frac{1}{2\kappa_4} \int \mathrm{d}^4x \sqrt{|\det g_{\mu\nu}|} \left[R - 2\Lambda\right] + S_{\mathrm{M}} \end{aligned} \tag{1.0.1}$$

where $g_{\mu\nu}$, $\mu, \nu = 0, \ldots, D-1$ is the metric of our D-dimensional spacetime, κ_4 is a constant given by $\kappa_4 = 8\pi G c^{-4}$, R corresponds to the Ricci scalar, Λ is the so-called cosmological constant and S_{M} is an action which accounts for the matter content. The action S_{EH} is the Einstein-Hilbert action. Einstein's celebrated field equations computed from this action are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.0.2}$$

where $R_{\mu\nu}$ is the Ricci tensor written in terms of the Christoffel symbols ${\Gamma^\lambda}_{\mu\nu}$ and $T_{\mu\nu}$ is the energy-momentum tensor. The Ricci tensor written in terms of the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ and the Christoffel symbols corresponding to the Levi-Civita connection are given as follows

$$\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}, \\ R_{\mu\nu} &= R^{\rho}{}_{\mu\rho\nu}, \\ \Gamma^{\lambda}{}_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right) \end{aligned} \tag{1.0.3}$$

Even though the accomplishments of GR cannot be overstated, it is not a complete theory of gravity – it is rather an effective theory. Indeed, when we explore high energies beyond the Planck mass $M_{\mathrm{Pl}}^2 = 1/8\pi G$ we find that we need to add higher curvature terms *ad infinitum* to S_{EH}. This is a signature that the theory is not-renormalizable and that it breaks down at the Planck mass scale. For a pedagogical note on this see for instance [12].

Another observation that signals that GR is an effective theory comes from the fact that the Universe is expanding at an accelerated rate. The observations of supernovae type Ia in 1998 suggested the existence of the so-called "dark energy" [13]. Further observations of the cosmic microwave background [14] and baryonic acoustic oscillations [15] confirm its existence. From a theoretical side, a natural choice would be to interpret dark energy as the gravitational constant Λ. However, we find that its predicted value is several orders of magnitude larger than its measured value [16, 17], even as high as 10^{120} times [18]. As of today, the origin of dark energy remains a mystery.

This gives us a motivation to test gravity from a theoretical side and generalize GR by modifying the action (1.0.1). These modifications might provide us with clues about the nature of gravity and dark energy on its own, even if they turn out to be not viable by experiments. This modifications, however, cannot be arbitrary: first, the modified theory must remain Lorentz-invariant and second, we want to make sure that the equations of motion are at most of second order in the metric $g_{\mu\nu}$[1]. The last requirement is needed in order to avoid the so-called Ostrogradsky instabilities [19]. Ostrogradsky's theorem [20] states that a physical system described by a non-degenerate Lagrangian dependent on time derivatives higher than one will present ghost-like instabilities. For a extended discussion on this, see for instance [21, 22].

Several approaches to generalize Einstein's gravity avoiding Ostrograsky instabilities have been proposed. One of them is a generalization of the Einstein-Hilbert action in higher dimensions, called Lovelock's gravity. Another possible approach would be to add extra degrees of freedom to the theory. The most simple of such additions would be a scalar field ϕ, and we could explore different couplings between ϕ with gravity. For non-minimal couplings between the scalar field and the curvature, we find a class of theories called *scalar-tensor theories* of gravity. Such theories suggest a different

[1]Notice that standard GR is a theory whose equations of motion are of second order in the derivatives on the metric, as we can see from (1.0.2) and (1.0.3)

origin of the gravitational force: Instead of having its origin from the curvature of spacetime alone, it originates in part from the value of the scalar field ϕ at each point in spacetime. An example of such theories is Brans-Dicke theory [23]. On the other hand, if we have a theory featuring minimal couplings between the scalar field with gravity we find theories such as quintessence [24, 25] and k-essence [26, 27], both of which have found applications in cosmology.

These modified theories of gravity can be thought of particular cases of the four dimensional Horndeski's theory[2] [32]. This is the most general, four-dimensional theory of gravity with non-minimal couplings between a scalar field ϕ and gravity whose equations of motion are up to second order in the derivatives. Such Lagrangian has had an extensive presence in the literature. Around the 2010s there was a surge of interest in Horndeski's theory when it was shown that the Horndeski's Lagrangian featured in the original paper [32] could be rephrased as a Lagrangian for generalized galileons [33–35]. Recently, the detection of the merger of two neutrons stars via GWs (GW170817) and electromagnetic radiation (GRB 170817A) has provided a unique opportunity to test the speed of propagation of gravitational waves [36–38], setting stringent constraints on modified theories of gravity. Having input from the observational side of GWs and the upcoming improvements and future detectors will provide decisive evidence about the nature of gravity.

The first-order formalism of gravity

Considering a manifold M equipped with a metric g we can define a sense of parallel transport of vectors along a curve with help of the connection, whose components in a coordinate basis are symbolized by $\Gamma^{\lambda}{}_{\mu\nu}$ [39]. Consider a pseudo-Riemannian manifold, which is the manifold for standard GR. Among the many connections that can be chosen, according to the Fundamental Theorem of Riemannian Geometry[3] there is a unique connection which is metric compatible and symmetric, which is the Levi-Civita connection as written in (1.0.3).

In standard GR, the concepts of metricity (given by the metric $g_{\mu\nu}$) and parallelism (encoded in the Christoffel symbols $\Gamma^{\lambda}{}_{\mu\nu}$ for the Levi-Civita connection) are not independent of one another. Indeed, according to the expression (1.0.3) we find that $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu}(g_{\rho\sigma})$. Let us consider now an an arbitrary connection $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$. By studying the commutator of covariant derivatives ∇_{μ} applied to a vector V^{ρ} we find [40]

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma} - T^{\lambda}{}_{\mu\nu}\,\nabla_{\lambda}V^{\rho}, \tag{1.0.4}$$

where[4] $T^{\lambda}{}_{\mu\nu} = 2\bar{\Gamma}^{\lambda}{}_{\underline{[\mu\nu]}}$ is called the *torsion* tensor [41] and $\nabla_{\mu}V^{\rho} = \partial_{\mu}V^{\rho} + \bar{\Gamma}^{\rho}{}_{\mu\sigma}V^{\sigma}$.

[2] Among quintessence, k-essence, Brans-Dicke and $f(R)$-gravity we find chameleon theory [28–30] and the convariant galileon as well [31].

[3] See Theorem 5.10 of [39]

[4] Our convention for the (anti-)symmetrization of indices contains a factor of $(1/n!)$. For better

Clearly, if we take the connection to be defined as in (1.0.3) the torsion vanishes. However, since we can freely choose a connection on our manifold, we can consider more general connections and in general the torsion might differ from zero. Such treatment can be studied within the formalism of *Cartan geometry*. The study of Einstein's gravity in four dimensions within the Cartan geometry framework is Einstein-Cartan-(Sciama-Kibble) (ECSK) gravity [41].

In the framework of Cartan geometry (see for instance [42]), the concepts of metricity and parallelism are encoded in the vielbein one-forms e^a and spin connection one-forms ω^{ab}, respectively. In here the indices $a, b = 0, \ldots, D-1$ belong to a frame such that they transform under local Lorentz transformations $\Lambda^a{}_b = \Lambda^a{}_b(p)$ for each point p of the manifold. This implies in particular that $\omega^{ab} = -\omega^{ba}$. Since e^a and ω^{ab} are one forms, they allow an expansion on the coordinate basis $\{\mathrm{d}X^\mu\}$, as follows

$$\begin{aligned} e^a &= e^a{}_\mu \, \mathrm{d}X^\mu, \\ \omega^{ab} &= \omega^{ab}{}_\mu \, \mathrm{d}X^\mu, \end{aligned} \tag{1.0.5}$$

where the components $e^a{}_\mu$ diagonalize the metric as $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu \, e^b{}_\nu$. Moreover, we can define the two-form torsion T^a and Lorentz curvature R^{ab} as follows

$$\begin{aligned} T^a &= \mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \\ R^{ab} &= \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}. \end{aligned} \tag{1.0.6}$$

Working in the first order formalism corresponds to consider metricity and parallelism to be independent. Together with the formalisms of Cartan geometry, we can have a clearer treatment and manipulation of the relevant quantities in differential geometry like the curvature, torsion, connection and the metric. This will be treated further in chapter .

1.1 Motivation

A non-vanishing torsion can lead to interesting phenomena. For instance, in [43] torsion has a dramatic effect on the very early Universe, generating repulsion from a finite radius and providing an alternative to cosmic inflation. In [44] the Randall-Sundrum model [45] was realized as a solution of a five-dimensional Chern-Simons gravitational theory with a torsion being zero along the usual four-dimensions but non-vanishing along the extra dimension. This torsion, along with the dilaton induced an expanding, accelerating universe.

In the ECSK theory, torsion is generated by fermions and affects only fermions. These effects are in general weak, since torsional terms are proportional to ψ^4 and become relevant when we have a large fermion density – see for instance section 8

distinction we highlight the (anti-)symmetrized indices by (under-) over-lining them.

of [46] and [47]. Furthermore, $U(1)$ gauge bosons do not generate and are not affected by torsion.

However, non-minimal couplings between the scalar field ϕ and the curvature as featured in Horndeski's theory might lead to a non-vanishing torsion, and there are reasons to justify this suspicion: In [48–51] couplings between the Euler four-form density and the scalar field $\phi\, \epsilon_{abcd}\, R^{ab} \wedge R^{cd}$ were studied. It was found that such term produced non-trivial dynamics with torsion.

In the first part of this doctoral work we will study Horndeski's theory within the first order formalism. We will consider the vielbein e^a, the spin connection ω^{ab} and the scalar field ϕ as independent degrees of freedom and study the corresponding equations of motion. Due to the inception of gravitational wave astronomy, we will study how torsion can couple to metric perturbations satisfying the wave equation. As stated above, the presence of the scalar field might support a non-vanishing torsion in this regime.

1.2 Outline of Part I

Including this introductory chapter, this first part consists of four chapters. With exception of chapter one, we outline their contents:

- In Chapter 2 we present a short review of modified theories of gravity. We mention first Lovelock's gravity and later we consider Horndeski's Lagrangian both in its original form and its equivalent description as a generalized Galileon theory. We present and discuss theories contained inside Horndeski's Lagrangian, namely modified gravity with minimally-coupled scalar field, Brans-Dicke theory and $f(R)$-gravity. Due to the recent breakthroughs in gravitational wave astronomy, we discuss as well the implications for Horndeski's theory and the constraints it has to meet.

- In Chapter 3 we study Horndeski's theory in the first order formalism within Cartan's formalism. We present a rewriting of the original Horndeski's Lagrangian in differential form language and study the equations of motion. Since we want to keep track of torsion across our treatment, we define useful torsion-aware differential operators. In the last section, we study Horndeski's Lagrangian at the linear perturbation regime on the vielbein, spin connection and scalar field and we explore how torsion couples with the metric perturbations.

- In Chapter 4 we present a summary of our findings and we close with conclusions.

Chapter 2

A primer on Horndeski's theory

In this chapter we will review and discuss Horndeski's theory and study some scalar-tensor theories that can be derived from it. We will explore them to get some insight about the motivations and consequences of such theories. Motivated by the recent experimental measurements of gravitational wave effects, we will review some constraints that arise from such observations and their impact on Horndeski's Lagrangian. This will serve as a good motivation to explore Horndeski's Lagrangian in the first order formalism, task which will be done in the next chapter.

This section is largely inspired from the reviews [3, 52, 53], with contributions taken from [54, 55]

On Lovelock gravity

One possible path to extend GR in the low energy regime would be to construct a generalization without adding extra degrees of freedom. Following Lovelock's theorem [56, 57], there is only one Lagrangian in $D = 4$ whose equations of motion are up to second order in its time derivatives that depends only on the metric $g_{\mu\nu}$, and it corresponds to the Einstein Hilbert action S_{EH} found in (1.0.1). In fact, Lovelock provides a generalization of Einstein's gravity depending only on the metric $g_{\mu\nu}$ for an arbitrary number of dimensions D, while ensuring that the EOMs will be up to second-order time derivatives in the equations of motion. Let $\mu, \nu = 0, 1, \ldots, D-1$ be spacetime indices. Considering the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ given by

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{2.0.1}$$

we can construct Lovelock's Lagrangian $\mathcal{L}_{\text{Lovelock}}$ as follows [56]

$$\mathcal{L}_{\text{Lovelock}} = \sum_{n=0}^{t} \alpha_n L^n, \tag{2.0.2}$$

where $t = (1/2)(D-2)$ for D even and $t = (1/2)(D-1)$ for D odd and α_n are arbitrary constants. L^n is defined as

$$L^n = \frac{1}{2^n}\delta^{\mu_1\nu_1\cdots\mu_n\nu_n}_{\alpha_1\beta_1\cdots\alpha_n\beta_n}\prod_{r=1}^{n} R^{\alpha_r\beta_r}{}_{\mu_r\nu_r}, \tag{2.0.3}$$

where $R^{\alpha\beta}{}_{\mu\nu} = g^{\beta\lambda}R^{\alpha}{}_{\lambda\mu\nu}$ and $\delta^{\mu_1\nu_1\cdots\mu_n\nu_n}_{\alpha_1\beta_1\cdots\alpha_n\beta_n}$ is the generalized Kronecker delta for n indices.

We find that the Einstein-Hilbert action is contained in Lovelock's Lagrangian for $D = 4$ $(t = 1)$. For $t = 2$ we find among the expansion of $\mathcal{L}_{\text{Lovelock}}$ the well-known Gauss-Bonnet term $\mathcal{R}^2$

$$\mathcal{R}^2 = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \tag{2.0.4}$$

which in $D = 4$ allows us to compute the Euler class $\chi(M_4)$ of our four dimensional manifold. Indeed, under the Chern-Gauss-Bonnet theorem[5] we find that

$$\chi(M_4) = \frac{1}{32\pi^2}\int_{M_4} \mathrm{d}^4x\, \mathcal{R}^2. \tag{2.0.5}$$

We mentioned that the Gauss-Bonnet term here presented appears at $t = 2$. Nothing stops us to take $\mathcal{R}^2$ and incorporate it directly into the Einstein Hilbert action S_{EH}. However, by computing the equations of motion with respect to $g_{\mu\nu}$ we find that in $D = 4$ this term does not make any contributions [58], reflecting its topological nature in four dimensions.

Lovelock's theorem and Lovelock gravity provides us with a theoretical framework to study possible generalizations of Einstein's gravity at the classical level. Its formulation guarantees us that no higher-order terms in the equations of motion for $g_{\mu\nu}$ will be present. In fact, any higher order derivative present in the variation of the Lovelock terms L^n conveniently end up as total divergent terms and thus do not contribute to the equations of motion [53]. If we want to consider other generalizations, then Lovelock's theorem states that we have to do one or more of the following [59]: (i) consider other fields rather than the metric tensor, (ii) allow higher-order derivatives in the equations of motion, (iii) consider dimensions other than four, (iv) accept non-locality, (v) give up on either rank $(2, 0)$ tensor field equations, symmetry of the field equations under exchange of indices or divergence-free field equations.

Our upcoming discussion on Horndeski's Lagrangian means that we will follow (i). We will consider the addition of an extra field into Einstein's formulation in such a way that still complies with the requirement of having second-order equations of motion for the metric field, thus making it a scalar-tensor theory. This will be the subject of our next section.

[5]For a textbook discussion on this, see for instance p.280 of [39].

2.1 Horndeski's theory

In the last section we considered a particular modification of standard GR, namely Lovelock gravity. In such setting we allowed the dimensionality of spacetime to be arbitrary and we did not introduce more fields into the picture; we just considered the metric itself. Now we will explore the case of a theory in $D = 4$ where we introduce an extra degree of freedom, a scalar field ϕ.

Scalar-tensor theories of gravity are non-fully geometrical, metric theories of gravity where the scalar field is non-minimally coupled to the curvature, i.e. we allow combinations of ϕ and its derivatives to multiply directly curvature terms in the full Lagrangian [3]. Of course, terms which depend solely on ϕ and its derivatives are allowed too, which is to say that we allow minimal couplings as well. This is fundamentally different as to only introduce the scalar field into the matter Lagrangian, since it would amount to introduce a certain matter field into the energy-momentum tensor $T_{\mu\nu}$ and work as a source for curvature in the field equations (1.0.2).

A scalar-tensor theory encompasses the introduction of non-minimal couplings between the curvature and a scalar field, leading to rethink the origin of gravitational phenomena: if we take for instance a coupling of the form $f(\phi)R$, with R the Ricci tensor and $f(\phi)$ a certain function of the scalar field, the effective gravitational coupling G will depend on $f(\phi)$ as $1/f$, changing the strength of the gravitational interactions at each point in spacetime [3]. More complicated couplings will lead to more involved modifications of the effective gravitational constant.

2.1.1 Horndeski's Lagrangian

Now we set to explore Horndeski's theory [32]. This corresponds to the most general scalar-tensor theory in four dimensions that leads to second-order field equations. Many of the well-known scalar-tensor theories of gravity which appear in the literature are contained inside Horndeski's theory.

Let $\alpha, \beta \ldots$ be spacetime indices such that $\alpha, \beta, \ldots = 0, 1, 2, 3$. Also, let us define X as

$$X \equiv -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi. \tag{2.1.1}$$

From the original work [32] we read Horndeski's Lagrangian as follows

$$
\begin{aligned}
\mathcal{L}_{\text{Horn}} = \; & \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left[\kappa_1 \nabla_\alpha \nabla^\mu \phi \, R_{\beta\gamma}{}^{\nu\sigma} + \frac{2}{3}\frac{\partial \kappa_1}{\partial X}\nabla_\alpha\nabla^\mu\phi\nabla_\beta\nabla^\nu\phi\nabla_\gamma\nabla^\sigma\phi \right. \\
& \left. + \kappa_3 \nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma} + 2\frac{\partial\kappa_3}{\partial X}\nabla_\alpha\phi\nabla^\mu\phi\nabla_\beta\nabla^\nu\phi\nabla_\gamma\nabla^\sigma\phi\right] \\
& + \delta^{\alpha\beta}_{\mu\nu}\left[(F+2W)R_{\alpha\beta}{}^{\mu\nu} + 2\frac{\partial F}{\partial X}\nabla_\alpha\nabla^\mu\phi\nabla_\beta\nabla^\nu\phi \right. \\
& \left. + 2\kappa_8 \nabla_\alpha\phi\nabla^\mu\phi\nabla_\beta\nabla^\nu\phi\right] \\
& - 6\left[\frac{\partial F}{\partial\phi} + 2\frac{\partial W}{\partial \phi} - X\kappa_8\right]\Box\phi + \kappa_9.
\end{aligned}
\tag{2.1.2}
$$

We recall that $\delta^{\alpha_1\alpha_2\cdots\alpha_n}_{\beta_1\beta_2\cdots\beta_n}$ is the generalized Kronecker delta. The quantities κ_i with $i = 1, 3, 8, 9$ are arbitrary functions of ϕ and X. We find $W = W(\phi)$ to be an arbitrary function of the scalar field and $F = F(\phi, X)$ to be a function of the scalar field and of X given in (2.1.1). This function F must satisfy the constraint

$$
\frac{\partial F}{\partial X} = 2\left[\kappa_3 + 2X\frac{\partial \kappa_3}{\partial X} - \frac{\partial \kappa_1}{\partial \phi}\right]. \tag{2.1.3}
$$

This is the form of Horndeski's Lagrangian which in next chapter will be formulated in differential form language and explored further in the first order formalism.

We would like to point out that we can think of Horndeski's theory rather as a *family* of theories. This echoes in a way the spirit of generality that binds this doctoral work together; by choosing the functions κ_i, F and W subject to the constraint (2.1.3) we are actually choosing in a way a *background* on which a test particle can move. In the next section, we will give some examples of theories contained inside Horndeski's Lagrangian.

As we mentioned above, the equations of motion that are derived from this Lagrangian are up to second-order in the derivatives. The construction that Horndeski provides in his original paper [32] can be sketched as follows [55, 53]: we consider first an action in four dimensions whose Lagrangian $\mathcal{L}$ depends on the metric $g_{\mu\nu}$, on the scalar field ϕ and on their respective derivatives up to order p and q for $p, q \geq 2$, i.e.

$$
\mathcal{L} = \mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \ldots, \partial^p g_{\mu\nu}; \phi, \partial\phi, \ldots \partial^q \phi). \tag{2.1.4}
$$

Let $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$ be the equations of motion for $g_{\mu\nu}$ and for ϕ, respectively. Due to diffeomorphism invariance, we find that the following Bianchi identity for $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$

holds[6]

$$\nabla^\mu \mathcal{E}_{\mu\nu} = -\nabla_\mu \phi \, \mathcal{E}_\phi. \tag{2.1.5}$$

In general, $\nabla^\mu \mathcal{E}_{\mu\nu}$ alone is an expression with third-order derivatives on $g_{\mu\nu}$ and ϕ. Let us assume for a moment that both $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_\phi$ contain at most second-order derivatives. The identity (2.1.5) implies then that $\nabla^\mu \mathcal{E}_{\mu\nu}$ is at most of second order in the derivatives due to the right hand side of (2.1.5). The rest of the proof goes on finding a suitable $\mathcal{E}_{\mu\nu}$ which satisfies (2.1.5). The final stages involve to find a suitable Lagrangian such that $\mathcal{E}_{\mu\nu} = 0$ and $\mathcal{E}_\phi = 0$ are the equations of motion.

For the rest of exposition in this section we will consider an equivalent formulation of Horndeski's Lagrangian (2.1.2). Let us take the Lagrangian $\mathcal{L}_{\text{Gal}}$ given by

$$\begin{aligned}
\mathcal{L}_{\text{Gal}} = & G_2 + G_3 \Box\phi + G_4 R + \frac{\partial G_4}{\partial X}\left[(\Box\phi)^2 - \nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi\right] \\
& + G_5 G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{3!}\frac{\partial G_5}{\partial X}\left[(\Box\phi)^3 - 3\Box\phi\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi \right. \\
& \left. + 2\nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi\right],
\end{aligned} \tag{2.1.6}$$

where $G_i = G_i(\phi, X)$ for $i = 2, 3, 4, 5$ and $G_{\mu\nu}$ is the usual Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. This formulation was achieved in [60,31], where the motivation was to generalize Galileon Lagrangians featured in [34]. The equivalence between the formulations (2.1.6) and (2.1.2) was proved in [35], where it was pointed out that the generalized galileon can be mapped to Horndeski's theory via the following identifications

$$\begin{aligned}
G_2 &= \kappa_9 + 4X \int^X \mathrm{d}X' \left[\frac{\partial \kappa_8}{\partial \phi} - 2\frac{\partial^2 \kappa_3}{\partial \phi^2}\right], \\
G_3 &= -6\frac{\partial F}{\partial \phi} + 2X\kappa_8 + 8X\frac{\partial \kappa_3}{\partial \phi} - 2\int^X \mathrm{d}X' \left[\kappa_8 - 2\frac{\partial \kappa_3}{\partial \phi}\right], \\
G_4 &= 2F - 4X\kappa_3, \\
G_5 &= -4\kappa_1.
\end{aligned} \tag{2.1.7}$$

Let us formulate Horndeski's action by including both $\mathcal{L}_{\text{Gal}}$ and a matter Lagrangian $\mathcal{L}_m$ as follows

$$S_{\text{Horn}} = \int \mathrm{d}^4x \sqrt{|g|}\, \mathcal{L}_{\text{Gal}} + \mathcal{L}_m. \tag{2.1.8}$$

[6] At the moment of construction of the Lagrangian (2.1.2), taking (2.1.5) and assuming $\nabla^\mu \mathcal{E}_{\mu\nu}$ to be up to second order in the derivatives imposes a series of constraints upon the form of the functions κ_i, which at the end prove to be not independent between each other. The expression (2.1.3) portrays in a compact form such constraints. For more details on this computation, see section 3 and equation (3.19) of [32]

We make $\kappa_4 = 8\pi G = 1$. By making appropriate choices for the functions G_i we can find some well-known examples of scalar-tensor theories found in the literature, as discussed in the next section.

2.1.2 Theories contained inside Horndeski's Lagrangian

In the previous section we presented an equivalent formulation of Horndeski's theory (2.1.2) in terms of the generalized galileon Lagrangian (2.1.7). From such formulation we will now discuss some theories contained within.

General relativity with a minimally coupled scalar field

By making the following choice for the functions G_i

$$G_2 = K(\phi, X), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}, \tag{2.1.9}$$

we find the action

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{2}R + K(\phi, X) + \mathcal{L}_m\right]. \tag{2.1.10}$$

In the particular case in which $K = -\Lambda = $ const. we recover standard GR with a cosmological constant, as seen in (1.0.1). For $K(\phi, X) = X - V(\phi)$ we find the *quintessence model*. For $K(\phi, X) = f(\phi)g(X)$ with f and g arbitrary we find the so-called *k-essence* models.

Quintessence models [24,61–64], involve the introduction of a scalar field ϕ minimally coupled to gravity, and is the simplest scalar-field scenario that addresses the problem of dark energy[7] [25].

These models can be classified in two classes [66]: in the first class of models, called *thawing models*, the scalar field ϕ is nearly frozen at the early cosmological epoch and it starts to evolve once the field mass drops below the Hubble expansion rate. In the second class of models, named *freezing models*, we find that the evolution of the field slows down at late cosmological times. The mass of quintessence, defined as

$$m_\phi^2 = \frac{d^2V(\phi)}{d\phi^2}, \tag{2.1.11}$$

has an upper bound of $|m_\phi| \lesssim H_0 \approx 10^{-33}$eV in order to account today's cosmological acceleration. Here, H_0 is the Hubble parameter [25].

Models of quintessence feature a so-called *tracker behavior*, which partly solves the cosmological constant problem [3]. Depending on the radiation and matter energy content during cosmological evolution, quintessence starts to resemble dark energy.

[7] For an introduction on this topic, see for instance [65]

On the other hand, k-essence models feature a scalar field ϕ also minimally coupled to gravity, but the dependence on the kinetic term X may be more involved. These models were first introduced to account for inflation [26]. Later, different versions of k-essence are used to model dark energy without introducing fine tuning and relying only on the dynamics of the fields – see for instance [67, 68]. In particular, in [69] a version of k-essence called uses purely the $K(X, \phi)$ function[8] in (2.1.10) as the full Lagrangian to account for a unified description of dark matter and dark energy.

This model also attempts to solve the *coincidence problem*; at the present time, the energy densities of dark matter and dark energy are of the same order, which under the assumptions of the standard cosmological model it suggests that the initial conditions in the early Universe were highly fine-tuned [70]. However, the viability of k-essence models to solve the coincidence problem was later questioned in [71].

Brans-Dicke theory

In this case we make the following choice

$$G_2 = \frac{1}{\phi}\omega_{\rm BD} X - V(\phi), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}\phi, \tag{2.1.12}$$

where $\omega_{\rm BD}$ is a constant. Plugging this into (2.1.7) we find the *Brans-Dicke* (BD) action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g|}\left[\phi R - \frac{1}{\phi}\omega_{\rm BD}\partial_\mu\phi\partial^\mu\phi - 2V(\phi) + \mathcal{L}_m\right]. \tag{2.1.13}$$

Brans-Dicke theory [23] is one of the simplest examples of a non-minimally-coupled modified theory of gravity. It should be noted that in the original work of Brans and Dicke in 1964 the parameter $\omega_{\rm BD}$ was not present [23]. In such paper, this theory is developed to incorporate Mach's principle, which (broadly speaking) states that inertial frames are determined by the large-scale structure of the universe[9].

BD theory has been used to in cosmology [76] and fluctuations of the gravitational constant in inflationary Brans-Dicke cosmology have been studied in [77]. Some forms of the $V(\phi)$ potential have been ruled out in light of the Planck 2015 data [78].

BD theory features predictions relevant to Cavendish-type experiments and phenomena at solar-system scales. As we mentioned above, the non-minimal coupling of ϕ to the curvature R modifies the gravitational constant. This can be computed by going at the weak field and low-velocity regime and compare with Newton's gravitational law. For the case in which $V(\phi) = 0$ we find that [3]

$$8\pi G_{\rm eff} = \frac{1}{\phi_0}\left[\frac{2\omega_{\rm BD} + 4}{2\omega_{\rm BD} + 3}\right], \tag{2.1.14}$$

[8] In the literature this is called *kinetic k-essence*.

[9] This idea can be traced back to Berkeley's *A Treatise Concerning the Principles of Human Knowledge* published in 1710 (see [72] for a modern introduction) to Mach's work *Die Mechanik in ihrer Entwicklung* published in 1883 (see [73] for a translated version to english). There has been several rephrasings of Mach's principle [74]. See [75] for a discussion of its many formulations.

where ϕ_0 is the value of ϕ at the weak field regime. The deflection of light around a massive object can be computed likewise [23,3], and it reads

$$\delta\theta = \frac{4G_{\text{eff}}M}{r_{\text{MA}}}\left[\frac{2\omega_{\text{BD}}+3}{2\omega_{\text{BD}}+4}\right]. \tag{2.1.15}$$

In here, M is the massive object and r_{MA} corresponds to the distance of closest approach to the object by the light ray. We can recover the usual standard GR predictions by taking the limit $\omega_{\text{BD}} \to \infty$. According to [79] however, this limit procedure is no longer valid if the trace of the energy-momentum tensor vanishes. The Cassini experiment results have led to the constraint $\omega_{\text{BD}} > 40000$ for BD theories with a scalar field ϕ having a negligible mass in relation to solar-system size scales [80].

$f(R)$-theory

By making the choice

$$G_2 = -\frac{1}{2}\big(\partial_R f\, R - f\big), \quad G_4 = \frac{1}{2}\partial_R f, \quad G_3 = G_5 = 0, \tag{2.1.16}$$

we get the action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g|}f(R). \tag{2.1.17}$$

$f(R)$-theory [81,82] corresponds to a modification of the Einstein-Hilbert action found in (1.0.1) without introducing a scalar field ϕ. This form of the Lagrangian allows to construct a higher-order gravity theory which does not present Ostrogradsky instabilities [83], making them excellent toy models.

A highlight of $f(R)$-theory is that it is able to produce acceleration of cosmic expansion with no need of invoking dark energy [84, 85]. Bouncing cosmologies scenarios have been addressed in [86].

One feature of $f(R)$ gravity is that it is *dynamically equivalent* to a Brans-Dicke theory for $\omega_{\text{BD}} = 0$ [3]. That is, by taking the modified gravity theory

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g|}\left[f(\psi) + \frac{\partial f}{\partial\psi}(R-\psi)\right], \tag{2.1.18}$$

and by looking at the equations of motion with respect to ψ we find that $\psi = R$. By replacing this result in (2.1.18) we find the usual $f(R)$-action. By defining a potential of the form $V(\phi) = \psi(\phi)\phi - f(\psi(\phi))$ we can write the action

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{|g|}\big[\phi\, R - V(\phi)\big], \tag{2.1.19}$$

which is the Brans-Dicke action (2.1.13) without the kinetic term. However, the case for $\omega_{\text{BD}} = 0$ has not been explored thoroughly since it has been ruled out by solar system experiments [3, 4]. On cosmological scales, $f(R)$ gravity has found as well evidence favoring the ΛCDM model instead [87].

2.2 Horndeski after LIGO

There has been a number of tests carried out to verify the feasibility of modified theories of gravity by looking at solar-sized experiments as deflection of light, Mercury's perihelion and strong equivalence principle – see for instance [4] for a thorough review. These observations have placed several constraints and accumulated evidence pointing out that certain models are to be ruled-out.

In August 2017, the LIGO and VIRGO collaborations along with the Fermi-Gamma Ray Burst monitor and the INTEGRAL collaborations observed gravitational waves (GW170817) and a gamma ray burst (GRB170817A) originating from a neutron star binary merger coming from the NGC 4993 galaxy [36,37]. For the first time, astronomers and astrophysicists had an opportunity to explore an extremely energetic event simultaneously from the electromagnetic and gravitational point of view. Both measurements set a bound on the speed of propagation of gravitational waves c_t. Under the normalization $c = 1$ it was found that [37]

$$-3 \times 10^{-15} < \, c_t - 1 \, < 7 \times 10^{-16}, \tag{2.2.1}$$

where c_t is the speed of propagation of gravitational waves. This result put stringent restrictions on the form of Horndeski's theory.

The $c_t = 1$ constraint

The bound (2.2.1) puts severe restrictions on Horndeski's Lagrangian. The way in which one can identify the speed of propagation of gravitational waves can be done by studying the behaviour of tensor perturbations $h_{\mu\nu}$ introduced around a background metric. This can be done by having first the set of solutions of the equations of motion of (2.1.6) for a specific background metric $\bar{g}_{\mu\nu}$ and for ϕ and by introducing a perturbation of the kind $\bar{g}_{\mu} \to \bar{g}_{\mu\nu} + h_{\mu\nu}$[10]. One studies the resulting Lagrangian at second order in the perturbations $h_{\mu\nu}$ and the equations of motion for such perturbations. One finds that the speed of propagation of the metric perturbations c_t satisfies [3]

$$c_t^2 = \frac{1}{q_t}\left[2G_4 - \dot{\phi}^2\frac{\partial G_5}{\partial \phi} - \dot{\phi}^2\ddot{\phi}\frac{\partial G_5}{\partial X}\right], \tag{2.2.2}$$

where q_t is given by

$$q_t = 2G_4 - 2\dot{\phi}^2\frac{\partial G_4}{\partial X} + \dot{\phi}^2\frac{\partial G_5}{\partial \phi} - H\dot{\phi}^3\frac{\partial G_5}{\partial X}. \tag{2.2.3}$$

The experimental bound (2.2.1) suggests the study of modified gravity theories for which $c_t^2 = 1$. By imposing this on (2.2.2) we find the condition

$$0 = \frac{\partial G_4}{\partial X} - 2\frac{\partial G_5}{\partial \phi} + \left[H\dot{\phi} - \ddot{\phi}\right]\frac{\partial G_5}{\partial X}. \tag{2.2.4}$$

[10] For a thorough review on the analysis of gravitational waves, see for instance [88]

By considering that each term of (2.2.4) vanishes, it implies that $G_4 = G_4(\phi)$ and that G_5 depends on neither X nor ϕ. Hence, the surviving Horndeski's Lagrangian is given by [89]

$$\mathcal{L}_{\text{Horn,sur}} = G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi)R. \tag{2.2.5}$$

The set of theories that survive the constraint are $f(R)$-gravity (2.1.17), Brans-Dicke (2.1.13), quintessence and k-essence (2.1.10). Other modified theories of gravity like chameleon models [28] and Kinetic Gravity Braiding models [90] remain viable as well [91]. The study of the consequences of the constraint (2.2.1) has still to be checked exhaustively: by taking $c_t = 1$ there is still the question of checking the status of compact objects in this modified theories of gravity, as pointed out in [92] and check the polarization of gravitational waves coming from surviving modified theories of gravity. We mention this in the next section.

Polarization content

Horndeski's Lagrangian in any of its forms (2.1.2) or (2.1.6) presents equations of motion for the metric $g_{\mu\nu}$ and the scalar field ϕ which both are up to second order in the derivatives. Following Ostrogradsky's result, we find that there are no ghosts that there are only three healthy degrees of freedom: two related to the polarization of $g_{\mu\nu}$ and one related to the scalar field, which can be massless or not.

The general result in standard GR is that gravitational waves propagate at the speed of light and possess two polarization states, the plus and cross modes – for a pedagogical review on gravitational waves, see for instance [88]. For generic modified theories of gravity, the total possible number of polarization states rises up to six [4]. For Horndeski theories in particular, it is expected that the extra degree of freedom coming from the scalar field will add an extra polarization state, thus having a total of three polarization states. Depending whether ϕ is massless or not, it excites either the so-called transverse breathing mode (case for $m_\phi = 0$) or the transverse-breathing and longitudinal modes (case for $m_\phi \neq 0$) [93]. Among the theories that survive the constraint $c_t = 1$, like Brans-Dicke theory and $f(R)$-theory, they feature gravitational waves with extra polarization states [94].

The study of additional polarization states in gravitational waves has been carried out even before GW170817: from the observation of binary pulsars, some evidence was found supporting GWs being described by standard tensor metric perturbations [95]. This was done in the context of specific beyond-GR theories [96,97]. From LIGO, using data collected between 2015 and 2016 a search for tensor, vector and scalar polarizations in the stochastic gravitational-wave background was done, finding no evidence for a background of any polarization [98]. The first measurement of polarization content done from a transient GW signal was GW170814 [93] and it was found that pure tensor polarizations are favored against pure vector and pure scalar polarizations [95, 93]. However, as pointed out in [98], LIGO and VIRGO still present limitations to discern polarization and future instrumentation like KAGRA [99] and LIGO-India would help to increase the precision of such measurements.

Chapter 3

Horndeski's Lagrangian with torsion

This chapter is based on our work [100]. Here we study how the non-minimal couplings between gravity and a scalar field produce torsion. To this end, we will consider the full Horndeski Lagrangian written conveniently in differential-form language. Since we want to work in first order formalism, we will consider the vierbein e^a and the spin connection ω^{ab} to be independent degrees of freedom.

We start exploring the first order formalism of the Horndeski's Lagrangian by doing some preliminary work: we will define some helpful operators that will be useful later on, together with expressing the full Horndeski's Lagrangian and studying the equations of motion derived from it.

3.1 First order formalism for Horndeski's theory

3.1.1 Preliminaries: contraction operators and vierbein

Let us consider a four-dimensional smooth manifold M equipped with a mostly-plus $(-,+,+,+)$ signature. Regarding the index structure, we will use throughout this chapter greek indices $\mu,\nu = 0,1,2,3$ to denote tensor components in the coordinate basis $\{\mathrm{d}X^\mu\}$, and lowercase Latin indices $a,b = 0,1,2,3$ for the orthonormal (Lorentz) basis $\{e^a\}$. Taking this into account, we are able to write down a change of basis matrix $e^a{}_\mu = e^a{}_\mu(X)$ such that

$$e^a = e^a{}_\mu \,\mathrm{d}X^\mu. \tag{3.1.1}$$

The vierbein components diagonalize the metric as $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu$ and thus allow us to write down the line element in a convenient way

$$ds^2 = g_{\mu\nu}\,\mathrm{d}X^\mu \otimes \,\mathrm{d}X^\nu = \eta_{ab}\, e^a \otimes \, e^b. \tag{3.1.2}$$

Please note that the orthonormal indices $a,b,\ldots$ are raised and lowered with the η^{ab} tensor and its inverse. Let us consider the space of $p-$forms defined on M by $\Omega^p(M)$

and define an operator $\mathrm{I}^{a_1\cdots a_q}$ that maps $p-$forms into $(p-q)-$forms, that is

$$\mathrm{I}^{a_1\cdots a_q} : \Omega^p(M) \to \Omega^{p-q}(M), \tag{3.1.3}$$

whose action on a $p-$form α is given by

$$\mathrm{I}^{a_1\cdots a_q}\alpha = (-1)^{p(p-q)+1} * \left(e^{a_1} \wedge \cdots \wedge e^{a_q} \wedge *\alpha\right), \tag{3.1.4}$$

with $*$ corresponding to the Hodge dual operator. Let α and β be $p-$ and $q-$forms respectively. We find that for the case $q = 1$ the operator I^a satisfies the following properties

$$\begin{aligned} &\text{Leibniz rule} \quad \mathrm{I}^a(\alpha \wedge \beta) = \mathrm{I}^a\, \alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{I}^a\, \beta, \\ &\text{Nilpotency} \qquad\quad \mathrm{I}^a \mathrm{I}_a = 0. \end{aligned} \tag{3.1.5}$$

The idea of working in the first order formalism is to consider the metric $g_{\mu\nu}$ and the connection $\Gamma^\lambda_{\mu\nu}$ to be independent quantities. In the Einstein-Cartan formalism it amounts to consider the vielbein e^a and the spin connection ω^{ab} to be independent from one another. More concretely for our case in $d = 4$, we will work with the set of vierbein $e^a = e^a{}_\mu\, \mathrm{d}X^\mu$ 1$-$forms, the spin connection $\omega^{ab} = \omega^{ab}{}_\mu\, \mathrm{d}X^\mu$ 2$-$form and a scalar field ϕ 0$-$form. From these fields, we can write down the torsion field T^a and the Lorentz curvature 2$-$forms as follows

$$\begin{aligned} T^a &= \mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \\ R^{ab} &= \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}. \end{aligned} \tag{3.1.6}$$

We will denote with a small circle above a certain quantity to be the *torsionless version* of that very same quantity. The usual Riemann differential geometry takes the torsion to be a vanishing, i.e. $T^a = 0$, landing in the usual differential quantities used in standard general relativity regarding the Levi-Civita connection and the Riemman curvature tensor[11]

$$\begin{aligned} \mathring{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \\ \mathring{R}^\rho{}_{\sigma\mu\nu} &= \partial_\mu \mathring{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \mathring{\Gamma}^\rho_{\mu\sigma} + \mathring{\Gamma}^\rho_{\mu\lambda}\mathring{\Gamma}^\lambda_{\nu\sigma} - \mathring{\Gamma}^\rho_{\nu\lambda}\mathring{\Gamma}^\lambda_{\mu\sigma}. \end{aligned} \tag{3.1.8}$$

We can separate systematically the torsional from the no-torsional part in the spin connection; it admits actually the splitting

$$\omega^{ab} = \mathring{\omega}^{ab} + \kappa^{ab}, \tag{3.1.9}$$

[11] This can be done by using the torsionless condition $T^a = 0$ and the so-called vielbein postulate (see p.158 of [46])

$$0 = \partial_\mu e^a{}_\nu + \omega^a{}_{b,\mu} e^b{}_\nu - \Gamma^\sigma_{\mu\nu} e^a{}_\sigma. \tag{3.1.7}$$

where $\mathring{\omega}^{ab}$ corresponds to the torsion-free 1−form spin connection derived from the vierbein and κ^{ab} is the 1−form *contorsion*. We notice that the torsion T^a 2−form can be neatly be written down with use of (3.1.9) as follows

$$\begin{aligned} T^a &= \mathrm{d}e^a + \omega^a{}_b \wedge e^b \\ &= \mathrm{d}e^a + \mathring{\omega}^a{}_b \wedge e^b + \kappa^a{}_b \wedge e^b \\ &= \qquad 0 \qquad + \kappa^a{}_b \wedge e^b. \end{aligned} \tag{3.1.10}$$

Let $\mathring{\mathrm{D}}$ be the covariant derivative with respect to the torsionless spin connection $\mathring{\omega}^{ab}$. With use of (3.1.9) we can express the Lorentz curvature 2−form as

$$R^{ab} = \mathring{R}^{ab} + \mathring{\mathrm{D}}\kappa^{ab} + \kappa^a{}_c \wedge \kappa^{cb}, \tag{3.1.11}$$

where $\mathring{R}^{ab}$ corresponds to the torsionless 2−form Riemann curvature given by

$$\mathring{R}^{ab} = \mathrm{d}\mathring{\omega}^{ab} + \mathring{\omega}^a{}_c \wedge \mathring{\omega}^{cb}. \tag{3.1.12}$$

We define some useful quantities regarding the scalar field ϕ and its derivatives in the first order formalism context. Let us first define the 0-form

$$Z^a = \mathrm{I}^a \mathrm{d}\phi. \tag{3.1.13}$$

along with the 1−forms

$$\begin{aligned} \pi^a &= \mathrm{D}Z^a, \\ \theta^a &= Z^a \mathrm{d}\phi. \end{aligned} \tag{3.1.14}$$

We can think of this quantities as follows: Z^a corresponds to the derivative of ϕ along the a−direction, while π^a and θ^a are equivalent to $\partial^2\phi$ and $(\partial\phi)^2$, respectively.

3.1.2 Horndeski's Lagrangian in differential form language

With help of the I^a operator, we will be able to work with Horndeski's Lagrangian in first order formalism in a manageable fashion. In terms of the variables we presented in the previous section, we present Horndeski's Lagrangian (2.1.2) written in differential form language as follows

$$
\begin{aligned}
\mathcal{L}_{\mathrm{Horn}}(e,\omega,\phi) = \epsilon_{abcd}\Big\{ & 2\kappa_1 R^{ab}\wedge e^c\wedge\pi^d + \frac{2}{3}\frac{\partial\kappa_1}{\partial X}\pi^a\wedge\pi^b\wedge\pi^c\wedge e^d \\
& +2\kappa_3 R^{ab}\wedge e^c\wedge\theta^d + 2\frac{\partial\kappa_3}{\partial X}\theta^a\wedge\pi^b\wedge\pi^c\wedge e^d \\
& +(F+2W)R^{ab}\wedge e^c\wedge e^d + \frac{\partial F}{\partial X}\pi^a\wedge\pi^b\wedge e^c\wedge e^d \\
& +\kappa_8\theta^a\wedge\pi^b\wedge e^c\wedge e^d - \left[\frac{\partial}{\partial\phi}(F+2W) - X\kappa_8\right]\pi^a\wedge e^b\wedge e^c\wedge e^d \\
& +\frac{1}{4!}\kappa_9 e^a\wedge e^b\wedge e^c\wedge e^d\Big\}.
\end{aligned}
\tag{3.1.15}
$$

From the definition in (2.1.2) we recall the functions $\kappa_i = \kappa_i(\phi, X)$ for $i = 1, 3, 8, 9$ and that $W = W(\phi)$, $F = F(\phi, X)$, which are subject to the constraint

$$
0 = \mathcal{C}(\phi, X) = \frac{\partial F}{\partial X} - 2\left[\kappa_3 + 2X\frac{\partial\kappa_3}{\partial X} - \frac{\partial\kappa_1}{\partial\phi}\right]. \tag{3.1.16}
$$

In the present context, we find that X defined already in (2.1.1) can be written in terms of Z^a given in (3.1.13) as

$$
X = -\frac{1}{2}Z_a Z^a. \tag{3.1.17}
$$

The Hodge operator only appears in the Horndeski action (3.1.15) only through the use of the operator I^a. It interesting to note that it allows us to cast the full Horndeski Lagrangian in a Lovelock-like fashion (see for instance [56, 101]), where π^a and θ^a play a role similar to the vierbein e^a.

In this work, we want to study how do the scalar couplings – craftily involved through the definitions for Z^a, π^a and θ^a – featured in Horndeski's Lagrangian (3.1.15) generate torsion. According to Horndeski's theorem [32], when torsion vanishes, (3.1.15) is the most general scalar-tensor Lagrangian that gives rise to second-order equations for the metric and Bianchi identities [53] – as long as the constraints (3.1.16) are satisfied. If we allow the torsion to be non-vanishing and to be explicitly introduced in (3.1.15), the theorem is no longer valid. The point being made here is that it might well happen that terms involving the torsion T^a explicitly might lead to second-order equations for the metric. We won't attempt to generalize Horndeski's theorem in this fashion, we will rather focus on how torsion may arise from the Lagrangian (3.1.15) when we consider the first order formalism. Having the first order formalism at hand, gives us the chance to have a glimpse of a more general construction where torsion might play a critical role.

3.1.3 Equations of motion

We present the equations of motion given the Lagrangian presented in the section above. Working within the first order formalism paradigm means that we need to find the equations of motion for e^a, ω^{ab} and ϕ. But we need to be careful: Z^a depends on e^a and the derivatives of ϕ through the I^a operator. This dependence must be taken into account when we perform the variations with respect to e^a and ϕ.

The full variation of the Lagrangian given in (3.1.15) can be written (modulo boundary terms) as $\delta\mathcal{L}_{\text{Horn}} = \delta\omega^{ab}\mathcal{E}_{ab} + \delta e^a\mathcal{E}_a + \delta\phi\mathcal{E}$. By setting $\delta\mathcal{L}_{\text{Horn}} = 0$ we will find the equations of motion $\mathcal{E}_{ab} = 0$, $\mathcal{E}_a = 0$, $\mathcal{E} = 0$ for ω^{ab}, e^a and ϕ respectively. In here, $\mathcal{E}_{ab}$ is given by

$$\mathcal{E}_{ab} = -\frac{1}{2}(Z_a\epsilon_{bcde} - Z_b\epsilon_{acde})I^{cde} + \epsilon_{abcd}e^c \wedge H^d - \epsilon_{abcd}T^c \wedge G^d. \tag{3.1.18}$$

where I^{cde}, H^d and G^d are given by

$$\begin{aligned} I^{cde} = &\Bigg[\kappa_1 R^{cd} + \pi^c \wedge \left(\frac{\partial\kappa_1}{\partial X}\pi^d + 2\frac{\partial\kappa_3}{\partial X} + \frac{\partial F}{\partial X}e^d\right) \\ &+\frac{1}{2}\left(\kappa_8\theta^c - \left[\frac{\partial}{\partial\phi}(F+2W) - X\kappa_8\right]e^c\right)\wedge e^d\Bigg] \end{aligned} \tag{3.1.19}$$

$$\begin{aligned} H^d = &\,\mathrm{d}\kappa_1 \wedge \pi^d + \kappa_1 R^d{}_e Z^e + \mathrm{d}\kappa_a \wedge \theta^d - \kappa_3 \mathrm{d}\phi \wedge \pi^d \\ &+\frac{1}{2}\mathrm{d}(F+2W)\wedge e^d \end{aligned} \tag{3.1.20}$$

$$G^d = \kappa_1\pi^d + \kappa_3\theta^d + (F+2W)e^d \tag{3.1.21}$$

On the other hand, $\mathcal{E}_a$ and $\mathcal{E}$ take the form

$$\begin{aligned} \mathcal{E}_a &= E_a + \mathrm{I}^b(\mathcal{S}_b + \mathcal{T}_b + \mathcal{U}_b)Z_a, \\ \mathcal{E} &= E + \mathcal{Z} - \mathrm{dI}^b(\mathcal{S}_b + \mathcal{T}_b + \mathcal{U}_b), \end{aligned} \tag{3.1.22}$$

where E_d, E and $\mathcal{Z}$ are given by

$$\begin{aligned} E_d = \epsilon_{abcd}\Bigg[&\, 2\kappa_1 R^{ab}\wedge\pi^c + \frac{2}{3}\frac{\partial\kappa_1}{\partial X}\pi^a\wedge\pi^b\wedge\pi^c \\ &+2\kappa_3 R^{ab}\wedge\theta^c + 2\frac{\partial\kappa_3}{\partial X}\theta^a\wedge\pi^b\wedge\pi^c \\ &+2(F+2W)R^{ab}\wedge e^c + 2\frac{\partial F}{\partial X}\pi^a\wedge\pi^b\wedge e^c \\ &+2\kappa_8\theta^a\wedge\pi^b\wedge e^c + \frac{1}{3!}\kappa_9 e^a\wedge e^b\wedge e^c \\ &-3\left[\frac{\partial}{\partial\phi}(F+2W) - X\kappa_8\right]\pi^a\wedge e^b\wedge e^c\Bigg], \end{aligned} \tag{3.1.23}$$

$$
\begin{aligned}
E = \epsilon_{abcd}\Bigg[& 2\left(\frac{\partial\kappa_1}{\partial\phi} - \kappa_3\right)R^{ab}\wedge e^c\wedge\pi^d + 2\left(\frac{1}{3}\frac{\partial^2\kappa_1}{\partial\phi\partial X} - \frac{\partial\kappa_3}{\partial X}\right)\pi^a\wedge\pi^b\wedge\pi^c\wedge e^d \\
& +2\frac{\partial\kappa_3}{\partial\phi}R^{ab}\wedge e^c\wedge\theta^d + 2\frac{\partial^2\kappa_3}{\partial\phi\partial X}\theta^a\wedge\pi^b\wedge\pi^c\wedge e^d \\
& +\frac{\partial}{\partial\phi}(F+2W)R^{ab}\wedge e^c\wedge e^d + \left(\frac{\partial^2 F}{\partial\phi\partial X} - \kappa_8\right)\pi^a\wedge\pi^b\wedge e^c\wedge e^d \\
& \frac{\partial\kappa_8}{\partial\phi}\theta^a\wedge\pi^b\wedge e^c\wedge e^d - \left[\frac{\partial^2}{\partial\phi^2}(F+2W) - X\frac{\partial\kappa_8}{\partial\phi}\right]\pi^a\wedge e^b\wedge e^c\wedge e^d \\
& +\frac{1}{4!}\frac{\partial\kappa_9}{\partial\phi}e^a\wedge e^b\wedge e^c\wedge e^d\Bigg],
\end{aligned}
\tag{3.1.24}
$$

$$
\begin{aligned}
\mathcal{Z} = \epsilon_{abcd}\Bigg[& 2\mathrm{d}\kappa_3\wedge R^{ab} + 2\mathrm{d}\frac{\partial\kappa_3}{\partial X}\wedge\pi^a\wedge\pi^b \\
& +\mathrm{d}\kappa_8\wedge\pi^a\wedge e^b + \mathrm{D}\pi^a\wedge\left(4\frac{\partial\kappa_3}{\partial X}\pi^b + \kappa_8 e^b\right)\Bigg]\wedge e^c Z^d \\
& +2\epsilon_{abcd}\left[\kappa_3 R^{ab} + \frac{\partial\kappa_3}{\partial X}\pi^a\wedge\pi^b + \kappa_8\pi^a\wedge e^b\right]\wedge T^c Z^d.
\end{aligned}
\tag{3.1.25}
$$

Finally we find that the 4−forms $\mathcal{S}_a$, $\mathcal{T}_a$ and U_a appearing in (3.1.22) are given by

$$
\begin{aligned}
\mathcal{S}_d = 2\epsilon_{abcd}\Bigg[& \mathrm{D}\pi^a\wedge e^b\wedge\left(2\frac{\partial\kappa_1}{\partial X}\pi^c + 2\frac{\partial\kappa_3}{\partial X}\theta^c + \frac{\partial F}{\partial X}e^c\right) \\
& +\pi^a\wedge e^b\wedge\mathrm{d}X\wedge\left(\frac{\partial^2\kappa_1}{\mathrm{d}X^2}\pi^c + 2\frac{\partial^2\kappa_3}{\partial X^2}\theta^c + \frac{\partial^2 F}{\partial X^2}e^c\right) \\
& +\frac{1}{2}e^a\wedge e^b\wedge\mathrm{d}X\wedge\left(\theta^c\frac{\partial\kappa_8}{\partial X} - e^c\frac{\partial}{\partial X}\left\{\frac{\partial F}{\partial\phi} - X\kappa_8\right\}\right)\Bigg],
\end{aligned}
\tag{3.1.26}
$$

$$
\begin{aligned}
\mathcal{T}_d = 2\epsilon_{abcd}\Bigg[& \kappa_1 R^{ab} + \frac{\partial\kappa_1}{\partial X}\pi^a\wedge\pi^b + 2\frac{\partial\kappa_3}{\partial X}\pi^a\wedge\theta^b \\
& +2\frac{\partial F}{\partial X}\pi^a\wedge e^b + \frac{1}{2}\kappa_8\, e^a\wedge\theta^b \\
& -\frac{3}{2}\left(\frac{\partial}{\partial\phi}(F+2W) - X\kappa_8\right)e^a\wedge e^b\Bigg]\wedge T^c,
\end{aligned}
\tag{3.1.27}
$$

$$
\begin{aligned}
\mathcal{U}_e = \epsilon_{abcd}\Bigg[& -R^{ab}\wedge e^c\wedge\left(C^d{}_e + 2\frac{\partial\kappa_1}{\partial X}\delta^{gd}_{ef}Z_g\pi^f\right) \\
& -\pi^a\wedge\pi^b\wedge e^c\wedge\left(\bar{C}^d{}_e + \frac{2}{3}\frac{\partial^2\kappa_1}{\partial X^2}\pi^d Z_e\right) \\
& +\pi^a\wedge e^b\wedge e^c\wedge M^d{}_e + e^a\wedge e^b\wedge e^c\wedge K^d{}_e\Bigg].
\end{aligned}
\tag{3.1.28}
$$

Lastly, we define the quantities $C^a{}_b$, $\bar{C}^a{}_b$, $K^a{}_b$ and $M^a{}_b$ found in (3.1.28)

$$
\begin{aligned}
C^a{}_b &= 2\mathrm{d}\phi\left[\frac{\partial\kappa_3}{\partial X}Z^a Z_b - \left(\kappa_3 - \frac{\partial\kappa_1}{\partial\phi}\delta^a_b\right)\right] \\
&+ e^a Z_b \frac{\partial F}{\partial X},
\end{aligned}
\tag{3.1.29}
$$

$$
\begin{aligned}
\bar{C}^a{}_b &= 2\mathrm{d}\phi\left[\frac{\partial^2\kappa_3}{\partial X^2}Z^a Z_b - \left(3\frac{\partial\kappa_3}{\partial X} - \frac{\partial^2\kappa_1}{\partial\phi\partial X}\delta^a_b\right)\right] \\
&+ e^a Z_b \frac{\partial^2 F}{\partial X^2},
\end{aligned}
\tag{3.1.30}
$$

$$
\begin{aligned}
K^a{}_b &= \left[\frac{\partial^2}{\partial\phi^2}(F + 2W) - X\frac{\partial\kappa_8}{\partial\phi}\right]\mathrm{d}\phi\delta^a_b \\
&- \frac{1}{4!}e^a Z_b \frac{\partial\kappa_9}{\partial X},
\end{aligned}
\tag{3.1.31}
$$

$$
\begin{aligned}
M^a{}_b &= \left[2\left(\kappa_8 - \frac{\partial^2 F}{\partial\phi\partial X}\right)\delta^a_b - \frac{\kappa_8}{\partial X}Z^a Z_b\right]\mathrm{d}\phi \\
&+ e^a Z_b \frac{\partial}{\partial X}\left[\frac{\partial F}{\partial\phi} - X\kappa_8\right].
\end{aligned}
\tag{3.1.32}
$$

As a last comment, we find that the 1−forms $C^a{}_b$ and $\bar{C}^a{}_b$ satisfy the properties

$$
\begin{aligned}
\mathrm{I}^b C^a{}_b &= Z^a\mathcal{C}, \\
\mathrm{I}^b \bar{C}^a{}_b &= Z^a\frac{\partial\mathcal{C}}{\partial X},
\end{aligned}
\tag{3.1.33}
$$

where $\mathcal{C}(\phi, X) = 0$ corresponds to the constraint (3.1.16). At first glance, the first order formalism approach to Horndeski's theory looks unmanageable, and trying to get meaningful conclusions from it looks like a highly-non trivial feat. This is why in the next section we will construct further differential operators which will make our analysis simpler and more effective. In any case, we can still draw some general remarks about the nature of torsion and the role of the ϕ field.

- We notice first that the definition of our operator I^a done in (3.1.3) and (3.1.4) allowed us to handle properly the Horndeski's Lagrangian (3.1.15) in the first order formalism, i.e. without imposing any restrictions on the torsion whatsoever.
- A quick look at the expressions (3.1.27) and (3.1.18) reveals how torsion appears naturally for Horndeski's theory in the first order formalism. This term arises generally for every non-minimal coupling with the scalar field, and from terms depending on $\pi^a = \mathrm{DI}^a\mathrm{d}\phi$.

- From eq. (3.1.27) we see that the dynamics of the torsion T^a and the scalar field ϕ are intertwined. In particular, we find that $T^a \sim \partial\phi$. This implies in turn that recovering the torsionless case proves to be a bit more tricky. Simply imposing $T^a = 0$ might freeze the dynamics of the scalar field, instead of recovering the usual torsionless equations of motion for ϕ.

The torsionless case

On regards of the last comment we did in the previous section, we need to think differently on the role of torsion and how it impacts the overall dynamics. Recovering the torsionless case in this formalism might be thought of a *reorganization* of the relations between the fields. This leads us to conclude that instead of simply imposing $T^a = 0$, we can recover consistently the torsionless case by imposing a constraint on the more general Cartan geometry framework[12]. This can be done easily with help of the I^a.

We introduce a light modification to the Lagrangian (3.1.15) by including the torsionless condition via a Lagrange multiplier Λ_a and integrating out it later. We consider the following Lagrangian

$$\mathcal{L}_{\text{Horn}} \to \bar{\mathcal{L}}_H = \mathcal{L}_{\text{Horn}} + \Lambda_a \wedge T^a, \tag{3.1.34}$$

whose equations of motion for e^a, ω^{ab} and ϕ are

$$\begin{aligned} \bar{\mathcal{E}} &= \mathcal{E} && = 0, \\ \bar{\mathcal{E}}_a &= \mathcal{E}_a - \mathrm{D}\Lambda_a && = 0, \\ \bar{\mathcal{E}}^{ab} &= \mathcal{E}^{ab} - \frac{1}{2}\delta^{ab}_{cd}\,\Lambda^c \wedge e^d && = 0, \\ T^a &= 0. \end{aligned} \tag{3.1.35}$$

We can integrate out this Lagrange multiplier first by solving $\bar{\mathcal{E}}^{ab} = 0$ for Λ^a and then by replacing it in the EOM $\bar{\mathcal{E}}_a = 0$. With help of the I_a and I_{ab} operator, we find that

$$\Lambda^a = 2\mathrm{D}\mathrm{I}_b\,\mathcal{E}^{ab} + \frac{1}{2}\,e^a \wedge \mathrm{I}_{bc}\,\mathcal{E}^{bc}, \tag{3.1.36}$$

and the standard field equations for the torsionless Horndeski's theory are recovered by setting

$$\begin{aligned} 0 &= \mathcal{E}^a - 2\mathrm{D}\mathrm{I}_b\,\mathcal{E}^{ab} + \frac{1}{2}e^a \wedge \mathrm{d}\mathrm{I}_{bc}\,\mathcal{E}^{bc}\Big|_{T^a=0}, \\ 0 &= \mathcal{E}\Big|_{T^a=0}. \end{aligned} \tag{3.1.37}$$

[12]This situation seems to have been acknowledged for a long time. See for instance [102] and section 1.7.1 of [103].

The first difference that strikes us is that, in contrast with the standard Einstein-Cartan scheme with minimally coupled fields the expression $T^a = 0$ is an equation of motion rather than a constraint. Introducing the a Lagrange multiplier in that case would be unnecessary. Moreover, in the Einstein-Cartan scheme, torsion is a non-propagating field where only fermionic fields can be a source of torsion [46].

Another aspect worth pointing out is how to make sense of the torsionless results known in standard Horndeski's theory with the first order formalism that we are dealing with here. Of course, known solutions in torsionless Horndeski's theory could be uplifted consistently into this framework and still be solutions, as it happens for instance in the refs. [104, 44]. Uplifting a torsionless Horndeskian solution to our first order formalism implies, in particular, that the Riemann curvature $\mathring{R}_{ab}$ in (3.1.12) and the scalar field ϕ must be the same in the torsionless Horndeski's theory. However, since the first order formalism reveals that torsion is sourced from scalar fields it might well happen that the Lorentz curvature won't be identical to the Riemann curvature $\mathring{R}^{ab}$. We are not always guaranteed that such uplift can always be done and it requires rather a case-by-case analysis.

We already have derived the equations of motion for the vierbein e^a, for the spin connection ω^{ab} and for the scalar field ϕ. To make the analysis more manageable, we ought to find better-suited wave operators when we approach the study of gravitational waves in our current scheme. These developments will be carried out in the next section.

3.2 Wave operators, torsion and Weitzenböck identities

As we mentioned earlier on, the objective is to study gravitational waves inside the first order formalism for Horndeski's Lagrangian (3.1.15). This means that we have to study small perturbations around a certain background configuration, which in turn implies that we ought to perturb the vierbein e^a and the spin connection ω^{ab}. These degrees of freedom are kept independent, and thus we require to construct torsion-aware wave operators capable to keep this independence untouched. Furthermore, these operators must be able to act on quantities carrying Lorentz indices, like the vierbein e^a for instance.

Case study I: scalar forms

We will consider first scalar $p-$forms, *scalar* meaning that these forms won't have any Lorentz indices. Let α be a $p-$form

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}\mathrm{d}X^{\mu_1}\wedge\ldots\wedge\mathrm{d}X^{\mu_p}. \tag{3.2.1}$$

Symbol	Definition	Change in form degree	Key property
d	$\mathrm{d}X^\mu\partial_\mu$	$+1$	$\mathrm{d}^2=0$
D	$\mathrm{d}+\omega$	$+1$	
$\mathring{\mathrm{D}}$	$\mathrm{d}+\mathring{\omega}$	$+1$	
$\mathrm{d}^\dagger$	$*\mathrm{d}*$	-1	
$\mathrm{D}^\dagger$	$*\mathrm{D}*$	-1	
$\mathring{\mathrm{D}}^\dagger$	$*\mathring{\mathrm{D}}*$	-1	
I^a	$-*(e^a\wedge *$	-1	$\mathrm{I}^a\mathrm{I}_a=0$
$\mathrm{D}^\ddagger$	$-\mathrm{I}^a\mathrm{D}\mathrm{I}_a$	-1	
$\mathring{\mathrm{D}}^\ddagger$	$-\mathrm{I}^a\mathring{\mathrm{D}}\mathrm{I}_a$	-1	
$\mathcal{D}_a$	$\mathrm{I}_a\mathrm{D}+\mathrm{D}\mathrm{I}_a$	0	
$\mathring{\mathcal{D}}_a$	$\mathrm{I}_a\mathring{\mathrm{D}}+\mathring{\mathrm{D}}\mathrm{I}_a$	0	$\mathring{\mathcal{D}}=e_a{}^\mu\mathring{\nabla}_\mu$
$\square_{\mathrm{dR}}$	$\mathrm{d}^\dagger\mathrm{d}+\mathrm{d}\mathrm{d}^\dagger$	0	
$\square_{\mathrm{B}}$	$-\mathring{\nabla}^\mu\mathring{\nabla}_\mu$	0	
$\blacksquare_{\mathrm{dR}}$	$\mathrm{D}\mathrm{D}^\ddagger+\mathrm{D}^\ddagger\mathrm{D}$	0	
$\blacksquare_{\mathrm{B}}$	$-\mathcal{D}^a\mathcal{D}_a$	0	

Table 3.1: Table containing a summary of the differential operators defined in this section and their most important properties

Inspired from Hodge theory, we consider one wave operator that could act on this form, the Laplace-de Rham operator $\square_{\mathrm{dR}}$ given by

$$\square_{\mathrm{dR}} = \mathrm{d}^\dagger\mathrm{d}+\mathrm{d}\mathrm{d}^\dagger. \tag{3.2.2}$$

In here, the exterior coderivative $\mathrm{d}^\dagger$is defined by $\mathrm{d}^\dagger = *\mathrm{d}*$[13]. If we apply this operator to our $p-$form α we find

$$\square_{\mathrm{dR}}\,\alpha = \square_{\mathrm{B}}\,\alpha + \mathrm{I}_a\left[\mathring{R}^a{}_b\wedge \mathrm{I}^b\alpha\right] \tag{3.2.3}$$

where $\square_{\mathrm{B}}$ corresponds to the Laplace-Beltrami operator defined as

$$\square_{\mathrm{B}} \equiv -\mathring{\nabla}^\mu\,\mathring{\nabla}_\mu \tag{3.2.4}$$

and $\mathring{\nabla}_\mu$ corresponds to the coordinate covariant derivative with the torsionless Christoffel symbols $\mathring{\Gamma}^\lambda_{\mu\nu}$. This writing of the Weitzenböck identity will prove to be useful later on.

The definition of the Laplace-de Rham operator is not aware of the presence of torsion; if it was, there should have been signs of contorsion or the full spin connection

[13]This definition holds for $d=4$ and for manifolds with Lorentzian signature. For other cases, a minus sign multiplying the definition (3.2.2) might be involved.

ω^{ab} on (3.2.2). The identity (3.2.3) confirms this since $\Box_{\mathrm{B}}$ works only with the torsionless Christoffel symbols, and we see that in the second term of the RHS of (3.2.3) the Riemann torsionless curvature $\mathring{R}^{ab}$ and the scalar form α are present.

We can see the utility of using the Laplace-de Rham operator with help of the Weitzenböck identity. Let us take for example the case for electromagnetism in a curved spacetime. Let us call A to the electromagnetic potential 1−form and F to its field intensity 2−form $F = \mathrm{d}A$. In vacuum –that is, when the current 1−form J is identically zero – we find that the non-homogeneous Maxwell equations take the form

$$0 = \mathrm{d}^{\dagger}\mathrm{d}A, \tag{3.2.5}$$

and by choosing the Lorenz gauge $\mathrm{d}^{\dagger}A \overset{!}{=} 0$ together with (3.2.3) we find

$$0 = \Box_{\mathrm{B}}\, A + \mathrm{I}_a \mathring{R}^a{}_b \mathrm{I}^b A, \tag{3.2.6}$$

which can be recasted in standard tensor language as

$$0 = -\mathring{\nabla}^{\lambda}\,\mathring{\nabla}_{\lambda} A_{\mu} + \mathring{R}_{\mu\nu} A^{\nu}. \tag{3.2.7}$$

Notice that $\mathring{R}_{\mu\nu}$ corresponds to the usual torsionless Ricci tensor. This small exercise shows us that U(1) gauge bosons for classical electromagnetism don't interact at all with torsion, even if the underlying spacetime geometry is such that $T^a \neq 0$[14]

Case study II: forms with indices

We explored the Laplace-de Rham operator $\Box_{\mathrm{dR}}$ and the Laplace-Beltrami $\Box_{\mathrm{B}}$ in the previous section with α as a test subject. We will explore wave differential operators for $p-$form with m Lorentz indices. Let us consider $\beta^{a_1\cdots a_m}$ be a $p-$ form with such indices, that is

$$\beta^{a_1\cdots a_m} = \frac{1}{p!}\beta^{a_1\cdots a_m}{}_{\mu_1\cdots\mu_p}\mathrm{d}X^{\mu_1}\wedge\cdots\wedge\mathrm{d}X^{\mu_p}. \tag{3.2.8}$$

Since this is a form that presents indices, we require to have a differential operator that takes them into account. This means that we require of a covariant derivative. A first approach would be to define a kind of de Rham Lorentz-covariant derivative as $\mathrm{D}^{\dagger} = *\mathrm{D}*$, completely analogous to $\mathrm{d}^{\dagger} = *\mathrm{d}*$. However, we will find that the following definition suits better

$$\mathrm{D}^{\ddagger} \equiv -\mathrm{I}^a \mathrm{D}\mathrm{I}_a. \tag{3.2.9}$$

[14] This is true when Yang-Mills bosons are described mathematically by connections on principal bundles, meaning in particular that the field strength reads $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{2}[A_\mu, A_\nu]$, not $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + \frac{1}{2}[A_\mu, A_\nu]$, which differs from the former when torsion is present. Both points of view have been studied in the literature. Some examples of the first can be found in [105–108]. Some examples of the second point of view, coupling YM bosons and torsion can be found in [109–111]. In here we will consider only the first approach.

Notice that for the torsionless case, $\mathring{\mathrm{D}}^{\ddagger}$ satisfies

$$\mathring{\mathrm{D}}^{\ddagger} = -\mathrm{I}^{a}\mathring{\mathrm{D}}\mathrm{I}_{a} = *\mathring{\mathrm{D}}* . \tag{3.2.10}$$

From this operator let us define the generalized Laplace-de Rham operator

$$\blacksquare_{\mathrm{dR}} \equiv \mathrm{D}\mathrm{D}^{\ddagger} + \mathrm{D}^{\ddagger}\mathrm{D}. \tag{3.2.11}$$

When we let $\blacksquare_{\mathrm{dR}}$ operate on a $p-$form with m Lorentz indices, it can be seen that this operator satisfies a generalized Weitzenböck identity, as follows

$$\begin{aligned} \blacksquare_{\mathrm{dR}}\, \beta^{a_1\cdots a_m} &= \blacksquare_{\mathrm{B}}\, \beta^{a_1\cdots a_m} + \mathrm{I}_c \mathrm{D}^2 \mathrm{I}^c \beta^{a_1\cdots a_m} \\ &= \blacksquare_{\mathrm{B}}\, \beta^{a_1\cdots a_m} + \mathrm{I}_c \big[R^{c}{}_{b} \mathrm{I}^b \beta^{a_1\cdots a_m} \\ &\quad + R^{a_1}{}_{b} \mathrm{I}^c \beta^{b a_2\cdots a_m} + \cdots + R^{a_m}{}_{b} \mathrm{I}^c \beta^{a_1\cdots a_{m-1} b} \big], \end{aligned} \tag{3.2.12}$$

where $\blacksquare_{\mathrm{B}}$ denotes the generalized Laplace-Beltrami operator

$$\blacksquare_{\mathrm{B}} \equiv -\mathcal{D}^a \mathcal{D}_a, \qquad \mathcal{D}_a \equiv \mathrm{I}_a \mathrm{D} + \mathrm{D}\mathrm{I}_a. \tag{3.2.13}$$

Some comments are in order. First, the Weitzenböck identity (3.2.12) regarding the operator $\blacksquare_{\mathrm{dR}}$ is capable to discern spacetime torsion since the Lorentz curvature R^{ab} containing the contorsion κ^{ab} is present. Second, the torsionless version of $\mathcal{D}_a$, namely $\mathring{\mathcal{D}}_a = \mathrm{I}_a\mathring{\mathrm{D}} + \mathring{\mathrm{D}}\mathrm{I}_a$ takes the simpler form $\mathring{\mathcal{D}}_a = e_a{}^{\mu}\mathring{\nabla}_{\mu}$. This means that $\mathring{\mathcal{D}}_a$ matches the usual torsionless covariant derivative $\mathring{\nabla} = \partial + \mathring{\Gamma}$ and we can recover the standard Weitzenböck identity (3.2.3) for multiple Lorentz indices

$$\begin{aligned} \mathring{\blacksquare}_{\mathrm{dR}}\, \beta^{a_1\cdots a_m} &= \Box_{\mathrm{B}}\, \beta^{a_1\cdots a_m} + \mathrm{I}_c \mathring{\mathrm{D}}^2 \mathrm{I}^c \beta^{a_1\cdots a_m} \\ &= \Box_{\mathrm{B}}\, \beta^{a_1\cdots a_m} + \mathrm{I}_c \big[\mathring{R}^{c}{}_{b} \mathrm{I}^b \beta^{a_1\cdots a_m} \\ &\quad + \mathring{R}^{a_1}{}_{b} \mathrm{I}^c \beta^{b a_2\cdots a_m} + \cdots + \mathring{R}^{a_m}{}_{b} \mathrm{I}^c \beta^{a_1\cdots a_{m-1} b} \big]. \end{aligned} \tag{3.2.14}$$

Let α and β be respectively $p-$ and $q-$forms. We find that the operator $\mathcal{D}_a$ satisfies the following properties

$$\mathcal{D}_a\,(\alpha \wedge \beta) = \mathcal{D}_a\,\alpha \wedge \beta + \alpha \wedge \mathcal{D}_a\,\beta, \tag{3.2.15}$$

$$[\mathrm{I}_a, \mathcal{D}_b] = -\mathrm{I}_{ab} T^c\, \mathrm{I}_c, \tag{3.2.16}$$

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b] &= \mathrm{D}^2\mathrm{I}_{ab} + \mathrm{I}_{ab}\mathrm{D}^2 + \mathrm{I}_a\mathrm{D}^2\mathrm{I}_b - \mathrm{I}_b\mathrm{D}^2\mathrm{I}_a \\ &\quad - (\mathrm{D}\mathrm{I}_{ab}T^c) \wedge \mathrm{I}_c - \mathrm{I}_{ab}T^c\mathcal{D}_c. \end{aligned} \tag{3.2.17}$$

From the previous discussion, it becomes clear that whenever we want gravitational waves to interact with the torsion sector of geometry, we require the field to have Lorentz indices; the differential operators defined in this section can discern the background spacetime torsion only when there are Lorentz indices to act on.

3.3 Gravitational waves and torsional modes

In this section we study torsional couplings to gravitational waves at first order in perturbation theory, without having an specific astrophysical process in mind. Second-order perturbations will be treated elsewhere.

3.3.1 Linear perturbations in first order formalism

Let us consider a background geometry configuration described by the fields $\bar{e}^a$, ω^{ab} and $\bar{\phi}$. We consider linear perturbations around this background following the prescription

$$\begin{aligned} \bar{e}^a \to \quad e^a &= \bar{e}^a + \frac{1}{2}h^a \quad |h^a| \ll 1, \\ \bar{\omega}^{ab} \to \omega^{ab} &= \bar{\omega}^{ab} + u^{ab} \ |u^{ab}| \ll 1, \\ \bar{\phi} \to \quad \phi &= \bar{\phi} + \varphi \qquad |\varphi| \ll 1. \end{aligned} \tag{3.3.1}$$

In here, we use the background vierbein $\bar{a}^a$ as a basis for one-forms written in the orthogonal frame. This implies that we can write h^a and u^{ab} as follows

$$\begin{aligned} h^a &= h^a{}_b\, \bar{e}^b, \\ u^{ab} &= u^{ab}{}_c\, \bar{e}^c. \end{aligned} \tag{3.3.2}$$

We find that the metric g with the above described prescription reads at first order in the perturbations as

$$\begin{aligned} g &= \eta_{ab}\, e^a \otimes e^b \\ &= \eta_{ab}\big[\bar{e}^a + h^a{}_c\, \bar{e}^c\big] \otimes \big[\bar{e}^b + h^b{}_d\, \bar{e}^d\big] \\ &= \eta_{ab}\, \bar{e}^a \otimes \bar{e}^b + \frac{1}{2}(h_{ab} + h_{ba})\, \bar{e}^a \otimes \bar{e}^b \\ &= (\bar{g}_{\mu\nu} + h^+_{\mu\nu})\, \mathrm{d}X^\mu \otimes \mathrm{d}X^\nu. \end{aligned} \tag{3.3.3}$$

In here we have made implicit use of the definition $h^\pm_{ab} = (1/2)(h_{ab} \pm h_{ba})$ which gives the symmetric and antisymmetric parts of h_{ab}, respectively. Notice that in the following analysis only the symmetric part of h_{ab} plays a role; it can be shown that h^-_{ab} amounts to nothing more than an infinitesimal local Lorentz transformation[15] Notice that the construction of the Lagrangian (3.1.15) is explicitly local Lorentz invariant,

[15] Under local Lorentz transformations realized by matrices $\Lambda^a{}_b$, the metric η_{ab} must remain invariant. This implies that $\Lambda^a{}_b = \delta^a_b + \lambda^a{}_b + O(\lambda^2)$, where $\lambda_{ab} = -\lambda_{ba}$. If the vierbein changes like $e^a \to e^a + \frac{1}{2}[h^-]^a{}_b e^b$, we identify $\frac{1}{2}[h^-]^a{}_b = \lambda^a{}_b$ and we find that such change amounts to an infinitesimal Lorentz transformation.

which means that the antisymmetric part of h_{ab} can always be gauged away. From now on, we will just assume that h_{ab} is a symmetric quantity.

We can do better and perform still a systematic separation of the metric-dependent part of the 1−form spin connection perturbation term u^{ab}. This goes as follows: we know that in standard general relativity the perturbation in the geometry is described in terms of the metric perturbations term $h_{\mu\nu}$ only. For us it is not surprising, since we know already the dependence of the connection on the metric itself. In the first order formalism, however, both the metric and the connection are independent degrees of freedom, which amounts to say the same for the vierbein e^a and the spin connection ω^{ab}. This implies in particular that the perturbations on the background vierbein and spin connection h^a and u^{ab} are independent of one another as well. We will see that it is always possible to split the linear perturbation 1−form u^{ab} in two pieces: one carrying all the dependence on h^a and one completely independent from it.

Let us consider the effects of the perturbation procedure (3.3.1) on the torsion $T^a = \mathrm{D}e^a$. We find that

$$\begin{aligned} \bar{T}^a \to T^a &= \bar{T}^a + \frac{1}{2}\bar{\mathrm{D}}h^a + u^a{}_b \wedge \bar{e}^b \\ &= \bar{T}^a + \frac{1}{2}\mathring{\bar{\mathrm{D}}}h^a + \frac{1}{2}\bar{\kappa}^a{}_b \wedge h^b + u^a{}_b \wedge \bar{e}^b. \end{aligned} \tag{3.3.4}$$

In the previous expression, $\mathring{\bar{\mathrm{D}}}$ denotes the exterior covariant derivative with respect to the torsionless piece of the background spin connection $\mathring{\bar{\omega}}^{ab}$.

Let us recall the writing of the torsion in terms of the contorsion 1−form κ^{ab} as $T^a = \kappa^a{}_b \wedge e^b$. Let us define as well a perturbation in the contorsion by q^{ab}, which works as

$$\bar{\kappa}^{ab} \to \kappa^{ab} = \bar{\kappa}^{ab} + q^{ab}. \tag{3.3.5}$$

We notice that the linear perturbations for the torsion read as

$$\bar{T}^a \to T^a = \bar{T}^a + \frac{1}{2}\kappa^a{}_b \wedge h^b + q^a{}_b \wedge \bar{e}^b. \tag{3.3.6}$$

If we take the relations (3.3.4) and (3.3.6) we avert an apparent contradiction. Notice that (3.3.4) possesses a derivative on h^a, whilst (3.3.6) does not. This annoyance can be cured precisely with help of the decomposition of the u^{ab} perturbation term we mentioned a while ago. Notice that u^{ab} must be of the form

$$u^{ab} = \mathring{u}^{ab} + q^{ab}, \tag{3.3.7}$$

We want still to have a sense of "torsionlessness"at the linear perturbation regime. By using the torsionless term in (3.3.7) we demand that – by taking the background vierbein $\bar{e}^a$ – the following torsionless relation holds

$$0 = \frac{1}{2}\mathring{\bar{\mathrm{D}}}h^a + \mathring{u}^a{}_b \wedge \bar{e}^b. \tag{3.3.8}$$

Indeed, if we take both (3.3.8) and (3.3.7) and put them to use in the expression (3.3.4) there is contradiction no more.

Further definitions

So far we have dealt with the basics for our perturbation scheme in the first-order formalism. This has been somewhat limited to study the metric and torsional perturbations. Before we study the perturbation regime for the Lagrangian (3.1.15), we need to carefully define some other useful quantities in order to keep our computations manageable. In particular, it can happen that the derivatives $\mathring{\bar{\mathrm{D}}}$ and $\bar{\mathrm{D}}$ might mix somewhere. For this, let us define the quantities

$$\begin{aligned}\mathcal{U}_{ab} &= \mathring{u}_{ab} - \frac{1}{2}\big[\bar{\mathrm{I}}_a(\bar{\kappa}_{bc}\wedge h^c) - \bar{\mathrm{I}}_b(\bar{\kappa}_{ac}\wedge h^c)\big],\\ \mathcal{V}_{ab} &= q_{ab} + \frac{1}{2}\big[\bar{\mathrm{I}}_a(\bar{\kappa}_{bc}\wedge h^c) - \bar{\mathrm{I}}_b(\bar{\kappa}_{ac}\wedge h^c)\big].\end{aligned} \tag{3.3.9}$$

where $\bar{\mathrm{I}}^a$ corresponds to the operator I^a using the background vierbein $\bar{e}^a$ and Hodge operator $\bar{*}$ related to the background metric structure, i.e. $\bar{\mathrm{I}}^a = -\bar{*}(\bar{e}^a \wedge \bar{*}$. We notice that $\mathcal{U}^{ab}$ and $\mathcal{V}$ satisfy

$$\begin{aligned}u^{ab} &= \mathring{u}^{ab} + q^{ab}\\ &= \mathcal{U}^{ab} + \mathcal{V}^{ab}.\end{aligned} \tag{3.3.10}$$

It can be proven that torsion T^a and contorsion κ^{ab} are related by

$$\kappa_{ab} = \frac{1}{2}\big(\mathrm{I}_a T_b - \mathrm{I}_b T_a + e^c\,\mathrm{I}_{ab} T_c\big), \tag{3.3.11}$$

and show that the torsionless condition (3.3.8) mentioned a bit ago can be written as

$$0 = \frac{1}{2}\bar{\mathrm{D}}h^a + \mathcal{U}^a{}_b \wedge \bar{e}^b + \frac{1}{2}\bar{\mathrm{I}}^a(h_b \wedge \bar{T}^b). \tag{3.3.12}$$

Having these results, we can prove that the linear perturbations for the torsion indicated in (3.3.6) can be written as

$$\bar{T}^a \to T^a = \bar{T}^a + \mathcal{V}^a{}_b \wedge \bar{e}^b - \frac{1}{2}\bar{\mathrm{I}}^a(h_b \wedge \bar{T}^b). \tag{3.3.13}$$

We can get a more compact expression for the linear perturbations in the Lorentz curvature R^{ab}. We see first that by using (3.3.12) we can cast U^{ab} as follows

$$\mathcal{U}^{ab} = -\frac{1}{2}\big(\bar{\mathrm{I}}^a\bar{\mathrm{D}}h^b - \bar{\mathrm{I}}^b\bar{\mathrm{D}}h^a\big), \tag{3.3.14}$$

and show that the linear perturbation of the Lorentz curvature can be simply written as

$$\bar{R}^{ab} \to R^{ab} = \bar{R}^{ab} + \bar{\mathrm{D}}\big(\mathcal{U}^{ab} + \mathcal{V}^{ab}\big). \tag{3.3.15}$$

Finally, one can show that the perturbations at first order for $Z^a = \mathrm{I}^a \mathrm{d}\phi$ can be written down as

$$\bar{Z}^a \to Z^a = \bar{Z}^a + \bar{\mathrm{I}}^a \mathrm{d}\varphi - \frac{1}{2} h^a{}_b \bar{Z}^b. \tag{3.3.16}$$

We finish this subsection with two comments: first, we were able to make a careful splitting of the spin connection perturbation u^{ab} in two parts, namely $\mathcal{U}^{ab}$ and $\mathcal{V}^{ab}$. We found that $\mathcal{U}^{ab}$ depends only on the vierbein perturbation, as featured in (3.3.14) and that linear perturbations on the torsion depend on both $\mathcal{V}^{ab}$ and h^a. Second, we derived that linear perturbations on the Lorentz curvature depend on a total covariant derivative of $\mathcal{U}^{ab} + \mathcal{V}^{ab}$.

In the following section we start performing the analysis of gravitational waves at first order for Horndeski's Lagrangian in the first-order formalism.

3.3.2 Gravitational waves in the first-order formalism

In the last section we saw that torsion is able to propagate through the modes $\mathcal{V}^{ab}$ we defined in (3.3.9) together with a coupling term between the metric perturbations h^a and the background torsion $\bar{T}^a$. One might think that propagations of torsion are only tied to terms in the Lagrangian (3.1.15) other than the standard Einstein-Hilbert (EH) term $\epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d$. An initial guess would say that performing perturbations around a background configuration described by (3.3.1), only the wave equation and interactions of h^a arise from the EH, just as in the torsionless standard case. This is no longer the case here: we will see that the EH term gives rise to both metric modes – which by the way may interact with the background torsion $\bar{T}^a$ – and propagating torsional modes.

The study of gravitational waves and propagation of torsion in the first order formalism for Horndeski's theory proves to be challenging and by no means a trivial task. Let us start then by studying first the Einstein-Hilbert term apart from the other terms in (3.1.15): consider a Lagrangian in the Horndeski family of the form

$$\mathcal{L}^{(4)}_{\mathrm{EH}}(e,\omega) = \mathcal{L}^{(4)}_{\mathrm{EH}} + (\text{other terms}). \tag{3.3.17}$$

In here, $\mathcal{L}^{(4)}_{\mathrm{EH}}$ corresponds to the usual Einstein-Hilbert four form

$$\mathcal{L}^{(4)}_{\mathrm{EH}}(e,\omega) = \frac{1}{4\kappa_4} \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d. \tag{3.3.18}$$

and the (other terms) correspond to quantities giving rise to torsion through non-minimal couplings and/or second-order derivatives of ϕ.

The field equations derived from (3.3.17) have the schematic form

$$\begin{aligned} \delta_e\mathcal{L}^{(4)} &= \delta_e\mathcal{L}^{(4)}_{\mathrm{EH}} + \delta_e(\text{other terms}) = 0, \\ \delta_\omega\mathcal{L}^{(4)} &= \delta_\omega\mathcal{L}^{(4)}_{\mathrm{EH}} + \delta_\omega(\text{other terms}) = 0, \\ \delta_e\mathcal{L}^{(4)} &= \qquad\qquad \delta_\phi(\text{other terms}) = 0, \end{aligned} \tag{3.3.19}$$

where we find that $\delta_e\mathcal{L}^{(4)}_{\mathrm{EH}}$ and $\delta_w\mathcal{L}^{(4)}_{\mathrm{EH}}$ are given by

$$\begin{aligned} \delta_e\mathcal{L}^{(4)}_{\mathrm{EH}} &= \frac{1}{2\kappa_4}\epsilon_{abcd}R^{ab}\wedge e^c\wedge\delta e^d, \\ \delta_\omega\mathcal{L}^{(4)}_{\mathrm{EH}} &= \frac{1}{2\kappa_4}\epsilon_{abcd}\delta\omega^{ab}\wedge T^c\wedge\delta e^d, \end{aligned} \tag{3.3.20}$$

The $\mathcal{G}$ term

Here we study the equations of motion for the linear perturbation fields. Let us then consider $\bar{e}^a$, $\bar{\omega}^{ab}$ and $\bar{\phi}$ to be a background configuration satisfying the field equations (3.3.19), and at the same time consider linear perturbations around that very same configuration as stated in (3.3.1). The wave behavior is found to be at the equation of motion for the vierbein as stated in (3.3.19). Taking this into account, we find that the following relation holds

$$\begin{aligned} 0 = \mathcal{G} + \frac{1}{4\kappa_4}\bar{R}^{ab}\wedge h^c\wedge\delta e^d + \frac{1}{2\kappa_4}\epsilon_{abcd}\bar{\mathrm{D}}\mathcal{V}^{ab}\wedge\bar{e}^c\wedge\delta e^d \\ + (\text{linear perturbations of other terms}). \end{aligned} \tag{3.3.21}$$

In here we have defined the 4−form $\mathcal{G}$

$$\mathcal{G} = \frac{1}{2\kappa_4}\epsilon_{abcd}\bar{\mathrm{D}}\,\mathcal{U}^{ab}\wedge\bar{e}^c\wedge\delta e^d. \tag{3.3.22}$$

We will see shortly that the term (3.3.22) generates gravitational waves coupled with torsion, process described by means of the generalized wave operator $\blacksquare_{\mathrm{dR}} = \mathrm{D}^\ddagger\mathrm{D}+\mathrm{D}\mathrm{D}^\ddagger$. By using the expression (3.3.14) in (3.3.22) we find the following rewriting of the $\mathcal{G}$-term

$$\begin{aligned} \mathcal{G} = -\frac{1}{4\kappa_4}\bar{*}\Big\{ & \bar{\mathcal{D}}^a\bar{\mathcal{D}}_a h_d - \bar{\mathcal{D}}_a\bar{\mathcal{D}}_d h^a \\ & -\bar{e}^c\bar{\mathcal{D}}_c\big(\bar{\mathrm{I}}_d\bar{\mathcal{D}}_a h^a - \bar{\mathcal{D}}_d h\big) \\ & -\frac{1}{2}\big[\bar{\mathrm{I}}_b\big(\bar{\mathcal{D}}^a\bar{\mathcal{D}}_a h^b - \bar{\mathcal{D}}_a\bar{\mathcal{D}}^b h^a\big) \\ & -\bar{\mathcal{D}}_b\big(\bar{\mathrm{I}}^b\bar{\mathcal{D}}_a h^a - \bar{\mathcal{D}}^b h\big)\big]\bar{e}_d\Big\}\wedge\delta e^d. \end{aligned} \tag{3.3.23}$$

Some definitions are in order here: The $\bar{\mathcal{D}}^a$ stands for the operator $\bar{\mathcal{D}}^a = \bar{\mathrm{I}}^a\bar{\mathrm{D}} + \bar{\mathrm{D}}\bar{\mathrm{I}}^a$ where $\bar{\mathrm{I}}^a$ was defined briefly in the previous subsection, and $h = \bar{\mathrm{I}}_a h^a$.

The Lorenz gauge

The discussion of standard gravitational waves involves the use of a gauge in order to simplify the equations describing wave propagation. We set out to find a suitable gauge choice in order to reach a manageable wave equation within the first-order formalism. To this end, we want to take advantage of the diffeomorphism invariance of the action to fix an appropriate gauge.

Let us then consider the following: Let ζ be a vector field. We want to study the infinitesimal diffeomorphism transformations for h_{ab} whenever ζ acts on the background geometry, which is equivalent to study the infinitesimal Lie dragging $1 - \pounds_\zeta$ on h_{ab}. We notice that h_{ab} responds to this infinitesimal dragging as

$$h_{ab} \to h'_{ab} = h_{ab} - \left(\bar{\mathcal{D}}_a\zeta_b + \bar{\mathcal{D}}_b\zeta_a\right) + \zeta^c\bar{\mathrm{I}}_c\left(\bar{\mathrm{I}}_a\bar{T}_b + \bar{\mathrm{I}}_b\bar{T}_a\right). \tag{3.3.24}$$

Let us do the standard change of variable $h^a \to \tilde{h}^a$, by saying

$$h^a = \tilde{h}^a - \frac{1}{2}\bar{e}^a\tilde{h}, \tag{3.3.25}$$

where $\tilde{h}$ corresponds to the trace of the components of $\tilde{h}^a$. By making use of the aforementioned operator $\bar{\mathrm{D}}_a$, one can show that the expression $\bar{\mathcal{D}}_a\tilde{h}^a$ under the dragging (3.3.24) transforms as $\bar{\mathcal{D}}_a\tilde{h}^a \to \bar{\mathcal{D}}_a\tilde{h}'^a$, where $\bar{\mathcal{D}}_a\tilde{h}'^a$ corresponds to

$$\begin{aligned}\bar{\mathcal{D}}_a\tilde{h}'^a = \Big[& -\mathring{\bar{\mathcal{D}}}_a\mathring{\bar{\mathcal{D}}}^a\zeta_b + \bar{\mathcal{D}}_a\bar{\mathrm{I}}_b\tilde{h}^a - \bar{\mathrm{I}}_{ab}\mathring{\bar{R}}^a{}_c\zeta^c \\ & \bar{\mathrm{I}}^{ac}\bar{T}_a\left(\mathring{\bar{\mathcal{D}}}_c\zeta_b + \mathring{\bar{\mathcal{D}}}_b\zeta_c - \eta_{cb}\mathring{\bar{\mathcal{D}}}_p\zeta^p\right) \\ & +\bar{\mathrm{I}}_{cb}\bar{T}_a\bar{\mathrm{I}}^c\tilde{h}^a\Big]\bar{e}^b,\end{aligned} \tag{3.3.26}$$

with $\mathring{\bar{\mathcal{D}}} = \bar{\mathrm{I}}_a\mathring{\bar{\mathrm{D}}} + \mathring{\bar{\mathrm{D}}}\bar{\mathrm{I}}_a$. This enables us to find a certain vector field ζ such that the RHS of (3.3.26) vanishes, thus fixing the so-called Lorenz gauge

$$\bar{\mathcal{D}}_a\tilde{h}^a = 0. \tag{3.3.27}$$

With help of this gauge fixing and by using the commutation relations found in (3.2.16) and (3.2.17) we can reformulate the expression (3.3.21) in terms of $\tilde{h}^a$ as

$$\begin{aligned}0 = \bar{\blacksquare}_{\mathrm{dR}}\,\tilde{h}_d + \bar{\mathrm{I}}_{ad}\left(\bar{R}^a{}_b \wedge \tilde{h}^b\right) & \\ -\left\{A_d + B_d + \frac{1}{2}\bar{e}_d\left[C - \bar{\mathrm{I}}_c\left(A^c + B^c\right)\right]\right\} & \\ +\epsilon_{abcd}\,\bar{*}\left(\bar{R}^{ab} \wedge h^c + 2\bar{\mathrm{D}}\mathcal{V}^{ab} \wedge \bar{e}^c\right) & \\ +(\text{linear perturbations of other terms}). &\end{aligned} \tag{3.3.28}$$

With help of the generalized Weitzenböck identity (3.2.12) we find $\bar{\blacksquare}_{\rm dR}\,\tilde{h}_a$ to be

$$\bar{\blacksquare}_{\rm dR}\,\tilde{h}_a = -\bar{\mathcal{D}}^b\bar{\mathcal{D}}_b\tilde{h}_a + \bar{\rm I}_b\big(\bar{R}^b{}_c\bar{\rm I}^c\tilde{h}_a - \bar{R}^b{}_a\bar{\rm I}^c\tilde{h}_c\big), \tag{3.3.29}$$

while the torsional terms A_a, B_a and C are given by

$$\begin{aligned} A_a &= \big(\bar{\rm I}_{ca}\bar{T}_b\big)\bar{D}^b\,\tilde{h}^c + \tilde{h}^{bc}\,\bar{\rm D}\bar{\rm I}_{ca}\bar{T}_b, \\ B_a &= \big(\bar{\rm I}^c\bar{T}_b\big)\bar{\rm I}^b\big[\bar{\mathcal{D}}_a\big(\tilde{h}\,\bar{e}^c\big) - \bar{\mathcal{D}}^c\big(\tilde{h}\,\bar{e}_a\big)\big] \\ &\quad + \frac{1}{2}\big\{\tilde{h}\,\bar{\rm D}^{\ddagger}\bar{T}_a + \bar{\rm I}^b\big[\bar{\rm D}\big(\tilde{h}\,\bar{\rm I}_a\bar{T}_b\big) - \bar{T}_b\bar{\rm I}_a\bar{\rm D}\tilde{h}\big]\big\}, \\ C &= \bar{\mathcal{D}}^c\big(\tilde{h}^{ab}\,\bar{\rm I}_{bc}\bar{T}_a\big) + \big(\bar{\rm I}_{bc}\bar{T}_a\big)\bar{\rm I}^a\bar{\rm D}^c\tilde{h}^b. \end{aligned} \tag{3.3.30}$$

Finally, we can find the equation for propagation of linear perturbations by taking (3.3.28) and plugging it in (3.3.21). We conclude this section with some remarks.

Remarks

- The metric wave $\tilde{h}_{ab}$ couples to both the background torsion $\bar{T}^a$ and background curvature $\bar{R}^{ab}$. This can be readily seen in the second term of (3.3.28) and in the C term in (3.3.30).
- Since $\tilde{h}^a$ satisfies a wave equation with source – and thus it is a propagating field, we can automatically conclude that torsion itself is a propagating field. This can be readily seen from the behavior of the torsion under linear perturbations (3.3.13) and from the definition of $\mathcal{V}_{ab}$ given in (3.3.9). At some point we ought to insert the propagating solution for $\tilde{h}^a$ in (3.3.9), thus obtaining wave behavior.
- Some couplings between $\tilde{h}_{ab}$ and the background torsion appear through the trace $\tilde{h}$. It turns out rather surprising that the "traceless"variable $\tilde{h}_{ab}$ does not lead to equations without the trace $\tilde{h}$.

3.3.3 Gravitational Waves and generic terms of Horndeski's Lagrangian

In the previous section we put ourselves to the task of exploring the arising of torsional modes from the well-known Einstein-Hilbert term, which corresponds to an specific model properly included among the family of theories belonging to Horndeski's Lagrangian. We found that the EH term can indeed produce gravitational waves interacting with the background torsion and propagating torsional modes. This leads us to think how torsion could arise from the other terms present in the Lagrangian (3.1.15).

Let us study – in a rather qualitative manner – how this behavior might arise similarly in other therms from the Horndeski family. We will find that generic terms from Horndeski's Lagrangian (3.1.15) will couple $\tilde{h}^a$, $\mathcal{V}^{ab}$ and φ with the background curvature $\bar{R}^{ab}$ and torsion $\bar{T}^a$. However, only some specific terms will contribute with second-order wave-like operators in the metric modes (terms of the form $\partial^2\tilde{h}^a$) and first order-operators on the torsional modes ($\partial\mathcal{V}^{ab}$ terms).

In order to get some clarity in our treatment, we will take as a simplifying assumption that propagations associated to the scalar field ϕ are turned off, i.e. we consider $\varphi = 0$. Having this in sight, we find that the linear perturbations on the fields e^a, R^{ab}, T^a, Z^a, θ^a and π^a are given by

$$
\begin{aligned}
\bar{e}^a \;\; &\rightarrow \;\; e^a = \bar{e}^a + \frac{1}{2}h^a, \\
\bar{R}^{ab} &\rightarrow R^{ab} = \bar{R}^{ab} + \bar{\mathrm{D}}\mathcal{V}^{ab} - \frac{1}{2}\bar{\mathrm{D}}\big(\bar{\mathrm{I}}^a\bar{\mathrm{D}}h^b - \bar{\mathrm{I}}^b\bar{\mathrm{D}}h^a\big), \\
\bar{T}^a \;\; &\rightarrow \;\; T^a = \bar{T}^a + \mathcal{V}^a{}_b \wedge \bar{e}^b - \frac{1}{2}\bar{\mathrm{I}}^a\big(h_b \wedge \bar{T}^b\big), \\
\bar{Z}^a \;\; &\rightarrow \;\; Z^a = \bar{Z}^a - \frac{1}{2}h^a{}_b\bar{Z}^b, \\
\bar{\theta}^a \;\; &\rightarrow \;\; \theta^a = \bar{\theta}^a - \frac{1}{2}h^a{}_b\bar{\theta}^b, \\
\bar{\pi}^a \;\; &\rightarrow \;\; \pi^a = \bar{\pi}^a - \frac{1}{2}h^a{}_b\bar{\pi}^b \\
&\qquad\qquad + \left[\mathcal{V}^{ab} - \frac{1}{2}\big(\bar{\mathrm{I}}^a\bar{\mathrm{D}}h^b - \bar{\mathrm{I}}^b\bar{\mathrm{D}}h^a - \bar{\mathrm{D}}h^{ab}\big)\right]\bar{Z}_b.
\end{aligned}
\tag{3.3.31}
$$

From the arrangement written above, we can draw already a couple of conclusions: only the perturbation of the Lorentz curvature includes second-order derivatives of h^a – due to the presence of the operator $\bar{\mathrm{D}}\bar{\mathrm{I}}^a\bar{\mathrm{D}}$ – and first-order derivatives of the torsional perturbation $\bar{\mathrm{D}}\mathcal{V}^{ab}$. In contrast, for π^a we find that only first-order derivatives for h^a and no derivatives for $\mathcal{V}^{ab}$ appear. Since the Hodge operator only appears in Horndeski's Lagrangian through the operator $\mathrm{I}^a = -*e^a \wedge (*$ together with the fact that $\mathrm{d}^2 = 0$, we find that in the equations of motion for the vielbein and spin connection $\mathcal{E}_a = 0$ and $\mathcal{E}_{ab} = 0$ (found in (3.1.18) and (3.1.22)) terms of the form $\partial^2\tilde{h}^a$ and $\partial\mathcal{V}^{ab}$ can arise only when the Lorentz curvature is present. This simplifies greatly our discussion here.

Indeed, for our Horndeski Lagrangian (3.1.15) we focus then in two kind of terms: those containing explicitly the Lorentz curvature R^{ab} and those containing π^a. Our interest on π^a comes from the fact that this term generates a Lorentz curvature through the Bianchi identity $\mathrm{D}\pi^a = \mathrm{D}^2 Z^a = R^a{}_b Z^b$[16].

[16]This can be seen more easily by considering the vierbein dependence of $Z^a = \mathrm{I}^a\mathrm{d}\phi$ and later by integrating by parts (see for instance (3.1.26)).

We ought to pay then our attention to the Lagrangian terms

$$
\begin{aligned}
(F + 2W)\, \epsilon_{abcd} R^{ab} \wedge e^c \wedge e^d, \\
\kappa_3\, \epsilon_{abcd} R^{ab} \wedge e^c \wedge \theta^d, \\
\kappa_1\, \epsilon_{abcd} R^{ab} \wedge e^c \wedge \pi^d,
\end{aligned}
\tag{3.3.32}
$$

and

$$
\begin{aligned}
\frac{\partial \kappa_1}{\partial X}\, \epsilon_{abcd}\, \pi^a \wedge \pi^b \wedge \pi^c \wedge e^d, \\
\frac{\partial \kappa_3}{\partial X}\, \epsilon_{abcd}\, \theta^a \wedge \pi^b \wedge \pi^c \wedge e^d, \\
\frac{\partial F}{\partial X}\, \epsilon_{abcd}\, \pi^a \wedge \pi^b \wedge e^c \wedge e^d.
\end{aligned}
\tag{3.3.33}
$$

Let us illustrate this point with an example. Consider the ferm $\mathcal{L}_\theta$ given by

$$
\mathcal{L}_\theta = \frac{1}{2}\epsilon_{abcd}\, R^{ab} \wedge e^c \wedge \theta^d. \tag{3.3.34}
$$

Consider the variation of $\mathcal{L}_\theta$ under an infinitesimal change in the vierbein, i.e. $\delta_e \mathcal{L}_\theta$ given by

$$
\delta_e \mathcal{L}_\theta = \left[\frac{1}{2}\epsilon_{abcd} R^{ab} \wedge \theta^c + \mathrm{I}^a (\mathcal{G}_a \wedge \theta_d)\right] \wedge \delta e^d, \tag{3.3.35}
$$

where the 3−form $\mathcal{G}_d$ corresponds to

$$
\mathcal{G}_d = \frac{1}{2}\epsilon_{abcd} R^{ab} \wedge e^c. \tag{3.3.36}
$$

Let's unfreeze the linear perturbations of ϕ (i.e. $\varphi \neq 0$) and implement the rest of them in (3.3.35). We find that $\delta_e \mathcal{L}_\theta$ behaves as

$$
\begin{aligned}
\delta_e \mathcal{L}_\theta = \delta_{\bar{e}} \mathcal{L}_{\bar{\theta}} + \Bigg\{ & \bar{\mathrm{I}}^m \big(\mathcal{W}_m \wedge \bar{\theta}_d + \bar{\mathcal{G}}_m \wedge \mathcal{Y}_d\big) \\
& + \frac{1}{2}\epsilon_{abcd} \bar{R}^{ab} \wedge \mathcal{Y}^c \\
& + \frac{1}{2}\epsilon_{abcd} \bar{\mathrm{D}} \left[\mathcal{V}^{ab} - \frac{1}{2}\big(\bar{\mathrm{I}}^a \bar{\mathrm{D}} h^b - \bar{\mathrm{I}}^b \bar{\mathrm{D}} h^a\big)\right] \wedge \bar{\theta}^c \\
& - \frac{1}{2} h^{mn} \bar{\mathrm{I}}_n \big(\bar{\mathcal{G}}_n \wedge \bar{\theta}_d\big) \Bigg\} \wedge \delta e^d.
\end{aligned}
\tag{3.3.37}
$$

In here we have defined $\mathcal{Y}^a$ as

$$\mathcal{Y}^a = -\frac{1}{2}h^a{}_b\bar{\theta}^b + \mathrm{d}\bar{\phi}\,\bar{\mathrm{I}}^a\mathrm{d}\varphi + \bar{Z}^a\mathrm{d}\varphi. \tag{3.3.38}$$

which actually corresponds to the linear perturbation of θ^a, while $\mathcal{W}_a$ corresponds to

$$\begin{aligned}\mathcal{W}_d &= -\frac{1}{4}\epsilon_{abcd}\bar{\mathrm{D}}\big(\bar{\mathrm{I}}^a\bar{\mathrm{D}}h^b - \bar{\mathrm{I}}^b\bar{\mathrm{D}}h^a\big)\wedge\bar{e}^c\\ &\quad+\frac{1}{2}\epsilon_{abcd}\left[\frac{1}{2}\bar{R}^{ab}\wedge h^c + \bar{\mathrm{D}}\mathcal{V}^{ab}\wedge\bar{e}^c\right]\\ &= +\frac{1}{4}\bar{*}\left[\bar{\blacksquare}_{\mathrm{dR}}\,\tilde{h}_d + \bar{\mathrm{I}}_{ad}\big(\bar{R}^a{}_b\wedge\tilde{h}^b\big)\right]\\ &\quad-\frac{1}{4}\bar{*}\left\{A_d + B_d + \frac{1}{2}\bar{e}_d\big[C - \bar{\mathrm{I}}_c\big(A^c + B^c\big)\big]\right\}\\ &\quad+\frac{1}{2}\epsilon_{abcd}\left[\frac{1}{2}\bar{R}^{ab}\wedge h^c + \bar{\mathrm{D}}\mathcal{V}^{ab}\wedge e^c\right].\end{aligned} \tag{3.3.39}$$

In here, A_a, B_a and C are the same torsion couplings defined in (3.3.30). One can find similar expressions on the linear perturbations for the rest of the terms written in (3.3.32) (3.3.33).

We finish this section by remarking some points. Interestingly enough, the appearance of terms of the form $\partial^2\tilde{h}^a$ is related to couplings with torsion. The treatment of linear perturbations in Horndeski's framework in first-order formalism states that this seems to be a rule, rather than an isolated occurrence.

Chapter 4

Summary and conclusions of Part I

This first part of this doctoral work addressed the study of Horndeski's Lagrangian within the first order formalism. Such Lagrangian entails all possible interactions between gravity and a scalar field in four-dimensional spacetime that yield equations of motion up to second order in the derivatives. We explored how these couplings within the first order formalism might give rise to a non-vanishing torsion. With help of torsion-aware differential operators we were able to study the linear perturbation regime around a certain background and see how the background torsion couples to the metric perturbations. Let us summarize this first part and findings therein.

Summary

In Chapter 2 we made a first presentation of certain modified theories of gravity. We discussed a generalization which depends solely on the metric, which is Lovelock's theory. This prompted the question of whether we can find further reasonable modifications for Einstein's gravity. Motivated by the current problem of dark energy, we discussed the possibility of including a scalar field into the picture. This lead us to introduce Horndeski's theory and give an account of its formulation, features and some of the theories contained. Given the recent developments and experimental results in gravitational wave astronomy, we dedicated a subsection to discuss how such data constraint the form of Horndeski's Lagrangian if we impose the speed of propagation of gravitational waves to be the speed of light. Other constraints coming from the polarization side were mentioned as well.

After having introduced Horndeski's theory, in Chapter 3 we studied Horndeski's Lagrangian within the first order formalism. Since torsion is a non-Riemannian feature of geometry, we find it convenient to work with the Cartan geometry formalism.

- In section one of chapter 3 we introduced some convenient definitions in order to explore Horndeski's Lagrangian in Cartan's first-order formalism. We wrote down the full Horndeski's Lagrangian in the language of p-forms and derived the field equations with respect to e^a, ω^{ab} and ϕ.

- Section two was the most mathematical-oriented portion of this chapter. In here, we defined torsion-aware differential operators as a preparation for our study of linear perturbations coupled to torsion. Such operators are able to discern the presence of torsion and can act on forms carrying Lorentz indices. We summarized such operators in Table 3.1. We provided a generalized version of the Weitzenböck identity that relates torsion-aware versions of the Laplace-Beltrami and the Laplace-de Rham operators. Such identities work for forms with and without Lorentz indices, as given by (3.2.12) and (3.2.3).

- Finally, in section three we dealt with the linear perturbation theory for a theory of gravity non-minimally coupled to a scalar field within the first-order formalism. We studied the most relevant parts of Horndeski's Lagrangian (3.1.15) which can lead to gravitational waves. Such perturbation theory is highly non-trivial. Thanks to the differential operators defined in section two we were able to discern and separate the torsional degrees of freedom from the metric perturbations, all within the assumption that our background configuration might have an arbitrary curvature and torsion. Taking the Einstein-Hilbert portion of Horndeski's Lagrangian as a case study, our analysis showed that the background torsion does indeed couple to the propagating metric degrees of freedom. This motivated us to study further terms of Horndeski's Lagrangian which might give rise to wave equations for the metric perturbations, concluding that the relevant terms one has to look for those containing the Lorentz curvature R^{ab} and π^a explicitly. As expected, we find a wave operator acting on the metric perturbations, coupled with the background curvature and torsion, analogous to the case for the Einstein-Hilbert term.

Discussion and conclusions

The exploration of classical theories of gravity can be further deepened within the first-order formalism of geometry. Indeed, by taking the concepts of metricity and parallelism as independent we relax our geometric assumptions and allow a broader account of phenomenological and theoretical aspects of our theory that might have been overlooked. This is the reason behind the appeal of studying Horndeski's theory within the first-order formalism: It is the most general description of a non-minimal coupling between a scalar field and gravity with no ghosts. Horndeski's Lagrangian contains several modified theories of gravity that address remaining problems in cosmology, such as inflation, dark energy and dark matter. As stated in 3.2, $U(1)$ gauge bosons are unaware of the background torsion. This motivated the study of torsion arising from other significant astrophysical phenomena, like gravitational waves.

In this first part of this doctoral work, we explored solutions to two important issues regarding torsion: First, how it can be sourced. Second, to check whether it arises from phenomena where torsion can be potentially falsified.

Regarding the first point, in the pure ECSK theory, torsion does not propagate in vacuum and fermions are its only (very weak) source. This makes it necessary to look for more general theories in four dimensions and new torsion sources. This prompted to take Horndeski's theory and allow torsion to be non-zero within the first-order formalism. We found that every non-minimal coupling of the geometry with the scalar field ϕ and every term containing second derivatives of ϕ are generic sources of torsion. This was expected from previous work, such as section 1.7.1 of [103] on Brans-Dicke theory and [51] on non-minimal couplings with the Gauss-Bonnet term. Regarding the second point, we approached gravitational waves as a way to probe torsion. We developed differential operators capable to discern torsion, and we showed that any Horndeski Lagrangian that includes the Einstein-Hilbert will generate –under linear perturbations– gravitational waves, as seen in (3.3.28). Such gravitational waves will contain interactions with the background torsion.

Even though we were able to show that torsion is present at the level of perturbations at linear level, there are still remaining issues that need to be addressed. For instance, with the input of observational evidence, modified theories of gravity have been heavily constrained. In particular, after GW170817 and its electromagnetic counterpart, observational evidence suggests to take the speed of propagation of gravitational waves to be the same as the speed of light if we explore alternative theories of gravitation. By doing so, we are limited only to a certain subset of theories contained within Horndeski's framework.

In this work, we explored the perturbation regime with no matter Lagrangian. The role of torsion in realistic astrophysical phenomena within surviving theories still needs to be modeled. Even for the vacuum case, a proper analysis of the equations of motion and possible solutions whenever torsion is presents needs to be done. It has been argued recently that torsion does have an effect on the polarization tensor of gravitational waves, without changing the speed of propagation [112]. New instrumentation, arrangements and upgrades for the current gravitational wave observatories might be able to discern the presence of torsion.

From a theoretical side, the question of adding torsional terms to Horndeski's Lagrangian still remains an open problem. Horndeski's Lagrangian is ensured to give equations of motion up to second order in the derivatives. Within the first order formalism and allowing torsion to be non-vanishing, it is still unclear how torsion might be added consistently to Horndeski's Lagrangian, i.e. how we could generalize the Lagrangian by adding torsional terms and guarantee that the equations of motion will be up to second order in the derivatives. The aforementioned remaining problems will be subject for future work.

Part II

T-duality and the Open String

Chapter 5

Introduction of Part II

Enter string theory

In the first part of this doctoral work we explored Horndeski's Lagrangian, allowing a non-vanishing torsion and studied linear perturbations for the scalar field, spin connection and vierbein. Horndeski's theory is a particular generalization of Einstein gravity, and allows us to explore possible extensions of it. Moreover, we worked inside a framework that does not force torsion to vanish. All in all, we relaxed our assumptions of geometry and found that torsion might play a significant role in the propagation of gravitational waves. This exercise is a small reflection of an aspect of the study of Nature: Throughout the history of science, we find that the search for generality and unified descriptions have driven the development of physics.

So far, our best framework to address gravitational phenomena is general relativity. Its predictive power cannot be overstated: The announcement of the detection of gravitational waves in February 2017 and the results of the Event Horizon Telescope released in April 2019 are the last of a long list of confirmations of Einstein's Theory. On the other hand, in March 2013 the CMS and ATLAS collaborations at CERN announced the detection of the Higgs boson, adding another success for the standard model of particles [113].

Despite their successes, there are still problems that the standard model and general relativity cannot address. For instance, the origin of dark matter and dark energy is still unknown. It is still unclear also if quantum chromodynamics and the electroweak theory can be unified under a Grand Unification Theory. Furthermore, if we want to incorporate gravity into the standard model we find that general relativity cannot be quantized since it is non-renormalizable.

It is widely regarded that the incorporation of general relativity into the Standard Model is needed. A theory of quantum gravity would allow – for instance – to have a unified description of all fundamental forces of nature. It could also allow the study of processes in early cosmology, where the quantum effects of the gravitational interaction become relevant [114]. The energy scale at which these effects become relevant is thought to be the Planck mass $M_P = \hbar c/G \simeq 10^{19}\,\text{GeV}/c^2$ [115]. A candidate framework for a quantum theory of gravity that unifies all fundamental forces and particles is string theory.

String theory is a framework in which we use strings instead of the idea of the point particle. These strings can be either opened or closed. Depending on the oscillation modes of the string, we can generate different particles. In fact, the graviton – which is thought to be the carrier particle of the gravitational interaction – appears naturally inside this framework from the oscillations of the closed bosonic string. In fact, source-free general relativity appears naturally when we study consistency conditions dictated by conformal invariance.

One of the many surprising consequences of string theory involves the dimensions of our spacetime. Bosonic string theory – that is, a string theory containing only bosons plus a tachyon – states that the number of dimensions is 26, whereas superstring theory – a string theory featuring fermions and bosons that incorporates supersymmetry – establishes the number of dimensions to 10. Since the world we live in has – up to our knowledge – only four spacetime directions, this feature of string theory has led to the study of compactification schemes that allow us to make contact with the four-dimensional spacetime. The treatment of compactification for the superstring lead to the study of interesting geometries, like Calabi-Yau manifolds. Regarding the bosonic case, we will explore compactification for the bosonic string in next chapter as a motivation to study T-duality transformations.

Another feature of string theory is that it contains extended objects called D-branes. These are the very same objects on which the endpoints of the open strings attach. An open string can stretch between two different D-branes or start and end on the same brane. The dimension of such D-branes is determined by the number of Dirichlet and Neumann boundary conditions that the open string satisfies. As an example, let d be the dimension of the target spacetime. If we have an open string with p directions X^a satisfying Neumann boundary conditions and $d - p$ directions X^i satisfying Dirichlet boundary conditions at its endpoints, we have a p-dimensional D-brane – or Dp-brane, for short[17].

Such objects are of crucial importance in string theory: D-branes are non-perturbative degrees of freedom and are allowed to have dynamics on their own. They were central to the discovery of an intricate web of dualities between all five superstring theories - see for instance section 3 of [117]. Intersecting D-branes play an important role in the construction of gauge bosons and matter fields with properties of those of the standard model. In this second part of this doctoral work, D-branes will be central to our treatment of T-duality transformations.

[17] There is an exotic case for a Dp-brane, which is the D(-1)-brane. This is a brane with Dirichlet boundary conditions in the time direction as well as all spatial directions. See for instance section 8.2 of [116].

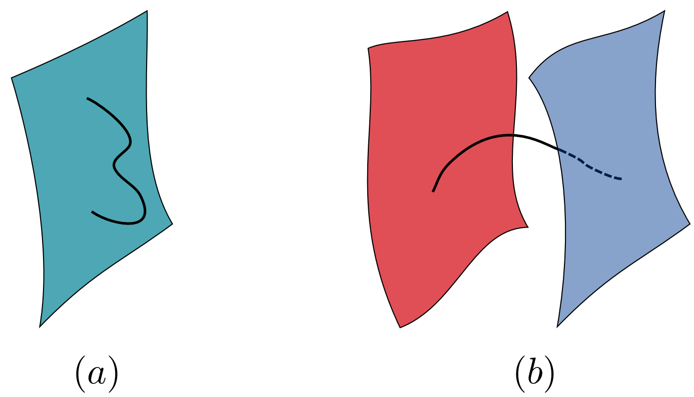

Figure 5.1: (a) An open string with endpoints attached to the same D-brane. (b) An open string stretching between two D-branes.

T-duality and non-geometric spaces

In string theory, T-duality is an equivalence between two seemingly different theories which actually describe the same physics. In its most simple case, it establishes that bosonic closed strings compactified on a circle of radius R have the same physics as theories compactified on a circle of radius α'/R - see for instance Section 10.2 of [115]. For the superstring, we find that it relates the type IIA and type IIB supersymmetric theories (see for instance [118, 119] and section 7.5 of [116]) and the heterotic $SO(32)$ and $E_8 \times E_8$ theories. For a thorough discussion on T-duality transformations for the heterotic string, see sections 2.1 and 2.2 of [120].

T-duality transformations can be used to construct non-geometric backgrounds in string theory. These non-geometric spaces cannot be described in terms of Riemannian geometry only: In order to have a proper globally well-defined patching of such spaces we need to incorporate duality transformations. This means that diffeomorphisms are not enough and we need to incorporate $O(D, D; \mathbb{Z})$ transformations as transition functions between local charts [121, 122]. Such spaces are of special interest in string theory, since they lead to non-associative [123–129] and non-commutative structures [130–139].

The standard construction of a non-geometric background is carried by applying successive or collective T-duality transformations on the three-torus with H-flux: This configuration considers a target space metric G of a product space $S^1 \times S^1 \times S^1$ together with the flux $H = \mathrm{d}B$ related to the antisymmetric Kalb-Ramond field B. By performing one T-duality transformation one reaches at a twisted three-torus [140, 141] which is a geometrical space with a geometric F-flux $F^i{}_{jk}$ associated to it. By performing a second T-duality transformation we reach the T-fold background [121], which carries the non-geometric Q-flux with components $Q^{ij}{}_k$. Even though this space allows a local geometric description in terms of the target-space metric and the Kalb-Ramond field B, it is globally non-geometric. Even though Buscher's rules do not allow it, we can still formally perform a third T-duality transformation and reach the so-called R-space [142, 143]. Such space carries the non-geometric R-flux R^{ijk} and does not even allow a local geometric description. The Q- and R-fluxes have found application in moduli-stabilization [144–149, 142, 143] and in construction of inflationary potentials [150, 151].

5.1 Motivation

The study of T-duality transformations for curved background configurations requires the use of the Buscher's procedure. This procedure employs the isometries of the background, where the corresponding isometry algebra can be abelian or non-abelian. On the other hand, the incorporation of D-branes into the picture requires to study T-duality transformations from the open-string point of view, hence the boundary conditions for such strings need to be taken into consideration. The study of T-duality transformations for the open string has been addressed in the literature. We mention some relevant works: in [152] T-duality transformations for the open string along one direction are discussed. This direction satisfies Neumann boundary conditions and a Lagrange multiplier is implemented. Such Lagrange-multiplier does not allow the generalization to Dirichlet directions for non-trivial worldsheet topologies. In [153–155] non-abelian T-duality transformations along one direction are studied from the path integral and canonical transformations point of view. The B-field is set to zero and only a trivial topology for the worldsheet is considered. In [156] the authors explore T-duality transformations for the open-string sigma model including the fermionic section on the worldsheet. The case for one T-duality transformation is discussed, taking into account a trivial topology for the worldsheet.

In this second part we study non-geometric backgrounds from the open-string point of view via Buscher's procedure, addressing missing details in the literature. In our discussion we will consider a non-trivial worldsheet topology and perform multiple T-duality transformations on a curved background. Since we include D-branes in this picture, we present treatments for directions which satisfy either Dirichlet or Neumann boundary conditions. We present the construction of a non-linear sigma model whose isometry algebra will be considered to be non-abelian and implement Buscher's rules, but main results will address the abelian case.

5.2 Outline of Part II

This second part consists of four chapters plus this introduction. We present an outline of their contents:

- In Chapter 6 we present an introductory treatment on T-duality and non-geometric spaces. We illustrate T-duality by presenting the case of the bosonic string compactified on a circle and on a toroidal lattice. We showcase the effect of T-duality transformations on the mass spectrum of the closed string and present the set of transformations behind them. We present as well a treatment on general backgrounds and introduce Buscher's procedure. In the next subsection we display the standard non-geometric configurations and some of their properties. In the final subsection of this chapter we present the so-called toroidal fibrations and show the globally well-definedness of non-geometric backgrounds when we use duality transformations.

- In Chapter 7 we present a treatment of T-duality transformations for the open string using Buscher's procedure. We construct a non-linear sigma model for the open string with non-trivial boundary and present in detail the boundary conditions for the string. We use Hodge's decomposition theorem for manifolds with boundary for this task. Later we study the global and local symmetry of the worldsheet action and implement Buscher's procedure. We study collective T-duality transformations along directions that satisfy either Dirichlet or Neumann boundary conditions and read the dual background configuration. We illustrate this formalism with the standard example of the three torus with H-flux and study the backgrounds of the dual configuration. Later, we discuss the application of the Freed-Witten anomaly cancelation condition on some of our examples, and later we study the global well-definedness of the D-branes on the dual configuration.

- In Chapter 8 we present a current development on non-abelian T-duality transformations. We consider a Wess-Zumino-Witten model for a Lie group manifold and apply Buscher's procedure. We integrate along all of the isometry directions of the diagonal subgroup of G, and present the corresponding change of basis. We illustrate this for the case $G = SU(2)$ and discuss preliminary results.

- In Chapter 9 we present a summary of our findings and we close with conclusions.

Chapter 6

T-duality and non-geometrical Spaces

In this chapter we will present the subject of T-duality as a motivation for our next chapter, when we study T-duality transformations for the open string.

Here, we review the study of a closed bosonic string defined on a Minkowskian target space background with S^1 as a compact dimension. We study its mass spectrum and the discrete transformations that leave it invariant. For completeness, we explore the general case of compactifications on tori $\mathbb{T}^D$. After these treatments, we go on the study of T-duality transformations for general curved backgrounds via Buscher's procedure. This will work as a preparation for the next chapter, when we extend these formalisms by adding consistently D-Branes in our configuration. We address the study of the geometrical spaces obtained after these transformations in the subsequent sections on non-geometric spaces and fibrations. Since this thesis deals with T-duality transformations for oriented strings, we won't do any treatment for the unoriented case. The interested reader can check section 4.11 of [116] and section 8.8 of [157] on this matter.

The following treatment is inspired from [158], [159] and [115]. The interested reader can further deepen on this subject with help of these references.

6.1 T-Duality

6.1.1 The case of S^1

Anomaly-free string theory for the bosonic string requires that $d = 26$, whereas the superstring requires $d = 10$. Since we want to establish contact with four-dimensional field theories and phenomenology, we can resort to compactification in order to get effective 4d-theories from higher dimensional ones. This idea has been long before explored: In 1921 and 1926 Kaluza and Klein put forward a compactification scheme to unify electromagnetism and general relativity by considering an extra dimension besides the usual four [160, 161].

Here we perform the simplest mechanism for compactification by allowing one direction to be along the circle S^1. We will consider the simplest case of a string moving on a Minkowskian target-space and study its mass spectrum.

To this end, we will consider the action for such string. Let $X^\mu = X^\mu(\tau,\sigma)$ denote each coordinate of the string which depends on parameters (τ,σ). The propagation of the string in spacetime generates a two-dimensional Lorentzian *worldsheet* Σ which can be parametrized precisely by (σ,τ). By performing the usual gauge fixings[18] we get the action for a worldsheet with intrinsic metric $h_{\mathsf{ab}} = \eta_{\mathsf{ab}} = \mathrm{diag}(-1,+1)$ given by

$$S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \frac{1}{2}\eta_{\mu\nu}\,\mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu, \tag{6.1.1}$$

where $\eta_{\mu\nu}$ is the metric of the *target space*. This action corresponds to the Polyakov action. We start by solving the equations of motion that can be derived from it.

Closed strings

Let us consider the expansion of a closed bosonic string moving on a Minkowski target space in $d = 26$ and derive the equations of motion derived from the (6.1.1). We find that all X^μ satisfy the wave equation $(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$ and we solve it for the closed string, which satisfies the periodic boundary conditions $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$, where ℓ corresponds to the length of the string. We find that X^μ allows an expansion in term of the left and right movers X_L and X_R as $X^\mu = X^\mu_L + X^\mu_R$, where

$$\begin{aligned} X^\mu_R(\sigma^a) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{\ell}p^\mu(\tau-\sigma) \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\frac{1}{n}\alpha^\mu_n \exp\left[-i\frac{2\pi n}{\ell}(\tau-\sigma)\right], \\ X^\mu_L(\sigma^a) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{\ell}p^\mu(\tau+\sigma) \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\frac{1}{n}\bar{\alpha}^\mu_n \exp\left[-i\frac{2\pi n}{\ell}(\tau+\sigma)\right], \end{aligned} \tag{6.1.2}$$

where c^μ is a constant, x^μ is the center of mass position of the string at $\tau = 0$ and p^μ is the total space-time momentum of the string. The quantities α^μ_n and $\bar{\alpha}^\mu_n$ correspond to the oscillators which follow the commutation relations after the usual quantization

[18] To get insight into the process of deriving the equations of motion, quantization procedure and light-cone gauge for the bosonic string, the reader can turn to Appendix A.

procedure

$$
\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= m\,\delta_{m+n}\,\eta^{\mu\nu} \qquad & [\bar\alpha_m^\mu, \bar\alpha_n^\nu] &= m\,\delta_{m+n}\,\eta^{\mu\nu} \\
[\alpha_m^\mu, \bar\alpha_n^\nu] &= 0 & [x^\mu, p^\nu] &= i\,\eta^{\mu\nu},
\end{aligned}
\tag{6.1.3}
$$

where the $\alpha's$ satisfy the reality (hermeticity) conditions $(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu$ and $(\bar\alpha_m^\mu)^\dagger = \bar\alpha_{-m}^\mu$.

From this point, we allow X^{25} to be periodic along a circle of radius R. This has two consequences: One, that the momentum p^{25} needs to be quantized so that the string states remain invariant under a translation along this circle, leading to

$$
p^{25} = \frac{n}{R} \tag{6.1.4}
$$

with $n \in \mathbb{Z}$. Second, in string theory the closed string is allowed to wind around this compact dimension more than once. This behavior is captured in numbers by

$$
X^{25}(\tau, \sigma + \ell) = X^{25}(\tau, \sigma) \,+\, 2\pi w R, \tag{6.1.5}
$$

where $w \in \mathbb{Z}$ is called the *winding* number. With these considerations, we find that the expansion for X^{25} can be written down as

$$
\begin{aligned}
X^{25}(\tau,\sigma) = x^{25} \,+\, &\frac{2\pi\alpha'}{\ell} p^{25}\tau \,+\, \frac{2\pi\alpha'}{\ell}\frac{wR}{\alpha'}\sigma + \\
&+ i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\,\left[\alpha_n^{25}\, e^{-i\frac{2\pi n}{\ell}(\tau-\sigma)} + \bar\alpha_n^{25}\, e^{-i\frac{2\pi n}{\ell}(\tau+\sigma)}\right],
\end{aligned}
\tag{6.1.6}
$$

where the oscillators α_n^μ and $\bar\alpha_n^\mu$ together with x^{25} and p^{25} satisfy the commutation relations

$$
[x^{25}, p^{25}] = i\,, \qquad
\begin{aligned}
[\alpha_m^{25}, \alpha_n^{25}] &= m\delta_{m+n}, \\
[\bar\alpha_m^{25}, \bar\alpha_n^{25}] &= m\delta_{m+n}, \\
[\alpha_m^{25}, \bar\alpha_n^{25}] &= 0.
\end{aligned}
\tag{6.1.7}
$$

It will be helpful to write $X^{25}(\tau,\sigma) = X_L^{25}(\tau,\sigma) + X_R^{25}(\tau,\sigma)$, where

$$
\begin{aligned}
X_L^{25}(\tau,\sigma) &= x_L^{25} \,+\, \frac{2\pi\alpha'}{\ell} p_L^{25}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\,\alpha_n^{25}\,\exp\left[-i\frac{2\pi n}{\ell}(\tau+\sigma)\right], \\
X_R^{25}(\tau,\sigma) &= x_R^{25} \,+\, \frac{2\pi\alpha'}{\ell} p_R^{25}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\,\bar\alpha_n^{25}\,\exp\left[-i\frac{2\pi n}{\ell}(\tau-\sigma)\right].
\end{aligned}
\tag{6.1.8}
$$

where we find the momenta $p^{25}_{L,R}$

$$\begin{aligned} p^{25}_L &= \frac{1}{2}\left[\frac{n}{R} + \frac{wR}{\alpha'}\right], \\ p^{25}_R &= \frac{1}{2}\left[\frac{n}{R} - \frac{wR}{\alpha'}\right]. \end{aligned} \tag{6.1.9}$$

Since we performed a compactification along the 26th spacetime direction, the idea is to have a sense of m^2 perceived by an 25−dimensional observer. This means that we ought to compute $\alpha' m^2 = -\alpha' p^\mu p_\mu$ for $\mu = 0, \ldots, 24$ via the light-cone gauge quantization procedure. By using the expressions for the light-cone gauge found at Appendix A we find that the mass squared has the form

$$\begin{aligned} \alpha' m^2 &= \alpha'\left[\frac{n^2}{R^2} + \frac{w^2R^2}{\alpha'^2}\right] + 2(\bar{N} + N - 2) \\ &= \alpha' m_R^2 + \alpha' m_L^2, \end{aligned} \tag{6.1.10}$$

where N and $\bar{N}$ are the number operators for the left and right movers respectively derived in the light-cone quantization, and the $\alpha' m_{L,R}$ are

$$\begin{aligned} \alpha' m_L^2 &= \frac{1}{2}\left[\frac{n}{R} + \frac{wR}{\alpha'}\right]^2 + 2N - 2, \\ \alpha' m_R^2 &= \frac{1}{2}\left[\frac{n}{R} - \frac{wR}{\alpha'}\right]^2 + 2\bar{N} - 2. \end{aligned} \tag{6.1.11}$$

Moreover, physical states must satisfy the constraints

$$m_L^2 = m_R^2 \qquad \longrightarrow \qquad N - \bar{N} = nw, \tag{6.1.12}$$

We notice something interesting in the mass formula (6.1.10). In the usual Kaluza-Klein schemes of compactification, we can get rid of higher dimensional effects on the mass of scalar fields by making them infinitely massive (thus being difficult to excite) by taking the extra dimension to be small. In here we see something quite different: In the limit $R \to \infty$ we find that the mass increases due to the winding modes and the compact momentum becomes continuous. This is expected for a non-compact direction, since in field theory the momentum for a particle in a non-compact direction is rather a continuous function of a continuous parameter. However, for $R \to 0$ one finds that while the momentum modes do become infinitely massive, the winding modes become at the same time continuous. This is a feature of stringy origin only.

The mass has more to tell us: From the mass-squared formula (6.1.10) we find that it remains invariant under the simultaneous discrete transformation

$$R \to \frac{\alpha'}{R}, \qquad n \longleftrightarrow w. \tag{6.1.13}$$

This illustrates a duality transformation for our bosonic string compactified on S^1 case, named T-duality. It states that the theory is indistinguishable from the original one under such transformation, as long we are concerned about the mass. This implies that the mass spectrum for this configuration is completely characterized by the values $R \geq \sqrt{\alpha'}$ or $0 < R \leq \sqrt{\alpha'}$, where evidently $R = \sqrt{\alpha'}$ is a fixed point of this transformation. By extending its action to all oscillator modes via the transformations we find

$$\begin{aligned} p_L &\to +p_L, \qquad X_L \to +X_L, \\ p_R &\to -p_R, \qquad X_R \to -X_R. \end{aligned} \tag{6.1.14}$$

By considering these transformations and using (6.1.8) we find that the action of these T-duality transformation corresponds to a $\mathbb{Z}_2$ asymmetric reflection, and the CFT content for both the left and right modes for X^{25} are left untouched. This shows that T-duality is a duality symmetry of the theory – in the sense that the physics of our system do not change.

As a brief comment, we mention that T-duality transformations are not limited to the bosonic string. The $\mathbb{Z}_2$ action acts on worldsheet fermions as well via the action $(+\psi_L, +\psi_R) \to (+\psi_L, -\psi_R)$. This maps the IIB theory on the circle of radius R into the type IIA theory on the dual circle with radius α'/R. This case will be explored elsewhere.

Open strings

The previous discussion focused on the treatment of the closed string. Since we will work with D-branes in the next chapter, it is a timely moment to mention the effect of a T-duality transformation on an open string.

Let us consider an open string satisfying Neumann boundary conditions at each endpoint, i.e. that

$$\left.\frac{\partial X^\mu}{\partial \sigma}\right|_{\sigma=0,\ell} = 0. \tag{6.1.15}$$

Solving the two-dimensional wave equation for X^μ with Neumann-Neumann conditions (NN) (6.1.15) we find

$$\begin{aligned} X^\mu_{\text{NN}}(\sigma) = x^\mu &+ \frac{2\pi\alpha'}{\ell} p^\mu \tau \\ &+ i\sqrt{2\alpha'} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n} \alpha^\mu_n \exp\left[-i\frac{\pi n \tau}{\ell}\right] \cos\left[\frac{\pi n \sigma}{\ell}\right]. \end{aligned} \tag{6.1.16}$$

Let us consider now the expansion of the NN string (6.1.16) on a circle of radius R, this time by writing its left and right moving expansion such that $X^{25}_{\text{NN}} = X^{25}_{\text{NN,L}} + X^{25}_{\text{NN,R}}$, and

$$
\begin{aligned}
X^{25}_{\text{NN,L}} &= x_{\text{NN,L}} + \frac{\pi\alpha'}{\ell}\frac{n}{R}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\frac{1}{n}\,\alpha^{25}_n\,\exp\left[-i\frac{\pi n}{\ell}(\tau+\sigma)\right],\\
X^{25}_{\text{NN,R}} &= x_{\text{NN,R}} + \frac{\pi\alpha'}{\ell}\frac{n}{R}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\frac{1}{n}\,\alpha^{25}_n\,\exp\left[-i\frac{\pi n}{\ell}(\tau-\sigma)\right],
\end{aligned}
\tag{6.1.17}
$$

where the momentum is quantized according to $p^{25}=n/R$ and $n\in\mathbb{Z}$. By taking the T-duality transformation $(X_L,X_R)\to(+X_L,-X_R)$ we find that the full expansion of the string transforms as $X^{25}_{\text{NN}}\to\tilde{X}^{25}_{\text{NN}}$, where

$$
\begin{aligned}
\tilde{X}^{25}_{\text{NN}} = x^{25}_0 &+ \frac{2\pi\alpha'}{\ell}\frac{n}{R}\sigma\\
&+\sqrt{2\alpha'}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\frac{1}{n}\alpha^{25}_n\,\exp\left[-i\frac{\pi n\tau}{\ell}\right]\sin\left[\frac{\pi n\sigma}{\ell}\right].
\end{aligned}
\tag{6.1.18}
$$

We find that $\tilde{X}^{25}_{\text{NN}}$ is an expansion of an open string with Dirichlet-Dirichlet boundary conditions. Notice that the string at $\sigma=0$ is located at $x^{25}_0 = x_{\text{NN,L}} - x_{\text{NN,R}}$ and it winds n times around the T-dual circle of radius $R'=\alpha'/R$ at $\sigma=\ell$. If we had taken a DD open string and performed the same T-duality transformation, we would have had this time a NN open string expansion. This means that a T-duality transformation on an open string exchanges boundary conditions.

This exchange of boundary conditions tells us that a T-duality transformation transforms a Dp−brane into a D(p ± 1)−brane depending on whether the transformation was done perpendicular to the brane (+) or parallel to it (−).

Let us get ahead and discuss the consequences of defining D-Branes. Having Dp−branes in our configuration requires that our worldsheet boundary must not be empty, i.e. $\partial\Sigma\neq\varnothing$. Let us remember that this object is characterized by the number of Dirichlet and Neumann boundary conditions imposed to the X^μ; we have $p+1$ Neumann directions and $d-p-1$ Dirichlet directions. At the same time, the study of the spectrum of Dp−brane states tells us the following: The massless level features a state of $p-1$ indices that transforms as a vector under of $SO(p-1)$ [115]. These are gauge bosons, and in fact we can stack several D-branes in such a way that these states correspond to Yang-Mills (non-abelian) boson. This lead us to interesting consequences when we approach the construction of curved backgrounds. In particular, we are allowed to define boundary gauge fields on $\partial\Sigma$, which (as the reader might suspect by now) are affected by T-duality transformations.

Let's consider a constant gauge field that begins and ends on the same brane. This means that we ought to work with a $U(1)$−valued gauge field one-form given by $a=$

$a_\mu(X)\mathrm{d}X^\mu$ on $\partial\Sigma$, whose writing on the action has the form

$$S_{\mathrm{g}} = -\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha'\, a, \tag{6.1.19}$$

where for illustrative purposes we will say $a = (\theta/2\pi R)\mathrm{d}X^{25}$. We consider as well the holonomy of our gauge field (or *Wilson line*[19]) to be

$$W_q = e^{iq\,S_{\mathrm{g}}} = e^{-iq\theta}. \tag{6.1.20}$$

The integral (6.1.19) with this particular choice of gauge field can be thought as an angle, and the exponential (6.1.20) as a group element of $U(1)$. In fact, we will see later on that if we have a Dp$-$brane wrapping along X^{25} – that is, that if one of the space directions of the brane is X^{25}, and if we perform a T-duality transformation along this very same direction, the resulting D(p $-$ 1)$-$brane will be located at θ on the dual circle. More intricate constructions can be done if we allow N branes to be separated, which leads to the notion of Chan-Paton factors[20], the construction of non-abelian gauge fields and even the realization of non-perturbative effects on the worldsheet [164]. We will illustrate with examples the role of boundary gauge fields whenever T-duality transformations are performed in the upcoming chapter.

This treatment has regarded only the case of one compact direction. It should not be a surprise that this procedure can be further generalized to the case for toroidal compactifications. By doing this we will have a better look on the group behind T-duality transformations.

6.1.2 On general background configurations

The non-linear sigma model

Let us study a particular generalization of the Polyakov action (6.1.1). When we study the different massless states arising from the closed bosonic string at level $n = 2$ – that is, by letting two creation operators act on the vacuum $|0; k^\mu\rangle$, there are three fields which arise: The target-space graviton $G_{\mu\nu}(X)$, the Kalb-Ramond field $B_{\mu\nu}(X)$ and the dilaton $\phi(X)$[21]. These fields in principle may depend on the bosonic target-space coordinates X^μ, and we can use them to write an action for a propagating string. From this point on we will work on a Euclidean worldsheet Σ by performing a Wick rotation

[19] This quantity measures the non-triviality of the gauged field by parallel-transporting it around a non-trivial loop. These objects have been used in non-perturbative QCD and involved as well in the so-called string confinement [162]. For a pedestrian approach on Wilson loops, see for instance [163].

[20] If we have a configuration of N Dp$-$branes on top of each other, the corresponding states can be written as $|k, l; p^i\rangle = \lambda^a{}_{kl}|a; p^i\rangle$, where $k, l = 1, \ldots, N$ encode the information on which brane the string starts and ends, $a = 1, \ldots, N^2$, and (i, a) denote D and N directions, respectively. The λ's are N^2 matrices which are the hermitian generators of $U(N)$ and are called the Chan-Paton factors.

[21] See section A.4.1 of Appendix A

$\sigma^0 \to -i\sigma^0$ and work with the corresponding Euclidean action S_E resulting from the rotation $S \to -iS_E$. Noticing that $*\,1$ is the volume element of the worldsheet (with $*$ the Hodge dual) we write

$$\begin{aligned} S_E = -\frac{1}{2\pi\alpha'} \int_\Sigma \Big[&\frac{1}{2} G_{\mu\nu}(X)\, \mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu \\ &+ \frac{i}{2} B_{\mu\nu}(X) \mathrm{d}X^\mu \wedge \mathrm{d}X^\nu + \frac{\alpha'}{2} \mathsf{R}\, \phi(X) * 1 \Big], \end{aligned} \tag{6.1.21}$$

where R corresponds to the Ricci scalar of the worldsheet. The action here accrued corresponds to a *non linear* σ*−model* for the string.

At first glance it seems odd that we establish a background with fields which are generated from the string itself. However, we can think of these fields as effective descriptions made out from the respective coherent states [116]. Consider for instance $G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$ to be an expansion of our target space metric and consider the action only with the gravitational coupling and without any Weyl gauge fixing

$$\begin{aligned} S_G &= -\frac{1}{2\pi\alpha'} \int_\Sigma \frac{1}{2} G_{\mu\nu}(X) \mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu \\ &= -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{\mathsf{ab}} G_{\mu\nu}(X) \partial_{\mathsf{a}} X^\mu \partial_{\mathsf{b}} X^\nu. \end{aligned} \tag{6.1.22}$$

Notice that exponentiation of this action including the expansion of the metric leads us to

$$e^{S_G} = e^{S_P} \left[1 - \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{\mathsf{ab}} \chi_{\mu\nu}(X) \partial_{\mathsf{a}} X^\mu \partial_{\mathsf{b}} X^\nu + \dots \right], \tag{6.1.23}$$

where S_P corresponds to the Polyakov action. We see that the term of order χ is actually a vertex operator for the graviton state, therefore stating that the metric itself is of "stringy"origin and allowing us to couple consistently the string with the curved background metric.

Let us briefly discuss the role of α'. The action (6.1.21) can be thought of as a 2D interacting quantum field theory. Since α' has units of (length)^2 we find that it counters the units of length found in the G and B fields, not being the case for the dilaton. In this respect, we can think of a dimensionless coupling constant $\lambda = \sqrt{\alpha'}/R_c$ where R_c is the characteristic radius of the target space – in fact, we are actually working in the $\lambda \ll 1$ regime, where the wavelengths are long compared to the string scale and massive string states are not created [157].

We conclude this section with a small remark. Looking at (6.1.21) we are at risk to jump quickly to the conclusion that choosing different values for $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ corresponds to choose different theories. This is not the case. What we do here actually is to choose different *backgrounds*. Let us not forget that we are working inside string theory, and that choosing different values for the aforementioned fields amount to look at different states inside the framework of ST.

Beta functions

Since the action (6.1.21) accrued from the generalization of the Polyakov action, we want to ensure Weyl invariance in order to define a consistent string theory. This amounts to enforce the tracelessness of the worldsheet energy-momentum tensor T_{ab}. Having in mind that $H = \mathrm{d}B$, we find that its trace is given by

$$T^{\mathsf{a}}{}_{\mathsf{a}} = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}h^{\mathsf{ab}}\,\partial_{\mathsf{a}}X^\mu\,\partial_{\mathsf{b}}X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{\mathsf{ab}}\,\partial_{\mathsf{a}}X^\mu\,\partial_{\mathsf{b}}X^\nu - \frac{1}{2}\beta^\phi R. \tag{6.1.24}$$

We find as well the so-called beta functions for the metric, Kalb-Ramond field and dilaton

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha'\left[R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\kappa\sigma}H_\nu{}^{\kappa\sigma}\right] + O(\alpha'^2),\\
\beta^B_{\mu\nu} &= \alpha'\left[-\frac{1}{2}\nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\phi H_{\kappa\mu\nu}\right] + O(\alpha'^2),\\
\beta^\phi &= \alpha'\left[\frac{D-D_c}{6\alpha'} - \frac{1}{2}\nabla^2\phi + \nabla_\kappa\phi\nabla^\kappa\phi - \frac{1}{24}H_{\kappa\mu\nu}H^{\kappa\mu\nu}\right] + O(\alpha'^2),
\end{aligned} \tag{6.1.25}$$

where D_c is the critical dimension of the theory – for the bosonic string here treated $D_c = 26$. These expressions required to vanish for Weyl invariance at first order in α'.

6.1.3 The case of $\mathbb{T}^D$

In this section we will consider a more general configuration for our propagating string and allow more compactified directions to exist. Taking (6.1.21), let us now consider the action for a bosonic, closed string

$$S = -\frac{1}{2\pi\alpha'}\int_\Sigma\left[\frac{1}{2}G_{\mu\nu}\,\mathrm{d}X^\mu\wedge *\mathrm{d}X^\nu + \frac{1}{2}B_{\mu\nu}\mathrm{d}X^\mu\wedge\mathrm{d}X^\nu\right], \tag{6.1.26}$$

with $G_{\mu\nu}$ and $B_{\mu\nu}$ constant components of the symmetric target space metric and the antisymmetric Kalb-Ramond field, respectively. For the moment we will consider the worldsheet metric to have a Lorentzian signature and $X^\mu = X^\mu(\tau,\sigma)$, $\mu = 0,\dots,25$ describes the motion of the string. We will explore the effect of T-duality transformations given this configuration with multiple compactified directions. Let us take for a moment $B_{\mu\nu} = 0$.

We let D bosonic coordinates to be compactified on a $D-$dimensional torus $\mathbb{T}^D$ via the identification

$$X^I \sim X^I + 2\pi L^I, \qquad I = 25-D,\dots,25 \tag{6.1.27}$$

with

$$L^I = \sum_{i=1}^D n^i\,e^I_i, \qquad n_i\in\mathbb{Z}. \tag{6.1.28}$$

Here the capital indices describe compact directions. The set of vectors L^I are vectors of a $D-$dimensional lattice which can be written as a linear combination of the lattice basis vectors $\{e_i^I\}$ – the vielbein basis. It will be useful to consider the dual basis of this lattice, given by $\{\tilde{e}_I^i\}$. Both bases satisfy the conditions

$$\begin{aligned} e_i^I\, \tilde{e}_I^j &= \delta_i^j, \\ e_i^I\, \tilde{e}_J^i &= \delta_J^I, \end{aligned} \tag{6.1.29}$$

from which we can write the metric of our lattice as $g_{ij} = e_i^I\, e_j^J\, \delta_{IJ}$ and its inverse as $g^{ij} = \tilde{e}_I^i\, \tilde{e}_J^j\, \delta^{IJ}$. This allows us to write the center of mass momentum p_I as $p_I = m_i\, \tilde{e}_I^i$ with $m_i \in \mathbb{Z}$, and we mention as well that the center of mass position and momentum satisfy the usual commutation relations

$$[x^I, p_J] = i\delta_J^I. \tag{6.1.30}$$

Taking into account the identification (6.1.27) we find that X^I allows an expansion of the form $X^I = X_L^I + X_R^I$, where

$$\begin{aligned} X_L^I &= x_L^I + \frac{2\pi\alpha'}{\ell} p_L^I(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\bar{\alpha}_n^I \exp\left[-i\frac{2\pi n}{\ell}(\tau+\sigma)\right], \\ X_R^I &= x_R^I + \frac{2\pi\alpha'}{\ell} p_R^I(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\bar{\alpha}_n^I \exp\left[-i\frac{2\pi n}{\ell}(\tau-\sigma)\right], \end{aligned} \tag{6.1.31}$$

and we find the center of mass position and momentum expansion

$$\begin{aligned} x_L^I &= \frac{1}{2}\left[x^I + c^I\right], \qquad p_L^I = \frac{1}{2}\left[p^I + \frac{L^I}{\alpha'}\right], \\ x_R^I &= \frac{1}{2}\left[x^I - c^I\right], \qquad p_R^I = \frac{1}{2}\left[p^I - \frac{L^I}{\alpha'}\right]. \end{aligned} \tag{6.1.32}$$

In fact, we find that the momentum $P^I = (p_L^I, p_R^I)$ is a lattice vector of an even, self-dual Lorentzian lattice $\Gamma_{D,D}$ whose scalar product is $P \cdot P' = \sum_I (p_L^I p_L'^J - p_R^I p_R'^J)\delta_{IJ}$.

Recalling the expression (6.1.11) we find

$$\begin{aligned} \alpha' m_L^2 &= \frac{\alpha'}{2} \sum_I \left[p^I + \frac{1}{\alpha'} L^I\right]^2 + 2\bar{N} - 2, \\ \alpha' m_R^2 &= \frac{\alpha'}{2} \sum_I \left[p^I - \frac{1}{\alpha'} L^I\right]^2 + 2N - 2. \end{aligned} \tag{6.1.33}$$

For the sake of simplicity we will consider our target space metric to be Euclidean along the compact directions (i.e. $G_{IJ} = \delta_{IJ}$) and turn on the Kalb-Ramond field components along these same directions only. By using (6.1.26) one can show that the presence of the Kalb-Ramond field generates a shift on the center of mass momentum π_I computed from the canonical momentum $\Pi_\mu = \partial L/\partial \dot{X}^\mu$. Turns out that π_I allows the writing $\pi_I = m_i \tilde{e}^i_I$, with $m_i \in \mathbb{Z}$ (since it generates translations on the lattice), which in turn allows the left and right components of the momenta p^I to be written as

$$\begin{aligned} p_{I,L} &= \frac{1}{2}\tilde{e}^i_I \left[m_i + \frac{1}{\alpha'}(g_{ij} - b_{ij})n^j \right], \\ p_{I,R} &= \frac{1}{2}\tilde{e}^i_I \left[m_i - \frac{1}{\alpha'}(g_{ij} + b_{ij})n^j \right]. \end{aligned} \tag{6.1.34}$$

where we have

$$\begin{aligned} g_{ij} &= G_{IJ}\, e^I_i e^J_j, \\ b_{ij} &= B_{IJ}\, e^I_i e^J_j. \end{aligned} \tag{6.1.35}$$

We will address this quantities later on when we discuss its role in the mass invariance of the string spectrum and when we comment on the moduli space of this compactified theory.

Mass invariance

In this section we explore the mass spectrum for our toroidal configuration. The purpose is to find a set of transformations that leave it invariant, just as we saw earlier for S^1. In matrix notation we find the square of the left and right momenta of the lattice to be

$$\begin{aligned} \alpha' p^2_{L,R} = & \frac{\alpha'}{2} m^T g^{-1} m + \frac{1}{2\alpha'} n^T \left[g - b g^{-1} b \right] n + \\ & + n^T b g^{-1} m \pm n^T m. \\ = & \frac{1}{2}\mathbf{n}^T \mathcal{H}\, \mathbf{n} \pm \frac{1}{2}\mathbf{n}^T \eta\, \mathbf{n}. \end{aligned} \tag{6.1.36}$$

where we have defined

$$\mathbf{n} = \begin{pmatrix} n^i \\ m_j \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \frac{1}{\alpha'}(g - b g^{-1} b) & b g^{-1} \\ -g^{-1} b & \alpha' g^{-1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1_{D\times D} \\ 1_{D\times D} & 0 \end{pmatrix}, \tag{6.1.37}$$

and of course we have $(+)$ for L and $(-)$ for R. The last expression in (6.1.36) – as seen for instance in [158,165] – is rather suggestive and it will prove to be quite useful. The quantity $\mathcal{H}$ defined above is called the *generalized metric.*

If we want to find the mass perceived by an $(25 - D)-$dimensional observer, we compute the mass squared under the light-cone gauge $\alpha' m^2 = -\alpha' \sum_{\mu=0}^{25-D} p_\mu p^\mu$.

Analogously to the S^1 case we find that the left and right mass squared and total mass squared are given by

$$\begin{aligned} \alpha' m_{L,R}^2 &= 2\alpha' p_{L,R}^2 + 2(N_{L,R} - 1), \\ \alpha' m^2 &= \alpha' m_L^2 + \alpha' m_R^2, \\ \alpha' m_L^2 &= \alpha' m_R^2 \qquad \text{(Level-matching condition)}, \end{aligned} \tag{6.1.38}$$

where regarding the notation in (6.1.11) we said $N_L \equiv N$ and $N_R \equiv \bar{N}$. We achieve invariance of the total mass squared by requiring that both $\alpha' m_L^2$ and $\alpha' m_R^2$ remain invariant under the action of a soon-to-be-determined transformation. This transformation must then leave the square of each p^2 of (6.1.36) invariant, as well as the number operators.[22] The reader can check that this invariance can be achieved under the transformations [158]

$$\mathbf{n} \to \tilde{\mathbf{n}} = \mathcal{O}\,\mathbf{n} \quad , \quad \mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{O}^{-T}\mathcal{H}\mathcal{O}^{-1}, \tag{6.1.39}$$

where the matrices $\mathcal{O}$ ought to satisfy $\mathcal{O}^T\,\eta\,\mathcal{O} = \eta$. The requirement that the components $\tilde{\mathbf{n}}$ must still belong to $\mathbb{Z}$ leads us to the conclusion that

$$\mathcal{O} \in O(D, D; \mathbb{Z}). \tag{6.1.40}$$

At this stage we have given a concrete realization of how T-duality transformations should act on $\mathbb{T}^D$-compactified configurations by starting from the example for S^1. Of course, the case for S^1 is contained in this kind of compactification; by setting $D = 1$, the group $O(D, D; \mathbb{Z})$ delivers us two copies of $\mathbb{Z}_2$, as expected.

We will take the opportunity to address the *moduli space* of this compactified setup. Basically, the moduli space is the space of parameters that determines an specific theory. In our case, we see that the theory is determined by g_{ij} and b_{ij} already defined in (6.1.35). Put together, we find that there are $D(D+1)/2 + D(D-1)/2 = D^2$ linearly independent parameters that determine our theory. These parameters live in a D^2-dimensional parameter space. The question that comes now is, what is the space of inequivalent physical theories? It can be proven that all possible self-dual lattices $\Gamma_{D,D}$ can be obtained from an $O(D, D; \mathbb{Z})$ rotation of some reference lattice Γ_0. But since the mass spectrum is invariant under independent rotations of the lattice momenta in (6.1.34) under either $O(D; \mathbb{Z})_L$ or $O(D; \mathbb{Z})_R$, then at first it seems that the moduli space is actually given by [115]

$$\mathcal{M}_0 = \frac{O(D, D; \mathbb{Z})}{O(D; \mathbb{Z}) \times O(D; \mathbb{Z})}. \tag{6.1.41}$$

[22]So far we are handling T-duality as a symmetry of the mass spectrum, meaning that we demand $p_{L,R}^2$ to be invariant only. In this respect, we can make it a duality symmetry between different CFTs. For that we require to write down the appropriate transformations for X and p.

However, taking into account the set of transformations $\mathcal{O} \in O(D, D; \mathbb{Z})$ that leave the mass square invariant, we need once more to divide by $O(D, D; \mathbb{Z})$ in order to distinguish the inequivalent physical configurations. Hence, the true moduli space is given by

$$\mathcal{M}_1 = \frac{O(D, D; \mathbb{R})}{O(D; \mathbb{R}) \times O(D; \mathbb{R})} \Big/ O(D, D; \mathbb{Z}). \tag{6.1.42}$$

Basic transformations

Having the $O(D, D; \mathbb{Z})$ group at our disposal opens a way for explicitly examining some specific transformations upon the g and b background fields, which can be read from $\tilde{\mathcal{H}}$. Before we move on towards the next section, we will state the basic transformations which will be used throughout our treatment for the open string. Note that we have not yet discussed the effect of these transformations on the momenta and bosonic coordinates – we will rather approach this matter concretely once we work out explicit examples in the next chapter. These transformations are further covered in the existing literature for instance in the review articles [166, 158] and in [167, 120], where in the latter the case for the heterotic string is thoroughly explored. We have then

1. **Diffeomorphisms:** Let $\mathsf{A} \in GL(D; \mathbb{Z})$. Diffeomorphism transformations are parametrized by a $2D \times 2D$ matrix $\mathcal{O}_A$ given by

$$\mathcal{O}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \mathsf{A}^{-T} \end{pmatrix}. \tag{6.1.43}$$

   This transformation belongs to the so-called geometric group.

2. **Integer B parameter shifts:** Let B be a $D \times D$ antisymmetric matrix with integer entries. This $2D \times 2D$ matrix $\mathcal{O}_{\mathsf{B}}$ given by

$$\mathcal{O}_{\mathsf{B}} = \begin{pmatrix} 1 & 0 \\ \mathsf{B} & 1 \end{pmatrix} \tag{6.1.44}$$

   shifts the b matrix as $b \to b + \alpha' \mathsf{B}$. Notice that if $\mathsf{B} = \mathrm{d}\Lambda$ (with Λ properly well defined) then this can be regarded as a gauge transformation for b. This transformation belongs to the geometric group as well.

3. **Factorized dualities:** Let D_i be a $D \times D$ matrix with null entries in all of its components except at the ii entry. We find that the $2D \times 2D$ matrices $\mathcal{O}_{\mathsf{D}_{\pm i}}$ given by

$$\mathcal{O}_{\mathsf{D}_{\pm i}} = \begin{pmatrix} 1 - \mathsf{D}_i & \pm\mathsf{D}_i \\ \pm\mathsf{D}_i & 1 - \mathsf{D}_i \end{pmatrix} \tag{6.1.45}$$

give us the generalization of the $R \to \alpha'/R$ circle duality. We mention that this kind of transformation is not a symmetry of the action, but a duality transformation instead.

4. **β-transformations:** Let's consider an antisymmetric $D \times D$ matrix β with integer entries. Consider now the transformations generated by the $2D \times 2D$ matrix $\mathcal{O}_\beta$ given by

$$\mathcal{O}_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{6.1.46}$$

As it happens with the factorized dualities, these transformations are not a symmetry of the action, but a duality transformation as well.

In the next section will explore the formalism for T-duality transformations whenever we find a curved background for our propagating string.

6.1.4 Buscher rules

The $\sigma-$model (6.1.21) offers the possibility to study intricate background configurations with fields that might depend on the coordinates of the target space X^μ. On the other hand, the cases we have explored so far for the S^1 and $\mathbb{T}^D$ have brought us insight about the structure of T-duality transformations for flat backgrounds. Can we extend our discussion and study T-duality transformations when we have an arbitrary, curved background then? The answer is yes. Instead of studying the conformal field theory for the curved case, we turn to study the Buscher's rules.

Such rules were first introduced in his papers [168, 169] and dealt with the case of performing a T-duality transformation along one direction only. In this doctoral work we will introduce instead the Buscher's rules for multiple directions right away. Our background configuration must first satisfy a set of conditions which we will mention later on. If these conditions are met we introduce then into the action a set of Lagrange multipliers and worldsheet gauge fields. We will see that the T-dual configuration can be reached finally by integrating-out the worldsheet gauge fields. The Lagrange multipliers are interpreted as coordinates of the T-dual target space.

The following discussion will take notation from [170] and marks the first approach to our work done in [171], with the difference that during the next sections we will address the gauging procedure taking in consideration the Wess-Zumino-Witten term.

The worldsheet action and its symmetries

We consider a slightly different $\sigma-$model action from the one showed in (6.1.21), where we work with the field intensity H instead of B[23]. Let's consider a compact

[23]By doing this, we avoid some issues regarding the well-definedness of B on the worldsheet itself, as can be seen in [172].

3−dimensional Euclidean worldsheet Ω such that $\partial\Omega = \Sigma$ is a 2−dimensional worldsheet. We consider the set of parameters σ^{a} to describe either Σ or Ω, when the situation requires it.We confine the metric G and dilaton ϕ to be on Σ and the Wess-Zumino-Witten (WZW) term H to be on Ω[24] We will consider the quantization condition for H

$$\frac{1}{2\pi\alpha'}\int_{\Omega} H \in 2\pi\mathbb{Z}. \tag{6.1.47}$$

Our action takes the form

$$\begin{aligned} S_{\text{WZW}} &= S_1 + S_2 \\ &= -\frac{1}{2\pi\alpha'}\int_{\Sigma}\left[\frac{1}{2!}G_{ij}\,\mathrm{d}X^i \wedge *\mathrm{d}X^j + \frac{\alpha'}{2}\mathsf{R}\,\phi * 1\right] \\ &\quad -\frac{i}{2\pi\alpha'}\int_{\Omega}\frac{1}{3!}H_{ijk}\,\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k, \end{aligned} \tag{6.1.48}$$

where the latin indices run as $i, j, k = 1, \dots, D$.

As we mentioned before, the gauging procedure requires for the background to satisfy some requirements. Let's consider an infinitesimal transformation on the coordinates X^i of the form

$$\delta_\epsilon X^i = \epsilon^\alpha k^i_\alpha, \tag{6.1.49}$$

where k^i_α are the components of the Killing vectors k_α, $\alpha = 1, \dots, N$ and ϵ^α are infinitesimal constant parameters. Here, the Killing vectors satisfy the Lie algebra

$$[k_\alpha, k_\beta] = f_{\alpha\beta}{}^{\gamma} k_\gamma. \tag{6.1.50}$$

We find that the action (6.1.48) is invariant under the transformations (6.1.49) if the background fields G, ϕ and H all satisfy the conditions

$$\mathcal{L}_{k_\alpha} G = 0, \qquad \mathcal{L}_{k_\alpha} H = 0, \qquad \mathcal{L}_{k_\alpha}\phi = 0, \tag{6.1.51}$$

where $\mathcal{L}_\xi$ is the Lie derivative $\mathcal{L}_\zeta = \mathrm{d}\,\iota_\zeta + \iota_\zeta\,\mathrm{d}$ acting on $p-$forms along a vector field ζ and ι_ξ is the contraction operator acting on $p-$forms along a vector field ξ. If all of the previous conditions are met, we say that the action features a global symmetry generated by the parameters ϵ. The next step is to promote these global symmetries to be local, and this can be achieved if we introduce worldsheet gauge fields.

We promote now the parameter ϵ^α to depend on the worldsheet parameters, that is $\epsilon^\alpha = \epsilon^\alpha(\sigma^{\mathsf{a}})$. Let us concretize the aforementioned conditions and say

$$\mathcal{L}_{k_\alpha} G = 0, \qquad \iota_{k_\alpha} H = \mathrm{d}v_\alpha, \qquad \mathcal{L}_{k_\alpha}\phi = 0, \tag{6.1.52}$$

[24] If H is not exact, then the Kalb-Ramond potential B satisfying $H = \mathrm{d}B$ can only be found locally and S_2 in (6.1.48) depends on the choice of manifold Ω.

which still satisfy (6.1.51) taking into account that $\mathrm{d}H = 0$. Here, $v_\alpha = v_\alpha(X)$ corresponds to a 1−form on Σ. On top of this, we introduce gauge fields A^α and Lagrange multipliers χ_α into the action (6.1.48) in the following way

$$\begin{aligned}\widehat{S}_{\text{WZW}} = &-\frac{1}{2\pi\alpha'}\int_\Sigma \left[\frac{1}{2!}G_{ij}\,(\mathrm{d}X^i + k^i_\alpha A^\alpha)\wedge *(\mathrm{d}X^j + k^j_\beta A^\beta) + \frac{\alpha'}{2}\mathsf{R}\,\phi * 1\right]\\ &-\frac{i}{2\pi\alpha'}\int_\Sigma \left[(v_\alpha + \mathrm{d}\chi_\alpha)\wedge A^\alpha + \frac{1}{2}(\iota_{[k_{\underline{\alpha}}}v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma\chi_\gamma)\,A^\alpha\wedge A^\beta\right]\\ &-\frac{i}{2\pi\alpha'}\int_\Omega \frac{1}{3!}H_{ijk}\,\mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k.\end{aligned} \qquad (6.1.53)$$

In here, A^α and χ_α are quantities which depend on the worldsheet parameters σ^{a} only.

It can be shown that the action (6.1.53) has a local symmetry provided that the conditions (6.1.52) are met along with the infinitesimal transformations and constraints

$$\begin{aligned}\delta_\epsilon X^i &= +\epsilon^\alpha k^i_\alpha\\ \delta_\epsilon A^\alpha &= -\mathrm{d}\epsilon^\alpha - \epsilon^\beta A^\gamma\,f_{\beta\gamma}{}^\alpha\\ \delta_\epsilon\chi_\alpha &= -\iota_{k_{(\overline{\alpha}}}v_{\overline{\beta})}\epsilon^\beta - f_{\alpha\beta}{}^\gamma\epsilon^\beta\chi_\gamma,\end{aligned} \qquad (6.1.54)$$

$$\mathcal{L}_{k_{[\underline{\alpha}}}v_{\underline{\beta}]} = f_{\alpha\beta}{}^\gamma v_\gamma, \qquad \iota_{k_{[\underline{\alpha}}}f_{\underline{\beta\gamma}]}{}^\delta v_\delta = \frac{1}{3}\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma}H. \qquad (6.1.55)$$

We have introduced worldsheet gauge fields in order to promote a global symmetry to a local symmetry. By doing this, we need to ensure that we haven't introduced additional degrees of freedom into the action. The Lagrange multipliers carry through their equation of motion the constraint that the worldsheet gauge fields A^α must satisfy, and in virtue of them we need to show that we can get back our original action (6.1.48). We show this in the next section.

Recovering the ungauged action

Our discussion must take a short digression and consider the Hodge decomposition[25] for $\mathrm{d}\chi_\alpha$. Since $\mathrm{d}\chi_\alpha$ is a closed form, it allows a decomposition in its exact and harmonic part as follows

$$\mathrm{d}\chi_\alpha = \mathrm{d}\chi_\alpha^{(0)} + \sum_{m=1}^{2g}\chi_\alpha^{(m)}\omega^m. \qquad (6.1.56)$$

Here we have used a basis of harmonic one-forms of Σ denoted by ω^m such that $\omega^m \in H^1(\Sigma,\mathbb{R})$ and $m = 1,\ldots,2g$, where g is the genus of Σ. We find the coefficients

[25] For a discussion on the Hodge decomposition theorem for p-forms, see for instance Theorem 7.7 of [173].

$\chi_\alpha^{(m)}$ to be real-valued. The one-forms ω^m are dual to the cycles γ_n, that is

$$\int_{\gamma_n} \omega^m = \delta_n^m. \tag{6.1.57}$$

We find as well that the forms ω^n satisfy

$$\int_\Sigma \omega^m \wedge \omega^n = J^{mn}, \tag{6.1.58}$$

where J^{mn} are the entries of a non-degenerate matrix [174]. To gain information about A^α we require to study both the equations of motion for the exact part and harmonic part of χ_α. Taking (6.1.53), we find the equations of motion for $\chi_\alpha^{(0)}$ to be

$$0 = \frac{i}{2\pi\alpha'} \int_\Sigma \delta\chi_\alpha^{(0)} \left[\mathrm{d}A^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \right]. \tag{6.1.59}$$

We consider the case for an abelian isometry algebra, i.e. $f_{\alpha\beta}{}^\gamma = 0$. The equations then tell us that A^α is a closed form, i.e. $\mathrm{d}A^\alpha = 0$ and in turn we can decompose A^α as follows.

$$A^\alpha = \mathrm{d}A^\alpha_{(0)} + \sum_{m=1}^{2g} A^\alpha_{(m)} \omega^m. \tag{6.1.60}$$

Let us derive now the equations of motion for $\chi_\alpha^{(m)}$. Taking the assumption that the isometry algebra is abelian we find that the relevant term $\int_\Sigma \mathrm{d}\chi_\alpha \wedge A^\alpha$ in the action (6.1.53) can be written using (6.1.56), (6.1.58) and (6.1.60) as follows

$$\begin{aligned} 2\pi i\alpha' S_{\text{rel}} &= \int_\Sigma \mathrm{d}\chi_\alpha \wedge A^\alpha \\ &= \int_\Sigma \left[\mathrm{d}\chi_\alpha^{(0)} + \sum_{m=1}^{2g} \chi_\alpha^{(m)} \omega^m \right] \wedge \left[\mathrm{d}A^\alpha_{(0)} + \sum_{n=1}^{2g} A^\alpha_{(n)} \omega^n \right] \\ &= \int_\Sigma \left[\sum_{m=1}^{2g} \chi_\alpha^{(m)} \omega^m \wedge \mathrm{d}A^\alpha_{(0)} \right] + \sum_{m,n=1}^{2g} \chi_\alpha^{(m)} A^\alpha_{(m)} J^{mn} \\ &\quad + \big(\text{terms depending on } \chi_\alpha^{(0)}\big). \end{aligned} \tag{6.1.61}$$

We dismiss the first term in the third line of (6.1.61) since it is a total derivative and does not contribute to the action. By performing the variation with respect to $\chi_\alpha^{(m)}$ we find that $A^\alpha_{(m)} = 0$, hence A^α is pure gauge. Finally, with help of the transformation for A^α in (6.1.54) we can set $A^\alpha = 0$ and recover finally the original, ungauged action (6.1.48).

The case for non-abelian isometry algebras proves to be a bit more involved. For instance, in [170] one can go back to the original action by taking $\mathrm{D}X^i \equiv \mathrm{d}X^i + k^i_\alpha A^\alpha$

and perform a change of basis in such a way that $\mathrm{D}X^i$ is closed, identify them as as set of vielbein valuated in a different set of coordinates Y^i, and finally express the gauged action in terms of $p-$forms valuated in the basis of $1-$forms $\mathrm{d}Y^i$, thus recovering the original action. However, this can be done since the worldsheet is considered to have an empty boundary, whereas the case for worldsheets with non-empty boundary cannot be properly addressed by means of the Hodge decomposition alone.

Dual action

We have treated the procedure to recover the original action from the gauged one. Now it is moment to take the next step and treat the procedure to find the dual action whenever T-duality transformations along different directions are done. The case for one T-duality transformation has become part of the canon of the standard textbooks, whereas the case for T-duality transformations along multiple directions has been done for instance in [172, 170, 175], with approaches from the point of view of doubled geometry in [176], and from the point of view of doubled field theory – in here, T-duality is treated as a symmetry of a field theory by doubling the configuration space (see for instance the review [177]) We won't include bla bla . This will be properly addressed in the next chapter of this work.

Let's retake our gauged action (6.1.53). To reach the dual model, we ought to integrate out the worldsheet gauge fields A^α. This can be done by taking the equations of motion related to A^α and replacing them directly into the action (6.1.53). Considering now the equations of motion given by the variation with respect to A^α we find the equation of motion

$$0 = *k_\alpha + \mathcal{G}_{\alpha\beta}A^\beta - i\,\xi_\alpha + i\,\mathcal{D}_{\alpha\beta}A^\beta, \tag{6.1.62}$$

where we have defined the following quantities

$$\begin{aligned} \mathsf{k}_\alpha &= k^i_\alpha G_{ij}\mathrm{d}X^j, & \mathcal{G}_{\alpha\beta} &= k^i_\alpha G_{ij} k^j_\beta, \\ \xi_\alpha &= v_\alpha + \mathrm{d}\chi_\alpha, & \mathcal{D}_{\alpha\beta} &= \iota_{[k_{\underline{\alpha}}} v_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma \chi_\gamma. \end{aligned} \tag{6.1.63}$$

Since the EOMs (6.1.62) do not involve derivatives of A^α, we can solve for A^α algebraically with help of the Hodge dual. We find in obvious matrix notation

$$A^\alpha = -\left[\left(\mathcal{G} - \mathcal{D}\mathcal{G}^{-1}\mathcal{D}\right]\right)^{\alpha\beta}\left(1 + i * \mathcal{D}\mathcal{G}^{-1}\right)_\beta{}^\gamma (\mathsf{k} + i * \xi)_\gamma\,. \tag{6.1.64}$$

Even though we have to work with the inverse of $\mathcal{G}$ for a while, we'll see later on that the construction of the dual model relies in the invertibility of $(\mathcal{G} \pm \mathcal{D})$.

Placing this last expression into the action (6.1.53) we obtain the dual action $\check{S}_{\mathrm{WZW}}$

$$\begin{aligned} \check{S}_{\mathrm{WZW}} = &-\frac{1}{2\pi\alpha'}\int_\Sigma \left[\check{G} + \frac{\alpha'}{2}\mathsf{R}\check{\phi}\right] \\ &-\frac{i}{2\pi\alpha'}\int_\Omega \check{H}, \end{aligned} \tag{6.1.65}$$

where the dual metric $\check{G}$ and the dual $H-$field $\check{H}$ are given by

$$\begin{aligned}\check{G} &= G - \frac{1}{2}(\mathsf{k}+\xi)_\alpha \left[(\mathcal{G}+\mathcal{D})^{-1}\right]^{\alpha\beta} \wedge * \, (\mathsf{k}-\xi)_\beta,\\ \check{H} &= H - \mathrm{d}\Big[\frac{1}{2}(\mathsf{k}+\xi)_\alpha \left[(\mathcal{G}+\mathcal{D})^{-1}\right]^{\alpha\beta} \wedge (\mathsf{k}-\xi)_\beta\Big].\end{aligned} \tag{6.1.66}$$

In here, we have of course

$$\begin{aligned}G &= \frac{1}{2!}G_{ij}\,\mathrm{d}X^i \wedge * \mathrm{d}X^j,\\ H &= \frac{1}{3!}H_{ijk}\,\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k,\end{aligned} \tag{6.1.67}$$

and a matrix multiplication being carried on the $\alpha, \beta = 1, \ldots, N$ indices. This points out that the new dual fields $\check{G}$ and $\check{H}$ are actually defined on an enlarged $(D+N)-$dimensional target space whose local coordinates are $\{X^i, \chi_\alpha\}$. As a final comment, the feasibility on reaching such dual fields relies whether the matrix combination $(\mathcal{G} \pm \mathcal{D})$ is invertible.

Notice that we haven't talked about $\check{\phi}$ so far. In order to do this and talk about non-geometric spaces in our upcoming sections, we must find first an appropriate change of basis. We set off to this task right away.

Change of basis

To find an appropriate change of basis, we first notice that the vector fields $\check{n}_\alpha$ given in components

$$\check{n}_\alpha = \begin{pmatrix} k^i_\alpha \\ \mathcal{D}_{\alpha\beta}\end{pmatrix} \tag{6.1.68}$$

conform a set of N null eigenvectors of $\check{G}$ and $\check{H}$, i.e. $\iota_{\check{n}_\alpha}\check{G} = 0$ and $\iota_{\check{n}_\alpha}\check{H} = 0$. Assuming for a moment that $\det k^\alpha_\beta \neq 0$, we can indeed find an useful change of basis matrix $\mathcal{T}$. Let us label collectively our extended basis of 1−forms by $\mathrm{d}X^A = \{\mathrm{d}X^i, \mathrm{d}\chi_\alpha\}$ and write accordingly

$$\begin{aligned}\check{G} &= \frac{1}{2!}\check{G}_{AB}\,\mathrm{d}X^A \wedge * \mathrm{d}X^B,\\ \check{H} &= \frac{1}{3!}\check{H}_{ABC}\,\mathrm{d}X^A \wedge \mathrm{d}X^B \wedge \mathrm{d}X^C.\end{aligned} \tag{6.1.69}$$

The components $\mathcal{T}^A{}_B$ of the change of basis matrix can be written as

$$\mathcal{T}^A{}_B = \left(\begin{array}{c|c|c} \multirow{2}{*}{k^i_α} & 0_{N\times(D-N)} & 0_{N\times N} \\ \cline{2-3} & 1_{(D-N)\times(D-N)} & 0_{(D-N)\times N} \\ \hline \mathcal{D}_{\alpha\beta} & 0_{N\times(D-N)} & 1_{N\times N}\end{array}\right), \tag{6.1.70}$$

where $0_{M\times N}$ and $1_{M\times N}$ denote the $M\times N$ null and identity matrix, respectively. The new 1−form basis corresponds to $e^A=(\mathcal{T}^{-1})^A{}_B\mathrm{d}X^B$, which can be split conveniently as $e^A=\{e^\alpha,e_\alpha,e^m\}$, where

$$\begin{aligned} e^\alpha &= (k^{-1})^\alpha{}_\beta\,\mathrm{d}X^\beta,\\ e_\alpha &= \mathcal{D}_{\alpha\beta}\,e^\beta+v_\alpha+\mathrm{d}\chi_\alpha,\\ e^m &= -k^m{}_\beta\,e^\beta+\mathrm{d}X^m. \end{aligned} \tag{6.1.71}$$

In here, $\alpha,\beta=1,\dots,N$, $m,n=N+1,\dots,D$ and $(k^{-1})^\alpha_\beta$ denotes the inverse of the matrix with components $k^\alpha{}_\beta$[26]. In general, each family of vielbein is not closed. Indeed, one can show that $\mathrm{d}e^A$ reads as

$$\begin{aligned} \mathrm{d}e^\alpha &= -\frac{1}{2}f_{\beta\gamma}{}^\alpha\,e^\beta\wedge e^\gamma-(k^{-1})^\alpha{}_\gamma\partial_m k^\gamma{}_\beta e^m\wedge e^\beta,\\ \mathrm{d}e_\alpha &= -f_{\alpha\beta}{}^\gamma\,e^\beta\wedge e_\gamma\\ &\quad+\big[\partial_m\iota_{k_{(\overline{\alpha}}}v_{\overline{\beta})}-(k^{-1})^\gamma{}_\delta\,\partial_m k^\delta{}_\beta\big(\iota_{k_{(\overline{\alpha}}}v_{\overline{\gamma})}+f_{\alpha\gamma}{}^\sigma\chi_\sigma\big)\big]\,e^m\wedge e^\beta,\\ \mathrm{d}e^m &= \big[k^m{}_\beta\,(k^{-1})^\beta{}_\gamma\,\partial_n k^\gamma{}_\alpha-\partial_n k^m{}_\alpha\big]\,e^n\wedge e^\alpha. \end{aligned} \tag{6.1.72}$$

The exterior derivative of a vielbein basis $\{e^A\}$ provides us with the structure constants $f_{BC}{}^A$ associated to the Lie algebra of the vector fields dual to $\{e^A\}$. This can be done if we assume a torsionless connection Γ on the manifold. The structure constants can be read as

$$\mathrm{d}e^A=-\frac{1}{2}f_{BC}{}^A\,e^B\wedge e^C. \tag{6.1.73}$$

This determines whether the vielbein bases $\{e^\alpha\}$, $\{e_\alpha\}$, and $\{e^m\}$ mix under the exterior derivative.

We see that in general the vielbein algebra does not close on itself under the exterior derivative. However, one can simplify this discussion and have $\mathrm{d}e^A=0$ if we work with an abelian isometry algebra ($f_{\alpha\beta}{}^\gamma=0$), with a v_α form such that $\iota_{k_{(\overline{\alpha}}}v_{\overline{\beta})}$ are constants and with a coordinate system such that $k^m{}_\alpha=0$.

On the other hand, the components of $\check{G}$ change as

$$\check{G}_{AB}\to\check{\mathsf{G}}_{AB}=\check{G}_{CD}\mathcal{T}^C{}_A\mathcal{T}^D{}_B, \tag{6.1.74}$$

while the components of $\check{H}$ change accordingly as

$$\check{H}_{ABC}\to\check{\mathsf{H}}_{ABC}=\check{H}_{CDE}\mathcal{T}^C{}_A\mathcal{T}^D{}_B\mathcal{T}^E{}_C. \tag{6.1.75}$$

[26] This can be done if the isometry group has no fixed point or only isolated points. However, if the isotropy of the isometry group is non-trivial, then the matrix of components k^α_β is not invertible (see for instance [178]) and we are required to find another suitable basis.

Under this change of basis, we find that the target-space metric can be written down as

$$\begin{aligned} \check{G} &= \frac{1}{2!}\check{\mathsf{G}}_{AB}\, e^A \wedge * e^B, \\ &= \frac{1}{2!}\check{\mathsf{G}}_{\mathsf{ab}}\, e^{\mathsf{a}} \wedge * e^{\mathsf{b}}, \end{aligned} \tag{6.1.76}$$

where $e^{\mathsf{a}} = \{e_\alpha, e^m\}$ and of course $\mathsf{a}, \mathsf{b} = 1, \ldots, D$. The reason for this is that $\check{G}$ possesses N null eigenvectors, which have been included in $\mathcal{T}$. The components $\check{\mathsf{G}}_{\mathsf{ab}}$ can be written down as follows

$$\begin{aligned} \check{\mathsf{G}}_{mn} =& G_{mn} - \mathsf{k}_{\alpha m}\big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{G}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta}\mathsf{k}_{\beta n} \\ & - \mathsf{k}_{\alpha m}\big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{D}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta} v_{\beta n} \\ & + v_{\alpha m}\big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{D}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta}\mathsf{k}_{\beta n} \\ & + v_{\alpha m}\big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{G}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta} v_{\beta n}, \\ \check{\mathsf{G}}_m{}^{\beta} =& - \mathsf{k}_{\alpha m}\big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{D}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta} \\ & + v_{\alpha m}\Big[\big(\mathcal{G}+\mathcal{D}\big)^{-1}\ \mathcal{G}\ (\mathcal{G}-\mathcal{D})^{-1}\,\Big]^{\alpha\beta}, \\ \check{\mathsf{G}}^{\alpha}{}_n =& + \big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{D}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta}\mathsf{k}_{\beta n} \\ & + \big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{G}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta} v_{\beta n}, \\ \check{\mathsf{G}}^{\alpha\beta} =& + \big[\, (\mathcal{G}+\mathcal{D})^{-1}\ \mathcal{G}\ (\mathcal{G}-\mathcal{D})^{-1}\,\big]^{\alpha\beta}. \end{aligned} \tag{6.1.77}$$

These correspond to the target-space metric components of the dual background after performing N collective T-duality transformations. We see that this transformations modify portions of the original target-space metric G in a non-trivial way, while generating completely new ones. We notice that this metric has no legs in the e^α vielbein as well, which is in a way an indication that we have departed from the original background configuration.

In a similar way we can compute the components of the new dual $H-$field components via the recipe indicated in (6.1.75). In particular, it can be shown that these components do not have legs in the set of e^α forms as well, that is, $\check{\mathsf{H}}_{\alpha JK} = 0$.

Dual dilaton

The form of the dual dilaton $\check{\phi}$ has to be determined via a one-loop computation along the lines of [169]. Equivalently (as put for instance in [115]) we can compute $\check{\phi}$ by enforcing that the combination $e^{-2\phi}\sqrt{\det G}$ remains invariant under T-duality transformations, that is, that

$$e^{-2\phi}\sqrt{\det G} \stackrel{!}{=} e^{-2\check{\phi}}\sqrt{\det \check{\mathsf{G}}}. \tag{6.1.78}$$

Solving for $\check{\phi}$ we find the transformation rule for the dilaton field

$$\check{\phi} = \phi - \frac{1}{4}\log\left[\frac{\det G}{\det \check{\mathsf{G}}}\right] , \tag{6.1.79}$$

where $\check{\mathsf{G}}$ has been already determined in (6.1.77). Having the dual fields $\check{\mathsf{G}}$, $\check{\mathsf{H}}$ and $\check{\phi}$ determined in (6.1.77), (6.1.75) and (6.1.79) respectively, we are able to reach the dual background configuration $(\check{\mathsf{G}}, \check{\mathsf{H}}, \check{\phi})$.

Final remarks

The dual metric $\check{G}$ dictates which coordinates are meaningful on the target-space manifold. Remember that in the T-dual configuration we are working with a metric with no legs in the e^α vielbein forms. In particular, if $e^\alpha = \mathrm{d}X^\alpha$ it means that the T-dual quantities $\check{G}$ and $\check{H}$ should not depend on the set of coordinates X^α. To address this one can show that

$$\check{\mathcal{L}}_{\check{n}_\alpha}\check{G} = 0, \qquad \check{\mathcal{L}}_{\check{n}_\alpha}\check{H} = 0, \qquad \check{\mathcal{L}}_{\check{n}_\alpha}\phi = 0 \tag{6.1.80}$$

These isometries in the enlarged target-space allow us to ‟drag″these fields along the isometry directions without changing their values. In particular, we can drag them to a convenient point and evaluate them there. For the case in which $e^\alpha = \mathrm{d}X^\alpha$ we can do it at $X^\alpha = 0$ and therefore they no longer depend on the original coordinates X^α [159].

6.2 Non-Geometric Spaces

In section 6.1.1 we studied the case for the closed bosonic string propagating on a 26−dimensional target space with one compact dimension. By choosing the 25^{th}−direction to be compact, the expansion of X^{25} given by (6.1.6) not only presents a discretized momentum number p^{25}, but also features a new discretized quantity called winding number w. This quantity accounts for the number of times a string can wrap around this compact direction. Wrapping around a certain direction is a capability that points particles do not have, which in turn tells us that strings can probe spaces in ways point particles cannot do. This can be used to gain access to other kind of spaces.

Let us take the WZW model described by the Lagrangian (6.1.48). Let us then to the gauging procedure and read the dual action described by (6.1.65) and consider the Buscher rules for the metric and the H−flux given in (6.1.77) and (6.1.75). By performing a T-duality transformation – that is, by gauging either along one or more directions – we actually map the background configuration described by the fields (G, H, ϕ) to a dual configuration $(\check{\mathsf{G}}, \check{\mathsf{H}}, \check{\phi})$. When it comes to the metric, it happens that the new background geometry described by $\check{\mathsf{G}}$ is no longer the same as the one described by G; even the topology could end up being totally different. For these geometries not even diffeomorphisms are enough to have a globally well-defined description. These kind of spaces are called *non-geometric spaces*.

To illustrate the points above we will take the WZW model and introduce the standard example of the three-torus ($\mathbb{T}^3$) with constant $H-$flux background, and keep the dilaton constant. We will perform T-duality transformations along one, two and three directions and discuss each of the new backgrounds we get. The way to proceed can be enlisted as follows:

1. Construct the starting background by defining explicitly the metric G, the $H-$flux H and the dilaton ϕ. In particular, the background metric must present isometries accounted by the Killing vectors k_α. These fields must satisfy the constraints (6.1.52).

2. Proceed to construct the gauged action as in (6.1.53) and find a 1−form v_α such that satisfies both (6.1.52) and the set of constraints (6.1.55).

3. Integrate-out the worldsheet gauge fields A^α, read the dual metric $\check{G}$, dual $H-$field $\check{H}$ and dual dilaton $\check{\phi}$ following (6.1.66) and (6.1.79). Perform the change of basis indicated by (6.1.71) and read the dual metric $\check{\mathsf{G}}$ belonging to the dual background $(\check{\mathsf{G}}, \check{\mathsf{H}}, \check{\phi})$.

We mention that several other non-geometric spaces can be studied, but for this doctoral work we will mention the T-duals of the $\mathbb{T}^3$ with $H-$flux only. For convenience, let us define

$$\mathrm{d}\check{\chi}_\alpha \equiv \frac{1}{\alpha'}\mathrm{d}\chi_\alpha. \tag{6.2.1}$$

The following section can be further deepened in [158].

6.2.1 $\mathbb{T}^3$ with $H-$flux

We describe the background configuration of a three-torus with $H-$flux through the metric G, the $H-$flux field $H = \mathrm{d}B$ and the dilaton field ϕ. By considering as well a basis of 1−forms $\{\mathrm{d}X^i\}$ for $i = 1, 2, 3$ we find that

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad H = \frac{\alpha'}{2\pi}\, h\, \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3 \,, \qquad \phi = \phi_0 \,, \tag{6.2.2}$$

where G_{ij} denotes the components of the metric G, $h \in \mathbb{Z}$ is constant and the dilaton has a constant value ϕ_0 as well. In here we find that the coordinates respect the identification $X^i \sim X^i + 2\pi$.

Let us denote the set of Killing vectors by k_α. The Killing vectors for the three-torus presented here satisfy an abelian Lie algebra, i.e. $[k_\alpha, k_\beta] = 0$ and we can write down the components of each vector in the basis $\{\partial_i\}$, $i = 1, 2, 3$ as follows

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.2.3}$$

Let us consider now a vielbein basis $\{e^a = e^a{}_i \mathrm{d}X^i\}$ for $a = 1, 2, 3$. In this case, the components of the vielbein $e^a{}_i$ correspond to those of a 3×3 matrix, and let $e_a{}^i$ correspond to the components of the inverse matrix. The vielbein components diagonalize the the metric components as $\delta_{ab} = e_a{}^i G_{ij} e_b{}^j$. We find that $e^a{}_i = \delta^a_i R_i$ with no summation convention.

In the mentioned vielbein basis, we find that the $H-$flux can be written as

$$H = \frac{\alpha'}{2\pi} \frac{h}{R_1 R_2 R_3} e^1 \wedge e^2 \wedge e^3. \tag{6.2.4}$$

We will take this background configuration and perform one, two and three simultaneous T-duality transformations along the isometry directions with help of the Buscher rules established in (6.1.75), (6.1.77) and (6.1.79).

6.2.2 Twisted torus

Let us take now the background configuration of the $\mathbb{T}^3$ with an $H-$flux described by (6.2.2) along with the identification $X^i \sim X^i + 2\pi$. We perform now a T-duality transformation along the direction of the Killing vector $k_1 = \partial_1$ by using the rules indicated in (6.1.77), (6.1.77) and (6.1.79).

By properly solving the constraints and going into adapted coordinates we can find the dual basis given by $e^{\mathsf{a}} \in \{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$. The metric components are given by

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \tag{6.2.5}$$

and the dual $H-$field and dilaton are

$$\begin{aligned} \check{\mathsf{H}} &= 0, \\ \check{\phi} &= \phi_0 - \log\left[\frac{R_1}{\sqrt{\alpha'}}\right]. \end{aligned} \tag{6.2.6}$$

This background is known as the twisted three-torus [140, 141]. Notice that in this background the flux $\check{\mathsf{H}}$ vanishes. We haven't lost information about the flux, however;

it has only migrated to the dual metric $\check{\mathsf{G}}$. We can recover it by studying the algebra of the vielbein $\tilde{e}^a$ that diagonalizes the metric. Consider then

$$\tilde{e}^1 = \frac{\alpha'}{R_1}\left[\mathrm{d}\check{\chi}_1 - \frac{h}{2\pi}X^3\mathrm{d}X^2\right], \quad \tilde{e}^2 = R_2\mathrm{d}X^2, \quad \tilde{e}^3 = R_3\mathrm{d}X^3. \tag{6.2.7}$$

One can check right away that $\check{\mathsf{G}}_{ij} = \tilde{e}^a_i\delta_{ab}\tilde{e}^b_j$. The structure constants $f_{bc}{}^a$ can be read from

$$\mathrm{d}\tilde{e}^a = \frac{1}{2}f_{bc}{}^a\,\tilde{e}^b\wedge\tilde{e}^c. \tag{6.2.8}$$

We find that the only non-vanishing structure constant component is $f_{23}{}^1$ and it is given by

$$f_{23}{}^1 = \frac{\alpha'}{2\pi}\frac{h}{R_1R_2R_3}. \tag{6.2.9}$$

This is the so-called $f-$flux, and since it can be extracted from the vielbein $\tilde{e}^a$ it corresponds to a *geometric flux.* The matter of global well-definedness of this space will be treated in the next section on toroidal fibrations.

6.2.3 T-fold

Now we will discuss a dual background coming from T-duality transformations performed along two directions indicated by the Killing vectors $k_1 = \partial_1$ and $k_2 = \partial_2$. By following the procedure indicated at the beginning of this section, we can find a dual basis $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}X^3\}$ and the dual background fields $(\check{\mathsf{G}}, \check{\mathsf{H}}, \check{\phi})$. The components of $\check{\mathsf{G}}$ are given by

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2R_2^2}{R_1^2R_2^2+\left[\frac{\alpha'}{2\pi}hX^3\right]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2R_1^2}{R_1^2R_2^2+\left[\frac{\alpha'}{2\pi}hX^3\right]^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \tag{6.2.10}$$

and the $\check{H}$ and $\check{\phi}$ field is given by

$$\begin{aligned} \check{\mathsf{H}} &= -\frac{h}{2\pi}\frac{\alpha'}{\rho^2}\left(\frac{R_1^2R_2^2}{\alpha'^2} - \left[\frac{hX^3}{2\pi}\right]^2\right)\mathrm{d}\check{\chi}_1\wedge\mathrm{d}\check{\chi}_2\wedge\mathrm{d}X^3, \\ \check{\phi} &= \phi_0 - \frac{1}{2}\log(\rho), \end{aligned} \tag{6.2.11}$$

where we have defined ρ as

$$\rho = \frac{R_1^2R_2^2}{\alpha'^2} + \left[\frac{hX^3}{2\pi}\right]^2. \tag{6.2.12}$$

This corresponds to the T-fold background [121]. The computation of this background marks a departure from what we call *geometric background*; in order to have a proper global description, we must ensure that going once around the circle along X^3 we come back to the same point. If we can do this up to a diffeomorphism transformation, then we would be successful in providing a geometrical description. However, there is no diffeomorphism transformation that can relate $\check{\mathsf{G}}|_{X^3=0}$ with $\check{\mathsf{G}}|_{X^3=2\pi}$. We will come to this issue in the following section.

We want to know now where the information of the original $H-$flux (6.2.4) went. For this, we require to know $\check{\mathsf{B}}$ such that $\check{\mathsf{H}} = \mathrm{d}\check{\mathsf{B}}$. We find that – up to a gauge transformation – the components of $\check{\mathsf{B}}$ can be written as

$$\check{\mathsf{B}}_{IJ} = \alpha'^2 \begin{pmatrix} 0 & \frac{-\frac{\alpha'}{2\pi}hX^3}{R_1^2R_2^2+\left[\frac{\alpha'}{2\pi}hX^3\right]^2} & 0 \\ \frac{+\frac{\alpha'}{2\pi}hX^3}{R_1^2R_2^2+\left[\frac{\alpha'}{2\pi}hX^3\right]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.2.13}$$

Let us now consider a certain metric g^{ij} and a bivector field β^{ij}, both arising from the following computation

$$\left[\check{\mathsf{G}} \pm \check{\mathsf{B}}\right]^{-1} = \mathsf{g} \pm \beta, \tag{6.2.14}$$

with g and β corresponding to the symmetric and antisymmetric parts of such matrix inverse. With this, we can compute the components of the $Q-$flux

$$Q_i{}^{jk} = \partial_i \beta^{jk}. \tag{6.2.15}$$

Taking then the dual metric in (6.2.10), the dual $B-$field in (6.2.13) and following the aforementioned steps for computing the $Q-$flux we find that the non-vanishing component is given by

$$Q_3{}^{12} = \frac{\alpha'}{2\pi}\frac{h}{R_1R_2R_3}. \tag{6.2.16}$$

Notice that the origin of this flux comes neither from the vielbein that diagonalizes $\check{\mathsf{G}}$ nor from the dual $H-$flux $\check{\mathsf{H}}$, quantities which are nor globally well-defined under diffeomorphism transformations; in the literature the $Q-$flux is referred as a *non-geometric flux*.

6.2.4 R space

We have reviewed spaces that comes from one and two T-duality transformations along the isometry directions of the flat three-torus $\mathbb{T}^3$ with a constant $H-$flux, being the twisted three-torus and the T-fold, respectively. The possibility of performing three

T-duality transformations on the $\mathbb{T}^3$ with constant H by making use of the Buscher rules is rather suggestive. However, a closer examination on the conditions (6.1.55) reveal that such procedure is forbidden. Indeed, since the isometry algebra is abelian, i.e. $f_{\alpha\beta}{}^{\gamma} = 0$ we find from (6.1.55) that

$$0 \overset{!}{=} \frac{1}{3}\iota_{k_1}\iota_{k_2}\iota_{k_3}H = \frac{1}{3}\frac{h}{2\pi}, \tag{6.2.17}$$

which immediately forces $h = 0$. Thus, performing three T-duality transformations on the three-torus with $H-$flux is forbidden, unless we turn off the $H-$flux by imposing $h = 0$.

The application of successive T-duality transformations on our background suggests that we have still a flux to be found. Indeed, by considering the decomposition (6.2.14) we can get an expression for the so-called *non-geometric $R-$flux* written with respect to the coordinate basis

$$R^{ijk} = 3\beta^{[\underline{i}m}\partial_m\beta^{\underline{jk}]}. \tag{6.2.18}$$

By employing the aformentioned definition, we see that the only non-vanishing component for R^{ijk} is

$$R^{123} = \frac{\alpha'}{2\pi}\frac{h}{R_1R_2R_3}. \tag{6.2.19}$$

The $R-$space is another example of a non-geometric space [143]. It is an example of a space in which we fail to provide even a local Riemannian geometrical description; in the next chapter we will argue that not even the description of a point in terms of Riemann geometry can be done.

6.3 Toroidal fibrations

In the previous section we discussed about different spaces that stem from the one, two and three T-duality transformations on the T^3 with constant $H-$flux. In this section we will review how we can study the matter of global well-definedness for the $\mathbb{T}^3$ with $H-$flux, the twisted three-torus and the T-fold. Please note that we will only discuss fibrations defined on S^1; it is possible to explore fibrations on other spaces, but we won't develop on this.

$\mathbb{T}^2$ fibrations on S^1

In order to discuss fibrations over the circle S^1 and the well-definedness of the (non-)geometrical spaces featured in the last section, the construction of the generalized metric $\mathcal{H}$ found in (6.1.37) will prove to be useful. For each space, we are required –up to a gauge choice– to find $\check{\mathsf{B}}$ such that $\check{\mathsf{H}} = \mathrm{d}\check{\mathsf{B}}$.

Let us consider the $\mathbb{T}^3$ with $H-$flux, the twisted torus and the T-fold backgrounds presented in the previous section. For each of them, we can write the components of the dual metric $\check{\mathsf{G}}$ and the dual $B-$field $\check{\mathsf{B}}$ schematically as

$$\check{\mathsf{G}}_{\mathsf{ab}} = \begin{pmatrix} g_{\mathsf{ij}}(X^3) & 0 \\ 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{\mathsf{ab}} = \begin{pmatrix} b_{\mathsf{ij}}(X^3) & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.3.1}$$

where $\mathsf{i}, \mathsf{j} = 1, 2$ corresponds to an index system related to the fiber and $\mathsf{a}, \mathsf{b} = 1, 2, 3$. We will provide for each background the respective dual metric and dual $B-$field in the upcoming sections.

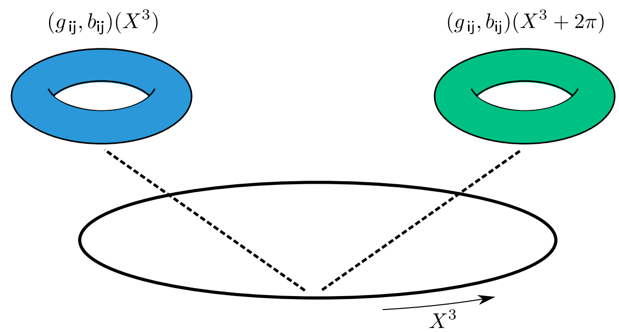

Figure 6.1: We can see toroidal fibrations over S^1 as tori defined on each point on a circle with parameter X^3. We can study the global behavior of such tori by transporting them once around this circle. For the backgrounds here presented, the patching between tori at X^3 and $X^3 + 2\pi$ can be done via $O(d, d; \mathbb{Z})$ transformations.

Taking inspiration from (6.1.37) and taking the metric and $B-$field indicated in (6.3.1) we construct the generalized metric $\mathcal{H}$ as follows

$$\mathcal{H} = \begin{pmatrix} -\frac{1}{\alpha'}\check{\mathsf{B}}\check{\mathsf{G}}^{-1}\check{\mathsf{B}} & \check{\mathsf{B}}\check{\mathsf{G}}^{-1} \\ -\check{\mathsf{G}}^{-1}\check{\mathsf{B}} & \alpha'\check{\mathsf{G}}^{-1} \end{pmatrix}. \tag{6.3.2}$$

Having the dual metric and dual $B-$field written as given in (6.3.1) delivers a more transparent picture of the following process: On top of each point of the circle parametrized by X^3 we define a toroidal space. Given that we haven't performed a T-duality transformation along X^3, we expect that under the shift $X^3 \to X^3 + 2\pi$ – that is, by transporting the $\mathbb{T}^2$ fiber around the circle – we can properly merge $\mathcal{H}$ at X^3 with $\mathcal{H}$ at $X^3 + 2\pi$. We will see that this is indeed possible, and furthermore, the merging is mediated by the $O(D, D; \mathbb{Z})$ transformations featured in section 6.1.3. These transformations contain gauge transformations on the $B-$field, diffeomorphisms and the so-called $\beta-$transformations.

The aim now is to study the behavior of $\mathcal{H}(X^3)$ under the shift $X^3 \to X^3 + 2\pi$ for each of the aforementioned spaces – except for the $R-$space.

$\mathbb{T}^3$ with $H-$flux

For this background configuration, we find that the components of the matrices g and b found in (6.3.1) can be written as

$$g_{\mathsf{ij}} = \begin{pmatrix} R_1^2 & 0 \\ 0 & R_2^2 \end{pmatrix}, \qquad b_{\mathsf{ij}} = \alpha' \begin{pmatrix} 0 & +\frac{hX^3}{2\pi} \\ -\frac{hX^3}{2\pi} & 0 \end{pmatrix}. \tag{6.3.3}$$

Let us now recall the set of transformations regarding a gauge transformation for the B field, indicated in (6.1.43). Let us take then the matrix $\mathcal{O}_{\mathsf{B}} \in O(2,2;\mathbb{Z}) \subset O(3,3;\mathbb{Z})$ denoted by

$$\mathcal{O}_{\mathsf{B}} = \begin{pmatrix} 1_{3\times3} & 0_{3\times3} \\ \mathsf{B} & 1_{3\times3} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 0 & +h & 0 \\ -h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.3.4}$$

It can be shown that the behavior of the generalized metric $\mathcal{H}$ in (6.3.2) – with g and b indicated in (6.3.3) – under the shift $X^3 \to X^3 + 2\pi$ can be expressed as

$$\mathcal{H}(X^3 + 2\pi) = \mathcal{O}_{\mathsf{B}}^{-T} \mathcal{H}(X^3) \mathcal{O}_{\mathsf{B}}^{-1}, \tag{6.3.5}$$

which indicates that by winding once around the X^3 direction of the circle we can patch consistently $(\check{\mathsf{G}}, \check{\mathsf{B}})$ at X^3 with $(\check{\mathsf{G}}, \check{\mathsf{B}})$ at $X^3 + 2\pi$ by performing a gauge transformation on the $B-$field. This is an example of a *geometrical* transformation.

The original identifications $X^i \sim X^i + 2\pi$ still hold, indicating that $\mathbb{T}^3$ is nothing more than the product of three circles, each with radius R_i.

Twisted three-torus

Now we take the background configuration of the twisted torus. Since the dual $H-$field in here vanishes, we can set a gauge such that $\check{\mathsf{B}} = 0$ and write the components of g and b found in (6.3.1) as follows

$$g_{\mathsf{ij}} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi} \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi} X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 \end{pmatrix}, \qquad b_{\mathsf{ij}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.3.6}$$

If we revolve around the base circle once we find that we can patch $(\check{\mathsf{G}}, \check{\mathsf{B}})$ via a diffeomorphism transformation. This transformation can be concretized via the matrix $\mathcal{O}_{\mathsf{A}} \in O(2,2;\mathbb{Z}) \subset O(3,3;\mathbb{Z})$ found in (6.1.43), namely

$$\mathcal{O}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0_{3\times3} \\ 0_{3\times3} & \mathsf{A}^{-T} \end{pmatrix}, \qquad \mathsf{A} = \begin{pmatrix} 1 & -h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.3.7}$$

We notice that under a shift $X^3 \to X^3 + 2\pi$, the generalized metric $\mathcal{H}$ constructed by taking g and b indicated in (6.3.6) behaves in the same fashion as (6.3.5), that is

$$\mathcal{H}(X^3 + 2\pi) = \mathcal{O}_{\mathsf{A}}^{-T}\mathcal{H}(X^3)\mathcal{O}_{\mathsf{A}}^{-1}. \tag{6.3.8}$$

This tells us that both the $\check{\mathsf{G}}$ and $\check{\mathsf{B}}$ fields at X^3 can be properly patched to their respective counterparts at $X^3 + 2\pi$ via a diffeomorphism transformation. This is again a *geometrical* transformation.

T-fold

Finally we consider the case of the T-fold background. In here we ought to consider the following g and b matrices characterized by their respective components

$$g_{\mathsf{ij}} = \frac{1}{\rho}\begin{pmatrix} R_2^2 & 0 \\ 0 & R_1^2 \end{pmatrix}, \qquad b_{\mathsf{ij}} = \frac{\alpha'}{\rho}\begin{pmatrix} 0 & -\frac{hX^3}{2\pi} \\ +\frac{hX^3}{2\pi} & 0 \end{pmatrix}, \tag{6.3.9}$$

where we recall that ρ is given by

$$\rho = \frac{R_1^2 R_2^2}{\alpha'^2} + \left[\frac{hX^3}{2\pi}\right]^2. \tag{6.3.10}$$

Now we would like to repeat the same procedure as before and merge consistently the $\check{\mathsf{G}}$ and $\check{\mathsf{B}}$ fields once we go around the circle by studying the behavior of the generalized metric when $X^3 \to X^3 + 2\pi$. We can indeed do this if we consider this time $\beta-$transformations, already pointed out in (6.1.46). Let us then consider $\mathcal{O}_\beta \in O(2,2;\mathbb{Z}) \subset O(3,3;\mathbb{Z})$, where

$$\mathcal{O}_\beta = \begin{pmatrix} 1_{3\times3} & \beta \\ 0_{3\times3} & 1_{3\times3} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & +h & 0 \\ -h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.3.11}$$

It can be proven that the generalized metric $\mathcal{H}$ constructed considering g and b in (6.3.9) transforms under the shift $X^3 \to X^3 + 2\pi$ as follows

$$\mathcal{H}(X^3 + 2\pi) = \mathcal{O}_\beta^{-T}\mathcal{H}(X^3)\mathcal{O}_\beta^{-1}. \tag{6.3.12}$$

Notice that the patching can be consistently done by considering neither gauge transformations on the $B-$field nor diffeomorphisms. Instead, we had to implement transformations that find no analog in the usual Riemannian geometry. This is the reason why these transformations are considered to be *non-geometric*.

Chapter 7

T-duality transformations for the Open String

This chapter is based on our work [171]. Here we study T-duality transformations for an open-string non-linear $\sigma-$ model with non-trivial boundary. While in the previous chapter we placed the foundations regarding the non-linear $\sigma-$model for the closed string, here we want to present a precise account for T-duality transformations whenever D-branes are included in the picture, addressing missing details in the literature.

This chapter starts by presenting the non-linear $\sigma-$model for the open string and establishing concretely the technical details in order to define consistently a D-brane on this setup.

7.1 The gauged non-linear sigma model

7.1.1 Worldsheet action

Following the same motivations and conventions that led the construction of (6.1.21), we consider now a $\sigma-$model on an Euclidean worldsheet Σ with non-empty boundary, i.e. $\partial\Sigma \neq \varnothing$. This gives us the chance of incorporating Wilson lines by introducing a boundary gauge field $a = a_i \, \mathrm{d}X^i$, along with the target-space metric G, the Kalb-Ramond field B and the dilaton ϕ. The field intensities for both a and B are to be taken as $F = \mathrm{d}a$ and $H = \mathrm{d}B$, respectively. Remembering that $i, j = 1, \ldots, D$, our action reads then as

$$S = -\frac{1}{2\pi\alpha'} \int_\Sigma \Big[\frac{1}{2} G_{ij}(X)\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \frac{i}{2} B_{ij}(X)\, \mathrm{d}X^i \wedge \mathrm{d}X^j + \frac{\alpha'}{2} \mathsf{R}\, \phi(X) * 1 \Big] - \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} \Big[2\pi i\, \alpha'\, a_i(X)\, \mathrm{d}X^i + \alpha'\, k(s)\, \phi\, \mathrm{d}s \Big]. \tag{7.1.1}$$

One thing to notice is that the gauge field a is meant to be restricted on the boundary. We recall that we called s the parameter of the boundary $\partial\Sigma$ and n^{a} and t^{a} the unit vectors normal and tangential to it, respectively. We find as well the incorporation of the extrinsic curvature of the boundary $k(s) = t^{\mathsf{a}} t^{\mathsf{b}} \nabla_{\mathsf{a}} n_{\mathsf{b}}$ coupled to the dilaton field. Finally, we remember that on the boundary $\mathrm{d}X^i|_{\partial\Sigma} = t^{\mathsf{a}} \partial_{\mathsf{a}} X^i \mathrm{d}s$.

By letting the fields G, B, ϕ and a to be present in this model we are actually studying the motion of a string moving on a background generated by states related to the open-oriented and closed-oriented string. As we have discussed in the previous chapter, the presence of a boundary gauge field (transforming in this case under $U(1)$) is due to excitations of oscillators on open string states.

Boundary conditions

In order to read the boundary conditions related to X^i we require to compute its equations of motion and read the boundary term that it arises. First, the EOMs can be read as

$$0 = \mathrm{d} * \mathrm{d}X^i + \Gamma^i_{mn}\, \mathrm{d}X^m \wedge *\mathrm{d}X^n - \frac{i}{2} G^{ik} H_{kmn}\, \mathrm{d}X^m \wedge \mathrm{d}X^n - \frac{\alpha'}{2} G^{im}\, \partial_m \phi\, \mathsf{R} * 1\,, \tag{7.1.2}$$

where Γ^i_{jk} corresponds to the usual Christoffel symbols related to the target-space metric and G^{ij} denotes obviously its inverse.

We take Dirichlet boundary conditions to be of the form $\delta X^i|_{\partial\Sigma} = 0$. The tangential and normal part of $\mathrm{d}X^i$ can be written as follows

$$\begin{aligned} \left(\mathrm{d}X^i\right)_{\mathrm{tan}} &\equiv t^{\mathsf{a}}\, \partial_{\mathsf{a}} X^i\, \mathrm{d}s\big|_{\partial\Sigma}, \\ \left(\mathrm{d}X^i\right)_{\mathrm{norm}} &\equiv n^{\mathsf{a}}\, \partial_{\mathsf{a}} X^i\, \mathrm{d}s\big|_{\partial\Sigma}. \end{aligned} \tag{7.1.3}$$

By splitting the target-space index $i = 1, \ldots, D$ as $i = \{\hat{\imath}, a\}$ to denote Dirichlet directions $\hat{\imath}$ and Neumann directions a we find the boundary conditions to be

$$\begin{aligned} &\text{Dirichlet} \quad 0 = \left(\mathrm{d}X^{\hat{\imath}}\right)_{\mathrm{tan}}\,, \\ &\text{Neumann} \quad 0 = G_{ai} \left(\mathrm{d}X^i\right)_{\mathrm{norm}} + 2\pi\alpha' i \mathcal{F}_{ab} \left(\mathrm{d}X^b\right)_{\mathrm{tan}} + \alpha'\, k(s)\, \partial_a \phi\, \mathrm{d}s\Big|_{\partial\Sigma}\,, \end{aligned} \tag{7.1.4}$$

where we have introduced the gauge-invariant open string field strength $\mathcal{F}_{ab}$ which satisfies $2\pi\alpha'\mathcal{F}_{ab} = 2\pi\alpha' F + B$.

Similar to the analysis we have done in the previous chapter, we need to handle properly the decomposition of forms, this time for manifolds with boundary. Let Ω^p be the space of smooth differential $p-$forms on Σ and $\mathrm{d}^\dagger \equiv *\mathrm{d}*$ the co-exterior derivative. The Hodge decomposition theorem for manifolds with boundary – as seen in [179], allows us to decompose the space of $p-$forms $C^p = \{\omega \in \Omega^p : \mathrm{d}\omega = 0\}$ as a direct sum of the space of exact $p-$forms $E^p = \{\omega \in \Omega^p : \omega = \mathrm{d}\eta, \eta \in \Omega^{p-1}\}$ and the space of closed and co-closed forms whose normal part vanishes on the boundary $CcC^p_N = \{\omega \in \Omega^p : \mathrm{d}\omega = 0, \mathrm{d}^\dagger\omega = 0, \omega_{\text{norm}} = 0\}$, i.e.

$$C^p = E^p \oplus CcC^p_N \,. \tag{7.1.5}$$

Moreover, it can be proven that if a closed and co-closed $p-$form defined on a connected, oriented, smooth Riemannian manifold vanishes on the boundary, then it is identically zero [179]. This implies in particular that $1-$forms $\mathrm{d}X^i$ that respect Dirichlet boundary conditions are exact on Σ.

Global symmetries

As we have seen previously, the action (7.1.1) must comply with a certain set of conditions before we can implement the Buscher's procedure. One of these is that it must remain invariant under infinitesimal transformations of the coordinates X^i of the form

$$\delta_\epsilon X^i = \epsilon^\alpha k^i_\alpha \tag{7.1.6}$$

where, as seen in (6.1.49) the ϵ^α are constant, infinitesimal parameters and the k^i_α are the components of the target-space vector fields k_α which satisfy the Lie algebra

$$[k_\alpha, k_\beta] = f_{\alpha\beta}{}^\gamma k_\gamma, \tag{7.1.7}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants and $\alpha = 1, \ldots, N$, where N corresponds to the dimension of the Lie algebra.

The action (7.1.1) features then a global symmetry under the infinitesimal transformations (7.1.6) as long the following conditions are met

$$\begin{aligned}
&\mathcal{L}_{k_\alpha} G = 0\,, \\
&\mathcal{L}_{k_\alpha} B = \mathrm{d}v_\alpha\,, \qquad\qquad 2\pi\alpha'\mathcal{L}_{k_\alpha} a\big|_{\partial\Sigma} = (-v_\alpha + \mathrm{d}\omega_\alpha)\big|_{\partial\Sigma}\,, \\
&\mathcal{L}_{k_\alpha}\phi \;= 0\,.
\end{aligned} \tag{7.1.8}$$

Here, v_α are globally well-defined $1-$forms on Σ, whereas ω_α are to be taken as globally well-defined functions on $\partial\Sigma$.

It might happen that the boundary conditions themselves change under this symmetry transformation of the action. By evaluating the infinitesimal transformation (7.1.6) on the boundary conditions stated in (7.1.4) we find a requirement for the Dirichlet boundary conditions, whereas for the Neumann conditions no requirement is necessary. These can be stated as

$$\begin{aligned} &\text{Dirichlet} \qquad && 0 = \partial_a k^{\hat{i}}_\alpha \Big|_{\partial\Sigma} \,, \\ &\text{Neumann} \qquad && \varnothing \,. \end{aligned} \tag{7.1.9}$$

In the next section we take care of its construction by promoting the global symmetries to local ones, mirroring the procedure we did for the WZW action in the previous chapter.

7.1.2 The gauged worldsheet action

As pointed out in the last section, we set off to promote the global symmetries to local symmetries by introducing worldsheet gauge fields. As stated before, we take the infinitesimal parameter ϵ^α to be coordinate-dependant, i.e. $\epsilon = \epsilon(\sigma^{\mathsf{a}})$. We read the gauged action to be then

$$\begin{aligned} \hat{S} = &-\frac{1}{2\pi\alpha'}\int_\Sigma \left[\frac{1}{2}G_{ij}\big(\mathrm{d}X^i + k^i_\alpha A^\alpha\big)\wedge *\big(\mathrm{d}X^j + k^j_\beta A^\beta\big) + \frac{\alpha'}{2}\mathsf{R}\,\phi * 1 \right] \\ &-\frac{i}{2\pi\alpha'}\int_\Sigma \left[\frac{1}{2}B_{ij}\,\mathrm{d}X^i\wedge \mathrm{d}X^j \right. \\ &\qquad\qquad \left. +(\tilde{v}_\alpha + \mathrm{d}\chi_\alpha)\wedge A^\alpha + \frac{1}{2}\Big(\iota_{k_{[\alpha}}\tilde{v}_{\beta]} + f_{\alpha\beta}{}^\gamma\chi_\gamma\Big)\,A^\alpha\wedge A^\beta\right] \\ &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\Big[2\pi i\,\alpha'\,a_a\,\mathrm{d}X^a - i\,\Omega_{\partial\Sigma} + \alpha'\,k(s)\,\phi\,\mathrm{d}s\Big]\,. \end{aligned} \tag{7.1.10}$$

Just like we did in (6.1.53), we have introduced a set of worldsheet gauge fields A^α and Lagrange multipliers χ_α, with $\alpha, \beta = 1, \ldots, N$. We have introduced here the one form Ω_Σ whose value depend on whether the gauging direction is along Dirichlet or Neumann directions. We will specify its value later on. Another field we have just introduced here is $\tilde{v}_\alpha$, which is given by

$$\tilde{v}_\alpha := v_\alpha - \iota_{k_\alpha}B. \tag{7.1.11}$$

Taking into account the constraints (7.1.8), we see that the action (7.1.10) is invariant under the local symmetry transformations

$$\begin{aligned} \hat{\delta}_\epsilon X^i &= \epsilon^\alpha k^i_\alpha \,, \\ \hat{\delta}_\epsilon A^\alpha &= -\mathrm{d}\epsilon^\alpha \qquad - f_{\beta\gamma}{}^\alpha\epsilon^\beta A^\gamma \,, \\ \hat{\delta}_\epsilon \chi_\alpha &= -\iota_{k_{(\overline{\alpha}}}v_{\overline{\beta})}\epsilon^\beta - f_{\alpha\beta}{}^\gamma\epsilon^\beta\chi_\gamma \,, \end{aligned} \tag{7.1.12}$$

provided that $\mathrm{d}\chi_\alpha$ is globally well-defined on all Σ and that the additional constraints are satisfied as well

$$\mathcal{L}_{k_{[\alpha}}\tilde{v}_{\beta]} = f_{\alpha\beta}{}^{\gamma}\tilde{v}_\gamma\,, \qquad\qquad \iota_{k_{[\alpha}}f_{\beta\gamma]}{}^{\delta}\tilde{v}_\delta = \frac{1}{3}\,\iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma}H\,. \tag{7.1.13}$$

Before we proceed, we need to address properly the matter of whether the boundary conditions (7.1.4) are preserved under the local infinitesimal transformation and constraints already presented. This is of uttermost importance; if we have a certain D-brane configuration in our system, it better remain the same under these transformations.

We see that the conditions (7.1.4) are indeed preserved if the infinitesimal transformation parameters ϵ^α meet the following conditions on the boundary

$$\begin{aligned} &\text{Dirichlet} \qquad && 0 = k_\alpha^{\hat{i}}\big(\mathrm{d}\epsilon^\alpha\big)_{\tan}\Big|_{\partial\Sigma}\,, \\ &\text{Neumann} \qquad && 0 = G_{ai}\,k_\alpha^i\,\big(\mathrm{d}\epsilon^\alpha\big)_{\text{norm}} + 2\pi\alpha' i\,\mathcal{F}_{ab}\,k_\alpha^b\big(\mathrm{d}\epsilon^\alpha\big)_{\tan}\Big|_{\partial\Sigma}\,, \end{aligned} \tag{7.1.14}$$

where we employed the restrictions (7.1.9) coming from the global symmetry transformations. Note however that the Dirichlet condition on the target-space coordinates $\delta X^{\hat{i}}|_{\partial\Sigma} = 0$ imply the somewhat stronger condition

$$\text{Dirichlet} \qquad 0 = k_\alpha^{\hat{i}}\,\epsilon^\alpha\Big|_{\partial\Sigma} \tag{7.1.15}$$

Since this is a statement about the behavior of the local infinitesimal parameter ϵ^α on the boundary along Dirichlet directions, it means that there will be an impact on the infinitesimal transformations (7.1.12) themselves on the boundary. In particular, we notice that

$$\hat{\delta}_\epsilon A^\alpha\Big|_{\partial\Sigma} = 0 \tag{7.1.16}$$

along the Dirichlet directions. For the Neumann directions, on the other hand, by studying the equations of motion related to X^i derived from the gauged action (7.1.10) we find an expression for the normal and tangential components of A^α. This motivates us to impose the following boundary conditions for A^α [156] as follows

$$\begin{aligned} &\text{Dirichlet} \qquad && 0 = k_\alpha^{\hat{i}}\big(A^\alpha\big)_{\tan}\Big|_{\partial\Sigma}\,, \\ &\text{Neumann} \qquad && 0 = G_{ai}\,k_\alpha^i\,\big(A^\alpha\big)_{\text{norm}} + 2\pi\alpha' i\,\mathcal{F}_{ab}\,k_\alpha^b\big(A^\alpha\big)_{\tan}\Big|_{\partial\Sigma}\,. \end{aligned} \tag{7.1.17}$$

It can be checked that these boundary conditions for the gauge field A^α are indeed preserved under the local infinitesimal transformations (7.1.12).

Having clarified the boundary conditions for our worldsheet gauge field A^α, we are set to concretize about the form of Ω_Σ. We will differentiate between two cases: when the infinitesimal transformations are done all along Dirichlet directions $X^{\hat{i}}$ or when are done all along Neumann directions X^{a} [27]

- For the Dirichlet case, we saw already in (7.1.15) that the variation parameters ϵ^α vanish on $\partial\Sigma$. This means that all of the terms at the boundary part of the action (7.1.1) remain invariant under the infinitesimal transformations (7.1.14), hence we introduce no term whatsoever to ensure invariance and set

$$\text{Dirichlet} \qquad \Omega_{\partial\Sigma} = 0\,. \tag{7.1.18}$$

- For Neumann directions we face a different situation. After we perform the infinitesimal transformations given in (7.1.12) we find that we need to add extra fields. Let ϕ_α to be a second set of Lagrange multipliers, where $\alpha = 1, \ldots, N$. We define Ω_Σ in this case to be

$$\text{Neumann} \qquad \Omega_{\partial\Sigma} = \left(\chi_\alpha + \phi_\alpha + \omega_\alpha - 2\pi\alpha'\iota_{k_\alpha} a\right) A^\alpha\,. \tag{7.1.19}$$

  Please note that by defining Ω_Σ on $\partial\Sigma$ we are allowing χ_α to be present on the boundary, meaning that χ_α must be a globally well defined $0-$form (or function) on $\partial\Sigma$ – when in principle $\mathrm{d}\chi_\alpha$ was only meant to be a globally well defined one-form on the bulk Σ. At the same time, ϕ_α must be a set of globally well defined constant functions on $\partial\Sigma$.

  The fact that χ_α is globally well defined on $\partial\Sigma$ brings us an useful fact: this means that $\mathrm{d}\chi_\alpha$ must be exact on $\partial\Sigma$, implying that its closed and co-closed component must vanish at $\partial\Sigma$. Therefore, in virtue of the Hodge decomposition theorem for manifolds with boundary, $\mathrm{d}\chi_\alpha$ must be exact on all of Σ.

  Finally, to ensure invariance of the action (7.1.1) in the case of Neumann boundary conditions we find that the following set of conditions must hold as well

$$\mathcal{L}_{k_{[\underline{\alpha}}}\omega_{\underline{\beta}]}\Big|_{\partial\Sigma} = \frac{1}{2}\left[f_{\alpha\beta}{}^\gamma\omega_\gamma + \iota_{k_{[\underline{\alpha}}}v_{\underline{\beta}]}\right]\Big|_{\partial\Sigma}\,, \qquad 0 = f_{\alpha\beta}{}^\gamma\phi_\gamma\Big|_{\partial\Sigma}\,. \tag{7.1.20}$$

Additional symmetries of the gauged action

Since we have introduced additional fields v_α, ϕ_α and ω_α into the game, it may happen that our action (7.1.1) might display new symmetries besides the ones generated by the local transformations (7.1.14). Indeed, the action features the following transformations[28] that render it invariant [153]:

[27] We consider these two cases only. Cases in which we perform infinitesimal transformations along Dirichlet and Neumann directions can also be treated with this formalism. However, for ease of the discussion we won't treat them here.

[28] Since $\Omega_\Sigma = 0$ for the case of Dirichlet boundary conditions, the last two transformations here listed become slightly modified and less stringent.

1 Gauge transformations of the Kalb-Ramond field:

$$\begin{aligned} B &\to B + \mathrm{d}\Lambda\,, \\ a &\to a - \tfrac{1}{2\pi\alpha'}\Lambda\,, \\ v_\alpha &\to v_\alpha + \iota_{k_\alpha}\mathrm{d}\Lambda\,, \\ \omega_\alpha &\to \omega_\alpha - \iota_{k_\alpha}\Lambda\,. \end{aligned} \tag{7.1.21}$$

with a globally well-defined one-form on the world-sheet Σ denoted by Λ.

2 Shifts of the one-forms v_α:

$$\begin{aligned} v_\alpha &\to v_\alpha + \mathrm{d}\lambda_\alpha\,, \\ \chi_\alpha &\to \chi_\alpha - \lambda_\alpha\,, \\ \omega_\alpha &\to \omega_\alpha + \lambda_\alpha\,, \end{aligned} \qquad \mathcal{L}_{k_{[\underline{\alpha}}}\lambda_{\underline{\beta}]} = f_{\alpha\beta}{}^{\gamma}\lambda_\gamma\,. \tag{7.1.22}$$

where λ_α are well defined functions on Σ.

3 Gauge transformations of the open-string gauge field a:

$$\begin{aligned} a &\to a + \mathrm{d}\lambda\,, \\ \omega_\alpha &\to \omega_\alpha + 2\pi\alpha'\iota_{k_\alpha}\mathrm{d}\lambda\,. \end{aligned} \tag{7.1.23}$$

where λ is a globally well-defined function on the boundary $\partial\Sigma$.

4 Shifts of the functions ω_α:

$$\begin{aligned} \chi_\alpha &\to \chi_\alpha + \theta_\alpha\,, \\ \omega_\alpha &\to \omega_\alpha - \theta_\alpha\,, \end{aligned} \qquad f_{\alpha\beta}{}^{\gamma}\theta_\gamma = 0\,. \tag{7.1.24}$$

where θ_α are constant quantities.

5 Shifts of the functions ϕ_α:

$$\begin{aligned} \phi_\alpha &\to \phi_\alpha + \Theta_\alpha\,, \\ \omega_\alpha &\to \omega_\alpha - \Theta_\alpha\,. \end{aligned} \tag{7.1.25}$$

where Θ_α are constants.

Thus far we have prepared the grounds to start implementing the Buscher rules. As we saw in the previous chapter, we need to ensure that we have not artificially introduced new degrees of freedom when we gauged the worldsheet action: We need to be able to recover the original, ungauged action. This can be done with help of the equations of motion of the Lagrange multipliers. We address this matter in the next section.

7.1.3 Recovering the original action

In this section we discuss how from the action (7.1.10) we can recover the original, ungauged action (7.1.1). To this end, we need to compute the equations of motion for the Lagrange multipliers χ_α and ϕ_α [180, 178, 181].

The equations of motion for χ_α and ϕ_α

Since the 1−form Ω_Σ defined on $\partial\Sigma$ is sensitive to the boundary conditions, we need to make a distinction and derive the equations of motion for ϕ_α and χ_α for both Dirichlet and Neumann conditions.

- Dirichlet conditions: Recalling (7.1.18), we have $\Omega_\Sigma = 0$ and therefore there's no EOMs for ϕ_α to look at. Taking this into account and performing the variation of the action (7.1.10) with respect to χ_α[29] we find

$$\delta_\chi \hat{S} = \frac{i}{2\pi\alpha'} \int_\Sigma \delta\chi_\alpha \left(\mathrm{d}A^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \right) . \tag{7.1.26}$$

  Recalling the Dirichlet boundary conditions for A^α (7.1.17) and taking $\delta_\chi \hat{S} = 0$ we find

$$0 = F^\alpha = \mathrm{d}A^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma , \qquad 0 = A^\alpha \Big|_{\partial\Sigma} . \tag{7.1.27}$$

- Neumann conditions: In contrast to the Dirichlet case, Ω_Σ given by (7.1.19) switches on and we must find the variations with respect to both χ_α and ϕ_α. This gives us

$$\begin{aligned} \delta_\chi \hat{S} &= \frac{i}{2\pi\alpha'} \int_\Sigma \delta\chi_\alpha \left(\mathrm{d}A^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma \right) , \\ \delta_\phi \hat{S} &= \frac{i}{2\pi\alpha'} \int_{\partial\Sigma} \delta\phi_\alpha \, A^\alpha . \end{aligned} \tag{7.1.28}$$

  This naturally leads us to the equations of motion

$$0 = F^\alpha = \mathrm{d}A^\alpha - \frac{1}{2} f_{\beta\gamma}{}^\alpha A^\beta \wedge A^\gamma , \qquad 0 = A^\alpha \Big|_{\partial\Sigma} . \tag{7.1.29}$$

[29] In the previous chapter we set off to recover the ungauged action as well. We found that we ought to systematically separate $\mathrm{d}\chi_\alpha$ (given by (6.1.56)) to read properly the equations of motion for χ_α. With our current setup, however, we know from the start that $\mathrm{d}\chi_\alpha$ is exact and thus $\chi_\alpha \equiv \chi_\alpha^{(0)}$; there are no equations of motion for its CcC^1_N component to be found since it is identically zero.

We notice that in both cases we make sure that A^α vanishes on the boundary [153], either as a consequence of its own the boundary conditions or as an actual equation of motion, hence the importance of introducing ϕ_α in the second case.

Now we explore in which cases we can actually go back to our original, ungauged action (7.1.1) depending on whether we have an abelian or non–abelian isometry algebra.

Ungauged action I: Abelian isometry algebra

Let's start with the case for an abelian isometry algebra, i.e. $f_{\alpha\beta}{}^{\gamma} = 0$. The equations of motion for χ_α derived previously in both Dirichlet and Neumann cases imply that A^α is closed on Σ. By applying then the Hodge decomposition theorem we find that A^α in this case can be written as

$$A^\alpha = \mathrm{d}a^\alpha_{(0)} + \sum_{\mathsf{m}} a^\alpha_{(\mathsf{m})} \varphi^{\mathsf{m}} \,. \tag{7.1.30}$$

In here, $a^\alpha_{(0)}$ are globally well-defined functions on Σ, $a^\alpha_{(\mathsf{m})}$ are real coefficients and φ^{m} conform a basis of the space of closed and co-closed 1−forms on Σ. The EOMs we have derived in (7.1.27) and (7.1.29) state that A^α must vanish on the boundary $\partial\Sigma$. This means then that the CcC^1_N component of A^α must vanish, that is $a^\alpha_{(\mathsf{m})} = 0$, leading us to the conclusion that A^α is an exact form.

Given that A^α is exact, we can make use of the infinitesimal transformations (7.1.12) and choose $\epsilon^\alpha = a^\alpha_{(0)}$ (together with $f_{\alpha\beta}{}^{\gamma} = 0$) to gauge it away and set $A^\alpha = 0$. By doing this, we recover immediately the original action (7.1.1) from the gauged one (7.1.12).

Ungauged action II: Non-abelian isometry algebra

Applying Hodge's decomposition theorem to closed forms allows us to systematically separate them into an exact part and closed and co-closed part, as indicated in (7.1.5). For the abelian case, we found that A^α complies with this decomposition and we were able to gauge it away, recovering the original model. As we mentioned in the previous chapter, for the non-abelian case the situation is rather different: A^α satisfies (7.1.29) with $f_{\beta\gamma}{}^{\alpha} \neq 0$ and is no longer closed, thus spoiling the decomposition (7.1.5).

We could follow an approach similar to the one in [170] and indicated in and set us to the task of redefining $\mathrm{D}X^i = \mathrm{d}X^i + k^i_\alpha A^\alpha$ as an element of a closed, 1−form basis $\{\mathrm{d}Y^i\}$ and thus recovering the form of the original model in the case of non-abelian isometries. The downside is though, that this approach does not take into account the topological non-triviality of the worldsheet.

A more accurate account would be to start from the cohomology of the gauge-covariant derivative and determine precisely the Hodge decomposition theorem for manifolds with non-empty boundary. This task is beyond the scope of this doctoral work and it will be treated elsewhere.

7.2 T-duality

We discuss now about T-duality transformations for the open string. As we mentioned at the beginning of this chapter, we are studying a microscopic model of a string moving in a background generated by the excitations on open-oriented and closed-oriented states. This means in particular that we must be able to tell the effects of T-duality upon the closed- and open-string sectors. We will talk about these two aspects of T-duality in the next sections.

7.2.1 The closed-string sector

Simply put, exploring the effects of T-duality transformations on the closed-string sector means studying how does the metric $G_{\mu\nu}$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton ϕ transform under their action. To this end, we will follow the aforementioned Buscher's procedure of gauging target-space isometries and integrating-out the world-sheet gauge fields A^α.

The equations of motion for A^α

We take the action (7.1.10) and perform the variation with respect to A^α. Let's address first the equations of motion for the bulk Σ. We get an analogous expression to (6.1.62), and we can solve algebraically for A^α. We get them using obvious matrix notation

$$A^\alpha = -\left[\left(\mathcal{G} - \mathcal{D}\mathcal{G}^{-1}\mathcal{D}\right)\right]^{\alpha\beta}\left(1 + i * \mathcal{D}\mathcal{G}^{-1}\right)_\beta{}^\gamma \left(\mathsf{k} + i * \xi\right)_\gamma , \tag{7.2.1}$$

where we recall that $\alpha, \beta = 1, \ldots, N$. The auxiliary quantities are almost the same as (6.1.63); in this case we find

$$\begin{aligned} \mathcal{G}_{\alpha\beta} &= k^i_\alpha G_{ij} k^j_\beta , & \xi_\alpha &= \mathrm{d}\chi_\alpha + \tilde{v}_\alpha , \\ \mathcal{D}_{\alpha\beta} &= \iota_{k_{[\underline{\alpha}}} \tilde{v}_{\underline{\beta}]} + f_{\alpha\beta}{}^\gamma \chi_\gamma , & \mathsf{k}_\alpha &= k^i_\alpha G_{ij} \mathrm{d}X^j , \end{aligned} \tag{7.2.2}$$

where we recall as well that $\tilde{v}_\alpha = v_\alpha - \iota_{k_\alpha} B$. Analogously to the WZW case, we need invertibility of $(\mathcal{G} \pm \mathcal{D})$ in order to construct the dual model, so we dismiss any concerns regarding the invertibility of $\mathcal{G}$.

Note that there is a contribution coming from the boundary $\partial\Sigma$ depending on whether Dirichlet or Neumann boundary conditions are considered. Since $\Omega_\Sigma = 0$ for the Dirichlet case, there are no A^α in $\partial\Sigma$ and we get no information. For the

Neumann case, on the other hand, we find an expression that has to be imposed as a constraint. In summary, we find

$$\begin{aligned} &\text{Dirichlet} \qquad && \varnothing\,, \\ &\text{Neumann} \qquad && 0 = 2\pi\alpha'\,\iota_{k_\alpha} a - (\chi_\alpha + \phi_\alpha + \omega_\alpha)\Big|_{\partial\Sigma}\,. \end{aligned} \tag{7.2.3}$$

Integrating-out the A^α fields

Having both (7.2.1) and (7.2.3) in mind, we can now take our gauged action (7.1.12) and evaluate it. We read this evaluated action to be

$$\begin{aligned} \check{S} = &-\frac{1}{2\pi\alpha'}\int_\Sigma \left[\check{G} + i\check{B} + \frac{\alpha'}{2}\mathsf{R}\,\phi * 1\right] \\ &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma}\left[2\pi i\,\alpha'\,a_a\,\mathrm{d}X^a + \alpha'\,k(s)\,\phi\,\mathrm{d}s\right], \end{aligned} \tag{7.2.4}$$

where $\check{G}$ and $\check{B}$ are given by

$$\begin{aligned} \check{G} &= G - \frac{1}{2}\,(\mathsf{k}+\xi)^T\left(\mathcal{G}+\mathcal{D}\right)^{-1}\wedge *(\mathsf{k}-\xi)\,, \\ \check{B} &= B - \frac{1}{2}\,(\mathsf{k}+\xi)^T\left(\mathcal{G}+\mathcal{D}\right)^{-1}\wedge \;\;(\mathsf{k}-\xi)\,, \end{aligned} \tag{7.2.5}$$

where the matrix multiplication here is understood. This multiplication is carried on the $\alpha,\beta = 1,\ldots,N$, and G and B are the usual two forms for the metric and Kalb-Ramond field given by

$$\begin{aligned} G &= \frac{1}{2!}G_{ij}\,\mathrm{d}X^i\wedge *\mathrm{d}X^j, \\ B &= \frac{1}{2!}B_{ij}\,\mathrm{d}X^i\wedge \;\;\mathrm{d}X^j. \end{aligned} \tag{7.2.6}$$

As it happens in our dual action for the WZW model (6.1.65), we find that the tensor quantities $\check{G}$ and $\check{B}$ are defined on an enlarged $(D+N)-$dimensional target-space as well, locally parametrized by the set of coordinates $\{\mathrm{d}X^i, \chi_\alpha\}$ [182].

In order to read properly the dual background fields for both the open and closed string sector we require to perform a suitable change of basis. We explore this subject right away.

The change of basis

The enlarged, symmetric target-space metric $\check{G}$ possesses N null-eigenvectors. We can readily express these eigenvectors in the $\{\mathrm{d}X^i, \chi_\alpha\}$ basis as follows

$$\check{n}_\alpha = \begin{pmatrix} k^i_\alpha \\ \mathcal{D}_{\alpha\beta} - \iota_{k_\alpha}\tilde{v}_\beta \end{pmatrix}\,. \tag{7.2.7}$$

From the last chapter we know that these vectors can be used to construct a change of basis matrix $\mathcal{T}$ of the form as (6.1.70) and find a vielbein basis of the form $e^A = (\mathcal{T}^{-1})^A_B \, \mathrm{d}X^B$, for coordinates $A = \{i, \alpha\}$. Now, by assuming that the k_α vectors are linearly independent we can always find a coordinate system in which the $N \times N$ matrix of components k^α_β has an inverse, while all of the other remaining components vanish[30].

$$
\begin{aligned}
e^\alpha &= \left(k^{-1}\right)^\alpha_\beta \mathrm{d}X^\beta \,, \\
e^m &= \mathrm{d}X^m \,, \\
e_\alpha &= \mathrm{d}\chi_\alpha + \left[\iota_{k_{(\overline{\alpha}}} v_{\overline{\beta})} + f_{\alpha\beta}{}^\gamma \chi_\gamma\right] \left(k^{-1}\right)^\beta{}_\gamma \, \mathrm{d}X^\gamma \,.
\end{aligned}
\tag{7.2.8}
$$

As always, $\alpha, \beta = 1, \ldots, N$ and $m, n = N+1, \ldots, D$. The simplification from (6.1.72) is evident; in particular, we notice that $\mathrm{d}e^m = 0$. In fact, to address properly the upcoming examples the present formalism we ought to take some simplifying assumptions to render our basis closed.

Having this new basis at hand, we are able now to express properly the dual G and B fields. Since the symmetric two-form $\check{G}$ has N zero eigenvalues (as we previously mentioned), it can be brought into the simpler form

$$
\check{G} = \frac{1}{2} \check{\mathsf{G}}_{IJ} \, e^I \wedge * e^J, \tag{7.2.9}
$$

where we designate the vielbein e^I as $e^I = \{e^m, e_\alpha\}$ and $I = 1, \ldots, D$. Following the definitions stated in (7.2.2) and (7.1.11) we find that the components $\check{\mathsf{G}}_{IJ}$ can be written down as

$$
\begin{aligned}
\check{\mathsf{G}}_{mn} = \; & G_{mn} - \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{G} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& - \mathsf{k}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{D} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n} \\
& + \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{D} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& + \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{G} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}, \\[1ex]
\check{\mathsf{G}}_m{}^\beta = & - \mathsf{k}_{\beta m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{D} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \\
& + \tilde{v}_{\alpha m} \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{G} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}, \\[1ex]
\check{\mathsf{G}}^\alpha{}_n = & + \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{D} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& + \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{G} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}, \\[1ex]
\check{\mathsf{G}}^{\alpha\beta} = & + \left[(\mathcal{G}+\mathcal{D})^{-1} \mathcal{G} \, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}.
\end{aligned}
\tag{7.2.10}
$$

These expressions for the dual metric metric match exactly those of (6.1.77), except that in this case we use $\tilde{v}_\alpha$ instead of v_α. Hence, these are the components of the metric

[30] See Footnote 26 on page 78.

of the dual background after performing a collective T-duality transformation along N directions. And naturally, for the case of a T-duality transformation along one direction these expressions reduce to the usual Buscher rules [168].

Now we turn to read off the components related to the $\check{B}$ field under the basis (7.2.8). We find that this field has the form

$$\check{B} = \frac{1}{2}\, \check{\mathsf{B}}_{IJ}\, e^I \wedge e^J + \check{B}^{\text{res.}}\,, \tag{7.2.11}$$

where the antisymmetric components $\check{\mathsf{B}}_{IJ}$ can be read as

$$\begin{aligned}
\check{\mathsf{B}}_{mn} = \; & B_{mn} + \mathsf{k}_{\alpha m}\, \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{D}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& + \mathsf{k}_{\alpha m}\, \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{G}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n} \\
& - \tilde{v}_{\alpha m}\, \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{G}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& - \tilde{v}_{\alpha m}\, \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{D}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}, \\
\check{\mathsf{B}}_m{}^{\beta} = \; & + \mathsf{k}_{\beta m}\left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{G}\;\; (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \\
& - \tilde{v}_{\alpha m}\left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{D}\;\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}, \\
\check{\mathsf{B}}^{\alpha}{}_{n} = \; & - \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{G}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \mathsf{k}_{\beta n} \\
& - \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{D}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta} \tilde{v}_{\beta n}, \\
\check{\mathsf{B}}^{\alpha\beta} = \; & - \left[(\mathcal{G}+\mathcal{D})^{-1}\, \mathcal{D}\, (\mathcal{G}-\mathcal{D})^{-1}\right]^{\alpha\beta}.
\end{aligned} \tag{7.2.12}$$

These expressions give the B-field of the T-dual background, which in the case of a single T-duality comply with the Buscher rules.

The residual $B-$field $\check{B}^{\text{res.}}$ is a quantity which will prove later on to be essential while exploring the effect of T-duality transformations on the open-string background. It can be written in terms of the basis (7.2.8) in the following way

$$\check{B}^{\text{res.}} = e^{\alpha} \wedge \left[\mathrm{d}\chi_{\alpha} + v_{\alpha} + \frac{1}{2}\big(\iota_{k_{[\alpha}} v_{\beta]} + f_{\alpha\beta}{}^{\gamma}\chi_{\gamma}\big)\, e^{\beta} \right]. \tag{7.2.13}$$

Notice that this field still depends on the coordinates of the original background through the one forms $e^{\alpha} = (k^{-1})^{\alpha}_{\beta}\, \mathrm{d}X^{\beta}$. Depending on whether we work with Dirichlet or Neumann boundary conditions, its form will change when we study its interplay with quantities related to the open string sector.

Dual dilaton

In the previous chapter we already discussed the transformation rule for the dilaton field in (6.1.79). Along with the expressions (7.2.10) and (7.2.12), we will use the the form of $\check{\phi}$ given by (6.1.79) to describe the dual background of the closed string sector $(\check{\mathsf{G}}, \check{\mathsf{B}}, \check{\phi})$. To this end, we first need to clarify the the closure of the basis to ensure that we are working with a sound 1−form basis.

Closure of the basis e^A

Having the basis e^A under the assumption that we can find a coordinate system in which the components k^m_α, $m = N+1, \ldots, D$ of the linearly independent Killing vectors k_α vanish makes the discussion about its closure easier – we saw already in (6.1.72) that the exterior derivative of e^A is rather convoluted. Indeed, by having $k^m_\alpha = 0$ we find that $\mathrm{d}e^\alpha$, $\mathrm{d}e^m$ and $\mathrm{d}e_\alpha$ are given as follows

$$\begin{aligned} \mathrm{d}e^\alpha &= -\frac{1}{2} f_{\beta\gamma}{}^\alpha e^\beta \wedge e^\gamma - \left(k^{-1}\right)^\alpha_\beta \left[\partial_m k^\beta_\gamma\right] e^m \wedge e^\gamma \,, \\ \mathrm{d}e^m &= 0 \,, \\ \mathrm{d}e_\alpha &= -f_{\alpha\beta}{}^\gamma e^\beta \wedge e_\gamma \\ &\quad + \Big(\partial_m \iota_{k_{(\overline{\alpha}}} v_{\overline{\beta})} - \left[\iota_{k_{(\overline{\alpha}}} v_{\overline{\gamma})} + f_{\alpha\gamma}{}^\delta \chi_\delta\right] \left(k^{-1}\right)^\gamma_\epsilon \left[\partial_m k^\epsilon_\beta\right]\Big) e^m \wedge e^\beta . \end{aligned} \tag{7.2.14}$$

In particular, we notice that in general the basis $e^I = \{e^m, e_\alpha\}$ is not closed. This means that the dual background might implicitly depend on the original coordinates, a property expected from a non-geometric background. We can circumvent this caveat by restricting ourselves to one of the following cases

$$\left\{ \begin{aligned} 0 &= f_{\alpha\beta}{}^\gamma \\ 0 &= \partial_m \iota_{k_{(\overline{\alpha}}} v_{\overline{\beta})} \\ 0 &= \partial_m k^\beta_\alpha \end{aligned} \right\} \,, \qquad \left\{ \begin{aligned} 0 &= f_{\alpha\beta}{}^\gamma \\ 0 &= \iota_{k_{(\overline{\alpha}}} v_{\overline{\beta})} \end{aligned} \right\} . \tag{7.2.15}$$

The reader can verify that by taking either of both scenarios here displayed, the basis of 1−forms $e^I = \{e^m, e_\alpha\}$ is closed under the exterior derivative. We will stick to either of one of the two cases when we present our findings on T-duality transformations done over specific configurations in the upcoming examples.

Remarks on non-geometric fluxes

This is a good moment to do a small pause and make contact with what we have said in section 6.2. Having the expressions for the dual metric (7.2.10) and B−field (7.2.12) we take the chance to talk about the possible non-geometric fluxes that might arise given an initial background configuration. Given the components $\check{\mathsf{G}}_{IJ}$ and $\check{\mathsf{B}}_{IJ}$, we can compute the quantity

$$\left(\check{\mathsf{G}} \pm \check{\mathsf{B}}\right)^{-1} = \mathsf{g} \pm \beta \,, \tag{7.2.16}$$

where g and β are the symmetric and antisymmetric parts of this decomposition. In here, g is a metric field with (symmetric) components g^{IJ} while β is a bivector field with (antisymmetric) components β^{IJ}. From these quantities we are able to compute the non-geometric fluxes Q and R as follows

$$Q_I{}^{JK} = \partial_I \beta^{JK}\,, \qquad R^{IJK} = 3\,\beta^{[\underline{I}M}\partial_M \beta^{\underline{JK}]}\,. \tag{7.2.17}$$

Let us consider a D dimensional configuration and perform a collective T-duality transformation along all the D directions. We obtain a dual D dimensional configuration with a vielbein basis e_α which is not necessarily closed, as stated in (7.2.14). Indeed, the exterior derivative for e_α is given by

$$\mathrm{d}e_\alpha = -f_{\alpha\beta}{}^{\gamma}\, e^\beta \wedge e_\gamma\,. \tag{7.2.18}$$

Let us put this fact aside for a while and proceed on computing the $Q-$ and $R-$fluxes for our dual configuration. Taking the components (7.2.10) and (7.2.12) we find that g^{IJ} and β^{IJ} are given as follows

$$\mathsf{g}_{\alpha\beta} = \mathcal{G}_{\alpha\beta}\,, \qquad \beta_{\alpha\beta} = \mathcal{D}_{\alpha\beta}\,, \tag{7.2.19}$$

where the index structure is changed due to the index structure associated to the coordinates of the original background configuration X^α. For our dual configuration, we find $\partial_I = \partial^\alpha = \partial/\partial\chi_\alpha$ and recalling the definitions for $\mathcal{D}_{\alpha\beta}$ in (7.2.2) and using the Jacobi identity we find

$$Q^\alpha{}_{\beta\gamma} = f_{\beta\gamma}{}^{\alpha}\,, \qquad R_{\alpha\beta\gamma} = \iota_{k_\alpha}\iota_{k_\beta}\iota_{k_\gamma} H\,. \tag{7.2.20}$$

This is a result we expected on general grounds; by having an isometry group with its respective non-vanishing structure constants we are told that the target-space metric is non-trivial, and in general will present a non-geometric flux related somehow to $f_{\alpha\beta}{}^{\gamma}$. By performing a collective T-duality transformation along all directions this non-triviality migrates and gets mapped into the non-geometric $Q-$flux. The $H-$flux data migrates as well towards the $R-$ flux.

7.2.2 The open-string sector: Neumann directions

Now that we have elucidated what happens for the closed string sector under T-duality transformations we turn now to explore the open string sector whenever these transformations are performed along Neumann directions X^a only.[31] In the next subsection we will address the case for T-duality transformations along Dirichlet directions.

The procedure for both Neumann and Dirichlet cases starts similarly by integrating out the worldsheet gauge fields A^α. The difference now is that we need to take into consideration its boundary conditions, as given in (7.1.17).

[31] In this doctoral work we won't talk about T-duality transformations performed along Neumann and Dirichlet boundary conditions, even though the formalism allows us to do it.

Integrating-out I: gauge fields A^α

As we saw in the previous sections, we require to integrate-out the gauge fields A^α to reach the dual background. This gives us the dual metric field $\check{G}$ and the dual $B-$field $\check{B}$ along with the residual $B-$field given in (7.2.13). Let us not forget that by doing this process of integrating-out we solved the equations of motion for A^α, giving us a contribution coming from the boundary $\partial\Sigma$ which has to be enforced as a constraint. This is summarized in (7.2.3). For the Neumann case, we implement this constraint as a delta function into the path integral given by

$$\delta\big(\phi_\alpha - \tilde{\chi}_\alpha\big)_{\partial\Sigma}\,, \qquad \tilde{\chi}_\alpha = \chi_\alpha + \omega_\alpha - 2\pi\alpha' \iota_{k_\alpha} a\,. \tag{7.2.21}$$

The Neumann boundary conditions for A^α must be evaluated taking into account the solution found in (7.2.1). In equations, we find that A^α must satisfy on the boundary

$$0 = \Big[G_{ai} k^i_\alpha * A^\alpha\big|_{(7.2.1)} + 2\pi\alpha' i \mathcal{F}_{ab} k^b_\alpha A^\alpha\big|_{(7.2.1)}\Big]_{\partial\Sigma}\,, \tag{7.2.22}$$

where $A^\alpha\big|_{(7.2.1)}$ corresponds to the solution (7.2.1).

The worldsheet gauge field A^α harbors information about the boundary conditions of the dual coordinates χ_α. For general cases, reading the boundary conditions turns out to be rather convoluted. However, if we consider the case of abelian isometries (that is, $f_{\alpha\beta}{}^\gamma = 0$) it can be shown that

$$0 = \mathrm{d}\tilde{\chi}_\alpha\big|_{\partial\Sigma}\,. \tag{7.2.23}$$

These expressions correspond precisely to Dirichlet boundary conditions for the dual coordinates χ_α, which is expected having seen the CFT analysis.

Integrating-out II: Lagrange multipliers ϕ_α

The process of integrating out a certain field φ belonging to a certain action $S[\varphi, \psi^A]$, where ψ^A denote a collection of other fields, amounts to perform the path integral on φ. This process was done already with the worldsheet gauge fields A^α and now we would like to implement the constraints (7.2.21) directly in the path integral. Taking into account both processes, the path integral takes the form

$$\mathcal{Z} = \int \frac{[\mathcal{D}X^i]\,[\mathcal{D}\chi_\alpha]}{\mathcal{V}_{\text{gauge}}} \int [\mathcal{D}\phi_\alpha]\, \delta\big(\phi_\alpha - \tilde{\chi}_\alpha\big)_{\partial\Sigma}\, \exp\, \check{S}[X^i, \chi_\alpha]\,. \tag{7.2.24}$$

Here, $\mathcal{V}_{\text{gauge}}$ corresponds to the volume of local gauge symmetry (7.1.12), the delta function along with $\tilde{\chi}_\alpha$ has been defined on (7.2.21) and the action $\check{S}$ corresponds to the one defined in (7.2.4).

Now, given that the action does not depend on the Lagrange multipliers ϕ_α (since the Neumann condition given in (7.2.3) is currently underway) it turns out that the integration can be performed trivially and the delta integration in (7.2.24) gives exactly one.

Integrating-out III: The coordinates X^α

As we mentioned in section 6.1.4, it may happen that after the application of the Buscher rules, the action $\check{S}$ still depends on the original coordinates X^α through the vielbein e^α present at both the boundary $\partial\Sigma$ and in the residual $B-$field $\check{B}^{\text{res.}}$. Let us not forget that these coordinates X^α satisfy Neumann boundary conditions, which in particular imply that the $\mathrm{d}X^\alpha$ forms may have a non-vanishing closed and co-closed part, as stated by the Hodge decomposition theorem. The local symmetry (7.1.12) alone is not enough to remove completely the $\mathrm{d}X^\alpha$'s. It turns out that the residual $B-$field provide the required terms to remove successfully these terms.

For ease of the discussion, let us restrict now to the abelian case and make the assumption that the components k^α_β are constant. For more general cases, a case-by-case study is necessary.

In this point the residual $B-$ field, the constraints (7.1.8) and the boundary fields all play together. The constraint for the open string gauge field a in (7.1.8) establishes a relation between a, ω_α and v_α in $\partial\Sigma$, which are quantities that depend on the target space coordinates X^i only. The key here is that X^i possess a unique continuation from the boundary to the bulk, and therefore we can assume that the relations (7.1.8) and (7.1.13) still hold on Σ.

This allows us to write down $\check{B}^{\text{res.}}$as follows

$$\check{B}^{\text{res.}} = \mathrm{d}\Big[-\tilde{\chi}_\alpha e^\alpha - 2\pi\alpha' a\Big] + 2\pi\alpha'\big(\tfrac{1}{2}F_{mn}e^m \wedge e^n\big)\,. \tag{7.2.25}$$

Let us remember that $F = \mathrm{d}a$ corresponds to the open-string field strength and that $\tilde{\chi}_\alpha = \chi_\alpha + \omega_\alpha - 2\pi\alpha'\iota_{k_\alpha}a$. Notice that the boundary gauge field a and the residual B-field (both in the dual action (7.2.4)) combine in the following way

$$\begin{aligned}
&-\frac{i}{2\pi\alpha'}\int_\Sigma \check{B}^{\text{res.}} - \frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' a \\
&= +\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}\tilde{\chi}_\alpha e^\alpha - \frac{i}{2\pi\alpha'}\int_\Sigma 2\pi\alpha'\big(\tfrac{1}{2}F_{mn}e^m \wedge e^n\big)\,.
\end{aligned} \tag{7.2.26}$$

The reader can verify that the second term from (7.2.26) corresponds to the open-string gauge field intensity along the directions which were not dualized. Its components combine with the B_{mn} of $\check{B}$ into the gauge-invariant open-string field strength $2\pi\alpha'\mathcal{F}_{mn} = B_{mn} + 2\pi\alpha' F_{mn}$.

Let's have a look on the $\mathrm{d}X^\alpha$ forms now. These forms are still present in the action through the one forms e^α at the boundary. Since the forms $\mathrm{d}X^\alpha$ are closed, we are allowed to expand them accordingly as follows

$$\mathrm{d}X^\alpha = \mathrm{d}X^\alpha_{(0)} + \sum_{\mathsf{m}} X^\alpha_{(\mathsf{m})}\varphi^{\mathsf{m}}\,. \tag{7.2.27}$$

In here, $X^\alpha_{(0)}$ are globally well-defined functions on Σ, $X^\alpha_{(\mathsf{m})}$ are real constants and $\varphi^{\mathsf{m}} \in CcN^1_N$ denote a basis of closed and co-closed one-forms with vanishing normal part at the boundary $\partial\Sigma$, which respects a normalization

$$\int_{\gamma_{\mathsf{m}}} \varphi^{\mathsf{n}} = \delta^{\mathsf{n}}_{\mathsf{m}} \tag{7.2.28}$$

where we have denoted by γ_{m} a basis of the first homology on $\partial\Sigma$.

The exact part of $\mathrm{d}X^\alpha$ can be set to zero by use of the local symmetries given in (7.1.12), whereas the closed and co-closed part of it requires a bit more of care. We can distinguish two situations that might happen:

- A first possibility is that the coefficients $X^\alpha_{(\mathsf{m})}$ respect a quantization related to momentum quantization along a compact direction. Here, such coefficients are proportional to a certain $\mathbb{Z}-$valued number such that respects the normalization

$$\oint_{\gamma_{\mathsf{m}}} \mathrm{d}X^\alpha = X^\alpha_{(\mathsf{m})} = 2\pi\, n^\alpha_{(\mathsf{m})}\,, \qquad\qquad n^\alpha_{(\mathsf{m})} \in \mathbb{Z}\,. \tag{7.2.29}$$

The $n^\alpha_{(\mathsf{m})}$ could be either winding or momentum numbers; for a compactification of X^α on a circle or a flat torus without $H-$flux these winding/momentum always exist, as we see for instance in the expansions (6.1.6), (6.1.17) and (6.1.31). On more general backgrounds though, these may be absent or not be quantized at all.

Let's pick our path-integral computation done in (7.2.24) after we integrated out ϕ_α. This computation contains the terms

$$\begin{aligned}
\mathcal{Z} &\supset \int \frac{[\mathcal{D}X^\alpha]}{\mathcal{V}_{\text{gauge}}} \exp\left[\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} \tilde{\chi}_\alpha e^\alpha\right] \\
&\supset \int \frac{[\mathcal{D}X^\alpha_{(0)}]}{\mathcal{V}_{\text{gauge}}} \sum_{n^\alpha_{(\mathsf{m})}\in\mathbb{Z}} \exp\left[\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} \tilde{\chi}_\beta \left(k^{-1}\right)^\beta_{\ \alpha} \mathrm{d}X^\alpha\right] \\
&\supset \sum_{n^\alpha_{(\mathsf{m})}\in\mathbb{Z}} \exp\left[\frac{i}{\alpha'}\tilde{\chi}_\beta \left(k^{-1}\right)^\beta_{\ \alpha} n^\alpha_{(\mathsf{m})}\right]_{\partial\Sigma} \\
&\supset \sum_{m_{\alpha(\mathsf{m})}\in\mathbb{Z}} \delta\left[\frac{1}{2\pi\alpha'}\tilde{\chi}_\beta \left(k^{-1}\right)^\beta_{\ \alpha} - m_{\alpha(\mathsf{m})}\right]_{\partial\Sigma}\,.
\end{aligned} \tag{7.2.30}$$

We see that the partition function $\mathcal{Z}$ contains a quantization condition for the $\tilde{\chi}_\alpha$ coordinates; we go from the second to the third line of this computation by removing the exact part via the local symmetries, and from the third to the fourth line the definition of the periodic delta Kronecker symbol was employed [180].

The partition function is meaningful as long as the deltas blow up, leading to the condition

$$\frac{1}{\alpha'}\,\tilde{\chi}_\beta\left(k^{-1}\right)^\beta{}_\alpha\Big|_{\partial\Sigma}\in 2\pi\mathbb{Z}\,. \tag{7.2.31}$$

- The other possibility is that the real coefficients $X^\alpha_{(\mathsf{m})}$ appearing in the expansion of $\mathrm{d}X^\alpha$ are not quantized and therefore are arbitrary. It turns out that the sum in (7.2.30) becomes an integral over the variables $X^\alpha_{(\mathsf{m})}$. This finally leads to the condition

$$\tilde{\chi}_\alpha\Big|_{\partial\Sigma}=0\,. \tag{7.2.32}$$

Bear in mind that the coordinate $\tilde{\chi}_\alpha$ actually correspond to the shifted coordinate χ_α by $\pi\alpha'\iota_{k_\alpha}a-\omega_\alpha$. Aside from the presence of ω_α, the shift caused by the boundary gauge field is expected on general grounds. This is indeed what we already discussed in 6.1.1 for the case of a direction compactified on a circle.

7.2.3 The open-string sector: Dirichlet directions

Now we will perform a collective T-duality transformation along entirely Dirichlet directions. Since in this case we find no contributions related to the EOMs for A^α coming from the boundary $\partial\Sigma$, we find that there are no ϕ_α Lagrange multipliers and the procedure differs from that of Neumann boundary conditions.

Integrating-out I: gauge fields A^α

The procedure is done similarly as the closed-string case and the Neumann open-string case: we solve the equations of motion for A^α and evaluate the action on it, leading to the dual metric and dual $B-$field given in (7.2.10) and (7.2.12). Later, we implement the Dirichlet boundary conditions stated in (7.1.17).

The form of the residual B field is given in (7.2.13) and we know already that there are no constraints coming from the equations of motion for A^α in the Dirichlet boundary conditions case, as stated in (7.2.3).

Now we need to impose the boundary conditions on our worldsheet gauge field A^α, according to (7.1.17). For the Dirichlet case, it means that A^α must vanish identically on the boundary, this is

$$0=\Big[\,A^\alpha\big|_{(7.2.1)}\Big]_{\partial\Sigma}\,. \tag{7.2.33}$$

Using the explicit form of A^α given in (7.2.1) and imposing such condition in the basis $e^I=\{e^m,e_\alpha\}$ we find

$$0=\check{\mathsf{G}}^\alpha{}_I\big(e^I\big)_{\text{norm}}+i\,\check{\mathsf{B}}^\alpha{}_I\big(e^I\big)_{\text{tan}}\,. \tag{7.2.34}$$

These relations correspond to Neumann boundary conditions for the dual coordinates, encoded in the normal part of e^I. This is again expected on general grounds.

Integrating-out II: removing the X^α coordinates

As happened before for the Neumann case, the dual action (7.2.4) still has depends on the dualized coordinates X^α through the 1−forms e^α present in the residual $B-$ field. Nonetheless, we can still get rid of them as follows.

Let's assume for simplicity that the Killing vectors are constant and that $v_\alpha = 0$[32]. Since the original coordinates X^α satisfy Dirichlet boundary conditions, then the $\mathrm{d}X^\alpha$ forms are exact on $\partial\Sigma$ and we can write down the residual $B-$field (7.2.13) as follows

$$\check{B}^{\text{res.}} = e^\alpha \wedge \mathrm{d}\chi_\alpha = \mathrm{d}\Big[X^\alpha \left(k^{-1}\right)^\beta_\alpha e_\beta\Big] \,. \tag{7.2.35}$$

By using Stokes' theorem we can pull the residual $B-$field to the boundary $\partial\Sigma$, obtaining

$$\begin{aligned} -\frac{i}{2\pi\alpha'}\int_\Sigma \check{B}^{\text{res.}} = & -\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' \left[\frac{X^\alpha (k^{-1})^\beta_\alpha}{2\pi\alpha'}\, e_\beta\right] \\ & -\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' \check{\mathsf{a}}^\alpha e_\alpha . \end{aligned} \tag{7.2.36}$$

This means that the position of the D−brane in the original theory $X^\alpha|_{\partial\Sigma}$ determines a constant gauge field in the T-dual theory given by

$$\check{\mathsf{a}}^\alpha = \frac{1}{2\pi\alpha'}\left(k^{-1}\right)^\alpha_\beta X^\beta \Big|_{\partial\Sigma} \,. \tag{7.2.37}$$

Using the local transformations given in (7.1.12) we can fix X^α in the bulk Σ to a convenient value and later trivially perform the path integral, reaching thus the T-dual theory.

The results in the open string sector agree with the expected CFT results for both the Neumann and the Dirichlet boundary condition cases. In short: a T-duality transformation along a Neumann direction gives us a Dirichlet direction, and if there's a non-trivial Wilson line (i.e. boundary gauge field) it leads to a shift of the dual coordinates of the boundary. On the other hand, a T-duality transformation along a Neumann direction results in a Neumann direction in the dual theory, where the position of the original brane leads to a constant open-string boundary gauge field in the T-dual configuration.

Now that we have this construction, the next step is to illustrate this formalism via some known examples. In particular, we will consider a three-torus with constant H-flux and perform T-duality transformations along multiple directions.

[32]For more general configurations what follows can be treated in a similar manner. However, depending on the configuration they have to be treated case by case.

7.3 Examples: $\mathbb{T}^3$ with $H-$flux

In this section we will illustrate our formalism and work out different examples with the three-torus with $H-$flux as a starting background. We will consider different Dp$-$brane setups and non-trivial background fields, exploiting the power of Buscher rules for curved background configurations.

Our setup is given as follows: we consider the target-space metric of a flat-three torus $\mathbb{T}^3$ with an $H-$flux. The coordinates on the three-torus will be labeled by X^i, $i = 1, 2, 3$ equipped with the identifications $X^i \sim X^i + 2\pi$. We choose the contangent basis of the $1-$forms to be given by the $\mathrm{d}X^i$. We consider then the target-space metric components G_{ij}, the B field B and the dilaton to be given as follows

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B = \frac{\alpha'}{2\pi}\, h X^3 \, \mathrm{d}X^1 \wedge \mathrm{d}X^2 \,, \qquad \phi = \phi_0 \,. \tag{7.3.1}$$

In here, $h \in \mathbb{Z}$ responds to the flux-quantization condition. The units of dimension are carried by the metric components; the R_i have the dimension of the string-length ℓ_s whereas the coordinates X^i are dimensionless. Finally, the dilaton ϕ is taken to be constant.

Notice that the Killing vectors k_α of $\mathbb{T}^3$ are the tangent vectors to each circle, which are linearly independent and satisfy an abelian Lie algebra, i.e. the vectors k_α in (7.1.7) satisfy

$$[k_\alpha, k_\beta] = 0. \tag{7.3.2}$$

Therefore, we can write down the components of each Killing vector k_α as

$$k_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.3.3}$$

For the open-string sector, we will consider that a Dp$-$brane has Neumann boundary conditions along the time direction and along p spatial directions. The remaining directions will have Dirichlet boundary conditions. For ease of our discussion, we will work with the dimensionless dual coordinates $\check{\chi}_\alpha$ given by

$$\check{\chi}_\alpha = \frac{1}{\alpha'}\, \chi_\alpha \,. \tag{7.3.4}$$

7.3.1 One T-duality transformation

In the following cases, we will perform one T-duality transformation for the configuration given above. The T-duality transformations will be done along the X^1 direction for simplicity.

D1-brane along X^1

Let's consider then a D1−brane along the X^1 (Neumann) direction. If we choose a constant open-string gauge field a we find that $F = \mathrm{d}a = 0$ and together with the background fields given above we read that the boundary conditions (7.1.4) are

$$0 = \left(\mathrm{d}X^1\right)_{\mathrm{norm}}\,, \qquad 0 = \left(\mathrm{d}X^2\right)_{\tan}\,, \qquad 0 = \left(\mathrm{d}X^3\right)_{\tan}\,. \tag{7.3.5}$$

Let us remember that in order to apply the Buscher rules we require to find fields v_α and $\omega - \alpha$ that satisfy the set of constraints (7.1.8), (7.1.13) and (7.1.20). The reader can check that the following set of fields does indeed satisfy them

$$\begin{aligned} a &= a_1 \mathrm{d}X^1\,, \\ v_1 &= 0\,, \qquad\qquad a_1 = \text{const.}\,. \\ \omega_1 &= 0\,, \end{aligned} \tag{7.3.6}$$

We compute the dual metric and dual B−field components $\check{\mathsf{G}}_{IJ}$ and $\check{\mathsf{B}}_{IJ}$ from (7.2.10) and (7.2.12). What we obtain is

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,. \tag{7.3.7}$$

This background is known as a twisted-three torus, whose dual basis is given by $\{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$. For the dual coordinate $\mathrm{d}\check{\chi}_1$ we can read off its boundary conditions from (7.2.22). We find at the end that the boundary conditions for the dual basis is given by

$$0 = \left(\mathrm{d}\check{\chi}_1\right)_{\tan}\,, \qquad 0 = \left(\mathrm{d}X^2\right)_{\tan}\,, \qquad 0 = \left(\mathrm{d}X^3\right)_{\tan}\,. \tag{7.3.8}$$

This means that the dual background contains a $D0$−brane, just what we expected. The residual B−field takes the form $\check{B}^{\text{res.}} = \mathrm{d}X^1 \wedge \mathrm{d}\chi_1$, and by performing the path-integral procedure described in (7.2.30) we find

$$\left[\check{\chi}_1 - 2\pi a_1\right]_{\partial\Sigma} \in 2\pi\mathbb{Z}\,. \tag{7.3.9}$$

As we pointed out in 7.2.2, depending on the background configuration we might have a clear quantization scheme, which will be reflected in the winding or momentum modes present in the expansion of the X^i. In particular, we do not know how to quantize the theory in presence of a non-trivial H−flux and thus we have no information about the momentum/winding numbers of the original coordinate X^1. We face then the second situation stated in the path-integral procedure in 7.2.2 and we set the RHS of (7.3.9). Finally, we see that the dual background corresponds to a twisted torus with a $D0$−brane, whose position is specified by the Wilson line a_1.

D2-brane along X^1 and X^2

We consider now a D2-brane along the directions X^1 and X^2. We choose now to have a non-trivial boundary gauge field a such that its field intensity $F = \mathrm{d}a$ is constant whose non-zero component is $F_{12} = f = \text{const}$. We find the boundary conditions to be

$$\begin{aligned} 0 &= R_1^2 \left(\mathrm{d}X^1\right)_{\text{norm}} + 2\pi\alpha' i \left(f + \tfrac{h}{4\pi^2} X^3\right) \left(\mathrm{d}X^2\right)_{\text{tan}} , \\ 0 &= R_2^2 \left(\mathrm{d}X^2\right)_{\text{norm}} - 2\pi\alpha' i \left(f + \tfrac{h}{4\pi^2} X^3\right) \left(\mathrm{d}X^1\right)_{\text{tan}} , \\ 0 &= \left(\mathrm{d}X^3\right)_{\text{tan}} . \end{aligned} \tag{7.3.10}$$

The fields that satisfy the constraints (7.1.8), (7.1.13) and (7.1.20) for a single T-duality transformation along X^1 are

$$\begin{aligned} a &= a_1 \mathrm{d}X^1 + a_2 \mathrm{d}X^2 + f\, X^1 \mathrm{d}X^2 , \\ v_1 &= -2\pi\alpha' f\, \mathrm{d}X^2 , \qquad\qquad a_1, a_2, f = \text{const.} , \\ \omega_1 &= 0 . \end{aligned} \tag{7.3.11}$$

Given this, we find that the T-dual G and B field given by (7.2.10) and (7.2.12) give

$$\begin{aligned} \check{\mathsf{G}}_{IJ} &= \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'}{R_1^2}\left[2\pi f + \frac{\alpha'}{2\pi} hX^3\right] & 0 \\ -\frac{\alpha'}{R_1^2}\left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right] & R_2^2 + \frac{1}{R_1^2}\left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} , \\ \check{\mathsf{B}}_{IJ} &= 0 . \end{aligned} \tag{7.3.12}$$

We notice that the background dual metric has the form of that of a twisted three-torus. Surprisingly, the field intensity of the boundary gauge field a appears through the gauge-invariant open-string field strength

$$2\pi\alpha' \mathcal{F}_{12} = 2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3 . \tag{7.3.13}$$

Now to determine the boundary conditions of the dual background basis. Following (7.2.8) we find again the dual basis to be $\{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$, and from (7.2.22) we find the boundary conditions

$$\begin{aligned} 0 &= \left(\mathrm{d}\check{\chi}_1\right)_{\text{tan}} , \\ 0 &= \check{\mathsf{G}}_2{}^1 \left(\mathrm{d}\check{\chi}_1\right)_{\text{norm}} + \check{\mathsf{G}}_{22} \left(\mathrm{d}X^2\right)_{\text{norm}} , \\ 0 &= \left(\mathrm{d}X^3\right)_{\text{tan}} . \end{aligned} \tag{7.3.14}$$

These boundary conditions describe Dirichlet boundary conditions for $\check{\chi}_1$ and X^3 and a Neumann boundary condition for X^2, thus giving D1$-$brane in the dual configuration.

Now the residual $B-$field comes in. Using (7.2.13) we find that $\check{B}^{\text{res.}}$can be written down as

$$\check{B}^{\text{res.}} = \mathrm{d}X^1 \wedge (\mathrm{d}\chi_1 - 2\pi\alpha'\mathrm{d}X^2) \tag{7.3.15}$$

which cancels the $fX^1\mathrm{d}X^2$ component of the open string gauge field a through the mechanism presented in (7.2.26). Since X^1 corresponds to the direction being T-dualized, we ought to perform the path integral on it. Remembering that we are in a background on which a proper quantization procedure is not clear, then according to (7.2.32) we find

$$\left[\check{\chi}_1 - 2\pi\, a_1\right]_{\partial\Sigma} = 0\,. \tag{7.3.16}$$

Lastly, the Wilson line $a_2\mathrm{d}X^2$ is the only surviving component of the original open-string boundary gauge field, hence the dual boundary gauge field $\check{\mathsf{a}}$ has the form

$$\check{\mathsf{a}} = a_2\,\mathrm{d}X^2\,. \tag{7.3.17}$$

In short, we found that the T-dual for this configuration is a twisted three-torus with a tilted D1$-$brane along the $\check{\chi}_1 - X^2$ direction, and with a constant dual boundary gauge field along X^2.

D3-brane along X^1, X^2 and X^3

Now we set to discuss a T-duality transformation along all directions of $\mathbb{T}^3$. This is more of a formal computation, since the Freed-Witten anomaly cancellation condition does not allow such starting configuration [183]. We will perform nonetheless the computation here and in the next section we will discuss it further. For the three-torus with $H-$flux, it states that the $H-$flux pulled back to the boundary must vanish in cohomology, i.e. taking into account the boundary conditions we must be able to write it down as an exact form, which in this case it does not happen. Nevertheless, nothing stops us to formally compute the dual G and B fields.

For simplicity, let us consider the same D2$-$ brane setup presented above, this time with X^3 as a Neumann direction and $a = 0$. We obtain then

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,, \tag{7.3.18}$$

with boundary conditions

$$\begin{aligned} 0 &= \left(\mathrm{d}\check{\chi}_1\right)_{\text{tan}}\,, \\ 0 &= \check{\mathsf{G}}_2{}^1\left(\mathrm{d}\check{\chi}_1\right)_{\text{norm}} + \check{\mathsf{G}}_{22}\left(\mathrm{d}X^2\right)_{\text{norm}}\,, \\ 0 &= \left(\mathrm{d}X^3\right)_{\text{norm}}\,. \end{aligned} \tag{7.3.19}$$

As it was expected, this configurations describe a D2−brane with Dirichlet boundary conditions along the $\check{\chi}_1$ direction and Neumann boundary conditions along X^2 and X^3. It turns out that since we started with an inconsistent configuration, the dual configuration is inconsistent as well. We will treat this shortly in the next section.

D0-brane

So far we have tested our formalism for cases in which we always had to T-dualize along a Neumann direction. Let us now try other configurations for which X^1 is a Dirichlet direction and perform a T-duality transformation along it. We start then with the D0−brane, which is a point-like object on the three-torus. From (7.1.4) we read the boundary conditions for this starting configuration to be

$$0 = \big(\mathrm{d}X^1\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^2\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^3\big)_{\tan}\,. \tag{7.3.20}$$

The constraints (7.1.8) and (7.1.13) are readily solved with the choice of the following fields

$$a = 0\,, \qquad v_1 = 0\,, \qquad \omega_1 = 0\,. \tag{7.3.21}$$

Via the expressions (7.2.10) and (7.2.12) we find that the dual metric and $B-$field are given by

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,. \tag{7.3.22}$$

We find the dual background to be again a twisted three-torus. We get the dual basis via (7.2.8) and reads $\{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$, which satisfies the boundary conditions (7.2.34)

$$\begin{aligned} 0 &= \check{\mathsf{G}}^{11}\big(\mathrm{d}\check{\chi}_1\big)_{\mathrm{norm}} + \check{\mathsf{G}}^{1}{}_{2}\big(\mathrm{d}X^2\big)_{\mathrm{norm}}\,, \\ 0 &= \big(\mathrm{d}X^2\big)_{\tan}\,, \\ 0 &= \big(\mathrm{d}X^3\big)_{\tan}\,. \end{aligned} \tag{7.3.23}$$

The boundary conditions of the dual basis describe a D1−brane; we find a Neumann boundary condition for the direction $\check{\chi}_1$ and Dirichlet boundary conditions for X^2 and X^3. Moreover, the residual $B-$field $\check{B}^{\mathrm{res.}}$ takes the form

$$\check{B}^{\mathrm{res.}} = \mathrm{d}X^1 \wedge \mathrm{d}\chi_1 = \mathrm{d}(X^1\mathrm{d}\chi_1), \tag{7.3.24}$$

where the last step can be done since $\mathrm{d}X^1$ is exact. This means that we can write down

$$-\frac{i}{2\pi\alpha'}\int_\Sigma \check{B}^{\text{res.}} = -\frac{i}{2\pi\alpha'}\int_{\partial\Sigma} 2\pi\alpha' \left[\tfrac{X^1}{2\pi}\,\mathrm{d}\check{\chi}_1\right]. \tag{7.3.25}$$

This describes a constant Wilson line along the direction $\check{\chi}_1$, whose component contains the one of the coordinates of the position of the original D0−brane, i.e.

$$\check{\mathsf{a}} = \frac{X^1|_{\partial\Sigma}}{2\pi}\,\mathrm{d}\check{\chi}_1\,. \tag{7.3.26}$$

D1-brane along X^2

Let's set now a D1−brane placed along the X^2 direction with a constant Wilson line along this same direction too. We find that the boundary conditions are given by

$$0 = \big(\mathrm{d}X^1\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^2\big)_{\text{norm}}\,, \qquad 0 = \big(\mathrm{d}X^3\big)_{\tan}\,, \tag{7.3.27}$$

and that the constraints (7.1.8) and (7.1.13) are satisfied by

$$\begin{aligned} a &= a_2\,\mathrm{d}X^2\,, \\ v_1 &= 0\,, \qquad\qquad a_2 = \text{const.}\,. \\ \omega_1 &= 0\,, \end{aligned} \tag{7.3.28}$$

Applying the Buscher rules we find the dual G and B fields

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,, \tag{7.3.29}$$

with dual basis $\{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$. The boundary conditions (7.2.34) for this dual basis are evaluated as

$$\begin{aligned} 0 &= \check{\mathsf{G}}^{11}\big(\mathrm{d}\check{\chi}_1\big)_{\text{norm}} + \check{\mathsf{G}}^1{}_2\big(\mathrm{d}X^2\big)_{\text{norm}}\,, \\ 0 &= \check{\mathsf{G}}_2{}^1\big(\mathrm{d}\check{\chi}_1\big)_{\text{norm}} + \check{\mathsf{G}}_{22}\big(\mathrm{d}X^2\big)_{\text{norm}}\,, \\ 0 &= \big(\mathrm{d}X^3\big)_{\tan}\,. \end{aligned} \tag{7.3.30}$$

These expressions describe a D2−brane along the directions $\check{\chi}_1$ and X^2. By working out the residual B−field term $\check{B}^{\text{res.}}$ we find the dual open-string gauge field

$$\check{\mathsf{a}} = \frac{X^1|_{\partial\Sigma}}{2\pi}\,\mathrm{d}\check{\chi}_1 + a_2\,\mathrm{d}X^2\,. \tag{7.3.31}$$

D2-brane along X^2 and X^3

We finish this section for one T-duality transformation with the case of a D2−brane placed along the X^2 and X^3 directions. For the sake of simplicity, we consider a vanishing open string gauge field, making the analysis similar to the case of a D1−brane along X^2 seen already above; by using the expressions for the G and B field (7.2.10) and (7.2.12) we find their duals to be

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,. \tag{7.3.32}$$

The dual basis corresponds to $\{\mathrm{d}\check{\chi}_1, \mathrm{d}X^2, \mathrm{d}X^3\}$ as well, and its boundary conditions (7.2.34) are

$$\begin{aligned} 0 &= \check{\mathsf{G}}^{11}\big(\mathrm{d}\check{\chi}_1\big)_{\text{norm}} + \check{\mathsf{G}}^{1}{}_{2}\big(\mathrm{d}X^2\big)_{\text{norm}}\,, \\ 0 &= \check{\mathsf{G}}_{2}{}^{1}\big(\mathrm{d}\check{\chi}_1\big)_{\text{norm}} + \check{\mathsf{G}}_{22}\big(\mathrm{d}X^2\big)_{\text{norm}}\,, \\ 0 &= \big(\mathrm{d}X^3\big)_{\text{norm}}\,, \end{aligned} \tag{7.3.33}$$

which describe a D3−brane along the twisted three-torus. The boundary gauge field in the dual picture is given by

$$\check{\mathsf{a}} = \frac{X^1|_{\partial\Sigma}}{2\pi}\,\mathrm{d}\check{\chi}_1\,. \tag{7.3.34}$$

In view of the Freed-Witten anomaly cancellation condition, this background is consistently well defined.

7.3.2 Two T-duality transformations

Let us now employ the capabilities of this formalism a bit further and perform two collective T-duality transformations for our three-torus with $H-$flux setup given in (7.3.1).

For the next set of examples we will perform our T-duality transformations along the X^1 and X^2 directions, which both will have the same boundary conditions.

D2-brane along X^1 and X^2

We consider first a D2−brane placed along the X^1 and X^2 directions, with an open-string field a such that its field intensity is constant, i.e. $F = \mathrm{d}a$ and $F_{12} = f = \text{const.}$

We read the boundary conditions (7.1.4) for the $\mathrm{d}X^i$ elements of the original basis to be

$$\begin{aligned} 0 &= R_1^2 \left(\mathrm{d}X^1\right)_{\text{norm}} + 2\pi\alpha' i \left(f + \tfrac{h}{4\pi^2} X^3\right) \left(\mathrm{d}X^2\right)_{\tan}, \\ 0 &= R_2^2 \left(\mathrm{d}X^2\right)_{\text{norm}} - 2\pi\alpha' i \left(f + \tfrac{h}{4\pi^2} X^3\right) \left(\mathrm{d}X^1\right)_{\tan}, \\ 0 &= \left(\mathrm{d}X^3\right)_{\tan}. \end{aligned} \tag{7.3.35}$$

The fields that solve the constraints (7.1.8) and (7.1.13) for T-duality transformations along two directions are given by

$$\begin{aligned} a &= a_1 \mathrm{d}X^1 + a_2 \mathrm{d}X^2 + \tfrac{1}{2} f \left(X^1 \mathrm{d}X^2 - X^2 \mathrm{d}X^1\right), \\ v_1 &= -2\pi\alpha' f \,\mathrm{d}X^2, \\ v_2 &= +2\pi\alpha' f \,\mathrm{d}X^1, \qquad\qquad a_1, a_2, f = \text{const.} \\ \omega_1 &= -\ \pi\alpha' f \ \ X^2, \\ \omega_2 &= +\ \pi\alpha' f \ \ X^1. \end{aligned} \tag{7.3.36}$$

The dual metric $\check{G}$ and the dual B field $\check{B}$ are computed using of course (7.2.10) and (7.2.12). These take the form

$$\begin{aligned} \check{\mathsf{G}}_{IJ} &= \begin{pmatrix} \frac{\alpha'^2 R_2^2}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2 R_1^2}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \\ \check{\mathsf{B}}_{IJ} &= \begin{pmatrix} 0 & \frac{-\alpha'^2 \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ \frac{+\alpha'^2 \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{7.3.37}$$

which are the metric and the Kalb-Ramond field B for the T-fold background [121]. We find once more that the open-string sector has an effect on the closed-string sector of the theory; the field intensity $F = \mathrm{d}a$ appears via the gauge invariant field strength

$$2\pi\alpha' \mathcal{F}_{12} = 2\pi\alpha' f + \frac{\alpha'}{2\pi} h X^3, \tag{7.3.38}$$

We find the dual basis via (7.2.8) and is given by $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}X^3\}$. The boundary conditions for this dual basis can be determined from (7.2.22), which are

$$0 = \left(\mathrm{d}\check{\chi}_1\right)_{\tan}, \qquad 0 = \left(\mathrm{d}\check{\chi}_2\right)_{\tan}, \qquad 0 = \left(\mathrm{d}X^3\right)_{\tan}. \tag{7.3.39}$$

This conditions tell us that the dual background contains a D0$-$brane. The residual $B-$field (7.2.13) is found to be

$$\check{B}^{\text{res.}} = \mathrm{d}X^1 \wedge \mathrm{d}\chi_1 + \mathrm{d}X^2 \wedge \mathrm{d}\chi_2 - 2\pi\alpha' f \mathrm{d}X^1 \wedge \mathrm{d}X^2, \tag{7.3.40}$$

which via the computation below (7.2.26) fixes the positions of the $\check{\chi}_\alpha$ as

$$\big[\check{\chi}_\alpha - 2\pi\, a_\alpha\big]_{\partial\Sigma} = 0\,, \tag{7.3.41}$$

with use of (7.2.32).

D3-brane along X^1, X^2 and X^3

For illustrative purposes, let us consider a D3$-$brane extending on our three-torus with $H-$flux. Such configuration is inconsistent, according to the Freed-Witten anomaly cancelation condition [183]. The situation is analog to the case for a D2-brane along X^1 and X^2. Consider then the background (7.3.36). By performing a collective T-duality transformation along the directions X^1 and X^2 we find the same background configuration as in (7.3.37), i.e.

$$\begin{aligned}
\check{\mathsf{G}}_{IJ} &= \begin{pmatrix} \frac{\alpha'^2 R_2^2}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2 R_1^2}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \\
\check{\mathsf{B}}_{IJ} &= \begin{pmatrix} 0 & \frac{-\alpha'^2 \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]^2} & 0 \\ \frac{+\alpha'^2 \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]}{R_1^2 R_2^2 + \left[2\pi\alpha' f + \frac{\alpha'}{2\pi} hX^3\right]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{7.3.42}$$

The dual basis is $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}X^3\}$, which satisfies the boundary conditions

$$0 = \big(\mathrm{d}\check{\chi}_1\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}\check{\chi}_2\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^3\big)_{\text{norm}}\,. \tag{7.3.43}$$

These conditions point out that the dual background contains a D1$-$brane along X^3. We will mention this setup later on when we discuss the Freed-Witten anomaly condition.

D0-brane

Now we turn to explore T-duality transformations along Dirichlet directions, these being X^1 and X^2. For this D0$-$brane configuration we find the boundary conditions for the 1$-$form basis of the original background to be

$$0 = \big(\mathrm{d}X^1\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^2\big)_{\tan}\,, \qquad 0 = \big(\mathrm{d}X^3\big)_{\tan}\,, \tag{7.3.44}$$

and the constraints (7.1.8) and (7.1.13) are solved by

$$a = 0\,, \qquad v_{1,2} = 0\,, \qquad \omega_{1,2} = 0\,. \tag{7.3.45}$$

Notice that since all of the $\mathrm{d}X^i$ 1−forms vanish at the boundary, our setup does not support an open-string gauge field, hence $a = 0$. The dual background can be easily computed with help of (7.2.10) and (7.2.12), which give

$$\begin{aligned}
\check{\mathsf{G}}_{IJ} &= \begin{pmatrix} \frac{\alpha'^2 R_2^2}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2 R_1^2}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \\
\check{\mathsf{B}}_{IJ} &= \begin{pmatrix} 0 & \frac{-\alpha'^2 \frac{\alpha'}{2\pi} h X^3}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ \frac{+\alpha'^2 \frac{\alpha'}{2\pi} h X^3}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{7.3.46}$$

This is again the G and $B-$ background fields for a T-fold. The boundary conditions for the dual basis $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}X^3\}$ can be read from (7.2.33) as

$$\begin{aligned}
0 &= \check{\mathsf{G}}^{11}\big(\mathrm{d}\check{\chi}_1\big)_{\text{norm}} + i\,\check{\mathsf{B}}^{12}\big(\mathrm{d}\check{\chi}_2\big)_{\text{tan}}\,, \\
0 &= \check{\mathsf{G}}^{22}\big(\mathrm{d}\check{\chi}_2\big)_{\text{norm}} + i\,\check{\mathsf{B}}^{21}\big(\mathrm{d}\check{\chi}_1\big)_{\text{tan}}\,, \\
0 &= \big(\mathrm{d}X^3\big)_{\text{tan}}\,.
\end{aligned} \tag{7.3.47}$$

These boundary conditions indicate that our background has a D2-brane along the $\check{\chi}_1$ and $\check{\chi}_2$ directions. The dual boundary gauge field has the form

$$\check{\mathsf{a}} = \frac{X^1|_{\partial\Sigma}}{2\pi}\,\mathrm{d}\check{\chi}_1 + \frac{X^2|_{\partial\Sigma}}{2\pi}\,\mathrm{d}\check{\chi}_2\,. \tag{7.3.48}$$

D1-brane along X^3

Let us exhaust the possible cases and explore finally the case of a D1−brane along the X^3 direction with no boundary gauge field. Turns out that this case is similar to the case of the D0−brane we just reviewed: by performing a T-duality transformation along the X^1 and X^2 directions we get the T-fold background as seen in (7.3.46), i.e.

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2 R_2^2}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2 R_1^2}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} ,$$
$$\check{\mathsf{B}}_{IJ} = \begin{pmatrix} 0 & \frac{-\alpha'^2 \frac{\alpha'}{2\pi} h X^3}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 \\ \frac{+\alpha'^2 \frac{\alpha'}{2\pi} h X^3}{R_1^2 R_2^2 + \left[\frac{\alpha'}{2\pi} h X^3\right]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{7.3.49}$$

The basis for this dual space is given again by $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}X^3\}$, whose boundary conditions are again given by

$$\begin{aligned} 0 &= \check{\mathsf{G}}^{11} \big(\mathrm{d}\check{\chi}_1\big)_{\mathrm{norm}} + i\, \check{\mathsf{B}}^{12} \big(\mathrm{d}\check{\chi}_2\big)_{\mathrm{tan}} , \\ 0 &= \check{\mathsf{G}}^{22} \big(\mathrm{d}\check{\chi}_2\big)_{\mathrm{norm}} + i\, \check{\mathsf{B}}^{21} \big(\mathrm{d}\check{\chi}_1\big)_{\mathrm{tan}} , \\ 0 &= \big(\mathrm{d}X^3\big)_{\mathrm{tan}} . \end{aligned} \tag{7.3.50}$$

These conditions tell us that the dual background contains a D3$-$brane. The dual boundary gauge field is given by

$$\check{\mathsf{a}} = \frac{X^1|_{\partial\Sigma}}{2\pi} \mathrm{d}\check{\chi}_1 + \frac{X^2|_{\partial\Sigma}}{2\pi} \mathrm{d}\check{\chi}_2 . \tag{7.3.51}$$

7.3.3 Three T-dualities

In this final section we discuss the case for T-duality transformations done along each direction of our three-torus, i.e. X^1, X^2 and X^3. In this case, however, in order to comply with the set of conditions given in (7.1.13), we need to turn off the $H-$field in (7.3.1) by setting $h = 0$.

Since the directions to be T-dualized must be either all Dirichlet or all Neumann, we have two cases at our disposal: A D3$-$brane or a D0$-$brane. Let's start then with an starting background containing D3$-$brane.

D3-brane along X^1, X^2 and X^3

In this situation we perform T-duality transformations along all directions of the three-torus. Since this configuration supports with no problem an open-string gauge field along all directions, we consider a such that $F = \mathrm{d}a$ is constant, with non-zero component $F_{12} = f = \text{const.}$. Since $h = 0$, we find $B = 0$. We find then the following boundary conditions for the $\mathrm{d}X^i$ basis

$$\begin{aligned} 0 &= R_1^2 \left(\mathrm{d}X^1\right)_{\text{norm}} + 2\pi\alpha' i\, f \left(\mathrm{d}X^2\right)_{\tan}, \\ 0 &= R_2^2 \left(\mathrm{d}X^2\right)_{\text{norm}} - 2\pi\alpha' i\, f \left(\mathrm{d}X^1\right)_{\tan}, \\ 0 &= R_3^2 \left(\mathrm{d}X^3\right)_{\text{norm}}. \end{aligned} \tag{7.3.52}$$

The constraints (7.1.8) and (7.1.13) are solved with the following fields

$$\begin{aligned} a &= a_1 \mathrm{d}X^1 + a_2 \mathrm{d}X^2 + a_3 \mathrm{d}X^3 + \tfrac{1}{2} f \left(X^1 \mathrm{d}X^2 - X^2 \mathrm{d}X^1\right), \\ v_1 &= -2\pi\alpha' f\, \mathrm{d}X^2, \\ v_2 &= +2\pi\alpha' f\, \mathrm{d}X^1, \\ v_3 &= 0, \qquad\qquad a_1, a_2, a_3, f = \text{const.}, \\ \omega_1 &= -\ \pi\alpha' f\ \ X^2, \\ \omega_2 &= +\ \pi\alpha' f\ \ X^1, \\ \omega_3 &= 0. \end{aligned} \tag{7.3.53}$$

By using the expressions (7.2.10) and (7.2.12) we find the dual metric field and the dual Kalb-Ramond field to be

$$\begin{aligned} \check{\mathsf{G}}_{IJ} &= \begin{pmatrix} \frac{\alpha'^2 R_2^2}{R_1^2 R_2^2 + [2\pi\alpha' f]^2} & 0 & 0 \\ 0 & \frac{\alpha'^2 R_1^2}{R_1^2 R_2^2 + [2\pi\alpha' f]^2} & 0 \\ 0 & 0 & \frac{\alpha'^2}{R_3^2} \end{pmatrix}, \\ \check{\mathsf{B}}_{IJ} &= \begin{pmatrix} 0 & \frac{-2\pi\alpha'^3 f}{R_1^2 R_2^2 + [2\pi\alpha' f]^2} & 0 \\ \frac{+2\pi\alpha'^3 f}{R_1^2 R_2^2 + [2\pi\alpha' f]^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{7.3.54}$$

This dual background corresponds to that of a T-fold, once more. We find as well that the open-string sector interacts with the closed-string sector of the theory via the open-string field intensity $F = \mathrm{d}a$.

By using (7.2.8) we obtain that the dual basis is given by $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}\check{\chi}_3\}$. The boundary conditions for this dual basis can be determined from (7.2.22) and are

$$0 = \left(\mathrm{d}\check{\chi}_1\right)_{\tan}, \qquad 0 = \left(\mathrm{d}\check{\chi}_2\right)_{\tan}, \qquad 0 = \left(\mathrm{d}\check{\chi}_3\right)_{\tan}, \tag{7.3.55}$$

This describes a D0−brane in the dual background. The location of each dual coordinate at the boundary is given by

$$\left[\check{\chi}_\alpha - 2\pi\, a_\alpha\right]_{\partial\Sigma} = 0. \tag{7.3.56}$$

D0-brane

To close this section, we deal with the D0−brane. This configuration does not support an open-string gauge field at the boundary and we have no $B-$field whatsoever. After performing a T-duality transformation along all three directions we find the dual background

$$\check{\mathsf{G}}_{IJ} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & 0 & 0 \\ 0 & \frac{\alpha'^2}{R_2^2} & 0 \\ 0 & 0 & \frac{\alpha'^2}{R_3^2} \end{pmatrix}, \qquad \check{\mathsf{B}}_{IJ} = 0\,. \tag{7.3.57}$$

This background is actually a three-torus with inverted radii; three independent circles each of radius R_i being T-dualized into dual circles of radii α'/R_i, echoing the simple examples we featured in sections 6.1.1 and 6.1.3.

The dual basis is determined via (7.2.8) and found to be $\{\mathrm{d}\check{\chi}_1, \mathrm{d}\check{\chi}_2, \mathrm{d}\check{\chi}_3\}$. We find that the dual background contains a D3−brane, which is described by the boundary conditions

$$0 = \big(\mathrm{d}\check{\chi}_1\big)_{\mathrm{norm}}\,, \qquad 0 = \big(\mathrm{d}\check{\chi}_2\big)_{\mathrm{norm}}\,, \qquad 0 = \big(\mathrm{d}\check{\chi}_3\big)_{\mathrm{norm}}\,. \tag{7.3.58}$$

We find a dual open-string gauge field, and with help of (7.2.37) we get that its components are

$$\check{\mathsf{a}}^{\alpha} = \frac{X^{\alpha}}{2\pi\alpha'}\Big|_{\partial\Sigma}\,. \tag{7.3.59}$$

Summary

We have taken the flat three-torus with H-flux background and illustrated how to implement Buscher's rules whenever we have D-branes in our setup, according to the procedure described in section 7.2. In this section we have presented some examples for certain backgrounds with Dp-branes defined on them and performed T-duality transformations along X^1 (one transformation), X^1 and X^2 (two transformations) or X^1, X^2 and X^3 (three transformations). Depending on whether the initial Dp-brane is parallel or perpendicular to the direction of such transformations, the brane in the dual configurations complies with the known results in conformal field theory.

We present now a summary of our findings. Please note that the components of the boundary gauge fields are written down according to the corresponding 1−form basis. The D-brane configurations are indicated according to the direction on which they extend. For instance, a D1-brane along X^2 will be indicated by "X^2". A D2-brane along $\check{\chi}^1$ and X^2 in the dual configuration will be indicated by "$\check{\chi}^1 - X^2$" and so on. We highlight in red those configurations not allowed by the Freed-Witten anomaly cancellation condition. We leave the discussion of the dilation for the last section of this chapter.

Original configuration		Dual configuration		
D-brane configuration	Boundary gauge field	$\breve{G}$ and $\breve{B}$	D-brane configuration	Boundary gauge field
D0	none	Twisted $\mathbb{T}^3$ (7.3.22)	$\check{\chi}_1$	$(\frac{X^1}{2\pi}, 0, 0)$
X^1	$(a_1, 0, 0)$	Twisted $\mathbb{T}^3$ (7.3.7)	D0	none
X^2	$(0, a_2, 0)$	Twisted $\mathbb{T}^3$ (7.3.29)	$\check{\chi}_1 - X^2$	$(\frac{X^1}{2\pi}, a_2, 0)$
$X^1 - X^2$	$(a_1, a_2 + fX^1, 0)$	Twisted $\mathbb{T}^3$ with f shift(7.3.12)	X^2	$(0, a_2, 0)$
$X^2 - X^3$	none	Twisted $\mathbb{T}^3$ (7.3.32)	$\check{\chi}_1 - X^2 - X^3$	$(\frac{X^1}{2\pi}, 0, 0)$
D3	none	Twisted $\mathbb{T}^3$ (7.3.18)	$X^2 - X^3$	none

Table 7.1: Examples for one T-duality transformation. We call $f-$shift to the shift experienced by the X^3 coordinate as $\frac{\alpha'}{2\pi}hX^3 \to 2\pi\alpha' f + \frac{\alpha'}{2\pi}hX^3$.

Original configuration		Dual configuration		
D-brane configuration	Boundary gauge field	$\breve{G}$ and $\breve{B}$	D-brane configuration	Boundary gauge field
D0	none	T-fold (7.3.46)	$\check{\chi}_1 - \check{\chi}_2$	$(\frac{X^1}{2\pi}, \frac{X^2}{2\pi}, 0)$
X^3	none	T-fold (7.3.49)	$\check{\chi}_1 - \check{\chi}_2 - X^3$	$(\frac{X^1}{2\pi}, \frac{X^2}{2\pi}, 0)$
$X^1 - X^2$	$(a_1 - \frac{1}{2}fX^2, a_2 + \frac{1}{2}fX^1, 0)$	T-fold with f shift(7.3.37)	D0	none
D3	$(a_1 - \frac{1}{2}fX^2, a_2 + \frac{1}{2}fX^1, 0)$	T-fold with f shift(7.3.42)	X^3	none

Table 7.2: Examples for two T-duality transformations. As before, we call $f-$shift to the shift experienced by the X^3 coordinate as $\frac{\alpha'}{2\pi}hX^3 \to 2\pi\alpha' f + \frac{\alpha'}{2\pi}hX^3$.

Original configuration		Dual configuration		
D-brane configuration	Boundary gauge field	$\breve{G}$ and $\breve{B}$	D-brane configuration	Boundary gauge field
D0	none	Inverted $\mathbb{T}^3$ (7.3.57)	$\check{\chi}_1 - \check{\chi}_2 - \check{\chi}_3$	$(\frac{X^1}{2\pi}, \frac{X^2}{2\pi}, \frac{X^3}{2\pi})$
D3	$(a_1 - \frac{1}{2}fX^2, a_2 + \frac{1}{2}fX^1, a_3)$	T-fold(7.3.54)	D0	none

Table 7.3: Examples for three T-duality transformations.

7.4 The Freed-Witten Anomaly and boundary conditions

This section will explore the well-definedness of some examples carried out in the last section. More concretely, we will talk about the Freed-Witten anomaly cancellation condition and study if the setups featured in the previous section comply with it. After this, we will study in a general manner whether the boundary conditions are properly well defined on the T-dual configurations.

7.4.1 The Freed-Witten anomaly

Whenever we have a background with a non-vanishing $H-$flux and D-branes placed in it, the Freed-Witten anomaly cancellation condition [183] gives a criterion to determine whether the background is consistently well-defined.

In particular, we find that the pullback of the field strength $H = \mathrm{d}B$ to the D-brane has to vanish in cohomology. In formulas, if we label the cycle wrapped by the D-brane by Γ and it's Poincaré dual by $[\Gamma]$ we can express this condition as

$$H \wedge [\Gamma] = 0\,. \tag{7.4.1}$$

This is for the case with $H-$flux only. If we want to consider geometries with $F-$flux and non-geometries with $Q-$ and $R-$ fluxes we ought to generalize this condition. This issue has been addressed in [184, 145, 185–187, 151]. We find the expression

$$\big(\mathrm{d} - H\wedge\ - F\circ\ - Q\bullet\ - R\llcorner\big)\,[\Gamma] = 0\,. \tag{7.4.2}$$

We ought to consider each flux as an operator acting on the dual Poincaré $[\Gamma]$. We can write down each flux operator as listed below

$$\begin{aligned} H\wedge &= \tfrac{1}{3!}\,H_{ijk}\ \mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\mathrm{d}X^k\ ,\\ F\circ\ &= \tfrac{1}{2!}\,F_{ij}{}^{k}\ \mathrm{d}X^i\wedge\mathrm{d}X^j\wedge\ \iota_k\ ,\\ Q\bullet\ &= \tfrac{1}{2!}\,Q_i{}^{jk}\ \mathrm{d}X^i\wedge\ \iota_j\ \wedge\ \iota_k\ ,\\ R\llcorner\ &= \tfrac{1}{3!}\,R^{ijk}\ \ \iota_i\ \wedge\ \iota_j\ \wedge\ \iota_k\ . \end{aligned} \tag{7.4.3}$$

In here, ι_i corresponds to the contraction operator along the direction of the ∂_i vectors dual to the $\mathrm{d}X^i$ basis, that is $\iota_i \equiv \iota_{\partial_i}$.

We have now a way to tell whether a certain background is consistently defined whenever D-branes are involved. In particular, it is expected that (non-)consistent configurations are mapped into (non-)consistent configurations under T-duality transformations. Indeed, let us review some examples developed in the previous section.

Let us start first with the case of the three-torus with H-flux. Let's recall the case of one T-duality transformation along X^1 for a D3$-$brane in section 7.3.1. We mentioned that placing a D3$-$brane with such background configuration is not allowed according to the Freed-Witten condition. Indeed, the Poincaré dual related to the cycles wrapped by this brane $[\Gamma_{\mathrm{D3}}]$ corresponds to a point on our three-torus. Evaluating then the condition (7.4.1) on any point of our $\mathbb{T}^3$ leads us to $H \overset{!}{=} 0$; the $H-$flux must be zero if we intend to place a D3$-$brane on our three-torus.

Now let's consider the background of a twisted torus with a geometric $F-$flux in section 7.3.1 as well. With help of the generalization of the Freed-Witten anomaly cancellation condition (7.4.2) we can determine which D-brane configuration is allowed or not. The components of the geometric $F-$flux ${F_{ij}}^k$ can be easily computed from the vielbein $E^a = {E^a}_b\, e^b$ whose components diagonalize the dual target space metric $\check{\mathsf{G}}$ as $E^{-T}\check{\mathsf{G}}E^{-1} = \mathbf{1}$ We find that

$$\mathrm{d}E^k = \frac{1}{2}\, {F_{ij}}^k\, E^i \wedge E^j. \tag{7.4.4}$$

We read then that the only non vanishing component of the geometric flux ${F_{ij}}^k$ is given by

$$ {F_{23}}^1 = \frac{\alpha'}{2\pi}\frac{h}{R_1 R_2 R_3}. \tag{7.4.5}$$

This tells us that a D2$-$brane placed along the directions X^2 and X^3 is not allowed in this case; the Poincaré dual of such brane has a component along $\mathrm{d}X^3$ which is picked up by the contraction operator, thus forcing that ${f^1}_{23}$ must be zero. This agrees with our conclusions regarding this case in section 7.3.1.

As a last case, let us consider the T-fold backgrounds in section 7.3.2. In this case we can compute the non-geometric $Q-$flux via the expressions found in (7.2.17). What we can find is that the only non-vanishing component of the $Q-$flux is given by

$$ {Q_3}^{12} = \frac{\alpha'}{2\pi}\, h, \tag{7.4.6}$$

which tells us that a D1$-$brane placed along the X^3 direction on this background is forbidden. This agrees with our findings, once again.

On a side note, we can ask if the open-string gauge field intensity $F = \mathrm{d}a$ can mix with the F- and Q- fluxes, since it can mix with the metric and B-field as seen in (7.3.12) and (7.3.37). Since $F_{12} = f$ is constant, it will not affect such fluxes.

7.4.2 Well-definedness of boundary conditions

With the use of the Buscher rules, we are able to start from a certain background configuration with D$-$branes defined on it and end up in the T-dual background, where the form of the dual D$-$brane will depend on the direction and amount of T-duality transformations. We saw as well in section 6.1.3 that the $O(D, D; \mathbb{Z})$ transformations keep the mass squared of the string invariant for toroidal-compactified configurations.

We saw already in section 6.3 that the geometric and non-geometric spaces we reviewed here so far can be treated as toroidal fibrations over a circle, giving us the chance of exploring the global behavior and well-definedness of these spaces. This was done regarding the closed-string sector fields G and B. Since we are studying the open string sector of the theory as well, we can ask ourselves whether the D-branes are globally well defined on these (non-)geometric backgrounds as well. It turns out that we can study the well-definedness of the boundary conditions fiberwise as well with help of the $O(D, D; \mathbb{Z})$ set of transformations.

In this section we will study the global well-definedness of D-branes on the different geometric and non-geometric backgrounds we have seen so far. In order to do this, we require to be concrete about the specification of boundary conditions in matrix language.

Boundary conditions

Let's remember that the metric G and Kalb-Ramond field B for the examples worked out in the last section can be brought into the form

$$G_{ij} = \begin{pmatrix} G_{\mathsf{ij}}(X^3) & 0 \\ 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} B_{\mathsf{ij}}(X^3) & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.4.7}$$

where $i,j = 1,2,3$ and the fiber directions are labeled by $\mathsf{i},\mathsf{j} = 1,2$. Let us recall the boundary conditions we found in (7.1.4) too. We can express them properly in $2D-$dimensional matrix notation as

$$\begin{pmatrix} \mathrm{D} \\ \mathrm{N} \end{pmatrix} = \begin{pmatrix} \alpha' & 0 \\ 2\pi\alpha'\mathcal{F} & G \end{pmatrix} \begin{pmatrix} i\left(\mathrm{d}X\right)_{\text{tan}} \\ \left(\mathrm{d}X\right)_{\text{norm}} \end{pmatrix}. \tag{7.4.8}$$

Notice that the dilaton has not been included in this set of transformations; we will address it separately in the following subsection. We understand this expression to be evaluated at the boundary; G and $\mathcal{F}$ in this case are considered to be restricted to $\partial\Sigma$.

To specify a determined D-brane configuration we need to agree on which directions we them to be Dirichlet or Neumann. This can be done it we let a projection operator act on (7.4.8). Let Π be this operator given by

$$\Pi = \begin{pmatrix} \Delta & 0 \\ 0 & 1-\Delta \end{pmatrix}, \qquad \Delta^2 = \Delta\,, \tag{7.4.9}$$

where Δ is a diagonal matrix whose components are either zero or one. For example, a $\mathrm{D}2-$brane along the X^1 and X^2 directions is specified by choosing $\Delta = \mathrm{diag}\,(0,0,1,\ldots,1)$. This projection operator will come in handy later on.

Now we want to see what happens when we transport the D-branes along the circle S^1. Intuition says that if the boundary conditions are globally well-defined, once we go around the circle we should be able to consistently glue the resulting $D-$brane configuration with the original one via a set of appropriate transformations. This transformations should be the same that leave the generalized metric $\mathcal{H}$ invariant. This tells us that we need to see how do the boundary transformations (7.4.8) behave under $X^3 \longrightarrow X^3 + 2\pi$, taking into account the form of the G and B fields given in (7.4.7).

We first see that the tangential and normal part of $\mathrm{d}X^i$ under the action of $O(D,D;\mathbb{Z})$ behave fiber-wise as follows

$$\begin{pmatrix} i\left(\mathrm{d}X\right)_{\text{tan}} \\ \left(\mathrm{d}X\right)_{\text{norm}} \end{pmatrix} \xrightarrow{\ \mathcal{O}\ } \begin{pmatrix} i\left(\mathrm{d}\tilde{X}\right)_{\text{tan}} \\ \left(\mathrm{d}\tilde{X}\right)_{\text{norm}} \end{pmatrix} = \Omega \begin{pmatrix} i\left(\mathrm{d}X\right)_{\text{tan}} \\ \left(\mathrm{d}X\right)_{\text{norm}} \end{pmatrix}. \tag{7.4.10}$$

In here, we find that for the three-torus with $H-$flux, the twisted three-torus and the T-fold the corresponding matrices are given by

$$\begin{aligned} &\mathbb{T}^3 \text{ with } H\text{-flux:} && \Omega_{\mathsf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &\text{twisted } \mathbb{T}^3\text{:} && \Omega_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \mathsf{A}^{-1} \end{pmatrix}, \\ &\text{T-fold:} && \Omega_{\beta} = \begin{pmatrix} 1+2\pi\beta\mathcal{F} & \frac{1}{\alpha'}\beta G \\ \frac{1}{\alpha'}\beta G & 1+2\pi\beta\mathcal{F} \end{pmatrix}. \end{aligned} \tag{7.4.11}$$

Here, we use the form of the matrices A and β indicated in (6.3.7) and (6.3.11), respectively. Now we want to see how does the boundary conditions (7.4.8) change when $X^3 \longrightarrow X^3 + 2\pi$. Both matrices at the RHS of (7.4.8) must be evaluated at $X^3 + 2\pi$, and taking into account the examples of section 7.3 we then find

$$\begin{aligned} \begin{pmatrix} \mathrm{D} \\ \mathrm{N} \end{pmatrix}_{X^3+2\pi} &= \begin{pmatrix} \alpha' & 0 \\ 2\pi\alpha'\mathcal{F} & G \end{pmatrix}_{X^3+2\pi} \begin{pmatrix} i\left(\mathrm{d}\tilde{X}\right)_{\tan} \\ \left(\mathrm{d}\tilde{X}\right)_{\mathrm{norm}} \end{pmatrix} \\ &= \mathcal{O}_\star \begin{pmatrix} \alpha' & 0 \\ 2\pi\alpha'\mathcal{F} & G \end{pmatrix}_{X^3} \Omega_\star^{-1} \begin{pmatrix} i\left(\mathrm{d}\tilde{X}\right)_{\tan} \\ \left(\mathrm{d}\tilde{X}\right)_{\mathrm{norm}} \end{pmatrix} \\ &= \mathcal{O}_\star \begin{pmatrix} \mathrm{D} \\ \mathrm{N} \end{pmatrix}_{X^3}. \end{aligned} \tag{7.4.12}$$

The subscript $\star = (\mathsf{B}, \mathsf{A}, \beta)$ labels the matrices to be used for the three-torus with $H-$flux, the twisted three-torus and the T-fold. We keep in mind that the $\mathrm{d}X^i$ forms transform as in (7.4.10) and that these relations make sense as long as we evaluate them at the boundary. We conclude that the boundary conditions are globally-well defined, meaning that D-branes can be consistently glued by using gauge transformations, diffeomeorphisms and $\beta-$transformations.

We mentioned early that a specific D-brane configuration can be determined via the projection operator Π defined in (7.4.9). The question that now arises is whether the configuration remains the same after we transport it once around the circle. We find that the transformations must to be performed first, and then projected by using Π and not the other way around, this is

$$\Pi\left[\begin{pmatrix} \mathrm{D} \\ \mathrm{N} \end{pmatrix}_{X^3+2\pi}\right] = \Pi\left[\mathcal{O}_\star \begin{pmatrix} \mathrm{D} \\ \mathrm{N} \end{pmatrix}_{X^3}\right], \tag{7.4.13}$$

where $\star$ labels the $O(D,D;\mathbb{Z})$ transformations related to the three torus with $H-$flux, twisted torus and the T-fold. We emphasize once more that the configurations are well preserved as long as the respective projection is carried out after we transport the boundary conditions once around S^1.

The dual dilaton

To finish this chapter, we finally touch the subject of the transformation of the dilaton. We mentioned early in this chapter that under T-duality transformations, the dilaton transforms according to (6.1.79); we are required to keep the combination $e^{-2\phi}\sqrt{\det G}$ invariant. By using then (6.1.79) we find for each background that the dilaton is given by

$$
\begin{aligned}
&\mathbb{T}^3 \text{ with } H\text{-flux:} && \phi = \phi_0\,, \\
&\text{twisted } \mathbb{T}^3: && \phi = \phi_0 - \log\left[\frac{R_1}{\sqrt{\alpha'}}\right], \\
&\text{T-fold:} && \phi = \phi_0 - \frac{1}{2}\log\left[\frac{R_1^2 R_2^2}{\alpha'^2} + \left(2\pi f + \tfrac{h}{2\pi}X^3\right)^2\right],
\end{aligned}
\tag{7.4.14}
$$

where for the T-fold we have included the case for a constant open-string gauge field intensity $f = \text{const}$.

The breakdown for each case goes as follows. For the three-torus with $H-$flux we find that a gauge transformation leaves the (constant) metric components invariant. Therefore, under the action of the $\mathcal{O}_{\mathsf{B}}$ transformation the combination $e^{-2\phi}\sqrt{\det G}$ remains invariant. At the boundary, since the dilaton is constant it remains unaffected under $X^3 \to X^3 + 2\pi$ and thus the contribution to the boundary is well defined.

Let us consider now see the case for the twisted three-torus. By applying the transformation rule (6.1.79) we find that the dual dilaton is still constant and that the combination $e^{-2\phi}\sqrt{\det G}$ remains invariant under the action of diffeomorphisms $\mathcal{O}_{\mathsf{A}}$. The contribution of the dilaton at the boundary is therefore well defined.

Finally, for the T-fold case we find that the resulting dual dilaton is non-constant and that it depends on the X^3 coordinate. However, the dilaton behaves properly under the action of the $\beta-$transformations.

To illustrate why this is the case, let's consider the generalized metric $\mathcal{H}$ of the T-fold background given for instance in (7.3.37). We find that $\mathcal{H}$ transforms under the action of $\mathcal{O}_{\mathsf{B}}$ according to (6.1.37), and we find that it adds a shift $2\pi f \to 2\pi f + h$ when $X^3 \to X^3 + 2\pi$. This is exactly how the dilaton for the T-fold in (7.4.14) behaves under the very same shift $X^3 \to X^3 + 2\pi$; $\mathcal{O}_{\mathsf{B}}$ acts rather in an abstract way and not as a matrix multiplication on the dilaton, that is

$$
\phi(X^3 + 2\pi) = \mathcal{O}_\beta\left[\phi(X^3)\right]. \tag{7.4.15}
$$

We can finally conclude that the dilaton is globally well-defined under $X^3 \to X^3 + 2\pi$ using a $\beta-$transformation, and that the contribution to the boundary conditions is well defined.

Chapter 8

Developments on non-abelian T-Duality Transfomations

In the previous chapter we studied T-duality transformations via the implementation of Buscher's rules on a concrete background configuration – namely, the three-torus $\mathbb{T}^3$ with $H-$flux – whenever D-branes are present. In the literature, this is a well-known example of a background with abelian isometry algebra. On the other hand, Buscher's procedure allows to explore T-duality transformations for background configurations whose isometry algebra is not abelian, as stated in (7.1.7).

In this chapter we report on some developments regarding non-abelian T-duality transformations for a background with $H-$flux and a background geometry corresponding to a Lie group manifold. We realize this configuration via a WZW model and perform T-duality transformations along all isometry directions of the diagonal subgroup. We circumvent the problem of finding an invertible change of basis, as mentioned in Footnote 26 in page 78. The matter of consistently including D-branes in this picture is not addressed.

We start this chapter by discussing the WZW model for group manifolds, following and borrowing notation from [176].

8.1 On non-abelian T-duality transformations

In previous chapters we presented Buscher's procedure in order to perform T-duality transformations along the isometry directions of the underlying geometry. The setup is constructed for an arbitrary isometry algebra, i.e. $f_{\alpha\beta}{}^{\gamma}$ not necessarily zero. It is then natural to study T-duality transformations on backgrounds whose isometry algebra is non-abelian, according to Buscher's procedure.

Non-abelian T-duality transformations (NATDTs) proves to be a challenging problem. For instance, it is not clear whether the gauged model –that is, the model we get after integrating out the gauge fields A^{α} – is equivalent to the original model [158]. Another caveat is that the transformation is not invertible, thus one cannot go back since the isometries of the original model are broken once the transformations are performed [188, 189]. Another issue is that the change of basis matrix $\mathcal{T}$ displayed in (6.1.70) might be singular.

Nonetheless, NATDTs have been used as a solution-generating technique in supergravity [190, 191] and there have been developments that circumvent some of these issues. For instance, NATDTs have been understood in the context of principal chiral models in [192, 188] – for a review, see for instance [193]. Other approaches to NATDTs can be found in the context of Poisson-Lie T-duality, as in [194–196].

In the current context, we would like to study the implementation of Buscher's rules presented in chapter 6 on a background configuration realized via the WZW action.

8.2 The WZW model for Lie group manifolds

Let $g \in G$ be an element of a $D-$dimensional Lie group G with Lie algebra $\mathfrak{g}$. The WZW action (6.1.48) – without the dilaton term – featured in section 6.1.4 is given by

$$\begin{aligned} S_{\mathrm{WZW}} = &+\frac{1}{2\pi\alpha'}\int_{\Sigma} \mathrm{tr}\left[\frac{k}{4}\, g^{-1}\mathrm{d}g \wedge * g^{-1}\mathrm{d}g\right] \\ &-\frac{i}{2\pi\alpha'}\int_{\Omega} \mathrm{tr}\left[\frac{k}{3!}\, g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right], \end{aligned} \tag{8.2.1}$$

where $\partial\Omega = \Sigma$ with Σ our two-dimensional worldsheet, k corresponds to the level, $*$ denotes the Hodge dual operator and tr() is the trace operator.

Let us use $\mathsf{a}, \mathsf{b}, \mathsf{c}\ldots$ as indices labelling $\mathfrak{g}$-valued elements, and by $i, j, k\ldots$ the indices of a coordinate basis. We can define the left- and right-invariant forms ω_L and ω_R given by

$$\omega_L = g^{-1}\mathrm{d}g = {\omega_L}^{\mathsf{a}} t_{\mathsf{a}}, \qquad \omega_R = \mathrm{d}g\, g^{-1} = {\omega_R}^{\mathsf{a}} t_{\mathsf{a}}, \tag{8.2.2}$$

where the t_{a} form a set of generators for the Lie algebra $\mathfrak{g}$. These generators satisfy[33]

$$[t_{\mathsf{a}}, t_{\mathsf{b}}] = i {f_{\mathsf{ab}}}^{\mathsf{c}} t_{\mathsf{c}}, \qquad \mathrm{tr}\big(t_{\mathsf{a}} t_{\mathsf{b}}\big) = 2\delta_{\mathsf{ab}}. \tag{8.2.3}$$

Having ${\omega_L}^{\mathsf{a}}$ and ${\omega_R}^{\mathsf{a}}$ specified in (8.2.2) we can construct the metric G and the $H-$field for our theory as follows

$$\begin{aligned} G &= -k\delta_{\mathsf{ab}}\, {\omega_L}^{\mathsf{a}} \wedge * {\omega_L}^{\mathsf{b}} &&= -k\delta_{\mathsf{ab}}\, {\omega_R}^{\mathsf{a}} \wedge * {\omega_R}^{\mathsf{b}}, \\ H &= -\frac{ik}{3!} f_{\mathsf{abc}}\, {\omega_L}^{\mathsf{a}} \wedge {\omega_L}^{\mathsf{b}} \wedge {\omega_L}^{\mathsf{c}} &&= -\frac{ik}{3!} f_{\mathsf{abc}}\, {\omega_R}^{\mathsf{a}} \wedge {\omega_R}^{\mathsf{b}} \wedge {\omega_R}^{\mathsf{c}}. \end{aligned} \tag{8.2.4}$$

[33] We assume that the structure constants $f_{\mathsf{abc}} = {f_{\mathsf{ab}}}^{\mathsf{d}}\delta_{\mathsf{dc}}$ are antisymmetric in its indices.

In particular, with (8.2.4) we can bring (8.2.1) in a suggestive fashion. To this end, let us now consider a coordinate basis $\{\mathrm{d}X^i\}$ with $i = 1, \dots, D$ such that ${\omega_L}^{\mathsf{a}}$ and ${\omega_R}^{\mathsf{a}}$ as follows

$$\begin{aligned} {\omega_L}^{\mathsf{a}} &= {\omega_L}^{\mathsf{a}}{}_i \, \mathrm{d}X^i, \\ {\omega_R}^{\mathsf{a}} &= {\omega_R}^{\mathsf{a}}{}_i \, \mathrm{d}X^i. \end{aligned} \tag{8.2.5}$$

By using (8.2.5) together with (8.2.4) we can cast (8.2.1) as

$$\begin{aligned} S_{\text{WZW}} = &+\frac{1}{2\pi\alpha'}\int_\Sigma \frac{1}{2} G_{ij}(X)\mathrm{d}X^i \wedge *\mathrm{d}X^j \\ &-\frac{i}{2\pi\alpha'}\int_\Omega \frac{1}{3!} H_{ijk}(X)\mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k. \end{aligned} \tag{8.2.6}$$

We now state some definitions regarding the underlying differential geometry. The left- and right-invariant forms satisfy Maurer-Cartan equations given by

$$\begin{aligned} 0 &= \mathrm{d}{\omega_L}^{\mathsf{a}} + \frac{i}{2} {f_{\mathsf{bc}}}^{\mathsf{a}} {\omega_L}^{\mathsf{b}} \wedge {\omega_L}^{\mathsf{c}}, \\ 0 &= \mathrm{d}{\omega_R}^{\mathsf{a}} - \frac{i}{2} {f_{\mathsf{bc}}}^{\mathsf{a}} {\omega_R}^{\mathsf{b}} \wedge {\omega_R}^{\mathsf{c}}. \end{aligned} \tag{8.2.7}$$

Having this we can define vector fields dual to ${\omega_L}^{\mathsf{a}}$ and ${\omega_R}^{\mathsf{a}}$ such that

$$\iota_{\xi_{L\mathsf{b}}} {\omega_L}^{\mathsf{a}} = \iota_{\xi_{R\mathsf{b}}} {\omega_R}^{\mathsf{a}} = \delta^{\mathsf{a}}_{\mathsf{b}}, \tag{8.2.8}$$

where the vector fields $\xi_{L\mathsf{a}}$ and $\xi_{R\mathsf{a}}$ take the form

$$\xi_{L\mathsf{a}} = \left(\omega_L^{-1}\right)^i{}_{\mathsf{a}} \partial_i, \qquad \xi_{R\mathsf{a}} = \left(\omega_R^{-1}\right)^i{}_{\mathsf{a}} \partial_i. \tag{8.2.9}$$

Using (8.2.9) we define the R matrix and its inverse componentwise as follows

$$\begin{aligned} \iota_{\xi_{L\mathsf{a}}} {\omega_R}^{\mathsf{b}} &= {\omega_R}^{\mathsf{b}}{}_i \left(\omega_L^{-1}\right)^i{}_{\mathsf{a}} = R^{\mathsf{b}}{}_{\mathsf{a}} \\ \iota_{\xi_{R\mathsf{a}}} {\omega_L}^{\mathsf{b}} &= {\omega_L}^{\mathsf{b}}{}_i \left(\omega_R^{-1}\right)^i{}_{\mathsf{a}} = \left(R^{-1}\right)^{\mathsf{b}}{}_{\mathsf{a}}. \end{aligned} \tag{8.2.10}$$

We find that $\xi_{L\mathsf{a}}$ and $\xi_{R\mathsf{a}}$ satisfy the algebra

$$\begin{aligned} [\xi_{L\mathsf{a}}, \xi_{L\mathsf{b}}] &= +i {f_{\mathsf{ab}}}^{\mathsf{c}} \xi_{L\mathsf{c}}, \\ [\xi_{R\mathsf{a}}, \xi_{R\mathsf{b}}] &= -i {f_{\mathsf{ab}}}^{\mathsf{c}} \xi_{R\mathsf{c}}, \\ [\xi_{L\mathsf{a}}, \xi_{R\mathsf{b}}] &= 0, \end{aligned} \tag{8.2.11}$$

and moreover, $\xi_{L\mathsf{a}}$ and $\xi_{R\mathsf{a}}$ are Killing vector fields for the metric, that is

$$\mathcal{L}_{\xi_{L\mathsf{a}}} G = \mathcal{L}_{\xi_{R\mathsf{a}}} G = 0. \tag{8.2.12}$$

Gauging conditions

Now we take the action (8.2.6) and implement the gauging procedure found in section 6.1.4. Let us define the vector fields k_{a} and one-forms v_{a} as

$$k_{\mathsf{a}} = -i\big(\xi_{L\mathsf{a}} - \xi_{R\mathsf{a}}\big), \qquad v_{\mathsf{a}} = -i\,k\,\delta_{\mathsf{ab}}\big({\omega_L}^{\mathsf{b}} + {\omega_R}^{\mathsf{b}}\big). \tag{8.2.13}$$

We find that the vectors k_{a} span a Lie algebra $[k_{\mathsf{a}}, k_{\mathsf{b}}] = {f_{\mathsf{ab}}}^{\mathsf{c}} k_{\mathsf{c}}$ and that G and H satisfy

$$\mathcal{L}_{k_{\mathsf{a}}} G = 0, \qquad \iota_{k_{\mathsf{a}}} H = \mathrm{d}v_{\mathsf{a}}, \tag{8.2.14}$$

that is to say that k_{a} correspond to Killing vector fields of the metric G. The action (8.2.1) therefore features a global symmetry under infinitesimal transformations $\delta X^i = \epsilon^{\mathsf{a}} k^i_{\mathsf{a}}$ for ϵ^{a} constant.

Additionally, by taking the definitions (8.2.13) together with (8.2.4) it can be shown that v_{a} and H satisfy

$$\mathcal{L}_{k_{[\underline{\mathsf{a}}}} v_{\underline{\mathsf{b}}]} = {f_{\mathsf{ab}}}^{\mathsf{c}} v_{\mathsf{c}}, \qquad \iota_{k_{[\underline{\mathsf{a}}}} {f_{\underline{\mathsf{bc}}]}}^{\mathsf{d}} v_{\mathsf{d}} = \frac{1}{3}\iota_{k_{\mathsf{a}}}\iota_{k_{\mathsf{b}}}\iota_{k_{\mathsf{c}}} H, \qquad \iota_{k_{(\overline{\mathsf{a}}}} v_{\overline{\mathsf{b}})} = 0. \tag{8.2.15}$$

Since by construction the expressions (8.2.15) hold for our model, we can promote the global symmetries to local symmetries by introducing worldsheet gauge fields A^{a} and Lagrange multipliers χ_{a}. This allows us to do the isometry gauging procedure to find the dual model. The gauged WZW model takes the form

$$\begin{aligned}
\widehat{S}_{\text{WZW}} = &+\frac{1}{2\pi\alpha'}\int_\Sigma \frac{1}{2} G_{ij}(\mathrm{d}X^i + k^i_{\mathsf{a}} A^{\mathsf{a}}) \wedge *(\mathrm{d}X^i + k^j_{\mathsf{b}} A^{\mathsf{b}}) \\
&-\frac{i}{2\pi\alpha'}\int_\Sigma \Big[(v_{\mathsf{a}} + \mathrm{d}\chi_{\mathsf{a}}) \wedge A^{\mathsf{a}} + \frac{1}{2}(\iota_{[k_{\underline{\mathsf{a}}}} v_{\underline{\mathsf{b}}]} + {f_{\mathsf{ab}}}^{\mathsf{c}}\chi_{\mathsf{c}})\, A^{\mathsf{a}} \wedge A^{\mathsf{b}}\Big] \\
&-\frac{i}{2\pi\alpha'}\int_\Omega \frac{1}{3!} H_{ijk} \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k,
\end{aligned} \tag{8.2.16}$$

which is invariant under the local infinitesimal transformations (6.1.54) for $\epsilon^{\mathsf{a}} = \epsilon^{\mathsf{a}}(X)$

$$\begin{aligned}
\delta_\epsilon X^i &= +\epsilon^{\mathsf{a}} k^i_{\mathsf{a}} \\
\delta_\epsilon A^{\mathsf{a}} &= -\mathrm{d}\epsilon^{\mathsf{a}} - \epsilon^{\mathsf{b}} A^{\mathsf{c}}\, {f_{\mathsf{bc}}}^{\mathsf{a}} \\
\delta_\epsilon \chi_{\mathsf{a}} &= -{f_{\mathsf{ab}}}^{\mathsf{c}} \epsilon^{\mathsf{b}} \chi_{\mathsf{c}}.
\end{aligned} \tag{8.2.17}$$

Since in our setup we consider only closed strings we can always reach the ungauged model as indicated in section 6.1.4. At the same time, we can integrate-out the gauge fields A^{a} and find $\check{G}$ and $\check{H}$ according to (6.1.65) and (6.1.66). Since we want to study non-abelian T-duality transformations, according to (8.2.15) we have to perform at least three T-duality transformations along the isometry directions indicated by the Killing vectors k_{a}. Let us recall that in order to read the T-dual configuration we have to find an appropriate change of basis. We will address this in the next section.

8.3 T-duality

We now give an account now recent developments in studying non-abelian T-duality transformations for the three-sphere with $H-$flux, taking the WZW model presented in the previous section. The aim is to perform three T-duality transformations according to Buscher's rules and read out the dual configuration.

Before we present all of the details concerning this computation, we will make a digression on non-abelian T-duality transformations in general.

8.3.1 Dual action

Let us consider a Lie group G. Let us take the gauged WZW action (8.2.16) and perform T-duality transformations along all of the isometry directions. We integrate-out the D worldsheet gauge fields A^{a} following the procedure indicated in section 6.1.4. This implies in particular that $i, j \ldots = 1, \ldots, D$ and $\mathsf{a}, \mathsf{b}, \ldots = 1, \ldots, D$. We find the action

$$\check{S}_{\mathrm{WZW}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \check{G} - \frac{i}{2\pi\alpha'}\int_\Omega \check{H}, \tag{8.3.1}$$

where $\check{G}$ and $\check{H}$ take the usual form

$$\begin{aligned} \check{G} &= G - \quad \frac{1}{2}(\mathsf{k}+\xi)_{\mathsf{a}}\ \left[(\mathcal{G}+\mathcal{D})^{-1}\right]^{\mathsf{ab}} \wedge * \ (\mathsf{k}-\xi)_{\mathsf{b}}, \\ \check{H} &= H - \mathrm{d}\Big[\ \frac{1}{2}(\mathsf{k}+\xi)_{\mathsf{a}}\ \left[(\mathcal{G}+\mathcal{D})^{-1}\right]^{\mathsf{ab}} \wedge \quad (\mathsf{k}-\xi)_{\mathsf{b}}\Big]. \end{aligned} \tag{8.3.2}$$

where we recall the fields defined in (6.1.63)

$$\begin{aligned} \mathsf{k}_{\mathsf{a}} &= k^i_{\mathsf{a}} G_{ij}\mathrm{d}X^j, \qquad & \mathcal{G}_{\mathsf{ab}} &= k^i_{\mathsf{a}} G_{ij} k^j_{\mathsf{a}} \\ \xi_{\mathsf{a}} &= v_{\mathsf{a}} + \mathrm{d}\chi_{\mathsf{a}}, \qquad & \mathcal{D}_{\mathsf{ab}} &= \iota_{[k_{\underline{\mathsf{a}}}} v_{\underline{\mathsf{b}}]} + f_{\mathsf{ab}}{}^{\mathsf{c}}\chi_{\mathsf{c}}. \end{aligned} \tag{8.3.3}$$

Here, the one-form k_{a} is defined by the left- and right-invariant one-forms as

$$\mathsf{k}_{\mathsf{a}} = +ik\,\delta_{\mathsf{ab}}\big(\omega_L{}^{\mathsf{b}} - \omega_R{}^{\mathsf{b}}\big). \tag{8.3.4}$$

Change of basis

Here we have performed T-duality transformations along all directions indicated by each Killing vector field k_{a}. Let $A, B = 1, \ldots, 2D$ be indices denoting collectively the $\{i, \mathsf{a}\}$ indices and let $\mathrm{d}X^A$ label the basis of $1-$forms $\{\mathrm{d}X^i, \mathrm{d}\chi_{\mathsf{a}}\}$. We recall from (6.1.69) that $\check{G}$ and $\check{H}$ can be written as

$$\begin{aligned} \check{G} &= \frac{1}{2!}(\mathrm{d}X^A)^T \check{G}_{AB} \wedge * \,\mathrm{d}X^B, \\ \check{H} &= \frac{1}{3!}\check{H}_{ABC}\,\mathrm{d}X^A \wedge \mathrm{d}X^B \wedge \mathrm{d}X^C. \end{aligned} \tag{8.3.5}$$

Following the observations done in section 6.1.4 we ought to find a change of basis matrix $\mathcal{T}$ capable of block-diagonalizing the $2D \times 2D$ $\check{G}$ matrix and bring it into the form

$$\left(\mathcal{T}^T \check{G} \mathcal{T}\right)_{AB} = \begin{pmatrix} 0_{D\times D} & 0_{D\times D} \\ 0_{D\times D} & \check{\mathsf{G}}_{\mathsf{ab}}. \end{pmatrix} \tag{8.3.6}$$

The caveat here is that the change of basis matrix $\mathcal{T}$ we defined previously in (6.1.70) might be singular. It turns out that there exists an invertible change of matrix capable of block-diagonalize $\check{G}$ for a general Lie group G and bring it into the form (8.3.6). We write down the change of basis matrix blockwise as follows

$$\mathcal{T} = \frac{1}{i\sqrt{k}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{8.3.7}$$

Let us denote the components of each block by $A^i{}_{\mathsf{b}}$, $B^i{}_{\mathsf{b}}$, C_{ab} and D_{ab}. The leftmost index labels a row and the rightmost index denotes a column. Explicitly, each component is given by

$$\begin{aligned} A^i{}_{\mathsf{b}} &= -2\left(\omega_L^{-1}\right)^i{}_{\mathsf{c}}\left(1 - R^{-1}\right)^{\mathsf{c}}{}_{\mathsf{b}}, \\ B^i{}_{\mathsf{b}} &= +\left(\omega_R^{-1}\right)^i{}_{\mathsf{b}}, \\ C_{\mathsf{ab}} &= +2i\, f_{\mathsf{ab}}{}^{\mathsf{c}} \chi_{\mathsf{c}}, \\ D_{\mathsf{ab}} &= +2i\, k\, \delta_{\mathsf{ac}}\left(R^{-1}\right)^{\mathsf{c}}{}_{\mathsf{b}}. \end{aligned} \tag{8.3.8}$$

The existence of the matrix $\mathcal{T}$ relies on the invertibility of the matrices of components $(\omega_L)^{\mathsf{a}}{}_i$ and $(\omega_R)^{\mathsf{a}}{}_i$. The one-forms ρ^A in the new basis are obtained by $\rho^A = \left(\mathcal{T}^{-1}\right)^A{}_B \mathrm{d}X^B$. The new basis corresponds to the set of one-forms $\{\pi^{\mathsf{a}}, \rho^{\mathsf{a}}\}$ given by

$$\begin{aligned} \pi^{\mathsf{a}} &= -i\sqrt{k}\left[\left(\mathcal{G} + \mathcal{D}\right)^{-1}\right]^{\mathsf{ab}} \left[k\,\delta_{\mathsf{bc}}\,\omega_L{}^{\mathsf{c}} + \frac{i}{2}\mathrm{d}\chi_{\mathsf{b}}\right], \\ \rho^{\mathsf{a}} &= +i\sqrt{k}\left[\omega_R^{\mathsf{a}} - i\left(1 - R\right)^{\mathsf{a}}{}_{\mathsf{b}}\left[\left(\mathcal{G} + \mathcal{D}\right)^{-1}\right]^{\mathsf{bc}}\left(2i\,k\,\delta_{\mathsf{cd}}\,\omega_L^{\mathsf{d}} - \mathrm{d}\chi_{\mathsf{c}}\right)\right]. \end{aligned} \tag{8.3.9}$$

Taking (8.3.8) into account, $\check{G}$ in this new basis is given by

$$\begin{aligned} \check{G} &= \frac{1}{2!}(\mathrm{d}X^A)^T \check{G}_{AB} \wedge *\mathrm{d}X^B \\ &= \frac{1}{2!}(\rho^A)^T \left(\mathcal{T}^T \check{G} \mathcal{T}\right)_{AB} \wedge *\rho^B \\ &= \rho^{\mathsf{a}} \delta_{\mathsf{ab}} \wedge *\rho^{\mathsf{b}}, \end{aligned} \tag{8.3.10}$$

thus reaching a proper block-diagonalization as stated in (8.3.6). We find that ρ^{a} conforms a vielbein basis for $\check{G}$. As for the dual $H-$flux $\check{H}$, we evaluate its components in the same fashion as displayed in (6.1.75).

Notice that the diagonalization procedure can be done for any Lie group G with a Lie algebra with completely antisymmetric structure constants. This change of basis holds up for the model stated by the action (8.3.1) whenever we perform T-duality transformations along all isometry directions – indicated by the set of Killing vectors k_{a} – following Buscher's procedure.

8.3.2 Vielbein algebra

We compute now the exterior derivative of the vielbein basis. Let us recall sections 6.1.4 and 7.2.1. Once we perform the change of basis in order to block-diagonalize $\check{G}$ we ought to check that the vielbein basis $\{e^{\alpha}, e_m, e_{\alpha}\}$ closes properly. The exterior derivative of the vielbein basis that diagonalizes the metric provides of a sense of ‟twisting″of the underlying geometry. In the case of T-duality transformations, after performing the change of basis it may well happen that the exterior derivative of the $1-$forms conforming this new basis still depends on the original coordinates, indicating potential non-geometric properties of the background.

Taking the definitions given in (8.3.9) we find that

$$\begin{aligned} \mathrm{d}\pi^{\mathsf{a}} &= +\frac{1}{\sqrt{k}} f_{\mathsf{bc}}{}^{\mathsf{a}}\,\pi^{\mathsf{b}}\wedge\pi^{\mathsf{c}} + \frac{1}{2\sqrt{k}} F_{\mathsf{bc}}{}^{\mathsf{a}}\,\rho^{\mathsf{b}}\wedge\rho^{\mathsf{c}},\\ \mathrm{d}\rho^{\mathsf{a}} &= -\frac{2}{\sqrt{k}} f_{\mathsf{bc}}{}^{\mathsf{a}}\,\pi^{\mathsf{b}}\wedge\rho^{\mathsf{c}} + \frac{1}{2\sqrt{k}}\left[f_{\mathsf{bc}}{}^{\mathsf{a}} - 2k(1-R)^{\mathsf{a}}{}_{\mathsf{d}}F_{\mathsf{bc}}{}^{\mathsf{d}}\right]\rho^{\mathsf{b}}\wedge\rho^{\mathsf{c}}, \end{aligned} \tag{8.3.11}$$

where we have defined $F_{\mathsf{bc}}{}^{\mathsf{a}}$ as

$$F_{\mathsf{bc}}{}^{\mathsf{a}} = \left[\left(\mathcal{G}+\mathcal{D}\right)\right]^{\mathsf{ad}}\delta_{\mathsf{de}}\left(R^{-1}\right)^{\mathsf{e}}{}_{\mathsf{f}}f_{\mathsf{bc}}{}^{\mathsf{f}}. \tag{8.3.12}$$

Let us recall that the metric $\check{G}$ is written only in terms of ρ^{a}. The exterior derivative of ρ^{a} contains terms with π^{a}. Hence the algebras spawned by the vector fields dual to the bases $\{\rho^{\mathsf{a}}\}$ and $\{\pi^{\mathsf{a}}\}$ mix under the Lie bracket. This signals that the dual target-space might present non-geometric properties.

In contrast to what we discussed in sections 6.1.4 and 7.2.1, choosing suitable conditions for the closure under the exterior derivative becomes more involved. This prevents us to read properly the dual metric: There is a mixing with the old coordinates that prevents us from forming a clear picture of the dual basis. This issue will be left for future work.

8.3.3 The case of $SU(2)$

Let us gain some insight and illustrate parts of this procedure by studying the case for the three-sphere S^3. The idea is not new: the case for three T-duality transformations on S^3 with $H = 0$ using Buscher's procedure as presented here has been studied in [170], while other approaches for the case of three T-duality transformations for S^3 have been discussed for instance in [178, 181, 192, 189, 197–200, 194]. As for our case, the construction presented in the previous section allows us to perform three T-duality transformations, in contrast to the case seen for the $\mathbb{T}^3$ with constant $H-$flux.

Let us then consider $G = SU(2)$ as our Lie group. Let us use the set of coordinates $\{X^1, X^2, X^3\}$ to parametrize this space. We can express the components of the left- and right-invariant forms ${\omega_L}^{\mathsf{a}}$ and ${\omega_R}^{\mathsf{a}}$ in the coordinate basis $\{\mathrm{d}X^i\}$ as follows

$$
\begin{aligned}
\left(\omega_L\right)^{\mathsf{a}}{}_i &= i\begin{pmatrix} 0 & +\cos(X^1)\sin(X^3) & -\sin(X^1) \\ 0 & -\sin(X^1)\sin(X^3) & -\cos(X^1) \\ -1 & -\cos(X^3) & 0 \end{pmatrix}, \\
\left(\omega_R\right)^{\mathsf{a}}{}_i &= i\begin{pmatrix} -\cos(X^2)\sin(X^3) & 0 & +\sin(X^2) \\ -\sin(X^2)\sin(X^3) & 0 & -\cos(X^2) \\ -\cos(X^3) & -1 & 0 \end{pmatrix},
\end{aligned}
\tag{8.3.13}
$$

where $X^1, X^2 \in [0, 2\pi[$ and $X^3 \in [0, \pi[$. By using the definitions given in (8.2.4) we find the metric components to be

$$
G_{ij} = k\begin{pmatrix} 1 & \cos(X^3) & 0 \\ \cos(X^3) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.3.14}
$$

while the $H-$field is given by

$$
H = -\frac{k}{3!}\sin(X^3)\,\mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3. \tag{8.3.15}
$$

To implement Buscher's rules we construct the relevant quantities according to (8.2.13), gauge the model to get (8.2.16), integrate out the worldsheet gauge fields A^{a} to read $\check{G}$ and $\check{H}$ and perform the change of basis.

The dual metric cannot be given just yet, since the vielbein basis $\{\rho^{\mathsf{a}}\}$ does not provide us with an appropriate one-form basis. On the other hand, by computing the components of the dual $H-$field according to (6.1.75) we find that

$$
\check{H} = 0. \tag{8.3.16}
$$

This result might provide with clues about the geometric nature of the T-dual space. Let us recall the beta functions (6.1.25) and consider a non-vanishing dilaton from the very beginning. The form of the beta functions remains invariant under T-duality transformations. Let us imagine for a moment that we found an appropriate basis of $1-$forms $\{e^\alpha\}$, $\alpha = 1, 2, 3$ such that the metric is properly block-diagonalized. For our case in particular, $\check{H} = 0$ which in turn implies that the beta function for G found in (6.1.25) reads

$$0 = \check{\mathsf{R}}_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta \check{\phi}, \tag{8.3.17}$$

where $\check{\mathsf{R}}_{\alpha\beta}$ is the Ricci tensor computed from the dual metric and $\check{\phi}$ is the dual dilaton, computed from (6.1.79). It is clear that in our dual picture the curvature is related to the dual dilaton, which might provide some insight about the dual geometry. This is left for future work.

Chapter 9

Summary and conclusions of Part II

This second and final part of this doctoral work focused on the study of T-duality transformations from an open-string perspective. First, we presented a formalism to study T-duality transformations for the open string via Buscher's procedure. Afterwards, we approached the study of non-abelian T-duality transformations considering a Wess-Zumino-Witten model. Let us summarize this

Summary

- In Chapter 2 we presented an introduction on T-duality and non-geometric spaces. We studied the effect of T-duality on the compactified bosonic string on S^1 and on $\mathbb{T}^D$ and explored the invariance of the mass spectrum. Later, we introduced a generalization of the Polyakov action and studied Buscher's procedure for this background. Under a change of basis, we read off the dual background configuration. We presented in the section for non-geometric backgrounds the twisted three-torus, the T-fold and the R-space, with their respective fluxes. Finally, we presented a way to visualize the dual configurations as toroidal fibrations.

- In Chapter 3 we studied T-duality transformations for the open string via Buscher's procedure and extended previous analyses: We considered a non-trivial worldsheet topology, we included T-duality transformations for directions satisfying Dirichlet boundary conditions and we found expressions for the dual metric and Kalb-Ramond field for multiple T-duality transformations.

  We studied the dual open string sector as well. For curved backgrounds, we found the boundary conditions are exchanged under T-duality, as expected from CFT. Furthermore, if we do a T-duality transformation along a Neumann direction with a constant Wilson line, this Wilson line shifts the position of the D-brane along the dual Dirichlet direction. If we perform a T-duality transformation along Dirichlet direction, on the other hand, we find that the original position of the brane along the Dirichet direction becomes a constant Wilson line along the dual Neumann direction.

We presented an application of our formalism and studied T-duality transformations for the three-torus with H-flux. For the case of one and two T-duality transformations we found D-branes on the twisted-three torus and T-fold, respectively. We found that the open-string gauge field intensity $F = \mathrm{d}a$ can appear in the closed string sector of the dual theory.

Later, we reviewed some explicit examples applying the Freed-Witten anomaly cancellation condition. Later on we studied whether D-branes on the dual configuration were well-defined. We discovered that such D-branes are properly well-defined using $O(D, D; \mathbb{Z})$ transformations on each of the T-dual configurations. For the dual dilaton $\check{\phi}$ we found that it was properly well defined under such transformations as well.

- In Chapter 4 we studied non-abelian T-duality transformations for the WZW model. We found a non-singular change of basis matrix and the vielbein π^a, ρ^a. We found that the dual metric $\check{G}$ can be written in terms of ρ^a only. However, ρ^a and π^a mix under the action of the exterior derivative, suggesting that the dual configuration is non-geometric. Nonetheless, for the case $G = SU(2)$ we find that the dual H-flux vanishes.

Discussion and conclusions

String theory has driven the development of sophisticated frameworks and techniques. It has lead to the discovery of interesting structures, geometries, spaces and it has given us an interesting way to think about how Nature works at a fundamental level. It provides a unified way to view the constituents of matter and the fundamental forces: This can be attributed to the oscillation modes of the string. Even though the prediction it makes are still far to be testable, it provides us a first suggestion of how a quantum theory of gravity should look like.

Since the fundamental object of the theory is an extended string, it means that it probes geometry in a different way as the point particle does [158]. This lead to the discovery and development of the so-called non-geometric backgrounds. In this second part of this doctoral work, we focused on bosonic open string theory, and explored many technical aspects surrounding T-duality transformations which have not been addressed so far. The exploration of the abelian isometry algebra case and the treatment of our examples allowed us to make consistency checks with the literature, while finding an interplay between the open and closed string sector.

The next step is to consider non-abelian T-duality transformations. In the treatment of chapter three, we couldn't reach the original model for a non-abelian isometry algebra. Another issue we found is that the change of basis we presented in (7.2.8) is singular for some non-abelian cases.

In chapter four we presented some current developments using a non-linear sigma model. We considered a WZW with Lie group manifold G with Lie algebra $\mathfrak{g}$ and followed Buscher's procedure. We can construct the Killing vectors of the target space metric from the diagonal subalgebra of $\mathfrak{g}$ and perform T-duality transformations along all of them. We found a non-singular change of basis for this case, but we fail to provide the dual background since the vielbeins ρ^{a} and π^{a} mix under the action of exterior derivative. The problem stems from the presence of the coordinates of the original configuration X^i. Integrating-out these coordinates proves to be highly non-trivial. If we were to solve this problem, the next step would be to incorporate D-Branes into this picture. Nonetheless, the case for $G = SU(2)$ shows that the dual H-flux vanishes, pointing out that the curvature of the dual configuration depends solely on the dilaton. This approach to non-abelian T-duality transformations is still underway and will be subject for future work.

Part III

Close

General conclusion

In the history of science, theory and experiment have been constantly in interplay. The search for generality and unification has been one of the driving forces of the development of theories and frameworks that address certain domains of Nature. In the age of gigantic particle accelerators, gravitational wave observatories and world-wide radiotelescope networks, our most fundamental ideas of Nature are being put to test right now in an unprecedented scale.

It is likely that in the course of the next decades, new intriguing results and data regarding the large-scale structure of the universe, black holes, particle physics and gravitational wave astronomy will be uncovered. This implies that new developments for our current understanding of theoretical physics will be needed. During the 20th century, the unification of phenomena in physics was a major driving force behind such developments.

We explored in this thesis two of such developments: Horndeski theory and non-geometric backgrounds within string theory. With use of Cartan's first formalism, we studied Horndeski theory and discovered that non-minimal couplings between the scalar field and gravity are generic sources of torsion. The phenomenological implications of this still need to be worked-out, and the upcoming improvements in instrumentation of the new gravitational wave observatories will help us falsify torsion.

We studied non-geometric spaces from the open bosonic string perspective. Since bosonic string theory dictates that the number of dimensions of spacetime is 26, we need to implement new mechanisms to make contact with the usual four-dimensional spacetime. By doing this, we explored compactification schemes, and we found that T-duality relates different background configurations that describe the same physics. We presented a non-linear sigma model for the open string and explored non-geometric backgrounds via T-duality transformations, extending previous results found in the literature. By using the standard setup of the three-torus with H-flux, we studied the global well-definedness of D-branes on such non-geometric backgrounds. T-duality transformations for the on-abelian isometry algebra case were explored as well. Such treatment is left for future work.

The exploration of new structures and frameworks within physics is crucial to the pursuit a better understanding of the fundamental mechanisms of Nature. New improvements in the sensitivity and design of particle detectors and gravitational wave observatories will provide us with new clues about directions that need to be taken. The history of physics has repeatedly showed us that this is the case.

Appendix A

The bosonic string

A.1 The Polyakov action

Consider a string moving in a d dimensional Minkowski spacetime. This sweeps a two-dimensional surface, the *worldsheet* Σ, in such spacetime. This surface is parametrized by the coordinates[34] $\sigma^{\mathsf{a}} = (\sigma^0, \sigma^1) = (\tau, \sigma)$. We consider for generality that the boundary of Σ is not necessarily empty. To describe the motion of the string, we consider d coordinates $X^\mu = X^\mu(\tau, \sigma)$, where $\mu = 0, 1, \ldots, d-1$.

The most simple Poincaré invariant action for our string can be given in terms of the area of Σ, namely

$$\begin{aligned} S_{\mathrm{NG}} &= -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}A \\ &= -\frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det \gamma_{\mathsf{ab}}}. \end{aligned} \tag{A.1.1}$$

In here, we have defined γ_{ab} as the induced metric on the worldsheet

$$\gamma_{\mathsf{ab}} = \eta_{\mu\nu}\, \partial_{\mathsf{a}} X^\mu\, \partial_{\mathsf{b}} X^\nu, \tag{A.1.2}$$

and α' corresponds to the Regge slope, with units of $(\text{length})^2$ [35] This action is called the Nambu-Goto action whose equations of motion with respect to X^μ describe the motion of our string. However, it turns out that the square root makes a deeper analysis rather convoluted. We can remove this square root by introducing an auxiliary worldsheet-valued metric $h_{ab} = h_{ab}(\sigma^a)$ with signature $(-,+)$ and by writing the equivalent Polyakov action

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det h}\, h^{\mathsf{ab}}\, \eta_{\mu\nu}\, \partial_{\mathsf{a}} X^\mu\, \partial_{\mathsf{b}} X^\nu. \tag{A.1.3}$$

[34] We consider that these parameters take values $\sigma^1 \in]\sigma^1_{\mathrm{i}}, \sigma^1_{\mathrm{f}}[$, and $\sigma^2 \in [0, \ell[$

[35] We can relate the Regge slope to the tension of the string T via $T = 1/2\pi\alpha'$. The open and closed string tensions are the same.

It can be checked that this action is classically equivalent to the Nambu-Goto action: Consider the equations of motion of h_{ab}

$$0 = T_{\mathsf{ab}} \tag{A.1.4}$$

where T_{ab} is the usual energy-momentum tensor[36] given by

$$T_{\mathsf{ab}} = -\frac{1}{\alpha'}\left[\partial_{\mathsf{a}} X^{\mu}\,\partial_{\mathsf{b}} X_{\mu} - \frac{1}{2} h_{\mathsf{ab}} h^{\mathsf{cd}}\,\partial_{\mathsf{c}} X^{\mu}\,\partial_{\mathsf{d}} X_{\mu}\right]. \tag{A.1.6}$$

By using (A.1.4) and (A.1.6) and replacing in (A.1.3) we immediately recover S_{NG}.

This action features three symmetries[37]:

1. Poincaré invariance in d dimensions (global):

$$\begin{aligned} \delta X^{\mu} &= a^{\mu}{}_{\nu}\, X^{\nu} + b^{\mu}, \\ \delta h_{\mathsf{ab}} &= 0. \end{aligned} \tag{A.1.7}$$

where $a^{\mu}{}_{\nu}$ and b^{μ} are constants such that $a_{\mu\nu} = -a_{\nu\mu}$.

2. Diffeomorphism invariance (local):

$$\begin{aligned} \delta X^{\mu} &= -\zeta^{\mathsf{a}} \partial_{\mathsf{a}} X^{\mu}, \\ \delta h_{\mathsf{ab}} &= -(\zeta^{\mathsf{c}}\,\partial_{\mathsf{c}} h_{\mathsf{ab}} + \partial_{\mathsf{a}} \zeta^{\mathsf{c}}\, h_{\mathsf{cb}} + \partial_{\mathsf{b}} \zeta^{\mathsf{c}}\, h_{\mathsf{ac}}) \\ &= -(\nabla_{\mathsf{a}} \zeta_{\mathsf{b}} + \nabla_{\mathsf{b}} \zeta_{\mathsf{a}}), \end{aligned} \tag{A.1.8}$$

where ζ^{a} are arbitary infinitesimal functions of σ^{a}.

3. Weyl invariance in two dimensions (local):

$$\begin{aligned} \delta X^{\mu} &= 0, \\ \delta h_{\mathsf{ab}} &= 2\Omega\, h_{\mathsf{ab}}, \end{aligned} \tag{A.1.9}$$

where Ω is an arbitrary infinitesimal function of σ^{a}.

We find that Weyl invariance leads to constraints for the energy-momentum tensor T_{ab}

$$0 = h^{\mathsf{ab}} T_{\mathsf{ab}} = T^{\mathsf{a}}{}_{\mathsf{a}} \qquad \text{(Weyl)}, \tag{A.1.10}$$

[36] The energy momentum is defined by

$$T_{\mathsf{ab}} = \frac{4\pi}{\sqrt{-h}}\,\frac{\delta S_{\text{P}}}{\delta h^{\mathsf{ab}}} \tag{A.1.5}$$

[37] It is worth pointing out that the Nambu-Goto action does feature Poincaré and diffeomorphism invariance, as given for the Polyakov action.

while diffeomorphism invariance, on the other hand, leads to energy-momentum conservation on the worldsheet

$$0 = \nabla^{\mathsf{a}} T_{\mathsf{ab}} \quad \text{(Diffeo)} \tag{A.1.11}$$

Before going into the equations of motion for X^μ, it is worth to mention a few points. First, this action allows a natural generalization by replacing the flat Minkowski background by a general metric $\eta_{\mu\nu} \to G_{\mu\nu}(X)$, fact that will bring us to the construction of non-linear sigma models in the upcoming chapters. Second, let's consider the quantity

$$\chi = \frac{1}{4\pi} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det h}\,\mathsf{R} + \frac{1}{2\pi} \int_{\partial\Sigma} \mathrm{d}s\, k(s)\,, \tag{A.1.12}$$

where R is the curvature scalar for the worldsheet metric h_{ab} and $k(s)$ corresponds to the extrinsic curvature of the boundary $\partial\Sigma$, with parameter s. This is nothing more than the Euler number of the worldsheet. We can add this quantity to S_P and consider $S = S_P - a\chi$. Although (A.1.12) breaks Weyl symmetry, it is a crucial if we want to study string perturbation theory, and it plays a role later on when we explore the non-linear sigma model[38]. For now, it is enough to consider the action (A.1.3).

A.2 Conformal gauge and equations of motion for the string

Let us compute the equations of motion of the string and study the boundary conditions. Of course, we could have considered either the Nambu-Goto action (A.1.1) or the Polyakov action (A.1.3) and derive the equations of motion for X^μ. However, we can derive the equations of motion more easily if we turn to the conformal gauge.

Before we proceed, it is a good moment to write down the action (A.1.3) in a suggestive manner. Let $\mathrm{d}X^\mu = \partial_{\mathsf{a}} X^\mu \mathrm{d}\sigma^{\mathsf{a}}$ be a set of one-forms where the $X^\mu = X^\mu(\sigma)$ are the usual target space coordinates. Consider as well the Hodge star operator $*$ on Σ and the components of the unit tangential and normal vectors to $\partial\Sigma$ by t^{a} and n^{a},

[38]One can see the role of χ if we write the full path integral defining the string theory

$$\mathcal{Z} = \int \mathcal{D}X\,\mathcal{D}g\, e^{S}.$$

The resulting amplitudes are weighted by a factor of $e^{\lambda\chi} = g_s^\chi$, which tells us that the amplitude is to be expanded in powers of the string coupling $g_s = e^\lambda$. Considering the different processes of absorption and emission of open and closed strings, the Euler number related to a worldsheet can be written as $\chi = 2 - 2h - b - c$, where h, b and c are the number of holes, boundaries and crosscaps. Clearly, χ works as a bookkeeping device.

respectively. Clearly, $\mathrm{d}X^\mu|_{\partial\Sigma} = t^{\mathsf{a}}\,\partial_{\mathsf{a}}\,X^\mu \mathrm{d}s$ where s is the parameter of the boundary, as usual. We can rewrite then the Polyakov action as

$$S_{\mathrm{P}} = -\frac{1}{2\pi\alpha'}\int_\Sigma \frac{1}{2}\eta_{\mu\nu}\,\mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu. \tag{A.2.1}$$

No matter which gauge choice we do regarding the worldsheet metric, the Hodge dual takes care of the form of h_{ab} automatically.

The two dimensional worldsheet metric h_{ab} has 3 independent degrees of freedom (since it is a 2×2 symmetric matrix). With help of the local symmetry transformations (A.1.8)–(A.1.9) we have the right amount of parameters (two for diffeomorphism transformations and one for Weyl transformations) to fix the metric as

$$h_{\mathsf{ab}} = \eta_{\mathsf{ab}}. \tag{A.2.2}$$

With this gauge choice, we see that (A.2.1) can be written as

$$\begin{aligned} S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\eta_{\mu\nu}\left(-\partial_\tau\,X^\mu\,\partial_\tau\,X^\nu + \partial_\sigma\,X^\mu\,\partial_\sigma\,X^\nu\right) \\ &= -\frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\,\eta_{\mu\nu}\left(-\dot{X}^\mu\,\dot{X}^\nu + X'^\mu\,X'^\nu\right) \\ &= \int_\Sigma \mathrm{d}^2\sigma\,\mathcal{L}. \end{aligned} \tag{A.2.3}$$

where we have defined $\dot{X}^\mu \equiv \partial_\tau\,X^\mu$ and $X'^\mu \equiv \partial_\sigma\,X^\mu$. [39].

We find the equations of motion for X^μ by performing the variation of S_{P} with respect to X^μ. By using integation by parts we find

$$\begin{aligned} \delta_X S_{\mathrm{P}} = 0 = &+\frac{1}{2\pi\alpha'}\int_\Sigma \delta X^\mu\,\eta_{\mu\nu}\,\mathrm{d}*\mathrm{d}X^\nu \\ &-\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} \delta X^\mu\,\eta_{\mu\nu}\,*\mathrm{d}X^\nu. \end{aligned} \tag{A.2.4}$$

The vanishing of the variation lead us to the EOMs for the string[40]

$$0 = \mathrm{d}*\mathrm{d}X^\mu, \tag{A.2.5}$$

while the boundary term in (A.2.4) delivers us the behaviour of the string at $\partial\Sigma$. We need to distinguish two cases: for the closed string we find that Σ has the topology of a

[39] This fixing can always be done locally. However, in order to have a globally well-defined conformal gauge fixing we have to satisfy more involved conditions. See Pag. 18 and 19 [115] for more details.

[40] We consider the variation of X^μ at τ_i and τ_f to be $\delta X^\mu(\tau_i) = \delta X^\mu(\tau_f) = 0$.

cylinder and thus $\partial\Sigma = \varnothing$ and there's no contribution from the boundary. We impose then periodic boundary conditions of the form

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell). \tag{A.2.6}$$

On the other hand, for the open string we have $\partial\Sigma \neq \varnothing$. In this case, we will consider the usual boundary conditions for the classical string, i.e. Dirichlet and Neumann boundary conditions, which can be casted as

$$\begin{aligned} &\text{Dirichlet} \qquad 0 = \quad \delta X^\mu|_{\partial\Sigma}, \\ &\text{Neumann} \qquad 0 = * \, \mathrm{d}X^\mu|_{\partial\Sigma}. \end{aligned} \tag{A.2.7}$$

We can recover the usual boundary conditions given in the literature by choosing a parametrization of the worldsheet such that the unit normal and tangential vectors are parallel to the lines of constant σ and τ, respectively. Thus we recover

$$\begin{aligned} &\text{Dirichlet} \qquad 0 = \partial_\tau X^\mu|_{\sigma=0,\ell}, \\ &\text{Neumann} \qquad 0 = \partial_\sigma X^\mu|_{\sigma=0,\ell}. \end{aligned} \tag{A.2.8}$$

If we think for a moment, what are actually the objects to which the endpoints of the open string are attached? We haven't placed any object in the theory besides the string itself, much less a wall. This is actually the first indication of the existence of a fundamental object in string theory, called a *D-brane*. D-branes are determined by the number of Dirichlet boundary conditions and Neumann boundary conditions and correspond to objects on which the endpoints of open string are attached. Closed strings, on the other hand, are not attached to them.

The equation of motion (A.2.5) is nothing more than the wave equation on the worldsheet. For the closed string, we find that the general two-dimensional solution for this equation is given by $X^\mu(\sigma^a) = X^\mu_L(\sigma^a) + X^\mu_R(\sigma^a)$, where

$$\begin{aligned} X^\mu_R(\sigma^a) &= \frac{1}{2}(x^\mu - c^\mu) + \frac{\pi\alpha'}{\ell} p^\mu(\tau - \sigma) \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n} \alpha^\mu_n \exp\left[-i\frac{2\pi n}{\ell}(\tau - \sigma)\right], \\ X^\mu_L(\sigma^a) &= \frac{1}{2}(x^\mu + c^\mu) + \frac{\pi\alpha'}{\ell} p^\mu(\tau + \sigma) \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \frac{1}{n} \bar{\alpha}^\mu_n \exp\left[-i\frac{2\pi n}{\ell}(\tau + \sigma)\right], \end{aligned} \tag{A.2.9}$$

where c^μ is a constant, x^μ is the center of mass position of the string at $\tau = 0$ and p^μ is the total space-time momentum of the string[41]. The coefficients α_n^μ and $\bar{\alpha}_n^\mu$ are Fourier coefficients such that they satisfy the reality condition

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \qquad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^* \tag{A.2.12}$$

For the open string we have two possible boundary conditions for each endpoint, making up a total of 4 combinations. It will be enough to write down the expansions for the Dirichlet-Dirichlet (DD) and Neumann-Neumann (NN) string. Recalling the form of the Dirichlet boundary conditions in (A.2.8) and fixing the endpoints of the string via $X^\mu(\tau, \sigma = 0) = x_0^\mu$ and $X^\mu(\tau, \sigma = \ell) = x_1^\mu$ we find that for DD

$$\begin{aligned} X^\mu_{\text{DD}}(\sigma) &= x_0^\mu + \frac{1}{\ell}(x_1^\mu - x_0^\mu)\sigma \\ &\quad + \sqrt{2\alpha'} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n^\mu \exp\left[-i\frac{\pi n \tau}{\ell}\right] \sin\left[\frac{\pi n\sigma}{\ell}\right], \end{aligned} \tag{A.2.13}$$

we notice that there is no center of mass momentum. For the open string with NN boundary conditions we find

$$\begin{aligned} X^\mu_{\text{NN}}(\sigma) &= x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau \\ &\quad + i\sqrt{2\alpha'} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n^\mu \exp\left[-i\frac{\pi n \tau}{\ell}\right] \cos\left[\frac{\pi n\sigma}{\ell}\right]. \end{aligned} \tag{A.2.14}$$

As it happens for the closed string, x^μ and p^μ are the center of mass position and total spacetime momentum, respectively. The oscillators α_n^μ for the open string – regardless of the boundary conditions, satisfy the reality conditions (A.2.12).

A.3 Hamilton dynamics

The first setting on which one can see what the effects of a T-duality transformation are corresponds to the study of the mass spectrum of a quantized string with one compactified dimension. To get to that point, we need to have a look on the quantization procedure of the bosonic string. The canonical quantization procedure takes the relevant dynamical quantities of the classical description of our system and promotes them to operators, point which will address in the following pages.

[41] For the open and closed string, x^μ and p^μ are called the *zero modes*. We define for ease of exposition

$$\text{(closed)} \quad \alpha_0^\mu = \sqrt{\alpha'/2}\,p^\mu \tag{A.2.10}$$

$$\text{(open)} \quad \alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu \tag{A.2.11}$$

By considering the Lagrangian density $\mathcal{L}$ from (A.2.3) we find that the conjugate momentum to X^μ is

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \frac{1}{2\pi\alpha'}\,\dot{X}^\mu. \tag{A.3.1}$$

By using the usual Poisson brackets at equal τ[42]we find that both X^μ and Π^μ satisfy the relations

$$\begin{aligned} 0 &= [X^\mu(\sigma), X^\nu(\sigma')]_{\text{PB}} = [\Pi^\mu(\sigma), \Pi^\nu(\sigma')]_{\text{PB}}, \\ \eta^{\mu\nu}\delta(\sigma-\sigma') &= [X^\mu(\sigma), \Pi^\nu(\sigma')]_{\text{PB}}. \end{aligned} \tag{A.3.2}$$

Let us recall the closed and open string expansions (A.2.9), (A.2.13) and (A.2.14). Regarding the oscillator coefficients for both kind of strings, it can be shown using (A.3.2) that they satisfy the Poisson brackets

$$\begin{aligned} -im\,\delta_{m+n}\,\eta^{\mu\nu} &= [\alpha^\mu_m,\,\alpha^\nu_n] = [\bar{\alpha}^\mu_m,\,\bar{\alpha}^\nu_n], \\ 0 &= [\alpha^\mu_m,\,\bar{\alpha}^\nu_n], \\ \eta^{\mu\nu} &= [x^\mu,\,p^\nu]. \end{aligned} \tag{A.3.3}$$

The Hamiltonian for the open and the closed string can be computed by considering the Hamiltonian density $\mathcal{H}(\sigma) = \dot{X}^\mu \Pi_\mu - \mathcal{L}$ for each string and then by integrating along its length. For the closed string we get

$$\begin{aligned} H_{\text{closed}} &= \int_0^\ell \mathrm{d}\sigma\,\mathcal{H}_{\text{closed}}(\sigma) \\ &= \frac{\pi}{\ell}\sum_{n\in\mathbb{Z}}(\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n), \end{aligned} \tag{A.3.4}$$

where we have defined for simplicity $\alpha_m \cdot \alpha_n \equiv \alpha^\mu_n\,\alpha_{\mu,n}$. For the open string we distinguish between the (DD) and (NN) cases. Performing the same integration along the length of each string we find

$$\begin{aligned} H_{\text{DD}} &= \frac{1}{4\pi\alpha'\ell}(x^\mu_1 - x^\mu_0)^2 + \frac{\pi}{2\ell}\sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}}\alpha_{-n}\cdot\alpha_n, \\ H_{\text{NN}} &= \frac{\pi}{2\ell}\sum_{n\in\mathbb{Z}}\alpha_{-n}\cdot\alpha_n. \end{aligned} \tag{A.3.5}$$

So far we haven't touched the constraints (A.1.10). Since these conditions arise from a local symmetry treated as a gauge symmetry, it implies immediately that a proper

[42] This is analog to the case in classical mechanics when we study Poisson brackets between different phase-space functions at equal time.

dynamical description of our system – in terms of position and momenta, require to know all of the constraints for an appropriate account of the physical degrees of freedom and for a proper writing of the Hamiltonian[43]. Furthermore, the equations of motion for h_{ab} (A.1.4) are to be taken as constraints as well to ensure the equivalence between the Nambu-Goto action and the Polyakov action while keeping the conformal gauge fixing. We can bring these constraints into a helpful way. Tracelessness and vanishing of the energy-momentum tensor can be expressed as

$$\begin{aligned} T_{01} = T_{10} &= -\frac{1}{\alpha'}\,\dot{X}^\mu X'_\mu &&= 0, \\ T_{00} = T_{11} &= -\frac{1}{2\alpha'}\,\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) &&= 0. \end{aligned} \tag{A.3.6}$$

By using the light-cone change of coordinates $\sigma^\pm = \tau \pm \sigma$ we find

$$\begin{aligned} T_{++} &= -\frac{1}{\alpha'}\,\partial_+ X^\mu \partial_+ X_\mu = 0, \\ T_{--} &= -\frac{1}{\alpha'}\,\partial_- X^\mu \partial_- X_\mu = 0, \\ &\qquad T_{+-} = T_{-+} = 0 \end{aligned} \tag{A.3.7}$$

Implementing energy-momentum conservation (A.1.11) in these expressions for these coordinates together with the constraints (A.3.7), we find the conservation equations

$$\begin{aligned} 0 &= \partial_+ T_{--} \Rightarrow T_{--} = T_{--}(\sigma^-) \\ 0 &= \partial_- T_{++} \Rightarrow T_{++} = T_{++}(\sigma^+). \end{aligned} \tag{A.3.8}$$

which is a statement of the existence of infinite conserved charges of the form

$$\begin{aligned} L_f &= \frac{1}{\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\, f(\sigma^+) T_{++}(\sigma^+), \\ \bar{L}_g &= \frac{1}{\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\, g(\sigma^-) T_{--}(\sigma^-), \end{aligned} \tag{A.3.9}$$

with f and g arbitrary functions. We can show that under appropriate choices of f and g for the closed and open string, this family of conserved charges satisfy an infinite-dimensional algebra. Taking for instance the closed string and defining

$$\begin{aligned} L_n &= -\frac{\ell}{4\pi^2}\int_0^\ell \mathrm{d}\sigma\, \exp\left[-i\frac{2\pi n}{\ell}\sigma\right] T_{--} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m, \\ \bar{L}_n &= -\frac{\ell}{4\pi^2}\int_0^\ell \mathrm{d}\sigma\, \exp\left[+i\frac{2\pi n}{\ell}\sigma\right] T_{++} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \bar{\alpha}_{n-m}\cdot\bar{\alpha}_m. \end{aligned} \tag{A.3.10}$$

[43] For a detailed account on this, see for instance [201, 202].

we can see that implementing the set of constraints is equivalent to (classically) implementing $L_m = 0$ for each m. Notice that $H_{\text{closed}} = \frac{2\pi}{\ell}(L_0 + \bar{L}_0)$. Let us turn now to the case for the open NN string. Taking into account the mixture of left and right movers, we define

$$\begin{aligned} L_n = -\frac{\ell}{4\pi^2}\int_0^\ell \mathrm{d}\sigma \, \Bigg[\exp\left[-i\frac{2\pi n}{\ell}\sigma\right] T_{--} \\ + \exp\left[+i\frac{2\pi n}{\ell}\sigma\right] T_{++}\Bigg] = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\cdot\alpha_m. \end{aligned} \tag{A.3.11}$$

and similarly for the DD open string for $n \neq 0$. For both DD and NN we can find as well that $H = (\pi/\ell)L_0$. Collecting our results (A.3.10) and (A.3.11) together with (A.3.3) we find that the L_m and $\bar{L}_m$ are actually generators satisfying the Witt algebra

$$\begin{aligned} [L_m, L_n]_{\text{PB}} &= -i(m-n)L_{m+n}, \\ [\bar{L}_m, \bar{L}_n]_{\text{PB}} &= -i(m-n)\bar{L}_{m+n}, \\ [L_m, \bar{L}_n]_{\text{PB}} &= 0. \end{aligned} \tag{A.3.12}$$

Implementing the constraints (A.3.7) in this context is equivalent to say $L_m = \bar{L}_m = 0$ for each m, which in the quantized context becomes essential in order to define the physical states. We finish this part by remarking on a noteworthy property of these generators. Implementing the constraint $L_0 = \bar{L}_0 = 0$ for both the NN open string and closed string and recalling the definition of the zero modes (A.2.10) and (A.2.11) we find

$$\begin{aligned} M^2 &= \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n && \text{(open NN)} \\ M^2 &= \frac{2}{\alpha'}\sum_{n=1}^{\infty} (\alpha_{-n}\cdot\alpha_n + \bar{\alpha}_{-n}\cdot\bar{\alpha}_n) && \text{(closed)} \end{aligned} \tag{A.3.13}$$

This tells us that the mass of the string is given in terms of the string tension (through α') and the number of excited oscillators in the string. Obviously, this observation is valid only in this classical setting: We proceed now to quantize the string.

A.4 Quantization procedure

For this, let's consider the Poisson brackets (A.3.2) and (A.3.3) and implement the usual quantization procedure $[\ \ ,\ \ ]_{\text{PB}} \to -i[\ \ ,\ \ ]$. Our set of Poisson brackets for the position, canonical momenta and oscillators – which we now consider as operators, become the following set of commutators at equal τ

$$\begin{aligned} [X^\mu(\sigma), \Pi^\mu(\sigma')] &= i\,\eta^{\mu\nu}\,\delta(\sigma-\sigma') & [\alpha^\mu_m, \alpha^\nu_n] &= m\,\delta_{m+n}\,\eta^{\mu\nu} \\ [X^\mu(\sigma), X^\mu(\sigma')] &= 0 & [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] &= m\,\delta_{m+n}\,\eta^{\mu\nu} \\ [\Pi^\mu(\sigma), \Pi^\mu(\sigma')] &= 0 & [\alpha^\mu_m, \bar{\alpha}^\nu_n] &= 0 \\ & & [x^\mu, p^\nu] &= i\,\eta^{\mu\nu}. \end{aligned} \tag{A.4.1}$$

where the oscillators satisfy the reality (hermeticity) conditions $(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}$ and $(\bar{\alpha}^\mu_m)^\dagger = \bar{\alpha}^\mu_{-m}$. What naturally comes after is to define a state in a Fock space properly labeled with our available data. We start with a ground state labeled by its center of mass momentum k^μ given by $|0; k^\mu\rangle$. This state is defined in such a way that it is annihilated by all positive modes α^μ_m, $m > 0$, i.e. by the annihilation operators. Creation operators are those corresponding to the negative modes a^μ_{-m}, $m > 0$. This state satisfies then

$$\begin{aligned} \alpha^\mu_m\,|0; k^\mu\rangle &= 0, \\ p^\mu\,|0; k^\mu\rangle &= k^\mu|0; k^\mu\rangle. \end{aligned} \tag{A.4.2}$$

That is, this state is an eigenstate of the momentum operator p^μ.

We have so far considered the quantization procedures for the string expansion. However, the Witt generators L_n satisfying the algebra (A.3.12) come into play and carry the information about the conformal invariance of our worldsheet description of the string's evolution, and need to be properly quantized as well. This can be achieved by means of normal ordering – by keeping annihilation operators to the right. However, L_0 poses an issue since the zero modes a^μ_0 bring a ambiguity at the moment of ordering, which can be cured by introducing a constant term. We will address this issue later on when we define the physical states.

We define then the generators L_n to be

$$\begin{aligned} L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} :\alpha_{n-m}\cdot\alpha_m: \quad \text{for } n \neq 0 \\ L_0 &= \frac{1}{2}\alpha_0^2 + \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m \end{aligned} \tag{A.4.3}$$

with similar expressions for $\bar{L}_m$ and $\bar{L}_0$. Taking care of the ordering, we see that these objects satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}\,m(m^2-1)\delta_{m+n}. \tag{A.4.4}$$

This states that the Virasoro algebra is nothing more that the central extension of the Witt algebra. The $\mathbb{C}-$number c it is called the *central charge* and it accounts for the breaking of the Weyl invariance at the quantum level. Its value depends on the matter content in the 2d theory defined on the worldsheet; each free worldsheet boson X^μ contributes with one unit to the value of c, while worldsheet fermions ψ^μ each make a contribution of 1/2. At the moment of studying the path integral of the free boson theory and bringing the Faddeev-Popov procedure, the vanishing of the free anomaly can be achieved if $d = 26$, and we get $d = 10$ for the superstring. The references [203, 204] will serve as a primer on this matter for the interested reader.

Now we define the physical states. As it happens in constrained Hamiltonian systems, our theory is described by configurations respecting the constraints under which our system is restricted. This means that our physical states $|\phi\rangle$ must comply with set of constraints (A.3.7) via the Virasoro generators in a quantum context, i.e. in the form

$$L_n \,|\phi\rangle \,= 0 \qquad \forall n. \tag{A.4.5}$$

However, the presence of the central charge in the Virasoro algebra leads to inconsistencies if we implement this condition for all n. In light of this, we define the physical states as those which satisfy the set of constraints.

$$\begin{aligned} L_m \,|\phi\rangle \; &= 0 \quad n > 0, \\ (L_0 + a)\,|\phi\rangle \; &= 0. \end{aligned} \tag{A.4.6}$$

where a is a constant regarding the normal ordering ambiguity for L_0. These constraints apply as well for the $\bar{L}_m$'s when considering the closed string, along with the *level matching condition*

$$(L_0 - \bar{L}_0)\; |\phi\rangle = 0. \tag{A.4.7}$$

With these considerations, we can express the mass operator for both the closed and open string. It can be shown that the mass for the closed string can be written as

$$\begin{aligned} \alpha' m^2 \; &= -\alpha' p^\mu p_\mu \\ &= \alpha' m_L^2 + \alpha'\, m_R^2 \\ &= 2(\bar{N} + a) + 2(N + a). \end{aligned} \tag{A.4.8}$$

where a is the normal ordering constant in (A.4.6) and N is the eigenvalue for of the *number operator* given by

$$N \,= \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{A.4.9}$$

for the closed string, with an analog expression for $\bar{N}$ and $m_L^2 = m_R^2$. For the open string, on the other hand we find

$$N \,= \sum_{n=1}^{\infty} \left(\alpha^\mu_{-n}\, \alpha_{\mu,n} + \alpha^i_{-n}\, \alpha_{i,n}\right), \tag{A.4.10}$$

where μ labels NN directions and i the DD directions[44].It can be shown that the mass of the open string is given by.

$$\alpha' m^2 \;=\; N + \alpha'(T\Delta x)^2 + a. \tag{A.4.11}$$

In here, T is the tension of the string, and Δx corresponds to the distance between the endpoints in the DD directions and a corresponds to the normal ordering constant as seen in (A.4.6). Again, here N is the eigenvalue of the number operator.

A.4.1 The light-cone gauge

We can express the mass formulae (A.4.8) and (A.4.11) in terms of physical degrees of freedom with help of the light-cone gauge. This gauge is a conformal gauge which uses the residual gauge freedom $\tau \to \tilde{\tau} = f(\sigma^+) + g(\sigma^-)$ and $\sigma \to \tilde{\sigma} = f(\sigma^+) - g(\sigma^-)$. Let $X^\pm = (1/\sqrt{2})(X^0 \pm X^1)$. We find that the components target-space valued vector A given by (A^+, A^-, A^i), $i = 2, \ldots, D$ will be raised and lowered as $A^+ = -A_-$, $A^- = -A_+$ and $A^i = A_i$. We fix X^+ as

$$X^+ = \frac{2\pi\alpha'}{\ell} p^+ \tau. \tag{A.4.12}$$

We notice that with this choice, all oscillators α_n^μ and $\bar{\alpha}_n^\mu$ vanish in the expansion for X^+ with exception of the zero modes. We find them to be

$$\begin{aligned} \alpha_0^+ = \bar{\alpha}_0^+ \;&=\; \sqrt{\frac{\alpha'}{2}} p^+ \;\text{ (closed string)}, \\ \alpha_0^+ \;&=\; \sqrt{2\alpha'} p^+ \;\text{ (open string)}. \end{aligned} \tag{A.4.13}$$

By defining the center of mass position $q^- = (1/\ell) \int_0^\ell \mathrm{d}\sigma X^-$ we find that the Polyakov action in conformal gauge (A.2.3) can be written as

$$\begin{aligned} S_{\text{P,l.c.}} &= \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \left[(\dot{X}^i)^2 - (X'^i)^2 \right] - \int \mathrm{d}\tau p^+ \partial \dot{q}^- \\ &= \int \mathrm{d}\tau L_{\text{l.c.}}, \end{aligned} \tag{A.4.14}$$

where we recall that $i = 2, \ldots, D$.

[44]We haven't considered in this exposition the DN or ND boundary conditions, which contribute to an additional term in the number operator. See for instance equations (2.105), (2.106) and (3.51) of [115].

The Hamiltonian in light-cone gauge

In this section we derive the form of the Hamiltonian for the open and closed string in the light-cone gauge. For that, we find first the canonical momenta p_- and Π^i following the usual Hamiltonian procedure

$$\begin{aligned} p_- &= \frac{\partial L_{\text{l.c.}}}{\partial \dot{q}^-}, \\ \Pi^i &= \frac{\partial L_{\text{l.c.}}}{\partial \dot{X}^i} = \frac{1}{2\pi\alpha'}\dot{X}^i. \end{aligned} \tag{A.4.15}$$

With this, we are able to write down the Hamiltonian in light-cone gauge as follows

$$H_{\text{l.c.}} = \frac{1}{4\pi\alpha'}\int_0^\ell \mathrm{d}\sigma\big[(\dot{X}^i)^2 + (X'^i)^2\big]. \tag{A.4.16}$$

By using the constraints (A.3.6) and $p^\mu = \frac{1}{2\pi\alpha}\int \mathrm{d}\sigma \dot{X}^\mu$ we find that

$$p^- = \frac{\ell}{2\pi\alpha' p^+}H_{\text{l.c.}}. \tag{A.4.17}$$

Under this gauge, the dynamical variables are given by the quantities p_-, q^-, X^i and Π^i. In order to write down the expansion of the light-cone Hamiltonian $H_{\text{l.c.}}$ we impose the following commutation relations between these variables

$$\begin{aligned} [q^-, p^+] &= -i, & [q^i, p^j] &= i\delta^{ij}, \\ [\alpha_n^i, \alpha_m^j] &= n\delta^{ij}\delta_{n+m,0}, & [\bar{\alpha}_n^i, \bar{\alpha}_m^j] &= n\delta^{ij}\delta_{m+n,0}. \end{aligned} \tag{A.4.18}$$

We recall now the expansions for the open and closed string given by the expressions (A.2.9), (A.2.13) and (A.2.14) and consider only the directions for which $\mu \geq 2$. Taking into account normal ordering, we find for the closed string

$$H_{\text{closed, l.c.}} = \frac{2\pi}{\ell}\left[\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \bar{\alpha}_{-n}^i\bar{\alpha}_n^i\right] + \frac{2\pi}{\ell}\left[a+\bar{a}\right] + \frac{\pi\alpha'}{\ell}p^ip^i, \tag{A.4.19}$$

while for the open string we have

$$\begin{aligned} H_{\text{open, l.c.}} = &\frac{\pi}{\ell}\left[\sum_{n>0}\alpha_{-n}^i\alpha_n^i\right] + \frac{\pi}{\ell}a \\ &+\frac{\pi\alpha'}{\ell}\sum_{\text{NN}}p^ip^i + \frac{1}{4\pi\alpha'\ell}\sum_{\text{DD}}(x_1^i - x_2^i)^2, \end{aligned} \tag{A.4.20}$$

where a and $\bar{a}$ account for the unregularized expression $\sum_n n$ arising from the normal ordering prescription on the α and $\bar{\alpha}$ oscillators, respectively. Note that we consider only open strings of with NN or DD boundary conditions. By using the definition of the Riemann zeta function $\zeta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{A.4.21}$$

one can show that $a = \bar{a} = -(1/24)(d-2)$. The mass operator $m^2 = -p_\mu p^\mu$ in the light-cone gauge takes the form $m^2 = 2p^+p^- - p^ip^i$. With this, we can express the masses for the open and closed string in terms of physical degrees of freedom only. Using our previous definitions, we have for the closed string $m^2 = m_L^2 + m_R^2$, where

$$\begin{aligned} \alpha' m_L^2 &= 2\bar{N}_{\rm tr} - \frac{1}{12}(d-2) \\ \alpha' m_R^2 &= 2N_{\rm tr} - \frac{1}{12}(d-2), \end{aligned} \tag{A.4.22}$$

where

$$\begin{aligned} \bar{N}_{\rm tr} &= \sum_{n>0} \bar{\alpha}^i_{-n}\bar{\alpha}^i_n, \\ N_{\rm tr} &= \sum_{n>0} \alpha^i_{-n}\alpha^i_n. \end{aligned} \tag{A.4.23}$$

On the other hand, for the open string we find the mass to be

$$\alpha' m^2 = N_{tr} - \frac{1}{24}(d-2) + \alpha'(T\Delta X)^2, \tag{A.4.24}$$

where we recall that $T^{-1} = 2\pi\alpha'$.

The massless closed-string spectrum

In this subsection we discuss the massless states for the closed string. Consider for simplicity that this string propagates in a Minkowski spacetime. Let us consider the ground state $|0\rangle$. By using the mass operator $m^2 = m_L^2 + m_R^2$ along with (A.4.22) we find

$$\alpha' m^2|0\rangle = -\frac{1}{6}(d-2)|0\rangle, \tag{A.4.25}$$

which implies for $d > 2$ that its mass squared is negative. This state corresponds to a scalar tachyon. We compute now the mass for the next excited state, which is $\alpha^i_{-1}\bar{\alpha}^j_{-1}|0\rangle$. By using the mass operator we find that its mass is

$$\alpha' m^2 = 4 - \frac{1}{6}(d-2). \tag{A.4.26}$$

Now, the state $\alpha^i_{-1}\bar{\alpha}^j_{-1}|0\rangle$ transforms under the Euclidean group in $d-2$ dimensions $E(d-2)$. However, since we discuss propagation of strings on a Minkowski background, let us consider rather $SO(d-2)$ which is contained in $E(d-2)$. $SO(d-2)$ corresponds precisely to the group whose representations classify massless particles [205]. This implies that such state must be massless. In particular, this implies that $d = 26$, which is the number of dimensions in which the bosonic string propagates.

We can finally decompose this state into irreducible representations of $SO(d-2)$. We find the symmetric traceless, antisymmetric and trace part of the state as follows.

$$\begin{aligned}\alpha^i_{-1}\bar{\alpha}^j_{-1}|0,p\rangle = & \left[\alpha^{(\bar{i}}_{-1}\bar{\alpha}^{\bar{j})}_{-1} - \frac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}\right]|0,p\rangle \\ & +\alpha^{[\underline{i}}_{-1}\bar{\alpha}^{\underline{j}]}_{-1}|0,p\rangle \\ & +\frac{1}{24}\delta^{ij}\alpha^k_{-1}\bar{\alpha}^k_{-1}|0,p\rangle.\end{aligned} \tag{A.4.27}$$

This decomposition indicates that we can represent a massless spin-two particle, an antisymmetric tensor and a massless scalar. These fields can be identified as the metric field $G_{\mu\nu}$ (symmetric), the Kalb-Ramond field $B_{\mu\nu}$ (antisymmetric) and the dilaton field (scalar). These are precisely the fields which we use in part 2 to construct a particular $\sigma-$model.

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