Department of Strategy and Innovation
Working Paper No. 10/2020

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March 2020

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Anonymous, neutral, and resolute social choice revisited*

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March 2, 2020

Abstract

We revisit the incompatibility of anonymity and neutrality in singleton-valued social choice. We first analyze the irresoluteness structure these two axioms together with Pareto efficiency impose on social choice rules and deliver a method to refine irresolute rules without violating anonymity, neutrality, and efficiency. Next, we propose a weakening of neutrality called *consequential neutrality* that requires resolute social choice rules to assign each alternative to the same number of profiles. We explore social choice problems in which consequential neutrality resolves impossibilities that stem from the fundamental tension between anonymity, neutrality, and resoluteness.

Keywords: anonymity, efficiency, neutrality, resoluteness

JEL Codes: D71, D72, D82

We are grateful to Walter Bossert, Denis Cornaz, Fatih Demirkale, Lars Ehlers, Ayça Ebru Giritligil, Jeffrey Hatley, Sean Horan, Hervé Moulin, Jean Lainé, Clemens Puppe, Yves Sprumont, and William Zwicker for helpful discussions. The paper extensively benefited from the thoughtful comments of three anonymous referees to whom we are grateful. Our work is partly supported by the projects ANR-14-CE24-0007-01, CoCoRICo-CoDec, and IDEX ANR-10-IDEX-0001-02 PSL MIFID.

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1 Introduction

Equal treatment of individuals as well as of alternatives are among the core principles of democratic decision-making. Equal treatment of individuals is usually ensured by the *anonymity* condition, which requires the social choice to be invariant under renaming individuals. The typical condition to ensure equal treatment of alternatives, on the other hand, is *neutrality*, which requires the social choice to change in compliance with renaming of alternatives.

The logical incompatibility between anonymity and neutrality while ensuring an untied outcome is among the most well-known results in social choice theory. Moulin (1980, 1991) characterizes the sizes of social choice problems that admit anonymous and neutral social choice rules (SCRs) that are resolute, *i.e.*, that choose a unique alternative at any profile. More precisely, a social choice problem with n individuals and m alternatives admits an anonymous, neutral, and resolute SCR if and only if m cannot be written as the sum of some divisors of n that exceed 1 (Moulin, 1991). When (Pareto) efficiency is imposed on top of anonymity and neutrality, this requirement is strengthened to "n not having a prime divisor less than or equal to m " (Moulin, 1980).[1]

How severe is this tension between anonymity and neutrality? Campbell and Kelly (2015) show the rarity of cases where anonymous, neutral, and resolute SCRs exist: when the number of individuals is divisible by two or more distinct primes, only a finite number of social choice problems admit anonymous, neutral, and resolute SCRs. Also, when the number of alternatives exceeds the smallest prime dividing the number of individuals, a resolute SCR is anonymous and neutral only if it chooses alternatives that are in the bottom half of preferences of all individuals. Adding efficiency on top of anonymity and neutrality restricts the sizes of social choice problems that admit anonymous, neutral, and resolute SCRs even further.

Do these results leave any hope for guaranteeing equal treatment of voters and alternatives for untied collective choice? We reject pessimism by identifying a weakening of neutrality which allows a vast range of possibilities while pandering to a very significant aspect of neutral treatment of alternatives. This new condition that we call *consequential neutrality* requires that all alternatives are chosen at the same number of preference profiles. The moderated character consequential neutrality possesses puts forward an ex-ante fairness property that is more outcome-oriented compared to the classical neutrality approach that entails a more procedure-oriented equal treatment of alternatives.

We start by analyzing the structure of irresoluteness imposed by anonymity, neutrality, and efficiency, a previously overlooked matter. We generalize the characterization of Moulin

[1] Zwicker (2016) delivers an introduction to the theory of voting where major results regarding anonymity and neutrality are included.

(1980) by completely describing the sizes of unavoidable ties under these conditions (Theorem 2). This generalization paves the way to identifying a method to refine SCRs that are "more irresolute than necessary", while anonymity, efficiency, and neutrality are preserved (Theorem 3).

We then turn to our analysis of consequential neutrality for resolute SCRs. We start with counting the number of resolute SCRs that are neutral and those that are consequentially neutral (CN) as a function of the size of the social choice problem (Theorem 4). An analytical comparison of these two numbers seems beyond reach, so we take a computational approach where we calculate the numbers of resolute SCRs in each class for a small set of values of the size of the social choice problem. These numerical exercises that we report on show strong tendencies in the comparison of the numbers of CN and neutral SCRs, hence we conjecture that the class of resolute SCRs that are CN is considerably larger than those that are neutral.

Thereafter, we discuss the possibility of refining anonymous, efficient, and neutral SCRs by replacing neutrality with consequential neutrality and deliver a possibility result under certain conditions (Theorem 5). We also identify some cases where these conditions are not satisfied but there exist anonymous, CN, and resolute SCRs (Theorem 6). These positive results do not hold over all conceivable social choice problems: we point to instances where anonymity, consequential neutrality, efficiency, and resoluteness turn out to be incompatible (Theorem 7). These are instances where the incompatibility prevails even without efficiency. However, even in those cases, anonymous, CN, and resolute SCRs exist. In fact, we are able to identify a large class of social choice problems, namely those where the number of alternatives exceeds the number of individuals, for which anonymity, consequential neutrality, and resoluteness are compatible (Theorem 9).

The paper is organized as follows. Section 2 gives basic notation and notions. Section 3 delivers a generalization of the classical result on incompatibility of anonymity, neutrality, and efficiency with resoluteness and proposes a refinement method towards resoluteness. Section 4 introduces consequential neutrality and presents our more permissive results when it replaces neutrality. Section 5 concludes.

2 Basic notions and notation

Writing $\mathbb{N}$ for the set of natural numbers and picking $m, n \in \mathbb{N} \setminus \{1\}$, we conceive a *social choice problem* as a set A of alternatives with $\#A = m$ and a set N of individuals with $\#N = n$. We refer to (m, n) as the *size* of the social choice problem (A, N). Writing $\mathcal{L}(X)$ for the set of linear orders, *i.e.,* complete, antisymmetric, and transitive binary relations on a given set X, let $P_i \in \mathcal{L}(A)$ denote the *preference* of $i \in N$.[2] An $n-$tuple of such individual

[2] So, given any distinct $x, y \in A$ and $P_i \in \mathcal{L}(A)$, precisely one of xP_iy and yP_ix holds. Moreover, xP_iy and yP_iz implies xP_iz for all $x, y, z \in A$ and $P_i \in \mathcal{L}(A)$.

preferences indicates a x*(preference) profile* $P_N \in \mathcal{L}(A)^N$. A *social choice rule* (SCR) is a mapping $f : \mathcal{L}(A)^N \to \mathcal{A}$, where $\mathcal{A} = 2^A \diagdown \{\varnothing\}$ is the set of non-empty subsets of A.

Given any two sets S and T, we write $S \subseteq T$ whenever S is a subset of T and $S \subset T$ whenever S is a proper subset of T. We let $P_N|_B$ denote the restriction of $P_N \in \mathcal{L}(A)^N$ to those alternatives in $B \in \mathcal{A}$ so that $P_N|_B \in \mathcal{L}(B)^N$ and $xP_iy \iff xP_i|_By$ for all $x, y \in B$ and $i \in N$. Given any two SCRs f_1 and f_2, we define the composite SCR $f_1f_2 : \mathcal{L}(A)^N \to \mathcal{A}$ such that $f_1f_2(P_N) = f_1(P_N|_{f_2(P_N)})\ \forall P_N \in \mathcal{L}(A)^N$. Given any two SCRs f_1 and f_2, we say that f_2 *refines* f_1 iff $f_2(P_N) \subseteq f_1(P_N)\ \forall P_N \in \mathcal{L}(A)^N$ and $f_2(P'_N) \subset f_1(P'_N)$ for some $P'_N \in \mathcal{L}(A)^N$. An SCR f is *resolute* whenever $\#f(P_N) = 1\ \forall P_N \in \mathcal{L}(A)^N$. For a resolute SCR f, we write $f(P_N) = x$ in place of $f(P_N) = \{x\}$.

We now define two equal treatment conditions that are at the core of our anaysis. For any non-empty finite set X, a *permutation* on X is a bijection $\sigma : X \leftrightarrow X$. Let Σ_X be the set of all permutations on X. We write, by a slight abuse of notation, $\sigma(P_N) = (P_{\sigma(i)})_{i\in N}$ for the profile obtained from $P_N \in \mathcal{L}(A)^N$ by a permutation $\sigma \in \Sigma_N$ on N. An SCR is *anonymous* iff $f(P_N) = f(\sigma(P_N))\ \forall P_N \in \mathcal{L}(A)^N\ \forall \sigma \in \Sigma_N$. Again, by an abuse of notation, we write $\sigma(P_i)$ for the preference obtained from $P_i \in \mathcal{L}(A)$ by a permutation $\sigma \in \Sigma_A$ on A, *i.e.*, $x\ P_i\ y \iff \sigma(x)\ \sigma(P_i)\ \sigma(y)\ \forall x, y \in A$. Moreover, $\sigma(P_N) = (\sigma(P_i))_{i\in N}\ \forall P_N \in \mathcal{L}(A)^N$. An SCR is *neutral* iff $x \in f(P_N) \iff \sigma(a) \in f(\sigma(P_N))\ \forall P_N \in \mathcal{L}(A)^N$, $\forall x \in A$, and $\forall \sigma \in \Sigma_A$.

We close the section by noting that an SCR f is *efficient* iff given any $P_N \in \mathcal{L}(A)^N$ and any $x \in f(P_N)$, $\nexists y \in A\backslash\{x\}$ with $y\ P_i\ x\ \forall i \in N$.

3 Anonymous, neutral, and efficient social choice

Given any $k, l \in \mathbb{N}$, we write $k \mid l$ whenever k divides l, *i.e.*, $\frac{l}{k} \in \mathbb{N}$, and $k \nmid l$ otherwise. Let $\mathcal{D}(n) = \{k \in \mathbb{N} : k \mid n\}$ for the set of divisors of $n \in \mathbb{N}$ while $\mathcal{D}^*(n) = \{k \text{ is a prime} : k \mid n\} \cup \{1\}$ is the set consisting of prime divisors of n as well as 1. Thus, $\mathcal{D}^*(n) \subseteq \mathcal{D}(n)$ for all $n \in \mathbb{N}$. For any $m, n \in \mathbb{N}$, the set of divisors of n that do not exceed m is denoted $D_{n,m} = \{d \in \mathcal{D}(n) : d \leq m\}$ and similarly $D^*_{n,m} = \{d \in \mathcal{D}^*(n) : d \leq m\}$. Again, $D^*_{n,m} \subseteq D_{n,m}$ for all $m, n \in \mathbb{N}$. Imposing $D^*_{n,m} = \{1\}$ is shown by Moulin (1980) to be a necessary and sufficient condition for a social choice problem to admit an anonymous, neutral, efficient, and resolute SCR.

Theorem **1** (**Moulin** (**1980**)) *A social choice problem (A, N) with size (m, n) admits an anonymous, efficient, neutral, and resolute SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ if and only if $D^*_{n,m} = \{1\}$.*

The condition $D^*_{n,m} = \{1\}$ in Theorem 1 can be replaced by $D_{n,m} = \{1\}$.[3] We will refer

[3]Having already noted $D^*_{n,m} \subseteq D_{n,m}$, we now remark that $D^*_{n,m} = \{1\} \implies D_{n,m} = \{1\}$ for all $m, n \in \mathbb{N}$. To see this, let $k \in D_{n,m}\backslash\{1\}$. Thus, we have $k \in \mathcal{D}(n)$ and $k \leq m$. Due to the fundemantal theorem of arithmetics, k has a prime divisor k^*, which divides n as well, hence $k^* \in D^*_{n,m}$, impying $D^*_{n,m} \neq \{1\}$.

to $D_{n,m} = \{1\}$ as Condition $\mu(m,n)$.[4]

Theorem 1 gives a complete picture of the sizes of social choice problems where irresoluteness is inevitable but is silent about the structure of irresoluteness in such cases. To analyze this, we define $K_f = \{\#f(P_N) : P_N \in \mathcal{L}(A)^N\}$ as the *irresoluteness structure* of SCR f. So for any natural number $k \leq m$, we have $k \in K_f$ if and only if there exists a profile to which f assigns precisely k alternatives.

Theorem 2 *Take any social choice problem (A, N) with size (m, n).*

i. *An SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ is anonymous, efficient, and neutral only if $K_f \supseteq D_{n,m}$.*

ii. *There exists an anonymous, efficient, and neutral SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ with $K_f = D_{n,m}$.*

Proof:

i. Take any $d \in D_{n,m}$. As $d \in \mathcal{D}(n)$, $n = d \times t$ for some $t \in \mathbb{N}$. Take any set of alternatives $\{a_1, \ldots, a_d\} \subseteq A$ and any partition $\{S_1, \ldots, S_d\}$ of N with $\#S_i = t$ for all $i \in \{1, \ldots, d\}$. Let $X = A \backslash \{a_1, \ldots, a_d\}$. Note that X may be empty. Now construct a profile P_N as depicted below:[5]

S_1	S_2	S_3	$\cdots$	S_d
a_1	a_2	a_3	$\cdots$	a_d
.	.	.	$\cdots$	a_1
.	.	.	$\cdots$	.
.	.	.	$\cdots$	.
.	.	a_d	$\cdots$	a_{d-3}
.	a_d	a_1	$\cdots$	a_{d-2}
a_d	a_1	a_2	$\cdots$	a_{d-1}
X	X	X	$\cdots$	X

As f is efficient, $f(P_N) \subseteq \{a_1, \ldots, a_d\}$. Note that when $d = 1$, this implies $\#f(P_N) = 1$, hence $d \in K_f$. Now let $d > 1$. Consider the permutation σ of A such that

$$\begin{aligned} \sigma(a_i) &= a_{i+1} \ \forall i \in \{1, \ldots, d-1\}, \\ \sigma(a_d) &= a_1, \text{ and} \\ \sigma(x) &= x \ \forall x \in X. \end{aligned}$$

[4]This condition is equivalent to asking the greatest common divisor of $m!$ and n to be 1, as shown by Doğan and Giritligil (2015), who reconsider the problem through a group theoretic approach. Interestingly, as Doğan and Giritligil (2015) as well as Bubboloni and Gori (2014) show, $gcd(m!, n) = 1$ turns out to be necessary and sufficient for the existence of anonymous and neutral social welfare functions (*i.e.,* functions which assign to every preference profile a strict ranking of alternatives).

[5]P_N is such that $a_i P_j a_{i+1} \cdots a_d P_j a_1 \cdots a_{i-2} P_j a_{i-1} P_j x_1 P_j \ldots P_j x_{m-d}$ for all $j \in S_i$ and for all $i \in \{1, \ldots, d\}$, where $X = \{x_1, \ldots, x_{m-d}\}$.

Next, let $\sigma' : N \leftrightarrow N$ be such that $\sigma'(i) = i - t$ for all $i \in \{t+1, \ldots, dt = n\}$ and $\sigma'(i) = (d-1)t + i$ for all $i \in \{1, \ldots, t\}$. Note that $\sigma(P_N) = \sigma'(P_N)$.

Now, by neutrality, $a_i \in f(P_N)$ implies $\sigma(a_i) \in f(\sigma(P_N))$. By anonymity, $a_i \in f(P_N)$ implies $a_i \in f(\sigma(P_N))$. Clearly, this is only possible when $f(P_N) = \{a_1, \ldots, a_d\}$. Hence, $\#f(P_N) = d$, thus $d \in K_f$.

ii. Let $\tau(x, P_N) = \#\{i \in N : xP_iy \ \forall y \in A\backslash\{x\}\}$ denote the number of individuals that rank x on top of their preferences in the profile P_N. Define the *plurality rule* $\Upsilon : \mathcal{L}(A)^N \to \mathcal{A}$ so that

$$\Upsilon(P_N) = \{x \in A : \tau(x, P_N) \geq \tau(y, P_N) \ \forall y \in A\backslash\{x\}\}.$$

Let us now define the *iterative plurality rule* $\upsilon : \mathcal{L}(A)^N \to \mathcal{A}$, which selects the plurality winners after successive restriction of profiles to plurality winners. Let $\Upsilon^2(P_N) = \Upsilon(P_N|_{\Upsilon(P_N)})$, $\Upsilon^3(P_N) = \Upsilon(P_N|_{\Upsilon^2(P_N)})$, and so on. Thus, we have

$$\upsilon(P_N) = \Upsilon^k(P_N)$$

for the minimal $k \in \mathbb{N}$ that satisfies $\Upsilon^k(P_N) = \Upsilon^{k+1}(P_N)$. Such an integer always exists given the finiteness of m. It is straightforward to see that υ is anonymous, efficient, and neutral. We will show that $K_\upsilon = D_{n,m}$ for all $n, m \in \mathbb{N}$. By definition, for any $x, y \in A$ and any $P_N \in \mathcal{L}(A)^N$, $x, y \in \upsilon(P_N)$ implies $\tau(x, P_N|_{\Upsilon^k(P_N)}) = \tau(y, P_N|_{\Upsilon^k(P_N)}) = t$ for some $t \in \mathbb{N}$. Thus, we have $\#\upsilon(P_N) \times t = n$, implying $K_\upsilon \subseteq \mathcal{D}(n)$. As $\#\upsilon(P_N) \leq m$, in fact, $K_\upsilon \subseteq D_{n,m}$. Given part (***i***) of the theorem, we have $K_\upsilon = D_{n,m}$. ■

Theorem 2 generalizes Theorem 1, which now comes as a corollary: when Condition $\mu(m, n)$ fails, by Theorem 2.*i*, every anonymous, efficient, neutral, and resolute SCR has $\#f(P_N) > 1$ for some $P_N \in \mathcal{L}(A)^N$ and when Condition $\mu(m, n)$ is satisfied, the iterative plurality rule υ ensures the existence of an anonymous, efficient, and neutral f with $\#f(P_N) = 1$ for all $P_N \in \mathcal{L}(A)^N$.

As a matter of fact, Theorem 2 establishes that the irresoluteness structure K_υ of the iterative plurality rule is the best that an anonymous, efficient, and neutral SCR can achieve. To be sure, this does not mean that υ cannot be refined while preserving anonymity, neutrality, and efficiency. But surely, any anonymous, efficient, and neutral refinement of υ will have the same irresoluteness structure as υ. Moreover, as we formally state in the next theorem, any anonymous, efficient, and neutral SCR whose irresoluteness structure is a proper superset of K_υ can be refined while anonymity, neutrality, and efficiency are preserved.

THEOREM **3** *Given a social choice problem (A, N) with size (m, n), an anonymous, efficient, and neutral SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ with $K_f \supset K_\upsilon$ admits an anonymous, efficient, and neutral refinement $g : \mathcal{L}(A)^N \to \mathcal{A}$ with $K_g = K_\upsilon$.*

Proof:

Take any social choice problem (A, N) and any anonymous, efficient, and neutral $f : \mathcal{L}(A)^N \to \mathcal{A}$ with $K_f \supset K_\upsilon$. By definition, we have $\upsilon f(P_N) \subseteq f(P_N)$ for all $P_N \in \mathcal{L}(A)^N$. Take any profile $P'_N \in \mathcal{L}(A)^N$ such that $\#f(P'_N) \notin K_\upsilon$. As $\#\upsilon f(P'_N) \in \mathcal{D}(n)$ by definition of υ, we cannot have $\upsilon f(P'_N) = f(P'_N)$. Thus, $\upsilon f(P'_N) \subset f(P'_N)$. Furthermore, as $\#\upsilon f(P'_N) \leq m$, we have $\#\upsilon f(P'_N) \in D_{n,m} = K_\upsilon$. ■

As an instance of Theorem 3, consider the social choice problem $A = \{x, y, z\}$, $N = \{1, 2\}$, and $P_N \in \mathcal{L}(A)^N$ with xP_1yP_1z and zP_2yP_2x. For the Borda rule β, which chooses the alternatives that have the minimal sum of ranks over individuals, we have $\beta(P_N) = \{x, y, z\}$, thus $3 \in K_\beta$, while one can check that $K_\upsilon = \{1, 2\}$, and, indeed, the composite rule $\upsilon\beta$, which gives $\upsilon\beta(P_N) = \{x, z\} \subset \beta(P_N)$, refines β and is anonymous, efficient, and neutral.

Theorem 3 points to the possibility of shrinking the irresoluteness structure of an anonymous, efficient, and neutral SCR f down to K_υ by composing f with υ, while υf preservers all three properties. However, this does not mean that υf can refine f at every $P_N \in \mathcal{L}(A)^N$ with $f(P_N) \supset \upsilon(P_N)$. To see this, let $m = 5$ and $n = 6$, and consider the following profile P_N.

P_1	P_2	P_3	P_4	P_5	P_6
x	x	t	r	y	y
y	y	z	z	x	x
z	z	x	y	r	z
r	t	y	x	z	t
t	r	r	t	t	r

We have $\upsilon(P_N) = \{x, y\}$ and $\varphi(P_N) = \{x, y, z\}$, where φ denotes the *fallback bargaining rule*.[6] However, the composite rule $\upsilon\varphi$ does not refine φ at this profile, *i.e.*, $\upsilon\varphi(P_N) = \{x, y, z\}$.

4 Consequential neutrality

Let $W_f(x) = \{P_N \in \mathcal{L}(A)^N : f(P_N) = x\}$ be the set of all profiles at which x is chosen under a resolute SCR f. A resolute SCR $f : \mathcal{L}(A)^N \to A$ is *consequentially neutral* (CN) iff $\#W_f(x) = \#W_f(y)$ for all $x, y \in A$. Given any $m, n \in \mathbb{N}\backslash\{1\}$, we write $\mathcal{F}^{CN}_{m,n}$ for the

[6]Define, for any $P_N \in \mathcal{L}(A)^N$, $\alpha(P_N, x, k) = \#\{i \in N : \#\{y \in A\backslash\{x\} : xP_iy\} \geq k\}$, which gives the number of individuals that rank x higher than at least k alternatives. Now, define the fallback bargaining rule $\varphi : \mathcal{L}(A)^N \to \mathcal{A}$ so that, $\forall x \in A$, $x \in \varphi(P_N)$ if and only if

$$\max_{k\in\{0,\ldots,m-1\}} \{k \in \mathbb{N} : \alpha(P_N, x, k) = n\} \geq \max_{k\in\{0,\ldots,m-1\}} \{k \in \mathbb{N} : \alpha(P_N, y, k) = n\}$$

for all $y \in A\backslash\{x\}$.

set of resolute SCRs that are CN; $\mathcal{F}_{m,n}^{NEUTRAL}$ for the set of resolute and neutral SCRs; and $\mathcal{F}_{m,n}^{RESOLUTE}$ for the set of resolute SCRs.

We say that P'_N is a *renaming* (for alternatives) of P_N iff there exists $\sigma \in \Sigma_A$ such that $P'_N = \sigma(P_N)$. We write $P_N \rho P'_N$ when P'_N is a renaming of P_N. Noting that $\rho \subseteq \mathcal{L}(A)^N \times \mathcal{L}(A)^N$ is an equivalence relation, we write $\mathcal{E}$ for the partition of $\mathcal{L}(A)^N$ provided by ρ. Thus, each profile P_N admits $m!$ renamings. Moreover, $\#\mathcal{L}(A)^N = m!^n$. Thus, $\mathcal{E}$ admits $m!^{n-1}$ equivalence classes, each of which contains $m!$ profiles. We write $\mathcal{E} = \{\mathcal{E}_i\}_{i\in\{1,\ldots,m!^{n-1}\}}$ with $\#\mathcal{E}_i = m!$ for all $i \in \{1, \ldots, m!^{n-1}\}$.

Proposition 1 *$\mathcal{F}_{m,n}^{NEUTRAL} \subset \mathcal{F}_{m,n}^{CN}$ for all $m, n \in \mathbb{N}\backslash\{1\}$.*

Proof: Let $A = \{x_1, \ldots, x_m\}$. Take any $f \in \mathcal{F}_{m,n}^{NEUTRAL}$, any $\mathcal{E}_t \in \mathcal{E}$, and any $P_N \in \mathcal{E}_t$, with $f(P_N) = x_i$ for some $i \in \{1, \ldots, m\}$. For any $i, j \in \{1, \ldots, m\}$, let

$$\mathcal{E}_t^{ij} = \{P'_N \in \mathcal{E}_t : P'_N = \sigma(P_N) \text{ for some } \sigma \in \Sigma_A \text{ with } \sigma(x_i) = x_j\}.$$

Note that for any $i \in \{1, \ldots, m\}$, $\{\mathcal{E}_t^{ij}\}_{j\in\{1,\ldots,m\}}$ partitions $\mathcal{E}_t$. By neutrality, we have $f(P'_N) = x_j$ for all $P'_N \in \mathcal{E}_t^{ij}$, which implies $\#\{P_N \in \mathcal{E}_t : f(P_N) = x_j\} = \#\{P_N \in \mathcal{E}_t : f(P_N) = x_k\}$ for all $j, k \in \{1, \ldots, m\}$. As $t \in \{1, \ldots, m!^{n-1}\}$ is chosen arbitrarily, $\#W_f(x_j) = \#W_f(x_k)$.

To show the strictness of the inclusion, fix a profile $\bar{P}_N \in \mathcal{L}(A)^N$. Take the permutation $\sigma \in \Sigma_A$ such that $\sigma(x_1) = x_2$, $\sigma(x_2) = x_1$, and $\sigma(x_j) = x_j$ for all $j \in \{3, \ldots, m\}$. Take any partition $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ of $\mathcal{L}(A)^N$ such that $\#\mathcal{P}_i = \#\mathcal{P}_j$ for all $i, j \in \{1, \ldots, m\}$ and $\bar{P}_N, \sigma(\bar{P}_N) \in \mathcal{P}_1$. Define $f : \mathcal{L}(A)^N \to A$ such that $f(P_N) = x_i$ for all $P_N \in \mathcal{P}_i$ and $i \in \{1, \ldots, m\}$. By construction, f is CN. However, it fails neutrality as $f(\bar{P}_N) = x_1 = f(\sigma(\bar{P}_N))$. ■

Proposition 1 raises the following two issues: How large is $\mathcal{F}_{m,n}^{CN}$ compared to $\mathcal{F}_{m,n}^{NEUTRAL}$ and which interesting resolute SCRs, if any, does it contain? We address the first question through a counting approach.

Theorem 4 *The following equalities hold.*

i. $\#\mathcal{F}_{m,n}^{RESOLUTE} = m^{(m!^n)}$.

ii. $\#\mathcal{F}_{m,n}^{NEUTRAL} = m^{(m!^{n-1})}$.

iii. $\#\mathcal{F}_{m,n}^{CN} = \frac{(m!^n)!m!}{(m!^{n-1}(m-1)!)!^m}$.

Proof:

i. This is straightforward, as we noted before that $\#\mathcal{L}(A)^N = m!^n$.

ii. Take any $\mathcal{E}_t \in \mathcal{E}$ and pick any $P_N \in \mathcal{E}_t$. Let $f(P_N) = x$ for some $x \in A$. Neutrality, together with the definition of $\mathcal{E}_t$, determines $f(P'_N)$ for any $P'_N \in \mathcal{E}_t$. Hence there are m

neutral and resolute SCRs that can be defined on $\mathcal{E}_t$. As $t \in \{1, \ldots, m!^{n-1}\}$, there are $m^{(m!^{n-1})}$ neutral and resolute SCRs altogether.

iii. First observe that, given any two natural numbers p and q, there are $\frac{(pq)!}{q!^p}$ ways to partition a set of cardinality pq into p sets, each with cardinality q. Hence, there are $\frac{(m!^n)!}{(m!^{n-1}(m-1)!)!^m}$ ways to partition $\mathcal{L}(A)^N$ with cardinality $m!^n$ into m sets, each with cardinality $m!^n/m = m!^{n-1}(m-1)!$. For each of these ways, $m!$ distinct resolute SCRs can be defined. As a result, $\frac{(m!^n)!}{(m!^{n-1}(m-1)!)!^m} \times m!$ resolute SCRs that satisfy consequential neutrality can be constructed. ■

From Theorem 4, one can compute $\frac{\#\mathcal{F}^{NEUTRAL}_{m,n}}{\#\mathcal{F}^{RESOLUTE}_{m,n}} = m^{(-1+\frac{1}{m!})m!^n}$. As $\frac{1}{m!} < 1$ for all $m > 1$, this ratio approaches to 0 as $m \to \infty$ or $n \to \infty$. Thus, we conclude that the ratio of the number of neutral and resolute SCRs to the number of all resolute SCRs is negligible in the limit. Although we do not have analytical solutions for the comparisons regarding consequential neutrality, we obtained some numerical observations through computations for small values of m and n that are provided in Appendix A. These indicate that both $\frac{\#\mathcal{F}^{CN}_{m,n}}{\#\mathcal{F}^{RESOLUTE}_{m,n}}$ and $\frac{\#\mathcal{F}^{CN}_{m,n}}{\#\mathcal{F}^{NEUTRAL}_{m,n}}$ converge to 0, as m or n increases. Thus, although consequential neutrality and neutrality are both hard to satisfy, neutrality seems to be considerably demanding compared to consequential neutrality.[7]

Now we address whether $\mathcal{F}^{CN}_{m,n} \backslash \mathcal{F}^{NEUTRAL}_{m,n}$ contains interesting SCRs and the answer is affirmative, at least for certain sizes of the social choice problem.

We say that (m, n) satisfies Condition $\gamma(m, n)$ iff $m \mid \binom{m}{k}$ for all $k \in D_{n,m}$. This condition, conjoined with $m \nmid n$, ensures the existence of anonymous, CN, efficient, and resolute SCRs, as shown in the following theorem. As a piece of notation we use in the proof of the theorem, let $\bar{W}_f(x, k) = \{P_N \in \mathcal{L}(A)^N : x \in f(P_N) \text{ and } \#f(P_N) = k\}$ denote the number of profiles at which x is chosen by f in a tie of k alternatives.

Theorem 5 *Let (A, N) be a social choice problem with size (m, n) where $m \nmid n$ and Condition $\gamma(m, n)$ is satisfied. Every anonymous and neutral SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ admits a resolute refinement which is anonymous and CN. Furthermore, when f is efficient, this refinement preserves efficiency.*

Proof: Fix $m, n \in \mathbb{N}$ as such, let $A = \{x_1, \ldots, x_m\}$, and take any anonymous and neutral $f : \mathcal{L}(A)^N \to \mathcal{A}$. Clearly, $\upsilon f(P_N) \in D_{n,m}$ for all $P_N \in \mathcal{L}(A)^N$ due to Theorems 2 and 3. Given neutrality, we have $\#\bar{W}_{\upsilon f}(x, k) = \#\bar{W}_{\upsilon f}(y, k)$ for all $x, y \in A$. As $m \nmid n$, there does not exist a profile P_N with $\upsilon f(P_N) = A$. For any $k \in D_{n,m}\backslash\{1\}$, denote with $\mathcal{A}_k$ the subsets of A with precisely k elements. So we have $\#\mathcal{A}_k = \binom{m}{k} = m \times \left(\binom{m-1}{k-1}/k\right)$. Condition $\gamma(m, n)$

[7] We are providing computational results for only some small values of m and n because as m and n increase, these values grow dramatically. As diminution in the ratios are also fast, these values appear to be sufficient for this conclusion.

ensures that $\binom{m-1}{k-1}/k$ ($= t$ from now on) is a natural number. Now, we need to show that to each $x_i \in A$, we can assign t distinct sets of cardinality k that contains x_i. To do that, we introduce the following iterative approach:

Step 0.

Take the first t sets in the lexicographic order of $\mathcal{A}_k$ (so starting with $\{x_1, \ldots, x_k\}$, $\{x_1, \ldots, x_{k-1}, x_{k+1}\}$, and so on). We denote with $\mathcal{A}_{k,1}$ the set of these t sets.

Step 1.

Define $\sigma : A \leftrightarrow A$ such that $\sigma(x_i) = x_{i-1}$ for all $i \in \{2 \ldots, m\}$ and $\sigma(x_i) = x_m$. Now, order the set $\mathcal{A}_k \backslash \mathcal{A}_{k,1}$ lexicographically based on its image under σ and denote the set of the first t of these sets with $\mathcal{A}_{k,2}$.

Step 2.

Now, order the set $\mathcal{A}_k \backslash \mathcal{A}_{k,1} \cup \mathcal{A}_{k,2}$ lexicographically based on its image under σ^2 (two-time application of σ, so that, for instance $\sigma^2(x_1) = x_{m-1}$, $\sigma^2(x_2) = x_m$, and so on) and denote the set of the first t of these sets with $\mathcal{A}_{k,3}$.

Steps 3 to $m-1$ (if $m \geq 3$) complete the process for assigning each of m alternatives to all sets with k alternatives chosen under vf. We now ascertain that $x_i \in S$ for all $S \in \mathcal{A}_{k,i}$ and for all $i \in \{1, \ldots, m\}$, which we show in the following lemma.

Lemma **1** *$x_i \in S$ for all $S \in \mathcal{A}_{k,i}$ for all $i \in \{1, \ldots, m\}$.*

Proof:

Take any $x_i \in A$. The number of sets with k elements that include x_i is $\binom{m-1}{k-1}$. The maximum number of times x_i can appear in sets $S \in \mathcal{A}_{k,j}$ for all $j \in \{1, \ldots, i-1\}$ is $(i-1) \times t$, *i.e.,* when each of them contains x_i. We have $i \times t = i \times \left(\binom{m-1}{k-1}/k\right) \leq \binom{m-1}{k-1}$, for if each of $S \in \mathcal{A}_{k,j}$ for all $j \in \{1, \ldots, i-1\}$ contains x_i, then we have $i \leq k$. Thus, as $x_i \in S$ for all $S \in \mathcal{A}_{k,i}$ because x_i is on top of the lexicographic order of each set in $S \in \mathcal{A} \backslash \bigcup_{j \leq i-1} \mathcal{A}_{k,j}$ with $x_i \in S$, under the image of σ^{i-1}, we have the desired result. ■

We now define a resolute SCR $g : \mathcal{L}(A)^N \to A$ that refines vf. So, $g(P_N) = vf(P_N)$ for all $P_N \in \mathcal{L}(A)^N$ with $\#vf(P_N) = 1$. Moreover, $vf(P_N) \in \mathcal{A}_{k,i} \implies g(P_N) = x_i$ for all $i \in \{1, \ldots, m\}$ and $P_N \in \mathcal{L}(A)^N$. g is a resolute refinement of f. Naturally, g is anonymous as f is. Furthermore, by construction, it is CN. Finally, if f is efficient, g is efficient as well. ■

To see how the refinement in Theorem 5 can be constructed, consider the following example with $A = \{x_1, \ldots, x_5\}$ and $n = 6$. Take any f that is anonymous, efficient, and neutral. We have $D_{6,5} = \{1, 2, 3\}$. Take $k = 3$. We have $\binom{5}{3}/5 = 10$ and $t = 2$, that is, there are 10 sets with three alternatives and all profiles that result in three alternatives under vf will be assigned to each of the 5 alternatives so that each will be chosen in 2 different sets. So in Step 0, we set $\mathcal{A}_{3,1} = \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}\}$ and so on as below.

$$\begin{array}{lrcl} \textit{Step 0}: & \mathcal{A}_{3,1} & = & \{\{x_1,x_2,x_3\},\{x_1,x_2,x_4\}\} \\ \textit{Step 1}: & \mathcal{A}_{3,2} & = & \{\{x_2,x_3,x_4\},\{x_2,x_3,x_5\}\} \\ \textit{Step 2}: & \mathcal{A}_{3,3} & = & \{\{x_3,x_4,x_5\},\{x_3,x_4,x_1\}\} \\ \textit{Step 3}: & \mathcal{A}_{3,4} & = & \{\{x_4,x_5,x_1\},\{x_4,x_5,x_2\}\} \\ \textit{Step 4}: & \mathcal{A}_{3,5} & = & \{\{x_5,x_1,x_2\},\{x_5,x_1,x_3\}\} \end{array}$$

Note that, for instance, $\{\{x_3,x_4,x_5\},\{x_3,x_4,x_1\}\} = \mathcal{A}_{3,3}$, because in Step 2 we have

$$\sigma^2(x_3) = x_1, \sigma^2(x_4) = x_2, \sigma^2(x_5) = x_3,$$

$$\sigma^2(x_3) = x_1, \sigma^2(x_4) = x_2, \sigma^2(x_1) = x_4,$$

which makes them the top two sets in the lexicographic order based on the image of σ^2.

So for each $i \in \{1,\ldots,m\}$ we have $g(P_N) = x_i$ for all $P_N \in \mathcal{L}(A)^N$ such that $\upsilon f(P_N) \in \mathcal{A}_{3,i}$. To check that this can be done for $k=2$ as well is left to the reader.

Note that Condition $\gamma(m,n)$ ensures the existence of anonymous, CN, efficient, and resolute SCRs by asking $m \mid \binom{m}{k}$ to hold for every $k \in D_{n,m}$. On the other hand, as it follows from the proof of Theorem 5, a weaker version of Condition $\gamma(m,n)$ that asks $m \mid \binom{m}{k}$ to hold for some $k \in D_{n,m}$, although does not ensure resoluteness, allows the existence of anonymous, CN, and efficient refinements. We state this formally in the following remark and leave out its proof as it follows from the proof of Theorem 5.

An SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ is $\overline{CN}$ iff for all $k \in \{1,\ldots,m\}$, we have $\#\bar{W}_f(x,k) = \#\bar{W}_f(y,k)$ for all $x,y \in A$.[8] Let us say that (m,n) satisfies Condition $\gamma'(m,n)$ iff $m \mid \binom{m}{k}$ for some $k \in D_{n,m}$.

Remark 1 *Let (A,N) be a social choice problem with size (m,n) where $m \nmid n$ and Condition $\gamma'(m,n)$ is satisfied. Every anonymous, efficient, and neutral SCR $f : \mathcal{L}(A)^N \to \mathcal{A}$ admits a refinement which is anonymous, efficient, and $\overline{CN}$.*

How restrictive is Condition $\gamma(m,n)$? Note that when m is a prime, Condition $\gamma(m,n)$ is satisfied.[9] Thus, anonymous, CN, efficient, and resolute SCRs exist when m is prime and does not divide n. Campbell and Kelly (2015) show that when n has at least two distinct prime factors, there can only be finitely many values of m for which there are anonymous, neutral, and resolute SCRs. Our result implies, for instance, that for such n, there are infinitely many values of m (such as all primes that are greater than n) for which there are anonymous, CN, and resolute SCRs. Furthermore, these SCRs can be efficient.

To expand the picture drawn by Theorem 5, we show that $m \nmid n$ and Condition $\gamma(m,n)$ are not necessary for the existence of anonymous, CN, efficient, and resolute SCRs. As a

[8]Note that for a resolute and CN SCR f, we have $\#\bar{W}_f(x,1) = \#\bar{W}_f(y,1)$ for all $x,y \in A$ and $\#\bar{W}_f(x,k) = 0$ for all $x \in A$ and for all $k \in \{2,\ldots,m\}$.

[9]To see this, note that the numerator in $\binom{m}{k} = \frac{m\times\cdots\times(m-k+1)}{k!}$ is divisible by m whereas none of $\{2,\ldots,k\}$ divides m.

matter of fact, the theorem below spans some instances where Condition $\gamma(m,n)$ fails, *e.g.*, $m = 4$ *and* $n = 2$ (note that $2 \in D_{2,4}$ and $4 \nmid \binom{4}{2}$).

Theorem 6 *Any social choice problem (A, N) with $m \geq 4$ and $n \in \{2, 3\}$ admits an anonymous, CN, efficient, and resolute SCR.*

Proof: Let $m \geq 4$ and $N = \{1, 2\}$. For any $x, y \in A$, let $T_{xy} \subset \mathcal{L}(A)^N$ denote the set of profiles where individual 1 ranks x first and individual 2 ranks y first. Hence $\{T_{xy}\}_{x \neq y}$ partitions the set of profiles where there is no unanimously top ranked alternative. Given any $x, y \in A$, note that

$$T_{yx} = \{(Q, P) \in \mathcal{L}(\mathcal{A})^N : (P, Q) \in T_{xy}\},$$

and hence, $\#T_{xy} = \#T_{yx}$. Now, let $A = \{x_1, \ldots, x_m\}$ and define for any distinct $i, j \in \{1, \ldots, m\}$, $D_m(i, j) =$

$$\big\{k, l \in \{1, \ldots, m\} \backslash \{i, j\} : k \neq l, k < t, \text{ and } l < t \; \forall t \in \{1, \ldots, m\} \backslash \{i, j, k, l\}\big\},$$

as the doubleton that contains the lowest two indices in A excluding i and j. Let $g : \mathcal{L}(A)^N \to A$ be a resolute SCR that picks at any profile the alternative that is ranked first by both individuals, if exists. Furthermore, let $g(P_N) = x_i$ if $x_k P_1 x_l \iff x_k P_2 x_l$ and $g(P_N) = x_j$ otherwise for all $P_N \in T_{x_i x_j}$ with $i < j$ and $D_m(i, j) = \{k, l\}$. Hence, we have $\#\{P_N \in T_{x_i x_j} : g(P_N) = x_i\} = \#\{P_N \in T_{x_i x_j} : g(P_N) = x_j\}$ for all $x_i, x_j \in A$ such that $i < j$. Furthermore, let $g(P, Q) = g(Q, P) \; \forall (P, Q) \in \mathcal{L}(A)^N$, so that g retains anonymity. Finally, as g picks an alternative only if it is ranked first by an individual, it is also efficient.

Now let $n = 3$. Under plurality rule, ties occur only when each individual has a distinct alternative as most preferred. This is the case for $((m-1)!)^3$ profiles, which is divisible by 3 for all $m \geq 4$, hence we can assign each of the three alternatives that appear on top to an equal number of profiles. ■

At this stage, one may be tempted to ask whether one can find an anonymous, CN, efficient, and resolute SCR at any (m, n). The following theorem advises caution on this.

Theorem 7 *There exists a social choice problem which admits no anonymous, CN, efficient, and resolute SCR.*

Proof: Let $A = \{x, y\}$ and $N = \{1, 2\}$. We have four possible profiles, P_N, P'_N, P''_N, P'''_N as shown below.

P_1	P_2		P'_1	P'_2		P''_1	P''_2		P'''_1	P'''_2
x	x		y	y		x	y		y	x
y	y		x	x		y	x		x	y

Efficiency implies choosing x at P_N and y at P'_N. Moreover, $f(P''_N) = f(P'''_N)$ by anonymity. Hence, $\#\{P_N \in \mathcal{L}(A)^N : f(P_N) = x\} \neq \#\{P_N \in \mathcal{L}(A)^N : f(P_N) = y\}$, a failure of consequential neutrality. ■

Nevertheless, the social choice problem in the proof of Theorem 7 admits an anonymous, CN, and resolute SCR.[10] This raises the question of how general is the compatibility between anonymity and consequential neutrality when we dispense with the efficiency condition.

Moulin (1991) introduces the following condition that we call $\psi(m,n)$. Let $\mathcal{D}(n)\backslash\{1\} = \{d_1, \ldots, d_K\}$ for some $K \in \mathbb{N}$.

Condition $\psi(m,n)$: $\nexists(a_1,\ldots,a_K) \in (\mathbb{N}\cup\{0\})^K$ such that $\sum_{i=1}^K a_i d_i = m$.

Remark **2** *$\mu(m,n) \implies \psi(m,n)$ for all $n, m \geq 2$.*

The following theorem states the cases of incompatibility of anonymity and neutrality in resolute social choice.

Theorem **8** (**Moulin (1991)**) *There exists an anonymous, neutral, and resolute SCR if and only if $\psi(m,n)$ holds.*

We are now ready to state and prove our final theorem, which shows that if $m > n$, there exist anonymous, CN, and resolute SCRs for any social choice problem with size (m,n). Note that for any n, there are infinitely many values of m that satisfy $m > n$.

Theorem **9** *For all social choice problems with $n < m$, there exists an anonymous, CN, and resolute SCR.*

Proof: Given any preference profile, we observe k distinct preferences for some $k \in \{1,\ldots,\min\{m!,n\}\}$. There are $\binom{m!}{k}$ ways to choose k preferences from $\mathcal{L}(A)$. Let $\mathbf{P} = \{p^1,\ldots,p^k\}$ be a set of k distinct preferences. Write V^k for the set of vectors $v = (v_1,\ldots,v_k)$ with $v_i \geq 1$ for all $i \in \{1,\ldots,k\}$ and $\sum_{i=1}^k v_i = n$. Each $v \in V^k$, combined with $\mathbf{P}$, induces a set of profiles $E_v^{\mathbf{P}}$ that consists of all profiles where p^i appears v_i times for all $i \in \{1,\ldots,k\}$. Let $E^{\mathbf{P}} = \bigcup_{v\in V^k} E_v^{\mathbf{P}}$. Three remarks are in order. First, $\#E^{\mathbf{P}}$ depends on k and not on the preferences in $\mathbf{P}$. Second, an SCR f that satisfies at any given $k \in \{1,\ldots,\min\{m!,n\}\}$ and $\mathbf{P}$ the invariance $f(P_N) = f(P'_N)$ for all $P_N, P'_N \in E^{\mathbf{P}}$ is anonymous. Third, $m \mid \binom{m!}{k}$ if $k < m$, which is ensured when $n < m$. Now, let $t_k = \binom{m!}{k}/m$. Write $A = \{x_1,\ldots,x_m\}$ and at each $k \in \{1,\ldots,\min\{m!,n\}\}$ assign to every x_i t_k distinct sets $\mathbf{P} = \{p^1,\ldots,p^k\}$ and let $f(P_N) = x_i$ for all $P_N \in E^{\mathbf{P}}$ at every $\mathbf{P}$ assigned to x_i for all $i \in \{1,\ldots,m\}$. By the three remarks, f is anonymous, CN, and resolute. ■

[10] To see this, consider $g : \mathcal{L}(A)^N \to \mathcal{A}$ such that $g(P_N) = g(P'_N) = x$ and $g(P''_N) = g(P'''_N) = y$, which is both anonymous and CN while not efficient.

5 Conclusion

With an irresolute SCR that prevails, one cannot reach a collective choice without referring to an additional mechanism that is external to the SCR. Therefore, Theorems 1 and 8 reflect the impossibility of making a collective choice by being confined to SCRs as collective choice procedures. We take two different but related approaches to address how severe this impossibility is.

First, we identify the minimal irresoluteness structure that would arise when anonymity, efficiency, and neutrality make ties inevitable. Based on this analysis, we deliver a method which, while preserving anonymity, efficiency, and neutrality, refines SCRs that deliver more ties than necessary.

Next, we introduce consequential neutrality as a weakening of neutrality. As expected, we obtain results that are more permissive than the (im-)possisibilities announced by Theorems 1 and 8. We identify a large class of social choice problems where resoluteness becomes possible just because consequential neutrality replaces neutrality. Nevertheless, when efficiency is preserved, we know that this possibility does not hold for every social choice problem.

Dispensing with efficiency presents a case of interest. We show that anonymous, CN, and resolute social choice is possible when $m > n$. Although this condition is logically independent of the necessary and sufficient condition of Theorem 8, it opens the door of resoluteness to a large class of social choice problems that are doomed to irresoluteness by Theorem 8. Moreover, we are not able to find any social choice problem where anonymity, consequential neutrality, and resoluteness are incompatible. This provokes to ask whether these three conditions are compatible for any size of the social choice problem, which we leave as a –combinatorically difficult– open question.

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A Appendix: Observations on the numbers of CN, neutral, and resolute SCRs

Tables 1 and 2 below show ratios the observations made in Section 4 are based on. In both tables, $\underline{\mathbf{0}}$ represents numbers smaller than 10^{-10mn}.

$\#\mathcal{F}_{m,n}^{NEUTRAL}/\#\mathcal{F}_{m,n}^{CN}$	$n=2$	$n=3$	$n=4$
$m=2$	0.333333	0.114286	0.00994561
$m=3$	3.58965×10^{-14}	5.73212×10^{-85}	$\underline{\mathbf{0}}$
$m=4$	$\underline{\mathbf{0}}$	$\underline{\mathbf{0}}$	$\underline{\mathbf{0}}$

Table 1. The ratio of $\#\mathcal{F}_{m,n}^{NEUTRAL}/\#\mathcal{F}_{m,n}^{CN}$ for different values of (m,n).

$\#\mathcal{F}_{m,n}^{CN}/\#\mathcal{F}_{m,n}$	$n=2$	$n=3$	$n=4$	$n=5$
$m=2$	0.75	0.546875	0.392761	0.2799
$m=3$	0.135304	0.0229012	0.0038267	0.000638057
$m=4$	0.00175989	0.0000149993	1.27583×10^{-7}	1.08512×10^{-9}
$m=5$	8.19334×10^{-7}	5.69061×10^{-11}	3.95181×10^{-15}	$\underline{\mathbf{0}}$
$m=6$	8.12216×10^{-12}	$\underline{\mathbf{0}}$	$\underline{\mathbf{0}}$	$\underline{\mathbf{0}}$

Table 2. The ratio of $\#\mathcal{F}_{m,n}^{CN}/\#\mathcal{F}_{m,n}$ for different values of (m,n).