Department of Strategy and Innovation
Working Paper No. 09/2020

Inclusive Cognitive Hierarchy

Yukio Koriyama
Ali Ozkes

March 2020

WIRTSCHAFTS
UNIVERSITÄT
WIEN VIENNA
UNIVERSITY OF
ECONOMICS
AND BUSINESS

Inclusive Cognitive Hierarchy

Yukio Koriyama[*] Ali Ozkes[†]

January 28, 2020

Abstract

Cognitive hierarchy theory, a collection of structural models of non-equilibrium thinking, in which players' best responses rely on heterogeneous beliefs on others' strategies including naïve behavior, proved powerful in explaining observations from a wide range of games. We introduce an *inclusive* cognitive hierarchy model, in which players do not rule out the possibility of facing opponents at their own thinking level. Our theoretical results show that inclusiveness is crucial for asymptotic properties of deviations from equilibrium behavior in expansive games. We show that the limiting behaviors are categorized in three distinct types: naïve, Savage rational with inconsistent beliefs, and sophisticated. We test the model in a laboratory experiment of collective decision-making. The data suggests that inclusiveness is indispensable with regard to explanatory power of the models of hierarchical thinking.

JEL classification: C91, C92, D72, D91

Keywords: cognitive hierarchy, collective decision-making, level-k model, strategic thinking

[*]CREST, Ecole Polytechnique, Institute Polytechnique of Paris. E-mail: yukio.koriyama@polytechnique.edu.

[†]Wirtschaftsuniversität Wien, Institute for Markets and Strategy. E-mail: ali.ozkes@wu.ac.at.

1 Introduction

Models of non-equilibrium strategic thinking have been proposed to explain structural deviations from equilibrium in a variety of games. A sizable part of bounded rationality literature is devoted to the models of cognitive hierarchy, starting with the *level-k* (L) model due to Nagel (1995) and Stahl and Wilson (1995), which allow for heterogeneity among players in levels of strategic thinking. In the level-k model, a foundational level (level-0) represents a strategically naïve initial approach to a game, and a level-k player (hereafter Lk), where $k \geq 1$, is assumed to best respond to others with level $k-1$.[1]

Closely related, the *cognitive hierarchy* (CH) model introduced by Camerer et al. (2004) allows for heterogeneity in the beliefs on others' levels. For each $k \geq 1$, a level-k player best responds to a mixture of strictly lower levels, induced by the truncation up to level $k-1$ from the underlying level distribution, which is either obtained from maximum likelihood estimations applied to data, or calibrated from previous estimates. Experimental studies provide evidence that the CH model delivers a better fit for explaining the actual behavior of players in certain games.[2] See Crawford et al. (2013) for a review of applications of L and CH models.

Common to these models is the assumption that level-k players do not assign any probability to levels higher than or equal to k, which is proposed as an embodiment of the understanding that the cognitive limits have a hierarchical structure. This entails that the possibility for players to assign positive probability to the events in which other players have the same cognitive level as themselves is ruled out. That is, *self-awareness*, as described in Camerer et al. (2004), is precluded.[3]

In this paper, we propose an *inclusive cognitive hierarchy* (ICH) model, in which players are allowed to project themselves to others in regard to their cognitive level. The ICH model thus allows for inclusiveness while maintaining the hierarchical structure of cognitive levels and the partial consistency induced by truncation of the underlying level distribution. We study the role of inclusiveness both theoretically and empirically. We provide theorems that describe asymptotic properties of the deviation from rational behavior and present results from a laboratory experiment on collective decision-making.

There are at least three reasons why we believe the consequences of inclusiveness should be studied. First, the observed limitations of existing models in the extant literature on strategic thinking[4] call for studies with an explicit focus on inclusiveness, for which our

[1] Betting games (Brocas et al., 2014), auctions (Crawford and Iriberri, 2007b), and coordination games (Crawford et al., 2008) are examples of the games for which L model explains well the experimental data.

[2] Lottery games (Östling et al., 2011), coordination games (Costa-Gomes et al., 2009), action commitment games (Carvalho and Santos-Pinto, 2014), and minimizer game (Berger et al., 2016) are among the games where CH model proved powerful in explaining the experimental data.

[3] Although it is the term used in the literature, "self-awareness" should not be confounded with the capacity for introspection often meant by in, for example, psychology. What is meant here is the capacity for projecting the self to others, which has little to do with introspection.

[4] Camerer et al. (2004, fn. 15) points to that CH fails in certain games in Stahl and Wilson (1995) and

experimental analysis provides a clear support. Second, our analytical results imply that inclusiveness lead to substantially different predictions in a certain class of games (Theorems 1 and 2). A novel finding of this paper is that without inclusiveness, the deviation from rational behavior asymptotically diverges away without a bound, which, we argue, is not coherent with the spirit of the cognitive hierarchy models.[5] Third, our post-experimental questionnaire also suggests that inclusiveness is indispensable.[6] Figure 3(a) in Appendix A shows that a fairly large proportion (96%) of our subjects exhibit a positive degree of inclusiveness: 'sometimes', 'most of the time', or 'always'. When asked for a subjective estimation of the percentage among others who used the 'same reasoning' (Figure 3(b)), their responses vary and a majority are far from 0%, which suggests that ruling out inclusiveness would be too extreme as an assumption.[7]

We show in Section 3 that inclusiveness matters the most in games in which the best response function is expansive. On the other hand, most of the remarkable results in the literature of strategic thinking are based on games such as the beauty contest games, market entry games, coordination games, centipede games, and so on, that fall outside of this class. Dominance solvable games can be considered within the class of non-expansive games, in the sense that the infinite iteration of applying the best-response function leads to a convergence to the Nash equilibrium. When this iteration leads to the unique Nash equilibrium, it corresponds to the high-level strategies converging to it. Such a property is often considered to fit well the idea that the deviation from Nash due to limited cognitive ability of the low-level players dissipates as the cognitive limit goes to infinity.[8] Our results in Section 3 show that inclusiveness proves crucial in this regard for expansive games. As we demonstrate empirically for the case of beauty contest games, it makes a difference in the prediction power in other games as well.

The idea of a hybrid model with hierarchical thinking and equilibrium approaches as in ICH is not entirely novel. Stahl and Wilson (1995) proposed "worldly" types, who best responds to a prior based on a belief that there are level-0, level-1, and equilibrium types. In fact, including naïve players next to sophisticated ones can be dated back to the "gang of four" (Kreps et al., 1982) type of models, such as Akerlof and Yellen (1985), Conlisk

suggest inclusiveness. Colman et al. (2014) point to the weak performance of the CH model in common interest games. Battaglini et al. (2010) conclude that the L model does poorly, in fact, worse than equilibrium, in explaining data from collective decision-making experiments that are similar to our experiments. Georganas et al. (2015) argue that level-k models do not perform well in some games. See Breitmoser (2012) for comparative studies over different models.

[5]Evidence from experimental studies on public good games suggest that as group size increases, individual behavior bears convergent and stabilizing tendencies. See Isaac et al. (1994).

[6]As the questionnaire is not incentivized, the interpretation should be done with an extra care.

[7]Heterogeneity of the answers implies that the participants' degree of inclusiveness was also heterogeneous. This observation is coherent with the assumption of hierarchical belief structure that we describe in detail in Section 2.

[8]Camerer et al. (2004) suggest inclusiveness for a set of games CH fails as iteration does not lead to Nash equilibrium.

(1980), Haltiwanger and Waldman (1985, 1991), and Russell and Thaler (1985). ICH picks up this intuition and provides a structural framework based on CH.[9] Furthermore, our implementation of ICH is closely related to "equilibrium plus noise" approaches, which entail either having equilibrium types next to naïve types or replacing best-responses with *better-responses, i.e.*, noisy or biased best-responses. As theoretically demonstrated in Section 3 and further discussed in Section 6, ICH can be seen as a model in line of the former type of models, *e.g.,* the "instinctive" and "contemplative" typology of Rubinstein (2016) or inconsistent beliefs in existence of irrational types held by rational types as in Kreps et al. (1982).[10]

Our empirical analysis is based on lab experiments we have run on collective decision-making. Particularly, subjects in our experiments were involved in a jury voting setup due to Condorcet (1785), in which each juror receives an imperfectly informative signal about an unknown binary state in order to collectively make the best decision given the aggregated information. Since Austen-Smith and Banks (1996), the strategic considerations based on pivotal-voting have been formally studied and Guarnaschelli et al. (2000) are the first among the many-to-come to experimentally study how the formal model fares in explaining behavior observed in the lab.

In our experiments, the size of the jury varied as in previous studies.[11]

Collective performances are correlated across challenges, as demonstrated in Woolley et al. (2010), hence a good knowledge about the behavioral basis in collective decision-making processes is essential in understanding the more general phenomena.

The paper proceeds as follows. We introduce the inclusive cognitive hierarchy model formally in the following section. In Section 3, we present our main theorems, which provide a characterization of games according to the asymptotic properties of the strategic thinking, by focusing on the linear quadratic games. We furthermore provide a numerical comparison of the individual behaviors and the performance of collective decisions under different specifications of the cognitive hierarchy in a game of information aggregation. In Section 4, we

[9]Note that best-responding to others who might play equilibrium is different than inclusiveness. Similarly, Levin and Zhang (2019) allow level-k players in the L model to believe that there might be other level-k players, whereas Jimenez-Gomez (2019), building on Strzalecki (2014), allows for this possibility only if they have the exact same beliefs. Both of these papers refer to the "false consensus effect" à la Ross et al. (1977). Alaoui and Penta (2016), on the other hand, introduce a model of strategic thinking that endogenizes individuals' cognitive bounds as a result of a cost-benefit analysis. Their framework allows players to reason about opponents whom they regard as more sophisticated as well.

[10]Our Nash estimation with the logistic error is also a better-response model. Thus, we do not expect models such as quantal response equilibrium (QRE) due to McKelvey and Palfrey (1995) would do much better than this Nash estimation. We discuss QRE further in Section 5. However, comparing CH approach and QRE is beyond the scope of this paper.

[11]See Palfrey (2016) for a review of experiments in political economy, including experiments based on the Condorcet jury model. Battaglini et al. (2008) document an increase in irrational, non-equilibrium play as the size increases. As Camerer (2003) stresses (Ch. 7), the effect of group size on behavior in strategic interactions is a persistent phenomenon, especially towards coordination.

introduce our experimental design that features novelties due to our modeling concerns and signal setup. Section 5 provides the results of the experiment, and the models are compared in terms of the fit to the data. We conclude by summarizing our findings and presenting further research questions in Section 6. The proofs are relegated to an appendix.

2 The Model

Let (N, X, u) be a symmetric normal-form game where $N = \{1, \ldots, n\}$ is the set of players, $X \subset \mathbb{R}$ is a convex set of pure strategies, and $u : X^n \to \mathbb{R}^n$ is the payoff function, twice differentiable. Each player forms a belief on the cognitive levels of the other players. Let $g_k(h)$ denote the probability that a k^{th}-level player assigns independently for each of the other players to belong to the h^{th}-level.

In the standard *level-k model*, a naïve, non-strategic behavior is specified as the initial level (*level*-0, or $L0$). For $k \geq 1$, a level-k (Lk) player holds the belief that all of the other players belong to exactly one level below herself:

$$g_k(h) = \begin{cases} 1 & \text{if } h = k - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{L}$$

In the cognitive hierarchy model, each k^{th}-level (CHk) player best responds to a mixture of lower levels. Let $f = (f_0, f_1, \ldots)$ be a distribution over $\mathbb{N}$ which represents the composition of levels. Each k^{th}-level player holds a belief on the distribution of the other players' levels that is a truncation up to one level below herself:

$$g_k(h) = \frac{f_h}{\sum_{m=0}^{k-1} f_m}, \text{ for } 0 \leq h \leq k - 1 \text{ and } k \geq 1. \tag{CH}$$

Thus, these two models share the following assumption:

Assumption 1 (Strong overconfidence) $g_k(h) = 0$ *for all* $h \geq k$.

Assumption 1 enables us to say that what we call "levels" here indeed have a hierarchical structure. To see this, consider $g_k(h)$ in the form of a k-h matrix. Assumption 1 implies that the upper-diagonal entries are all zeros, and thus the remaining non-zero elements have a pyramid structure with strictly lower-diagonal entries. Each level-k player assigns non-zero probabilities only to the levels strictly lower than herself. In that sense, players are assumed to be *overconfident*.[12] Alternatively, the following assumption can be considered:

Assumption 2 (Weak overconfidence) $g_k(h) = 0$ *for all* $h > k$.

[12] Camerer and Lovallo (1999) report on experimental evidence for overconfident behavior in the case of the market entry game. When ability is a payoff-relevant variable in a strategic interaction, evidence shows that players tend to be overconfident (see Benoît and Dubra, 2011). On the other hand, Azmat et al. (2019) find an underestimation of students' grades in the absence of feedback.

Assumption 2 is weaker than Assumption 1. As in Assumption 1, zero probability is assigned for all strictly upper-diagonal entries, and thus a hierarchical structure among levels is still preserved. However, the diagonal entries are not restricted to be zero. A level-k player is allowed to assign a non-zero possibility for the other players to have the same level as herself.

In what follows, we formally introduce the *inclusive cognitive hierarchy (ICH) model*, in which the strong overconfidence condition is weakened to allow for inclusiveness.

2.1 Inclusive Cognitive Hierarchy

Fix an integer $K > 0$ that prescribes the highest level considered in the model.[13] In the ICH model, we consider a sequence of mixed strategies $\sigma = (\sigma_0, \ldots, \sigma_K)$, in which for each $k \in \{1, ..., K\}$, $\sigma_k \in \Delta(X)$ is a best reply, assuming that the other players' levels are drawn from the truncation of the underlying distribution f *up to level* k. As in the other models, we focus on level-symmetric profiles in which all players of the same level use the same mixed strategy.

Definition 1 *A sequence of level-symmetric strategies $\sigma = (\sigma_0, \ldots, \sigma_K)$ is called **inclusive cognitive hierarchy strategies** when there exists a distribution f over $\mathbb{N}$ under which*

$$supp\left(\sigma_k\right) \subset \arg\max_{x_i \in X} \mathbb{E}_{x_{-i}}\left[u\left(x_i, x_{-i}\right) | g_k, \sigma\right], \ \forall k \in \{1, \cdots, K\},$$

where g_k is the truncated distribution induced by f such that

$$g_k\left(h\right) = \frac{f_h}{\sum_{m=0}^{k} f_m} \text{ for } h \in \{0, \cdots, k\}, \tag{ICH}$$

and the expectation over x_{-i} is drawn, for each player $j \neq i$, from a distribution

$$\gamma_k\left(\sigma\right) := \sum_{m=0}^{k} g_k\left(m\right) \sigma_m.$$

Definition 1 is analogous to the definitions used in the standard level-k model and the cognitive hierarchy model. It simply replaces the assumptions on beliefs, (L) and (CH), with (ICH). Note that the sequence of truncated distributions $g = (g_1, \ldots, g_K)$ is uniquely defined from the underlying distribution f of levels. Building on previous studies (as developed by Camerer et al., 2004), we maintain the assumption that f follows a Poisson distribution with coefficient τ:

$$f_k = \frac{\tau^k}{k!} e^{-\tau}.$$

The expectation of the distribution is τ, which thus represents the overall expected level among the players.

[13] We assume $f_i > 0$ for all $i \leq K$. For the truncated distribution to be well-defined, it is sufficient to assume $f_0 > 0$, but we restrict ourselves to the cases where all levels are present with a positive probability.

2.2 Existence and Uniqueness

Since the definition of ICH involves the best response to a strategy which coincides with the own strategy with a positive probability, the strategy is defined as a solution of a fixed-point problem. We provide a set of sufficient conditions for the existence and uniqueness.

Let $X = [\underline{x}, \overline{x}] \subset \mathbb{R}$ be the strategy space allowing $\underline{x} = -\infty$ and/or $\overline{x} = \infty$.

Assumption 3 *$\forall x_i$ and $\forall x_{-i}$,*

$$\frac{\partial^2 u}{\partial x_i^2}(x_i, x_{-i}) < 0, \ \frac{\partial u}{\partial x_i}(\underline{x}, x_i) > 0 \ \text{and} \ \frac{\partial u}{\partial x_i}(\overline{x}, x_i) < 0.$$

Assumption 4 *$\forall x_i$ and $\forall x_j$ such that $i \neq j$,*

$$\frac{\partial^2 u}{\partial x_i^2} + (n-1)\frac{\partial^2 u}{\partial x_i \partial x_j} < 0.$$

Assumption 3 guarantees that the best response function is well-defined, and Assumption 4 assures that its slope to be less than +1 (including any negative value). The former provides boundary conditions for the best response function and the latter is sufficient for uniqueness, which is closely related to the condition imposed in Angeletos and Pavan (2007) for linear quadratic games that we investigate thoroughly in the following section. Under these conditions, the ICH exists uniquely. The proof of the following Proposition is relegated to the Appendix.

Proposition 1 *Under Assumptions 3 and 4, the sequence of ICH strategies $(\sigma_k)_{k=1}^{\infty}$ is uniquely determined for any $g = (g_k)_{k=1}^{\infty}$ and σ_0.*

3 Asymptotic Properties

In this section, we provide theoretical insights into implications of inclusiveness. We show below that the distance of the L and CH strategies from the Nash equilibrium diverges, while that of the ICH strategy is bounded (Theorem 1) for the games in which the best-reply functions are asymptotically expanding. On the other hand, the strategies in all of those models are bounded (Theorem 2) for the games in which the best-reply functions are not asymptotically expanding. These analytical results suggest that whether or not the inclusiveness condition matters in describing the asymptotic behavior depends on the asymptotic property of the best-reply functions.

For the sake of tractability, we focus on linear quadratic games (Currarini and Feri (2015)) which have desirable features for our analysis. First, they are fully aggregative games (Cornes and Hartley (2012)), in which the payoff of each player is affected by the action profile of the players through the aggregate of the strategies of all players and her own strategy. This fits well to our current objective, as our goal here is to understand analytically how the optimal strategy of a player would be affected by the belief over the type of the other players. The fact that the strategy of the other players appears explicitly as a term in an aggregative form allows us to obtain straightforward insights on the relationship between the shape of the best-reply functions and the players' beliefs over the strategies of the other players. Second, in a more technical convenience, linear quadratic games have a property such that, when a player holds a stochastic belief over the strategies of the other players, the maximizer of her expected payoff coincides with the best reply against the pure strategy which takes the expected value of the aggregate. This is because the linearity of the derivative allows us to switch the order of the partial derivative and the expectation. Then, facing heterogeneous beliefs over the other players' strategies, our analysis can simply focus on the best reply against the expectation of the beliefs, which provides us with a high tractability of the models.[14]

Consider n individuals, each of whom takes an action $x_i \in \mathbb{R}$. The payoff of player i in a linear quadratic game is a function of her own action x_i and the aggregate of the other players' actions $X_{-i} = \sum_{j \neq i} x_j$ in the following form:

$$u_i(x_i, X_{-i}) = \lambda^t x + x^t \Gamma x \tag{1}$$

where $x = (x_i \quad X_{-i})^t$, $\lambda = (\lambda_x \quad \lambda_X)^t$ and

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{12} & \gamma_{22} \end{pmatrix}.$$

There are several games of interest which fall into the class of linear quadratic games.

Example **1** *(A simple quadratic game) Suppose* $u_i(x_i, x_{-i}) = -\left(\sum_j x_j\right)^2$. *Then,* $\lambda^t = (0 \quad 0)$ *and*

$$\Gamma = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

Example **2** *(Cournot competition) Consider a Cournot competition. Suppose that the inverse demand function is linear* $P(Q) = a - bQ$, *and each firm has a constant marginal cost* c_i. *Let* q_i *be the quantity produced by firm* i *and* $Q_{-i} := \sum_{j \neq i} q_j$. *The profit of firm* i *is:*

$$\Pi_i = q_i (a - b(q_i + Q_{-i}) - c_i).$$

Then, $\lambda^t = (a - c_i \quad 0)$ *and*

$$\Gamma = \begin{pmatrix} -b & -\frac{b}{2} \\ -\frac{b}{2} & 0 \end{pmatrix}.$$

[14] Obtained insights could be extended to a game with more general payoff functions, to the extent that the second-degree Taylor expansion of the payoff function with respect to the aggregate strategy provides an approximation.

Example 3 *(Keynesian beauty contest games) Suppose that each of n players chooses a number x_i simultaneously, and each player's payoff is quadratic with respect to the distance between her own choice and the average of all players' choices multiplied by a constant $p \in (0,1)$. Then,*

$$u_i(x_i, X_{-i}) = -\left(x_i - p\left(\frac{x_i + X_{-i}}{n}\right)\right)^2.$$

Then, $\lambda^t = (0 \quad 0)$ and

$$\Gamma = \begin{pmatrix} -\left(1-\frac{p}{n}\right)^2 & \left(1-\frac{p}{n}\right)\frac{p}{n} \\ \left(1-\frac{p}{n}\right)\frac{p}{n} & -\left(\frac{p}{n}\right)^2 \end{pmatrix}.$$

Example 4 *(Public good provision game) Suppose that each agent contributes x_i to a public good and the cost is quadratic:*

$$u_i(x_i, X_{-i}) = \theta_i(x_i + X_{-i}) - c_i x_i^2.$$

Then, $\lambda^t = (\theta_i \quad \theta_i)$ and

$$\Gamma = \begin{pmatrix} -c_i & 0 \\ 0 & 0 \end{pmatrix}.$$

We impose some regularity conditions on the linear quadratic game in the form (1). First, we assume $\gamma_{11} < 0$. This implies that u_i has a unique maximizer for any X_{-i} and thus the best-reply function is well-defined. It is straightforward to show that the game defined by (1) has a unique symmetric Nash equilibrium:

$$x^* := -\frac{\lambda_x}{2(\gamma_{11} + (n-1)\gamma_{12})}.$$

We assume that the denominator is non-zero so that the symmetric Nash equilibrium is well-defined. By applying a parallel transformation $y_i := x_i - x^*$, (1) becomes:

$$u_i = \lambda^t x + x^t \Gamma x = \lambda_y^t y + y^t \Gamma y + c$$

where $Y_{-i} = \sum_{j \neq i} y_j$, $y = (y_i \quad Y_{-i})^t$, $\lambda_y = (0 \quad \lambda_Y)^t$, and λ_Y and c are independent of y. As the terms $\lambda_y^t y = \lambda_Y Y_{-i}$ and c have no strategic consequence on player i's behavior (i.e. the best-reply function of player i is unaffected), we can assume $\lambda_Y = 0$ and $c = 0$ without loss of generality. Therefore, in the following, we focus our attention on the games with the payoff function:

$$u_i = y^t \Gamma y, \tag{2}$$

with $\gamma_{11} < 0$ and $\gamma_{11} + (n-1)\gamma_{12} \neq 0$ (as in Angeletos and Pavan (2007)). Notice that there is a unique symmetric Nash equilibrium $y_i^* = 0$ for all i.

The first-order condition of player i is:

$$\frac{\partial u_i}{\partial y_i} = 2\gamma_{11} y_i + 2\gamma_{12} Y_{-i}.$$

When player i holds a stochastic belief over the strategies of the other players, the aggregate of the other players' strategies is a random variable $\tilde{Y}_{-i}$. Since the first-order condition is linear in Y_{-i} in quadratic games, the best reply against a mixed-strategy profile coincides with the best reply against the aggregate strategy which takes deterministically the expected value of the random variable:

$$BR_i\left(\tilde{Y}_{-i}\right) = -\frac{\gamma_{12}}{\gamma_{11}}\mathbb{E}\left[\tilde{Y}_{-i}\right]. \tag{3}$$

In order to describe asymptotic properties, we consider a sequence of linear quadratic games in which the number of players increases. More precisely, let $G(n) = \langle n, \mathbb{R}, (u_i^n)_{i=1}^n \rangle$ be a normal-form game with n players where the set of pure strategies is fixed as the set of real numbers $\mathbb{R}$,[15] and u_i^n is the payoff function of player i which satisfies (2). We analyze asymptotic properties of the strategies under the sequence of games $\{G(n)\}_{n=2}^{\infty}$.

Remember that the three models under our scrutiny here, L, CH and ICH, differ only in the assumption imposed on players' beliefs on the types of the other players. For each model, the strategy in each level is defined in the same way as in Definition 1. The only difference is that the frequency $g_k(h)$, assigned in the belief of a level-k player to the event in which each of the other players should be level-h, is specified by the equation (ICH) in Section 2 in the ICH model, but it is replaced by (L) (resp. (CH)) in the L (resp. CH) model. It is worth emphasizing that our results in this section do not hinge on the Poisson assumption concerning the underlying distribution f_k. We consider a sequence of level-symmetric strategies $\sigma = (\sigma_k)_{k\geq 0}$ where for each $k \geq 1$, σ_k maximizes the expected payoff under the belief $g_k(h)$ (Definition 1).

For each game $G(n)$ and in each of the three models, the level-0 strategy σ_0 is exogenously given, allowing the possibility of a mixed strategy. In order to make the comparison explicit across the models for $k \geq 1$, we add a superscript which represents the model, such as σ_k^L, σ_k^{CH}, and σ_k^{ICH}. Note that, by (3), $\sigma_k^M(n)$ are all pure strategies for $k \geq 1$ for each model $M \in \{L, CH, ICH\}$.

We assume that the following limit exists, allowing infinity:

$$A := \lim_{n\to\infty}\left|\frac{\gamma_{12}}{\gamma_{11}}n\right| \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

Remember that $-\frac{\gamma_{12}}{\gamma_{11}}$ is the slope of the best-reply function (3). Since $\tilde{Y}_{-i}$ is the sum of the strategies of the other players, A is the limit of the slope of player i's best-reply function, as a function of the *average* of the other players.

First, we consider the case $A = \infty$. In such games, we say that the sequence of the games is *asymptotically expanding*, denoting the property that the sensitivity of one's strategy to

[15]The assumption of the one-dimensional, unbounded strategy space allows us to obtain clear insights on the convergence and/or divergence of the strategies. In the games with a compact, one-dimensional strategy space, these insights could be inherited with some adjustments, e.g. divergence corresponds to a bang-bang corner solution.

the average strategy of the other players increases without a bound. We call such games as *expansive games*. We show that the strategies diverge from the Nash equilibrium in the L and the CH models, while it is bounded in the ICH model.

Theorem 1 *Consider a sequence of games $\{G(n)\}_{n=2}^{\infty}$ in which the payoff functions satisfy (2) for each n. Consider any σ_0 and let $\mu := \mathbb{E}[\sigma_0]$. Suppose $A = \infty$. For any $\mu \neq 0$, $\lim_{n\to\infty} \left|\sigma_k^L(n)\right| = \infty$ and $\lim_{n\to\infty} \left|\sigma_k^{CH}(n)\right| = \infty$, while $\lim_{n\to\infty} \left|\sigma_1^{ICH}(n)\right| < \infty$ and $\lim_{n\to\infty} \left|\sigma_k^{ICH}(n)\right| = 0$ for all $k \geq 2$.*

Among the examples described above, the sequence $\{G(n)\}_{n=2}^{\infty}$ is asymptotically expanding ($A = \infty$) in the simple quadratic game (Example 1) and in the linear Cournot competition (Example 2). A common feature of these games is that the aggregate of all players' strategies enters into each player's payoff in a way that the aggregate term does not dissipate for large n. When $A = \infty$, we show that the behaviors in the ICH model show a stark contrast with those in the L or in the CH model. The presence of the inclusiveness condition thus leads to an intrinsic difference in the prediction. Moreover, we show that the ICH strategy converges to the Nash equilibrium for any level $k \geq 2$.

Theorem 1 implies that for asymptotically expanding games, there are fundamentally three degrees of strategic sophistication: naïve (level-0), partially sophisticated (level-1), and highly sophisticated (level-2 or more). Since the strategies of level-2 or higher all converge to the Nash equilibrium, behaviors in this class of games fall into one of the following three cases asymptotically: (i) naïve strategy that does not maximize the expected payoff, (ii) level-1 strategy that maximizes the payoff under an inconsistent belief, and (iii) fully sophisticated strategy that maximizes the payoff under the consistent belief. This is in line with Camerer et al. (2004) and Crawford et al. (2013), who point to that in many games there are no more than three levels of hierarchical thinking observed, and with Rubinstein (2016), who argues for prevalence of a two-type (instinctive and contemplative) typology. In a sense, ICH is a middle-way integrating these two sets of observations.

Now, consider a sequence of games which satisfies the same conditions assumed in Theorem 1, except for that on A.

Theorem 2 *Suppose $A < \infty$. For any μ, $\left|\sigma_k^L(n)\right|$, $\left|\sigma_k^{CH}(n)\right|$, and $\left|\sigma_k^{ICH}(n)\right|$ are all bounded as $n \to \infty$, for all $k \geq 1$.*

In the standard Keynesian beauty contest games (Example 3), we have:

$$A = \lim_{n\to\infty} \left| \frac{\left(1 - \frac{p}{n}\right)\frac{p}{n} n}{-\left(1 - \frac{p}{n}\right)^2} \right| = p < \infty.$$

In the games with $A < \infty$, the slope of the best-reply function is bounded as n goes to infinity. Hence, even in a game with a large number of players, the optimal strategy of a player does not diverge. In the beauty contest games, we see that the aggregate term is relevant in each player's payoff to the degree of the *average* of all players. Table 1 shows the comparison of the L, CH, and ICH models fitting to the data in beauty contest games studied in Bosch-Domènech et al. (2002).[16] We observe that the ICH model fits better than L and CH by comparing second moments (that is, matching the means to estimate models lead to standard deviations that are closer to the data for ICH compared to L or CH).

		Data		L		CH		ICH	
Name	N	mean	s.d.	τ	s.d.	τ	s.d.	τ	s.d.
Lab	86	35.1	19.6	1.06	12.4	1.31	9.6	0.90	12.9
Classroom	138	26.8	17.7	1.90	12.7	2.53	8.6	1.51	14.1
Take-home	119	25.2	17.0	2.11	12.5	2.85	8.3	1.65	14.2
Theorists	54	17.8	24.3	3.46	10.4	5.15	6.0	2.42	14.3
Conference	92	16.8	20.1	3.72	9.9	5.64	5.6	2.54	14.3
E-mail	150	22.2	20.7	2.56	11.9	3.58	7.5	1.93	14.3
Newspapers	7893	23.1	20.2	2.41	12.1	3.34	7.7	1.85	14.3

Table 1: Method of moments estimation of beauty contest games studied in Bosch-Domènech et al. (2002). Following Camerer et al. (2004), the Poisson coefficient τ is estimated by matching the first moments.

3.1 Condorcet Jury Theorem

In order to highlight the difference of the asymptotic behaviors between the games with $A = \infty$ and $A < \infty$, we now pay specific attention to a game of collective decision-making in a standard setting of the Condorcet Jury Theorem. The game fits well to the objectives of our analysis here, because (i) it falls in the class in which the best response function is an expansion mapping, therefore allows us to provide a contrast to the analysis we provide above for the beauty contest games, (ii) asymptotic behavior of the players as n becomes large is often the center of the analysis and is thus well-documented in the literature, and (iii) with an obvious real-life application, the game is simple enough to be understood by non-economists, and therefore it is opt for experimental studies at laboratory, which we provide in Section 4.

A group of n individuals makes a binary collective decision $d \in \{-1, 1\}$. The true state of the world is also binary, $\omega \in \{-1, 1\}$, with a common prior of equal probabilities. The payoff is a function of the realized state and the collective decision as follows:

$$\tilde{u}(\omega, d) = \begin{cases} 0 & \text{if } \omega \neq d, \\ q & \text{if } \omega = d = 1, \\ 1-q & \text{if } \omega = d = -1, \end{cases}$$

[16] We express our gratitude to Rosemarie Nagel for providing us with the data.

with $q \in (0,1)$ for all individuals.[17] Each individual $i \in \{1, \cdots, n\}$ receives a private signal $s_i \in S$, distributed independently conditional on the true state ω. A collective decision is made by the majority rule. Upon receiving signal s_i, individual i casts a vote $v_i \in \{0,1\}$, and the collective decision is determined by the sign of $(\sum_i v_i - n/2)$.

We do not restrict the signal space S to be binary.[18] We assume $S \subset \mathbb{R}$ so that S is an ordered set, and we assume that the commonly known distribution satisfies the monotone likelihood ratio property (MLRP), that is, the posterior distribution $\Pr[\omega = 1|s_i]$ is increasing in s_i.

A strategic Condorcet Jury Theorem claims that asymptotic efficiency is obtained among the rational individuals with homogeneous preferences and costless information acquisition, as described above. More precisely, it claims that, under the Nash equilibrium behavior, the probability of making a right decision converges to one as n goes to infinity. In the following subsections we characterize the best response function and investigate whether the asymptotic collective efficiency would be obtained under the cognitive hierarchy models in which individuals may show systematic deviations from the Nash behavior.

3.1.1 Best response function

Suppose that the private signal of each voter is independently and identically drawn from a normal distribution conditional on the true state: $s_i \sim \mathcal{N}(\omega, \sigma)$. Since the distribution satisfies the MLRP, the posterior belief is monotone with respect to the obtained signal. Therefore, given the startegies taken by the other players, the best response of a voter is a cutoff strategy, that is, voting for $v_i = 0$ (resp. 1) if the obtained signal is smaller (resp. larger) than a threshold. Let $\bar{s}$ be the threshold in a symmetric Nash equilibrium. Then, the voter should be indifferent between voting for $v_i = 0$ and 1 when she receives the signal $\bar{s}$. Since the expected payoff should be equal,

$$\mathbb{E}\left[\tilde{u}(\omega, d)\,|v_i = 1, \bar{s}\right] = \mathbb{E}\left[\tilde{u}(\omega, d)\,|v_i = 0, \bar{s}\right]. \tag{4}$$

The expected payoffs differ only on the pivotal events. Suppose that n is odd and $n = 2m+1$ with an integer m. A voter is pivotal if the votes from the other players are split exactly to

[17]The assumption of symmetric prior is without loss of generality, since we allow the payoffs of the two types of right decisions to be heterogeneous. Although the preferences are often represented by a loss function for wrong decisions in the standard CJT models, we equivalently use a gain function for right decisions in accordance with our experiment, which awards positive points to right decisions, rather than subtracting points for wrong decisions.

[18]Even though a number of CJT models assume a binary signal space, we believe that this is not the right assumption for information aggregation problems. Even under the binary state space, there are uncountably many ways to update the prior belief, and thus the set of possible beliefs spans a continuous space. Assuming the signal space to be binary implies that there are only two ways of Bayesian update, which is far from innocuous.

m votes for $v_i = 0$ and 1. Therefore, (4) is equivalent to:

$$
\begin{aligned}
& \tilde{u}(1,1)\Pr[\omega = 1|\bar{s}, \text{piv}] = \tilde{u}(-1,-1)\Pr[\omega = -1|\bar{s}, \text{piv}] \\
\Leftrightarrow\ & q\Pr[\omega = 1, \bar{s}, \text{piv}] = (1-q)\Pr[\omega = -1, \bar{s}, \text{piv}] \\
\Leftrightarrow\ & q\frac{1}{2}\Pr[\bar{s}, \text{piv}|\omega = 1] = (1-q)\frac{1}{2}\Pr[\bar{s}, \text{piv}|\omega = -1] \\
\Leftrightarrow\ & q\phi\left(\frac{\bar{s}-1}{\sigma}\right)\binom{2m}{m}\left(\Phi\left(\frac{\bar{s}-1}{\sigma}\right)\right)^m\left(1-\Phi\left(\frac{\bar{s}-1}{\sigma}\right)\right)^m \\
& = (1-q)\phi\left(\frac{\bar{s}+1}{\sigma}\right)\binom{2m}{m}\left(\Phi\left(\frac{\bar{s}+1}{\sigma}\right)\right)^m\left(1-\Phi\left(\frac{\bar{s}+1}{\sigma}\right)\right)^m \\
\Leftrightarrow\ & \frac{q}{1-q}\frac{\phi\left(\frac{\bar{s}-1}{\sigma}\right)}{\phi\left(\frac{\bar{s}+1}{\sigma}\right)} = \left(\frac{\Phi\left(\frac{\bar{s}+1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}+1}{\sigma}\right)\right)}{\Phi\left(\frac{\bar{s}-1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}-1}{\sigma}\right)\right)}\right)^m
\end{aligned}
$$

where ϕ and Φ are respectively the pdf and the cdf of the standard normal distribution $\mathcal{N}(0,1)$. The left-hand side of the last line is increasing in $\bar{s}$, taking values from 0 to ∞, while the right-hand side is decreasing in $\bar{s}$, taking values from ∞ to 0. Hence, the symmetric Nash equilibrium $\bar{s}$ is uniquely defined for each (q, σ, n). For example, let $q = 9/11$, $\sigma = 1$. For the values of $n = 5, 9, 19$, we have $\bar{s} = -0.217, -0.126, -0.062$, respectively.

Now, consider the best response function of player i. Suppose that all other players use a strategy biased by $b \in \mathbb{R}$ with respect to the symmetric Nash equilibirum, that is, the threshold is $\bar{s} + b$ for all players other than i. Then, the best response of player i is given by:

$$
\begin{aligned}
& \frac{q}{1-q}\frac{\phi\left(\frac{s_i-1}{\sigma}\right)}{\phi\left(\frac{s_i+1}{\sigma}\right)} = \left(\frac{\Phi\left(\frac{\bar{s}+b+1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}+b+1}{\sigma}\right)\right)}{\Phi\left(\frac{\bar{s}+b-1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}+b-1}{\sigma}\right)\right)}\right)^m \\
\Leftrightarrow\ & s_i = \frac{\sigma^2}{2}\log\left(\frac{1-q}{q}\left(\frac{\Phi\left(\frac{\bar{s}+b+1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}+b+1}{\sigma}\right)\right)}{\Phi\left(\frac{\bar{s}+b-1}{\sigma}\right)\left(1-\Phi\left(\frac{\bar{s}+b-1}{\sigma}\right)\right)}\right)^m\right).
\end{aligned}
$$

We can verify that the right-hand side is decreasing in b. By letting $b_i = s_i - \bar{s}$, we can rewrite the best response of player i as a function from b to b_i.

The best response functions for different values of n are plotted in Figure 1. The model parameters are chosen so that the game coincides with the one with asymmetric payoffs in our experiments.[19] We observe that the best response functions are fairly close to a linear function. Our analysis using the linear quadratic game in this section can be applied to this game to the extent that the second-order Taylor expansion approximates the payoff functions. From the figure, we can expect that the approximation by a linear quadratic game is sufficiently precise.

[19]The only difference is that the signals are normally distributed here, while they are discrete in the experiments. This is due to the difficulty of describing normally distributed signal in the laboratory. Variances are chosen to be the same. The Poisson coefficient is calibrated here as the average of the estimated values obtained in the experimental analysis in Section 5.

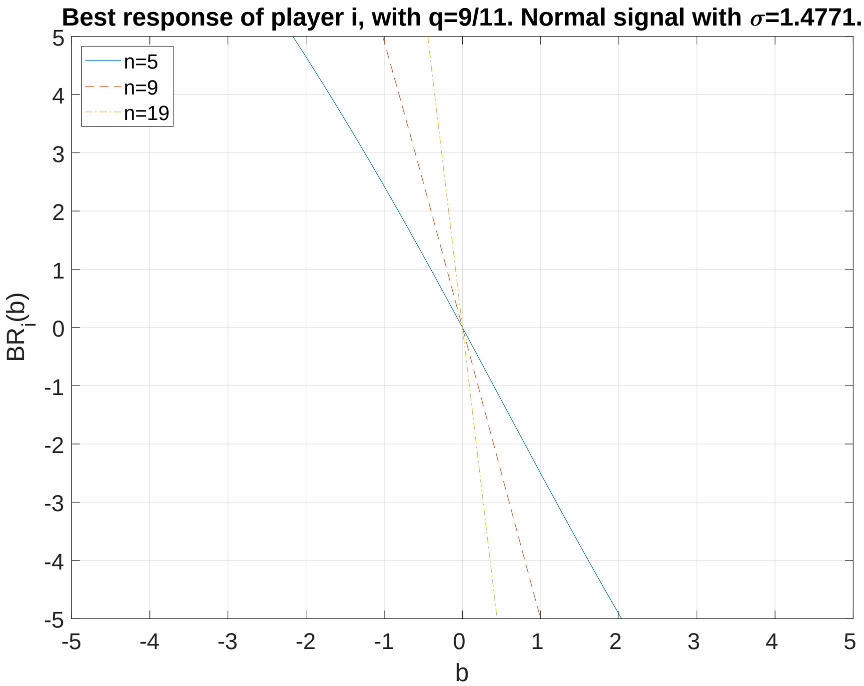


Figure 1: Best response function of player i, given the equal bias b by all other players. Payoff parameter is $q = 9/11$. Normal signal $\mathcal{N}(\omega, \sigma)$ with $\sigma = 1$.

We also observe from the figure that the slope is negative and it becomes steep as n increases. The idea behind the monotonicity is that, when the same bias is added by all other players, the best response of the player is to correct the total bias by going to the opposite direction. When the signals are normally distributed, the total bias is well approximated by simply adding all biases, which makes the best response approximately a linear function. Since the absolute value of the slope increases without a bound as n becomes large, our game falls in the class with $A = \infty$.

3.1.2 Numerical comparison

In order to underline the differences implied by different behavioral assumptions, we provide numerical computation results using the game described above. Our aim here is to highlight numerically the behavioral consequences of the inclusiveness condition, by comparing the individual strategies predicted by each model. Four different behavioral specifications are compared: Nash equilibrium (NE), the level-k model (L), the cognitive hierarchy model (CH), and the inclusive cognitive hierarchy model (ICH).

Table 2 shows the level-1 and level-2 strategies under the L, CH, ICH models, and the Nash equilibrium, as a function of the group size n.[20] In the L model, level-1 (L1) strategy

[20] The level-0 strategy is set as $\sigma_0 = -5$. See subsection 5.4 for a detailed discussion.

n	NE	L1, CH1	L2	CH2	ICH1	ICH2
3	−0.345	14.5	−45.1	−1.651	−0.136	−0.321
5	−0.226	25.3	−128	−1.657	0.042	−0.204
7	−0.168	36.0	−254	−1.664	0.136	−0.149
9	−0.133	46.8	−423	−1.670	0.194	−0.117
11	−0.111	57.5	−635	−1.677	0.234	−0.096
13	−0.095	68.3	−890	−1.683	0.263	−0.081
15	−0.083	79.1	−1188	−1.690	0.285	−0.071
17	−0.073	89.8	−1529	−1.696	0.302	−0.062
19	−0.066	101	−1913	−1.703	0.316	−0.056

Table 2: Level-1 and level-2 strategies under the L, CH, ICH models, and the Nash equilibrium, as a function of the group size n, when the level-0 strategy is set as $\sigma_0 = -5$.

diverges to infinity, while Level-2 (L2) diverges to minus infinity, exhibiting the expanding nature of the best response function. In the CH model, the level-1 strategy (CH1) is the same as in the L model, by definition. The level-2 strategy (CH2) is slightly decreasing, meaning a divergence from the Nash behavior. In the ICH model, strategies are increasing in both level-1 and level-2, in accordance with the Nash behavior. ICH2 strategy is converging to the Nash equilibrium and ICH1 is not. We can observe that the ICH strategies are classified asymptotically to the following types: intermediary (ICH1) and rational (ICH2), aside from naïve (L0). The numerical computations are coherent with the theoretical properties we obtained above.

The numerical example suggests a contrast among the behavioral assumptions under our consideration. In particular, it suggests that the inclusiveness condition plays a key role in describing the asymptotic behavior. In what follows, we show the results from our stylized laboratory experiment which provides statistical evidence for our scrutiny of the models.

4 Experimental Design

All of our computerized experimental sessions were held at the Ecole Polytechnique Experimental Economics Laboratory.[21] In total we had 140 actual participants in 7 sessions, in addition to the pilot sessions with more than 60 participants. In each session, 20 participants took part in 4 phases (together with a short trial phase) which lasted about one hour in total. Earnings were expressed in experimental currency units (ECUs) and exchanged for cash, to be paid immediately following the session. Participants earned an average of about 21 Euros, including a default 5 Euros for participation. Complete instructions and details

[21] Both the z-Tree program (Fischbacher (2007)) and the website for participant registration were developed by Sri Srikandan, to whom we are very much grateful.

can be found in our online appendix.[22] The instructions pertaining to the entire experiment were read aloud at the beginning of each session. Before each phase, the changes from the previous phase were read aloud, and an information sheet providing the relevant details of the game was distributed. These sheets were exchanged with new ones before each phase.

We employed a within-subject design where each participant played all 4 phases consecutively in a session. Each phase contained 15 periods of play, and thus each participant played for a total of 60 periods under a direct-response method. Since the question of our research relates to the strategic aspects of group decisions, our experiment was presented to participants as an abstract group decision-making task where neutral language was used to avoid any reference to voting or elections of any sort.

In the beginning of each period, the computer randomly formed groups of participants, of a size that was commonly known and predetermined for each phase (either $n = 5$, 9, or 19).[23] Then, a box was shown to each participant with one hundred squares (to be referred as *cards* from now on), all colorless (gray in *z-Tree*). At the same time, the unknown true color of the box for each group was determined randomly by the computer. The participants were informed that the color of the box would be either blue or yellow, with equal probability. It was announced that the blue box contained 60 blue and 40 yellow cards, whereas the yellow box contained 60 yellow and 40 blue cards.

After confirming to proceed to the next screen, 10 cards drawn by the computer with random locations in the box were shown to the participants, this time with the true colors. These draws were independent among all participants but were drawn from the same box in the same group. Having observed the 10 randomly-drawn cards, the participants were required to choose either blue or yellow by clicking on the corresponding button. Then, the decision for the group was reached by majority rule, which was resolute every time, since we only admitted odd numbers for group sizes and abstention was not allowed. Once all participants in a group had made their choices, the true color of the box, the number of members who chose blue, the number of members who chose yellow, and the earnings for that period were revealed on the following screen. A new period started after everyone confirmed.

In one of the four phases, the group size was set at $n = 5$ and the payoffs were symmetric. Each participant earned 500 ECUs for any correct group decision (i.e., a blue decision when true color of the box was blue, or a yellow decision when true color of the box was yellow). In the case of an incorrect decision, no award was earned. In the other three phases, each treatment differed only in the size of the groups (5, 9, or 19) where asymmetric payoffs were fixed. The correct group decision when the true color of the box was blue awarded

[22] The online appendix can be found at http://sites.google.com/site/ozkesali.

[23] In the phase with $n = 9$, two groups of 9 randomly-chosen members were formed at each period. Having 20 participants in total in each session, 2 randomly-chosen participants were 'on hold' during the period. The same method is applied in the phase with $n = 19$. A group of 19 was formed and thus one randomly-chosen participant was on hold.

each participant in the group with 900 ECUs, whereas the correct group decision when the true color of the box was yellow awarded them with 200 ECUs.[24] Lastly, we implemented a random-lottery incentive system where the final payoffs at each phase were determined by the payoffs from a randomly-drawn period.[25]

Let us underline that asymmetry in remuneration is introduced in our experiment in order to observe the effect of a prior bias on the participants' behavior. It is not surprising that an informative strategy (i.e. voting for the choice favored by the signal), or one close to it, is employed by a large majority of the participants under symmetric awards.[26] When it is commonly known that one of the alternatives may provide a larger award, in addition to the change of the symmetric Nash equilibrium shifting towards the ex ante preferable alternative, each individual's behavior may shift, and furthermore, such shifts may be heterogeneous across individuals. Consequentially, each individual may hold heterogeneous beliefs over the strategies employed by the other individuals in the group. The accumulated effects of such heterogeneous belief formation may hamper the performance of group decision-making, which is one of our main concerns in this paper.

At the beginning of each session, as part of the instructions, participants played through two mandatory trial periods. Each session concluded after a short questionnaire. According to the anonymously-recorded questionnaire, 44% of the participants were female. The age distribution was as follows: 31% between the age of 19 and 22, 26% between 23 and 29, 14% between 30 and 39, and 29% between 40 and 67. Heterogeneity in their professions was relatively high: 46% administrative staff, 37% undergraduate students ("Polytechniciens"), 12% Ph.D. students, 1% master students, and 3% researchers. 6% of the participants had previously taken an advanced course in game theory, while 14% had taken an introductory course. 39% said that they had some notions about game theory, while 41% claimed to have no knowledge of game theory.

5 Experimental Results

5.1 Cutoff Strategies

Under our experimental design, a pure strategy of an individual is a function from the realized signal to a binary vote. It is straightforward to show that the best reply of an individual, given any belief on the strategies of other group members, is a cutoff strategy. There exists

[24] We also conducted pilot sessions with the rewards of 800 ECUs and 300 ECUs. As there was less of a marked contrast in the observed deviations from Nash behavior, we decided to run the rest of the sessions with the rewards 900 ECUs and 200 ECUs.

[25] Participants were told both verbally and through info sheets that in the case where the lottery picked a period for remuneration in which a participant had been on hold, the payoff in that phase for this participant was set at 500 ECUs, which is about the average of the winning points.

[26] Behaviors close to the informative strategy are indeed observed in our experiments with unbiased payoffs. See histogram in Figure 3(a).

a threshold for each individual such that she votes for blue if and only if her signal induces a higher posterior probability of a blue state than the threshold. Since the posterior belief over the two states varies monotonically as a function of the number of blue cards among the 10 revealed ones, a cutoff strategy in our experiment is that each individual votes for blue when the number of observed blue cards is higher than the cutoff value, and for yellow otherwise. Special cases include voting for one of the colors regardless of the signal. The cutoff value is considered as an extreme value (either 0 or 10) for such a behavior.

We have observed in our data that a majority of participants used a cutoff strategy with randomization. Randomization occurs with two or more realized values of the signal, with the degree of randomization varying monotonically in the right direction (i.e. a higher probability of voting for blue given more blue cards in the signal). We regard such a behavior as a consequence of decision-making with an error or other uncertainties which are not explicitly formalized in the model.[27]

More precisely, let $x_i \in [0, 10]$ be the (continuous) cutoff strategy of voter i, and let $s_i^t \in \{0, 1, \cdots, 10\}$ be the (discrete) signal realized in period t, i.e. the number of blue cards out of the ten revealed ones. We assume that the probability that i casts a vote $v_i^t \in \{0(\text{yellow}), 1(\text{blue})\}$ depends on the distance between the signal and the cutoff value, following the logistic distribution:

$$\varphi\left(v_i^t | s_i^t, x_i, \varepsilon_i\right) = \frac{\left(\exp \varepsilon_i \left(s_i^t - x_i\right)\right)^{v_i^t}}{1 + \exp \varepsilon_i \left(s_i^t - x_i\right)} \tag{5}$$

where ε_i is the error coefficient. As φ pins down a distribution over voting profiles $(v_i)_{i=1}^n$, the social decision d made by the simple majority rule is a function of the realized signals $s = (s_i)_{i=1}^n$, cutoff values $x = (x_i)_{i=1}^n$, and the error coefficients $\varepsilon = (\varepsilon_i)_{i=1}^n$. Hence, for fixed ε, we can write the expected payoff as a function of the cutoff strategy profile $x \in [0, 10]^n$:[28]

$$u(x) = \mathbb{E}_{\omega, s}\left[\tilde{u}\left(\omega, d\left(s, x, \varepsilon\right)\right)\right].$$

Let T be the set of periods under consideration. Provided the observation $(s_i^t, v_i^t)_{t \in T}$, our estimation of the cutoff strategies is obtained by maximizing the likelihood:

$$\mathcal{L}\left(\left(s_i^t, v_i^t\right)_{t \in T} \middle| x_i, \varepsilon_i \right) = \prod_{t \in T} \varphi\left(v_i^t | s_i^t, x_i, \varepsilon_i\right).$$

In Figure 2, histograms of the estimated cutoff values are shown for each phase. Several remarks are in order. First, we see a clear shift in the distribution from the symmetric

[27] In the post-experiment questionnaire, a few participants expressed reasonings which seemed to have no clear connection with any Bayesian update, such as "I chose yellow when I saw three or more yellow cards aligned in a row, since I thought it was a strong sign that the box is yellow." We assume that such deviations from rationality are accounted for by the error term.

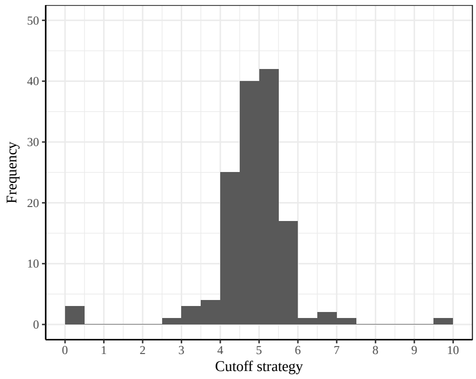
[28] Technically, we can extend the strategy space beyond the interval $[0, 10]$, given our specification of the errors in (5). However, we chose not to do so, as the main intuition of the model is unchanged, and we prefer to avoid any possible confusion caused by a cutoff value defined outside of the signal space.

payoffs to the asymmetric ones. Most notably, for each of the group sizes of 5, 9 and 19 with asymmetric payoffs, a peak in the frequencies is clearly visible on the intervals $[0, 1)$, representing 6%, 9% and 9% of all cutoff values, respectively. As the cutoff value 0 corresponds to the behavior of voting for blue regardless of the obtained signal, the presence of the peaks suggests that a certain amount of participants used the signal-independent voting strategy, or at least one close to it. Second, about half of the estimated cutoff values are included in the interval $[4, 5)$ with asymmetric payoffs. The percentages in this interval for group sizes of 5, 9, and 19 are 54%, 51%, and 66%, respectively. Note that the unbiased strategy is represented by the cutoff value of 5. A cutoff value lower than 5 corresponds to a strategy biased in favor of voting for blue, the ex ante optimal choice. According to our estimation, roughly between one half and two thirds of participants used a cutoff strategy slightly biased towards the ex ante optimal choice. Third, no single player used a cutoff strategy higher than 8 in any phase with asymmetric payoffs. It is worth underlining that no signal-independent voting behavior to the other extreme direction (i.e. a cutoff value of 10, which corresponds to voting regardless of the signal for yellow, the ex ante suboptimal alternative) is observed with asymmetric payoffs. Fourth, a non-negligible amount of voting behaviors in favor of yellow are observed, even though they are rather a minority. The frequencies of cutoff values higher than 5 are 15%, 17% and 9%, respectively, in the three phases with asymmetric payoffs.

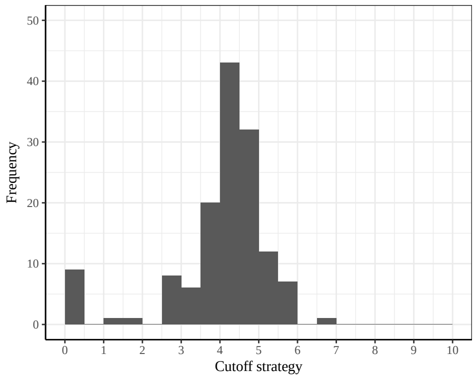
5.2 Robustness Check

Among the robustness checks we carried out, two of them are worth mentioning. First, we checked whether learning over the periods occurred. During each period t, what we observe for each individual i is only one realized signal s_i^t and the casted vote v_i^t. Therefore, a cutoff estimation requires a pool of observations over periods. We checked estimations over different intervals of periods: $T = \{1, \cdots, 15\}$, $\{1, \cdots, 10\}$, $\{6, \cdots, 15\}$, $\{1, \cdots, 7\}$, $\{9, \cdots, 15\}$, $\{1, \cdots, 5\}$, $\{6, \cdots, 10\}$, and $\{11, \cdots, 15\}$. The t-tests do not reject the null hypothesis for any pairs of intervals at the $p = 0.10$ level, suggesting that no change in strategy occurred over time, ruling out the possibility of learning. We thus conclude that the estimation is robust including observations over all 15 periods. We also counted the number of votes consistent with the cutoff estimation obtained from $T = \{1, \cdots, 15\}$. Overall, 90.3% of actions are consistent. Inconsistent actions are spread across periods, and the t-statistics of the comparison between the first and last 7 periods are 1.06, 1.32 and 1.39 for the number of inconsistent blue actions, yellow actions and the sum, respectively, none statistically significant at the $p = 0.10$ level.

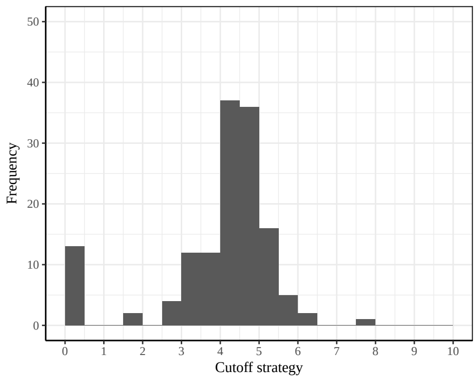
We also tested different assumptions on ε_i: (i) common across phases for each individual, (ii) common across individuals for each phase and session, (iii) common across individuals and sessions for each phase, and (iv) common for individuals, sessions and phases. Comparing the estimated cutoff values between any pair of different assumptions on ε_i, the R^2 values for



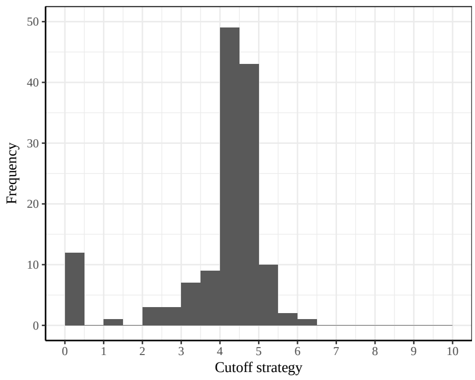
(a) $n = 5$, 500:500. Ave. 4.84



(b) $n = 5$, 900:200. Ave. 4.03



(c) $n = 9$, 900:200. Ave. 4.01



(d) $n = 19$, 900:200. Ave. 3.98

Figure 2: Histogram of the estimated cutoff strategies, given 140 observations, together with average values.

the 6 pairs are respectively, 0.90, 0.89, 0.89, 0.99, 0.99 and 0.99, implying that the estimation is robust. In the following, we use the estimated cutoff values under assumption (iii), as our main concern here is the change in subjects' behavior as a function of the group size.

5.3 The model fit

In what follows, we evaluate three models: level-k (L), cognitive hierarchy (CH), and inclusive cognitive hierarchy (ICH), estimating the parameters which fit best to our experimental data.

Our aim is to find out the sequence of level strategies $\sigma = (\sigma_1, \cdots, \sigma_K)$ and the level distribution $f = (f_0, \cdots, f_K)$, which best fits the observed cutoff values $(x_i)_{i \in N}$, provided an exogenously fixed level-0 strategy σ_0.[29] Throughout the paper, we maintain the assumption that f follows the truncated Poisson distribution with coefficient τ. Estimation of the level

[29]Remember that K is the highest level under consideration. Our estimation sets $K = 2$ due to a heavy computational burden for $K \geq 3$ and large n.

distribution f thus boils down to finding the best-fitting τ.

We assume that the observed cutoff values $(x_i)_{i\in N}$ are drawn from the distribution $\sum_{k=0}^{K} f_k \sigma_k$ with a logistic error. More precisely, the probability that the realized cutoff strategy is x_i is:

$$\phi\left(x_i|f,\sigma,\rho\right) = \sum_{k=0}^{K} f_k \mathbb{E}_y\left[\ell\left(x_i|y,\rho\right)|\,\sigma_k\right],$$

where expectation over y is drawn from the distribution σ_k,[30] and

$$\ell\left(x_i|y,\rho\right) = \frac{\exp\left(\frac{x_i-y}{\rho}\right)}{\rho\left(1+\exp\left(\frac{x_i-y}{\rho}\right)\right)^2}$$

is the density function of the logistic distribution with mean y and the error coefficient ρ. Our estimation maximizes the likelihood, choosing the Poisson distribution f and the error coefficient ρ:

$$\prod_{i\in N}\phi\left(x_i|f,\sigma,\rho\right).$$

Given σ_0, the variables to be estimated are thus τ and ρ.

5.4 Level-0 Strategy

Before proceeding to the comparison of the models, we briefly discuss the choice of the level-0 strategy, which can be supported by the idea of *saliency*. As discussed in Crawford and Iriberri (2007a), *inter alia*, some naturally occurring landscapes that are focal across the strategy space may constitute salient non-strategic features of a game and attract naïve assessments.[31] For instance, a strategy space represented by a real interval, say $[m, M]$, may have its minimal point m, its maximal point M, and its midpoint $\frac{m+M}{2}$ as salient locations. In our game, two of these deserve close attention. First, if we expect that a non-strategic, level-0 player would evaluate her choices while disregarding others' strategic incentives, such a behavior corresponds to a strategy of the ex ante favored choice, i.e. always voting for blue. A salient location would then be 0. Second, if we expect that a non-strategic player would choose a strategy which maximizes the probability of making a right decision regardless of the winning point (and thus is not payoff-maximizing), then a salient strategy is the midpoint 5. Furthermore, a uniform randomization over all available pure strategies is often chosen as the level-0 strategy in the literature (see discussion in Camerer et al. (2004)).

Table 3 provides a comparison based on the maximum likelihood estimation described in the previous subsection, under the ICH model with the group size $n = 5$. We used a grid

[30] σ_k could be a mixed strategy, as is often assumed for $k = 0$.

[31] We abstract from cognitive foundations that might raise the issue of "strategic awareness" as in Fehr and Huck (2016), as we do not have data on subjects' cognitive abilities.

	τ^*	ρ^*	$ICH1$ (σ_1)	$ICH2$ (σ_2)	LL
$\sigma_0 = 0$	3.9	2.5	4.45	4.28	−190.7
$\sigma_0 = 5$	10	1.6	4.17	4.23	−222.9
$\sigma_0 = 10$	10	1.6	3.84	4.21	−225.6
$\sigma_0 = \mathcal{U}[0, 10]$	4.3	2.3	4.00	4.20	−213.3

Table 3: Comparison of level-0 specifications for $n = 5$, together with the maximum log-likelihood attained under the ICH model.

search, with both τ and ρ varied from 0.1 to 10 with an increment of 0.1. Among the four salient strategies of σ_0 we specified above, the highest log-likelihood is attained at $\sigma_0 = 0$, followed by the uniform distribution, with the lowest at $\sigma_0 = 5$ and 10.[32] Note that the estimated τ^* hits the upper bound grid value 10 for $\sigma_0 = 5$ and 10. Provided the Poisson assumption imposed on f, a high value of τ^* means that the model can best fit the data by assigning the largest possible probability to the highest level-K (and the smallest to level-0). Therefore, one can expect at best the model to provide as a good fit as the symmetric Nash equilibrium. We thus conclude that setting such values of the level-0 strategy is of limited interest for describing the deviation from rational behavior, since these values of σ_0 can provide only a limited capacity of explaining our data beyond the Nash model.[33] Between the uniform σ_0 and $\sigma_0 = 0$, the latter fits better, which is coherent with the observations from our experiments with asymmetric payoffs that (i) there is a peak at 0 for all n, and (ii) no single player used a cutoff strategy higher than 8 in any phase. In what follows, we continue our analysis by setting the level-0 strategy to be $\sigma_0 = 0$. It is important to note that, this choice is made independently from how ICH performs with respect to other models. As seen in Table 4, ICH performs better than L and CH, regardless of the choice of level-0, as the lowest maximum log-likelihood obtained by ICH is higher than the highest maximum log-likelihood obtained by L or CH.

$\sigma_0 =$	0	5	10	$\mathcal{U}[0, 10]$
L	−415.3	−243.8	−438.6	−340.6
CH	−366.4	−241.2	−304.3	−240.4
ICH	−190.7	−222.9	−225.6	−213.3

Table 4: Comparison of maximum log-likelihoods obtained for different level-0 specifications with $n = 5$. The cases $n = 9$ and $n = 19$ are in Tables 7 and 8 in Appendix C.

[32] This holds for $n = 9$ as well, whereas for $n = 19$, the ICH model with uniform random level-0 performs better than 0 (see Table 8 in Appendix C).

[33] Similarly, we observed that τ^* hits the boundary value at $\sigma_0 = 5$ and $\sigma_0 = 10$ for $n = 9$ and $n = 19$, as well as under the L and the CH models, which implies that the prediction power of the models is quite limited under the assumption of $\sigma_0 = 5$ or $\sigma_0 = 10$.

5.5 Results

Table 5 summarizes the comparison of the models by maximum likelihood estimation.

		τ^*	ρ^*	σ_1	σ_2	LL
$n = 5$	L	0.1	0.4	10	0	-415.3
	CH	8.0	0.6	10	1.76	-366.4
	ICH	3.9	2.5	4.45	4.28	-190.7
$n = 9$	L	0.1	0.4	10	0	-416.3
	CH	10.0	0.7	10	2.52	-334.3
	ICH	5.0	2.3	4.91	4.67	-210.6
$n = 19$	L	0.1	0.4	10	0	-414.8
	CH	9.1	0.7	10	2.40	-338.0
	ICH	6.7	2.4	5.11	4.84	-210.7

Table 5: Comparison of the models by maximum log-likelihood, using 140 observations, under the assumption $\sigma_0 = 0$. Grid search is run for both the Poisson coefficient τ and the error coefficient ρ, from 0.1 to 10 with an increment of 0.1.

5.5.1 Level-k Model

An Lk player (level-k player in the L model) maximizes her payoff holding a belief that all other individuals play the $L(k-1)$ strategy. Since each individual has an incentive of correcting biases caused by all other players, the best reply of a Lk player is biased toward the opposite direction with respect to the Nash equilibrium as compared to the $L(k-1)$ strategy, and the degree of amplification increases as n increases.[34] We see in Table 5 that the cutoff strategy of the $L1$ player hits $\sigma_1 = 10$ as a response to the $L0$ strategy $\sigma_0 = 0$ for all values of $n = 5$, 9, and 19. A similar argument applies to $L2$ in the opposite direction, implying $\sigma_2 = 0$. Such an oscillation continues in the L model, and a bang-bang solution is obtained perpetually as k increases.

For all n, we observe that the most likely value τ^* hits the lower bound 0.1. This is not surprising, given the bang-bang strategies $\sigma_1 = 10$ and $\sigma_0 = \sigma_2 = 0$. As the truncated Poisson density of level 1, $g_2(1) = \frac{\tau}{1+\tau+\tau^2/2}$, is minimized at $\tau = 0$, the model fits best to the data observed in Figure 2, at the lowest value of τ.

[34]Note that the best reply is always well-defined, since the probability of being pivotal is always non-zero, given the probabilistic voting action of each player specified by the logistic error function (5).

5.5.2 CH Model

The CH model stipulates that a CHk player (a level-k player in the CH model) maximizes her expected payoff holding a belief that other $n-1$ players have levels up to $k-1$. In particular, a $CH1$ player holds the belief that all other players have level 0, which is exactly the same as the belief of an $L1$ player. In our game, the $CH1$ strategy is $\sigma_1 = 10$ for all $n = 5$, 9 and 19.

In Table 5, we observe that the values of τ^* are high for all n. This means that the best-fitting Poisson distribution assigns the highest density to level 2. Since the CH model attributes the corner values $\sigma_0 = 0$ and $\sigma_1 = 10$ for levels 0 and 1, the model fits to the data by assigning minimal densities to these corner strategies. We do not observe either a particular trend in τ^*, or a convergence of the $CH2$ strategies to Nash equilibrium, as n increases.[35] A key observation is an increasing sensitivity of the $CH2$ strategy for large n. Not only is the best-reply function in our game decreasing, but the *slope* of the best-reply function becomes steeper as n increases. Thus, the sensitivity of the best reply to the belief over the other players' strategies also increases as n increases. As we show in Theorem 1, such an increasing sensitivity leads to a divergence of the CH strategy in asymptotically expanding games, as n increases.

5.5.3 ICH Model

We observe that the estimated strategies of $ICH1$ and $ICH2$ are both increasing in n.[36] Comparing these values with the Nash equilibrium, the increase of both $ICH1$ and $ICH2$ is in line with the increase of the Nash equilibrium strategy with respect to n (Table 2). The intuition is that the Nash strategy monotonically converges to the unbiased strategy (i.e. 5), since all individuals equally share the prior bias caused by the asymmetric payoffs, and such an individual share converges to zero as n increases. As we discuss in Section 3, $ICH1$ would converge to a value opposite to the prior bias with respect to the Nash equilibrium, and $ICH2$ would approach to the Nash equilibrium (Theorem 1). Our ICH estimations from the data are consistent with these theoretical predictions.

The estimated values of τ^* are 3.9, 5.0 and 6.7, for $n = 5$, 9 and 19, respectively. For these values of τ, it is interesting to compute the probability assigned to their own level by an inclusive player. For a level-2 player, it is $g_2(2) = \frac{\tau^2/2}{1+\tau+\tau^2/2}$, which is equal to 61%, 68%

[35] As a robustness check, the session-wise estimation shows that the difference of σ_2 is not statistically significant between $n = 5$ and $n = 9$. We observe that an increase from $n = 9$ to $n = 19$ is significant at the $p = 0.10$ level by the t-test, although such an increase is not observed in the histogram of all estimated strategies in Figure 2.

[36] This observation is robust with the session-wise estimations. The differences are statistically significant at the $p = 0.01$ level under the Wilcoxon test. The t-statistics for $ICH1$ are 13.9 and 6.94, and for $ICH2$ are 9.65 and 9.74, from $n = 5$ to $n = 9$ and from $n = 9$ to $n = 19$, respectively, implying that all differences are statistically significant at the $p = 0.01$ level.

and 74%, respectively. For a level-1 player, it is $g_1(1) = \frac{\tau}{1+\tau}$, which is equal to 80%, 83% and 87%, respectively. These values are consistent with the responses we observed in Figure 3(b).

Another observation is that the best-fitting τ values are increasing as the group size increases. Given that τ is the expectation of the level drawn from the Poisson distribution, a larger τ corresponds, ceteris paribus, to a higher value in the expected cognitive levels. Therefore, an increase in the estimated values of τ may be interpreted as evidence that the average cognitive level increases as the group size grows larger. This observation is not consistent with the findings of Guarnaschelli et al. (2000), in which evidence of lower expected levels with larger groups is reported, reflecting a larger cognitive load in large groups.

5.5.4 Comparison

Our maximum likelihood estimations show that the ICH model best fits the data for all $n = 5$, 9 and 19. The result is robust for the estimations done by each session separately: the log-likelihood is improved under the ICH model and the differences are significant under the Wilcoxon test ($p < 0.001$) against CH. From the observations above, we conclude that our laboratory experiments provide clear evidence that the inclusive cognitive hierarchy model (ICH) fits better to the data as compared with the standard level-k model (L) and the cognitive hierarchy model (CH).

Nash equilibrium does not seem to explain aggregate behavior. To see this note that the equilibrium cutoff strategy increases with group size, whereas as seen in Figure 2, average observed cutoffs decrease with group size. Our data also provides us with statistics that allow a comparison of the fits between the above models and Nash equilibrium plus noise (Table 6). The Bayesian information criterion suggests that the ICH model fits better than Nash equilibrium, followed by the L and CH models. Our observation that Nash equilibrium fits better than the L and CH models is consistent with the findings of Battaglini et al. (2010) in an experiment on the swing voter's curse.

$n =$	5	9	19
NE	-222.9	-246.2	-244.2
ICH	-190.7	-210.6	-210.7

Table 6: Comparison of maximum log-likelihoods obtained for ICH and NE (with logistic errors $\rho^* = 1.6$, 1.3 and 1.3, for $n = 5$, 9 and 19, respectively).

We now briefly discuss the QRE approach. We argue that, since the best response of a player is a cutoff strategy in our experimental game, the expected payoffs should be single-peaked around the best response, when the belief over other players' strategies is perturbed. Thus, the QRE strategy would also be a single-peaked distribution around the best reply, similar to our Nash with logistic error model. Hence, considering QRE would not add much more than our Nash estimations. Also, equilibrium-plus-noise models such as QRE often miss systematic patterns in participants' deviations from equilibrium such as the one observed in our experimental data. We are aware that there are ways of embedding heterogeneities that resonate with thinking hierarchies into QRE approach, as in the case of the "truncated quantal response equilibrium" (TQRE) due to Rogers et al. (2009). However, the relationship and comparison of ICH and TQRE (or other QRE specifications) are beyond the scope of this paper and would be a possible future work.

6 Conclusion

We have introduced a cognitive hierarchy model which allows inclusiveness in the belief over the cognitive levels of other players. We find that asymptotic properties of the group decision-making, especially asymptotic efficiency, exhibit a stark contrast depending on whether the inclusiveness condition is admitted in the model or not. Results from our laboratory experiment provide evidence for (i) systematic deviations from Nash equilibrium behavior, and (ii) a better fit to the data under the model with inclusiveness.

Our theoretical analysis implies that the asymptotic property of the slope of the best-reply function is the key ingredient to determining whether the asymptotic properties differ between the models with and without inclusiveness. Even with an increasing sensitivity of the best-reply function to the beliefs, inclusiveness prevents divergence of the strategies from fully rational behavior. Since the same property is shared by the best-reply function of the collective decision-making studied in this paper, the presence of the inclusiveness prevents strategies from diverging away from the symmetric Nash equilibrium, and hence provides asymptotic efficiency as the group size increases.

Most of the games studied with cognitive hierarchy models share the property that the best-reply function is asymptotically non-expansive (i.e. $A < \infty$). In such games, our analysis implies that the presence of inclusiveness has little relevance, at least asymptotically. Therefore, inclusiveness does not imperil the insights obtained in the existing cognitive hierarchy models, even though its explanatory power may vary as a function of model settings and parameters. A major example is the classical Keynesian beauty contest game. The data provides us with evidence that the model with inclusiveness fits the data toward the same direction as those without inclusiveness. Similar insights are inherited in the games with best-reply functions that are 'contractive' in a broader sense, such as dominance-solvable games, coordination games, and market entry games, among others. We hence do not insist on an intrinsic improvement of the predictive power of the cognitive hierarchy model with the presence of the inclusiveness condition in this class of games. The main message of this paper is that there are games in the other class in which the presence of inclusiveness matters. We think there are interesting games in this class, e.g. Cournot competitions, that are worth pursuing in further analysis.

Our interests go beyond the analytical results obtained in this paper. A crucial difference induced by the presence of inclusiveness is the existence of 'sophisticated' players. A highest-level player in the ICH model best replies holding the correct belief concerning the distribution of the levels of other players. This is not the case in the cognitive hierarchy models without inclusiveness. Players are supposed to be Savage rational, but full consistency of their beliefs is not postulated even for the highest level. In that sense, simply the existence of fully-sophisticated players may suffice to convey our message. However, our model consists of players who are naïve (level-0), best-replying but with inconsistent beliefs (level-1), and sophisticated (level-2). Rubinstein (2016) proposes a typology of players with two types, e.g. "instinctive" and "contemplative". The three types emerging from our approach based on cognitive hierarchy theory can be seen as a richer typology in the same vein that allows for an intermediary type. And our experiments show that these intermediary types can indeed be necessary, as a significant number of subjects play cutoff strategies that are higher than 5 (symmetric strategy), which can only be explained as a response to the belief that other players will be overly biased towards blue. The beauty of the cognitive hierarchy models lies, we believe, in the heterogeneous degrees of belief inconsistency that can be explicitly accommodated. We would like to further understand the role of heterogeneous degrees of inconsistent beliefs under the existence of fully-sophisticated players. We leave this for future research.

Acknowledgments

We are thankful to Colin Camerer, Nobuyuki Hanaki, Guillaume Hollard, Keigo Inukai, Nagore Iriberri, Jean-François Laslier, Uğur Özdemir, Thomas Palfrey, Eduardo Perez-Richet, Phil Reny, Ariel Rubinstein, seminar participants at Ecole Polytechnique, Paris School of Economics, Toulouse School of Economics, GATE at Saint-Etienne, University of Chicago, Caltech, University of Bayreuth, Hitotsubashi University, Keio University, Kyoto University, Osaka University, 5th Murat Sertel Workshop in Istanbul, SAET Conference 2014, Transatlantic Theory Workshop 2014, French Symposium on Games 2015, ASFEE 2015, GAM-SOC Workshop 2015, Social Choice and Welfare Conference 2016, Game Theory Congress 2016, and BEAM Conference for valuable comments. Financial support by Investissements d'Avenir, ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047, DynaMITE: Dynamic Matching and Interactions: Theory and Experiments, ANR-13-BSHS1-0010 and IDEX ANR-10-IDEX-0001-02 PSL* MIFID is gratefully acknowledged.

References

Akerlof, G. A. and J. L. Yellen (1985): "A near-rational model of the business cycle, with wage and price inertia," *The Quarterly Journal of Economics*, 100, 823–838.

Alaoui, L. and A. Penta (2016): "Endogenous depth of reasoning," *The Review of Economic Studies*, 83, 1297–1333.

Angeletos, G.-M. and A. Pavan (2007): "Efficient use of information and social value of information," *Econometrica*, 75, 1103–1142.

Austen-Smith, D. and J. S. Banks (1996): "Information aggregation, rationality, and the Condorcet jury theorem," *American Political Science Review*, 34–45.

Azmat, G., M. Bagues, A. Cabrales, and N. Iriberri (2019): "What You Don't Know... Can't Hurt You? A Natural Field Experiment on Relative Performance Feedback in Higher Education," *Management Science*.

Battaglini, M., R. B. Morton, and T. R. Palfrey (2008): "Information aggregation and strategic abstention in large laboratory elections," *The American Economic Review*, 194–200.

——— (2010): "The swing voter's curse in the laboratory," *The Review of Economic Studies*, 77, 61–89.

Benoît, J.-P. and J. Dubra (2011): "Apparent overconfidence," *Econometrica*, 79, 1591–1625.

Berger, U., H. De Silva, and G. Fellner-Röhling (2016): "Cognitive hierarchies in the minimizer game," *Journal of Economic Behavior & Organization*, 130, 337–348.

Bosch-Domènech, A., J. G. Montalvo, R. Nagel, and A. Sattora (2002): "One, Two, (Three), Infinity, ...: Newspaper and Lab Beauty-Contest Experiments," *The American Economic Review*, 92(5), 1687–1701.

Breitmoser, Y. (2012): "Strategic reasoning in p-beauty contests," *Games and Economic Behavior*, 75, 555–569.

Brocas, I., J. D. Carrillo, S. W. Wang, and C. F. Camerer (2014): "Imperfect choice or imperfect attention? Understanding strategic thinking in private information games," *The Review of Economic Studies*.

Camerer, C. (2003): *Behavioral game theory: Experiments in strategic interaction*, Princeton University Press.

Camerer, C., T.-H. Ho, and J.-K. Chong (2004): "A cognitive hierarchy model of games," *The Quarterly Journal of Economics*, 861–898.

Camerer, C. and D. Lovallo (1999): "Overconfidence and Excess Entry: An Experimental Approach," *The American Economic Review*, 306–318.

Carvalho, D. and L. Santos-Pinto (2014): "A cognitive hierarchy model of behavior in the action commitment game," *International Journal of Game Theory*, 43, 551–577.

Colman, A. M., B. D. Pulford, and C. L. Lawrence (2014): "Explaining strategic coordination: Cognitive hierarchy theory, strong Stackelberg reasoning, and team reasoning." *Decision*, 1, 35.

Condorcet, M. J. A. N. C. M. d. (1785): *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Chelsea.

Conlisk, J. (1980): "Costly optimizers versus cheap imitators," *Journal of Economic Behavior & Organization*, 1, 275–293.

Cornes, R. and R. Hartley (2012): "Fully aggregative games," *Economics Letters*, 116, 631–633.

Costa-Gomes, M. A., V. P. Crawford, and N. Iriberri (2009): "Comparing models of strategic thinking in Van Huyck, Battalio, and Beil's coordination games," *Journal of the European Economic Association*, 7, 365–376.

Crawford, V. P., M. A. Costa-Gomes, and N. Iriberri (2013): "Structural models of nonequilibrium strategic thinking: Theory, evidence, and applications," *Journal of Economic Literature*, 51, 5–62.

Crawford, V. P., U. Gneezy, and Y. Rottenstreich (2008): "The power of focal points is limited: Even minute payoff asymmetry may yield large coordination failures," *American Economic Review*, 98, 1443–58.

Crawford, V. P. and N. Iriberri (2007a): "Fatal attraction: Salience, naivete, and sophistication in experimental Hide-and-Seek games," *The American Economic Review*, 1731–1750.

——— (2007b): "Level-k auctions: Can a nonequilibrium model of strategic thinking explain the winner's curse and overbidding in private-value auctions?" *Econometrica*, 75, 1721–1770.

Currarini, S. and F. Feri (2015): "Information sharing networks in linear quadratic games," *International Journal of Game Theory*, 44, 701–732.

Fehr, D. and S. Huck (2016): "Who knows it is a game? On strategic awareness and cognitive ability," *Experimental Economics*, 19, 713–726.

Fischbacher, U. (2007): "z-Tree: Zurich toolbox for ready-made economic experiments," *Experimental Economics*, 10, 171–178.

Georganas, S., P. J. Healy, and R. A. Weber (2015): "On the persistence of strategic sophistication," *Journal of Economic Theory*, 159, 369–400.

Guarnaschelli, S., R. D. McKelvey, and T. R. Palfrey (2000): "An experimental study of jury decision rules," *American Political Science Review*, 407–423.

Haltiwanger, J. and M. Waldman (1985): "Rational expectations and the limits of rationality: An analysis of heterogeneity," *American Economic Review*, 75, 326–340.

——— (1991): "Responders versus non-responders: a new perspective on heterogeneity," *The Economic Journal*, 101, 1085–1102.

Isaac, R. M., J. M. Walker, and A. W. Williams (1994): "Group size and the voluntary provision of public goods: experimental evidence utilizing large groups," *Journal of Public Economics*, 54, 1–36.

Jimenez-Gomez, D. (2019): "False Consensus in Games: Embedding Level-k Models into Games of Incomplete Information," *Available at SSRN 3216040*.

Kreps, D. M., P. Milgrom, J. Roberts, and R. Wilson (1982): "Rational cooperation in the finitely repeated prisoners' dilemma," *Journal of Economic theory*, 27, 245–252.

Levin, D. and L. Zhang (2019): "Bridging level-k to nash equilibrium," *Available at SSRN 2934696*.

McKelvey, R. D. and T. R. Palfrey (1995): "Quantal response equilibria for normal form games," *Games and economic behavior*, 10, 6–38.

Nagel, R. (1995): "Unraveling in guessing games: An experimental study," *The American Economic Review*, 1313–1326.

Östling, R., J. T.-y. Wang, E. Y. Chou, and C. F. Camerer (2011): "Testing game theory in the field: Swedish LUPI lottery games," *American Economic Journal: Microeconomics*, 3, 1–33.

Palfrey, T. R. (2016): "Experiments in political economy," *The handbook of experimental economics*, 2, 347–434.

Rogers, B. W., T. R. Palfrey, and C. F. Camerer (2009): "Heterogeneous quantal response equilibrium and cognitive hierarchies," *Journal of Economic Theory*, 144, 1440–1467.

Ross, L., D. Greene, and P. House (1977): "The false consensus effect: An egocentric bias in social perception and attribution processes," *Journal of experimental social psychology*, 13, 279–301.

Rubinstein, A. (2016): "A typology of players: Between instinctive and contemplative," *The Quarterly Journal of Economics*, 131, 859–890.

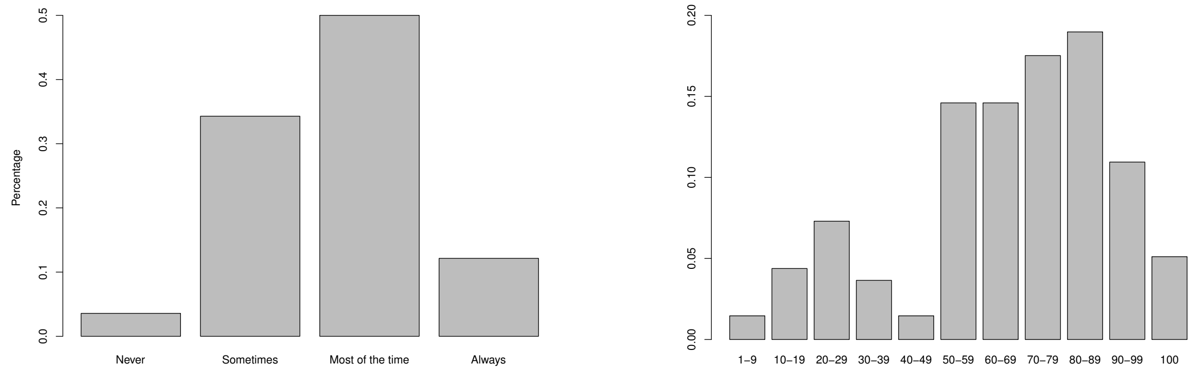
Russell, T. and R. Thaler (1985): "The relevance of quasi rationality in competitive markets," *The American Economic Review*, 75, 1071–1082.

Stahl, D. O. and P. W. Wilson (1995): "On Players Models of Other Players: Theory and Experimental Evidence," *Games and Economic Behavior*, 10, 218–254.

Strzalecki, T. (2014): "Depth of reasoning and higher order beliefs," *Journal of Economic Behavior & Organization*, 108, 108–122.

Woolley, A. W., C. F. Chabris, A. Pentland, N. Hashmi, and T. W. Malone (2010): "Evidence for a collective intelligence factor in the performance of human groups," *Science*, 330, 686–688.

A Questionnaire



(a) "When you made decisions, did you think that the other participants in your group used exactly the same reasoning as you did?"

(b) "What is the percentage of the other participants using the same reasoning, according to your estimation?"

Figure 3: Responses in the post-experimental questionnaire.

B Proofs

B.1 Proof of Proposition 1

We first prove the following lemma:

Lemma **1** *Let $a_i, b_i, p_i \in \mathbb{R}$ for $i = 1, \cdots n$. Suppose that $b_i < 0$, $p_i > 0$, $a_i/b_i < 1$ for all i and $\sum_i p_i = 1$. Then, $\left(\sum_i p_i a_i\right) / \left(\sum_i p_i b_i\right) < 1$.*

Proof: $\forall i$, $a_i > b_i$. Since $p_i > 0$, $\sum_i p_i a_i > \sum_i p_i b_i$. Since the RHS is negative, we have the result. ■

The following is the proof of Proposition 1.

Proof: Fix any $k \geq 1$ and suppose that the statement is true up to $k-1$. By definition, σ_k solves

$$\max_{x_i} \mathbb{E}_{x_{-i}} \left[u \left(x_i, x_{-i} \right) \right]. \tag{6}$$

By Assumption 3 and by linearity of expectation, the second-order partial derivative $\frac{\partial^2}{\partial x_i^2} \left\{ \mathbb{E}_{x_{-i}} \left[u \left(x_i, x_{-i} \right) \right] \right\}$ is strictly negative $\forall x_i$. Hence, the first-order partial derivative is strictly decreasing in x_i. Again by Assumption 3, the first-order derivative should satisfy the following inequalities at the boundary:

$$\frac{\partial}{\partial x_i} \left\{ \mathbb{E}_{x_{-i}} \left[u \left(\underline{x}, x_{-i} \right) \right] \right\} > 0 \text{ and } \frac{\partial}{\partial x_i} \left\{ \mathbb{E}_{x_{-i}} \left[u \left(\overline{x}, x_{-i} \right) \right] \right\} < 0.$$

Therefore, the solution of (6) is uniquely determined by the first-order condition:

$$\frac{\partial}{\partial x_i}\left\{\mathbb{E}_{x_{-i}}\left[u\left(x_i, x_{-i}\right)\right]\right\} = 0. \tag{7}$$

In particular, the solution is unique when x_{-i} is deterministic. Now, let $\varphi\left(x_{-i}\right)$ be the unique solution of $\frac{\partial u}{\partial x_i}\left(x_i, x_{-i}\right) = 0$ for a fixed x_{-i}. Pick one $j \neq i$ and consider $\frac{\partial \varphi}{\partial x_j}$. As φ is defined by an implicit function, its derivate should satisfy:

$$\frac{\partial \varphi}{\partial x_j} = -\frac{\frac{\partial^2 u}{\partial x_i \partial x_j}}{\frac{\partial^2 u}{\partial x_i^2}}.$$

By Assumptions 3 and 4, we have $\frac{\partial \varphi}{\partial x_j} < \frac{1}{n-1}\ \forall x_{-i}$. In other words, if we fix one $j\ (\neq i)$ and increase x_j by a small amount δ, the solution $\varphi\left(x_{-i}\right)$ "increases at most $\delta / \left(n-1\right)$."[37]

Now, suppose that among $n-1$ players other than i, $m\,(\in\{0,\cdots,n-1\})$ of them use the strategy $\hat{x} \in \mathbb{R}$, while the rest of them use a strategy profile $\tilde{x} \in \mathbb{R}^{n-1-m}$. That is, the strategy profile including player i is $\left(x_i, \underbrace{\hat{x},\cdots,\hat{x}}_{\in\mathbb{R}^m}, \underbrace{\tilde{x}}_{\in\mathbb{R}^{n-1-m}}\right)$. For such a profile,

$$-\frac{\frac{\partial^2 u}{\partial x_i \partial \hat{x}}}{\frac{\partial^2 u}{\partial x_i^2}} = -\frac{m\frac{\partial^2 u}{\partial x_i \partial x_j}}{\frac{\partial^2 u}{\partial x_i^2}} < \frac{m}{n-1} < 1. \tag{8}$$

In other words, if any number of players other than i increase simultaneously their strategies by the same, small amount δ, the solution for x_i increases at most δ.

Remember that the expectation over each $x_j\ (j \neq i)$ in (6) is taken independently across j according to i's belief, which assigns probabilities that sum up to $1 - g_k\left(k\right)$ to a combination of $\left(\sigma_h\right)_{k=0}^{k-1}$, and probability $g_k\left(k\right)$ to σ_k. We now consider the change in the solution of (6) caused by the change in σ_k, fixing $g_k\left(k\right)$ and $\left(\sigma_h\right)_{k=0}^{k-1}$. Let $\psi\left(\sigma_k\right)$ be the solution, which is well-defined since (7) has a unique solution. By using ψ, the ICH startegy σ_k can be described as the solution of the fixed-point problem $\sigma_k = \psi\left(\sigma_k\right)$.

Among $n-1$ players other than i, let m be the number of players for whom the realization of the random variable x_j coincides with σ_k (which happens with probability $\binom{n-1}{m} g_k\left(k\right)^m \left(1 - g_k\left(k\right)\right)^{n-1-m}$). Then by (8), for each of such a realization, we have:

$$-\frac{\frac{\partial^2 u}{\partial x_i \partial \sigma_k}}{\frac{\partial^2 u}{\partial x_i^2}} < 1.$$

Now, since $\psi\left(\sigma_k\right)$ is the unique solution of (7), we have:

$$\frac{d\psi}{d\sigma_k} = -\frac{\mathbb{E}_{x_{-i}}\left[\frac{\partial^2}{\partial x_i \partial \sigma_k} u\left(x_i, x_{-i}\right)\right]}{\mathbb{E}_{x_{-i}}\left[\frac{\partial^2}{\partial x_i^2} u\left(x_i, x_{-i}\right)\right]}.$$

[37] The partial derivative can take any negative value, while it is less than $1/(n-1)$ for positive values. Hence, by this expression, we mean that φ may decrease.

By Lemma 1, we have $\frac{d\psi}{d\sigma_k} < 1$. Therefore, the fixed-point problem $\sigma_k = \psi(\sigma_k)$ has a unique solution.

Finally, by induction, σ_k is uniquely determined for all k. ■

B.2 Proof of Theorem 1

Proof: Let

$$\alpha_n := -\frac{\gamma_{12}}{\gamma_{11}}(n-1).$$

By (3), α_n is the slope of the best-reply function with respect to the *average* of the other players' strategies. Using α_n, we can explicitly write the level-k strategy under each of the three models, L, CH, and ICH. Note that these models differ only in the belief held by each player, specified in equations (L), (CH) and (ICH) in Section 2.[38] By definition, $\lim_{n\to\infty}|\alpha_n| = A$.

In the L model, the strategy of the level-$(k+1)$ player is defined as the best reply to the level-k player. By (3) and (L),

$$\sigma_{k+1}^L(n) = \alpha_n \sigma_k^L(n) \quad \text{for } k \geq 0.$$

Hence,

$$\sigma_k^L(n) = (\alpha_n)^k \mu \quad \text{for } k \geq 1.$$

Therefore, for any $\mu \neq 0$ and any $k \geq 1$, we have $\lim_{n\to\infty}\left|\sigma_k^L(n)\right| = \infty$ if $A = \infty$, and bounded if $A < \infty$.

In the CH model, by (3) and (CH),

$$\sigma_k^{CH}(n) = \alpha_n \left(\sum_{h=0}^{k-1} g_k^{CH}(h)\, \sigma_h^{CH}(n)\right). \tag{9}$$

Especially, $\sigma_1^{CH}(n) = \alpha_n \mu$. For the sake of induction, assume that $\sigma_h^{CH}(n)$ is a polynomial of degree h with respect to α_n for $h \leq k-1$. Then, by (9), $\sigma_k^{CH}(n)$ is a polynomial of degree k with respect to α_n. Therefore, we have:

$$\sigma_k^{CH}(n) = \varphi_k(\alpha_n)\, \mu$$

where $\varphi_k(\cdot)$ is a polynomial of degree k. Therefore, for any $\mu \neq 0$ and any $k \geq 1$, we have $\lim_{n\to\infty}\left|\sigma_k^{CH}(n)\right| = \infty$ if $A = \infty$, and bounded if $A < \infty$.

[38] In the proof, we write the (possibly mixed) strategy of a level-0 player as $\sigma_0 = \mu$, identifying it with its expected value, since expectation is the only relevant term which determines the best reply in the linear quadratic games.

In the ICH model, by (3) and (ICH),

$$\sigma_k^{ICH}(n) = \alpha_n \left(\sum_{h=0}^{k} g_k^{ICH}(h)\, \sigma_h^{ICH}(n) \right).$$

Hence,

$$\sigma_k^{ICH}(n) = \frac{\alpha_n \sum_{h=0}^{k-1} g_k^{ICH}(h)\, \sigma_h^{ICH}(n)}{1 - \alpha_n g_k^{ICH}(k)}. \tag{10}$$

Now, suppose $A = \infty$. For $k = 1$,

$$\sigma_1^{ICH}(n) = \frac{\alpha_n g_1^{ICH}(0)\, \sigma_0}{1 - \alpha_n g_1^{ICH}(1)}.$$

As $\lim_{n\to\infty} |\alpha_n| = \infty$, we have $\lim_{n\to\infty} \sigma_1^{ICH} = -\frac{g_1^{ICH}(0)}{g_1^{ICH}(1)}\mu = -\frac{f_0}{f_1}\mu$.[39]

For $k = 2$, by (10),

$$\sigma_2^{ICH}(n) = \frac{\alpha_n \left(g_2^{ICH}(0)\, \sigma_0 + g_2^{ICH}(1)\, \sigma_1^{ICH}(n) \right)}{1 - \alpha_n g_2^{ICH}(2)}.$$

As $\lim_{n\to\infty} |\alpha_n| = \infty$, we have:

$$\lim_{n\to\infty} \sigma_2^{ICH}(n) \quad = \quad -\frac{g_2^{ICH}(0)\, \mu + g_2^{ICH}(1) \left(-\frac{f_0}{f_1}\mu \right)}{g_2^{ICH}(2)}.$$

Since $\frac{g_2^{ICH}(0)}{g_2^{ICH}(1)} = \frac{f_0}{f_1}$, we have $\lim_{n\to\infty} \sigma_2^{ICH}(n) = 0$. For $k > 2$,

$$\begin{aligned} & \lim_{n\to\infty} \sigma_k^{ICH}(n) \\ = \; & \lim_{n\to\infty} \left(\frac{\sum_{h=0}^{k-1} g_k^{ICH}(h)\, \sigma_h^{ICH}(n)}{\frac{1}{\alpha_n} - g_k^{ICH}(k)} \right) \\ = \; & -\frac{1}{g_k^{ICH}(k)} \left(g_k^{ICH}(0)\, \mu + g_k^{ICH}(1) \left(-\frac{f_0}{f_1}\mu \right) + \sum_{h=2}^{k-1} g_k^{ICH}(h) \lim_{n\to\infty} \sigma_h^{ICH}(n) \right). \end{aligned}$$

The first two terms in the bracket cancel out, since $\frac{g_k^{ICH}(0)}{g_k^{ICH}(1)} = \frac{f_0}{f_1}$. For the sake of induction, assume $\lim_{n\to\infty} \sigma_h^{ICH}(n) = 0$ for $2 \le h \le k-1$. Then, $\lim_{n\to\infty} \sigma_k^{ICH}(n) = 0$. ∎

[39] Remember that g_k is the truncated distribution induced by f, the underlying distribution over levels defined in Definition 1.

B.3 Proof of Theorem 2

Proof: Suppose $A < \infty$. Then by (10), for $k \geq 1$,

$$\lim_{n\to\infty} \sigma_k^{ICH}(n) = \frac{A\sum_{h=0}^{k-1} g_k^{ICH}(h)\,\sigma_h^{ICH}(n)}{1 - Ag_k^{ICH}(k)}.$$

Especially, for $k = 1$,

$$\lim_{n\to\infty} \sigma_1^{ICH}(n) = \frac{Ag_1^{ICH}(0)\,\mu}{1 - Ag_1^{ICH}(1)} < \infty.$$

For the sake of induction, assume $\lim_{n\to\infty} \left|\sigma_h^{ICH}(n)\right| =: s_h < \infty$ for $1 \leq h \leq k-1$. Then, for $k \geq 2$,

$$\lim_{n\to\infty} \left|\sigma_k^{ICH}(n)\right| \leq \left|\frac{A\sum_{h=0}^{k-1} g_k^{ICH}(h)\, s_h}{1 - Ag_k^{ICH}(k)}\right| < \infty.$$

■

C Tables

$\sigma_0 =$	0	5	10	$\mathcal{U}[0,10]$
L	−416.3	−252.9	−438.6	−342.4
CH	−334.4	−253.9	−288.9	−286.1
ICH	−210.6	−250.3	−250.6	−217.4

Table 7: Comparison of maximum log-likelihoods obtained for different level-0 specifications with $n = 9$.

$\sigma_0 =$	0	5	10	$\mathcal{U}[0,10]$
L	−414.8	−242.3	−438.8	−342.0
CH	−338.0	−247.8	−250.5	−306.2
ICH	−210.7	−248.1	−247.1	−195.1

Table 8: Comparison of maximum log-likelihoods obtained for different level-0 specifications with $n = 19$.