Operational interpretation of weight-based resource quantifiers in convex quantum resource theories

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We introduce the resource quantifier of weight of resource for convex quantum resource theories of states and measurements with arbitrary resources. We show that it captures the advantage that a resourceful state (measurement) offers over all possible free states (measurements), in the operational task of exclusion of subchannels (states). Furthermore, we introduce information-theoretic quantities related to exclusion for quantum channels, and find a connection between the weight of resource of a measurement, and the exclusion-type information of quantum-to-classical channels. The results found in this article apply to the resource theory of entanglement, in which the weight of resource is known as the best-separable approximation or Lewenstein-Sanpera decomposition, introduced in 1998. Consequently, the results found here provide an operational interpretation to this 21 year-old entanglement quantifier.

The 21st-century is currently witnessing a second quantum revolution which, broadly speaking, aims at harnessing different quantum phenomena for the development of quantum technologies. Quantum phenomena can then be seen as resources for fuelling quantum information protocols. In this regard, the framework of *Quantum Resource Theories (QRTs)* has been put forward in order to address these phenomena within a common unifying framework, see the recent review [1]. There are several QRTs of different quantum '*objects*' addressing different properties (of the object) as a *resource*. We can then broadly classify QRTs by first specifying the objects of the theory, followed by the property to be harnessed as a resource. In this broad classification there are QRTs addressing quantum objects like: states [2, 3], addressing resources such as entanglement [2, 3], coherence [4, 5], asymmetry [4], and athermality [6], among many others [7–10]; measurement, addressing resources such as as projective simulability [11, 12] and informativeness [13, 14] among others [15]; correlations [16–19], steering assemblages [20], and channels [21–23].

One of the main goals within the framework of QRTs is to define *resource quantifiers* for abstract QRTs, so that resources of different objects can be quantified and compared in a fair manner. There are different measures for quantifying resources, depending on the type of QRT being considered [1]. In particular, when considering *convex* QRTs, well-studied geometric quantifiers include the so-called *robustness-based* [24–31] and *weight-based* [32–37] quantifiers, both of which can be defined for general convex QRTs. This has allowed for the cross-fertilisation across QRTs, in which results and insights from a particular QRT with an specific resource are being extended to additional resources and families of QRTs [1, 38, 39].

In addition to quantifying the amount of resource present in a quantum object, it is also of interest to develop practical applications in the form of *operational tasks* that explicitly take advantage of specific given resources, as well as to identify adequate resources and quantifiers characterising already existing operational tasks. In this regard, a general correspondence between robustness-based measures and *discrimination-based* operational tasks has recently been established: steering for subchannel discrimination [26], incompatibility for ensemble discrimination [40–42], coherence for unitary discrimination [27] and informativeness for state discrimination [13]. This correspondence initially considered for specific QRTs and resources, has been extended to QRT of states, measurements and channels with *arbitrary* resources [38, 39]. Furthermore, it turns out that when considering QRTs of measurements there exists an additional correspondence to single-shot information-theoretic quantities [13]. This three-way correspondence, initially considered for the resource of informativeness [13], has been extended to convex QRTs of measurements with *arbitrary* resources [39].

It is then natural to ask whether operational tasks exist in which, *weight-based* quantifiers play the relevant role. Surprisingly, in this work we prove that if one studies situations where excluding possibilities is the relevant goal – which we refer to as exclusion-based tasks – then this is precisely related to weight-based quantifiers. Furthermore, we prove that weight-based quantifiers for the QRTs of measurements also happen to satisfy a stronger three-way correspondence, establishing a link to single-shot information-theoretic quantities associated to information exclusion.

This parallel three-way correspondence establishes that, in addition to robustness-based quantifiers, weight-based quantifiers also play a relevant role in the characterisation of operational tasks. This raises questions regarding whether there is a much more general connection to be uncovered, linking resource quantifiers and operational tasks by focusing on different forms of information.

Convex quantum resource theories and resource *quantifiers.*—A general *resource theory* consists of: a set of objects O, the identification of a property of these objects to be considered as a resource, and a consequent bipartition of the set of objects into *resourceful* and *free* objects. If the set of free objects forms a convex set, we say that we have a convex resource theory. In this work we focus on the convex QRTs of states and measurements with arbitrary resources.

Definition 1: (Convex QRTs of states and measurements) Consider the set of quantum states in a finite dimensional Hilbert space of dimension d. Consider a property of these states defining a closed convex set which we will call the set of free states and denote as F. We say a state $\rho \in \mathrm{F}$ is a *free state*, and $\rho \notin \mathrm{F}$ is a *resourceful* state. Consider also the set of Positive-Operator Valued Measures (POVMs) acting on a finite dimensional Hilbert space. A POVM $\mathbb{M}$ is a collection of POVM elements $\mathbb{M} = \{M_a\}$ with $a \in \{1, ..., o\}$ satisfying $M_a \geq 0 \; \forall a$ and $\sum_a M_a = \mathbb{1}$. Similarly, we consider a property of measurements defining a closed convex set of free measurements and denote it as $\mathbb{F}$. We say a POVM $\mathbb{M} \in \mathbb{F}$ is a *free* measurement and it is *resourceful* otherwise.

It will be useful to introduce the notion of simulability of measurements.

Definition 2: (Simulability of measurements [43]) We say that a measurement $\mathbb{N} = \{N_x\}$, $x \in \{1, ..., k\}$ is simulable by the measurement $\mathbb{M} = \{M_a\}$, $a \in \{1, ..., o\}$ when there exists a conditional probability distribution $\{q(x|a)\}$ such that $N_x = \sum_a q(x|a) M_a$. One can check that the simulability of measurements defines a partial order for the set of measurements and therefore we use the notation $\mathbb{N} \preceq \mathbb{M}$, meaning that $\mathbb{N}$ is simulable by $\mathbb{M}$. Simulability as defined here can be understood as a post-processing of the measurement.

We now define a weight-based quantifier for arbitrary resources of states and measurements. The idea is to geometrically quantify the amount of resource contained in an object. This quantifier was originally introduced in [32] in the context of nonlocality and it was later independently rediscovered in [33] in the context of entanglement. This quantifier has several different names such as: part, content, cost and weight. In order to keep consistency with recent notation in the literature, we adopt *weight* in this work.

Definition 3: (Weight of resource for states and measurements) The weight of resource of a state and a measurement are given by:

$$\mathrm{W}_{\mathrm{F}}(\rho) = \min_{\substack{w \geq 0 \\ \sigma \in \mathrm{F} \\ \rho^G}} \left\{ w \,\middle|\, \rho = w\rho^G + (1-w)\sigma \right\}, \tag{1}$$

$$\mathrm{W}_{\mathbb{F}}(\mathbb{M}) = \min_{\substack{w \geq 0 \\ \mathbb{N} \in \mathbb{F} \\ \mathbb{M}^G}} \left\{ w \,\middle|\, M_a = wM_a^G + (1-w)N_a \right\}. \tag{2}$$

The weight quantifies the minimal amount with which some resourceful state ρ^G (measurement $\mathbb{M}^G$) needs to be used in order to reproduce the state ρ (measurement $\mathbb{M}$). Evaluating the weight of resource is a convex optimisation problem [44] and hence it can in general be solved efficiently numerically.

Exclusion-based operational tasks.—We consider a game first formalised in [45] for analysing the Pusey-Barrett-Rudolph (PBR) theorem [46]. The property considered by PBR has been addressed under different names like antidistinguishability [47] or not-Post-Peierls compatibility (Post-Peierls incompatibility) [48, 49]. It has been studied in a number of contexts, e.g. under noisy channels [47], and its communication complexity properties [50, 51].

Game 1: (State exclusion [45]) A referee has a collection of states $\{\rho_x\}$, $x \in \{1, ..., k\}$, and promises to send a player the state ρ_x with probability $p(x)$. The goal is for the player to output a guess $g \in \{1, ..., k\}$ of a state that was *not* sent. That is, the player succeeds at the game if $g \neq x$ and fails when $g = x$. This game can be seen as being opposite of state discrimination, in which the goal is to correctly identify the state that was sent. Since the goal is to guess the state that was *not* sent, this game is referred to as excluding, rather than discriminating.

A given state exclusion game is fully specified by an ensemble $\mathcal{E} = \{\rho_x, p(x)\}$. When the player uses the measurement $\mathbb{M}$ to play the game, we take as our figure of merit the average error probability, i.e. the total probability of incorrectly guessing $g = x$,

$$\mathbb{P}_{\mathrm{err}}^{\mathrm{Q}}(\mathcal{E}, \mathbb{M}) = \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x p(x) \mathrm{Tr}[N_x \rho_x], \tag{3}$$

Note that given only the measurement $\mathbb{M}$, the player can still simulate any measurement $\mathbb{N} \preceq \mathbb{M}$, and we assume the use the best such simulable measurement in order to minimise their error probability.

We will be interested in comparing a fixed resourceful measurement $\mathbb{M}$ to the best free measurement. We can thus define the "classical" error probability as the error probability of this best free measurement, i.e.

$$\mathbb{P}_{\mathrm{err}}^{\mathrm{C}}(\mathcal{E}) = \min_{\mathbb{N} \in \mathbb{F}} \sum_x p(x) \mathrm{Tr}[N_x \rho_x], \tag{4}$$

Note that we don't need the minimisation over simulations here, since this is automatically encompassed in the minimisation over all free measurements.

We will also consider a closely related game, that of subchannel exclusion.

Game 2: (Subchannel exclusion) The player sends a quantum state ρ to the referee who has a collection of subchannels $\Psi = \{\Psi_x\}$, $x \in \{1, ..., k\}$. The subchannels Ψ_x are completely-positive (CP) trace-nonincreasing maps such that $\Lambda = \sum_x \Psi_x$ forms a completely-positive trace-preserving (CPTP) map. The referee promises to apply one of these subchannels on the state ρ and the transformed state is then sent back to the player. The player then has access to the ensemble $\mathcal{E}_\Psi = \{\rho_x, p(x)\}$ with $p(x) = \mathrm{Tr}[\Psi_x(\rho)]$, $\rho_x = \Psi_x(\rho)/p(x)$. The goal is for the player to output a guess $g \in \{1, ..., k\}$ for a subchannel that did *not* take place.

Using a fixed state ρ and fixed measurement $\mathbb{M}$, the average error probability of the player is given by

$$P^{\rm Q}_{\rm err}(\Psi,\mathbb{M},\rho)=\sum_x {\rm Tr}\big[M_x\Psi_x(\rho)\big], \tag{5}$$

If we allow for the optimisation over all measurements, in order to understand how well the state ρ performs, we can also define $P^{\rm Q}_{\rm err}(\Psi,\rho)=\min_{\mathbb{M}}\sum_x {\rm Tr}\big[M_x\Psi_x(\rho)\big]$. We will again be interested in comparing how a fixed resourceful state ρ performs at subchannel exclusion compared to the best resourceless state. The error probability with a fixed measurement in this case is

$$P^{\rm C}_{\rm err}(\Psi,\mathbb{M})=\min_{\sigma\in F}\sum_x {\rm Tr}\big[M_x\Psi_x(\sigma)\big], \tag{6}$$

and, as above, if we optimise over all measurements we can define $P^{\rm C}_{\rm err}(\Psi)=\min_{\mathbb{M}}P^{\rm C}_{\rm err}(\Psi,\mathbb{M})$.

For both games, in general one expects the player should not have a larger error probability when using a resourceful object (state or measurement) compared to a resourceless one, and hence $\mathbb{P}^{\rm Q}_{\rm err}(\mathcal{E},\mathbb{M})/\mathbb{P}^{\rm C}_{\rm err}(\mathcal{E})\leq 1$ and $P^{\rm Q}_{\rm err}(\Psi,\rho)/P^{\rm C}_{\rm err}(\Psi)\leq 1$. We will be interested in the optimal *advantage* that can be obtained in each game when comparing resourceful and resourceless objects, e.g. in how small the ratio between quantum and classical error probabilities can be made. In the next section we will show that this is precisely characterised by the weight of informativeness.

All quantum resources are useful for an exclusion task.— Before proving our main result, we first show a preliminary result, which formalises the above intuition.

Result 1: For any resourceful state $\rho\notin{\rm F}$ (measurement $\mathbb{M}\notin\mathbb{F}$), there exists a subchannel exclusion game Ψ^ρ (state exclusion game $\mathcal{E}^{\mathbb{M}}$) for which playing with the state ρ (measurement $\mathbb{M}$) has small error probability when compared with any free state (measurement). These two statements are represented by the strict inequalities:

$$P^{\rm Q}_{\rm err}(\Psi^\rho,\rho)<P^{\rm C}_{\rm err}(\Psi^\rho), \tag{7}$$
$$\mathbb{P}^{\rm Q}_{\rm err}(\mathcal{E}^{\mathbb{M}},\mathbb{M})<\mathbb{P}^{\rm C}_{\rm err}(\mathcal{E}^{\mathbb{M}}). \tag{8}$$

The proof of these results can be found in the supplementary material, with both based around the hyperplane separation theorem. These results shows that every resourceful state (or measurement) is better that any possible free state (or measurement) when playing a tailored exclusion game. We now address how to *quantify* the performance of a resourceful object using exclusion games.

Weight of resource as the advantage in exclusion games.— We are now interested in quantifying the performance of a resourceful state (measurement) in comparison to all free states (measurements) when playing subchannel exclusion (state exclusion) games. Our main result is the following:

Result 2:

$$\min_{\Psi,\mathbb{M}}\frac{P_{\rm err}(\Psi,\mathbb{M},\rho)}{P^{\rm C}_{\rm err}(\Psi,\mathbb{M})}=1-{\rm W}_{\rm F}(\rho), \tag{9}$$

$$\min_{\mathcal{E}}\frac{\mathbb{P}^{\rm Q}_{\rm err}(\mathcal{E},\mathbb{M})}{\mathbb{P}^{\rm C}_{\rm err}(\mathcal{E})}=1-{\rm W}_{\mathbb{F}}(\mathbb{M}). \tag{10}$$

The proof, given in the Supplemental Material [52], is similar in each case, and consists of two parts. First we prove that the weight lower bounds the advantage for all tasks. We then prove that this lower bound can be achieved, by extracting an optimal game out of the relevant dual formulation of the weight.

This theorem shows two things: that for all exclusion games the weight bounds the decrease in error probability that can be obtained; and that there exists a game where this decrease is given precisely by the weight. This theorem establishes for the first time an operational interpretation of weight-based quantifiers, making a link to exclusion tasks, and thus establishing a connection between these two previously unrelated concepts. In particular, we remark that concerning states, our result holds for the weight of entanglement, a. k. a. the best separable approximation or Lewenstein-Sanpera decomposition [33], and for the weight of asymmetry [37].

Relaxing the measurement constraint.— In the subchannel game in (9), both the quantum and classical players are required to use the same measurement. The measurement is thus more like part of the game than part of the strategy of the player. It is therefore natural to allow the measurements to be chosen independently by the different players. Although in general this does not seem to lead to an operational interpretation, for a subset of resource theories, we can obtain the following result:

Result 3: *Consider a state ρ and the associated optimal dual variable $Y^\rho=\sum y_i\,|e_i\rangle\langle e_i|$ from the weight ${\rm W}_{\rm F}(\rho)$. If there exist a set of unitaries $\{U_x\}$ satisfying: (i) $\sum_x U_x\,|e_j\rangle\langle e_j|\,U_x^\dagger=\mathbb{1},\forall j$; (ii) $U_i\sigma U_i^\dagger=U_j\sigma U_j^\dagger\ \forall\sigma\in{\rm F}$, $\forall i,j$, then, the weight quantifies the advantage of the state ρ over all free states in quantum subchannel exclusion games with independent measurements:*

$$\min_{\Psi}\frac{P^{\rm Q}_{\rm err}(\Psi,\rho)}{P^{\rm c}_{\rm err}(\Psi)}=1-{\rm W}_{\rm F}(\rho). \tag{11}$$

Importantly, the resource theories of coherence and asymmetry satisfy the conditions of Result 3.

Weight of resource and single-shot information theory.— We now introduce an exclusion-based quantity closely related to the accessible information of a channel, and show that it too relates to the weight of resource of a measurement. We are interested in the ability of a channel Λ to be useful for sending *exclusion-type information*. This is a type of information where identifying is not the relevant task, but excluding, e.g. the information of the statement 'do not cut the blue wire' in a bomb-defusing situation. Two possibilities for conveying this information are either to communicate the wire to be avoided, or to communicate a wire that should be cut. If there is a noisy communication channel, it could be advantageous to use one type of encoding over the other.

Formally, we assume that the information to be excluded is represented by a random variable X, with probability distribution $p(x)$. This is encoded into a quantum ensemble as $\mathcal{E}=\{\rho_x,p(x)\}$. The quantum state is sent through a channel

Λ, and then an optimal decoding measurement $\mathbb{D} = \{D_g\}_g$ is performed, in order to make the best prediction for a value $x' \neq x$, which will always be $\arg\min_x p(x|g)$, i.e. the least likely value of x given the observed g, where $p(x,g) = p(x)\,\mathrm{Tr}[D_g \Lambda(\rho_x)]$. The error probability is $P_{\mathrm{err}}(X|G) = \min_{\mathbb{D}} \sum_g \min_x p(x,g)$ and the associated conditional entropy, which we call the 'exclusion conditional entropy' is

$$H_{-\infty}(X|G)_{\mathcal{E},\Lambda} = -\log P_{\mathrm{err}}(X|G), \tag{12}$$

which is the order minus-infinity conditional Rényi entropy, and where we have explicitly denoted the dependence on the quantum encoding $\mathcal{E}$ and the channel Λ.

We are now interested in comparing how different channels perform with the same quantum encoding. In particular, we are interested in how much larger the exclusion conditional entropy is for a given fixed channel Λ compared to a set of free channels $\mathcal{F}$ for sending the exclusion information stored in $\mathcal{E}$. Note that since the exclusion entropy is associated to an error probability, having a larger exclusion entropy signifies having a smaller average error probability. We thus define the gain in exclusion conditional entropy as

$$G^{\mathrm{exc}}_{-\infty}(\mathcal{E},\Lambda) = H_{-\infty}(X|G)_{\mathcal{E},\Lambda} - \max_{\Omega\in\mathcal{F}} H_{-\infty}(X|G)_{\mathcal{E},\Omega} \tag{13}$$

We think of this quantity as being a generalisation of the accessible information of a channel, in two ways: first we consider here exclusion-type information, instead of standard 'discrimination-type' information; second, we compare to a general set of free channels, rather than relative to a single free channel – the completely noisy channel. In the latter case, the second term would become simply $H_{-\infty}(X)_{\mathcal{E}} = -\log P_{\mathrm{err}}(X)$, the 'exclusion entropy' associated with the random variable X, and the definition would reduce to a mutual information-type quantity.

We now focus on quantum-to-classical channels which arise by the action of a measurement. In particular, to any measurement $\mathbb{M}$ we can define the associated channel $\Lambda_{\mathbb{M}}$ such that $\Lambda_{\mathbb{M}}(\rho) = \sum_a \mathrm{Tr}[M_a\rho]\,|a\rangle\langle a|$, where $\{|a\rangle\}$ forms an orthonormal basis, and records the measurement outcome. The conditional probability distribution that this channel leads to is $p(g|x) = \sum_a \mathrm{Tr}[M_a\rho_x]\,\langle a| D_g |a\rangle$.

We will then compare the fixed channel $\Lambda_{\mathbb{M}}$ associated with the measurement $\mathbb{M}$ with all of the channels $\Lambda_{\mathbb{N}}$ that can arise from a free measurement $\mathbb{N} \in \mathbb{F}$. We find the following result:

Result 4: The weight of resource of a measurement $\mathbb{M}$ quantifies the biggest gain in exclusion information of the associated measurement channel $\Lambda_{\mathbb{M}}$ relative to the set of free measurement channels $\mathcal{F} = \{\Lambda_{\mathbb{N}} | \mathbb{N} \in \mathbb{F}\}$

$$\max_{\mathcal{E}} G^{\mathrm{exc}}_{-\infty}(\mathcal{E},\Lambda_{\mathbb{M}}) = -\log\left[1 - \mathrm{W}_{\mathbb{F}}(\mathbb{M})\right] \tag{14}$$

with the maximisation over all quantum encodings $\mathcal{E} = \{\rho_x, p(x)\}$ of X.

The proof of this result is in the Supplementary Material. This result, which mirrors the results found in [39], establishes, for the QRT of measurements with arbitrary resources, a new *three-way correspondence* between weight-based resource quantifiers, exclusion-based tasks, and single-shot information-theoretic quantities. This supports the conjecture that we make here, that whenever there is a robustness-discrimination correspondence, there is a weight-exclusion correspondence.

Complete set of monotones.—As a final result, we show that for QRTs of measurements, the error probability in all exclusion games forms a complete set of monotones for the partial order of measurement simulation:

Result 5: For any two measurements $\mathbb{M}$ and $\mathbb{N}$, $\mathbb{M}$ can simulate the measurement $\mathbb{N}$, $\mathbb{M} \succeq \mathbb{N}$ if and only if

$$\mathbb{P}^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E},\mathbb{M}) \le \mathbb{P}^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E},\mathbb{N}) \quad \forall\,\mathcal{E} = \{p(x),\rho_x\}. \tag{15}$$

That is, a measurement $\mathbb{M}$ can simulate a measurement $\mathbb{N}$ if and only if it is never worse in any state exclusion game $\mathcal{E}$. The proof of this result is in the Supplementary Material.

This result shows then that the error probabilities over all state exclusion games form a complete set of (decreasing) monotones for the partial order of measurement simulation. It is interesting to note that it was previously shown that the probability of succeeding in state discrimination also forms a complete set of (increasing) monotones for measurement simulation [13, 39]. Hence, we now have a second, independent, complete set of monotones.

Conclusions.— In this work we have uncovered an intimate connection between weight-based quantifiers and exclusion tasks, which intimately parallels the connection between robustness-based quantifiers and discrimination tasks. We have shown that this connection holds both for convex resource theories of states and of measurements. For the latter, we have also shown a connection to single-shot information theory, and shown that the error probability in all exclusion games constitute a complete set of monotones for measurement simulation.

What is remarkable about these results is just how closely they parallel the corresponding results for robustness and discrimination. Previously this was the only known connection of its type. Now, we have uncovered a second connection, relating a second type of a very well known geometrical quantifier – the weight – and an interesting task in information theory – exclusion. Moreover, given that these two quantifiers are related to two limiting Renyi entropies (the order ∞ and $-\infty$ respectively), through the guessing probability and the error probability, the results presented here raise a fascinating possibility that there is a full spectrum of connections between resource quantifiers and tasks, with only the two ends currently uncovered. We believe this is an exciting line of enquiry for future research, which could lead to far-reaching insight into quantum information theory and in particular the fruitful resource-theory approach to quantum physics.

Note added.— During the development of this work we became aware of a complementary work by R. Uola et. al. [53].

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