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UNIQUE PSEUDO-EXPECTATIONS FOR C^*-INCLUSIONS

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Dedicated to E. G. Effros on the occasion of his 80th birthday

Abstract. Given an inclusion $\mathcal{D} \subseteq \mathcal{C}$ of unital C^*-algebras (with common unit), a unital completely positive linear map Φ of $\mathcal{C}$ into the injective envelope $I(\mathcal{D})$ of $\mathcal{D}$ which extends the inclusion of $\mathcal{D}$ into $I(\mathcal{D})$ is a pseudo-expectation. Pseudo-expectations are generalizations of conditional expectations, but with the advantage that they always exist. The set $\operatorname{PsExp}(\mathcal{C}, \mathcal{D})$ of all pseudo-expectations is a convex set, and when $\mathcal{D}$ is Abelian, we prove a Krein–Milman type theorem showing that $\operatorname{PsExp}(\mathcal{C}, \mathcal{D})$ can be recovered from its set of extreme points. In general, $\operatorname{PsExp}(\mathcal{C}, \mathcal{D})$ is not a singleton. However, there are large and natural classes of inclusions (e.g., when $\mathcal{D}$ is a regular MASA in $\mathcal{C}$) such that there is a unique pseudo-expectation. Uniqueness of the pseudo-expectation typically implies interesting structural properties for the inclusion. For general inclusions of C^*-algebras with $\mathcal{D}$ Abelian, we give a characterization of the unique pseudo-expectation property in terms of order structure; and when $\mathcal{C}$ is Abelian, we are able to give a topological description of the unique pseudo-expectation property.

As applications, we show that if an inclusion $\mathcal{D} \subseteq \mathcal{C}$ has a unique pseudo-expectation Φ which is also faithful, then the C^*-envelope of any operator space $\mathcal{X}$ with $\mathcal{D} \subseteq \mathcal{X} \subseteq \mathcal{C}$ is the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{X}$; we also show that for many interesting classes of C^*-inclusions, having a faithful unique pseudo-expectation implies that $\mathcal{D}$ norms $\mathcal{C}$, although this is not true in general.

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1. Introduction

The goal of this paper is to investigate the unique pseudo-expectation property for C^*-inclusions. A *C^*-inclusion* is a pair $(\mathcal{C},\mathcal{D})$ of unital C^*-algebras with $\mathcal{D}\subseteq\mathcal{C}$ and which have the same unit. For any unital C^*-algebra $\mathcal{D}$, there exists an *injective envelope* $I(\mathcal{D})$ for $\mathcal{D}$ [13]. That is, $I(\mathcal{D})$ is an injective object in the category OSys_1 of operator systems and unital completely positive (ucp) maps, which contains $\mathcal{D}$, and which is minimal with respect to these two properties. In fact, $I(\mathcal{D})$ is a C^*-algebra and $\mathcal{D}\subseteq I(\mathcal{D})$ is a C^*-subalgebra. A *pseudo-expectation* is a ucp map $\Phi:\mathcal{C}\to I(\mathcal{D})$ which extends the identity map on $\mathcal{D}$.

Pseudo-expectations are natural generalizations of conditional expectations, and due to injectivity, have the distinct advantage that they are guaranteed to exist for any C^*-inclusion. Pseudo-expectations were introduced by Pitts in [28] and were used there as a replacement for conditional expectations in settings where no conditional expectation exists.

One significant difference between conditional expectations and pseudo-expectations arises when one attempts to iterate these maps. For a conditional expectation $E:\mathcal{C}\to\mathcal{D}$, we have that $E\circ E=E$ (i.e., a conditional expectation is an idempotent map). For a pseudo-expectation $\Phi:\mathcal{C}\to I(\mathcal{D})$, the composition $\Phi\circ\Phi$ is typically undefined, since $I(\mathcal{D})$ is usually not contained in $\mathcal{C}$. This technical difficulty of pseudo-expectations is far outweighed by the aforementioned benefit, that pseudo-expectations always exist for any C^*-inclusion.

We view the uniqueness and faithfulness properties of pseudo-expectations as giving a measure of the relative size of a subalgebra inside the containing algebra. To orient the reader with this philosophy, we begin by explaining how the unique pseudo-expectation property fits with the program of deciding when a C^*-subalgebra is large/substantial/rich in its containing C^*-algebra.

1.1. Large subalgebras. Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. There are many ways of expressing that $\mathcal{D}$ is "large" (or "substantial", or "rich") in $\mathcal{C}$. For example:

(ARC) The *relative commutant* $\mathcal{D}^c=\mathcal{D}'\cap\mathcal{C}$ is Abelian.

(Reg) $\mathcal{D}$ is *regular* in $\mathcal{C}$, meaning that $\overline{\operatorname{span}}(N(\mathcal{C},\mathcal{D}))=\mathcal{C}$, where

$$N(\mathcal{C},\mathcal{D})=\left\{x\in\mathcal{C}:x\mathcal{D}x^*\subseteq\mathcal{D},x^*\mathcal{D}x\subseteq\mathcal{D}\right\}$$

are the *normalizers* of $\mathcal{D}$ in $\mathcal{C}$.

(Ess) $\mathcal{D}$ is *essential* in $\mathcal{C}$, meaning that every nontrivial closed two-sided ideal of $\mathcal{C}$ intersects $\mathcal{D}$ nontrivially.

(UEP) $\mathcal{C}$ has the *unique extension property* relative to $\mathcal{D}$, meaning that every pure state on $\mathcal{D}$ extends uniquely to a pure state on $\mathcal{C}$.

(Norming) $\mathcal{D}$ *norms* $\mathcal{C}$, meaning that for all $X \in M_{d\times d}(\mathcal{C})$,

$$\|X\| = \sup\big\{\|RXC\| : R \in \mathrm{Ball}\big(M_{1\times d}(\mathcal{D})\big), C \in \mathrm{Ball}\big(M_{d\times 1}(\mathcal{D})\big)\big\}.$$

Some of these conditions are purely algebraic, others purely analytic, and yet others somewhere in between. Each of them has advantages and disadvantages, and their relative merits vary by context. Indeed, two desirable properties for any condition which "measures" the largeness of $\mathcal{D}$ in $\mathcal{C}$ are the following:

- *Hereditary from above*: If $\mathcal{D}$ is large in $\mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ is a C^*-algebra, then $\mathcal{D}$ is large in $\mathcal{C}_0$.
- *Hereditary from below*: If $\mathcal{D}$ is large in $\mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq \mathcal{C}$ is a C^*-algebra, then $\mathcal{D}_0$ is large in $\mathcal{C}$.

Table 1 shows which of these hereditary properties the various types of inclusions possess (an entry marked "?" indicates we do not know whether the property holds). Only conditions (ARC) and (Norming) are known to the authors to be both hereditary from above and below, see Table 1.

On the other hand, as the following example shows, condition (Ess) works the best for the particular class of Abelian inclusions, in spite of its general shortcomings.

Example 1.1. Suppose $(\mathcal{A},\mathcal{D}) = (C(Y), C(X))$ is an Abelian inclusion, with corresponding continuous surjection $j : Y \to X$. Then

- $(\mathcal{A},\mathcal{D})$ always satisfies (ARC).
- $(\mathcal{A},\mathcal{D})$ always satisfies (Reg).
- $(\mathcal{A},\mathcal{D})$ satisfies (Ess) $\iff$ the only closed set $K \subseteq Y$ such that $j(K) = X$ is Y itself.
- $(\mathcal{A},\mathcal{D})$ satisfies (UEP) $\iff$ $\mathcal{A} = \mathcal{D}$.
- $(\mathcal{A},\mathcal{D})$ always satisfies (Norming).

1.2. Unique expectations. If $\mathcal{D}$ is large in $\mathcal{C}$, then there should not be many ways to project $\mathcal{C}$ onto $\mathcal{D}$. The most natural way to project a C^*-algebra $\mathcal{C}$ onto a C^*-subalgebra $\mathcal{D}$ is via a *conditional expectation*. Recall that a conditional expectation for $(\mathcal{C},\mathcal{D})$ is a ucp map $E : \mathcal{C} \to \mathcal{D}$ such that $E|_{\mathcal{D}} = \mathrm{id}$.

Table 1. Hereditary properties of inclusions

Condition	Hereditary from above	Hereditary from below
ARC	yes	yes
Reg	?	no
Ess	no	yes
UEP	yes	?
Norming	yes	yes

A conditional expectation $E : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* if $E(x^*x) = 0$ implies $x = 0$ (i.e., if E is faithful as a ucp map). Any convex combination of conditional expectations for $(\mathcal{C}, \mathcal{D})$ is again a conditional expectation for $(\mathcal{C}, \mathcal{D})$. Thus a C^*-inclusion has either zero, one, or uncountably many conditional expectations, and all three possibilities can occur.

In light of the previous discussion, it is reasonable to propose the following property as yet another expression of the largeness of $\mathcal{D}$ in $\mathcal{C}$:

(!CE) There is at most one conditional expectation $E : \mathcal{C} \to \mathcal{D}$.

The utility of this property is seriously limited in two ways. First, for many naturally arising C^*-inclusions, there are no conditional expectations at all; and second, as the next two examples show, (!CE) fails to be hereditary from above or below.

Example 1.2. Consider the C^*-inclusions

$$C[0,1] \subseteq C\big([0,1] \times [0,1]\big) \subseteq B\big(L^2\big([0,1] \times [0,1]\big)\big),$$

where the first inclusion corresponds to the continuous surjection

$$j : [0,1] \times [0,1] \to [0,1] : (s,t) \mapsto s.$$

Then there are no conditional expectations for the inclusion $(B(L^2([0,1] \times [0,1])), C[0,1])$, since $C[0,1]$ is not injective (in the category OSys_1). But there are infinitely many conditional expectations for the inclusion $(C([0,1] \times [0,1]), C[0,1])$. Indeed,

$$E_t : C\big([0,1] \times [0,1]\big) \to C[0,1] : g \mapsto g(\cdot, t)$$

is a conditional expectation for each $t \in [0,1]$. Thus, (!CE) is not hereditary from above.

Example 1.3. Likewise, consider the C^*-inclusions

$$C[0,1] \subseteq L^\infty[0,1] \subseteq B\big(L^2[0,1]\big).$$

There are no conditional expectations for the inclusion $(B(L^2[0,1]), C[0,1])$, but infinitely many conditional expectations for $(B(L^2[0,1]), L^\infty[0,1])$, see [20]. Thus, (!CE) is not hereditary from below.

1.3. Unique pseudo-expectations. Recall that a ucp map $\Phi : \mathcal{C} \to I(\mathcal{D})$ is a pseudo-expectation if it extends the inclusion of $\mathcal{D}$ into $I(\mathcal{D})$. Clearly every conditional expectation for $(\mathcal{C}, \mathcal{D})$ is a pseudo-expectation for $(\mathcal{C}, \mathcal{D})$, so pseudo-expectations generalize conditional expectations. But pseudo-expectations always exist for any C^*-inclusion.

With the discussion of the previous section in mind, we are led to replace condition (!CE) there by the following stronger condition:

(!PsE) There exists a unique pseudo-expectation for $(\mathcal{C}, \mathcal{D})$.

Or perhaps by the even stronger condition:

(f!PsE) There exists a unique pseudo-expectation for $(\mathcal{C}, \mathcal{D})$, which is faithful.

We will see shortly that both of these conditions are hereditary from above (Proposition 2.6). Compelling evidence that (!PsE) and (f!PsE) are closely related to the largeness of $\mathcal{D}$ in $\mathcal{C}$ is provided by a striking result from [28]:

THEOREM 1.4 (Pitts). *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion with $\mathcal{D}$ a MASA in $\mathcal{C}$.*

(i) *Then there exists a unique pseudo-expectation $\Phi:\mathcal{C}\to I(\mathcal{D})$.*
(ii) *If $\mathcal{L}_\Phi=\{x\in\mathcal{C}:\Phi(x^*x)=0\}$ is the left kernel of Φ, then $\mathcal{L}_\Phi$ is the unique maximal $\mathcal{D}$-disjoint ideal in $\mathcal{C}$.*
(iii) *If Φ is faithful (i.e., if $\mathcal{L}_\Phi=0$), then $\mathcal{D}$ norms $\mathcal{C}$.*

Rephrasing Theorem 1.4 using the notation of this section, statement (i) says that for a C^*-inclusion $(\mathcal{C},\mathcal{D})$, with $\mathcal{D}$ maximal Abelian,

$$(\mathsf{Reg}) \implies (\mathsf{!PsE}).$$

Statements (ii) and (iii) imply that under the same hypotheses,

$$(\mathsf{Reg})\wedge(\mathsf{f!PsE}) \implies (\mathsf{Ess})\wedge(\mathsf{Norming}).$$

This paper is a systematic attempt to generalize Theorem 1.4. We characterize the unique pseudo-expectation property for various important classes of C^*-inclusions, and we relate the unique pseudo-expectation property for a C^*-inclusion $(\mathcal{C},\mathcal{D})$ to other measures of the largeness of $\mathcal{D}$ in $\mathcal{C}$, in particular conditions (ARC), (Reg), (Ess), (UEP), and (Norming) above. Necessarily, we significantly develop the general theory of pseudo-expectations along the way.

2. The unique pseudo-expectation property

2.1. Definitions and basic properties. In this section, we formally define pseudo-expectations and explore their basic properties. Before doing so, we remind the reader of a few facts about injective envelopes and establish some standing assumptions used throughout the paper. All C^*-algebras are assumed unital, and homomorphisms between C^*-algebras will always be $*$-homomorphisms which preserve the units. We will denote by OSys_1 the category whose objects are operator systems and whose morphisms are ucp (unital completely positive) maps. A C^*-algebra is *injective* if it is injective when viewed as an object in OSys_1. Let AbC^* be the category of Abelian C^*-algebras and homomorphisms. Clearly every object in AbC^* is also an object in OSys_1. An important observation found in [14] and [12] is that an Abelian C^*-algebrais injective in AbC^* if and only if it is injective in OSys_1.

THEOREM 2.1 (See [10] or [27]). *Let $\mathcal{D}$ be a unital C^*-algebra. Then there exists a unital C^*-algebra $\mathcal{A}$ and a unital $*$-monomorphism $\iota:\mathcal{D}\to\mathcal{A}$ with the following properties*:

(i) *$\mathcal{A}$ is injective*;

(ii) *if $\mathcal{S}$ is an injective object in* OSys_1 *and $\tau : \mathcal{D} \to \mathcal{S}$ is a unital complete isometry, then there exists a unital complete isometry $\tau_1 : \mathcal{A} \to \mathcal{S}$ such that $\tau = \tau_1 \circ \iota$.*

The pair $(\mathcal{A}, \iota)$ is called an *injective envelope* for $\mathcal{D}$, and it is "nearly" unique. The ambiguity arises from the fact that in general, the choice of τ_1 in Theorem 2.1 is not unique. However, in the sequel, we will assume that for a given C^*-algebra $\mathcal{D}$ under discussion, a choice of injective envelope $(I(\mathcal{D}), \iota)$ has been made. Furthermore, we will regard ι as an inclusion map and suppress writing it. Thus, we will always regard $\mathcal{D}$ as a C^*-subalgebra of $I(\mathcal{D})$.

Definition 2.2. A *pseudo-expectation* for the C^*-inclusion $(\mathcal{C}, \mathcal{D})$ is a ucp map $\Phi : \mathcal{C} \to I(\mathcal{D})$ such that $\Phi|_{\mathcal{D}} = \mathrm{id}$. We denote by $\mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ the collection of all pseudo-expectations for $(\mathcal{C}, \mathcal{D})$.

Proposition 2.3. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-inclusion, $\Phi \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$, and*

$$\mathcal{L}_\Phi = \{x \in \mathcal{C} : \Phi(x^*x) = 0\}$$

be the left kernel of Φ. Then the following statements hold:

(i) *Φ is a $\mathcal{D}$-bimodule map. That is, $\Phi(d_1 x d_2) = d_1 \Phi(x) d_2$ for all $x \in \mathcal{C}$, $d_1, d_2 \in \mathcal{D}$.*

(ii) *$\mathcal{L}_\Phi$ is a closed left ideal in $\mathcal{C}$ which intersects $\mathcal{D}$ trivially. Furthermore, $\mathcal{L}_\Phi$ is a right $\mathcal{D}$-module.*

Proof. The first statement follows from Choi's lemma ([27, Corollary 3.19]); the second is straightforward. ☐

Proposition 2.4. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-inclusion.*

(i) *The family $\mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ of all pseudo-expectations for $(\mathcal{C}, \mathcal{D})$ forms a nonempty convex subset of $\mathrm{UCP}(\mathcal{C}, I(\mathcal{D}))$, the ucp maps from $\mathcal{C}$ into $I(\mathcal{D})$. In fact, $\mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ is a face of $\mathrm{UCP}(\mathcal{C}, I(\mathcal{D}))$. Thus, any extreme point of $\mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ is an extreme point of $\mathrm{UCP}(\mathcal{C}, I(\mathcal{D}))$.*

(ii) *If $\mathrm{CE}(\mathcal{C}, \mathcal{D})$ denotes the collection of all conditional expectations for $(\mathcal{C}, \mathcal{D})$, then $\mathrm{CE}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$. Of course it can happen that $\mathrm{CE}(\mathcal{C}, \mathcal{D}) = \emptyset$, whereas $\mathrm{PsExp}(\mathcal{C}, \mathcal{D}) \neq \emptyset$, by injectivity.*

Proof. We only prove that $\mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ is a face of $\mathrm{UCP}(\mathcal{C}, I(\mathcal{D}))$. Indeed, suppose $\Phi \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$ and $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$, where Φ_1, Φ_2 belong to $\mathrm{UCP}(\mathcal{C}, I(\mathcal{D}))$ and $\lambda \in (0,1)$. For any $u \in U(\mathcal{D})$ (the unitary group of $\mathcal{D}$), we have that

$$u = \Phi(u) = \lambda\Phi_1(u) + (1-\lambda)\Phi_2(u).$$

Since $\Phi_1(u), \Phi_2(u) \in \mathrm{Ball}(I(\mathcal{D}))$ and

$$u \in U(\mathcal{D}) \subseteq U\big(I(\mathcal{D})\big) \subseteq \mathrm{Ext}\big(\mathrm{Ball}\big(I(\mathcal{D})\big)\big),$$

we conclude that $\Phi_1(u) = \Phi_2(u) = u$. It follows that $\Phi_1(d) = \Phi_2(d) = d$ for all $d \in \mathcal{D}$, so that $\Phi_1, \Phi_2 \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$. □

Definition 2.5. We say that a C^*-inclusion $(\mathcal{C}, \mathcal{D})$ has the *unique pseudo-expectation property* (!PsE) if there exists a unique $\Phi \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$. If, in addition, Φ is faithful, then we say that $(\mathcal{C}, \mathcal{D})$ has the *faithful unique pseudo-expectation property* (f!PsE).

As in the Introduction, we say that a property of C^*-inclusions is *hereditary from above* if whenever $(\mathcal{C}, \mathcal{D})$ has the property and $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ is a C^*-algebra, then $(\mathcal{C}_0, \mathcal{D})$ has the property.

Proposition 2.6. *The unique pseudo-expectation property is hereditary from above, as is the faithful unique pseudo-expectation property.*

Proof. Suppose $\mathrm{PsExp}(\mathcal{C}, \mathcal{D}) = \{\Phi\}$. Let $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ be a C^*-algebra, and fix $\theta \in \mathrm{PsExp}(\mathcal{C}_0, \mathcal{D})$. By injectivity, there exists a ucp map $\Theta : \mathcal{C} \to I(\mathcal{D})$ such that $\Theta|_{\mathcal{C}_0} = \theta$. Then $\Theta|_{\mathcal{D}} = \theta|_{\mathcal{D}} = \mathrm{id}$, so that $\Theta \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$. It follows that $\Theta = \Phi$, which implies $\theta = \Theta|_{\mathcal{C}_0} = \Phi|_{\mathcal{C}_0}$. Thus $\mathrm{PsExp}(\mathcal{C}_0, \mathcal{D}) = \{\Phi|_{\mathcal{C}_0}\}$. If Φ is faithful, then so is $\Phi|_{\mathcal{C}_0}$. □

On the other hand, if $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property and $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq \mathcal{C}$ is a C^*-algebra, then $(\mathcal{C}, \mathcal{D}_0)$ may not have the unique pseudo-expectation property (see Example 4.1). That is, the unique pseudo-expectation property is not *hereditary from below.*

2.2. Elementary examples. In this section, we give some examples of C^*-inclusions with (and without) the unique pseudo-expectation property. These examples are "elementary", insofar as we can prove that they are actually examples without any additional technology. Later, after we have developed some general theory for pseudo-expectations, we will give a number of "advanced" examples.

Example 2.7 (Regular MASA inclusions). Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion. Then $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property, by Pitts' Theorem 1.4. Two classes of regular MASA inclusions which appear in the literature are *C^*-diagonals* in the sense of Kumjian [22], and *Cartan subalgebras* in the sense of Renault [32].

Example 2.8 (Atomic MASA). The inclusion $(B(\ell^2), \ell^\infty)$ has the faithful unique pseudo-expectation property. Indeed, ℓ^∞ is injective (since it is an Abelian W^*-algebra) and there exists a unique conditional expectation $E : B(\ell^2) \to \ell^\infty$, which is faithful [20, Theorem 1].

Example 2.9 (Diffuse MASA). The inclusion $(B(L^2[0,1]), L^\infty[0,1])$ has infinitely many pseudo-expectations, none of which are faithful. However, the inclusion, $(L^\infty[0,1] + K(L^2[0,1]), L^\infty[0,1])$ has a unique pseudo-expectation, which is not faithful.

Proof. Since $L^\infty[0,1]$ is injective, conditional expectations and pseudo-expectations for $(B(L^2[0,1]), L^\infty[0,1])$ coincide. By Theorem 2 and Remark 5 of [20], there are infinitely many conditional expectations $B(L^2[0,1]) \to L^\infty[0,1]$, all of which annihilate $K(L^2[0,1])$. Now suppose $E : L^\infty[0,1] + K(L^2[0,1]) \to L^\infty[0,1]$ is a conditional expectation. Then E extends to a conditional expectation $\tilde{E} : B(L^2[0,1]) \to L^\infty[0,1]$. Thus, by the previous discussion,

$$E(d+h) = \tilde{E}(d+h) = d$$

for all $d \in L^\infty[0,1]$, $h \in K(L^2[0,1])$. □

Remark 2.10. Let $(\mathcal{C}, \mathcal{D})$ be a C^*-inclusion. Then we have C^*-inclusions $\mathcal{D} \subseteq \mathcal{C} \subseteq I(\mathcal{C})$. By Theorem 2.1, it follows that we have an operator system inclusion $I(\mathcal{D}) \subseteq I(\mathcal{C})$. If $\mathcal{D}$ is Abelian, then in fact we have a C^*-inclusion $I(\mathcal{D}) \subseteq I(\mathcal{C})$ [12, Thm. 2.21]. In that case, if $\Phi : \mathcal{C} \to I(\mathcal{D})$ is a pseudo-expectation for $(\mathcal{C}, \mathcal{D})$, then it is not hard to see that any ucp extension $\tilde{\Phi} : I(\mathcal{C}) \to I(\mathcal{D})$ of Φ is a conditional expectation for $(I(\mathcal{C}), I(\mathcal{D}))$. As the previous example shows, this extension need not be unique. Indeed, by [14, Ex. 5.3], $I(L^\infty[0,1] + K(L^2[0,1])) = B(L^2[0,1])$.

Next, we consider C^*-inclusions $(\mathcal{C}, \mathcal{D})$ such that there is a monomorphism of $\mathcal{C}$ into $I(\mathcal{D})$. By [13, Lemma 4.6], these are precisely the operator space essential inclusions. A C^*-inclusion $(\mathcal{C}, \mathcal{D})$ is *operator space essential* (OSE) if every complete contraction $u : \mathcal{C} \to B(\mathcal{H})$ which is completely isometric on $\mathcal{D}$ is actually completely isometric on $\mathcal{C}$.

Example 2.11 (OSE inclusions). Let $\mathcal{D}$ be an arbitrary unital C^*-algebra. Then $(I(\mathcal{D}), \mathcal{D})$ has the faithful unique pseudo-expectation property. More generally, if $\mathcal{D} \subseteq \mathcal{C} \subseteq I(\mathcal{D})$ are C^*-inclusions, then $(\mathcal{C}, \mathcal{D})$ has the faithful unique pseudo-expectation property, by Proposition 2.6.

Proof. Let $\Phi \in \mathrm{PsExp}(I(\mathcal{D}), \mathcal{D})$. Then $\Phi : I(\mathcal{D}) \to I(\mathcal{D})$ is a ucp map such that $\Phi|_{\mathcal{D}} = \mathrm{id}$. By the *rigidity* of the injective envelope, $\Phi = \mathrm{id}$. □

Example 2.12 (UEP MASA inclusions). Let $(\mathcal{C}, \mathcal{D})$ be a C^*-inclusion, with $\mathcal{D}$ Abelian. Assume that $(\mathcal{C}, \mathcal{D})$ has the *unique extension property* (UEP), meaning that every pure state on $\mathcal{D}$ extends uniquely to a pure state on $\mathcal{C}$. (This forces $\mathcal{D}$ to be a MASA in $\mathcal{C}$.) Then $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property. In fact, the unique pseudo-expectation is a conditional expectation.

Proof. By [3, Cor. 2.7], we have the direct sum decomposition

$$\mathcal{C} = \mathcal{D} + \overline{\mathrm{span}}\big\{[\mathcal{C}, \mathcal{D}]\big\}.$$

If $\Phi \in \mathrm{PsExp}(\mathcal{C}, \mathcal{D})$, then by Proposition 2.3 and the fact that $I(\mathcal{D})$ is Abelian,

$$\Phi(xd - dx) = \Phi(xd) - \Phi(dx) = \Phi(x)d - d\Phi(x) = 0, \quad x \in \mathcal{C}, d \in \mathcal{D}.$$

The result follows. □

REMARK 2.13. Initially the study of UEP inclusions $(\mathcal{C},\mathcal{D})$ focused on the case $\mathcal{D}$ Abelian, and there has been substantial work in this direction. Later work has made progress in the general setting [6]. It would be interesting to know whether a general UEP inclusion $(\mathcal{C},\mathcal{D})$ has the unique pseudo-expectation property. The inclusion $(C_r^*(\mathbb{F}_m), C_r^*(\mathbb{F}_n))$ for $m > n \geq 2$, may provide a possible test case [2, Thm. 2.6].

EXAMPLE 2.14. Let $\mathcal{M}$ be a II_1 factor with separable predual and $\mathcal{D} \subseteq \mathcal{M}$ be a MASA. More generally, let $\mathcal{M}$ be any II_1 factor and $\mathcal{D} \subseteq \mathcal{M}$ be a singly-generated MASA. Then $(\mathcal{M},\mathcal{D})$ does not have the unique pseudo-expectation property [1, Thm. 4.4].

3. Some general theory

In this section, we prove some general results about pseudo-expectations, which we will use later to analyze more complicated examples than those considered so far.

3.1. Left kernel. Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. We say that a closed two-sided ideal $\mathcal{J} \lhd \mathcal{C}$ is *$\mathcal{D}$-disjoint* if $\mathcal{D} \cap \mathcal{J} = 0$. It is not hard to prove that every $\mathcal{D}$-disjoint ideal of $\mathcal{C}$ is contained in a maximal $\mathcal{D}$-disjoint ideal of $\mathcal{C}$.

As seen in Theorem 1.4, if $(\mathcal{C},\mathcal{D})$ is a regular MASA inclusion, then there exists a unique maximal $\mathcal{D}$-disjoint ideal in $\mathcal{C}$, namely the left kernel $\mathcal{L}_\Phi$ of the unique pseudo-expectation $\Phi : \mathcal{C} \to I(\mathcal{D})$. In general, for a C^*-inclusion $(\mathcal{C},\mathcal{D})$ with unique pseudo-expectation Φ, the left kernel $\mathcal{L}_\Phi$ is only a left ideal of $\mathcal{C}$, rather than a two-sided ideal (see Example 3.10 below). Nevertheless, we have the following structural result for general C^*-inclusions with the unique pseudo-expectation property.

PROPOSITION 3.1. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. If $(\mathcal{C},\mathcal{D})$ has unique pseudo-expectation Φ, then there exists a unique maximal $\mathcal{D}$-disjoint ideal $\mathcal{I} \lhd \mathcal{C}$. Furthermore, $\mathcal{I} \subseteq \mathcal{L}_\Phi$, the left kernel of Φ.*

Proof. Let $\mathcal{J} \lhd \mathcal{C}$ be a $\mathcal{D}$-disjoint ideal. Then the map $\mathcal{D} + \mathcal{J} \to \mathcal{D} : d + h \mapsto d$ is a unital $*$-homomorphism, which extends by injectivity to a pseudo-expectation for $(\mathcal{C},\mathcal{D})$, necessarily Φ. Thus $\mathcal{J} \subseteq \ker(\Phi)$. If $h \in \mathcal{J}$, then $h^*h \in \mathcal{J}$, which implies $\Phi(h^*h) = 0$, which in turn implies $h \in \mathcal{L}_\Phi$. Thus $\mathcal{J} \subseteq \mathcal{L}_\Phi$. It follows that

$$\bigcup \{\mathcal{J} : \mathcal{J} \lhd \mathcal{C}, \mathcal{D} \cap \mathcal{J} = 0\} \subseteq \mathcal{L}_\Phi,$$

and so

$$\mathcal{I} = \overline{\operatorname{span}}\left(\bigcup \{\mathcal{J} : \mathcal{J} \lhd \mathcal{C}, \mathcal{D} \cap \mathcal{J} = 0\} \right) \subseteq \mathcal{L}_\Phi.$$

Thus $\mathcal{I}$ is the unique maximal $\mathcal{D}$-disjoint ideal of $\mathcal{C}$. $\square$

3.2. Characterization: Every pseudo-expectation is faithful. In this section, we characterize the property "every pseudo-expectation is faithful" for arbitrary C^*-inclusions $(\mathcal{C},\mathcal{D})$ in terms of the (hereditary) $\mathcal{D}$-disjoint ideal structure of $\mathcal{C}$ (Theorem 3.5). Formally, the property "every pseudo-expectation is faithful" is weaker than the faithful unique pseudo-expectation property. On the other hand, we have no examples showing that it is strictly weaker. So in principle, Theorem 3.5 could be a characterization of the faithful unique pseudo-expectation property. We list this as an open problem.

QUESTION 3.2. Does the property "every pseudo-expectation is faithful" imply the faithful unique pseudo-expectation property?

To proceed with our characterization, we will need two notions from earlier in the paper. First, recall that a closed two-sided ideal $\mathcal{J} \lhd \mathcal{C}$ is *$\mathcal{D}$-disjoint* if $\mathcal{D}\cap\mathcal{J}=0$. Second, recall that a C^*-inclusion $(\mathcal{C},\mathcal{D})$ is *essential* (Ess) if every nontrivial closed two-sided ideal of $\mathcal{C}$ intersects $\mathcal{D}$ nontrivially. The following proposition relates these two notions with each other, as well as to a useful mapping property.

PROPOSITION 3.3. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. Then the following are equivalent*:

(i) *$(\mathcal{C},\mathcal{D})$ is essential.*
(ii) *The only $\mathcal{D}$-disjoint ideal of $\mathcal{C}$ is* 0.
(iii) *Whenever $\pi : \mathcal{C} \to B(\mathcal{H})$ is a unital $*$-homomorphism such that $\pi|_{\mathcal{D}}$ is faithful, then π itself is faithful.*

Proof. (i) $\iff$ (ii) Tautological.

(ii) $\implies$ (iii) Suppose the only $\mathcal{D}$-disjoint ideal of $\mathcal{C}$ is the trivial ideal. Let $\pi : \mathcal{C} \to B(\mathcal{H})$ be a unital $*$-homomorphism such that $\pi|_{\mathcal{D}}$ is faithful. Then $\ker(\pi) \lhd \mathcal{C}$ is a $\mathcal{D}$-disjoint ideal. By assumption, $\ker(\pi)=0$, so π is faithful.

(iii) $\implies$ (ii) Conversely, suppose that for every unital $*$-homomorphism $\pi : \mathcal{C} \to B(\mathcal{H})$, π is faithful whenever $\pi|_{\mathcal{D}}$ is faithful. Let $\mathcal{J} \lhd \mathcal{C}$ be a $\mathcal{D}$-disjoint ideal. Then $q : \mathcal{C} \to \mathcal{C}/\mathcal{J} : x \mapsto x+\mathcal{J}$ is a unital $*$-homomorphism such that $q|_{\mathcal{D}}$ is faithful. By assumption, q is faithful, so $\mathcal{J}=0$. ☐

As we saw in the Introduction, the condition (Ess) is not hereditary from above. Indeed, $(M_{2\times 2}(\mathbb{C}),\mathbb{C}I)$ satisfies (Ess), since $M_{2\times 2}(\mathbb{C})$ is simple, and $\mathbb{C}I \subseteq \mathbb{C}\oplus\mathbb{C} \subseteq M_{2\times 2}(\mathbb{C})$ is a C^*-algebra, but $(\mathbb{C}\oplus\mathbb{C},\mathbb{C}I)$ fails (Ess). To resolve this issue, we introduce the following stronger condition:

DEFINITION 3.4. We say that a C^*-inclusion $(\mathcal{C},\mathcal{D})$ is *hereditarily essential* if $(\mathcal{C}_0,\mathcal{D})$ is essential whenever $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ is a C^*-algebra.

Now comes the promised characterization.

THEOREM 3.5. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. Then the following are equivalent*:

(i) *Every pseudo-expectation* $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$ *is faithful.*
(ii) $(\mathcal{C},\mathcal{D})$ *is hereditarily essential.*

Proof. (i) $\implies$ (ii) Suppose that every pseudo-expectation $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$ is faithful. Let $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ be a C^*-algebra and $\mathcal{J}_0 \lhd \mathcal{C}_0$ be a $\mathcal{D}$-disjoint ideal. Then $\Phi_0 : \mathcal{D}+\mathcal{J}_0 \to \mathcal{D} : d+h \mapsto d$ is a unital $*$-homomorphism. By injectivity, there exists $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$ such that such that $\Phi|_{\mathcal{D}+\mathcal{J}_0} = \Phi_0$. Since Φ is faithful, so is Φ_0, which implies $\mathcal{J}_0 = 0$. It follows that $(\mathcal{C}_0,\mathcal{D})$ is essential, which implies $(\mathcal{C},\mathcal{D})$ is hereditarily essential.

(ii) $\implies$ (i) Conversely, suppose that $(\mathcal{C},\mathcal{D})$ is hereditarily essential. Let $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$ and $x \in \mathcal{L}_\Phi$ (the left kernel of Φ). Define $\mathcal{C}_0 = C^*(\mathcal{D},|x|)$, so that $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$, and let $\mathcal{J}_0 \lhd \mathcal{C}_0$ be the closed two-sided ideal generated by $|x|$. We claim that $\mathcal{J}_0 \subseteq \mathcal{L}_\Phi$. Indeed, $\mathcal{J}_0 = \overline{\mathrm{span}}\{w|x|d : w \in \mathcal{C}_0, d \in \mathcal{D}\}$ and $\mathcal{L}_\Phi$ is both a closed left ideal and a right $\mathcal{D}$-module in $\mathcal{C}$ containing $|x|$ (Proposition 2.3). Since $\mathcal{D} \cap \mathcal{L}_\Phi = 0$, $\mathcal{D} \cap \mathcal{J}_0 = 0$, and since $(\mathcal{C}_0,\mathcal{D})$ is essential by assumption, $\mathcal{J}_0 = 0$. Thus $|x| = 0$, which implies $x = 0$. Hence $\mathcal{L}_\Phi = 0$, so Φ is faithful. ☐

3.3. Quotients. We next examine the behavior of the unique pseudo-expectation property with respect to quotients. Specifically, for a closed two-sided ideal $\mathcal{J} \lhd \mathcal{C}$, we are interested to know when the unique pseudo-expectation property for $(\mathcal{C},\mathcal{D})$ passes to $(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$. If $\mathcal{J}\cap\mathcal{D} = 0$, then the answer is “always”, and faithfulness is preserved.

PROPOSITION 3.6. *Let* $(\mathcal{C},\mathcal{D})$ *be a* C^**-inclusion and* $\mathcal{J} \lhd \mathcal{C}$ *be a* $\mathcal{D}$*-disjoint ideal. If* $(\mathcal{C},\mathcal{D})$ *has the unique pseudo-expectation property, then so does* $(\mathcal{C}/\mathcal{J},\mathcal{D})$. *If* $(\mathcal{C},\mathcal{D})$ *has the faithful unique pseudo-expectation property, then so does* $(\mathcal{C}/\mathcal{J},\mathcal{D})$ *(trivially, because* $\mathcal{J} = 0$*).*

Proof. Suppose $\mathrm{PsExp}(\mathcal{C},\mathcal{D}) = \{\Phi\}$. Let $\theta \in \mathrm{PsExp}(\mathcal{C}/\mathcal{J},\mathcal{D})$. Then $\theta \circ q \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$, where $q : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ is the quotient map. Thus $\theta \circ q = \Phi$, which implies $\theta(x+\mathcal{J}) = \Phi(x)$, $x \in \mathcal{C}$. Hence, $\mathrm{PsExp}(\mathcal{C}/\mathcal{J},\mathcal{D}) = \{\theta\}$. If Φ is faithful, then $\mathcal{J} = 0$, by Theorem 3.5. ☐

REMARK 3.7. If $\mathcal{D}\cap\mathcal{J} \neq 0$, then it is entirely possible that $(\mathcal{C},\mathcal{D})$ has a unique pseudo-expectation but $(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ does not (see Example 4.2).

In order to obtain a positive result when $\mathcal{D}\cap\mathcal{J} \neq 0$, we require $\mathcal{J}\cap\mathcal{D} \lhd \mathcal{D}$ to be regular. Recall that if $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I} \lhd \mathcal{A}$, then

$$\mathcal{I}^\perp = \{a \in \mathcal{A} : a\mathcal{I} = \mathcal{I}a = 0\} \lhd \mathcal{A}.$$

Also, $\mathcal{I}$ is *regular* if $\mathcal{I}^{\perp\perp} = (\mathcal{I}^\perp)^\perp = \mathcal{I}$. Combining [16, Lemma 1.3(iii)] with [15, Theorem 6.3], one finds that given a regular ideal $\mathcal{I} \lhd \mathcal{A}$, there exists a unique projection $p \in Z(I(\mathcal{A}))$ such that $\mathcal{I} = \{a \in \mathcal{A} : ap = a\}$. In that case, the unital $*$-isomorphism $\mathcal{A}/\mathcal{I} \to \mathcal{A}p^\perp : a+\mathcal{I} \mapsto ap^\perp$ extends uniquely to a unital $*$-isomorphism $I(\mathcal{A}/\mathcal{I}) \cong I(\mathcal{A})p^\perp$.

THEOREM 3.8. *Let* $(\mathcal{C},\mathcal{D})$ *be a* C^**-inclusion and* $\mathcal{J} \lhd \mathcal{C}$. *If* $(\mathcal{C},\mathcal{D})$ *has the unique pseudo-expectation property and* $\mathcal{J}\cap\mathcal{D} \lhd \mathcal{D}$ *is regular*, *then the inclusion* $(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ *has the unique pseudo-expectation property.*

Proof. Let $p \in Z(I(\mathcal{D}))$ be the unique projection such that $\mathcal{J}\cap\mathcal{D} = \{d \in \mathcal{D} : dp = d\}$. Then the unital $*$-isomorphism $\mathcal{D}/(\mathcal{J}\cap\mathcal{D}) \to \mathcal{D}p^\perp : d + (\mathcal{J}\cap\mathcal{D}) \mapsto dp^\perp$ extends uniquely to a unital $*$-isomorphism $I(\mathcal{D}/(\mathcal{J}\cap\mathcal{D})) \cong I(\mathcal{D})p^\perp$. Now suppose $\operatorname{PsExp}(\mathcal{C},\mathcal{D}) = \{\Phi\}$ and let $\theta \in \operatorname{PsExp}(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$. Then $\theta : \mathcal{C}/\mathcal{J} \to I(\mathcal{D})p^\perp$ is a ucp map such that $\theta(d+\mathcal{J}) = dp^\perp$, $d \in \mathcal{D}$. Define $\Theta : \mathcal{C} \to I(\mathcal{D})$ by

$$\Theta(x) = \theta(x+\mathcal{J}) + \Phi(x)p, \quad x \in \mathcal{C}.$$

Then $\Theta \in \operatorname{PsExp}(\mathcal{C},\mathcal{D})$, which implies $\Theta = \Phi$, which in turn implies

$$\theta(x+\mathcal{J}) = \Phi(x)p^\perp, \quad x \in \mathcal{C}.$$

Thus $(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ has the unique pseudo-expectation property. □

REMARK 3.9. If $(\mathcal{C},\mathcal{D})$ has the faithful unique pseudo-expectation property and $\mathcal{J} \lhd \mathcal{C}$, then $(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ need not have the faithful unique pseudo-expectation property, even if $\mathcal{J}\cap\mathcal{D} \lhd \mathcal{D}$ is regular (see Example 4.3).

A very interesting example not covered by the results of this section occurs when $\mathcal{C} = B(\ell^2)$, $\mathcal{D} = \ell^\infty$, and $\mathcal{J} = K(\ell^2)$, so that

$$\big(\mathcal{C}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D})\big) = \big(B(\ell^2)/K(\ell^2),\ell^\infty/c_0\big).$$

Indeed, $c_0 \lhd \ell^\infty$ is not regular, since $c_0^{\perp\perp} = \ell^\infty$. Our analysis of this example is greatly simplified by the recent remarkable affirmative solution to the Kadison–Singer problem [26].

EXAMPLE 3.10 (Calkin algebra). The inclusion $(B(\ell^2)/K(\ell^2),\ell^\infty/c_0)$ has the unique pseudo-expectation property. In fact, the unique pseudo-expectation is a conditional expectation which is <u>not</u> faithful.

Proof. By [26], the inclusion $(B(\ell^2),\ell^\infty)$ has the unique extension property (UEP). By [3, Lemma 3.1], $(B(\ell^2)/K(\ell^2),\ell^\infty/c_0)$ has (UEP) as well. Thus, $(B(\ell^2)/K(\ell^2),\ell^\infty/c_0)$ has a unique pseudo-expectation $\tilde{E}$, which is actually a conditional expectation, by Example 2.12. In fact,

$$\tilde{E}\big(x + K(\ell^2)\big) = E(x) + c_0, \quad x \in B(\ell^2),$$

where $E : B(\ell^2) \to \ell^\infty$ is the unique conditional expectation. Letting $h \in B(\ell^2)_+$ be the *Hilbert matrix* [7], we see that $\tilde{E}(h + K(\ell^2)) = 0$, but $h + K(\ell^2) \neq 0$. □

REMARK 3.11. Example 3.10 furnishes an instance of a C^*-inclusion $(\mathcal{C},\mathcal{D})$ with a unique pseudo-expectation Φ, such that $\mathcal{L}_\Phi$ is not a two-sided ideal of $\mathcal{C}$. Indeed, $\mathcal{C}$ is simple but $\mathcal{L}_\Phi \neq 0$ in Example 3.10. This should be compared with Theorem 1.4.

3.4. Abelian relative commutant. As mentioned in the Introduction, the unique pseudo-expectation property for a C^*-inclusion $(\mathcal{C},\mathcal{D})$ can be thought of as an expression of the fact that $\mathcal{D}$ is "large" in $\mathcal{C}$. A more familiar algebraic expression of the largeness of $\mathcal{D}$ in $\mathcal{C}$ is that $\mathcal{D}^c = \mathcal{D}' \cap \mathcal{C}$, the *relative commutant* of $\mathcal{D}$ in $\mathcal{C}$, is "small" (Abelian). In Corollary 3.14 below, we show that the faithful unique pseudo-expectation property implies that the relative commutant is Abelian, symbolically

$$(\mathsf{f!PsE}) \implies (\mathsf{ARC}).$$

We expect that the hypothesis of faithfulness is not needed for this result, but we have not been able to eliminate it.

THEOREM 3.12. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. Assume that there exists a faithful pseudo-expectation $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$. If $\mathcal{D}^c$ is not Abelian, then there exist infinitely many pseudo-expectations for $(\mathcal{C},\mathcal{D})$, some of which are not faithful.*

Proof. We may assume that $\mathcal{C} \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If $\mathcal{D}^c$ is not Abelian, then there exists $x \in \mathcal{D}^c$ with $\|x\| = 1$ and $x^2 = 0$ ([9, p. 288]). Let $x = u|x|$ be the polar decomposition, so that $u \in \mathcal{D}'$ is a partial isometry with initial space $\overline{\mathrm{ran}}(|x|)$ and final space $\overline{\mathrm{ran}}(x)$. Since

$$\overline{\mathrm{ran}}(x) \subseteq \ker(x) = \ker\big(|x|\big) = \mathrm{ran}\big(|x|\big)^{\perp},$$

we find that $u^2 = 0$. It follows that

$$p_1 = u^*u, \qquad p_2 = uu^*, \quad \text{and} \quad p_3 = 1 - u^*u - uu^*$$

are orthogonal projections in $\mathcal{D}'$. For $\lambda \in [0,1]$ define $\theta_\lambda : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$\theta_\lambda(t) = \lambda p_1 t p_1 + (1-\lambda)u^*tu + \lambda utu^* + (1-\lambda)p_2tp_2 + p_3tp_3.$$

Then θ_λ is a ucp map such that

$$\theta_\lambda|_{\mathcal{D}} = \mathrm{id}, \qquad \theta_\lambda\big(x^*x\big) = \lambda\big(x^*x + xx^*\big), \quad \text{and} \quad \theta_\lambda\big(xx^*\big) = (1-\lambda)\big(x^*x + xx^*\big).$$

Consider the operator system

$$\mathcal{S} := \mathcal{D} + \mathbb{C}x^*x + \mathbb{C}xx^* \subseteq \mathcal{C}.$$

Since $\theta_\lambda(\mathcal{S}) \subseteq \mathcal{S}$,

$$\Phi_\lambda^0 := \Phi \circ \theta_\lambda|_{\mathcal{S}} : \mathcal{S} \to I(\mathcal{D})$$

is a well-defined ucp map such that

$$\Phi_\lambda^0|_{\mathcal{D}} = \mathrm{id}, \qquad \Phi_\lambda^0\big(x^*x\big) = \lambda\Phi\big(x^*x + xx^*\big), \quad \text{and}$$
$$\Phi_\lambda^0\big(xx^*\big) = (1-\lambda)\Phi\big(x^*x + xx^*\big).$$

By injectivity, there exists $\Phi_\lambda \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$ such that $\Phi_\lambda|_{\mathcal{S}} = \Phi_\lambda^0$. Since Φ is faithful, $\Phi(x^*x + xx^*) \neq 0$, and so $\Phi_\lambda \neq \Phi_\mu$ if $\lambda \neq \mu$. Consequently, $\{\Phi_\lambda : \lambda \in [0,1]\}$ is an infinite family of pseudo-expectations for $(\mathcal{C},\mathcal{D})$, some of which are not faithful (namely Φ_0 and Φ_1). ∎

REMARK 3.13. In Theorem 3.12, we may remove the hypothesis that there exists a faithful pseudo-expectation $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$, provided we strengthen the hypothesis on $\mathcal{D}^c$. For example, we could ask that $\mathcal{D}^c$ contain a halving projection. In that case, the proof simplifies substantially.

COROLLARY 3.14. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion. If $(\mathcal{C},\mathcal{D})$ has the faithful unique pseudo-expectation property, then $\mathcal{D}^c$ is Abelian.*

REMARK 3.15. As indicated earlier, we expect Corollary 3.14 to remain true without the assumption of faithfulness. At this point, however, we do not have a proof, even in the case $\mathcal{D}$ Abelian. On the other hand, the case of W^*-inclusions is completely settled in the affirmative (Corollary 5.3).

3.5. Characterization: Unique pseudo-expectation property for Abelian subalgebras. In this section, we give an order-theoretic characterization of the unique pseudo-expectation property for C^*-inclusions $(\mathcal{C},\mathcal{D})$, with $\mathcal{D}$ Abelian. We remind the reader that if $\mathcal{D}$ is a unital Abelian C^*-algebra, then $I(\mathcal{D})$ is *order complete*, meaning that every nonempty set $S \subseteq I(\mathcal{D})_{sa}$ with an upper bound has a supremum [36, Prop. III.1.7].

THEOREM 3.16. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion, with $\mathcal{D}$ Abelian. Then the following are equivalent:*
(i) *$(\mathcal{C},\mathcal{D})$ has the unique pseudo-expectation property.*
(ii) *For all $x \in \mathcal{C}_{sa}$,*

$$\sup_{I(\mathcal{D})}\{d \in \mathcal{D}_{sa} : d \leq x\} = \inf_{I(\mathcal{D})}\{d \in \mathcal{D}_{sa} : d \geq x\}.$$

Proof. For $x \in \mathcal{C}_{sa}$, set

$$\ell(x) = \sup_{I(\mathcal{D})}\{d \in \mathcal{D}_{sa} : d \leq x\} \quad \text{and} \quad u(x) = \inf_{I(\mathcal{D})}\{d \in \mathcal{D}_{sa} : d \geq x\}.$$

It is easy to see that $\ell(x) \leq u(x)$. (If $e \in \mathcal{D}_{sa}$ and $e \geq x$, then $d \leq e$ for all $d \in \mathcal{D}_{sa}$ such that $d \leq x$. Thus, $\ell(x) \leq e$. Since the choice of e was arbitrary, $\ell(x) \leq u(x)$.) Clearly, $\ell(d) = d = u(d)$ for all $d \in \mathcal{D}_{sa}$.

(i) $\implies$ (ii) Let $x \in \mathcal{C}_{sa} \setminus \mathcal{D}_{sa}$ and suppose $a \in I(\mathcal{D})_{sa}$ satisfies $\ell(x) \leq a \leq u(x)$. Since $\mathcal{D} \cap \mathbb{C}x = 0$,

$$\Phi_0 : \mathcal{D} + \mathbb{C}x \to I(\mathcal{D}) : d + \lambda x \mapsto d + \lambda a$$

is a well-defined linear map such that $\Phi_0|_{\mathcal{D}} = \mathrm{id}$. Suppose $d + \lambda x \geq 0$, so that $d \in \mathcal{D}_{sa}$ and $\lambda \in \mathbb{R}$.

Case 1: If $\lambda = 0$, then $d \geq 0$, which implies $d + \lambda a = d \geq 0$.
Case 2: If $\lambda > 0$, then $x \geq -\frac{1}{\lambda}d$, which implies $-\frac{1}{\lambda}d \leq \ell(x) \leq a$, which in turn implies $d + \lambda a \geq 0$.
Case 3: If $\lambda < 0$, then $x \leq -\frac{1}{\lambda}d$, which implies $a \leq u(x) \leq -\frac{1}{\lambda}d$, which in turn implies $d + \lambda a \geq 0$.

The preceding analysis shows that Φ_0 is positive, and since $I(\mathcal{D})$ is Abelian, it is actually completely positive. By injectivity, there exists a ucp map $\Phi : \mathcal{C} \to I(\mathcal{D})$ such that $\Phi|_{\mathcal{D}+\mathbb{C}x} = \Phi_0$. Then Φ is a pseudo-expectation for $(\mathcal{C},\mathcal{D})$ such that $\Phi(x) = a$. It follows that if there exists $x \in \mathcal{C}_{sa}$ such that $\ell(x) \neq u(x)$, then $(\mathcal{C},\mathcal{D})$ admits multiple pseudo-expectations.

(ii) $\implies$ (i) Conversely, suppose $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$. Let $x \in \mathcal{C}_{sa}$. If $d \in \mathcal{D}_{sa}$ and $d \leq x$, then $d = \Phi(d) \leq \Phi(x)$, which implies $\ell(x) \leq \Phi(x)$. Likewise if $d \in \mathcal{D}_{sa}$ and $d \geq x$, then $d = \Phi(d) \geq \Phi(x)$, which implies $u(x) \geq \Phi(x)$. Thus if $\ell(x) = u(x)$ for all $x \in \mathcal{C}_{sa}$, then Φ is uniquely determined on $\mathcal{C}_{sa}$, therefore on $\mathcal{C}$. □

3.6. A Krein–Milman theorem for pseudo-expectations when the subalgebra is Abelian. The purpose of this section is to prove a Krein–Milman theorem for the pseudo-expectation space $\mathrm{PsExp}(\mathcal{C},\mathcal{D})$, valid for C^*-inclusions $(\mathcal{C},\mathcal{D})$, with $\mathcal{D}$ Abelian. Our goal is to show that there is a rich supply of extreme points in $\mathrm{PsExp}(\mathcal{C},\mathcal{D})$. It will then follow that uniqueness of pseudo-expectations is equivalent to uniqueness of extreme pseudo-expectations. One approach to this type of result might be the following: first, introduce an appropriate locally convex topology on the set of all bounded linear maps from $\mathcal{C}$ into $I(\mathcal{D})$; second, show that $\mathrm{PsExp}(\mathcal{C},\mathcal{D})$ is compact in this topology; and finally, apply the usual Krein–Milman theorem. While this may be a viable approach, it is not clear (at least to us) how to define such a topology, so we proceed instead using a route through convexity theory, which is perhaps less well-traveled.

Our key tool is Kutateladze's Krein–Milman theorem for subdifferentials of sublinear operators into Kantorovich spaces [23]. Let V and W be real vector spaces. Assume further that W is a *Kantorovich space*, meaning that W is a vector lattice such that every nonempty subset with an upper bound has a supremum. Suppose $Q : V \to W$ a *sublinear operator*, meaning that

- $Q(\alpha v) = \alpha Q(v)$ for all $v \in V$, $\alpha \geq 0$;
- $Q(v_1 + v_2) \leq Q(v_1) + Q(v_2)$ for all $v_1, v_2 \in V$.

Let ∂Q be the *subdifferential* of Q:

$$\partial Q = \{T \in \mathrm{Lin}(V,W) : T(v) \leq Q(v), v \in V\}.$$

(Here $\mathrm{Lin}(V,W)$ denotes the set of all real linear maps from V to W.) Kutateladze's version of the Krein–Milman theorem is the following.

Theorem 3.17 (Kutateladze [23]). *Let V and W be real vector spaces with W a Kantorovich space, and suppose $Q : V \to W$ is a sublinear operator. Then the following statements hold:*

(i) $\mathrm{Ext}(\partial Q) \neq \emptyset$.
(ii) *For $v \in V$, define $P(v) = \sup_W\{T(v) : T \in \mathrm{Ext}(\partial Q)\}$. Then $P : V \to W$ is a sublinear operator and $\partial Q = \partial P$.*

We are now ready to apply Kutateladze's theorem to our setting.

THEOREM 3.18. *Let* $(\mathcal{C},\mathcal{D})$ *be a* C^**-inclusion, with* $\mathcal{D}$ *Abelian. Then the following statements hold*:

(i) $\mathrm{Ext}(\mathrm{PsExp}(\mathcal{C},\mathcal{D}))\neq\emptyset$.

(ii) *For* $x\in\mathcal{C}_{sa}$, *define* $P(x)=\sup_{I(\mathcal{D})}\{\Psi(x):\Psi\in\mathrm{Ext}(\mathrm{PsExp}(\mathcal{C},\mathcal{D}))\}$. *Then*

$$\mathrm{PsExp}(\mathcal{C},\mathcal{D})=\big\{\Phi\in\mathrm{UCP}\big(\mathcal{C},I(\mathcal{D})\big):\Phi(x)\leq P(x)\ \textit{for all}\ x\in\mathcal{C}_{sa}\big\}.$$

In particular,

$$\exists!\Phi\in\mathrm{PsExp}(\mathcal{C},\mathcal{D})\iff\exists!\Psi\in\mathrm{Ext}\big(\mathrm{PsExp}(\mathcal{C},\mathcal{D})\big).$$

Proof. Since $\mathcal{D}$ is Abelian, $I(\mathcal{D})_{sa}$ is a Kantorovich space. For all $x\in\mathcal{C}_{sa}$, define

$$Q(x)=\sup_{I(\mathcal{D})}\big\{\Phi(x):\Phi\in\mathrm{PsExp}(\mathcal{C},\mathcal{D})\big\}.$$

It is easy to see that $Q:\mathcal{C}_{sa}\to I(\mathcal{D})_{sa}$ is a sublinear operator. We claim that

$$\mathrm{PsExp}(\mathcal{C},\mathcal{D})=\big\{\Phi\in\mathrm{UCP}\big(\mathcal{C},I(\mathcal{D})\big):\Phi|_{\mathcal{C}_{sa}}\in\partial Q\big\}=\{\tilde{T}:T\in\partial Q\},$$

where $\tilde{T}:\mathcal{C}\to I(\mathcal{D})$ is the *complexification* of $T:\mathcal{C}_{sa}\to I(\mathcal{D})_{sa}$. Indeed, the inclusions of the first set into the second, and the second set into the third, are tautological. Now let $T\in\partial Q$. Then

$$x\in\mathcal{C}_+\quad\Longrightarrow\quad -T(x)=T(-x)\leq Q(-x)\leq 0\quad\Longrightarrow\quad T(x)\geq 0.$$

Thus $\tilde{T}$ is positive, and since $I(\mathcal{D})$ is Abelian, completely positive. Also

$$d\in\mathcal{D}_{sa}\quad\Longrightarrow\quad \pm T(d)=T(\pm d)\leq Q(\pm d)=\pm d\quad\Longrightarrow\quad T(d)=d.$$

Therefore, $\tilde{T}\in\mathrm{PsExp}(\mathcal{C},\mathcal{D})$.

Invoking Kutateladze's Krein–Milman theorem, we have that

$$\mathrm{Ext}\big(\mathrm{PsExp}(\mathcal{C},\mathcal{D})\big)\neq\emptyset,$$

and in fact

$$\mathrm{PsExp}(\mathcal{C},\mathcal{D})=\big\{\Phi\in\mathrm{UCP}\big(\mathcal{C},I(\mathcal{D})\big):\Phi|_{\mathcal{C}_{sa}}\in\partial P\big\},$$

where for all $x\in\mathcal{C}_{sa}$,

$$P(x)=\sup_{I(\mathcal{D})}\big\{\tilde{T}(x):T\in\mathrm{Ext}(\partial Q)\big\}=\sup_{I(\mathcal{D})}\big\{\Psi(x):\Psi\in\mathrm{Ext}\big(\mathrm{PsExp}(\mathcal{C},\mathcal{D})\big)\big\}.$$

If there exists a unique $\Phi\in\mathrm{PsExp}(\mathcal{C},\mathcal{D})$, then clearly there exists a unique $\Psi\in\mathrm{Ext}(\mathrm{PsExp}(\mathcal{C},\mathcal{D}))$. Conversely, suppose there exists a unique $\Psi\in\mathrm{Ext}(\mathrm{PsExp}(\mathcal{C},\mathcal{D}))$. Then for all $\Phi\in\mathrm{PsExp}(\mathcal{C},\mathcal{D})$,

$$\begin{aligned}x\in\mathcal{C}_{sa}\quad&\Longrightarrow\quad \pm\Phi(x)=\Phi(\pm x)\leq P(\pm x)=\Psi(\pm x)=\pm\Psi(x)\\&\Longrightarrow\quad \Phi(x)=\Psi(x),\end{aligned}$$

and so $\Phi=\Psi$. ∎

3.7. Abelian inclusions. In this section, we consider the unique pseudo-expectation property for C^*-inclusions $(\mathcal{A},\mathcal{D})$, with $\mathcal{A}$ Abelian. By Gelfand duality, these are precisely the C^*-inclusions $(C(Y),C(X))$, where X and Y are compact Hausdorff spaces. We recall that unital $*$-monomorphisms $\pi : C(X) \to C(Y)$ correspond bijectively to continuous surjections $j : Y \to X$. Indeed, if $j : Y \to X$ is a continuous surjection, then $\pi_j : C(X) \to C(Y) : f \mapsto f \circ j$ is a unital $*$-monomorphism. We may identify $\pi_j(C(X))$ with the continuous functions on Y which are constant on the fibers $j^{-1}(x)$, $x \in X$. Conversely, if $\pi : C(X) \to C(Y)$ is a unital $*$-monomorphism, then for each $y \in Y$ there exists a unique $j(y) \in X$ such that $\delta_y \circ \pi = \delta_{j(y)}$, and it is easy to verify that $j : Y \to X$ is a continuous surjection such that $\pi_j = \pi$.

In light of Theorem 3.18, to characterize when $\mathrm{PsExp}(\mathcal{A},\mathcal{D})$ is a singleton, it suffices to characterize when $\mathrm{Ext}(\mathrm{PsExp}(\mathcal{A},\mathcal{D}))$ is a singleton. As we saw in Proposition 2.4,

$$\mathrm{Ext}\big(\mathrm{PsExp}(\mathcal{A},\mathcal{D})\big) \subseteq \mathrm{Ext}\big(\mathrm{UCP}\big(\mathcal{A}, I(\mathcal{D})\big)\big).$$

On the other hand, $\Psi \in \mathrm{Ext}(\mathrm{UCP}(\mathcal{A}, I(\mathcal{D})))$ iff Ψ is multiplicative (i.e., a unital $*$-homomorphism) [35, Cor. 3.1.6]. Thus, the extreme pseudo-expectations for $(\mathcal{A},\mathcal{D})$ are precisely the *multiplicative pseudo-expectations*:

$$\mathrm{Ext}\big(\mathrm{PsExp}(\mathcal{A},\mathcal{D})\big) = \mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D}).$$

THEOREM 3.19. *Let $(\mathcal{A},\mathcal{D})$ be an Abelian inclusion. Then the mapping*

$$\mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D}) \to \left\{ \begin{matrix} \textit{maximal } \mathcal{D}\textit{-disjoint} \\ \textit{ideals of } \mathcal{A} \end{matrix} \right\} \quad \textit{given by} \quad \Psi \mapsto \ker(\Psi)$$

is a bijection. In particular, $\mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D})$ *is a singleton iff there exists a unique maximal $\mathcal{D}$-disjoint ideal $\mathcal{I} \lhd \mathcal{A}$.*

Proof. Let $\Psi \in \mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D})$. Then $\ker(\Psi)$ is a $\mathcal{D}$-disjoint ideal of $\mathcal{A}$, and the map $\mathcal{I} \mapsto \Psi(\mathcal{I})$ is an order-preserving bijection between the $\mathcal{D}$-disjoint ideals of $\mathcal{A}$ containing $\ker(\Psi)$ and the $\mathcal{D}$-disjoint ideals of $\Psi(\mathcal{A})$. Since $(I(\mathcal{D}),\mathcal{D})$ has the faithful unique pseudo-expectation property (Example 2.11), it is hereditarily essential, by Theorem 3.5. Thus $(\Psi(\mathcal{A}),\mathcal{D})$ is essential, so that the only $\mathcal{D}$-disjoint ideal of $\Psi(\mathcal{A})$ is 0. It follows that the only $\mathcal{D}$-disjoint ideal of $\mathcal{A}$ containing $\ker(\Psi)$ is $\ker(\Psi)$ itself, which says that $\ker(\Psi)$ is a maximal $\mathcal{D}$-disjoint ideal of $\mathcal{A}$.

Conversely, suppose $\mathcal{I} \subseteq \mathcal{A}$ is a maximal $\mathcal{D}$-disjoint ideal. The map $\Psi_0 : \mathcal{I} + \mathcal{D} \to \mathcal{D} : h + d \mapsto d$ is a unital $*$-homomorphism. Since $I(\mathcal{D})$ is an injective unital Abelian C^*-algebra, there exists a unital $*$-homomorphism $\Psi : \mathcal{A} \to I(\mathcal{D})$ such that $\Psi|_{\mathcal{I}+\mathcal{D}} = \Psi_0$ [12]. Clearly, $\Psi \in \mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D})$ and $\mathcal{I} \subseteq \ker(\Psi) \subseteq \mathcal{A}$ is a $\mathcal{D}$-disjoint ideal. Thus, $\ker(\Psi) = \mathcal{I}$, by maximality.

Finally, suppose $\Psi_1, \Psi_2 \in \mathrm{PsExp}^{\times}(\mathcal{A},\mathcal{D})$, with $\ker(\Psi_1) = \ker(\Psi_2)$. Define $\iota : \Psi_1(\mathcal{A}) \to \Psi_2(\mathcal{A})$ by the formula $\iota(\Psi_1(x)) = \Psi_2(x)$, $x \in \mathcal{A}$. Then ι is a unital $*$-isomorphism which fixes $\mathcal{D}$. By injectivity, there exists an unital $*$-homomorphism $\bar{\iota}: I(\mathcal{D}) \to I(\mathcal{D})$ such that $\bar{\iota}|_{\Psi_1(\mathcal{A})} = \iota$. By the rigidity of the injective envelope, $\bar{\iota} = \text{id}$, so that $\Psi_1 = \Psi_2$. □

REMARK 3.20. Taking $\mathcal{D} = \mathbb{C}$ in Theorem 3.19 above, one recovers the well-known bijective correspondence between the characters and the maximal ideals of $\mathcal{A}$.

COROLLARY 3.21. *Let $(C(Y), C(X))$ be an Abelian inclusion with corresponding continuous surjection $j: Y \to X$. The following four conditions are equivalent*:

(i) *There exists a unique pseudo-expectation for $(C(Y), C(X))$.*
(ii) *There exists a unique multiplicative pseudo-expectation for $(C(Y), C(X))$.*
(iii) *There exists a unique maximal $C(X)$-disjoint ideal in $C(Y)$.*
(iv) *There exists a unique minimal closed set $K \subseteq Y$ such that $j(K) = X$.*

Proof. (i) $\iff$ (ii) follows from Theorem 3.18.

(ii) $\iff$ (iii) Theorem 3.19.

(iii) $\iff$ (iv) The map $K \mapsto \{g \in C(Y) : g|_K = 0\}$ is an order-reversing bijection between the closed sets $K \subseteq Y$ such that $j(K) = X$ and the $C(X)$-disjoint ideals in $C(Y)$. □

COROLLARY 3.22. *Let $(C(Y), C(X))$ be an Abelian inclusion with corresponding continuous surjection $j: Y \to X$. Then the following are equivalent*:

(i) *There exists a unique pseudo-expectation for $(C(Y), C(X))$, which is faithful.*
(ii) *There exists a unique multiplicative pseudo-expectation for $(C(Y), C(X))$, which is faithful.*
(iii) *$(C(Y), C(X))$ is essential.*
(iv) *If $K \subseteq Y$ is closed and $j(K) = X$, then $K = Y$.*

Proof. Same proof as Corollary 3.21. □

4. Examples

Now we provide additional examples of C^*-inclusions with (and without) the unique pseudo-expectation property (resp. the faithful unique pseudo-expectation property). In proving that these examples are actually examples, we will take advantage of some of the general theory developed so far.

In Section 2.1, we mentioned that the unique pseudo-expectation property is not hereditary from below. Equipped with the results of the previous section, it is easy to give an example which demonstrates this.

EXAMPLE 4.1. There exist Abelian inclusions $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq \mathcal{A}$ such that $(\mathcal{A}, \mathcal{D})$ has the unique pseudo-expectation property, but $(\mathcal{A}, \mathcal{D}_0)$ does not. That is, the unique pseudo-expectation property is not hereditary from below.

Proof. Let $X=[0,1]$, $X_0=[0,1]\cup\{2\}$, and $Y=[0,1]\cup\{2,3\}$. Define continuous surjections $j:X_0\to X$ and $k:Y\to X_0$ by the formulas

$$j(t)=\begin{cases}t, & t\in[0,1],\\ 1, & t=2\end{cases} \quad\text{and}\quad k(t)=\begin{cases}t, & t\in[0,1],\\ 2, & t\in\{2,3\}.\end{cases}$$

Then $i=j\circ k:Y\to X$ is the continuous surjection

$$i(t)=\begin{cases}t, & t\in[0,1],\\ 1, & t\in\{2,3\}.\end{cases}$$

Clearly there exists a unique minimal closed set $K\subseteq Y$ such that $i(K)=X$, namely $K=[0,1]$. On the other hand, there are multiple minimal closed sets $L\subseteq Y$ such that $k(L)=X_0$, for example both $L=[0,1]\cup\{2\}$ and $L=[0,1]\cup\{3\}$. Thus by Corollary 3.21, we have inclusions $C(X)\subseteq C(X_0)\subseteq C(Y)$ such that $(C(Y),C(X))$ has the unique pseudo-expectation property, but $(C(Y),C(X_0))$ does not. □

We can also provide examples of the poor behavior of the unique pseudo-expectation property with respect to quotients described in Section 3.3. To that end, let $(C(Y),C(X))$ be an Abelian inclusion, with corresponding continuous surjection $j:Y\to X$. Suppose $Z\subseteq Y$ is closed and $\mathcal{J}=\{g\in C(Y): g|_Z=0\}\lhd C(Y)$. Then $\mathcal{J}\cap C(X)=\{f\in C(X): f|_{j(Z)}=0\}\lhd C(X)$. Thus

$$\big(C(Y)/\mathcal{J},C(X)/\big(\mathcal{J}\cap C(X)\big)\big)\cong\big(C(Z),C\big(j(Z)\big)\big),$$

with corresponding continuous surjection $j|_Z:Z\to j(Z)$. Furthermore, $\mathcal{J}\cap C(X)\lhd C(X)$ is regular iff $\overline{j(Z)^\circ}=j(Z)$, where the interior and closure are calculated in X.

Example 4.2. There exists an Abelian inclusion $(\mathcal{A},\mathcal{D})$ and $\mathcal{J}\lhd\mathcal{A}$ such that $(\mathcal{A},\mathcal{D})$ has the unique pseudo-expectation property, but $(\mathcal{A}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ does not. Of course, $\mathcal{J}\cap\mathcal{D}\lhd\mathcal{D}$ is not a regular ideal.

Proof. Let $Y=([0,1]\times\{0\})\cup(\{1\}\times[0,1])\subseteq[0,1]\times[0,1]$, $X=[0,1]$, and $j:Y\to X$ be defined by the formula $j(s,t)=s$, $(s,t)\in Y$. Then there exists a unique minimal closed set $K\subseteq Y$ such that $j(K)=X$, namely $K=[0,1]\times\{0\}$. Thus, $(C(Y),C(X))$ has the unique pseudo-expectation property, by Corollary 3.21. Now let $Z=\{1\}\times[0,1]$, a closed subset of Y. Then $j(Z)=\{1\}$, and there does not exist a unique minimal closed set $L\subseteq Z$ such that $j(L)=j(Z)$. Thus, $(C(Z),C(j(Z))$ does not have the unique pseudo-expectation property. Of course $\overline{j(Z)^\circ}=\emptyset\subsetneq j(Z)$. □

Example 4.3. There exists an Abelian inclusion $(\mathcal{A},\mathcal{D})$ and $\mathcal{J}\lhd\mathcal{A}$ such that $(\mathcal{A},\mathcal{D})$ has the faithful unique pseudo-expectation property and $\mathcal{J}\cap\mathcal{D}$ is a regular ideal in $\mathcal{D}$, but $(\mathcal{A}/\mathcal{J},\mathcal{D}/(\mathcal{J}\cap\mathcal{D}))$ does not have the faithful unique pseudo-expectation property.

Proof. Let $Y = ([0,1/2]\times\{0\})\cup([1/2,1]\times\{1\})\subseteq[0,1]\times[0,1]$, $X=[0,1]$, and $j:Y\to X$ be defined by the formula $j(s,t)=s$, $(s,t)\in Y$. Then there exists a unique minimal closed set $K\subseteq Y$ such that $j(K)=X$, namely $K=Y$. Thus $(C(Y),C(X))$ has the faithful unique pseudo-expectation property, by Corollary 3.22. Now let $Z=([0,1/2]\times\{0\})\cup\{(1/2,1)\}$, a closed subset of Y. Then $j(Z)=[0,1/2]$, so that $\overline{j(Z)^\circ}=j(Z)$. There exists a unique minimal closed set $L\subseteq Z$ such that $j(L)=j(Z)$, namely $L=[0,1/2]\times\{0\}$. Since $L\subsetneq Z$, $(C(Z),C(j(Z)))$ has the unique pseudo-expectation property, but not the faithful unique pseudo-expectation property. ☐

In the Introduction we mentioned that the inclusion $(B(L^2[0,1]),C[0,1])$ admits no conditional expectations (Example 1.3). An interesting question (posed to us by Philip Gipson) is whether or not $(B(L^2[0,1]),C[0,1])$ has a unique pseudo-expectation. It turns out that even the Abelian inclusion $(L^\infty[0,1],C[0,1])$ admits multiple pseudo-expectations. We found it difficult to fit this example into the context of Corollary 3.21, so we utilize Theorem 3.16 instead.

Example 4.4. The Abelian inclusion $(L^\infty[0,1],C[0,1])$ has infinitely many pseudo-expectations, none of which are faithful.

Proof. Let $B^\infty[0,1]$ be the C^*-algebra of bounded complex-valued Borel functions on $[0,1]$. Let $N[0,1]\lhd B^\infty[0,1]$ be the Lebesgue-null functions, so that $B^\infty[0,1]/N[0,1]=L^\infty[0,1]$. Likewise, let $M[0,1]\lhd B^\infty[0,1]$ be the meager functions, so that $B^\infty[0,1]/M[0,1]=D[0,1]$, the *Dixmier algebra*. Recall that $D[0,1]=I(C[0,1])$ [12].

Now let $A\subseteq[0,1]$ be a Borel set such that both A and A^c are measure dense, meaning that $|V\cap A|>0$ and $|V\cap A^c|>0$ for every open set $V\subseteq[0,1]$. (Here $|\cdot|$ stands for Lebesgue measure.) One possible construction of A can be found in [33].

A measure-theoretic argument shows that if $f\in C[0,1]_{sa}$ and $f+N[0,1]\le\chi_A+N[0,1]$, then $f\le 0$. Likewise, if $f\in C[0,1]_{sa}$ and $f+N[0,1]\ge\chi_A+N[0,1]$, then $f\ge 1$.

It follows that

$$\sup_{D[0,1]}\left\{f+M[0,1]: f\in C[0,1]_{sa}, f+N[0,1]\le\chi_A+N[0,1]\right\}=0+M[0,1]$$

and

$$\inf_{D[0,1]}\left\{f+M[0,1]: f\in C[0,1]_{sa}, f+N[0,1]\ge\chi_A+N[0,1]\right\}=1+M[0,1].$$

By Theorem 3.16, $(L^\infty[0,1],C[0,1])$ does not have the unique pseudo-expectation property.

It remains to show that no pseudo-expectation for $(L^\infty[0,1],C[0,1])$ is faithful. By [12, Thm. 2.21], there are C^*-inclusions $C[0,1]\subseteq I(C[0,1])\subseteq L^\infty[0,1]$. Suppose Φ is a faithful pseudo-expectation for $(L^\infty[0,1],C[0,1])$. Then

$\Phi : L^\infty[0,1] \to I(C[0,1])$ is a ucp map such that $\Phi|_{C[0,1]} = \mathrm{id}$. By the rigidity of the injective envelope, $\Phi|_{I(C[0,1])} = \mathrm{id}$. Thus, Φ is a faithful conditional expectation of $L^\infty[0,1]$ onto $I(C[0,1])$. It follows from [17, Lemma 1] that $D[0,1] = I(C[0,1])$ is a W^*-algebra, contradicting [19, Exercise 5.7.21]. □

Transformation group C^*-algebras. Let Γ be a discrete group acting on a compact Hausdorff space X by homeomorphisms, and let $C(X) \rtimes_r \Gamma$ be the corresponding reduced crossed product (see [5, Ch. 4] for more details). In this section, we examine the unique pseudo-expectation property for the inclusions $(C(X) \rtimes_r \Gamma, C(X))$ and $(C(X) \rtimes_r \Gamma, C(X)^c)$. We recall that elements of $C(X) \rtimes_r \Gamma$ have formal series representations $\sum_{t\in\Gamma} a_t\lambda_t$, where $a_t \in C(X)$ for all $t \in \Gamma$, and that there exists a faithful conditional expectation $E : C(X) \rtimes_r \Gamma \to C(X)$, namely

$$E\left(\sum_{t\in\Gamma} a_t\lambda_t\right) = a_e.$$

For $s \in \Gamma$, write $F_s = \{x \in X : sx = x\}$ for the *fixed points* of s, and for $x \in X$, let

$$H^x := \{s \in \Gamma : x \in (F_s)^\circ\}.$$

The condition that H^x is Abelian for every $x \in X$ is equivalent to the condition that $C(X)^c$ is Abelian [28, Theorem 6.6]. The following result shows that when either of these equivalent conditions hold, then $(C(X) \rtimes_r \Gamma, C(X)^c)$ has the faithful unique pseudo-expectation property.

Proposition 4.5 ([28, Theorem 6.10]). *If $C(X)^c$ is Abelian, then $(C(X) \rtimes_r \Gamma, C(X)^c)$ has the faithful unique pseudo-expectation property. In particular, this happens when Γ is Abelian.*

In general, we do not know a characterization of when $(C(X) \rtimes_r \Gamma, C(X)^c)$ has the faithful unique pseudo-expectation property, or even the unique pseudo-expectation property. However, we do have the following result for the inclusion $(C(X) \rtimes_r \Gamma, C(X))$.

Theorem 4.6. *The following are equivalent:*

(i) *$(C(X) \rtimes_r \Gamma, C(X))$ has the unique pseudo-expectation property.*
(ii) *$(C(X) \rtimes_r \Gamma, C(X))$ has the faithful unique pseudo-expectation property.*
(iii) *$C(X)^c = C(X)$ (i.e., $C(X)$ is a MASA in $C(X) \rtimes_r \Gamma$).*
(iv) *The action of Γ on X is topologically free (i.e., $(F_t)^\circ = \emptyset$ for all $e \neq t \in \Gamma$).*

If in addition, $C(X) \rtimes_r \Gamma = C(X) \rtimes \Gamma$ (e.g. when Γ is amenable), conditions (i) *through* (iv) *are equivalent to,*

(v) *$(C(X) \rtimes_r \Gamma, C(X))$ is essential.*

Proof. (i) $\Longrightarrow$ (ii) If $(C(X) \rtimes_r \Gamma, C(X))$ has the unique pseudo-expectation property, then

$$\mathrm{PsExp}\big(C(X) \rtimes_r \Gamma, C(X)\big) = \{E\},$$

and E is faithful.

(ii) $\implies$ (iii) Suppose $(C(X)\rtimes_r\Gamma, C(X))$ has the faithful unique pseudo-expectation property. By Corollary 3.14, $C(X)^c$ is Abelian. By Proposition 2.6, $E|_{C(X)^c}$ is the unique pseudo-expectation for $(C(X)^c, C(X))$, and so $E|_{C(X)^c}$ is multiplicative by Corollary 3.21. It follows that $E|_{C(X)^c} = \mathrm{id}$, so that $C(X)^c = C(X)$.

(iii) $\implies$ (i) Since $C(X)^c = C(X)$, $(C(X)\rtimes_r\Gamma, C(X))$ is a regular MASA inclusion. Thus $(C(X)\rtimes_r\Gamma, C(X))$ has the unique pseudo-expectation property, by Pitts' Theorem 1.4.

(iii) $\iff$ (iv) By [28, Prop. 6.3]

$$C(X)^c = \left\{\sum_{t\in\Gamma} a_t\lambda_t \in C(X)\rtimes_r\Gamma : \mathrm{supp}(a_t)\subseteq (F_t)^\circ, t\in\Gamma\right\}.$$

Thus $C(X)^c = C(X)$ if and only if $(F_t)^\circ = \emptyset$ for all $e\neq t\in\Gamma$, if and only if the action of Γ on X is topologically free.

Finally, assume that $C(X)\rtimes\Gamma = C(X)\rtimes_r\Gamma$. Theorem 3.5 shows that (ii) implies (v), and [21, Thm. 4.1] gives (v) implies (iv). □

5. W^*-inclusions

In this section, we investigate the unique pseudo-expectation property for W^*-inclusions $(\mathcal{M},\mathcal{D})$. This means that $(\mathcal{M},\mathcal{D})$ is a C^*-inclusion such that $\mathcal{M}$ is a W^*-algebra and $\mathcal{D}$ is $\sigma(\mathcal{M},\mathcal{M}_*)$-closed.

First we consider Abelian W^*-inclusions. Corollary 3.21 above shows that there exist nontrivial Abelian C^*-inclusions $(C(Y), C(X))$ with the unique pseudo-expectation property. Not so for Abelian W^*-inclusions, due to the abundance of normal states.

THEOREM 5.1. *Let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion.*

(i) *Suppose $\mathcal{M}$ is Abelian. Then $(\mathcal{M},\mathcal{D})$ has the unique pseudo-expectation property iff $\mathcal{M}=\mathcal{D}$.*
(ii) *More generally, let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion, with $\mathcal{D}$ Abelian (and $\mathcal{M}$ possibly non-Abelian). If $(\mathcal{M},\mathcal{D})$ has the unique pseudo-expectation property, then $\mathcal{D}$ is a MASA in $\mathcal{M}$.*

Observe that because $\mathcal{D}$ is an Abelian von Neumann algebra, the pseudo-expectations in the theorem are conditional expectations.

Proof. (i) Suppose $\mathrm{PsExp}(\mathcal{M},\mathcal{D}) = \{E\}$. Let $a\in\mathcal{M}_{sa}$. Since $\mathcal{M}$ is Abelian,

$$\{d\in\mathcal{D}_{sa} : d\leq a\}$$

is an increasing net indexed by itself. Indeed, if $f,g\leq h$ are continuous functions, then $\max\{f,g\}\leq h$. By Theorem 3.16, we have that

$$E(a) = \sup_{\mathcal{D}}\{d\in\mathcal{D}_{sa} : d\leq a\}.$$

Now let $\phi \in (\mathcal{D}_*)_+$ and $\overline{\phi} \in (\mathcal{M}^*)_+$ be an extension. Then

$$\phi\big(E(a)\big) = \sup\big\{\phi(d) : d \leq a\big\},$$

by normality. On the other hand, if $d \leq a$, then $\phi(d) = \overline{\phi}(d) \leq \overline{\phi}(a)$, which implies $\phi(E(a)) \leq \overline{\phi}(a)$. Replacing a by $-a$, we conclude that $\phi(E(a)) = \overline{\phi}(a)$, and so $\overline{\phi} = \phi \circ E$. Thus if $a \in \mathcal{M}$ and $\psi \in (\mathcal{M}_*)_+$, then $\psi = \psi|_{\mathcal{D}} \circ E$, which implies $\psi(a) = \psi(E(a))$. Since the choice of ψ was arbitrary, $a = E(a) \in \mathcal{D}$.

(ii) Let $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$ be a MASA. Since $(\mathcal{M}, \mathcal{D})$ has the unique pseudo-expectation property, so does $(\mathcal{A}, \mathcal{D})$, by Proposition 2.6. Then $\mathcal{A} = \mathcal{D}$, by (i) above. □

Our next objective is to generalize Theorem 5.1, by showing that for an arbitrary W^*-inclusion $(\mathcal{M}, \mathcal{D})$, the unique pseudo-expectation property implies that $\mathcal{D}^c = Z(\mathcal{D})$ (Corollary 5.3). Our proof relies on a nice bijective correspondence between the conditional expectations $\mathcal{D}^c \to Z(\mathcal{D})$ and the conditional expectations $C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$ (Theorem 5.2). Theorem 5.2 is related to [8, Thm. 5.3], but to our knowledge, is new. Our proof of Theorem 5.2 uses some fairly recent technology, which we now describe.

Let $\mathcal{M}$, $\mathcal{N}$ be W^*-algebras and $\mathcal{Z} \subseteq Z(\mathcal{M}) \cap Z(\mathcal{N})$ be a W^*-subalgebra. By [4], [11], there exist on the $\mathcal{Z}$-balanced algebraic tensor product $\mathcal{M} \otimes_{\mathcal{Z}} \mathcal{N}$ both a minimal C^*-norm $\|\cdot\|_{\min}$ and a maximal C^*-norm $\|\cdot\|_{\max}$, which coincide if either $\mathcal{M}$ or $\mathcal{N}$ is Abelian. When $\mathcal{Z} = \mathbb{C}$, this fact is now classical, see [36, Chapter IV.4]. Now suppose that for $i = 1, 2$, $\mathcal{M}_i$, $\mathcal{N}_i$ are W^*-algebras and $\mathcal{Z} \subseteq Z(\mathcal{M}_i) \cap Z(\mathcal{N}_i)$ is a W^*-subalgebra. If $u : \mathcal{M}_1 \to \mathcal{M}_2$ and $v : \mathcal{N}_1 \to \mathcal{N}_2$ are completely contractive $\mathcal{Z}$-bimodule maps, then the unique $\mathcal{Z}$-bimodule map $u \otimes v : (\mathcal{M}_1 \otimes_{\mathcal{Z}} \mathcal{N}_1, \|\cdot\|_{\min}) \to (\mathcal{M}_2 \otimes_{\mathcal{Z}} \mathcal{N}_2, \|\cdot\|_{\min})$ such that $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$ for all $x \in \mathcal{M}_1$, $y \in \mathcal{N}_1$ is a contraction. Furthermore, if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $\mathcal{N}_1 \subseteq \mathcal{N}_2$, then $(\mathcal{M}_1 \otimes_{\mathcal{Z}} \mathcal{N}_1, \|\cdot\|_{\min}) \subseteq (\mathcal{M}_2 \otimes_{\mathcal{Z}} \mathcal{N}_2, \|\cdot\|_{\min})$.

THEOREM 5.2. *Let $(\mathcal{M}, \mathcal{D})$ be a W^*-inclusion. Then the map $E \mapsto E|_{\mathcal{D}^c}$ is a bijective correspondence between the conditional expectations $C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$ and the conditional expectations $\mathcal{D}^c \to Z(\mathcal{D})$. In particular, there exists a conditional expectation $C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$.*

Proof. Let $E : C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$ be a conditional expectation. For all $d' \in \mathcal{D}^c$ and $d \in \mathcal{D}$, we have that

$$dE\big(d'\big) = E\big(dd'\big) = E\big(d'd\big) = E\big(d'\big)d,$$

which implies $E(d') \in Z(\mathcal{D})$. Conversely, suppose $\theta : \mathcal{D}^c \to Z(\mathcal{D})$ is a conditional expectation. By [19, Thm. 5.5.4],

$$\mathcal{D} \otimes_{Z(\mathcal{D})} \mathcal{D}^c \to \mathrm{Alg}\big(\mathcal{D}, \mathcal{D}^c\big) \subseteq \mathcal{M} : \sum_{i=1}^{n} d_i \otimes d_i' \mapsto \sum_{i=1}^{n} d_i d_i'$$

is a $*$-isomorphism. Thus, (with the notation of the previous paragraph)

$$\left\|\sum_{i=1}^n d_i \otimes d_i'\right\|_{\min} \leq \left\|\sum_{i=1}^n d_i d_i'\right\| \leq \left\|\sum_{i=1}^n d_i \otimes d_i'\right\|_{\max}, \quad \sum_{i=1}^n d_i \otimes d_i' \in \mathcal{D} \otimes_{Z(\mathcal{D})} \mathcal{D}^c.$$

Furthermore, since $Z(\mathcal{D})$ is Abelian,

$$\left\|\sum_{i=1}^n d_i \otimes z_i\right\|_{\min} = \left\|\sum_{i=1}^n d_i z_i\right\|, \quad \sum_{i=1}^n d_i \otimes z_i \in \mathcal{D} \otimes_{Z(\mathcal{D})} Z(\mathcal{D}).$$

Thus for all $\sum_{i=1}^n d_i \otimes d_i' \in \mathcal{D} \otimes_{Z(\mathcal{D})} \mathcal{D}^c$,

$$\left\|\sum_{i=1}^n d_i \theta\big(d_i'\big)\right\| = \left\|\sum_{i=1}^n d_i \otimes \theta\big(d_i'\big)\right\|_{\min} \leq \left\|\sum_{i=1}^n d_i \otimes d_i'\right\|_{\min} \leq \left\|\sum_{i=1}^n d_i d_i'\right\|.$$

It follows that the map

$$\mathrm{Alg}\big(\mathcal{D}, \mathcal{D}^c\big) \to \mathrm{Alg}\big(\mathcal{D}, Z(\mathcal{D})\big) = \mathcal{D} : \sum_{i=1}^n d_i d_i' \mapsto \sum_{i=1}^n d_i \theta\big(d_i'\big)$$

extends uniquely to a conditional expectation $\Theta : C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$. Clearly the maps $E \mapsto E|_{\mathcal{D}^c}$ and $\theta \mapsto \Theta$ described above are inverse to one another. $\square$

Recall that in the case of a C^*-inclusion, the faithful unique pseudo-expectation property implies that $\mathcal{D}^c$ is Abelian, but we do not know whether the faithfulness assumption can be dropped. However, the following corollary to Theorem 5.2 shows that in the W^*-case, faithfulness is not necessary to conclude $\mathcal{D}^c$ is Abelian. In fact, more is true.

THEOREM 5.3. *Let $(\mathcal{M}, \mathcal{D})$ be a W^*-inclusion. If $(\mathcal{M}, \mathcal{D})$ has the unique pseudo-expectation property, then $\mathcal{D}^c = Z(\mathcal{D})$.*

Proof. If $(\mathcal{M}, \mathcal{D})$ has the unique pseudo-expectation property, then so does $(C^*(\mathcal{D}, \mathcal{D}^c), \mathcal{D})$, by Proposition 2.6. By Theorem 5.2, there exists a unique conditional expectation $C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$, therefore a unique conditional expectation $\mathcal{D}^c \to Z(\mathcal{D})$. By Theorem 5.1, $Z(\mathcal{D})' \cap \mathcal{D}^c = Z(\mathcal{D})$. But

$$Z(\mathcal{D})' \cap \mathcal{D}^c = Z(\mathcal{D})' \cap \mathcal{D}' \cap \mathcal{M} = \mathcal{D}' \cap \mathcal{M} = \mathcal{D}^c. \qquad \square$$

We now turn to our main purpose in this section—characterizing the unique pseudo-expectation property for various classes of W^*-inclusions (see Theorems 5.5 and 5.6).

The statement of Theorem 5.6 involves the tracial ultrapower construction, which we recall for the reader. Let $\mathcal{M}$ be a II_1 factor with trace τ, and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. The *tracial ultrapower* of $\mathcal{M}$ with respect to ω is defined to be $\mathcal{M}^\omega = \ell^\infty(\mathcal{M})/\mathcal{I}_\omega$, where

$$\mathcal{I}_\omega = \Big\{(x_n) \in \ell^\infty(\mathcal{M}) : \lim_\omega \tau\big(x_n^* x_n\big) = 0\Big\} \lhd \ell^\infty(\mathcal{M}).$$

It can be shown that $\mathcal{M}^\omega$ itself is a II_1 factor with trace

$$\tau_\omega\big((x_n)+\mathcal{I}_\omega\big)=\lim_\omega \tau(x_n).$$

The map $\mathcal{M}\to\mathcal{M}^\omega : x\mapsto (x)+\mathcal{I}_\omega$ is an embedding. If $\mathcal{D}\subseteq\mathcal{M}$ is a MASA, then $\mathcal{D}^\omega=(\ell^\infty(\mathcal{D})+\mathcal{I}_\omega)/\mathcal{I}_\omega\subseteq\mathcal{M}^\omega$ is a MASA. See [34, Appendix A] for more details.

The proofs of Theorems 5.5 and 5.6 require some standard facts about conditional expectations, which we collect into a proposition for the reader's convenience.

Proposition 5.4.

(i) *Let I be an index set and for $i\in I$, let $\mathcal{M}_i\subseteq B(\mathcal{H}_i)$ be a W^*-algebra. Then there exists a bijective correspondence between families of conditional expectations $\{E_i : B(\mathcal{H}_i)\to\mathcal{M}_i\}_{i\in I}$ and conditional expectations $\theta : B(\bigoplus_{i\in I}\mathcal{H}_i)\to\bigoplus_{i\in I}\mathcal{M}_i$. Namely*

$$\theta\big([x_{ij}]\big)=\bigoplus_{i\in I}E_i(x_{ii}).$$

We have that θ is normal (resp. faithful) iff every E_i is normal (resp. faithful).

(ii) *For $i=1,2$, let $(\mathcal{M}_i,\mathcal{D}_i)$ be a W^*-inclusion and $E_i:\mathcal{M}_i\to\mathcal{D}_i$ be a conditional expectation. Then there exists a conditional expectation $E:\mathcal{M}_1\,\overline{\otimes}\,\mathcal{M}_2\to\mathcal{D}_1\,\overline{\otimes}\,\mathcal{D}_2$ such that $E(x_1\otimes x_2)=E_1(x_1)\otimes E_2(x_2)$ for all $x_1\in\mathcal{M}_1$, $x_2\in\mathcal{M}_2$. If E_1 and E_2 are normal, then there exists a unique normal conditional expectation E as above.* [38, Thm. 4]

(iii) *Let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion. Then there exists a bijective correspondence between conditional expectations $E:\mathcal{M}\to\mathcal{D}$ and conditional expectations $\theta : B(\mathcal{K})\,\overline{\otimes}\,\mathcal{M}\to B(\mathcal{K})\,\overline{\otimes}\,\mathcal{D}$. Namely*

$$\theta\big([x_{ij}]\big)=\big[E(x_{ij})\big]\,.$$

We have that E is normal (resp. faithful) iff θ is normal (resp. faithful).

Now we come to the main results of this section. The first characterizes the unique pseudo-expectation property for W^*-inclusions of the form $(\mathcal{B}(\mathcal{H}),\mathcal{D})$, and the second characterizes the unique pseudo-expectation property for W^*-inclusions $(\mathcal{M},\mathcal{D})$ when $\mathcal{M}_*$ is separable and $\mathcal{D}$ is Abelian. In the latter result, the separability hypothesis cannot be removed.

Theorem 5.5.

(i) *Let $\mathcal{A}\subseteq B(\mathcal{H})$ be an Abelian W^*-algebra. Then $(B(\mathcal{H}),\mathcal{A})$ has the unique pseudo-expectation property iff $\mathcal{A}$ is an atomic MASA. The unique pseudo-expectation is a normal faithful conditional expectation.*

(ii) *Generalizing* (i), *let* $\mathcal{M} \subseteq B(\mathcal{H})$ *be a* W^**-algebra. Then* $(B(\mathcal{H}), \mathcal{M})$ *has the unique pseudo-expectation property iff* $\mathcal{M}'$ *is Abelian and atomic. The unique pseudo-expectation is a normal faithful conditional expectation. In particular*, $\mathcal{M}$ *is type* I (*and injective*).

Proof. (i) By Theorem 5.1, we may assume that $\mathcal{A} \subseteq B(\mathcal{H})$ is a MASA. We have the unitary equivalence

$$\mathcal{A} = \mathcal{A}_{\text{atomic}} \oplus \mathcal{A}_{\text{diffuse}},$$

where $\mathcal{A}_{\text{atomic}}$ is spatially isomorphic to $\ell^\infty(\kappa)$ acting on $\ell^2(\kappa)$ for some index set κ, and $\mathcal{A}_{\text{diffuse}}$ is spatially isomorphic to $\bigoplus_{i \in I} L^\infty([0,1]^{\alpha_i})$ acting on $\bigoplus_{i \in I} L^2([0,1]^{\alpha_i})$ for some index set I and cardinals α_i, $i \in I$ (see [25]). It is easy to see that for κ nonempty, there exists a unique conditional expectation $B(\ell^2(\kappa)) \to \ell^\infty(\kappa)$, which is normal and faithful. On the other hand, for any nonzero cardinal α, there are multiple conditional expectations $B(L^2([0,1]^\alpha)) \to L^\infty([0,1]^\alpha)$. Indeed, this is well known when $\alpha = 1$ (Example 2.9), and follows from Proposition 5.4 (ii) and the unitary equivalence

$$L^\infty\big([0,1]^\alpha\big) = L^\infty\big([0,1]\big) \mathbin{\overline{\otimes}} L^\infty\big([0,1]^\beta\big) \subseteq B\big(L^2\big([0,1]\big)\big) \mathbin{\overline{\otimes}} B\big(L^2\big([0,1]^\beta\big)\big)$$

when $\alpha > 1$ (here $\beta = \alpha - 1$ if α is finite, and $\beta = \alpha$ if α is infinite). The result now follows from Proposition 5.4(i).

(ii) By Theorem 5.3, we may assume that $\mathcal{M}' = Z(\mathcal{M})$, so that

$$\mathcal{H} = \bigoplus_m \ell^2_m \otimes \mathcal{H}_m, \qquad \mathcal{M}' = \bigoplus_m I_m \mathbin{\overline{\otimes}} \mathcal{A}_m, \quad \text{and} \quad \mathcal{M} = \bigoplus_m B\big(\ell^2_m\big) \mathbin{\overline{\otimes}} \mathcal{A}_m,$$

where $\mathcal{A}_m \subseteq B(\mathcal{H}_m)$ is a MASA for each m. In particular, $\mathcal{M}$ is injective, so that pseudo-expectations for $(B(\mathcal{H}), \mathcal{M})$ are actually conditional expectations. By Proposition 5.4(i) and (iii), there exists a unique conditional expectation $B(\mathcal{H}) \to \mathcal{M}$ iff there exist unique conditional expectations $B(\mathcal{H}_m) \to \mathcal{A}_m$ for each m, iff $\mathcal{A}_m$ is atomic for each m (by part (i) above). The result now follows. □

THEOREM 5.6.

(i) *Let* $(\mathcal{M}, \mathcal{D})$ *be a* W^**-inclusion, with* $\mathcal{M}_*$ *separable and* $\mathcal{D}$ *Abelian. Then* $(\mathcal{M}, \mathcal{D})$ *has the unique pseudo-expectation property iff* $\mathcal{M}$ *is type* I, $\mathcal{D}$ *is a MASA, and there exists a family* $\{p_t\}$ *of Abelian projections for* $\mathcal{M}$ *such that* $\{p_t\} \subseteq \mathcal{D}$ *and* $\sum_t p_t = 1$. *The unique pseudo-expectation is a normal faithful conditional expectation.*

(ii) *Let* $(\mathcal{M}, \mathcal{D})$ *be a* W^**-inclusion, with* $\mathcal{M}$ *a* II_1 *factor and* $\mathcal{D}$ *a singular MASA. If* $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, *then* $(\mathcal{M}^\omega, \mathcal{D}^\omega)$ *has the faithful unique pseudo-expectation property. The unique pseudo-expectation is a normal, trace-preserving conditional expectation.*

Proof. (i) ($\Rightarrow$) Suppose there exists a unique expectation $E:\mathcal{M}\to\mathcal{D}$. Then $\mathcal{D}$ is a MASA, by Theorem 5.1. Since $\mathcal{M}_*$ is separable, $\mathcal{D}$ is singly-generated. Then E is normal and faithful, by [1, Cor. 3.3]. It follows that $\mathcal{M}$ is type I, by [31, Thm. 3.3]. Thus there exist Abelian projections $\{p_t\}$ for $\mathcal{M}$ such that $\{p_t\}\subseteq\mathcal{D}$ and $\sum_t p_t=1$, by [1, Thm. 4.1].

($\Leftarrow$) Conversely, if $\mathcal{M}$ is type I, $\mathcal{D}$ is a MASA, and there exist Abelian projections $\{p_t\}$ for $\mathcal{M}$ such that $\{p_t\}\subseteq\mathcal{D}$ and $\sum_t p_t=1$, then there exists a unique conditional expectation $E:\mathcal{M}\to\mathcal{D}$, by [1, Thm. 4.1].

(ii) By [30, Thm. 0.1], $(\mathcal{M}^\omega,\mathcal{D}^\omega)$ has the unique extension property (UEP). By Example 2.12, $(\mathcal{M}^\omega,\mathcal{D}^\omega)$ has the unique pseudo-expectation property, and the unique pseudo-expectation is a conditional expectation, necessarily normal, faithful, and trace-preserving. $\square$

Remark 5.7. Contrasting statements (i) and (ii) of Theorem 5.6 above, we see that separability plays a role in the unique pseudo-expectation property.

6. Applications

In this section, we show that the faithful unique pseudo-expectation property can substantially simplify *C^*-envelope* calculations, and we relate the faithful unique pseudo-expectation property to *norming* in the sense of Pop, Sinclair, and Smith.

6.1. C^*-envelopes. Let $\mathcal{C}$ be a unital C^*-algebra and $\mathcal{X}\subseteq\mathcal{C}$ be a unital operator space such that $C^*(\mathcal{X})=\mathcal{C}$. There exists a unique maximal closed two-sided ideal $\mathcal{J}\lhd\mathcal{C}$ such that quotient map $q:\mathcal{C}\to\mathcal{C}/\mathcal{J}$ is completely isometric on $\mathcal{X}$ [14]. Then $C^*_e(\mathcal{X})=\mathcal{C}/\mathcal{J}$ is the *C^*-envelope* of $\mathcal{X}$, the (essentially) unique minimal C^*-algebra generated by a completely isometric copy of $\mathcal{X}$. In general, determining $C^*_e(\mathcal{X})$ can be quite challenging. However, if $\mathcal{D}\subseteq\mathcal{X}\subseteq\mathcal{C}$, and $(\mathcal{C},\mathcal{D})$ has the faithful unique pseudo-expectation property, then determining $C^*_e(\mathcal{X})$ is not hard at all.

Theorem 6.1. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion with the faithful unique pseudo-expectation property (more generally, such that every pseudo-expectation is faithful). If $\mathcal{D}\subseteq\mathcal{X}\subseteq\mathcal{C}$ is an operator space, then $C^*_e(\mathcal{X})=C^*(\mathcal{X})$. That is, the C^*-envelope equals the generated C^*-algebra.*

Proof. By the previous discussion, $C^*_e(\mathcal{X})=C^*(\mathcal{X})/\mathcal{J}$, where $\mathcal{J}\lhd C^*(\mathcal{X})$ is the unique maximal closed two-sided ideal such that $q:C^*(\mathcal{X})\to C^*(\mathcal{X})/\mathcal{J}$ is completely isometric on $\mathcal{X}$. Since $\mathcal{D}\subseteq\mathcal{X}$, $\mathcal{J}$ must be $\mathcal{D}$-disjoint. But then $\mathcal{J}=0$, since $(\mathcal{C},\mathcal{D})$ is hereditarily essential, by Theorem 3.5. $\square$

We say that a C^*-inclusion $(\mathcal{C},\mathcal{D})$ is *C^*-envelope determining* if $C^*_e(\mathcal{X})=C^*(\mathcal{X})$ for every operator space $\mathcal{D}\subseteq\mathcal{X}\subseteq\mathcal{C}$. With this terminology, Theorem 6.1 becomes the implication

$$\text{every pseudo-expectation faithful}\implies C^*\text{-envelope determining.}$$

The converse is false.

EXAMPLE 6.2. Let $\mathcal{C} = M_{2\times 2}(\mathbb{C})$ and $\mathcal{D} = \mathbb{C}I$. Then $(\mathcal{C}, \mathcal{D})$ is C^*-envelope determining, but admits multiple pseudo-expectations, some of which are not faithful.

Proof. Let $\mathbb{C} \subseteq \mathcal{X} \subseteq M_{2\times 2}(\mathbb{C})$ be an operator space. Then $\dim(C^*(\mathcal{X})) \in \{1, 2, 4\}$. If $\dim(C^*(\mathcal{X})) \in \{1, 2\}$, then $C^*(\mathcal{X}) = \mathcal{X}$, which implies $C^*_e(\mathcal{X}) = \mathcal{X}$. Otherwise, if $\dim(C^*(\mathcal{X})) = 4$, then $C^*(\mathcal{X}) = M_{2\times 2}(\mathbb{C})$, which implies $C^*_e(\mathcal{X}) = C^*(\mathcal{X})$, since $M_{2\times 2}(\mathbb{C})$ is simple. □

6.2. Norming. According to Pitts' Theorem 1.4, if $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion with the faithful unique pseudo-expectation property, then $\mathcal{D}$ *norms* $\mathcal{C}$ in the sense of Pop, Sinclair, and Smith [29]. In this section, we investigate the relationship between the faithful unique pseudo-expectation property and norming, for arbitrary C^*-inclusions. We show that the faithful unique pseudo-expectation is conducive to norming (Theorem 6.8), but does not imply it (Example 6.9).

We begin by recalling the definition of norming, and proving some general norming results which we will need later. Some of these results may be of independent interest.

DEFINITION 6.3. We say that an inclusion $(\mathcal{C}, \mathcal{D})$ is *norming* if for any $X \in M_{d\times d}(\mathcal{C})$, we have that

$$\|X\| = \sup\big\{\|RXC\| : R \in \mathrm{Ball}\big(M_{1\times d}(\mathcal{D})\big), C \in \mathrm{Ball}\big(M_{d\times 1}(\mathcal{D})\big)\big\}.$$

PROPOSITION 6.4. *Let $(\mathcal{M}, \mathcal{D})$ be a W^*-inclusion and $\{p_t\} \subseteq \mathcal{D}$ be an increasing net of projections such that* $\sup_t p_t = 1$. *If $(p_t\mathcal{M}p_t, p_t\mathcal{D}p_t)$ is norming for all t, then $(\mathcal{M}, \mathcal{D})$ is norming.*

Proof. Let $\mathcal{H}$ be the Hilbert space on which $\mathcal{M}$ acts. Fix $X \in M_{d\times d}(\mathcal{M})$ and $\varepsilon > 0$. There exist $\xi, \eta \in \mathrm{Ball}(\mathcal{H}^d)$ such that

$$\big|\langle X\xi, \eta\rangle\big| > \|X\| - \varepsilon.$$

Since $\sup_t p_t = 1$, there exists t such that

$$\big|\big\langle X(I_d \otimes p_t)\xi, (I_d \otimes p_t)\eta\big\rangle\big| > \big|\langle X\xi, \eta\rangle\big| - \varepsilon.$$

Set $\tilde{X} = (I_d \otimes p_t)X(I_d \otimes p_t) \in M_{d\times d}(p_t\mathcal{M}p_t)$. Since $p_t\mathcal{D}p_t$ norms $p_t\mathcal{M}p_t$, there exist $R \in \mathrm{Ball}(M_{1\times d}(p_t\mathcal{D}p_t))$, $C \in \mathrm{Ball}(M_{d\times 1}(p_t\mathcal{D}p_t))$ such that

$$\|R\tilde{X}C\| > \|\tilde{X}\| - \varepsilon.$$

Then $R \in \mathrm{Ball}(M_{1\times d}(\mathcal{D}))$, $C \in \mathrm{Ball}(M_{d\times 1}(\mathcal{D}))$, and

$$\begin{aligned}\|RXC\| &= \big\|R(I_d \otimes p_t)X(I_d \otimes p_t)C\big\| = \|R\tilde{X}C\| \\ &> \|\tilde{X}\| - \varepsilon \geq \big|\langle \tilde{X}\xi, \eta\rangle\big| - \varepsilon\end{aligned}$$

$$= \left|\left\langle X(I_d \otimes p_t)\xi, (I_d \otimes p_t)\eta\right\rangle\right| - \varepsilon > \left|\langle X\xi, \eta\rangle\right| - 2\varepsilon$$
$$> \|X\| - 3\varepsilon. \qquad \square$$

Corollary 6.5. *For $i \in I$, let $(\mathcal{M}_i, \mathcal{D}_i)$ be a W^*-inclusion. If $(\mathcal{M}_i, \mathcal{D}_i)$ is norming for all $i \in I$, then $(\bigoplus_{i\in I} \mathcal{M}_i, \bigoplus_{i\in I} \mathcal{D}_i)$ is norming.*

Lemma 6.6. *Let $(\mathcal{C}, \mathcal{D})$ be a C^*-inclusion and $\mathcal{I} \lhd \mathcal{C}$. If $(\mathcal{D} + \mathcal{I})/\mathcal{I}$ norms $\mathcal{C}/\mathcal{I}$, then for every $X \in M_{d\times d}(\mathcal{C})$, there exist $R \in \mathrm{Ball}(M_{1\times d}(\mathcal{D}))$ and $C \in \mathrm{Ball}(M_{d\times 1}(\mathcal{D}))$ such that*

$$\|RXC\| > \left\|X + M_{d\times d}(\mathcal{I})\right\| - \varepsilon.$$

Proof. Let $\pi : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ be the quotient map. Fix $X \in M_{d\times d}(\mathcal{C})$ and $\varepsilon > 0$. By assumption, there exist $R \in M_{1\times d}(\mathcal{D})$ and $C \in M_{d\times 1}(\mathcal{D})$ such that $\|\pi_{1\times d}(R)\| < 1$, $\|\pi_{d\times 1}(C)\| < 1$, and

$$\left\|\pi_{1\times d}(R)\pi_{d\times d}(X)\pi_{d\times 1}(C)\right\| > \left\|\pi_{d\times d}(X)\right\| - \varepsilon.$$

Now the map

$$\mathcal{D}/(\mathcal{D} \cap \mathcal{I}) \to (\mathcal{D} + \mathcal{I})/\mathcal{I} : d + \mathcal{D} \cap \mathcal{I} \mapsto d + \mathcal{I}$$

is a unital $*$-isomorphism, in particular a complete isometry. Thus

$$\left\|\pi_{1\times d}(R)\right\| = \left\|R + M_{1\times d}(\mathcal{I})\right\| = \left\|R + M_{1\times d}(\mathcal{D} \cap \mathcal{I})\right\|.$$

It follows that there exists $\tilde{R} \in M_{1\times d}(\mathcal{D})$ such that $\|\tilde{R}\| < 1$ and $\pi_{1\times d}(\tilde{R}) = \pi_{1\times d}(R)$. Likewise, there exists $\tilde{C} \in M_{d\times 1}(\mathcal{D})$ such that $\|\tilde{C}\| < 1$ and $\pi_{d\times 1}(\tilde{C}) = \pi_{d\times 1}(C)$. Then

$$\|\tilde{R}X\tilde{C}\| \geq \left\|\pi_{1\times d}(\tilde{R})\pi_{d\times d}(X)\pi_{d\times 1}(\tilde{C})\right\| > \left\|\pi_{d\times d}(X)\right\| - \varepsilon. \qquad \square$$

Proposition 6.7. *Let $\mathcal{M}$ be a II_1 factor, $\mathcal{D} \subseteq \mathcal{M}$ be a MASA, and $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. If $(\mathcal{M}^\omega, \mathcal{D}^\omega)$ is norming, then so is $(\mathcal{M}, \mathcal{D})$.*

Proof. By assumption, $\mathcal{D}^\omega = (\ell^\infty(\mathcal{D}) + \mathcal{I}_\omega)/\mathcal{I}_\omega$ norms $\mathcal{M}^\omega = \ell^\infty(\mathcal{M})/\mathcal{I}_\omega$. Let $X \in M_{d\times d}(\mathcal{M})$ and $\varepsilon > 0$. Then $(X) \in M_{d\times d}(\ell^\infty(\mathcal{M})) = \ell^\infty(M_{d\times d}(\mathcal{M}))$. By Lemma 6.6, there exist $(R_n) \in M_{1\times d}(\ell^\infty(\mathcal{D})) = \ell^\infty(M_{1\times d}(\mathcal{D}))$ and $(C_n) \in M_{d\times 1}(\ell^\infty(\mathcal{D})) = \ell^\infty(M_{d\times 1}(\mathcal{D}))$ such that $\|(R_n)\| < 1$, $\|(C_n)\| < 1$, and

$$\left\|(R_n)(X)(C_n)\right\| > \left\|(X) + M_{d\times d}(\mathcal{I}_\omega)\right\| - \varepsilon.$$

Thus,

$$\sup_n \|R_n\| < 1, \qquad \sup_n \|C_n\| < 1, \quad \text{and} \quad \sup_n \|R_n X C_n\| > \|X\| - \varepsilon. \qquad \square$$

Now we list some classes of C^*-inclusions for which (f!PsE) $\implies$ (Norming).

Theorem 6.8. *For the following classes of C^*-inclusions, the faithful unique pseudo-expectation property implies norming:*

(i) *Regular MASA inclusions $(\mathcal{C}, \mathcal{D})$. In particular, C^*-inclusions $(C(X) \rtimes_r \Gamma, C(X))$ for a discrete group Γ acting on a compact Hausdorff space X by homeomorphisms.*

(ii) *Abelian inclusions* $(C(Y), C(X))$.
(iii) C^**-inclusions* $(\mathcal{C}, \mathcal{D})$ *with* $\mathcal{D} \subseteq \mathcal{C} \subseteq I(\mathcal{D})$ (*i.e., operator space essential inclusions*).
(iv) W^**-inclusions* $(B(\mathcal{H}), \mathcal{M})$.
(v) W^**-inclusions* $(\mathcal{M}, \mathcal{D})$, *with* $\mathcal{M}_*$ *separable and* $\mathcal{D}$ *Abelian.*

Proof. (i) This is Pitts' Theorem 1.4.

(ii) Abelian inclusions $(C(Y), C(X))$ are norming, whether or not they have the faithful unique pseudo-expectation property [29, Ex. 2.5].

(iii) Let $(\mathcal{C}, \mathcal{D})$ be an operator space essential inclusion (see the discussion preceding Example 2.11). For $X \in M_{d\times d}(\mathcal{C})$, define

$$\gamma_d(X) = \sup\big\{\|RXC\| : R \in \mathrm{Ball}\big(M_{1\times d}(\mathcal{D})\big), C \in \mathrm{Ball}\big(M_{d\times 1}(\mathcal{D})\big)\big\}.$$

By [24, Thm. 2.1], $\gamma = (\gamma_d)_{d=1}^\infty$ is an operator space structure on $\mathcal{C}$ with the following properties:

- $\gamma_1(x) = \|x\|$, $x \in \mathcal{C}$;
- $\gamma_d(X) \le \|X\|$, $X \in M_{d\times d}(\mathcal{C})$;
- $\gamma_d(D) = \|D\|$, $D \in M_{d\times d}(\mathcal{D})$.

It follows that the identity map $\iota : (\mathcal{C}, \|\cdot\|) \to (\mathcal{C}, \gamma)$ is a complete contraction which is completely isometric on $\mathcal{D}$. Since $(\mathcal{C}, \mathcal{D})$ is operator space essential, ι is actually completely isometric on $\mathcal{C}$, so $(\mathcal{C}, \mathcal{D})$ is norming.

(iv) By Theorem 5.6, $\mathcal{M}'$ is Abelian. Let $\mathcal{M}' \subseteq \mathcal{A} \subseteq B(\mathcal{H})$ be a MASA. Then $\mathcal{A} = \mathcal{A}' \subseteq \mathcal{M}'' = \mathcal{M}$. Since $\mathcal{A}$ norms $B(\mathcal{H})$, $\mathcal{M}$ norms $B(\mathcal{H})$ as well.

(v) By Theorem 5.6, $\mathcal{M}$ is type I, $\mathcal{D}$ is a MASA, and there exist Abelian projections $\{p_t\} \subseteq \mathcal{D}$ for $\mathcal{M}$ such that $\sum_t p_t = 1$. Letting $p_F = \sum_{t\in F} p_t$ for every finite set of indices F, we obtain an increasing net $\{p_F\} \subseteq \mathcal{D}$ of finite projections for $\mathcal{M}$ such that $\sup_F p_F = 1$. By Proposition 6.4, to prove that $(\mathcal{M}, \mathcal{D})$ is norming it suffices to prove that $(p_F\mathcal{M}p_F, \mathcal{D}p_F)$ is norming for each F. Thus, we may assume that $\mathcal{M}$ is finite type I. By Corollary 6.5, we may further assume that $\mathcal{M}$ is type I_n for some $n \in \mathbb{N}$. There is a (non-spatial) $*$-isomorphism $\mathcal{M} = M_{n\times n}(\mathcal{A}) \subseteq B(\mathcal{H}^n)$, where $\mathcal{A} \subseteq B(\mathcal{H})$ is a MASA and $\mathcal{H}$ is separable. By [18, Thm. 3.19], there exists a unitary $u \in \mathcal{M}$ such that $u\mathcal{D}u^* = \ell_n^\infty(\mathcal{A})$. It follows that $\mathcal{D} \subseteq B(\mathcal{H}^n)$ is a MASA, which implies that $\mathcal{D}$ norms $B(\mathcal{H}^n)$ [29, Thm. 2.7]. Thus, $\mathcal{D}$ norms $\mathcal{M}$. □

EXAMPLE 6.9. There exists a II_1 factor $\mathcal{M}$ and a singular MASA $\mathcal{D} \subseteq \mathcal{M}$ such that $(\mathcal{M}, \mathcal{D})$ has the faithful unique pseudo-expectation property, but $\mathcal{D}$ does not norm $\mathcal{M}$. Of course $\mathcal{M}_*$ is non-separable.

Proof. Let $\mathbb{F}_2$ be the free group on two generators u and v, and let $\mathcal{D} = W^*(u) \subseteq W^*(\mathbb{F}_2) = \mathcal{M}$. Then $\mathcal{M}$ is a II_1 factor, $\mathcal{D}$ is a singular MASA, and $\mathcal{D}$ does not norm $\mathcal{M}$ [29, Thm. 5.3]. Now let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Then $\mathcal{M}^\omega$ is a II_1 factor, $\mathcal{D}^\omega$ is a singular MASA, and $(\mathcal{M}^\omega, \mathcal{D}^\omega)$ has the faithful unique pseudo-expectation property, by Theorem 5.6. But $\mathcal{D}^\omega$ does not norm $\mathcal{M}^\omega$, by Proposition 6.7. □

7. Conclusion

We conclude this paper with a list of questions and partial progress toward some answers.

7.1. Questions.

- Q1 Is the condition (Reg) hereditary from above? That is, if $(\mathcal{C}, \mathcal{D})$ is a regular inclusion and $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$ is a C^*-algebra, is $(\mathcal{C}_0, \mathcal{D})$ a regular inclusion? (We expect that the answer is "no".)
- Q2 Is the condition (UEP) hereditary from below? That is, if $(\mathcal{C}, \mathcal{D})$ has the unique extension property and $\mathcal{D} \subseteq \mathcal{D}_0 \subseteq \mathcal{C}$ is a C^*-algebra, does $(\mathcal{C}, \mathcal{D}_0)$ have the unique extension property? (Again, we expect that the answer is "no".)
- Q3 Is there a C^*-inclusion $(\mathcal{C}, \mathcal{D})$ with a unique conditional expectation, but multiple pseudo-expectations?
- Q4 If $(\mathcal{C}, \mathcal{D})$ has the unique extension property (UEP), does $(\mathcal{C}, \mathcal{D})$ have the unique pseudo-expectation property? (By Example 2.12, the answer is "yes" if $\mathcal{D}$ is Abelian.)
- Q5 If $(\mathcal{C}, \mathcal{D})$ has the unique pseudo-expectation property, is $\mathcal{D}^c$ Abelian? What if $\mathcal{D}$ is Abelian? (By Corollary 3.14, the answer is "yes" if $(\mathcal{C}, \mathcal{D})$ has the faithful unique pseudo-expectation property.)
- Q6 If every pseudo-expectation is faithful, is there a unique pseudo-expectation? Equivalently, by Theorem 3.5, does $(\mathcal{C}, \mathcal{D})$ have the faithful unique pseudo-expectation property iff $(\mathcal{C}, \mathcal{D})$ is hereditarily essential?
- Q7 Let Γ be a discrete group acting on a compact Hausdorff space X by homeomorphisms. Find a condition on the action equivalent to $(C(X) \rtimes_r \Gamma, C(X)^c)$ having the unique pseudo-expectation property.
- Q8 Is the C^*-inclusion $(B(\ell^2)/K(\ell^2), \ell^\infty/c_0)$ norming?
- Q9 Is there a condition on a C^*-inclusion $(\mathcal{C}, \mathcal{D})$ which together with the faithful unique pseudo-expectation property implies norming? In particular, is the separability of $\mathcal{C}$ such a condition?
- Q10 Is there a condition on a W^*-inclusion $(\mathcal{M}, \mathcal{D})$ which together with the faithful unique pseudo-expectation property implies norming? In particular, is the separability of $\mathcal{M}_*$ such a condition? (By Theorem 6.8, the answer is "yes" if $\mathcal{D}$ is Abelian.)

7.2. Progress on Questions 5 and 6.

Question 5. We are able to show that if $(\mathcal{C}, \mathcal{D})$ is a C^*-inclusion with $\mathcal{D}$ Abelian, such that there exists a unique pseudo-expectation Φ, then Φ is multiplicative on $\mathcal{D}^c$ (Proposition 7.1). We regard this as partial progress toward proving that $\mathcal{D}^c$ is Abelian. Indeed, by Corollary 3.21, if $\mathcal{D}^c$ is Abelian and Φ

is the unique pseudo-expectation for $(\mathcal{C},\mathcal{D})$, then Φ necessarily is multiplicative on $\mathcal{D}^c$.

PROPOSITION 7.1. *Let $(\mathcal{C},\mathcal{D})$ be a C^*-inclusion, with $\mathcal{D}$ Abelian. If $(\mathcal{C},\mathcal{D})$ has unique pseudo-expectation $\Phi \in \mathrm{PsExp}(\mathcal{C},\mathcal{D})$, then Φ is multiplicative on $\mathcal{D}^c$.*

Proof. Let

$$M_\Phi = \left\{x \in \mathcal{C} : \Phi\big(x^*x\big) = \Phi(x)^*\Phi(x), \Phi\big(xx^*\big) = \Phi(x)\Phi(x)^*\right\}$$

be the *multiplicative domain* of Φ, the largest C^*-subalgebra of $\mathcal{C}$ on which Φ is multiplicative [27, Thm. 3.18]. Suppose $x \in (\mathcal{D}^c)_{sa}$. Then $\mathcal{C}_x = C^*(\mathcal{D},x)$ is a unital Abelian C^*-algebra containing $\mathcal{D}$. By Proposition 2.6, $(\mathcal{C}_x,\mathcal{D})$ has unique pseudo-expectation $\Phi|_{\mathcal{C}_x}$, which is multiplicative by Corollary 3.21. It follows that $x \in M_\Phi$. ☐

Question 6. We show that if $(\mathcal{M},\mathcal{D})$ is a W^*-inclusion with $\mathcal{D}$ injective, such that every pseudo-expectation is faithful, then there exists a unique pseudo-expectation (Proposition 7.5).

Recall that a bounded linear map T between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ is *singular* if $f \circ T \in (\mathcal{M}_*)^\perp$ whenever $f \in \mathcal{N}_*$.

LEMMA 7.2. *Let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion and $\theta : \mathcal{M} \to \mathcal{D}$ be a completely positive $\mathcal{D}$-bimodule map. If θ is singular, then for every projection $0 \neq p \in Z(\mathcal{D})$, there exists a projection $0 \neq e \in \mathcal{M}$ such that $e \leq p$ and $\theta(e) = 0$.*

Proof. Let $\{\phi_i\} \subseteq (\mathcal{D}_*)_+$ be a maximal family with mutually orthogonal supports $\{s(\phi_i)\} \subseteq \mathcal{D}$. Then $\sum_i s(\phi_i) = 1$, and so there exists j such that $s(\phi_j)p \neq 0$. Since θ is singular, $\phi_j \circ \theta \in (\mathcal{M}_*)^\perp_+$. Thus there exists a projection $0 \neq e \in \mathcal{M}$ such that $e \leq s(\phi_j)p$ and $\phi_j(\theta(e)) = 0$ [36, Thm. III.3.8]. It follows that $s(\phi_j)\theta(e)s(\phi_j) = 0$, which implies

$$\theta(e) = \theta\big(s(\phi_j)es(\phi_j)\big) = s(\phi_j)\theta(e)s(\phi_j) = 0. \qquad ☐$$

LEMMA 7.3. *Let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion and $\theta : \mathcal{M} \to \mathcal{D}$ be a completely positive $\mathcal{D}$-bimodule map. If $\mathrm{CE}(\mathcal{M},\mathcal{D}) \neq \emptyset$, then there exists $E \in \mathrm{CE}(\mathcal{M},\mathcal{D})$ such that $\theta(x) = \theta(1)E(x)$, $x \in \mathcal{M}$. Furthermore, if $0 \leq x \leq s(\theta(1))$ and $\theta(x) = 0$, then $E(x) = 0$.*

Proof. Fix $E_0 \in \mathrm{CE}(\mathcal{M},\mathcal{D})$. Let $p = s(\theta(1)) \in Z(\mathcal{D})$ and define

$$E(x) = \lim_{k\to\infty} \big(\theta(1) + 1/k\big)^{-1}\theta(x) + p^\perp E_0(x), \quad x \in \mathcal{M},$$

where the limit exists in the strong operator topology (see [10, Lemma 5.1.6]). Then $E \in \mathrm{CE}(\mathcal{M},\mathcal{D})$ and $\theta(x) = \theta(1)E(x)$, $x \in \mathcal{M}$. If $0 \leq x \leq p$ and $\theta(x) = 0$, then $0 \leq E(x) \leq p$. But $E(x) = p^\perp E_0(x)$, which implies $E(x) = 0$. ☐

LEMMA 7.4. *Let $(\mathcal{M},\mathcal{D})$ be a W^*-inclusion. If every conditional expectation $\mathcal{M} \to \mathcal{D}$ is faithful, then every conditional expectation $\mathcal{M} \to \mathcal{D}$ is normal.*

Proof. Let $E : \mathcal{M} \to \mathcal{D}$ be a conditional expectation. By [37], there exist completely positive $\mathcal{D}$-bimodule maps $\theta_n, \theta_s : \mathcal{M} \to \mathcal{D}$ such that θ_n is normal, θ_s is singular, and $E = \theta_n + \theta_s$. Assume that $\theta_s \neq 0$. By Lemma 7.2, there exists a projection $0 \neq e \in \mathcal{M}$ such that $e \leq s(\theta_s(1))$ and $\theta_s(e) = 0$. By Lemma 7.3, there exists a conditional expectation $E_s : \mathcal{M} \to \mathcal{D}$ such that $E_s(e) = 0$, a contradiction. Thus, $E = \theta_n$ is normal. □

PROPOSITION 7.5. *Let $(\mathcal{M}, \mathcal{D})$ be a W^*-inclusion, with $\mathcal{D}$ injective. If every pseudo-expectation $\Phi \in \mathrm{PsExp}(\mathcal{M}, \mathcal{D})$ is faithful, then $(\mathcal{M}, \mathcal{D})$ has the unique pseudo-expectation property. In this situation, the unique pseudo-expectation is a conditional expectation which is faithful and normal.*

Proof. Since $\mathcal{D}$ is injective, $\mathrm{PsExp}(\mathcal{M}, \mathcal{D}) = \mathrm{CE}(\mathcal{M}, \mathcal{D})$. By Lemma 7.4, every conditional expectation $\mathcal{M} \to \mathcal{D}$ is faithful and normal. By [8, Thm. 5.3], the map $E \mapsto E|_{\mathcal{D}^c}$ is a bijection between the faithful normal conditional expectations $\mathcal{M} \to \mathcal{D}$ and the faithful normal conditional expectations $\mathcal{D}^c \to Z(\mathcal{D})$. Thus to show that there exists a unique conditional expectation $\mathcal{M} \to \mathcal{D}$, it suffices to show that there exists a unique faithful normal conditional expectation $\mathcal{D}^c \to Z(\mathcal{D})$. In fact, we will show that $\mathcal{D}^c = Z(\mathcal{D})$.

By Theorem 3.12, $\mathcal{D}^c$ is Abelian. Since every conditional expectation $\mathcal{M} \to \mathcal{D}$ is faithful, every conditional expectation $C^*(\mathcal{D}, \mathcal{D}^c) \to \mathcal{D}$ is faithful, which implies every conditional expectation $\mathcal{D}^c \to Z(\mathcal{D})$ is faithful, by Theorem 5.2. In particular, every multiplicative conditional expectation $\mathcal{D}^c \to Z(\mathcal{D})$ is faithful, which implies $\mathcal{D}^c = Z(\mathcal{D})$. □

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