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CARTAN TRIPLES

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ABSTRACT. We introduce the class of Cartan triples as a generalization of the notion of a Cartan MASA in a von Neumann algebra. We obtain a one-to-one correspondence between Cartan triples and certain Clifford extensions of inverse semigroups. Moreover, there is a spectral theorem describing bimodules in terms of their support sets in the fundamental inverse semigroup and, as a corollary, an extension of Aoi's theorem to this setting. This context contains that of Fulman's generalization of Cartan MASAs and we discuss his generalization in an appendix.

1. INTRODUCTION

As observed in the seminal work of Feldman-Moore [14, 15], when a von Neumann algebra contains a Cartan MASA, strong structural results about the algebra may be obtained. However, many von Neumann algebras do not contain a Cartan MASA; the first examples were found in [32]. Determining which von Neumann algebras have a Cartan MASA and when it is unique is an important question and has attracted significant attention; for two examples, see [27, 28]. Part of the interest is that Cartan MASAs are closely connected to crossed product decompositions, as indeed is clear from the work of Feldman-Moore.

Recall a Cartan MASA $\mathcal{D}$ in a von Neumann algebra $\mathcal{M}$ is a maximal abelian subalgebra with two additional properties: it is regular, that is, the span of its normalizers is weak-$*$ dense in $\mathcal{M}$; and there is a faithful, normal conditional expectation from $\mathcal{M}$ to $\mathcal{D}$.

In this paper, we study a much larger family of regular abelian von Neumann subalgebras of von Neumann algebras. Specifically, if $\mathcal{M}$ is a von Neumann algebra, we consider an abelian and regular subalgebra $\mathcal{D} \subseteq \mathcal{M}$ such that there is a faithful normal conditional expectation onto the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{M}$. Because $\mathcal{D}^c$ plays an important role in the structure of the algebras, we name it $\mathcal{N}$ and call $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ a *Cartan triple*.

In our previous work [12], we showed that Cartan MASAs can be described in terms of certain extensions of inverse semigroups. In the setting of Cartan triples, our main result is a correspondence between Cartan triples and a larger class of extensions of inverse semigroups

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}.$$

Further, we obtain a Spectral Theorem for $\mathcal{N}$-Bimodules and a version of Aoi's theorem in this context of Cartan triples. Although some of the methods from [12] extend naturally, significant modifications are needed. For example, for Cartan triples, $\mathcal{P}$ is not usually an abelian inverse semigroup, but rather is Clifford, that is, the idempotents of $\mathcal{P}$ commute with all elements of $\mathcal{P}$.

Various generalizations of MASAs have been considered in the literature. For example in [13], Ruy Exel connects the existence of a suitable non-abelian generalization of a MASA in a *separable* C^*-algebra to a reduced crossed product decomposition of the containing algebra. Instead of the inverse semigroup approach considered here, Exel considers a Fell bundle over an inverse semigroup as the classifying structure. The appropriate variant of Exel's notion of a generalized Cartan subalgebra in the von Neumann algebra setting is a full Cartan triple, meaning $\mathcal{D}$ is the center of

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$\mathcal{N}$ (see Definition 2.3). Our approach using extensions of inverse semigroups for classification, while related to Fell bundles, is rather different.

Another generalization, also related to crossed product decompositions, has been considered by Igor Fulman in [17]. Fulman's generalization of a Cartan MASA is, in our terms, a Cartan triple with an additional condition, the existence of a subgroup of the unitaries in $\mathcal{M}$ that normalizes $\mathcal{D}$, contains the unitaries of $\mathcal{N}$, and has a suitable fixed point property. We show in Appendix A that this additional condition can be characterized as the existence of a lift of an inverse semigroup homomorphism from $\mathcal{S}$ into the partial automorphisms of the Cartan triple. Crossed products by inverse semigroups were first introduced by Nándor Sieben in [29] using such a semigroup homomorphism into the partial automorphisms of a C*-algebra. Thus, Fulman's condition can be interpreted naturally as saying that the containing algebra is a crossed product by a suitable inverse semigroup. Fulman's starting point was to generalize the Feldman-Moore characterization of Cartan subalgebras [14, 15] using measured equivalence relations. While some of our results resemble Fulman's, ours are more general, perhaps because of the comparative simplicity of the inverse semigroup approach used here.

We now discuss our results and their motivation in more detail. We associate to each Cartan triple an extension of inverse semigroups, $\mathcal{P} \hookrightarrow \mathcal{G} \to \mathcal{S}$ where $\mathcal{S}$ is a fundamental inverse semigroup, that is, the only elements commuting with the idempotents of $\mathcal{S}$, denoted $\mathcal{E}(\mathcal{S})$, is $\mathcal{E}(\mathcal{S})$ itself, and $\mathcal{P}$ is Clifford, meaning all elements of $\mathcal{P}$ commute with $\mathcal{E}(\mathcal{S})$. To be an extension, the restriction to idempotents of the maps above must be isomorphisms. It is well known that every inverse semigroup $\mathcal{G}$ may be represented as such an (idempotent-separating) extension; see [21, p. 141].

To construct the extension from a Cartan triple, take $\mathcal{P}$ to be the partial isometries in $\mathcal{N}$ that normalize $\mathcal{D}$ and $\mathcal{G}$ to the partial isometries in $\mathcal{M}$ that normalize $\mathcal{D}$, with $\mathcal{P} \hookrightarrow \mathcal{G}$ the inclusion map. To construct $\mathcal{S}$, we identify elements with the same action on the idempotents, i.e., we quotient by the Munn congruence.

In [12], the inverse semigroup $\mathcal{P}$ was abelian and we required that the character space of $\mathcal{E}(\mathcal{P})$ was hyperstonean. In that case, it was easy to recover $\mathcal{D}$ from $\mathcal{P}$, as the continuous functions on the character space of $\mathcal{E}(\mathcal{P})$.

Here, we need a condition that allows us to again recover $\mathcal{D}^c = \mathcal{N}$ from $\mathcal{P}$: $\mathcal{P}$ arises as the partial isometries in a von Neumann algebra $\mathcal{N}$ which normalize a fixed von Neumann subalgebra of the center of $\mathcal{N}$. In this case, we say that $\mathcal{P}$ is an $\mathcal{N}$-Clifford inverse monoid (Definition 2.7).

To see the need for this condition, consider the (degenerate) Cartan triple, $(\mathcal{M}, \mathcal{M}, \mathbb{C}I)$. In this case, the associated extension has the form $\mathcal{U}(\mathcal{M}) \xrightarrow{\text{id}} \mathcal{U}(\mathcal{M}) \to \mathbb{C}I$. However, there are von Neumann algebras not isomorphic to their opposite algebras [10]. The unitary groups of such an algebra and its opposite are isomorphic, so if the extension was defined purely in terms of inverse semigroups (and without our stronger condition) it would be possible for non-isomorphic triples to produce the same extension.

With these definitions in hand and the construction of an extension from a triple (as outlined above), we show that Cartan triples are isomorphic if and only if their extensions are (in a suitable sense) isomorphic, Theorem 2.22.

To obtain the converse, we construct a Cartan triple from an extension in Section 3. This is more subtle, and we build on the strategy of our previous paper. In particular, we use the order structure of $\mathcal{S}$ to construct a reproducing kernel Hilbert $\mathcal{N}$-bimodule, $\mathfrak{A}$. We then define a representation, λ, of $\mathcal{G}$, Theorem 3.2, by partial isometries on $\mathfrak{A}$. After tensoring this representation with a faithful normal representation of $\mathcal{N}$ to obtain a suitable representation of $\mathcal{G}$ (Theorem 3.2), we define the Cartan triple of an extension in terms of the double commutants of $\mathcal{G}$, $\mathcal{P}$, and their (common) idempotents (Definition 4.1). In Theorem 4.11 we complete the circle of ideas by showing that the extension associated to the Cartan triple constructed is (isomorphic to) the original extension.

In Section 5 we begin a study of the $\mathcal{N}$-bimodules in a Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$. For our strongest results we require that $\mathcal{D}$ be as large as possible, that is, $\mathcal{D}$ is the center $\mathcal{Z}(\mathcal{N})$ of $\mathcal{N}$. We call such a Cartan triple *full*. When $(\mathcal{M}, \mathcal{D})$ is a Cartan pair, $(\mathcal{M}, \mathcal{D}, \mathcal{D})$ is a full Cartan triple, and so the class of full Cartan triples properly includes Cartan pairs. Different examples arise when $\mathcal{M}$ is type I, Section 6.1, and when $\mathcal{M} = \mathcal{N} \rtimes_\alpha G$ is a crossed product of $\mathcal{N}$ by a discrete group G which acts by properly outer automorphisms on $\mathcal{N}$ and $\mathcal{Z}(\mathcal{N})$, Theorem 6.3.

Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a full Cartan triple, with associated extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$. We show in Theorem 5.2 that if $B \subseteq \mathcal{M}$ is a non-zero weak-$*$ closed $\mathcal{N}$-bimodule, then $\mathcal{G} \cap B \neq \{0\}$. Thus, every weak-$*$ closed $\mathcal{N}$-bimodule gives rise to a non-trivial subset $q(B \cap \mathcal{G})$ of $\mathcal{S}$. We call such sets *spectral sets*, (Definition 5.1). If $A \subseteq \mathcal{S}$ is a spectral set, then $\operatorname{span}\{q^{-1}(A)\}$ is an $\mathcal{N}$-bimodule in $\mathcal{M}$.

Of course, it is conceivable that distinct weak-$*$ closed $\mathcal{N}$-bimodules have the same spectral sets. To study this, we use the Bures-topology on $\mathcal{M}$, induced by the conditional expectation $E\colon \mathcal{M} \to \mathcal{N}$. Whilst the weak-$*$ and Bures topologies on $\mathcal{M}$ are not, in general, comparable, the Bures-closed $\mathcal{N}$-bimodules are weak-$*$ closed [6, Lemma 3.1]. The advantage of the Bures topology over the weak-$*$ topology is that certain Fourier-type series often converge in the Bures topology, while they need not converge in the weak-$*$ topology (or in any other "natural" topology). Indeed, Mercer showed in [23] that the Fourier series of elements in crossed-product von Neumann algebra converge in the Bures-topology, but need not converge in the weak-$*$ topology. Analogously, when $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple, we show in Theorem 5.7 that if $x \in \mathcal{M}$, then x is the Bures-limit of the net of finite sums

$$\sum_{u \in F \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})} uE(u^* x).$$

Similar results for x in a Cartan pair are given in [5, Proposition 2.4.4] and [24, Theorem 4.4].

In Proposition 5.8 we show that if B is a weak-$*$ closed $\mathcal{N}$-bimodule, $B_0 = \overline{\operatorname{span}}^{wk^*}\{B \cap \mathcal{G}\}$, and $B_1 = \overline{\operatorname{span}}^{\text{Bures}}\{B \cap \mathcal{G}\}$, then $B_0 \subseteq B \subseteq B_1$ and each of B_0, B and B_1 give the same spectral sets. We do not address when the bimodules B_0, B and B_1 are necessarily equal. In [5], if all weak-$*$ bimodules are necessarily Bures closed, the Cartan pair is said to satisfy *spectral synthesis*. Even in the case of Cartan pairs, whether $B_0 = B_1$ remains an open problem. There are some special cases when the result is known. If $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple of the type studied by Fulman [17] discussed above, with the added condition that $\mathcal{M}$ is constructed from a hyperfinite equivalence relation, then it can be deduced from Theorem 5.10 and [17, Theorem 15.18] that all weak-$*$ closed $\mathcal{N}$-bimodules are necessarily Bures closed. Cameron and Smith [6, 8] studied a related problem in crossed-products. They showed that if G is a discrete group satisfying the AP condition, acting on a von Neumann algebra $\mathcal{N}$ by properly outer automorphisms, then the weak-$*$ closed $\mathcal{N}$-bimodules in $\mathcal{N} \rtimes_\alpha G$ are necessarily Bures closed.

We give a Spectral Theorem for Bimodules in Theorem 5.10, which gives a one-to-one correspondence between the Bures-closed $\mathcal{N}$-bimodules and the spectral sets in $\mathcal{S}$. We find it striking that Theorem 5.10 depends only on $\mathcal{S}$ and not on the extension $\mathcal{G}$. Fuller and Pitts [16] had previously observed a similar phenomenon: non-isomorphic Cartan pairs that have isomorphic lattices of Bures-closed bimodules. Theorem 5.10 generalizes the Spectral Theorem for Cartan pairs found in [12]; see also [5]. It should be noted that the study of bimodules in Cartan pairs was initiated in the seminal work of Muhly, Saito and Solel [25]. They present a spectral theorem for weak-$*$ closed bimodules. Their work, however, has a gap. Though not explicitly stated as such, the gap in [25] amounts to assuming that the weak-$*$ closed bimodules are necessarily Bures closed, see [5].

A class of $\mathcal{N}$-bimodules of particular interest are the von Neumann algebras $\mathcal{L}$ such that $\mathcal{N} \subseteq \mathcal{L} \subseteq \mathcal{M}$. In Theorem 5.12 we show that if $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a full Cartan triple and $\mathcal{N} \subseteq \mathcal{L} \subseteq \mathcal{M}$, then $(\mathcal{L}, \mathcal{N}, \mathcal{D})$ is again a Cartan triple. This extends Aoi's result for intermediate von Neumann algebras in Cartan pairs [1]. A key step in the proof is showing that an intermediate subalgebra $\mathcal{L}$ is necessarily closed in the Bures topology. Thus, Theorem 5.12 together with Theorem 5.10 immediately give a one-to-one correspondence between the intermediate von Neumann subalgebras containing $\mathcal{N}$, and the sub-inverse Cartan monoids of $\mathcal{S}$, Corollary 5.14. We view this as a Galois correspondence-type result; although we do not have a group to hand, there is the Cartan inverse semigroup. Corollary 5.14 should be compared with the following well-known result: If $\mathcal{N}$ is a factor and G is a discrete group acting on $\mathcal{N}$ by (properly) outer automorphisms, then Izumi, Longo and Popa [18] show that if $\mathcal{L}$ is a von Neumann algebra satisfying $\mathcal{N} \subseteq \mathcal{L} \subseteq \mathcal{N} \rtimes_\alpha G$ then there is a subgroup H of G such that $\mathcal{L} = \mathcal{N} \rtimes_\alpha H$; see also [6, 9]. That is, there is a one-to-one correspondence between subgroups of G and the von Neumann algebras $\mathcal{M}$ with $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{N} \rtimes_\alpha G$. A similar Galois correspondence-type theorem without an explicit group structure has been obtained in [2] for von Neumann algebras generated by a measured equivalence relation and an appropriate cocycle.

Cameron and Smith have considered similar questions in [6, 7, 8]. They study crossed products by discrete groups and the bimodule and intermediate algebra structure therein, amongst other things. There is overlap with our work and [8], with neither work subsuming the other. There they let $\mathcal{N}$ be any von Neumann algebra and let G be a discrete group acting on $\mathcal{N}$ by properly outer automorphisms. If $\mathcal{N}$ is abelian, then $\mathcal{N}$ is a Cartan MASA in $\mathcal{N} \rtimes_\alpha G$ and so both our settings cover this case. If $\mathcal{N}$ is not abelian, but G also acts on $\mathcal{Z}(\mathcal{N})$ by properly outer automorphisms then it is shown in Theorem 6.3 $(\mathcal{N} \rtimes_\alpha G, \mathcal{N}, \mathcal{Z}(\mathcal{N}))$ is a Cartan triple.

2. Cartan triples and their extensions

Our main goals in this section are the construction of the extension associated to a Cartan triple, Proposition 2.13, and the result that two such extensions are isomorphic if and only if they arise from isomorphic Cartan triples, Theorem 2.22.

We begin by fixing some notation. For $\mathcal{M}$ a von Neumann algebra, $\mathcal{Z}(\mathcal{M})$ denotes its center and $\mathcal{U}(\mathcal{M})$ the unitary elements. For $\mathcal{X} \subseteq \mathcal{M}$, $\mathcal{X}^c$ denotes the relative commutant, that is,

$$\mathcal{X}^c := \{m \in \mathcal{M} : xm = mx \text{ for all } x \in \mathcal{X}\}.$$

Definition 2.1. Suppose $\mathcal{M}$ and $\mathcal{L}$ are von Neumann algebras with $\mathcal{L} \subseteq \mathcal{M}$. The *groupoid normalizer of $\mathcal{L}$ in $\mathcal{M}$* is the set,

$$\mathcal{GN}(\mathcal{M}, \mathcal{L}) := \{v \in \mathcal{M} : v \text{ is a partial isometry and } v^*\mathcal{L}v \cup v\mathcal{L}v^* \subseteq \mathcal{L}\}.$$

If the linear span of $\mathcal{GN}(\mathcal{M}, \mathcal{L})$ is weak-$*$ dense in $\mathcal{M}$, we say $\mathcal{L}$ is a *regular subalgebra of* $\mathcal{M}$ or that $\mathcal{L}$ is *regular* in $\mathcal{M}$.

Remark 2.2. When $\mathcal{L} \subseteq \mathcal{M}$ is an abelian von Neumann subalgebra of $\mathcal{M}$, it is more common to say $\mathcal{L}$ is regular in $\mathcal{M}$ if $\operatorname{span}\{U \in \mathcal{U}(\mathcal{M}) : U\mathcal{D}U^* = \mathcal{D}\}$ is weak-$*$ dense. However, if $\mathcal{L}$ is abelian, then

$$\operatorname{span}\{U \in \mathcal{U}(\mathcal{M}) : U\mathcal{D}U^* = \mathcal{D}\} = \operatorname{span} \mathcal{GN}(\mathcal{M}, \mathcal{D}), \tag{2.1}$$

thus the two definitions coincide in this case. For a proof of this statement see [5, p. 479, Inclusion 2.8].

We now introduce our main topic of study.

Definition 2.3. A *Cartan triple* is a triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ consisting of three von Neumann algebras satisfying:

(a) $\mathcal{D}$ is an abelian and regular von Neumann subalgebra of $\mathcal{M}$;
(b) $\mathcal{N}$ is the relative commutant of $\mathcal{D}$ in $\mathcal{M}$; and
(c) there exists a faithful normal conditional expectation $E : \mathcal{M} \to \mathcal{N}$.

A Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is *full* when $\mathcal{D} = \mathcal{Z}(\mathcal{N})$.

Remarks 2.4.

(a) For any Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$, $\mathcal{M} \supseteq \mathcal{N} \supseteq \mathcal{D}$ because $\mathcal{D}$ is abelian.

(b) If $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple, then $(\mathcal{M}, \mathcal{N}, \mathcal{Z}(\mathcal{N}))$ is a full Cartan triple. Indeed, $\mathcal{N} = \mathcal{D}^c$ implies $\mathcal{N} = (\mathcal{Z}(\mathcal{N}))^c$, and since $\mathcal{GN}(\mathcal{M}, \mathcal{D}) \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{Z}(N))$, $\mathcal{Z}(\mathcal{N})$ is regular in $\mathcal{M}$.

Section 6 is devoted to examples of Cartan triples. Here we content ourselves with making two simple observations regarding what occurs when two of the von Neumann algebras in a Cartan triple coincide.

Examples 2.5.

(a) Suppose $(\mathcal{M}, \mathcal{D}, \mathcal{D})$ is a Cartan triple. Then $(\mathcal{M}, \mathcal{D}, \mathcal{D})$ is full and $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$. In this sense, the class of Cartan triples includes the class of Cartan pairs.

(b) Now suppose $(\mathcal{N}, \mathcal{N}, \mathcal{D})$ is a Cartan triple. Then $\mathcal{D}^c = \mathcal{N}$. When $\mathcal{N}$ has separable predual, we may write $\mathcal{D} = L^\infty(X, \mu)$ and write $\mathcal{N} = \int_X^\oplus \mathcal{N}_x \, d\mu(x)$ as a direct integral. When this is done, $\mathcal{GN}(\mathcal{N}, \mathcal{D})$ may be identified with the set of all functions $f \in \int_X^\oplus \mathcal{N}_x \, d\mu(x)$ such that for almost every $x \in X$, $f(x) \in \mathcal{U}(\mathcal{N}_x) \cup \{0\}$.

Example 2.5(b) shows how the inverse semigroup $\mathcal{GN}(\mathcal{N}, \mathcal{D})$ can be used to describe a direct integral. Further, this inverse semigroup approach allows one to work with von Neumann algebras which do not have separable predual. This example discussed further in Example 2.15.

We fix some notation for inverse semigroups next. For the most part, our notation follows Section 2 of [12], which also gives much of the inverse semigroup theory we will use. For an in-depth text on inverse semigroups see [21]. Throughout the paper:

- $\mathcal{E}(\mathcal{S})$ will denote the idempotents of the inverse semigroup $\mathcal{S}$;
- we use $s^\dagger$ to denote the inverse of the element s in an abstract inverse semigroup; however, for an inverse semigroup of partial isometries on a Hilbert space, the adjoint v^* is the inverse of the element v and we typically use v^* instead of $v^\dagger$ in this setting.

For our extensions, we need two special classes of inverse monoids: Cartan inverse monoids, defined in [12], and $\mathcal{N}$-Clifford inverse monoids, which are new.

Definition 2.6 ([12, Definition 2.11])**.** We call an inverse semigroup $\mathcal{S}$ a *Cartan inverse monoid* if

(a) $\mathcal{S}$ is fundamental;

(b) $\mathcal{S}$ is a complete Boolean inverse monoid; and

(c) the character space $\widehat{\mathcal{E}(\mathcal{S})}$ of the complete Boolean lattice $\mathcal{E}(\mathcal{S})$ is a hyperstonean topological space.

Definition 2.7. Let $\mathcal{N}$ be a von Neumann algebra. An $\mathcal{N}$-*Clifford inverse monoid* is an inverse monoid $\mathcal{P}$ such that $\mathcal{P} = \mathcal{GN}(\mathcal{N}, \mathcal{D})$, where $\mathcal{D}$ is a von Neumann subalgebra of $\mathcal{Z}(\mathcal{N})$. If in addition $\mathcal{D} = \mathcal{Z}(\mathcal{N})$, we say $\mathcal{P}$ is a *full* $\mathcal{N}$-*Clifford inverse monoid.*

Remark 2.8. Suppose $\mathcal{P} = \mathcal{GN}(\mathcal{N}, \mathcal{D})$ is an $\mathcal{N}$-Clifford inverse monoid. It is not difficult to show that

$$\mathcal{P} = \{v \in \mathcal{N} : v \text{ is a partial isometry with } vv^* = v^*v \in \mathcal{D}\} \quad \text{and} \quad \mathcal{E}(\mathcal{P}) = \operatorname{proj}(\mathcal{D}).$$

In particular, $\mathcal{P}$ is a Clifford inverse semigroup of partial isometries.

Since $\mathcal{U}(\mathcal{N}) \subseteq \mathcal{P}$, every element of $\mathcal{N}$ is a linear combination of at most four elements of $\mathcal{P}$.

We need an appropriate notion of isomorphism of such inverse monoids.

Definition 2.9. If for $i = 1, 2$, $\mathcal{P}_i$ are $\mathcal{N}_i$-Clifford inverse monoids, a map $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ is an *extendible isomorphism* if there exists a normal $*$-isomorphism $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\alpha = \theta|_{\mathcal{P}_1}$; equivalently, there exists a normal isomorphism $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

Obviously, any extendible isomorphism is an isomorphism of inverse semigroups.

Definition 2.10. For $i = 1, 2$, let $\mathcal{S}_i$ be Cartan inverse monoids, let $\mathcal{P}_i$ be $\mathcal{N}_i$-Clifford inverse monoids, and suppose $\mathcal{P}_i \overset{\iota_i}{\hookrightarrow} \mathcal{G}_i \overset{q}{\twoheadrightarrow} \mathcal{S}_i$ are extensions of $\mathcal{S}_i$ by $\mathcal{P}_i$. These extensions are *equivalent* if there are semigroup isomorphisms $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$, $\tilde{\alpha} : \mathcal{S}_1 \to \mathcal{S}_2$ and an extendible isomorphism $\underline{\alpha} : \mathcal{P}_1 \to \mathcal{P}_2$ such that

$$\alpha \circ \iota_1 = \iota_2 \circ \underline{\alpha} \quad \text{and} \quad q_2 \circ \alpha = \tilde{\alpha} \circ q_1.$$

Remark 2.11. When $\mathcal{P}_i$ is the set of partial isometries in $C(\widehat{\mathcal{E}(\mathcal{S}_i)})$ and $\mathcal{N}_i := C(\widehat{\mathcal{E}(\mathcal{S}_i)})$, then this definition reduces to the notion of equivalence for extensions found in [12].

Definition 2.12. Let $\mathcal{G}$ be an inverse semigroup. The *Munn congruence* (also called the *Munn relation*) on $\mathcal{G}$ is the set

$$R_M := \{(v_1, v_2) \in \mathcal{G} \times \mathcal{G} : v_1 e v_1^\dagger = v_2 e v_2^\dagger \text{ for all } e \in \mathcal{E}(\mathcal{G})\}.$$

Then R_M is the maximal idempotent separating congruence on $\mathcal{G}$ and the set of R_M-equivalence classes equipped with the product $[v][w] = [vw]$ and inverse $[v]^\dagger = [v^\dagger]$ form a fundamental inverse semigroup [21, Proposition 5.2.5].

We now show how a Cartan triple gives rise to an extension of inverse semigroups.

Proposition 2.13. *Let* $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *be a Cartan triple and set*

$$\mathcal{G} := \mathcal{GN}(\mathcal{M}, \mathcal{D}) \quad \textit{and} \quad \mathcal{P} := \mathcal{GN}(\mathcal{N}, \mathcal{D}).$$

Then $\mathcal{G}$ *and* $\mathcal{P}$ *are inverse semigroups with* $\mathcal{P} \subseteq \mathcal{G}$ *and*

$$\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{G}) = \mathit{proj}(\mathcal{D}).$$

Moreover, the following statements hold.

(a) $\mathcal{P}$ *is a* $\mathcal{N}$*-Clifford inverse monoid.*
(b) $\mathcal{P}$ *is the set of elements of* $\mathcal{G}$ *Munn-related to an idempotent.*
(c) If $\mathcal{S}$ *is the quotient of* $\mathcal{G}$ *by the Munn congruence, then* $\mathcal{S}$ *is a Cartan inverse monoid.*

Proof. Obviously, $\mathcal{P} \subseteq \mathcal{G}$ and by definition, $\mathcal{P}$ is an $\mathcal{N}$-Clifford inverse monoid. If $v \in \mathcal{G}$, then $v^*v \in \mathcal{D}$, so every idempotent of $\mathcal{G}$ is a projection in $\mathcal{D}$. Also, every projection in $\mathcal{D}$ is an idempotent in $\mathcal{G}$. It follows that $\mathcal{G}$ and $\mathcal{P}$ are von Neumann regular monoids for which the idempotents commute, so both are inverse monoids and

$$\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{G}) = \mathrm{proj}(\mathcal{D}).$$

If $v \in \mathcal{P}$, then $(v, vv^*) \in R_M$, so every element of $\mathcal{P}$ is Munn-related to an idempotent. On the other hand, suppose $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ is Munn-related to the idempotent e. Then $vv^* = e$. Let p be a projection in $\mathcal{D}$. Then

$$vp = vpv^*v = epv = pev = pv.$$

Hence $v \in \mathcal{N}$, and so $v \in \mathcal{P}$. Thus, $\mathcal{P}$ is the set of elements of $\mathcal{G}$ Munn related to an idempotent.

We have already observed that $\mathcal{S}$ is a fundamental inverse monoid, and it clearly contains a zero element 0. As the Munn congruence is idempotent separating, $\mathcal{E}(\mathcal{S})$ is isomorphic to $\mathcal{E}(\mathcal{P})$. The proof that $\mathcal{S}$ is a Cartan inverse monoid now follows exactly as in the proof of [12, Proposition 3.5]. $\square$

As noted in the introduction, any inverse semigroup G may be represented as an extension of a fundamental inverse semigroup S by a Clifford inverse semigroup C. Indeed, C may be taken to be the set of elements of G which are Munn-related to an idempotent, and S is the quotient of G by the Munn relation. We apply this construction to the inverse semigroup $\mathcal{G}$ of Proposition 2.13 to obtain the class of extensions studied in this paper.

Definition 2.14. Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple. Put $\mathcal{G} := \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $\mathcal{P} := \mathcal{GN}(\mathcal{N}, \mathcal{D})$, $\mathcal{S} := \mathcal{G}/R_M$, and let $q : \mathcal{G} \to \mathcal{S}$ the quotient map. The extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}. \tag{2.2}$$

is called the *extension associated to* $(\mathcal{M}, \mathcal{N}, \mathcal{D})$.

Example 2.15. We return to the context of Example 2.5(b), that is, of a Cartan triple having the form $(\mathcal{N}, \mathcal{N}, \mathcal{D})$. In this setting, $\mathcal{P} = \mathcal{G}$ consists of the partial isometries in $\mathcal{N}$ whose initial and final spaces coincide and belong to $\mathcal{D}$; $\mathcal{S}$ is the projection lattice of $\mathcal{D}$; and q is the map $v \mapsto v^*v$. Note that $\mathcal{N}$ is the linear span of $\mathcal{P}$. When $\mathcal{N}_*$ is separable and $\mathcal{N}$ is identified as the direct integral $\int_X^{\oplus} \mathcal{N}_x \, d\mu(x)$, we may view the extension $\mathcal{P} \hookrightarrow \mathcal{P} \overset{q}{\twoheadrightarrow} \mathcal{S}$ as giving a description of the direct integral in terms of the linear span of

$$\left\{ f \in \int_X^{\oplus} \mathcal{M}_x \, d\mu(x) : f(x) \in \mathcal{U}(\mathcal{M}_x) \cup \{0\} \text{ for almost every } x \right\}.$$

The extension approach encodes the measure theory into $\mathcal{D}$, and is a more operator theoretic view of $\mathcal{N}$ as opposed to the point based view of $\mathcal{N}$ as a direct integral.

In the study of extensions, it is often useful to choose a section j for the quotient map q, that is, j is a map such that $q \circ j = \mathrm{id}|_{\mathcal{S}}$. In our context, we will frequently need a section which is *order preserving* in the sense that $j(1) = 1$ and whenever $s, t \in \mathcal{S}$ and $s \leq t$, we have $j(s) \leq j(t)$ (see [12, Definition 4.1]). Most of the following result was proved in [12] when $\mathcal{P}$ is the set of partial isometries in $C^*(\mathcal{E}(\mathcal{S}))$, but the same proof holds for extensions of $\mathcal{S}$ by $\mathcal{N}$-Clifford inverse monoids considered here.

Recall that $q|_{\mathcal{E}(\mathcal{G})}$ is a complete Boolean algebra isomorphism of $\mathcal{E}(\mathcal{G})$ onto $\mathcal{E}(\mathcal{S})$. Also, as observed in [12, Remark 4.8], for any $s, t \in \mathcal{S}$, $(s^\dagger t \wedge 1)$ is the source idempotent for $s \wedge t$, that is,

$$(s \wedge t)^\dagger (s \wedge t) = s^\dagger t \wedge 1. \tag{2.3}$$

Proposition 2.16 (c.f. [12, Proposition 4.6])**.** *Let $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ be an extension of the Cartan inverse monoid $\mathcal{S}$ by the $\mathcal{N}$-Clifford inverse monoid $\mathcal{P}$. The map $(q|_{\mathcal{E}(\mathcal{S})})^{-1}$ extends to an order preserving section $j : \mathcal{S} \to \mathcal{G}$ for q. This section has the property that for $s_1, s_2 \in \mathcal{S}$,*

$$j(s_1)^\dagger j(s_2) j(s_1^\dagger s_2 \wedge 1) = j(s_1^\dagger s_2 \wedge 1). \tag{2.4}$$

Proof. Equation (2.4) was not proved in [12], so we provide a proof here. Using (2.3), observe

$$\begin{aligned} j(s_1)^\dagger j(s_2) j(s_1^\dagger s_2 \wedge 1) &= j(s_1)^\dagger j(s_1 \wedge s_2) \\ &= \Big(j(s_1 \wedge s_2) j(s_1 \wedge s_2)^\dagger j(s_1) \Big)^\dagger j(s_1 \wedge s_2) \\ &= j(s_1 \wedge s_2)^\dagger j(s_1 \wedge s_2) = j(s_1^\dagger s_2 \wedge 1). \end{aligned}$$ □

Definition 2.17. We say that the Cartan triples $(\mathcal{M}_1, \mathcal{N}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{N}_2, \mathcal{D}_2)$ are *isomorphic* if there exists a normal $*$-isomorphism $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

Our goal is to show that Cartan triples, up to isomorphism, are uniquely determined by their associated extensions, up to equivalence. We do this in Theorem 2.22. We first need some technical lemmas.

Throughout the remainder of the section, fix an order-preserving section j for the extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ associated to the Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$.

Lemma 2.18. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple with conditional expectation $E : \mathcal{M} \to \mathcal{N}$. Let $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ be the associated extension. Then $E(\mathcal{G}) = \mathcal{P}$. Further, for $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$,*

$$E(v) = ve, \quad \textit{where} \quad e = j(q(v) \wedge 1) \in \mathcal{E}(\mathcal{G}).$$

Also, E preserves the natural inverse semigroup partial order on $\mathcal{G}$ and $\mathcal{P}$ in the sense that if $v, w \in \mathcal{G}$ with $v \leq w$, then $E(v) \leq E(w)$.

Proof. Each $v \in \mathcal{G}$ induces a normal $*$-isomorphism θ_v from $v^*v\mathcal{D}$ to $vv^*\mathcal{D}$, given by $\theta_v(d) = vdv^*$. Applying Frolík's Theorem [26, Proposition 2.11A] to θ_v we may find elements $e_0, e_1, e_2, e_3 \in \mathcal{E}(\mathcal{G})$ such that

(a) for $i \neq j$, $e_i \wedge e_j = 0$;
(b) $e_0 \vee e_1 \vee e_2 \vee e_3 = v^*v$;
(c) for $i = 1, 2, 3$, $(ve_i)^2 = 0$; and
(d) $q(ve_0) \in \mathcal{E}(\mathcal{S})$.

If $w \in \mathcal{G}$ and $w^2 = 0$, then

$$E(w) = ww^*E(w) = E(w)ww^* = E(www^*) = 0.$$

It follows that

$$E(v) = E(ve_0) = ve_0.$$

Let $e := e_0$. Then $q(e) = q(v) \wedge 1$ because e_0 corresponds to the ideal of $\mathcal{D}$ consisting of all elements fixed by θ_v.

Finally, if $v, w \in \mathcal{G}$ and $v \leq w$, we may find $f \in \mathcal{E}(\mathcal{G})$ so that $v = wf$. Then $E(v) = E(w)f$, so $E(v) \leq E(w)$. ☐

Lemma 2.19. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple, and suppose y belongs to the linear span of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$. Then there exists a finite set $\{w_k\}_{k=1}^m \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $E(w_j^*w_k) = 0$ for $j \neq k$ and*

$$y = \sum_{k=1}^{m} w_k E(w_k^* y). \tag{2.5}$$

Proof. Choose $\{v_j\}_{j=1}^N \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and scalars $\{c_j\}_{j=1}^N$ so that $c_jv_j \neq 0$ for each j and $y = \sum_{j=1}^N c_jv_j$. Let $s_i := q(v_i)$ and apply [12, Lemma 4.15] to obtain a finite set $\{t_k\}_{k=1}^m \subseteq \mathcal{S}$ satisfying

(a) for $1 \leq j \leq m$, $t_j \neq 0$;
(b) for $j \neq k$, $t_j \wedge t_k = 0$;
(c) for $1 \leq j \leq m$ and $1 \leq n \leq N$, $t_j \wedge s_n \in \{0, t_j\}$;
(d) for $1 \leq j \leq m$ there exists $1 \leq n \leq N$ such that $t_j \wedge s_n = t_j$; and
(e) for each $1 \leq n \leq N$, $s_n = \bigvee\{t_j : t_j \leq s_n\}$.

Let $w_k := j(t_k)$. Lemma 2.18 implies that $E(w_j^*w_k) = 0$ when $j \neq k$.

For $1 \leq n \leq N$, let $I_n := \{i : t_i \leq s_n\}$. Given n and $i \in I_n$, another application of Lemma 2.18 gives

$$w_iE(w_i^*v_n) = w_iw_i^*v_nj(t_i^\dagger s_n \wedge 1) = v_n(v_n^*w_iw_i^*v_n)j(t_i^\dagger s_n \wedge 1) = v_nj(t_j^\dagger s_n \wedge 1).$$

Since $s_n = \bigvee_{i \in I_n}(t_i \wedge s_n)$, we obtain,

$$v_n = \sum_{i=1}^{m} w_iE(w_i^*v_n).$$

Equation (2.5) follows. ☐

We now recall notation regarding weights on a von Neumann algebra used in [30]. Suppose $\mathcal{M}$ is a von Neumann algebra and ϕ is a weight on $\mathcal{M}$. Recall that

$$\mathfrak{p}_\phi := \{x \in \mathcal{M}_+ : \phi(x) < \infty\}, \quad \mathfrak{n}_\phi := \{x \in \mathcal{M} : \phi(x^*x) < \infty\} \text{ and}$$
$$\mathfrak{m}_\phi := \left\{ \sum_{k=1}^{N} y_k^* x_k : n \in \mathbb{N}, x_k, y_k \in \mathfrak{n}_\phi \right\}.$$

By [30, Lemma VII.1.2], $\mathfrak{p}_\phi$ is a hereditary convex cone in $\mathcal{M}_+$, $\mathfrak{n}_\phi$ is a left ideal of $\mathcal{M}$, $\mathfrak{m}_\phi$ is a hereditary $*$-algebra of $\mathcal{M}$, and every element of $\mathfrak{m}_\phi$ is a linear combination of four elements of $\mathfrak{p}_\phi$. The semi-cyclic representation π_ϕ of $\mathcal{M}$ on the Hilbert space $\mathcal{H}_\phi$ associated to ϕ will be denoted $(\pi_\phi, \mathcal{H}_\phi, \eta_\phi)$. When ϕ is faithful, normal and semi-finite, π_ϕ is a faithful, normal representation of $\mathcal{M}$.

Lemma 2.20. *Suppose $\mathcal{M}$ is a von Neumann algebra, $\mathcal{L} \subseteq \mathcal{M}$ is a von Neumann subalgebra, and there exists a faithful normal conditional expectation $E : \mathcal{M} \to \mathcal{L}$. Let ψ be a faithful, normal, semi-finite weight on $\mathcal{L}$ and let $\phi := \psi \circ E$. Then ϕ is a faithful normal semi-finite weight on $\mathcal{M}$.*

Proof. Since $\mathfrak{m}_\psi$ is a $*$-subalgebra of $\mathcal{L}$, a corollary of the Kaplansky density theorem shows there exists a net (x_λ) in $\mathfrak{p}_\psi$ with $0 \le x_\lambda \le I$ which converges σ-strongly to I. For any $z \in \mathcal{M}_+$, $x_\lambda z x_\lambda \le \|z\| p_\lambda^2 \in \mathfrak{p}_\psi$. Therefore, $x_\lambda z x_\lambda \in \mathfrak{p}_\phi$. Since $\lim^{\sigma\text{-strong}} x_\lambda z x_\lambda = z$, we have that $\mathfrak{p}_\phi$ generates $\mathcal{M}$. That is, ϕ is semi-finite on $\mathcal{M}$. $\square$

The following result is the key technical tool used in the proof of Theorem 2.22.

Lemma 2.21. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple, suppose ψ is a faithful normal semi-finite weight on $\mathcal{N}$ and let $\phi = \psi \circ E$. Then the linear span of $\{\eta_\phi(vn) : v \in \mathcal{GN}(\mathcal{M}, \mathcal{D}) \text{ and } n \in \mathfrak{n}_\psi\}$ is dense in $\mathcal{H}_\phi$.*

Proof. For $m \in \mathfrak{n}_\phi$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$,

$$\psi(E(m^*v)v^*vE(v^*m)) = \psi(E(m^*v)E(v^*m)) \le \psi(E(m^*vv^*m)) \le \psi(E(m^*m)) = \phi(m^*m) < \infty.$$

This yields the following.

(a) $E(\mathfrak{n}_\phi) = \mathfrak{n}_\psi$ (take $v = I$).
(b) For $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, the map $\mathcal{M} \ni m \mapsto vE(v^*m)$ is idempotent and leaves $\mathfrak{n}_\phi$ invariant; hence there is a projection $P_v \in \mathcal{B}(\mathcal{H}_\phi)$ whose action on $\eta_\phi(\mathfrak{n}_\phi)$ is given by $P_v\eta_\phi(m) = \eta_\phi(vE(v^*m))$. In addition, notice that range $P_v = \overline{\{\eta_\phi(vn) : n \in \mathfrak{n}_\psi\}}$.

To prove the lemma, it therefore suffices to show that if $\xi \in \mathcal{H}_\phi$ and $P_v\xi = 0$ for every $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, then $\xi = 0$. We begin with a preliminary fact about approximating norms of vectors in $\mathcal{H}_\phi$.

Let

$$\Phi := \{\tau \circ E : \tau \in \mathcal{N}_*^+ \text{ and } \tau(n) \le \psi(n) \text{ for all } 0 \le n \in \mathcal{N}\}.$$

Clearly $\Phi \subseteq \mathcal{M}_*^+$. For $\omega \in \Phi$, let $(\pi_\omega, \mathcal{H}_\omega, \eta_\omega)$ be the semi-cyclic representation of $\mathcal{M}$ arising from ω. This representation is actually cyclic and $\mathfrak{n}_\omega = \mathcal{M}$ because ω is a bounded positive linear functional on $\mathcal{M}$. Define $T_\omega : \eta_\phi(\mathfrak{n}_\phi) \to \mathcal{H}_\omega$ by $T_\omega\eta_\phi(m) = \eta_\omega(m)$. Write $\omega = \rho \circ E$ for some $\rho \in \mathcal{N}_*^+$. Then for $m \in \mathfrak{n}_\phi$,

$$\|\eta_\omega(m)\|^2 = \rho(E(m^*m)) \le \psi(E(m^*m)) = \|\eta_\phi(m)\|^2.$$

Thus T_ω extends to a contraction belonging to $\mathcal{B}(\mathcal{H}_\phi, \mathcal{H}_\omega)$, which we again denote by T_ω.

We claim that for any $\xi \in \mathcal{H}_\phi$,

$$\|\xi\| = \sup_{\omega \in \Phi} \|T_\omega \xi\|. \tag{2.6}$$

To see this, fix $\xi \in \mathcal{H}_\phi$ and choose a real number r such that $r < \|\xi\|$. Let $\varepsilon > 0$ satisfy $3\varepsilon < \|\xi\| - r$. Choose $m \in \mathfrak{n}_\phi$ such that $\|\xi - \eta_\phi(m)\| < \varepsilon$. By Haagerup's Theorem (see [30, Theorem VII.1.11]),

$$\psi(E(m^*m)) = \sup\{\tau(E(m^*m)) : \tau \in \mathcal{N}_*^+ \text{ and } \tau(n) \leq \psi(n) \text{ for all } 0 \leq n \in \mathcal{N}\}.$$

Hence there exists $\omega \in \Phi$ such that

$$\|T_\omega \eta_\phi(m)\| > \|\eta_\phi(m)\| - \varepsilon.$$

Then

$$\begin{aligned} \|\xi\| &\leq \|\xi - \eta_\phi(m)\| + \|\eta_\phi(m)\| < 2\varepsilon + \|\eta_\phi(m)\| - \varepsilon \\ &< 2\varepsilon + \|T_\omega \eta_\phi(m)\| \\ &\leq 3\varepsilon + \|T_\omega \xi\| < \|\xi\| - r + \|T_\omega \xi\|, \end{aligned}$$

whence $r < \|T_\omega \xi\|$. Thus (2.6) holds.

For each $\omega \in \Phi$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, let P_v^ω be the projection on $\mathcal{H}_\omega$ determined by $\eta_\omega(m) \mapsto \eta_\omega(vE(v^*m))$, $m \in \mathcal{M}$. A routine calculation shows that for every $\omega \in \Phi$, $m \in \mathfrak{n}_\phi$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $T_\omega P_v \eta_\phi(m) = P_v^\omega T_\omega \eta_\phi(m)$, so

$$T_\omega P_v = P_v^\omega T_\omega. \tag{2.7}$$

Fix $\omega \in \Phi$. We claim that if $\zeta \in \mathcal{H}_\omega$ and $P_v^\omega \zeta = 0$ for every $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, then $\zeta = 0$. Suppose $\zeta \in \mathcal{H}_\omega$ is such a vector. Given $\varepsilon > 0$, there exists $x \in \mathcal{M}$ such that $\|\zeta - \eta_\omega(x)\| < \varepsilon$. Since $\mathcal{D}$ is regular in $\mathcal{M}$, and $\omega \in \mathcal{M}_*^+$, there exists $y \in \operatorname{span} \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $\|\eta_\omega(x - y)\| < \varepsilon$. By Lemma 2.19, there exists a finite E-orthogonal set $\{v_k\}_{k=1}^m \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $y = \sum_{k=1}^m v_k E(v_k^* y)$. As $\{P_{v_k}^\omega\}_{k=1}^n$ is a pairwise orthogonal set of projections, $Q := \sum_{k=1}^m P_{v_k}^\omega$ is a projection. Therefore,

$$\eta_\omega(y) = \sum_{k=1}^m P_{v_k}^\omega \eta_\omega(y) = Q\eta_\omega(y).$$

So,

$$\begin{aligned} \|\zeta\| &\leq \|\zeta - \eta_\omega(x)\| + \|\eta_\omega(x - y)\| + \|\eta_\omega(y)\| \\ &< 2\varepsilon + \|\eta_\omega(y) - Q\zeta\| = 2\varepsilon + \|Q(\eta_\omega(y) - \zeta)\| \\ &\leq 2\varepsilon + \|\eta_\omega(y) - \zeta\| \\ &< 4\varepsilon. \end{aligned}$$

Thus the claim holds.

Now suppose $\xi \in \mathcal{H}_\phi$ satisfies $P_v \xi = 0$ for every $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Then for every $\omega \in \Phi$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$,

$$0 = T_\omega P_v \xi = P_v^\omega T_\omega \xi.$$

Hence $T_\omega \xi = 0$ for every $\omega \in \Phi$, so $\xi = 0$ by (2.6). The proof is now complete. □

We come now to the main theorem of this section.

Theorem 2.22. *The Cartan triples $(\mathcal{M}_1, \mathcal{N}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{N}_2, \mathcal{D}_2)$ are isomorphic if and only if their associated extensions, $\mathcal{P}_1 \hookrightarrow \mathcal{G}_2 \overset{q_1}{\twoheadrightarrow} \mathcal{S}_2$ and $\mathcal{P}_1 \hookrightarrow \mathcal{G}_2 \overset{q_2}{\twoheadrightarrow} \mathcal{S}_2$, are equivalent.*

Proof. It is easy to see that if the triples are isomorphic, then their associated extensions are equivalent.

Suppose now that the associated extensions are equivalent via the triple of maps $(\underline{\alpha}, \alpha, \tilde{\alpha})$. Then $\alpha|_{\mathcal{P}_1} = \underline{\alpha}$, $q_2 \circ \alpha = \tilde{\alpha} \circ q_1$ and $\underline{\alpha}$ is an extendible isomorphism, say $\underline{\alpha} = \theta|_{\mathcal{P}}$, where $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ is a normal isomorphism with $\theta(\mathcal{D}_1) = \mathcal{D}_2$. Let $E_i : \mathcal{M}_i \to \mathcal{N}_i$ be the conditional expectations. By Lemma 2.18,

$$E_2 \circ \alpha = (\alpha \circ E_1)|_{\mathcal{G}_1}, \quad \text{equivalently} \quad E_2 \circ \alpha = (\theta \circ E_1)|_{\mathcal{G}_1}.$$

Let ψ_1 be a faithful normal weight on $\mathcal{N}_1$ and let $\psi_2 = \psi_1 \circ \theta^{-1}$. Now let $\phi_i := \psi_i \circ E_i$. Then ϕ_i are faithful semi-finite normal weights on $\mathcal{M}_i$. Let $(\pi_i, \mathfrak{H}_i, \eta_i)$ be the associated semi-cyclic representations and let $\mathfrak{n}_i := \{x \in \mathcal{M}_i : \phi_i(x^*x) < \infty\}$. By Lemma 2.21, $\operatorname{span}\{\eta_{\phi_i}(vn) : v \in \mathcal{GN}(\mathcal{M}_i, \mathcal{D}_i)$ and $n \in \mathfrak{n}_{\psi_i}\}$ is dense in $\mathfrak{H}_i$.

Let $n \in \mathbb{N}$ and suppose $v_1, \ldots, v_n \in \mathcal{G}_1$ and $c_1, \ldots, c_n \in \mathfrak{n}_{\psi_1}$. Then $\alpha(v_j) \in \mathcal{G}_2$, and, since $(\alpha \circ E_1)|_{\mathcal{G}_1} = E_2 \circ \alpha$, it follows from the definition of ϕ_2 that

$$\begin{aligned}
\phi_2\left(\left(\sum_{i=1}^n \alpha(v_i)\theta(c_i)\right)^* \left(\sum_{i=1}^n \alpha(v_i)\theta(c_i)\right)\right) &= \phi_2\left(\sum_{i,j=1}^n \theta(c_i)^*\alpha(v_i^*v_j)\theta(c_j)\right) \\
&= \psi_2\left(E_2\left(\sum_{i,j=1}^n \theta(c_i)^*\alpha(v_i^*v_j)\theta(c_j)\right)\right) \\
&= \psi_2\left(\sum_{i,j=1}^n \theta(c_i)^*E_2(\alpha(v_i^*v_j))\theta(c_j)\right) \\
&= \psi_2\left(\sum_{i,j=1}^n \theta(c_i)^*\theta(E_1(v_i^*v_j))\theta(c_j)\right) \\
&= \psi_1\left(\sum_{i,j=1}^n E_1(c_i^*v_i^*v_jc_j)\right) \\
&= \phi_1\left(\left(\sum_{i=1}^n v_ic_i\right)^* \left(\sum_{i=1}^n v_ic_i\right)\right).
\end{aligned}$$

Hence the map

$$\eta_1\left(\sum_{i=1}^n v_ic_i\right) \mapsto \eta_2\left(\sum_{i=1}^n \alpha(v_i)\theta(c_i)\right)$$

extends to a unitary operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$. It is routine to verify that for $v \in \mathcal{G}_1$, $U\pi_1(v) = \pi_2(\alpha(v))U$. Therefore the map $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ given by $\theta(x) = \pi_2^{-1}(U\pi_1(x)U^*)$ is an isomorphism of $(\mathcal{M}_1, \mathcal{N}_1, \mathcal{D}_1)$ onto $(\mathcal{M}_2, \mathcal{N}_2, \mathcal{D}_2)$. $\square$

3. Representing an extension

In this section we will show how to represent an extension as partial isometries on a right Hilbert-module. In Section 4 we will show how this gives rise to a Cartan triple. Throughout this section:

- $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ will be an idempotent separating extension of the Cartan inverse monoid $\mathcal{S}$ by the $\mathcal{N}$-Clifford inverse monoid $\mathcal{P}$;
- $j : \mathcal{S} \to \mathcal{G}$ will be a fixed order-preserving section (see Proposition 2.16); and
- $\mathcal{D}$ is the von Neumann subalgebra of $\mathcal{Z}(\mathcal{N})$ generated by $\mathcal{E}(\mathcal{P})$. We will sometimes use the fact that viewed as a C^*-algebra, $\mathcal{D}$ is isomorphic to the universal C^*-algebra $C^*(\widehat{\mathcal{E}(\mathcal{S})})$ generated by the meet semilattice $\mathcal{E}(\mathcal{S})$, see [12, Proposition 2.2].

We now construct a right reproducing kernel Hilbert $\mathcal{N}$-module. We begin by using the construction of the right reproducing Hilbert $\mathcal{D}$-module as done in [12, Section 4.2]. Recall that $j|_{\mathcal{E}(\mathcal{S})}$ is a complete lattice isomorphism of $\mathcal{E}(\mathcal{S})$ onto $\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{G}) \subseteq \mathcal{D}$. Define $K : \mathcal{S} \times \mathcal{S} \to \mathcal{D}$ by

$$K(t, s) = j(s^\dagger t \wedge 1).$$

and for $s \in \mathcal{S}$, define $k_s : \mathcal{S} \to \mathcal{D}$ by

$$k_s(t) = K(t, s).$$

For $d \in \mathcal{D}$ and $s \in \mathcal{S}$, we use $k_s d$ to denote the map from $\mathcal{S}$ into $\mathcal{D}$ given by $\mathcal{S} \ni t \mapsto k_s(t)d$. Put

$$\mathfrak{A}_0 := \operatorname{span}\{k_s d : s \in \mathcal{S} \text{ and } d \in \mathcal{D}\}.$$

Let $u, v \in \mathfrak{A}_0$. Lemma 4.11 and Proposition 4.12 of [12] show that:

(a) if $u = \sum_{i=1}^n k_{s_i} d_i$ and $v = \sum_{j=1}^n k_{t_j} e_j$, then the formula

$$\langle u, v\rangle := \sum_{i,j=1}^n d_i^* K(s_i, t_j) e_j \tag{3.1}$$

is independent of the choice of the representations for u and v and determines a well-defined $\mathcal{D}$-valued inner product on $\mathfrak{A}_0$ which is conjugate linear in the first variable;

(b) for every $s \in \mathcal{S}$ and $u \in \mathfrak{A}_0$, $\langle k_s, u\rangle = u(s)$;

(c) the completion $\mathfrak{A}_{\mathcal{D}}$ of $\mathfrak{A}_0$ with respect to this inner product is a right Hilbert $\mathcal{D}$-module of functions from $\mathcal{S}$ to $\mathcal{D}$; and

(d) $\operatorname{span}\{k_s : s \in \mathcal{S}\}$ is dense in $\mathfrak{A}_{\mathcal{D}}$.

Next, we "fatten" $\mathfrak{A}_{\mathcal{D}}$ to incorporate the fact that $\mathcal{P}$ is an $\mathcal{N}$-Clifford semigroup, not a $\mathcal{D}$-Clifford semigroup as in [12]. View $\mathcal{N}$ as a right Hilbert $\mathcal{N}$-module, with $\langle x, y\rangle_{\mathcal{N}} := x^* y$. Define a $*$-monomorphism $\iota : \mathcal{D} \to \mathcal{L}(\mathcal{N})$ by $\iota(d)(x) = dx$ (where $d \in \mathcal{D}$ and $x \in \mathcal{N}$). Put

$$\mathfrak{A} := \mathfrak{A}_{\mathcal{D}} \otimes_\iota \mathcal{N}, \tag{3.2}$$

see [19, pages 38–44]. Then $\mathfrak{A}$ is a right Hilbert $\mathcal{N}$-module. This is the space on which we shall define a representation of $\mathcal{G}$. Note that the inner product on the algebraic tensor product $\mathfrak{A}_{\mathcal{D}} \odot_\iota \mathcal{N}$ is

$$\left\langle \sum_{i=1}^N k_{s_i} \otimes x_i, \sum_{i=1}^N k_{t_i} \otimes y_i \right\rangle_{\mathfrak{A}} = \sum_{i,j=1}^N x_i^* K(s_i, t_j) y_j. \tag{3.3}$$

We denote the bounded, adjointable operators on $\mathfrak{A}$ by $\mathcal{L}(\mathfrak{A})$.

We will presently describe the representation of $\mathcal{G}$ on $\mathfrak{A}$. First, we need a little more machinery derived from our extension. It is usual to describe idempotent-separating extensions in terms of a cocycle function $\gamma\colon \mathcal{S} \times \mathcal{S} \to \mathcal{P}$. In the case when $\mathcal{P}$ is a abelian this is done explicitly by Lausch [20], leading to a one-to-one correspondence between extensions and the cohomology group $H^2(\mathcal{S}, \mathcal{P})$. In the case when $\mathcal{P}$ is not abelian D'Alarcao [11] has studied extensions, modelled on the Schreier extensions of groups. Though no cocycle is explicitly given, the construction again relies on functions from $\mathcal{S} \times \mathcal{S}$ to $\mathcal{P}$. In our setting, where we are assuming we have an extension $\mathcal{P} \hookrightarrow \mathcal{G} \twoheadrightarrow \mathcal{S}$, we instead work with a *cocycle-like* function from $\mathcal{G} \times \mathcal{S}$ into $\mathcal{P}$. This leads to significant computational simplifications when we define our representation of $\mathcal{G}$. To our knowledge, there is not a cohomological description of extensions when $\mathcal{P}$ is not abelian. As our cocycle-like function includes $\mathcal{G}$ a priori, our approach is unlikely to shed further light on that question.

Definition 3.1. Define a cocycle-like function $\sigma\colon \mathcal{G} \times \mathcal{S} \to \mathcal{P}$ by

$$\sigma(v, s) = j(q(v)s)^\dagger v j(s) = j(s^\dagger q(v^\dagger)) v j(s).$$

Since

$$q(\sigma(v, s)) = s^\dagger q(v^\dagger v) s \in \mathcal{E}(\mathcal{S}),$$

$\sigma(v, s) \in \mathcal{P}$. Thus σ indeed maps $\mathcal{G} \times \mathcal{S}$ into $\mathcal{P}$. Observe also that

$$\sigma(v, s)^* \sigma(v, s) = j(s^\dagger q(v^\dagger v) s) = j(s)^\dagger v^\dagger v j(s). \tag{3.4}$$

The following result gives the definition of the representation of $\mathcal{G}$ in $\mathcal{L}(\mathfrak{A})$ and is the analog of [12, Theorem 4.16] suitable for our context. While the outline of the proof is the same as the proof of [12, Theorem 4.16], there are differences. Due to the importance of the result for our work, we provide most of the details of the proof.

Theorem 3.2. *For $v \in \mathcal{G}$, $s \in \mathcal{S}$ and $x \in \mathcal{N}$, the formula,*

$$\lambda(v)(k_s \otimes x) := k_{q(v)s} \otimes \sigma(v,s)x$$

determines a partial isometry $\lambda(v) \in \mathcal{L}(\mathfrak{A})$. Moreover, $\lambda : \mathcal{G} \to \mathcal{L}(\mathfrak{A})$ is a one-to-one representation of $\mathcal{G}$ as partial isometries in $\mathcal{L}(\mathfrak{A})$.

Proof. Fix $v \in \mathcal{G}$, and set $r := q(v)$. Given $s_1, \ldots, s_N \in \mathcal{S}$, apply [12, Lemma 4.15] to obtain $A \subseteq \mathcal{S}$ satisfying:

(a) $0 \notin A$;
(b) if $a, b \in A$ then $a \wedge b = 0$;
(c) if $a \in A$ then $a \wedge s_n \in \{0, a\}$ for $1 \leq n \leq N$; and there exists $1 \leq n \leq N$ such that $a \wedge s_n = a$;
(d) for each $1 \leq n \leq N$, $s_n = \bigvee\{a \in A \colon a \leq s_n\}$.

Choose $c_1, \ldots, c_N \in \mathcal{N}$.

For $a \in A$ and $1 \leq m \leq N$, put

$$A_m := \{b \in A : b \leq s_m\} \quad \text{and} \quad c_a := \sum\{c_n : a \leq s_n\}.$$

Since $A_m \subseteq A$, the elements of A_m are pairwise meet orthogonal. Further, $\bigvee A_m = s_m$.

As in the proof of [12, Theorem 4.16],

$$\sum_{n=1}^{N} k_{s_n} \otimes c_n = \sum_{a \in A} k_a \otimes c_a. \tag{3.5}$$

Secondly, with routine modifications to the proof of [12, Equation (4.4)], we obtain

$$\sum_{n=1}^{N} k_{rs_n} \otimes \sigma(v, s_n)c_n = \sum_{a \in A} k_{ra} \otimes \sigma(v,a)c_a. \tag{3.6}$$

Notice that if $a, b \in A$ are distinct, then ra and rb are orthogonal, so for $x, y \in \mathcal{N}$,

$$\begin{aligned}
\langle k_{ra} \otimes \sigma(v,a)x, k_{rb} \otimes \sigma(v,b)y\rangle &= x^*\sigma(v,a)^*K(ra,rb)\sigma(v,b)y \\
&= 0 \\
&= x^*K(a,b)y \\
&= \langle k_a \otimes x, k_b \otimes y\rangle .
\end{aligned}$$

Thus, as $\mathcal{D} \subseteq \mathcal{Z}(\mathcal{N})$ and using (3.6), then (3.5),

$$\begin{aligned}
\left\langle \sum_{n=1}^{N} k_{rs_n} \otimes \sigma(v, s_n)c_n, \sum_{n=1}^{N} k_{rs_n} \otimes \sigma(v, s_n)c_n \right\rangle &= \left\langle \sum_{a \in A} k_{ra} \otimes \sigma(v, a)c_a, \sum_{a \in A} k_{ra} \otimes \sigma(v, a)c_a \right\rangle \\
&= \sum_{a \in A} |c_a|^2 \sigma(v, a)^* j(a^\dagger r^\dagger r a)\sigma(v, a) \\
&= \sum_{a \in A} |c_a|^2 j(a^\dagger r^\dagger r a) \\
&\leq \sum_{a \in A} |c_a|^2 j(a^\dagger a) \\
&= \left\langle \sum_{a \in A} k_a \otimes c_a, \sum_{a \in A} k_a \otimes c_a \right\rangle \\
&= \left\langle \sum_{n=1}^{N} k_{s_n} \otimes c_n, \sum_{n=1}^{N} k_{s_n} \otimes c_n \right\rangle.
\end{aligned}$$

Therefore,

$$\left\| \sum_{n=1}^{N} \lambda(v)(k_{s_n} \otimes c_n) \right\| \leq \left\| \sum_{n=1}^{n} k_{s_n} \otimes c_n \right\|.$$

It follows that we may extend $\lambda(v)$ linearly to a contractive operator from the algebraic tensor product $\mathfrak{A}_0 \odot_\iota \mathcal{N}$ into $\mathfrak{A}$. Finally extend $\lambda(v)$ by continuity to a contraction in $\mathcal{B}(\mathfrak{A})$, the bounded operators on $\mathfrak{A}$.

We next show that $\lambda(v)$ is adjointable. As in the proof of the corresponding equality found in the proof of [12, Theorem 4.16], for $s, t \in \mathcal{S}$,

$$\sigma(v, s)^\dagger K(rs, t) = \sigma(v^\dagger, t)K(s, r^\dagger t).$$

Therefore for any $s, t \in \mathcal{S}$ and $x, y \in \mathcal{N}$,

$$\begin{aligned}
\langle \lambda(v)(k_s \otimes x), k_t \otimes y \rangle &= \langle k_{rs} \otimes \sigma(v, s)x, k_t \otimes y \rangle = x^* \sigma(v, s)^* K(rs, t)y \\
&= x^* \sigma(v^\dagger, t)K(s, r^\dagger t)y = x^* K(s, r^\dagger t)\sigma(v^\dagger, t)y \\
&= \langle k_s \otimes x, \lambda(v^\dagger)(k_t \otimes y) \rangle.
\end{aligned}$$

This equality implies that $\lambda(v)$ is adjointable and $\lambda(v)^* = \lambda(v^\dagger)$.

We now show that λ is a homomorphism. Suppose that $v_1, v_2 \in \mathcal{G}$, $x \in \mathcal{N}$ and $s \in \mathcal{S}$. Then

$$\begin{aligned}
\lambda(v_1)(\lambda(v_2)(k_s \otimes x)) &= \lambda(v_1)(k_{q(v_2)s} \otimes \sigma(v_2, s)x) \\
&= k_{q(v_1 v_2)s} \otimes \sigma(v_1, q(v_2)s)\ \sigma(v_2, s)x.
\end{aligned}$$

But

$$\begin{aligned}
\sigma(v_1, q(v_2)s)\ \sigma(v_2, s) &= j(q(v_1)q(v_2)s))^\dagger v_1 j(q(v_2)s)\ j(q(v_2)s)^\dagger v_2 j(s) \\
&= j(q(v_1 v_2)s))^\dagger v_1 j(q(v_2)s)\ j(s^\dagger q(v_2)^\dagger) v_2 j(s) \\
&= j(q(v_1 v_2)s))^\dagger v_1 (v_2 j(ss^\dagger) v_2^\dagger) v_2 j(s) \\
&= j(q(v_1 v_2)s))^\dagger v_1 v_2 v_2^\dagger v_2 j(ss^\dagger) j(s) \\
&= j(q(v_1 v_2)s))^\dagger v_1 v_2 j(s) = \sigma(v_1 v_2, s).
\end{aligned}$$

Hence $\lambda(v_1)\lambda(v_2)(k_s \otimes x) = \lambda(v_1v_2)(k_s \otimes x)$. As $\operatorname{span}\{k_s \otimes x : s \in \mathcal{S} \text{ and } x \in \mathcal{N}\}$ is dense in $\mathfrak{A}$, we conclude that $\lambda(v_1v_2) = \lambda(v_1)\lambda(v_2)$. It follows that for every $e \in \mathcal{E}(\mathcal{G})$, $\lambda(e)$ is a projection. Furthermore, for $v \in \mathcal{G}$, $\lambda(v)$ is a partial isometry because $\lambda(v)^* = \lambda(v^\dagger)$.

It remains to show that λ is one-to-one. We first show that $\lambda|_{\mathcal{E}(\mathcal{G})}$ is one-to-one. Suppose $e, f \in \mathcal{E}(\mathcal{S})$ and $\lambda(j(e)) = \lambda(j(f))$. Then for every $s \in \mathcal{S}$, $\sigma(j(e), s) = j(s^\dagger es) \in \mathcal{D}$ and $\sigma(j(f), s) = j(s^\dagger fs) \in \mathcal{D}$. As the tensor product is balanced,

$$\begin{aligned} k_{es}j(s^\dagger es) \otimes I &= k_{es} \otimes \sigma(j(e), s) \\ &= \lambda(j(e))(k_s \otimes I) = \lambda(j(f))(k_s \otimes I) \\ &= k_{fs} \otimes \sigma(j(f), s) = k_{fs}j(s^\dagger fs) \otimes I, \end{aligned}$$

whence $k_{es}j(s^\dagger es) = k_{fs}j(s^\dagger fs)$. Taking $s = 1$ gives $k_e j(e) = k_f j(f)$. Evaluating these elements of $\mathfrak{A}_\mathcal{D}$ at $t = 1$ gives $j(e) = j(f)$, so $\lambda|_{\mathcal{E}(\mathcal{G})}$ is one-to-one.

Now suppose $v_1, v_2 \in \mathcal{G}$ and $\lambda(v_1) = \lambda(v_2)$. Then

$$\lambda(v_1^\dagger v_1) = \lambda(v_1^\dagger v_2) = \lambda(v_1^\dagger v_2)^* = \lambda(v_2^\dagger v_1) = \lambda(v_2^\dagger v_2).$$

Likewise,

$$\lambda(v_1v_1^\dagger) = \lambda(v_1v_2^\dagger) = \lambda(v_2v_1^\dagger) = \lambda(v_2v_2^\dagger).$$

Hence $v_1^\dagger v_1 = v_2^\dagger v_2$ and $v_1v_1^\dagger = v_2v_2^\dagger$. For any $e \in \mathcal{E}(\mathcal{S})$, we have

$$\begin{aligned} \lambda(v_1 j(e) v_1^\dagger) &= \lambda(v_1v_1^\dagger v_1 j(e) v_1^\dagger v_1 v_1^\dagger) = \lambda(v_1 v_2^\dagger v_2 j(e) v_2^\dagger v_2 v_1^\dagger) = \lambda(v_2 v_2^\dagger v_2 j(e) v_2^\dagger v_2 v_2^\dagger) \\ &= \lambda(v_2 j(e) v_2^\dagger). \end{aligned}$$

Hence $v_1 j(e) v_1^\dagger = v_2 j(e) v_2^\dagger$. Since this holds for every $e \in \mathcal{E}(\mathcal{S})$ and $\mathcal{S}$ is fundamental, we conclude that $q(v_1) = q(v_2)$.

Let $e := q(v_1^\dagger v_1)$ and $s := q(v_1)$. Since the functions $\lambda(v_1)k_e$ and $\lambda(v_2)k_e$ agree, we obtain, $k_s j(s)^\dagger v_1 = k_s j(s)^\dagger v_2$. Evaluating these functions at $t = s$ gives, $j(s)^\dagger v_1 = j(s)^\dagger v_2$. Multiplying each side of this equality on the left by $j(s)$, we obtain $v_1 = v_2$. ☐

Let $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ be a normal representation. Recall there is a $*$-representation $\pi_* : \mathcal{L}(\mathfrak{A}) \to \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$ given by

$$\pi_*(T)(u \otimes \xi) = (Tu) \otimes \xi. \tag{3.7}$$

This representation is strictly continuous on the unit ball of $\mathcal{L}(\mathfrak{A})$ and is faithful whenever π is faithful [19, p. 42]. As in [12, Corollary 4.17] we have the following corollary.

Corollary 3.3. *Let $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of $\mathcal{N}$ on the Hilbert space $\mathcal{H}$. Then $\lambda_\pi := \pi_* \circ \lambda$ is a representation of $\mathcal{G}$ by partial isometries on $\mathfrak{A} \otimes_\pi \mathcal{H}$. If π is faithful, then λ_π is one-to-one.*

Remark 3.4. Our construction of $\mathfrak{A} \otimes_\pi \mathcal{H}$ depends upon the order structure of $\mathcal{S}$. We show presently that the range of λ_π will generate a Cartan triple. When one starts with a Cartan triple and applies this construction to the extension associated to the pair, the Hilbert space $\mathfrak{A} \otimes_\pi \mathcal{H}$ can be recognized as arising from the representation associated to a faithful normal weight ϕ on $\mathcal{M}$ such that $\phi \circ E = \phi$. We will give the formal statement in Proposition 4.13 below.

We close this section with some results which will be needed when constructing a Cartan triple from an extension in Section 4. They will also be used in Section 5 when we study the $\mathcal{N}$-bimodule structure for a Cartan triple. Observe that the construction of the right Hilbert $\mathcal{D}$-module $\mathfrak{A}_\mathcal{D}$ above depends only upon $\mathcal{S}$ because $j|_{\mathcal{E}(\mathcal{S})}$ is the inverse of $q|_{\mathcal{E}(\mathcal{P})}$. The τ_1-topology described in the following definition has been considered by several authors, see Section 3.5 of the survey article [22].

Definition 3.5. Let $\mathfrak{M}$ be a right Hilbert module over the von Neumann algebra $\mathcal{N}$.

(a) The τ_1-*topology on* $\mathfrak{M}$ is the topology generated by the seminorms, $\xi \mapsto \phi(\langle \xi, \xi \rangle)^{1/2}$, where ϕ is a normal state on $\mathcal{N}$.

(b) The τ_1-*strict topology on* $\mathcal{L}(\mathfrak{M})$ is the topology generated by the seminorms, $T \mapsto \phi(\langle T\xi, T\xi \rangle)^{1/2}$ where $\xi \in \mathfrak{M}$ and ϕ is a normal state on $\mathcal{N}$.

Notice that a net $T_\alpha \to T$ in the τ_1-strict topology if and only if for every $\xi \in \mathfrak{M}$,

$$\langle (T_\alpha - T)\xi, (T_\alpha - T)\xi \rangle \to 0$$

in the σ-strong topology of $\mathcal{N}$.

The following result can be proved directly in the same way as Proposition 5.2 of [12], but it is simpler to apply [12, Theorem 4.16 and Proposition 5.2].

Proposition 3.6. *For $s \in \mathcal{S}$, the map $\mathcal{S} \ni t \mapsto k_{s\wedge t}$ extends to a projection $Q_s \in \mathcal{L}(\mathfrak{A}_\mathcal{D})$ whose range is $\overline{\operatorname{span}}\{k_t : t \leq s\}$. Furthermore, the following statements hold.*

(a) Let $s, t \in \mathcal{S}$. If $s \wedge t = 0$, then $Q_sQ_t = Q_tQ_s = 0$; if $s^\dagger t = st^\dagger = 0$, then $Q_t + Q_s = Q_{s\vee t}$.

(b) If $\mathcal{B} \subseteq \mathcal{S}$ is a maximal meet disjoint subset and Λ is the set of all finite subsets of $\mathcal{B}$ directed by inclusion, then the net $(\sum_{s\in F} Q_s)_{F\in\Lambda}$ converges τ_1-strictly to the identity operator in $\mathcal{L}(\mathfrak{A}_\mathcal{D})$.

Proof. By replacing $\sigma(v, t)$ with the identity operator throughout the proof of [12, Thoerem 4.16] (or Theorem 3.2 above) one finds that for every $s \in \mathcal{S}$, there exists a partial isometry $\lambda_0(s) \in \mathcal{L}(\mathfrak{A}_\mathcal{D})$ such that

$$\lambda_0(s)k_t = k_{st}.$$

Calculations show that for every $t \in \mathcal{S}$,

$$\lambda_0(s)P_\mathcal{D}\lambda_0(s)^*k_t = k_{s\wedge t},$$

where $P_\mathcal{D}k_t := k_{t\wedge 1}$ is the projection from [12, Proposition 5.2]. This establishes the existence of the projection Q_s, with the desired range.

The proof of (a) is routine and left to the reader. Let $\mathcal{B}$ be a maximal meet disjoint subset of $\mathcal{S}$. Since the net $(\sum_{s\in F} Q_s)_{F\in\Lambda}$ is an increasing net of projections, it suffices to show that for each $t \in \mathcal{S}$, the net

$$\left(\sum_{s\in F} Q_s k_t\right)_{F\in\Lambda} \quad \tau_1\text{-converges to } k_t.$$

For $F \in \Lambda$, let $Q_F := \sum_{s\in F} Q_s$. For $r \in \mathcal{S}$ and $s_1, s_2 \in F$, $r \wedge s_1$ and $r \wedge s_2$ are disjoint elements of $A_r := \{t \in \mathcal{S} : t \leq r\}$, so $(r \wedge s_1) \vee (r \wedge s_2)$ is defined. Let $t_F := \bigvee_{s\in F}(r \wedge s)$. Then $t_F \leq r$ and $Q_Fk_r = k_{t_F}$. Denote by $\neg$ the NOT operation in the Boolean algebra $\mathcal{E}(\mathcal{S})$. We have that

$$\langle Q_Fk_r - k_r, Q_Fk_r - k_r \rangle = \langle k_{t_F} - k_r, k_{t_F} - k_r \rangle = j(r^\dagger r \wedge \neg(t_F^\dagger t_F)).$$

Let $b := \bigvee_{F\in\Lambda}(t_F^\dagger t_F)$. Clearly $b \leq r^\dagger r$. Set $a := r^\dagger r \wedge (\neg b)$. Then for $s \in \mathcal{B}$,

$$a \wedge s^\dagger s = r^\dagger r \wedge s^\dagger s \wedge (\neg b) \leq r^\dagger r \wedge s^\dagger s \wedge \neg((r \wedge s)^\dagger (r \wedge s)) = 0.$$

Now $ra \wedge s = ra(r^\dagger r \wedge s^\dagger s) = 0$, so that ra is meet disjoint from every element of $\mathcal{B}$. By maximality of $\mathcal{B}$, we obtain $ra = 0$, whence $a = 0$. Thus $b = r^\dagger r$, from which it follows that $j(t_F^\dagger t_F)$ converges σ-strongly in $\mathcal{D}$ to $j(r^\dagger r)$. Therefore, Q_F converges τ_1-strictly to $I_{\mathcal{L}(\mathfrak{A}_\mathcal{D})}$. $\square$

We now have the following corollary to Proposition 3.6.

Corollary 3.7. *The net $Q_F \otimes I_\mathcal{N}$ converges τ_1-strictly to $I_{\mathcal{L}(\mathfrak{A})}$.*

Proof. For $n \in \mathcal{N}$ and $s \in \mathcal{S}$, we have

$$\begin{aligned}
\langle (Q_F \otimes I_{\mathcal{N}})(k_s \otimes n) - (k_s \otimes n), (Q_F \otimes I_{\mathcal{N}})(k_s \otimes n) - (k_s \otimes n)\rangle \\
&= \langle (Q_F k_s - k_s) \otimes n, (Q_F k_s - k_s) \otimes n\rangle \\
&= n^* \langle Q_F k_s - k_s, Q_F k_s - k_s\rangle_{\mathfrak{A}_{\mathcal{D}}} n \\
&= n^* n \langle Q_F k_s - k_s, Q_F k_s - k_s\rangle_{\mathfrak{A}_{\mathcal{D}}} .
\end{aligned}$$

As the last expression tends to zero in the σ-strong operator topology on $\mathcal{N}$, the result follows. ☐

Now let $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ be a faithful normal representation of $\mathcal{N}$ and for $s \in \mathcal{S}$ define projections on $\mathfrak{A} \otimes_\pi \mathcal{H}$ by

$$P_{s,\pi} := (Q_s \otimes I_{\mathcal{N}}) \otimes I_{\mathcal{H}}. \tag{3.8}$$

Proposition 3.8. *Let $\mathcal{B} \subseteq \mathcal{S}$ be a maximal meet disjoint subset. Then $\sum_{s\in\mathcal{B}} P_{s,\pi}$ converges strongly to $I \in \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$.*

Proof. For any $h \in \mathcal{H}$ and $\xi \in \mathfrak{A}$, the map

$$\mathcal{L}(\mathfrak{A}) \ni T \mapsto \langle ((T \otimes I_{\mathcal{H}})(\xi \otimes h), (T \otimes I_{\mathcal{H}})(\xi \otimes h)\rangle = \langle h, \pi(\langle T\xi, T\xi\rangle_{\mathfrak{A}})h\rangle_{\mathcal{H}}^{1/2}$$

is a τ_1-strict seminorm on $\mathcal{L}(\mathfrak{A})$. Therefore, if (T_α) is a bounded net in $\mathcal{L}(\mathfrak{A})$ which converges τ_1-strictly to $T \in \mathcal{L}(\mathfrak{A})$, $(T_\alpha \otimes I_{\mathcal{H}})$ converges strongly in $\mathcal{B}(\mathcal{H})$ to $T \otimes I$. An application of Corollary 3.7 completes the proof. ☐

Remark 3.9. Proposition 3.8 is similar to Lemma 2.21. The initial data for Proposition 3.8 is the extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ and the representation $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$; its conclusion may be interpreted as the statement that $\bigvee_{s\in\mathcal{S}} P_{s,\pi} = I_{\mathfrak{A}\otimes_\pi\mathcal{H}}$. On the other hand, Lemma 2.21 deals with the Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ and a semi-cyclic representation induced by a suitable weight; its conclusion may be interpreted as the statement that $\bigvee_{v\in\mathcal{GN}(\mathcal{M},\mathcal{D})} P_v = I_{\mathcal{H}_\phi}$, where P_v is defined in the proof of Lemma 2.21. There is a further relation between these results: when the extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ is the extension associated to the Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$, Proposition 4.13 below implies that for any $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, the projections P_v and $P_{q(v),\pi_\phi}$ are unitarily equivalent.

4. The Cartan Triple Associated to an Extension

Throughout this section we will consider the extension,

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S},$$

where $\mathcal{S}$ is a Cartan inverse monoid, and $\mathcal{P}$ is an $\mathcal{N}$-Clifford inverse monoid. Assume throughout that a fixed order-preserving section $j : \mathcal{S} \to \mathcal{G}$ is given. Our goal, achieved in Theorem 4.10, is to show how the representation of $\mathcal{G}$ constructed in Corollary 3.3 gives rise to a Cartan triple with associated extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$. In Theorem 4.11 we further show that the extension associated to the Cartan triple returns the original extension.

We denote by $\mathfrak{A}$ the right Hilbert $\mathcal{N}$-module as defined in Equation (3.2). Let π be a faithful, normal representation of $\mathcal{N}$, and let

$$\lambda_\pi : \mathcal{G} \to \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$$

be the representation of $\mathcal{G}$ by partial isometries, as constructed in Theorem 3.2 and Corollary 3.3.

Definition 4.1. Let

$$\mathcal{M}_q := (\lambda_\pi(\mathcal{G}))'', \quad \mathcal{N}_q := (\lambda_\pi(\mathcal{P}))'', \text{ and } \mathcal{D}_q := (\lambda_\pi(\mathcal{E}(\mathcal{G}))''.$$

We will show that $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ is a Cartan triple. The definitions of $\mathcal{M}_q$, $\mathcal{N}_q$ and $\mathcal{D}_q$ depend upon the choice of π and, because $\lambda : \mathcal{G} \to \mathcal{L}(\mathfrak{A})$ depends on the choice of j, $\mathcal{M}_q$, $\mathcal{N}_q$ and $\mathcal{D}_q$ also depend on j. However, we shall see in Theorem 4.11 that the *isomorphism class* of $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ depends only on the extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ and not upon π or j.

The first step is to show that there is a faithful normal conditional expectation from $\mathcal{M}_q$ onto $\mathcal{N}_q$. This will be used to show that $\mathcal{N}_q = \mathcal{D}_q^c$ in Proposition 4.9. As in [12], the expectation on $\mathcal{M}_q$ will be induced by the map $s \mapsto s \wedge 1$ on $\mathcal{S}$.

Define $\Delta : \mathcal{G} \to \mathcal{P}$ by

$$\Delta(v) := vj(q(v) \wedge 1),$$

for all $v \in \mathcal{G}$. First note that

$$q(\Delta(v)) = q(v)(q(v) \wedge 1) = q(v) \wedge 1 \in \mathcal{E}(\mathcal{S}).$$

Thus $\Delta(v) \in \mathcal{P}$ for all $v \in \mathcal{G}$. Further, if $v \in \mathcal{P}$ then $q(v) \in \mathcal{E}(\mathcal{S})$, thus

$$\Delta(v) = vj(q(v) \wedge 1) = vj(q(v)) = v.$$

We will show that, given $v \in \mathcal{G}$, the formula,

$$E_q(\lambda_\pi(v)) := \lambda_\pi(\Delta(v))$$

extends to a faithful conditional expectation $E_q : \mathcal{M}_q \to \mathcal{N}_q$.

Notation 4.2. Here is some notation.

(a) Let $P_{\mathcal{D}} \in \mathcal{L}(\mathfrak{A}_{\mathcal{D}})$ be the projection defined in Proposition 3.6, so that $P_{\mathcal{D}} k_s = k_{s\wedge 1}$. That is, $P_{\mathcal{D}} = Q_1$ as defined in Proposition 3.6. Since $\mathfrak{A} = \mathfrak{A}_{\mathcal{D}} \otimes_\iota \mathcal{N}$, the tensor product of $P_{\mathcal{D}}$ with the identity of $\mathcal{N}$ gives a projection $P \in \mathcal{L}(\mathfrak{A})$ so that, for $s \in \mathcal{S}$ and $x \in \mathcal{N}$,

$$P(k_s \otimes x) = k_{s\wedge 1} \otimes x. \tag{4.1}$$

(b) For $x \in \mathcal{N}$, and $y = \sum_{i=1}^n k_{s_i} \otimes n_i \in \mathfrak{A}_{\mathcal{D}} \odot \mathcal{N}$, $\|\sum_{i=1}^n k_{s_i} \otimes xn_i\|_{\mathfrak{A}} \le \|x\| \, \|y\|_{\mathfrak{A}}$. It follows that the map $k_s \otimes n \mapsto k_s \otimes xn$ extends to a bounded linear map $\pi_\ell(x)$ on $\mathfrak{A}$. A computation shows that $\pi_\ell(x)$ is adjointable, so there exists a faithful $*$-representation $\pi_\ell : \mathcal{N} \to \mathcal{L}(\mathfrak{A})$. Tensoring with the identity map, we obtain a faithful normal representation $\pi_{\ell *} = \pi_\ell \otimes I$ of $\mathcal{N}$ on $\mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$. To be explicit, for $x \in \mathcal{N}$, $\pi_{\ell *}(x)$ is defined on elementary tensors $(k_s \otimes n \otimes \xi) \in \mathfrak{A} \otimes_\pi \mathcal{H}$ by

$$\pi_{\ell *}(x)(k_s \otimes n \otimes \xi) = k_s \otimes xn \otimes \xi = k_s \otimes I_{\mathcal{N}} \otimes \pi(xn)\xi. \tag{4.2}$$

Lemma 4.3. *For $s \in \mathcal{S}$, let Q_s be the projection on $\mathcal{L}(\mathfrak{A}_{\mathcal{D}})$ defined in Proposition 3.6 and let $P_{s,\pi} := Q_s \otimes I_{\mathcal{N}} \otimes I_{\mathcal{H}} \in \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$ be the projection defined in Equation* (3.8)*. The following statements hold.*

(a) For $v \in \mathcal{P}$, $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $\xi \in \mathcal{H}$,

$$\lambda_\pi(v)(k_s \otimes n \otimes \xi) = k_s \otimes j(s)^* v j(s) n \otimes \xi.$$

(b) $P_{s,\pi} \in \mathcal{N}_q'$ and for every $v \in \mathcal{P}$, $\lambda_\pi(v) P_{s,\pi} = \pi_{\ell *}(j(s)^* v j(s)) P_{s,\pi}$.

Proof. Since $v \in \mathcal{P}$, $q(v) = q(vv^*)$, so $\sigma(v, s) = j(s^\dagger q(v)^\dagger) v j(s) = j(s)^* v j(s)$. Therefore,

$$\begin{aligned}
\lambda_\pi(v)(k_s \otimes n \otimes \xi) &= k_{q(v)s} \otimes j(s)^* v j(s) n \otimes \xi = k_{ss^\dagger q(v)s} \otimes j(s)^* v j(s) n \otimes \xi \\
&= k_s j(s^\dagger q(v) s) \otimes j(s)^* v j(s) n \otimes \xi = k_s \otimes j(s^\dagger q(v) s) j(s)^* v j(s) n \otimes \xi \\
&= k_s \otimes j(s)^* v j(s) n \otimes \xi,
\end{aligned}$$

where the third equality follows from [12, Corollary 4.9]. This gives part (a) and shows range$(P_{s,\pi})$ is invariant for every element of $\lambda_\pi(\mathcal{P})$. Thus, range$(P_{s,\pi})$ is invariant for $\mathcal{N}_q$. As $\mathcal{N}_q$ is a $*$-algebra, $P_{s,\pi}$ reduces $\mathcal{N}_q$, whence $P_{s,\pi} \in \mathcal{N}_q'$.

Now suppose $t \in \mathcal{S}$, $n \in \mathcal{N}$ and $\xi \in \mathcal{H}$. Note that $j(s)j(s^\dagger t \wedge 1) = j(s(s^\dagger t \wedge 1)) = j(s \wedge t)$. Then, again using [12, Corollary 4.9],

$$P_{s,\pi}(k_t \otimes n \otimes \xi) = k_{s\wedge t} \otimes n \otimes \xi = k_s \otimes j(s^\dagger t \wedge 1)n \otimes \xi. \tag{4.3}$$

A computation using part (a) and (4.3) now gives the formula in part (b). $\square$

With the obvious modifications to the proof of [12, Proposition 5.2], we obtain the following.

Lemma 4.4. *With P defined as in Equation* (4.1)*, the following properties hold:*

(a) $\operatorname{range} P = \overline{\operatorname{span}}\{k_e \otimes x : e \in \mathcal{E}(\mathcal{S})$ *and* $x \in \mathcal{N}\}$*; and*

(b) for $v \in \mathcal{G}$,

$$P\lambda(v)P = \lambda(\Delta(v))P.$$

Modifications to the proof of [12, Proposition 5.3] yield the following result.

Lemma 4.5. *The map* $V : \mathcal{H} \to \mathfrak{A} \otimes_\pi \mathcal{H}$ *given by* $V\xi = (k_1 \otimes I_{\mathcal{N}}) \otimes \xi$ *is an isometry. Moreover, the following properties hold:*

(a) for $s \in \mathcal{S}$, $x \in \mathcal{N}$ *and* $\xi \in \mathcal{H}$, $V^*(k_s \otimes x \otimes \xi) = \pi(j(s \wedge 1)x)\xi$*;*

(b) $VV^* = \pi_*(P)$, *where* $\pi_* : \mathcal{L}(\mathfrak{A}) \to \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$ *is defined by* $\pi_*(T)(u \otimes \xi) = (Tu) \otimes \xi$*;*

(c) for $v \in \mathcal{G}$, $V^*\lambda_\pi(v)V = \pi(\Delta(v))$.

Lemma 4.6. *We have*

$$V^*\mathcal{M}_qV = \pi(\mathcal{N}) = V^*\mathcal{N}_qV.$$

Proof. Lemma 4.5(c) shows that for $x \in \mathcal{M}_q$, $V^*xV \in \pi(\mathcal{N})$, so $V^*\mathcal{M}_qV \subseteq \pi(\mathcal{N})$. On the other hand, for $v \in \mathcal{P}$ we have

$$V^*\lambda_\pi(v)V = \pi(\Delta(v)) = \pi(v), \tag{4.4}$$

so $V^*\mathcal{N}_qV \subseteq \pi(\mathcal{N})$. Since every element of $\mathcal{N}$ is a linear combination of at most four elements of $\mathcal{P}$, we obtain the result. $\square$

Thus, the map $\mathcal{M}_q \ni x \mapsto \pi^{-1}(V^*xV)$ is a normal, completely positive contraction of $\mathcal{M}_q$ onto $\mathcal{N}$. We now show this map gives an isomorphism of $\mathcal{N}_q$ onto $\mathcal{N}$.

Lemma 4.7. *The map* $\alpha : \mathcal{N}_q \to \mathcal{N}$ *defined by* $\alpha(x) = \pi^{-1}(V^*xV)$ *is a normal isomorphism of* $\mathcal{N}_q$ *onto* $\mathcal{N}$.

Proof. The definition of α shows it is normal. Next we show that α is a homomorphism. For $v_1, v_2 \in \mathcal{P}$, Lemma 4.5(c) gives

$$V^*(\lambda_\pi(v_1))VV^*\lambda_\pi(v_2)V = \pi(v_1)\pi(v_2) = \pi(v_1v_2) = V^*\lambda_\pi(v_1v_2)V.$$

Thus α is multiplicative on $\lambda_\pi(\mathcal{P})$. It follows that α is multiplicative on $\operatorname{span}(\lambda_\pi(\mathcal{P}))$. As multiplication is σ-strongly continuous on bounded sets, the Kaplansky density theorem ensures α multiplicative. Lemma 4.6 now shows α is a $*$-epimorphism.

It remains to show α is one-to-one. To do this, we show α is isometric on $\operatorname{span}(\lambda_\pi(\mathcal{P}))$. Suppose $n \in \mathbb{N}$, $c_j \in \mathbb{C}$, $v_j \in \mathcal{P}$ and $x = \sum_{j=1}^n c_j\lambda_\pi(v_j)$. Put $y = \alpha(x)$ so that $y = \sum_{j=1}^n c_jv_j$ by Equation 4.4. By Lemma 4.3, for each $s \in \mathcal{S}$,

$$xP_{s,\pi} = \pi_{\ell*}(j(s)^*yj(s))P_{s,\pi},$$

and hence $\|xP_{s,\pi}\| \le \|y\|$. Now suppose $\mathcal{B}$ is a maximal meet-disjoint subset of $\mathcal{S}$. Then for distinct $s, t \in \mathcal{B}$, $P_{s,\pi}$ and $P_{t,\pi}$ are orthogonal projections. By Proposition 3.8,

$$\|x\| = \sup_{s\in\mathcal{B}} \|xP_{s,\pi}\| \le \|y\| = \|\alpha(x)\| \le \|x\|.$$

So α is isometric on $\operatorname{span}(\lambda_\pi(\mathcal{P}))$. $\square$

At last, we can define the conditional expectation E_q.

Proposition 4.8. *The formula,*

$$E_q(x) := \alpha^{-1}(\pi^{-1}(V^*xV)) \tag{4.5}$$

gives a faithful normal conditional expectation of $\mathcal{M}_q$ *onto* $\mathcal{N}_q$. *Furthermore, for* $v \in \mathcal{G}$,

$$E_q(\lambda_\pi(v)) = v\lambda_\pi(\Delta(v)). \tag{4.6}$$

Proof. Lemmas 4.6 and 4.7 imply E_q is a normal conditional expectation of $\mathcal{M}_q$ onto $\mathcal{N}_q$. It remains only to establish that E_q is faithful. The proof that E_q is faithful is modeled on the proof of [12, Proposition 5.9].

Let $\mathcal{C}$ denote the center of $\mathcal{M}_q$. We claim that $E_q|_\mathcal{C}$ is faithful. Let $x \in \mathcal{C}$ and suppose $E_q(x^*x) = 0$. The definition of E_q from Equation 4.5 shows that $V^*x^*xV = 0$ and hence $xV = 0$. Notice that $\sigma(j(s), 1) = j(s^\dagger s)$ so that for $n \in \mathcal{N}$, $\lambda(j(s))(k_1 \otimes n) = k_s j(s^\dagger s) \otimes n = k_s \otimes n$ (see [12, Corollary 4.9]). Hence for $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $\xi \in \mathcal{H}$,

$$x(k_s \otimes n \otimes \xi) = x\lambda_\pi(j(s))(k_1 \otimes n \otimes \xi) = \lambda_\pi(j(s))x(k_1 \otimes n \otimes \xi) = \lambda(j(s))xV\pi(n)\xi = 0.$$

Since the span of such vectors is a dense subspace of $\mathfrak{A} \otimes_\pi \mathcal{H}$, we conclude that $x = 0$.

Let $\mathfrak{J} := \{x \in \mathcal{M}_q : E_q(x^*x) = 0\}$. Then $\mathfrak{J}$ is a left ideal of $\mathcal{M}_q$. Compute as in the second part of [12, Lemma 5.8] to find that for $x \in \mathfrak{J}$ and $v \in \mathcal{G}$,

$$E_q(\lambda_\pi(v)^*x^*x\lambda_\pi(v)) = \lambda_\pi(v)^*E_q(x^*x)\lambda_\pi(v) = 0.$$

Thus, $x\lambda_\pi(v) \in \mathfrak{J}$. It now follows that $\mathfrak{J}$ is a two-sided ideal of $\mathcal{M}_q$ as well. Since $\mathfrak{J}$ is weak-$*$-closed, by [31, Proposition II.3.12], there is a projection $Q \in \mathcal{C}$ such that $\mathfrak{J} = Q\mathcal{M}_q$. As $Q \in \mathfrak{J}$ and $E_q|_\mathcal{C}$ is faithful, we obtain $Q = 0$. Thus $\mathfrak{J} = (0)$, that is, E_q is faithful. The equality (4.6) follows from Lemma 4.5. ☐

Proposition 4.9. *The algebra* $\mathcal{N}_q$ *is the relative commutant of* $\mathcal{D}_q$ *in* $\mathcal{M}_q$. *That is,* $\mathcal{N}_q = \mathcal{D}_q^c$.

Proof. Notice that $v \in \mathcal{G}$ commutes with every element of $\mathcal{E}(\mathcal{G})$ if and only if $v \in \mathcal{P}$. Since λ_π is a isomorphism of $\mathcal{G}$ onto $\lambda_\pi(\mathcal{G})$, we obtain $\lambda_\pi(\mathcal{G}) \cap \mathcal{D}_q^c = \lambda_\pi(\mathcal{P})$. Therefore, $\mathcal{N}_q \subseteq \mathcal{D}_q^c$.

Take $x \in \mathcal{D}_q^c$. Suppose $w \in \lambda_\pi(\mathcal{G})$ satisfies $w^2 = 0$. Then

$$E_q(w^*x) = w^*wE_q(w^*x) = E_q(w^*x)w^*w = E_q(w^*xw^*w) = E_q((w^*)^2wx) = 0. \tag{4.7}$$

Now choose an arbitrary $v \in \lambda_\pi(\mathcal{G})$. Our goal is to show that (again with $x \in \mathcal{D}_q^c$)

$$E_q(v^*x) = E_q(v^*)E_q(x). \tag{4.8}$$

As in Lemma 2.18, v defines a map θ_v on $\mathcal{D}_q$ ($d \mapsto vdv^*$). By Frolík's Theorem (see [26, Proposition 2.11a]) there are orthogonal projections $e_0,\ e_1,\ e_2, e_3 \in \mathcal{D}_q$ such that

$$v = \sum_{k=0}^{3} ve_k,$$

$\theta_v|_{\mathcal{D}e_0} = \mathrm{id}|_{\mathcal{D}e_0}$, and $\theta_v(e_k)e_k = 0$ for $k = 1, 2, 3$.

As e_0 is the largest projection in $\mathcal{D}_q$ on which $\theta_v|_{\mathcal{D}e_0} = \mathrm{id}_{\mathcal{D}e_0}$, it follows that $ve_0 = E_q(v)$. Also, as $\theta_v(e_k)e_k = 0$, it follows that $(ve_k)^2 = 0$, for $k = 1, 2, 3$. By (4.7), for $k = 1, 2, 3$, $E_q(e_kv^*x) = 0$. Thus

$$E_q(v^*x) = \sum_{k=0}^{3} E_q(e_kv^*x) = E_q(e_0v^*x) = E_q(v^*)E_q(x),$$

so (4.8) holds.

Since $\lambda_\pi(\mathcal{G})$ spans a weak*-dense subset of $\mathcal{M}_q$, it follows that for $x \in \mathcal{D}_q^c$ we have

$$E_q(x^*x) = E_q(x^*)E_q(x).$$

Hence, if $x \in \mathcal{D}_q^c$ we have

$$E_q((x - E_q(x))^*(x - E_q(x)) = E_q(x - E_q(x))^* E_q(x - E_q(x)) = 0.$$

Since E_q is faithful, it follows that $x = E_q(x) \in \mathcal{N}_q$. ☐

Proposition 4.8 and Proposition 4.9 now immediately give the first main theorem of this section.

Theorem 4.10. *$(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ is a Cartan triple.*

The second main theorem of this section is that the extension associated to $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ gives back the original extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$.

Theorem 4.11. *Let $\mathcal{P}$ be a $\mathcal{N}$-Clifford inverse monoid and suppose $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ is an extension of the Cartan inverse monoid $\mathcal{S}$ by $\mathcal{P}$. Let $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ be the Cartan triple constructed in Theorem 4.10. The extension associated to $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ is equivalent to the extension*

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$$

from which $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ was constructed.

Moreover, the isomorphism class of $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ depends only upon the equivalence class of the extension (and not on the choice of representation π or section j).

Remark 4.12. In the proof of Theorem 4.11 and also in the proof of Theorem 5.2 below, we shall utilize a result of Arveson, [3, Theorem 6.2.2]. In [3], Arveson makes the blanket assumption that all Hilbert spaces are separable (see [3, Section 1.2]). However the proof of [3, Theorem 6.2.2] does not require separability.

Proof. The argument below is a modification of the proof of [12, Theorem 5.12]. Let R_M and $R_{M,\pi}$ be the Munn congruences for $\mathcal{G}$ and $\lambda_\pi(\mathcal{G})$ respectively. Since λ_π is an isomorphism of $\mathcal{G}$ onto $\lambda_\pi(\mathcal{G})$, (v, w) belongs to R_M if and only if $(\lambda_\pi(v), \lambda_\pi(w))$ belongs to $R_{M,\pi}$. Let $q_\pi : \lambda_\pi(\mathcal{G}) \to \lambda_\pi(\mathcal{G})/R_{M,\pi}$ be the quotient map. The fact that $\mathcal{S}$ is fundamental implies that $\widetilde{\lambda}_\pi := q_\pi \circ \lambda_\pi \circ j$ is a multiplicative map of $\mathcal{S}$ onto $\lambda_\pi(\mathcal{G})/R_{M,\pi}$. In fact, $\widetilde{\lambda}_\pi$ is an isomorphism satisfying $\widetilde{\lambda}_\pi \circ q = q_\pi \circ \lambda_\pi$, and furthermore, $\lambda_\pi|_{\mathcal{P}}$ is an isomorphism of $\mathcal{P}$ onto $\lambda_\pi(\mathcal{P})$. Let $v \in \mathcal{P}$. By Equation (4.4) (see Lemma 4.6), $V^* \lambda_\pi(v) V = \pi(v)$. Thus in the notation of Lemma 4.7, $\alpha^{-1}(v) = \lambda_\pi(v)$. Therefore $\lambda_\pi|_{\mathcal{P}} = \alpha^{-1}|_{\mathcal{P}}$. It is now clear that the extensions

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$$

and

$$\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \overset{q_\pi}{\twoheadrightarrow} \widetilde{\lambda}_\pi(\mathcal{S})$$

are equivalent. For later use, note that in particular, $\lambda_\pi|_{\mathcal{E}(\mathcal{G})}$ is an isomorphism of $\mathcal{E}(\mathcal{G})$ onto $\mathcal{E}(\lambda_\pi(\mathcal{G}))$.

Our next task is to show that

$$\lambda_\pi(\mathcal{G}) = \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q). \tag{4.9}$$

It will then follow immediately that $\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \overset{q_\pi}{\twoheadrightarrow} \widetilde{\lambda}_\pi(\mathcal{S})$ is the extension associated to $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$.

Claim 1: If $u \in \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$, then $uE_q(u^*)$ is a projection in $\mathcal{D}_q$, and

$$uE_q(u^*) = E_q(uE_q(u^*)) = E_q(u)E_q(u^*). \tag{4.10}$$

Let Λ be an invariant mean on the abelian group $\mathcal{U}(\mathcal{D}_q)$. By [3, Theorem 6.2.2],

$$uE_q(u^*) = \bigwedge_{g \in \mathcal{U}(\mathcal{D}_q)} (ugu^*)g^* \in \mathcal{D}_q.$$

Next,

$$uE_q(u^*)uE_q(u^*) = uE_q(u^*uE_q(u^*)) = uu^*uE_q((E_q(u^*)) = uE_q(u^*),$$

so $uE_q(u^*)$ is a projection in $\mathcal{D}_q$. The equality (4.10) is now obvious, so Claim 1 holds.

By construction, $\lambda_\pi(\mathcal{G}) \subseteq \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$. To establish the reverse inclusion, fix $v \in \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$; without loss of generality, assume $v \neq 0$.

Claim 2: There exists $p \in \lambda_\pi(\mathcal{E}(\mathcal{G}))$ such that: a) $vp \in \lambda_\pi(\mathcal{G})$, b) $p \leq v^*v$, and c) $vp \neq 0$.

Since $\lambda_\pi(\mathcal{G})'' = \mathcal{M}_q$, it follows (as in the proof of [5, Proposition 1.3.4]) that there exists $w \in \lambda_\pi(\mathcal{G})$ such that $wE_q(w^*v) \neq 0$. Let $p = v^*wE_q(w^*v)$. By Claim 1, $p \in \mathcal{D}_q$ is a projection, so in particular, $p \in \lambda_\pi(\mathcal{E}(\mathcal{G}))$. It is evident that $p \leq v^*v$. By (4.10),

$$E_q(v^*w)E_q(w^*v) = p,$$

so $x := E_q(w^*v)$ is a partial isometry in $\mathcal{N}_q$ with source projection $p \in \mathcal{D}_q$. On the other hand, let $p' := w^*vE_q(v^*w)$. Claim 1 gives $p' \in \mathcal{D}_q$ and $p' = E_q(w^*v)E_q(v^*w)$. We have thus shown that both the source and range projections for x belong to $\mathcal{D}_q \subseteq \mathcal{Z}(\mathcal{N}_q)$. Therefore,

$$p = x^*x = x^*(xx^*)x = (x^*x)(xx^*) = x(x^*x)x^* = xx^* = p'.$$

Hence $E_q(w^*v)$ is a partial isometry in $\mathcal{N}_q$ whose source and range projections both equal $p \in \mathcal{D}_q$. Thus, $E_q(w^*v) \in \mathcal{GN}(\mathcal{N}_q, \mathcal{D}_q) = \lambda_\pi(\mathcal{P})$. This gives $wE_q(w^*v) \in \lambda_\pi(\mathcal{G})$. Since $E_q(w^*v) = w^*v(v^*wE_q(w^*v))$, we obtain,

$$0 \neq wE_q(w^*v) = w(w^*v(v^*wE_q(w^*v))) = vv^*wE_q(w^*v) = vp.$$

Thus Claim 2 holds.

Now argue exactly as in the proof of [12, Theorem 5.12] to conclude that $v \in \lambda_\pi(\mathcal{G})$. Therefore, we have shown that $\lambda_\pi(\mathcal{G}) = \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$. Hence

$$\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \overset{q_\pi}{\twoheadrightarrow} q_\pi\widetilde{\lambda}_\pi(\mathcal{S})$$

is the extension for $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$.

Suppose that $\tilde{\pi}$ is a faithful normal representation of $\mathcal{N}$ and $\tilde{j} : \mathcal{S} \to \mathcal{G}$ is an order preserving section for q. Let $(\tilde{\mathcal{M}}_q, \tilde{\mathcal{N}}_q, \tilde{\mathcal{D}}_q)$ be the Cartan triple constructed using $\tilde{\pi}$ and $\tilde{j}$ as in Theorem 4.10. Then the previous paragraphs show that the extensions associated to $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ and $(\tilde{\mathcal{M}}_q, \tilde{\mathcal{N}}_q, \tilde{\mathcal{D}}_q)$ are equivalent extensions. By Theorem 2.22, $(\mathcal{M}_q, \mathcal{N}_q, \mathcal{D}_q)$ and $(\tilde{\mathcal{M}}_q, \tilde{\mathcal{N}}_q, \tilde{\mathcal{D}}_q)$ are isomorphic Cartan triples. The proof is now complete. ☐

Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple and let ϕ be a faithful normal semi-finite weight on $\mathcal{M}$ satisfying $\phi \circ E = \phi$. We end this section by relating the semi-cyclic representation $(\pi_\phi, \mathcal{H}_\phi, \eta_\phi)$ and the reproducing kernel Hilbert $\mathcal{N}$-module $\mathfrak{A} \otimes \mathcal{N}$ constructed from the extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ associated to $(\mathcal{M}, \mathcal{N}, \mathcal{D})$.

Proposition 4.13. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple, suppose ψ is a faithful, normal semi-finite weight on $\mathcal{N}$, and put $\phi := \psi \circ E$. Let $(\pi_\psi, \mathcal{H}_\psi, \eta_\psi)$ and $(\pi_\phi, \mathcal{H}_\phi, \eta_\phi)$ be the semi-cyclic representations of $\mathcal{N}$ and $\mathcal{M}$ associated with ψ and ϕ respectively. Let*

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$$

be the extension associated to $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ and let $j : \mathcal{S} \to \mathcal{G}$ be an order-preserving section for q. For $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $x \in \mathfrak{n}_\psi$, $j(s)nx \in \mathfrak{n}_\phi$, and the map

$$(\mathfrak{A}_\mathcal{D} \otimes_\iota \mathcal{N}) \otimes_{\pi_\psi} \mathcal{H}_\psi \ni k_s \otimes n \otimes \eta_\psi(x) \mapsto \eta_\phi(j(s)nx) \in \mathcal{H}_\phi$$

extends to a unitary operator $W : \mathfrak{A} \otimes_{\pi_\psi} \mathcal{H}_\psi \to \mathcal{H}_\phi$ such that for every $v \in \mathcal{G}$,

$$W\lambda_{\pi_\psi}(v)W^* = \pi_\phi(v).$$

Proof. For $i = 1, 2$, let $s_i \in \mathcal{S}$, $n_i \in \mathcal{N}$ and $x_i \in \mathfrak{n}_\psi$. Then
$$\phi((j(s_i)n_ix_i)^*(j(s_i)n_ix_i)) = \psi(x_i^*n_i^*E(j(s_i)^*j(s_i))n_ix_i) \leq \|n_i\|^2\,\psi(x_i^*x_i),$$
so $j(s_i)n_ix_i \in \mathfrak{n}_\phi$. Recall that for any $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $E(v) = vj(q(v) \wedge 1)$. In particular, using Proposition 2.16, we have
$$E(j(s_1)^*j(s_2)) = j(s_1)^*j(s_2)j(s_1^\dagger s_2 \wedge 1) = j(s_1^\dagger s_2 \wedge 1).$$
So
$$\begin{aligned}\langle (k_{s_1} \otimes n_1 \otimes \eta_\psi(x_1)), (k_{s_2} \otimes n_2 \otimes \eta_\psi(x_2))\rangle &= \psi(x_1^*n_1^*j(s_1^\dagger s_2 \wedge 1)n_2x_1)\\ &= \psi(x_1^*n_1^*E(j(s_1)^*j(s_2))n_2x_1)\\ &= \langle \eta_\phi(j(s_1)n_1x_1), \eta_\phi(j(s_2)n_2x_2)\rangle\,.\end{aligned}$$

As every element in $\operatorname{span}\{k_s \otimes n \otimes x : s \in \mathcal{S}, n \in \mathcal{N}, x \in \mathfrak{n}_\psi\}$ can be written as $\sum_{a\in A} k_a \otimes n_a \otimes x_a$ where $A \subseteq \mathcal{S}$ is a finite pairwise meet disjoint set, $\{n_a : a \in A\} \subseteq \mathcal{N}$ and $\{x_a : a \in A\} \subseteq \mathfrak{n}_\psi$, it follows that $k_s \otimes n \otimes x \mapsto \eta_\phi(j(s)nx)$ extends to an isometry $W : \mathfrak{A} \otimes_{\pi_\psi} \mathcal{H}_\psi \to \mathcal{H}_\phi$. By Proposition 2.21, $\operatorname{span}\{\eta_\phi(j(s)nx) : s \in \mathcal{S}, n \in \mathcal{N}, x \in \mathfrak{n}_\psi\}$ is dense in $\mathcal{H}_\phi$, so W is a unitary operator.

If $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $x \in \mathfrak{n}_\psi$,
$$\begin{aligned}W\lambda_{\pi_\psi}(v)(k_s \otimes n \otimes x) &= W(k_{q(v)s} \otimes \sigma(v, s)n \otimes x)\\ &= \eta_\phi(j(q(v)s)\sigma(v, s)nx)\\ &= \eta_\phi(vj(s)nx) = \pi_\phi(v)W(k_s \otimes n \otimes x).\end{aligned}$$
Thus, $W\lambda_{\pi_\psi}(v)W^* = \pi_\phi(v)$. $\square$

5. The spectral theorem for bimodules and Aoi's theorem

Throughout this section, $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ will be a Cartan triple with associated extension
$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S},$$
and $j : \mathcal{S} \to \mathcal{G}$ will be a fixed choice of an order-preserving section for q. The goal in this section is study the $\mathcal{N}$-bimodules in $\mathcal{M}$. Recall the following definition from [12].

Definition 5.1. A subset A of a Cartan inverse monoid $\mathcal{S}$ is a *spectral set* if

(a) $s \in A$ and $t \leq s$ implies that $t \in A$; and
(b) if $\{s_i\}_{i\in I}$ is a pairwise orthogonal family in A, then $\bigvee_{i\in I} s_i \in A$.

In Theorem 5.10 we prove a Spectral Theorem for Bimodules. Will show a one-to-one correspondence between the spectral sets in $\mathcal{S}$ and a large class of weak-$*$ closed $\mathcal{N}$-bimodules: the Bures-closed $\mathcal{N}$-bimodules (see Definition 5.5). We go on to study the intermediate von Neumann algebras $\mathcal{N} \subseteq \mathcal{L} \subseteq \mathcal{M}$, giving a generalization of Aoi's Theorem [1] in Theorem 5.12. Several of these theorems require that the Cartan triple is a full Cartan triple.

5.1. $\mathcal{N}$-bimodules.

We begin by showing that weak-$*$ closed $\mathcal{N}$-bimodules give rise to non-empty spectral sets. In particular, Theorem 5.2 shows that when $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a full Cartan triple, any weak-$*$ closed $\mathcal{N}$-bimodule contains an abundance of elements of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$. Example 5.4 below gives a simple example showing fullness is necessary.

Theorem 5.2. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a full Cartan triple. Suppose $(0) \neq \mathcal{B} \subseteq \mathcal{M}$ is a weak-$*$-closed $\mathcal{N}$-bimodule. Then*
$$\{0\} \neq \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}.$$
In fact, for every $x \in \mathcal{B}$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $vE(v^*x)$ is a linear combination of at most four elements of $\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}$.

Proof. Let $x \in \mathcal{B}$ be non-zero. Since $\mathcal{M}$ is the weak-$*$ closed span of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$, there exists $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $E(v^*x) \neq 0$, and hence $vE(v^*x) \neq 0$. By [3, Theorem 6.2.2] (see Remark 4.12 above), for any $y \in \mathcal{M}$,

$$E(y) = \bigwedge_{U \in \mathcal{U}(\mathcal{D})} U^*yU.$$

Therefore,

$$vE(v^*x) = \bigwedge_{U \in \mathcal{U}(\mathcal{D})} (vU^*v^*)xU \in \mathcal{B}.$$

Let J be the weak-$*$ closed, two-sided ideal in $\mathcal{N}$ generated by $E(v^*x)$. For any $n_1, n_2 \in \mathcal{N}$ we have

$$v(n_1E(v^*x)n_2) = (vn_1v^*)(vE(v^*x))n_2 \in \mathcal{B}.$$

Since $\mathcal{B}$ is weak-$*$ closed, it follows that $vJ \subseteq \mathcal{B}$. Let $p \in \mathcal{Z}(\mathcal{N}) = \mathcal{D}$ be such that $J = p\mathcal{N}$. Then, $vp \in \mathcal{B} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Since $vE(v^*x) = vpE(v^*x)$, $0 \neq vp$.

Since $p\mathcal{N}$ is a von Neumann algebra (with unit p), $E(v^*x)$ is a linear combination of four unitary elements of $p\mathcal{N}$. As $p \in \mathcal{Z}(\mathcal{N})$, $\mathcal{U}(p\mathcal{N}) = \{pw : w \in \mathcal{U}(\mathcal{N})\}$. Also, for any unitary $w \in \mathcal{U}(\mathcal{N})$ we have $vpw \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Thus, $vE(v^*x)$ is a linear combination of at most four elements of $\mathcal{GN}(\mathcal{M}, \mathcal{D})$. $\square$

The following corollary of the proof of Theorem 5.2 will be needed in the sequel.

Corollary 5.3. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a (not necessarily full) Cartan triple. For any $x \in \mathcal{M}$ and $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $vE(v^*x)$ belongs to the weak-$*$ closed $\mathcal{D}$-bimodule generated by x.*

Example 5.4. Let $\mathcal{M}$ be any von Neumann algebra with non-trivial center. Then $(\mathcal{M}, \mathcal{M}, \mathbb{C}I)$ is a Cartan triple which is not full. Let p be a central projection in $\mathcal{M}$ with $0 < p < I$. Then $\mathcal{M}p$ is a weak-$*$ closed $\mathcal{M}$-bimodule. However $\mathcal{M}p \cap \mathcal{GN}(\mathcal{M}, \mathbb{C}I) = \{0\}$. Thus, the condition of fullness in Theorem 5.2 is necessary.

A natural problem is to characterize the weak-$*$ closed $\mathcal{N}$-bimodules in $\mathcal{M}$. Given a weak-$*$ closed $\mathcal{N}$-bimodule $B \subseteq \mathcal{M}$, one might hope to use Theorem 5.2 to reconstruct a given element $x \in B$ from the elements of $B \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. However, for doing this, the weak-$*$ topology is not generally the appropriate topology. Instead, as Mercer shows in [23], the Bures-topology turns out to be the "right" topology to handle such reconstruction problems. This phenomenon was also observed in studying bimodules in the Cartan pair case by Cameron, Pitts and Zarikian [5] (see also [12]), and in the crossed-product von Neumann algebras by Cameron and Smith [6, 8]. Our next goal is Theorem 5.7, which gives a method for reconstructing $x \in B$ using $\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap B$ when $B \subseteq \mathcal{M}$ is a Bures-closed $\mathcal{N}$-bimodule. We begin with recalling the definition of the Bures-topology.

Definition 5.5. Let $\mathcal{L} \subseteq \mathcal{M}$ be an inclusion of von Neumann algebras and assume there is a faithful normal conditional expectation $E_{\mathcal{L}} : \mathcal{M} \to \mathcal{L}$. The *$E_{\mathcal{L}}$-Bures topology* (or simply *Bures topology* when the context is clear) is the locally convex topology determined by the family of seminorms,

$$\mathcal{M} \ni x \mapsto \rho(E(x^*x))^{1/2}, \rho \in \mathcal{L}_*^+.$$

The Bures topology was introduced in [4] in the case when $\mathcal{M}$ is a factor and $\mathcal{L}$ is abelian. By [6, Lemma 3.1], for any convex set $C \subseteq \mathcal{M}$, the Bures closure of C contains the weak-$*$ closure of C, that is,

$$\mathrm{cl}^{\text{weak-}*}(C) \subseteq \mathrm{cl}^{\text{Bures}}(C).$$

Take $x \in \mathcal{M}$. We showed in Theorem 5.2 and Corollary 5.3 that for each $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, $vE(v^*x)$ is in the $\mathcal{N}$-bimodule generated by x. We now show that, in the Bures topology, we can recover x from the elements of the form $vE(v^*x)$.

Definition 5.6. For a Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$, a subset $\mathcal{Y} \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ is *E-orthogonal* if whenever $v, w \in \mathcal{Y}$ with $v \neq w$, $E(v^*w) = 0$.

Theorem 5.7. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a (not necessarily full) Cartan triple and let $\mathcal{Y} \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ be a maximal E-orthogonal subset. Let Λ be the set of all finite subsets of $\mathcal{Y}$ directed by inclusion. For $x \in \mathcal{M}$ and $F \in \Lambda$, let $x_F := \sum_{u \in F} uE(u^*x)$. Then the net $(x_F)_{F \in \Lambda}$ converges in the Bures topology to x.*

Proof. Let $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ be the extension associated to $(\mathcal{M}, \mathcal{N}, \mathcal{D})$, let ψ be a faithful, normal semi-finite weight on $\mathcal{N}$, let $\phi = \psi \circ E$, and let $(\pi_\phi, \mathcal{H}_\phi, \eta_\phi)$ be the semi-cyclic representation of $\mathcal{M}$ associated to ϕ. For any $v \in \mathcal{G}$, the map $\mathcal{M} \ni x \mapsto vE(v^*x)$ leaves $\mathfrak{n}_\phi$ invariant and depends only on $s = q(v)$. Further, when $x \in \mathfrak{n}_\phi$, $\eta_\phi(x) \mapsto \eta_\phi(vE(v^*x))$ is contractive, and extends to a projection $P_s \in \mathcal{B}(\mathcal{H}_\phi)$. (In the notation of Lemma 4.3 and Proposition 4.13, $P_s = WP_{s,\pi_\phi}W^*$). When $s = 1$, write P instead of P_1.

Arguing as in [5, Lemma 2.2], we find that the two families of semi-norms on $\mathcal{M}$,

$$\{\mathcal{M} \ni m \mapsto \sqrt{\tau(E(m^*m))} : \tau \in \mathcal{N}_*^+\} \quad \text{and} \quad \{\mathcal{M} \ni m \mapsto \|\pi_\phi(m)\xi\| : \xi \in \text{range}(P)\}$$

coincide. These families of semi-norms define the Bures topology on $\mathcal{M}$ (see [5, Definition 2.2.3]).

We now argue exactly as in the proof of [5, Proposition 2.4.4]. Let $n \in \mathfrak{n}_\phi \cap \mathcal{N}$. Then

$$\begin{aligned}
\pi_\phi(x_F)\eta_\phi(n) &= \sum_{u \in F} \eta_\phi(uE(u^*xn)) \\
&= \sum_{u \in F} \pi_\phi(u)P\pi_\phi(u)^*\eta_\phi(xn) \\
&= \sum_{u \in F} P_{q(u)}\eta_\phi(xn) = \sum_{u \in F} P_{q(u)}\pi_\phi(x)\eta_\phi(n).
\end{aligned}$$

Hence for every $\xi \in \overline{\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{N})}$,

$$\pi_\phi(x_F)\xi = \sum_{u \in F} P_{q(u)}\pi_\phi(x)\xi.$$

By Proposition 3.8 and Proposition 4.13, $I = \sum_{u \in \mathcal{Y}} P_{q(u)}$ (where the sum converges strongly in $\mathcal{B}(\mathcal{H}_\phi)$). Thus for every $\xi \in \overline{\eta_\phi(\mathfrak{n}_\phi \cap \mathcal{N})}$,

$$\pi_\phi(x_F)\xi \to \pi_\phi(x)\xi.$$

Therefore, $x_F \overset{\text{Bures}}{\to} x$. ∎

We now show that the Bures closure of a weak-$*$ closed $\mathcal{N}$-bimodule $\mathcal{B}$ contains exactly the same groupoid normalizers as $\mathcal{B}$ itself. The reader should note that this result gives the versions of [5, Proposition 2.5.3 and Theorem 2.5.1] appropriate to our context.

Proposition 5.8. *Let $\mathcal{B} \subseteq \mathcal{M}$ be a weak-$*$ closed $\mathcal{N}$-bimodule, and set*

$$\mathcal{B}_0 = \overline{\text{span}}^{w*}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}) \quad \textit{and} \quad \mathcal{B}_1 := \overline{\text{span}}^{\textit{Bures}}(\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}),$$

Then $\mathcal{B}_0$ and $\mathcal{B}_1$ are weak-$*$ closed $\mathcal{N}$-bimodules satisfying $\mathcal{B}_0 \subseteq \mathcal{B}_1$ and

$$\mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}_0 = \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B} = \mathcal{GN}(\mathcal{M}, \mathcal{D}) \cap \mathcal{B}_1.$$

Furthermore, when $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a full Cartan triple,

$$\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \mathcal{B}_1 = \overline{\mathcal{B}}^{\textit{Bures}}.$$

Proof. Notice that if (x_λ) is a net in $\mathcal{M}$ which Bures-converges to $x \in \mathcal{M}$, then for any $a \in \mathcal{M}$ and $b \in \mathcal{N}$, $\lim^{\text{Bures}} ax_\lambda b = axb$. It follows that $\mathcal{B}_1$ is a weak-$*$ closed $\mathcal{N}$ bimodule. That $\mathcal{B}_0$ is an $\mathcal{N}$-bimodule follows from the fact that $\mathcal{U}(\mathcal{N})\mathcal{GN}(\mathcal{M},\mathcal{D})\mathcal{U}(\mathcal{N}) = \mathcal{GN}(\mathcal{M},\mathcal{D})$ and that $\operatorname{span}\mathcal{U}(\mathcal{N}) = \mathcal{N}$. Clearly $\mathcal{B}_0 \subseteq \mathcal{B}_1$.

Suppose $v \in \mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}_1$. If (x_λ) is a net in $\mathcal{B}$ with $\lim^{\text{Bures}} x_\lambda = v$, we find $\lim^{\text{Bures}} v^*x_\lambda = v^*v$. As E is Bures continuous, we have that $\lim^{\text{Bures}} E(v^*x_\lambda) = E(v^*v) = v^*v$. Since the relative Bures topology on $\mathcal{N}$ is the σ-strong topology on $\mathcal{N}$, $E(v^*x_\lambda)$ converges weak-$*$ to v^*v. By Corollary 5.3, $vE(v^*x_\lambda)$ is a net in $\mathcal{B}$ converging weak-$*$ to v, showing that $v \in \mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}$. Thus, $\mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}_0 = \mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B} = \mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}_1$.

Now suppose $(\mathcal{M},\mathcal{N},\mathcal{D})$ is a full Cartan triple. Clearly $\mathcal{B}_0 \subseteq \mathcal{B} \subseteq \mathcal{B}_1$. Let L be the linear span of $\mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}$ and choose $x \in \overline{\mathcal{B}}^{\text{Bures}}$. By Theorem 5.2, for each $u \in \mathcal{GN}(\mathcal{M},\mathcal{D})$, $uE(u^*x) \in L$, so Theorem 5.7 shows $x \in \mathcal{B}_1$, whence $\mathcal{B}_1 = \overline{\mathcal{B}}^{\text{Bures}}$. ∎

Notation 5.9. For a Bures-closed $\mathcal{N}$-bimodule $\mathcal{B} \subseteq \mathcal{M}$, let $\mathcal{GN}(\mathcal{B},\mathcal{D}) := \mathcal{GN}(\mathcal{M},\mathcal{D}) \cap \mathcal{B}$. Define $\Theta(\mathcal{B}) \subseteq \mathcal{S}$ by

$$\Theta(\mathcal{B}) = q(\mathcal{GN}(\mathcal{B},\mathcal{D})).$$

Further, define a map Ψ from the collection of spectral sets (see Definition 5.1) in $\mathcal{S}$ to Bures-closed $\mathcal{N}$-bimodules in $\mathcal{M}$ by

$$\Psi(A) = \overline{\operatorname{span}}^{\text{Bures}} q^{-1}(A) = \overline{\operatorname{span}}^{\text{Bures}}\{j(a)n : a \in A, n \in \mathcal{N}\},$$

which is necessarily a Bures-closed $\mathcal{N}$-bimodule.

When $(\mathcal{M},\mathcal{N},\mathcal{D})$ is full, Theorem 5.2 shows that $\mathcal{GN}(\mathcal{B},\mathcal{D})$ is non-zero whenever $\mathcal{B} \neq (0)$. We now extend the spectral theorem for bimodules in Cartan pairs (see [5, Theorem 2.5.8] and [12, Theorem 6.3]) to the context of Bures closed bimodules in a Cartan triple. Theorem 5.10 below should also be compared with [16, Theorem 4.3].

Suppose for $i = 1, 2$ that $\mathcal{P}_i$ are full $\mathcal{N}_i$-Clifford inverse monoids, $\mathcal{S}$ is a Cartan inverse monoid, $\mathcal{P}_i \hookrightarrow \mathcal{G}_i \overset{q_i}{\twoheadrightarrow} \mathcal{S}$ are extensions of $\mathcal{S}$ by $\mathcal{P}_i$, and let $(\mathcal{M}_i,\mathcal{N}_i,\mathcal{D}_i)$ be the corresponding Cartan triples. Theorem 5.10 implies the striking fact that the lattice structure of the Bures-closed $\mathcal{N}_i$-bimodules in $\mathcal{M}_i$ is isomorphic to the lattice of spectral sets in $\mathcal{S}$. Thus, $\mathcal{S}$ completely determines the lattice structure of the Bures-closed $\mathcal{N}_i$-bimodules regardless of the choice of extension of $\mathcal{S}$.

Theorem 5.10 (Spectral Theorem for Bimodules)**.** *Let $(\mathcal{M},\mathcal{N},\mathcal{D})$ be a full Cartan triple. The map Θ is a lattice isomorphism between the family of Bures-closed $\mathcal{N}$-bimodules in $\mathcal{M}$ and the family of spectral sets in $\mathcal{S}$. Moreover, $\Theta^{-1} = \Psi$.*

Proof. Let $\mathcal{B}$ be a Bures-closed $\mathcal{N}$-bimodule in $\mathcal{M}$ and let $A := \Theta(\mathcal{B})$. We will first show that A is a spectral set in $\mathcal{S}$. Suppose $s \in A$ and $t \leq s$. Then there exists an $e \in \mathcal{E}(\mathcal{S})$ such that $t = se$. Write $s = q(v)$ for some $v \in \mathcal{GN}(\mathcal{B},\mathcal{D})$, and $e = q(p)$ for some projection $p \in \mathcal{D}$, we find $t = q(vp)$, so $t \in A$. Next, suppose that $\{s_i\}_{i\in I}$ is a pairwise orthogonal family in A and let $s = \bigvee s_i$. For $i \neq k$, the orthogonality of s_i and s_k implies that $j(s_i)$ and $j(s_k)$ are partial isometries with orthogonal initial spaces and orthogonal range spaces. Therefore, the sum $\sum_{i\in I} j(s_i)$ converges strong-$*$ to an element $v \in \mathcal{GN}(\mathcal{M},\mathcal{D})$. As the Bures topology is weaker than the strong-$*$ topology, $v \in \mathcal{GN}(\mathcal{B},\mathcal{D})$. For every $i \in I$, $q(vj(s_i^\dagger s_i)) = s_i$, and it follows that $q(v) = s$. Thus $j(s) \in \mathcal{B}$, and hence $s \in A$. Therefore $A = \Theta(\mathcal{B})$ is a spectral set.

Proposition 5.8 shows that $\mathcal{B}$ is generated as a $\mathcal{N}$-bimodule by $\mathcal{B} \cap \mathcal{GN}(\mathcal{M},\mathcal{D})$. It follows that $\Psi(\Theta(\mathcal{B})) = \mathcal{B}$.

We now prove that $A = \Theta(\Psi(A))$. Clearly, $A \subseteq \Theta(\Psi(A))$. Choose $s \in \Theta(\Psi(A))$ and let $\mathcal{B} := \Psi(A)$. By definition, there exists $v \in \mathcal{GN}(\mathcal{B},\mathcal{D})$ such that $q(v) = s$. Let

$$r = \sup\{p \in \operatorname{proj}(\mathcal{D}) : q(vp) \in A\}.$$

Then $q(r)$ is the maximal idempotent in $\mathcal{E}(\mathcal{S})$ such that $s\,q(r) \in A$. Thus if $a \in A$, $s\,q(r^\perp) \wedge a = 0$. Therefore, for any $n \in \mathcal{N}$,

$$E((vr^\perp)^* j(a)) = 0 = E((vr^\perp)^* j(a)n).$$

Hence for any $x \in \operatorname{span}(q^{-1}(A))$, $E((vr^\perp)^* x) = 0$. As E is Bures continuous, we find that $E((vr^\perp)^* x) = 0$ for every $x \in \mathcal{B}$. As $vr^\perp \in \mathcal{B}$ and E is faithful, we obtain $vr^\perp = 0$. Hence $v = vr$. Applying q we obtain, $s = s\,q(r) \in A$, as desired.

Finally, the order preserving properties follow by the definitions of Θ and Ψ. $\square$

5.2. **Intermediate von Neumann algebras.** Our next goal is to give a version of Aoi's Theorem appropriate to our context. We first note the following technical result.

Proposition 5.11. *Let $\mathcal{M} \supseteq \mathcal{N}$ be an inclusion of von Neumann algebras and let $\mathcal{D} \subseteq \mathcal{Z}(\mathcal{N})$ be a von Neumann subalgebra. Assume further that there exists a faithful, normal conditional expectation $E : \mathcal{M} \to \mathcal{N}$. Let ψ be a faithful normal semi-finite weight on $\mathcal{N}$ and let $\phi = \psi \circ E$. Let σ_t^ϕ be the modular automorphism group for ϕ. The following statements hold.*

- *(a) The centralizer, $\mathcal{M}_\phi := \{x \in \mathcal{M} : \sigma_t^\phi(x) = x\ \forall\ t \in \mathbb{R}\}$, for σ_t^ϕ contains $\mathcal{D}$.*
- *(b) If $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$, then for every $t \in \mathbb{R}$, $\sigma_t^\phi(v) \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Further $\sigma_t^\phi(v)$ is Munn related to v;*
- *(c) If $\mathcal{A}$ is a von Neumann algebra such that $\mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{D}$ is regular in $\mathcal{A}$, then there is a unique faithful normal conditional expectation $E_\mathcal{A} : \mathcal{M} \to \mathcal{A}$ such that $\phi = \phi \circ E_\mathcal{A}$. In addition, $E_\mathcal{A}$ has the following properties:*
  - *(i) $E_\mathcal{A} E = E E_\mathcal{A} = E$; and*
  - *(ii) $E_\mathcal{A}$ is continuous when regarded as a map of $(\mathcal{M}, E\text{-Bures})$ into $(\mathcal{M}, E\text{-Bures})$.*

Proof. For $x \in \mathcal{M}$ and $d \in \mathcal{D}$, $E(x^* d^* dx) \le \|d\|^2 E(x^* x)$ and

$$E(d^* x^* x d) = E(x^* x)^{1/2} d^* d E(x^* x)^{1/2} \le \|d\|^2 E(x^* x).$$

Thus $\mathfrak{n}_\phi$ is a $\mathcal{D}$-bimodule. Recalling that

$$\mathfrak{m}_\phi := \left\{ \sum_{j=1}^n y_j^* x_j : n \in \mathbb{N}, x_j, y_j \in \mathfrak{n}_\phi \right\},$$

we see that $\mathfrak{m}_\phi$ is also a $\mathcal{D}$-bimodule. Furthermore, for any $d \in \mathcal{D}$ and $x \in \mathfrak{m}_\phi$, we have

$$\phi(xd) = \psi(E(xd)) = \psi(E(x)d) = \psi(dE(x)) = \psi(E(dx)) = \phi(dx).$$

An application of [30, Theorem VIII.2.6] now gives part (a).

Now let $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and let $w = \sigma_t^\phi(v)$. Using (a) we have

$$w^* d w = \sigma_t^\phi(v^* d v) = v^* d v.$$

Thus $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and w is Munn related to v, proving part (b).

The regularity of $\mathcal{D}$ in $\mathcal{A}$ and part (b) show that $\sigma_t^\phi(\mathcal{A}) \subseteq \mathcal{A}$ for every $t \in \mathbb{R}$. Lemma 2.20 gives $\phi|_\mathcal{A}$ is a faithful, semi-finite normal weight on $\mathcal{A}$. By [30, Theorem IX.4.2], there exists a unique normal conditional expectation $E_\mathcal{A} : \mathcal{M} \to \mathcal{A}$ such that $\phi \circ E_\mathcal{A} = \phi$. Since $\mathcal{N} \subseteq \mathcal{A}$, $E_\mathcal{A} \circ E = E$. Let $\Phi := E \circ E_\mathcal{A}$. Then Φ is a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ which satisfies $\phi \circ \Phi = \phi$. The uniqueness assertion of [30, Theorem IX.4.2] gives $\Phi = E$. We thus have the formula in part (c(i)). As E is faithful, so is $E_\mathcal{A}$.

Finally, suppose (x_λ) is a net in $\mathcal{M}$ converging to x in the E-Bures topology. Applying E to both sides of the inequality,

$$(E_\mathcal{A}(x_\lambda) - E_\mathcal{A}(x))^*(E_\mathcal{A}(x_\lambda) - E_\mathcal{A}(x)) = E_\mathcal{A}(x_\lambda - x)^* E_\mathcal{A}(x_\lambda - x) \le E_\mathcal{A}((x_\lambda - x)^*(x_\lambda - x))$$

and using the fact that $EE_{\mathcal{A}} = E$ shows that $E_{\mathcal{A}}(x_\lambda) \to E_{\mathcal{A}}(x)$ in the E-Bures topology. Thus $E_{\mathcal{A}}$ is E-Bures continuous. $\square$

Theorem 5.12 (Aoi's Theorem for Cartan Triples)**.** *Let* $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *be a Cartan triple and suppose* $\mathcal{A}$ *is a von Neumann algebra such that* $\mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{M}$*. Then* $\mathcal{A}$ *is Bures closed. Furthermore, if* $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *is full, then* $(\mathcal{A}, \mathcal{N}, \mathcal{D})$ *is a Cartan triple.*

Proof. Let $\mathcal{A}_0$ be the weak-$*$ closure of $\operatorname{span} \mathcal{GN}(\mathcal{A}, \mathcal{Z}(\mathcal{N}))$. Then $\mathcal{A}_0$ is a von Neumann algebra and, as $\mathcal{U}(\mathcal{N}) \subseteq \mathcal{GN}(\mathcal{A}, \mathcal{Z}(\mathcal{N}))$, $\mathcal{A}_0 \supseteq \mathcal{N}$. Thus, $\mathcal{A}_0$ is a weak-$*$ closed $\mathcal{N}$-bimodule.

By Proposition 5.11, there exists a faithful, normal conditional expectation $E_{\mathcal{A}_0} : \mathcal{M} \to \mathcal{A}_0$. Since $E_{\mathcal{A}_0}$ is E-Bures continuous, it follows that $\mathcal{A}_0$ is E-Bures closed. By Proposition 5.8,

$$\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \overline{\mathcal{A}}^{Bures} = \overline{\operatorname{span}}^{Bures} \mathcal{GN}(\mathcal{A}, \mathcal{Z}(\mathcal{N})) = \overline{\mathcal{A}_0}^{Bures} = \mathcal{A}_0,$$

so $\mathcal{A}$ is Bures closed.

When $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is full, that is, $\mathcal{D} = \mathcal{Z}(\mathcal{N})$, the previous paragraph shows that $\mathcal{D}$ is regular in $\mathcal{A}$, so $(\mathcal{A}, \mathcal{N}, \mathcal{D})$ is a Cartan triple. $\square$

Remark 5.13. With the notation of Theorem 5.12 and its proof, let $\mathcal{A}_{00}$ be the weak-$*$ closure of $\operatorname{span} \mathcal{GN}(\mathcal{A}, \mathcal{D})$. Then $\mathcal{N} \subseteq \mathcal{A}_{00} \subseteq \mathcal{A}_0$, and $\mathcal{A}_{00}$ is Bures closed. However, we have been unable to show $\mathcal{A}_{00} = \mathcal{A}_0$ in general, which is why we required the fullness hypothesis to conclude $(\mathcal{A}, \mathcal{N}, \mathcal{D})$ is a Cartan triple. However, this hypothesis is rather mild, and is satisfied when $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$. Thus Theorem 5.12 is indeed a generalization of Aoi's theorem for Cartan pairs.

As an immediate corollary, we use the Spectral Theorem for Bimodules to parametrize the intermediate von Neumann algebras for a full Cartan triple. As with Bures-closed bimodules, this parametrization depends only on the Cartan inverse monoid and not the extension.

Corollary 5.14. *Let* $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *be a full Cartan triple. Set*

$$\begin{aligned} vN(\mathcal{M}, \mathcal{N}, \mathcal{D}) &:= \{\mathcal{A} : \mathcal{A} \text{ is a von Neumann algebras such that } \mathcal{N} \subseteq \mathcal{A} \subseteq \mathcal{M}\} \text{ and} \\ sub(\mathcal{S}) &:= \{\mathcal{T} \subseteq \mathcal{S} : \mathcal{T} \text{ is a Cartan inverse submonoid of } \mathcal{S} \text{ with } \mathcal{E}(\mathcal{T}) = \mathcal{E}(\mathcal{S})\}. \end{aligned}$$

Then the restriction of Θ *to* $vN(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *gives a bijection between* $vN(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *and* $sub(\mathcal{S})$*.*

6. Examples

In this section, we give several examples of Cartan triples.

6.1. **Type I examples.** Suppose a Hilbert space $\mathcal{H}$ is decomposed as as a direct sum, $\mathcal{H} = \bigoplus_{i \in I} H_i$, where for all $i, j \in I$, $\dim \mathcal{H}_i = \dim \mathcal{H}_j$. Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{D}$ be the von Neumann algebra generated by $\{P_i : P_i \text{ is the projection onto } \mathcal{H}_i, i \in I\}$. Then $\mathcal{N} = \mathcal{D}' = \oplus_{i \in I} \mathcal{B}(\mathcal{H}_i)$ and $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a full Cartan triple. Indeed,

$$\mathcal{M} \cong \mathcal{B}(\ell^2(I)) \overline{\otimes} \mathcal{B}(\mathcal{H}_1), \mathcal{D} \cong D(\ell^2(I)) \overline{\otimes} \mathbb{C} I_{\mathcal{H}_1}, \text{ and } \mathcal{N} \cong D(\ell^2(I)) \overline{\otimes} \mathcal{B}(\mathcal{H}_1), \tag{6.1}$$

where $D(\ell^2(I))$ are the diagonal operators in $\mathcal{B}(\ell^2(I))$.

We now show every Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ with $\mathcal{M} = \mathcal{B}(\mathcal{H})$ has the form outlined above, and hence is necessarily full. Showing that $\mathcal{D}$ is atomic is the key step. To start, let P be the projection onto the closure of the span of the ranges of the minimal projections in $\mathcal{D}$. We argue by contradiction to show $P = I$. If $P \neq I$, fix a unit vector $\eta \in P^\perp \mathcal{H}$ and a positive integer n. Choose a maximal chain $\mathfrak{P}$ in $\operatorname{proj}(P^\perp \mathcal{D})$ (with respect to the ordering $\leq$ in $\operatorname{proj}(P^\perp \mathcal{D})$). The map from $\mathfrak{P}$ into $[0, 1]$ given by $\mathfrak{P} \ni R \mapsto \|R\eta\|$ is onto $[0, 1]$ since $\mathfrak{P}$ has no atoms. So for $0 \leq j \leq n$, let $R_j \in \mathfrak{P}$ be such that $\|R_j \eta\| = j/n$ and for $1 \leq j \leq n$ put $Q_j := R_j - R_{j-1}$. Thus $\|Q_j \eta\| = 1/n$ for every j.

If X is the rank-one projection onto the span of η, then

$$E(X) = E\left(\sum_{i=1}^{n} Q_i X\right) = \sum_{i=1}^{n} Q_i E(X) = E\left(\sum_{i=1}^{n} Q_i X Q_i\right).$$

As $\|Q_i X Q_i\| = 1/n$ for each i, $\|E(X)\| \leq 1/n$ for all choices of n and so $E(X) = 0$, contradicting faithfulness of E. Thus $P^\perp = 0$ and $\mathcal{D}$ is atomic.

Finally, since $\mathcal{GN}(\mathcal{M}, \mathcal{D})$ spans $\mathcal{M}$, any two atoms of $\mathcal{D}$, say A and B, must have the same dimension, since otherwise $A\mathcal{M}B \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) = \{0\}$, contradicting the regularity of $\mathcal{D}$ in $\mathcal{M}$.

By the previous paragraph, for every non-zero minimal projection $P \in \mathcal{Z}(\mathcal{M})$, $(\mathcal{M}P, \mathcal{N}P, \mathcal{D}P)$ is a full Cartan triple. As a consequence, we have the following observation.

Proposition 6.1. *If $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple with $\dim(\mathcal{M}) < \infty$, then $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is full.*

6.2. **Tensoring Cartan pairs.** Equations (6.1) decomposed a Cartan triple into tensor products, where $\mathcal{M} \cong \mathcal{B}(\ell^2(I))\overline{\otimes}\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{N} = D(\ell^2(I))\overline{\otimes}\mathcal{B}(\mathcal{H}_1)$. Note that $D(\ell^2(I))$ is a Cartan subalgebra of $\mathcal{B}(\ell^2(I))$. In fact, starting with any Cartan pair we can create a Cartan triple by tensoring with a von Neumann algebra.

Suppose $\mathcal{M}$ is a von Neumann algebra, $\mathcal{D} \subseteq \mathcal{M}$ is a Cartan MASA and let $\mathcal{N}$ be any von Neumann algebra. Consider the von Neumann algebras

$$\mathcal{D} \otimes I_{\mathcal{N}} \subseteq \mathcal{D}\overline{\otimes}\mathcal{N} \subseteq \mathcal{M}\overline{\otimes}\mathcal{N}.$$

Since $\mathcal{D}$ is regular in $\mathcal{M}$ it follows that $\mathcal{D} \otimes I_{\mathcal{N}}$ is regular in $\mathcal{M}\overline{\otimes}\mathcal{N}$. Further, the conditional expectation $E\colon \mathcal{M} \to \mathcal{D}$ induces a faithful conditional expectation $E\overline{\otimes}\mathrm{id}_{\mathcal{N}}\colon \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{D}\overline{\otimes}\mathcal{N}$. By [31, Theorem IV.5.9 and Corollary IV.5.10], $\mathcal{D}\overline{\otimes}\mathcal{N} = (\mathcal{D}\overline{\otimes}I_{\mathcal{N}})^c$. Thus $(\mathcal{M}\overline{\otimes}\mathcal{N}, \mathcal{D}\overline{\otimes}\mathcal{N}, \mathcal{D} \otimes I_{\mathcal{N}})$ is a Cartan triple. Further, if $\mathcal{N}$ is a factor $(\mathcal{M}\overline{\otimes}\mathcal{N}, \mathcal{D}\overline{\otimes}\mathcal{N}, \mathcal{D} \otimes I_{\mathcal{N}})$ is a full Cartan triple.

6.3. **Crossed products by discrete groups.** Cartan triples arise naturally as crossed product von Neumann algebras. In Section 6.3.1 we will show that if G is a discrete group acting on an abelian von Neumann algebra $\mathcal{D}$ then $(\mathcal{D} \rtimes_\alpha G, \mathcal{D}^c, \mathcal{D})$ will always give a Cartan triple. If a discrete group G acts on a (not necessarily abelian) von Neumann algebra $\mathcal{N}$, and $\mathcal{D} = \mathcal{Z}(\mathcal{N})$, we give necessary and sufficient conditions for $(\mathcal{N} \rtimes_\alpha G, \mathcal{N}, \mathcal{D})$ to be a Cartan triple in Section 6.3.2.

Let G be a discrete group acting on a von Neumann algebra $\mathcal{N}$ by automorphisms α. Let $\mathcal{M} = \mathcal{N} \rtimes_\alpha G$. The von Neumann algebra $\mathcal{M}$ is generated by a copy of $\mathcal{N}$ and a unitary representation of G, $\{u_g\}_{g\in G}$ such that $\alpha_g(d) = u_g d u_g^*$. There is a faithful, normal conditional expectation $E_{\mathcal{N}}$ from $\mathcal{M}$ onto $\mathcal{N}$. Each element $x \in \mathcal{M}$ is uniquely determined by a Fourier series

$$x = \sum_{g\in G} x_g u_g, \quad \text{where} \quad x_g := E_{\mathcal{N}}(x u_g^*) \in \mathcal{N}.$$

This series converges in the Bures-topology on $\mathcal{M}$ induced by $E_{\mathcal{N}}$ [23].

Cameron and Smith [6, 8] have studied Bures-closed bimodules and intermediate von Neumann algebras in a large class of crossed products. We will see in Theorem 6.3 that there is overlap in our work and theirs. However, neither work subsumes the other.

6.3.1. *Crossed products of abelian algebras.*

Theorem 6.2. *Let $\mathcal{D}$ be an abelian von Neumann algebra and let G be a discrete group acting on $\mathcal{D}$ by automorphisms α. Let $\mathcal{M} := \mathcal{D} \rtimes_\alpha G$ and $\mathcal{N} = \mathcal{D}^c$. Then $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple.*

Proof. Since $\mathcal{D}$ is clearly regular in $\mathcal{D} \rtimes_\alpha G$, we only need to note that there is a faithful normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Since there is a faithful, normal conditional expectation $E_{\mathcal{D}}$ from $\mathcal{M}$ onto $\mathcal{D}$, $\mathcal{D}$ is regular in $\mathcal{N}$, and $\mathcal{D} \subseteq \mathcal{N} \subseteq \mathcal{M}$, this follows from Proposition 5.11(c). Alternatively, the existence of the conditional expectation onto $\mathcal{N}$ also follows from the proof of

Theorem 3.2 of [8]. In [8] it is assumed that the action of G is by properly outer automorphisms, though this is not needed in the proof. $\square$

We give further details on the structure of this Cartan triple. For each $g \in G$, let p_g be the largest projection in $\mathcal{D}$ such that $\alpha_g|_{\mathcal{D}p_g}$ is the identity. We note p_g is the Frolík projection e_0 for ad_{u_g} on $\mathcal{D}$ described in the proofs of Lemma 2.18 and Proposition 4.9. By [26, Lemma 2.15], $u_g p_g = p_g u_g \in \mathcal{N}$ and $E_{\mathcal{N}}(u_g) = u_g p_g$.

Since $E_{\mathcal{N}}$ is Bures continuous, we can explicitly describe $E_{\mathcal{N}}$ by

$$E_{\mathcal{N}}\left(\sum_{g\in G} x_g u_g\right) = \sum_{g\in G} x_g p_g u_g.$$

6.3.2. *Crossed products of non-abelian algebras.* An automorphism α on a von Neumann algebra $\mathcal{N}$ is *properly outer* if there are no nonzero central projections $z \in \mathcal{Z}(\mathcal{N})$ such that $\alpha|_{\mathcal{N}z}$ is inner. Equivalently α is properly outer if and only if

$$yx = x\alpha(y)$$

for all $y \in \mathcal{N}$ implies that $x = 0$. In [8] crossed products by properly outer automorphisms are studied and the Bures-closed bimodules and intermediate von Neumann algebras are characterized. We show now that the crossed products studied in [8] give rise to full Cartan triples under the assumption that the restriction of the action to the center $\mathcal{Z}(\mathcal{N})$ is also properly outer.

Theorem 6.3. *Let $\mathcal{N}$ be a von Neumann algebra and let G be a discrete group acting on $\mathcal{N}$ by properly outer automorphisms α. Let $\mathcal{M} = \mathcal{N} \rtimes_\alpha G$. Then $(\mathcal{M}, \mathcal{N}, \mathcal{Z}(\mathcal{N}))$ is a Cartan triple if and only if the action of G restricted to the center $\mathcal{Z}(\mathcal{N})$ is properly outer.*

Proof. Suppose $x \in \mathcal{Z}(\mathcal{N})' \cap \mathcal{M}$. Let $x = \sum_{g\in G} x_g u_g$ be the (Bures convergent) Fourier series for x. Since $x \in \mathcal{Z}(\mathcal{N})'$ it follows that if $x_g \neq 0$ then for $d \in \mathcal{Z}(\mathcal{N})$,

$$dx_g = E_{\mathcal{N}}(dxu_g^*) = E_{\mathcal{N}}(xu_g^*(u_g d u_g^*)) = x_g \alpha_g(d). \tag{6.2}$$

Let J_g be the two-sided ideal in $\mathcal{N}$ generated by x_g. It follows from (6.2) that $xd = x\alpha(d)$ for all $x \in J_g$ and all $d \in \mathcal{Z}(\mathcal{N})$. Since J_g is a two-sided ideal, there is a central projection $z_g \in \mathcal{Z}(\mathcal{N})$ such that $J_g = \mathcal{N}z_g$. Thus $z_g d = z_g \alpha_g(d)$ for all $d \in \mathcal{Z}(\mathcal{N})$. That is, $\alpha_g|_{\mathcal{Z}(\mathcal{N})z_g} = \mathrm{id}|_{\mathcal{Z}(\mathcal{N})z_g}$.

It follows that $\mathcal{N} = \mathcal{Z}(\mathcal{N})^c$ if and only if for all $g \neq e$, $\alpha_g|_{\mathcal{Z}(\mathcal{N})}$ is properly outer. $\square$

6.4. **Crossed products by equivalence relations.** Igor Fulman in [17] studied a class of Cartan triples which he called crossed products by an equivalence relation. A crossed product by an equivalence relation is a Cartan triple satisfying the condition in Definition 6.4 below, which we also call *Fulman's condition.* In Appendix A we provide a conceptual framework in terms of inverse semigroups for Fulman's condition and show that Fulman's condition amounts to a lifting problem. Here we give a class of Cartan triples which satisfy Fulman's condition.

Suppose $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple with associated extension,

$$\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$$

and fixed order preserving section j.

Definition 6.4. A *regularizer* is a subgroup $R \subseteq \mathcal{U}(\mathcal{M})$ satisfying:

(a) $\mathcal{U}(\mathcal{N}) \subseteq R \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$;
(b) $\operatorname{span} R$ is weak-$*$ dense in $\mathcal{M}$;
(c) there is a homomorphism $\alpha\colon R \to \mathrm{Aut}(\mathcal{N})$ such that
  (a) if p is a projection in $\mathcal{D}$ such that $\alpha_u|_{\mathcal{D}p} = \mathrm{id}|_{\mathcal{D}p}$ then $\alpha_u|_{\mathcal{N}p} = \mathrm{id}|_{\mathcal{N}p}$.
  (b) $\alpha_u(d) = udu^*$ for each $u \in R$ and $d \in \mathcal{D}$.

We will call a map α satisfying conditions (i) and (ii) of part (c) a *regularizing map* for R.

When the Cartan triple $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ has a regularizer, we say $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ *satisfies Fulman's condition.*

Note that if R is a regularizer with regularizing map α, then $\ker \alpha = \mathcal{U}(\mathcal{N})$ ([17, Remark, pg. 41]).

Example 6.5. Let $\mathcal{N}$ be a von Neumann algebra and let $\mathcal{D} = \mathcal{Z}(\mathcal{N})$. Let G be a discrete group acting on $\mathcal{N}$ by properly outer automorphisms. Further assume that the restriction of the action of G to $\mathcal{D}$ is properly outer. Let $\mathcal{M} = \mathcal{N} \rtimes_\alpha G$. Then by Theorem 6.3 $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ is a Cartan triple. Let R be the group generated by

$$\{u_g \colon g \in G\} \cup \{u \in \mathcal{N} \colon u \text{ unitary}\}.$$

Let $R := \{u_g : g \in G\}$ and let $\alpha : R \to \operatorname{Aut}(\mathcal{N})$ be $u_g \mapsto \operatorname{ad}_{u_g}$. Since G acts by properly outer automorphisms on $\mathcal{Z}(\mathcal{N})$, α is a regularizing map for R so that R is a regularizer. Thus $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ satisfies Fulman's condition.

Appendix A. An Inverse Semigroup Description of Fulman's Condition

Fulman's condition as stated in Definition 6.4, is mysterious. Our goal in this appendix is to establish Theorem A.8, which shows Fulman's condition is equivalent to the statement that a rather natural lifting problem for inverse semigroups (Diagram A.1) has a positive solution. We begin with a definition.

Definition A.1. Let $(\mathcal{N}, \mathcal{D})$ be a pair of von Neumann algebras with $\mathcal{D}$ a von Neumann subalgebra of $\mathcal{Z}(\mathcal{N})$. A *partial automorphism of* $(\mathcal{N}, \mathcal{D})$ is a triple (e, α, f) consisting of projections $e, f \in \mathcal{D}$ and a normal $*$-isomorphism $\alpha : f\mathcal{N} \to e\mathcal{N}$ satisfying $\alpha(e\mathcal{D}) = f\mathcal{D}$. We will use $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ for the set of all partial automorphisms of $(\mathcal{N}, \mathcal{D})$. If $f = 0$ (or $e = 0$) we say (e, α, f) is the zero element of $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$. Further, define an involution and a product in $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ via,

$$(e, \alpha, f)^\dagger := (f, \alpha^{-1}, e) \quad \text{and} \quad (e_1, \alpha_1, f_1)(e_2, \alpha_2, f_2) := (\alpha_1(f_1 e_2), (\alpha_1 \circ \alpha_2)|_{\alpha_2^{-1}(e_2 f_1)}, \alpha_2^{-1}(f_1 e_2)).$$

Then $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ is an inverse monoid with 0. Also

$$\mathcal{E}(\operatorname{pAut}(\mathcal{N}, \mathcal{D})) = \{(e, \operatorname{id}|_{e\mathcal{N}}, e) : e \in \operatorname{proj}(\mathcal{D})\}$$

and hence may be identified with $\operatorname{proj}(\mathcal{D})$. For $\gamma = (e, \alpha, f) \in \operatorname{pAut}(\mathcal{N}, \mathcal{D})$ and $x \in f\mathcal{N}$, we write $\gamma(x)$ for the value of α at x.

We require the following notions for an inverse semigroup $\mathcal{R}$.

- Two elements $s, t \in \mathcal{R}$ are *compatible* if $st^\dagger$ and $s^\dagger t$ are idempotents; a subset $A \subseteq \mathcal{R}$ is *compatible* ([21, page 26] if every pair of elements of A is compatible.
- $\mathcal{R}$ is *infinitely distributive* ([21, page 28]) if whenever I is an index set and $\{r_i\}_{i \in I} \subseteq \mathcal{R}$ is such that $\bigvee_{i \in I} r_i$ exists then for any $s \in \mathcal{R}$,
$$\bigvee_{i \in I} s r_i \text{ and } \bigvee_{i \in I} r_i s \text{ exist and } s \left(\bigvee_{i \in I} r_i \right) = \bigvee_{i \in I} s r_i, \quad \left(\bigvee_{i \in I} r_i \right) s = \bigvee_{i \in I} r_i s.$$
- $\mathcal{R}$ is *complete* ([21, page 27]) if whenever $A \subseteq \mathcal{R}$ is a compatible subset, $\bigvee A$ exists.

Lemma A.2. *The inverse semigroup* $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ *is infinitely distributive and complete.*

Proof. As $\operatorname{proj}(\mathcal{D})$ is a complete Boolean algebra, [21, Proposition 1.4.20] shows $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ is an infinitely distributive inverse semigroup.

We turn now to showing $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$ is complete. Given $a = (e_a, \alpha_a, f_a) \in \operatorname{pAut}(\mathcal{N}, \mathcal{D})$, identify $a^\dagger a$ with f_a and $aa^\dagger$ with e_a, so that the source and range of a belong to $\operatorname{proj}(\mathcal{D})$.

First suppose that $A \subseteq \mathrm{pAut}(\mathcal{N},\mathcal{D})$ is a finite and orthogonal set. Let $e = \bigvee_{a\in A} aa^\dagger$ and $f = \bigvee_{a\in A} a^\dagger a$. For $n \in \mathcal{N}f$, $n = \sum_{a\in A} na^\dagger a$ and define

$$\alpha(n) := \sum_{a\in A} \alpha_a(na).$$

Then $(e,\alpha,f) \in \mathrm{pAut}(\mathcal{N},\mathcal{D})$. For $a \in A$, $(e,\alpha,f)(f_a, \mathrm{id}|_{\mathcal{N}f_a}, f_a) = a$ so $a \leq (e,\alpha,f)$. Further, if for every $a \in A$, $a \leq (e',\alpha',f')$, then $(e,\alpha,f) = (e',\alpha',f')(f,\mathrm{id}|_{\mathcal{N}f}, f) = (e,\alpha,f)$. Thus, $(e,\alpha,f) = \bigvee A$. So joins exist for finite orthogonal sets.

Next, suppose A is a finite compatible set. Let $\mathcal{B}$ be the (finite) Boolean algebra generated by $\{a^\dagger a : a \in A\}$. The identity of $\mathcal{B}$ is $f := \bigvee_{a\in A} a^\dagger a$. Let $\mathrm{atom}(\mathcal{B})$ be the (finite) set of atoms of $\mathcal{B}$. Let $C := \{ap : a \in A, p \in \mathrm{atom}(\mathcal{B})\}$. Then C is a finite orthogonal set of elements in $\mathrm{pAut}(\mathcal{N},\mathcal{D})$. Let $(e,\alpha,f) := \bigvee C$. Let $a \in A$ and $P_a := \{p \in \mathrm{atom}(\mathcal{B}) : ap \neq 0\}$. Then $a = \sum_{p\in P_a} ap$ and $a^\dagger a = f_a = \sum_{p\in P_a} p \leq f$. So for every $p \in P_a$, $(e,\alpha,f)(p,\mathrm{id}|_{\mathcal{N}p}, p) = ap$. Thus

$$(e,\alpha,f)(p,\mathrm{id}|_{\mathcal{N}p},p) = ap, \quad \text{whence} \quad (e,\alpha,f)(f_a, \mathrm{id}_{\mathcal{N}f_a}, f_a) = a.$$

This shows that for every $a \in A$, $a \leq (e,\alpha,f)$. On the other hand, if $a \leq (e',\alpha',f')$ for every $a \in A$, then $(e',\alpha',f')(f,\mathrm{id}|_{\mathcal{N}f}, f) = (e,\alpha,f)$ so $(e,\alpha,f) = \bigvee A$, showing joins exist for any finite compatible set.

Finally, let $A \subseteq \mathrm{pAut}(\mathcal{N},\mathcal{D})$ be an arbitrary compatible subset. Let $\mathcal{F}$ be the set of all finite subsets of A ordered by inclusion and for $F \in \mathcal{F}$, let $a_F = \bigvee F$. Notice that if $F_1 \subseteq F_2$, then $a_{F_1} \leq a_{F_2}$. Write $a_F = (e_F, \alpha_F, f_F)$. Let $e = \bigvee\{aa^\dagger : a \in A\} = \bigvee_{F\in\mathcal{F}} e_F$ and $f = \bigvee\{a^\dagger a : a \in A\} = \bigvee_{F\in\mathcal{F}} f_F$. For $n \in \mathcal{N}f$, the net $\alpha_F(nf_F)$ converges strongly, and we define $\alpha(n) = \lim \alpha_F(nf_F)$. Then $(e,\alpha,f) = \bigvee A$. $\square$

The inverse semigroup $\mathrm{pAut}(\mathcal{N},\mathcal{D})$ may be written as an extension,

$$\mathrm{Cliff}(\mathrm{pAut}(\mathcal{N},\mathcal{D})) \hookrightarrow \mathrm{pAut}(\mathcal{N},\mathcal{D}) \overset{\pi}{\twoheadrightarrow} \mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$$

where $\mathrm{Cliff}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$ is the Clifford inverse subsemigroup of all elements of $\mathrm{pAut}(\mathcal{N},\mathcal{D})$ which are Munn related to an idempotent, and $\mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$ is the quotient of $\mathrm{pAut}(\mathcal{N},\mathcal{D})$ by the Munn relation.

Henceforth, fix a Cartan triple $(\mathcal{M},\mathcal{N},\mathcal{D})$ with associated extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ and order-preserving section j. We shall be interested in the semigroup $\mathrm{pAut}(\mathcal{N},\mathcal{D})$ arising from this Cartan triple.

The idempotents of $\mathrm{pAut}(\mathcal{N},\mathcal{D})$ (and hence those of $\mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$) may be identified with $\mathcal{E}(\mathcal{S})$. We shall show that for any Cartan triple, there is a one-to-one inverse semigroup homomorphism $\theta : \mathcal{S} \to \mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$ which fixes idempotents. Our goal in this section is to show that Fulman's condition is satisfied if and only if there is a lifting of θ to an inverse semigroup homomorphism α so that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{pAut}(\mathcal{N},\mathcal{D}) \arrow[d, "\pi"] \\ \mathcal{S} \arrow[ur, dashed, "\alpha"] \arrow[r, "\theta"'] & \mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D})). \end{tikzcd} \tag{A.1}$$

For $(e,\alpha,f) \in \mathrm{pAut}(\mathcal{N},\mathcal{D})$, let $[e,\alpha,f] \in \mathrm{Fund}(\mathrm{pAut}(\mathcal{N},\mathcal{D}))$ denote the Munn equivalence class of (e,α,f). It will be helpful to have an explicit description of the Munn relation on $\mathrm{pAut}(\mathcal{N},\mathcal{D})$.

Lemma A.3. *For $i = 1,2$, let $(e_i,\alpha_i,f_i) \in \mathrm{pAut}(\mathcal{N},\mathcal{D})$. The following are equivalent.*

(a) (e_1,α_1,f_1) is Munn related to (e_2,α_2,f_2);
(b) for every $d \in \mathrm{proj}(\mathcal{D})$, $\alpha_1(df_1) = \alpha_2(df_2)$;

(c) $e_1 = e_2$, $f_1 = f_2$ *and* $\alpha_1|_{f_1\mathcal{D}} = \alpha_2|_{f_2\mathcal{D}}$.

Proof. Suppose (a) holds. Then for any $d \in \operatorname{proj}(\mathcal{D})$, $(d, \operatorname{id}|_{d\mathcal{N}}, d) \in \mathcal{E}(\operatorname{pAut}(\mathcal{N}, \mathcal{D}))$, so

$$(e_i, \alpha_i, f_i)(d, \operatorname{id}|_{d\mathcal{N}}, d)(f_i, \alpha_i^{-1}, e_i) = (\alpha_i(df_i), \operatorname{id}_{\alpha_i(df_i)\mathcal{N}}, \alpha_i(df_i)), \tag{A.2}$$

which yields (b).

Now suppose (b) holds. Taking $d \in \{f_1, f_2, f_1f_2\}$ gives $\alpha_1(f_1) = \alpha_2(f_1f_2) = \alpha_1(f_1f_2) = \alpha_2(f_2)$, so that $f_1 = f_2$ and $e_1 = e_2$. Since $f_i\mathcal{D}$ is generated by $\operatorname{proj}(f_i\mathcal{D})$, we obtain (c).

Finally, assume (c) holds. Let $d \in \operatorname{proj}(\mathcal{D})$. Examining (A.2) we obtain

$$\begin{aligned}(e_1, \alpha_1, f_1)(d, \operatorname{id}_{d\mathcal{N}}, d)(f_1, \alpha_1^{-1}, e_1) &= (\alpha_1(df_1), \operatorname{id}_{\alpha_1(df_1)\mathcal{N}}, \alpha_1(df_1)) \\ &= (\alpha_2(df_2), \operatorname{id}_{\alpha_2(df_2)\mathcal{N}}, \alpha_2(df_2)) \\ &= (e_2, \alpha_2, f_2)(d, \operatorname{id}_{d\mathcal{N}}, d)(f_2, \alpha_2^{-1}, e_2).\end{aligned}$$

Thus (a) holds and the proof is complete. $\square$

We now observe that there is always a one-to-one inverse semigroup homomorphism of $\mathcal{S}$ into $\operatorname{Fund}(\operatorname{pAut}(\mathcal{N}, \mathcal{D}))$. Note that if $v \in \mathcal{G}$, then v defines a partial automorphism in $\operatorname{pAut}(\mathcal{N}, \mathcal{D})$. Indeed if we define ad_v by

$$\begin{aligned}\operatorname{ad}_v \colon v^*v\mathcal{N} &\to vv^*\mathcal{N} \\ v^*vx &\mapsto vxv^*,\end{aligned}$$

then $(vv^*, \operatorname{ad}_v, v^*v) \in \operatorname{pAut}(\mathcal{N}, \mathcal{D})$. We define a map $\theta \colon \mathcal{S} \to \operatorname{Fund}(\operatorname{pAut}(\mathcal{N}, \mathcal{D})))$ by

$$\theta(s) = [j(ss^\dagger), \operatorname{ad}_{j(s)}, j(s^\dagger s)].$$

By Lemma A.3, if $v, w \in \mathcal{G}$ and v and w are Munn equivalent, then $[vv^*, \operatorname{ad}_v, v^*v] = [ww^*, \operatorname{ad}_w, w^*w]$. Hence the map θ is independent of the choice of j. Indeed, for any $w \in q^{-1}\{s\}$, $\theta(s) = [ww^*, \operatorname{ad}_w, w^*w]$. Thus we may use any of

$$[j(ss^\dagger), \alpha_{j(s)}, j(s^\dagger s)], \quad [ss^\dagger, \alpha_s, s^\dagger s], \quad \text{or} \quad [j(ss^\dagger), \alpha_s, j(s^\dagger s)]$$

to denote $\theta(s)$.

Proposition A.4. *The map* $\theta : \mathcal{S} \to \operatorname{Fund}(\operatorname{pAut}(\mathcal{N}, \mathcal{D}))$ *given by*

$$\theta(s) := [j(ss^\dagger), \operatorname{ad}_s, j(s^\dagger s)]$$

is a one-to-one homomorphism of inverse semigroups such that $\theta|_{\mathcal{E}(\mathcal{S})}$ *is an isomorphism of* $\mathcal{E}(\mathcal{S})$ *onto* $\mathcal{E}(\operatorname{Fund}(\operatorname{pAut}(\mathcal{N}, \mathcal{D})))$.

Proof. For $e \in \mathcal{E}(\mathcal{S})$, $\theta(e) = [j(e), \operatorname{id}_{j(e)\mathcal{N}}, j(e)]$, so $\theta|_{\mathcal{E}(\mathcal{S})}$ is an isomorphism of $\mathcal{E}(\mathcal{S})$ onto $\mathcal{E}(\operatorname{Fund}(\operatorname{pAut}(\mathcal{N}, \mathcal{D})))$.

Take $s_1, s_2 \in \mathcal{S}$. Then

$$\theta(s_1s_2) = [j(s_1s_2s_2^\dagger s_1^\dagger), \operatorname{ad}_{s_1s_2}, j(s_2^\dagger s_1^\dagger s_1s_2)].$$

On the other hand, $\operatorname{ad}_{s_2}^{-1}(j(s_1^\dagger s_1s_2s_2^\dagger)) = j(s_2^\dagger s_1^\dagger s_1s_2)$ and $\operatorname{ad}_{s_1}(j(s_1^\dagger s_1s_2s_2^\dagger)) = j(s_1s_2s_2^\dagger s_1^\dagger)$, so

$$[j(s_1s_1^\dagger), \operatorname{ad}_{s_1}, j(s_1^\dagger s_1)][j(s_2s_2^\dagger), \operatorname{ad}_{s_2}, j(s_2^\dagger s_2)] = [j(s_1s_2s_2^\dagger s_1^\dagger), \operatorname{ad}_{s_1} \circ (\operatorname{ad}_{s_2}|_{j(s_2^\dagger s_1^\dagger s_1s_2)\mathcal{N}}), j(s_2^\dagger s_1^\dagger s_1s_2)].$$

Thus to show that θ is a homomorphism it suffices to show that

$$\operatorname{ad}_{s_1s_2} = \operatorname{ad}_{s_1} \circ (\operatorname{ad}_{s_2}|_{j(s_2^\dagger s_1^\dagger s_1s_2)\mathcal{N}}).$$

Note that for each $s \in \mathcal{S}$ and $e \in \mathcal{E}(\mathcal{S})$,

$$\theta(s)(j(e)) = j(s)j(e)j(s)^* = j(ses^\dagger).$$

Hence for $e \in \mathcal{E}(\mathcal{S})$, $\mathrm{ad}_{s_1s_2}(j(e)j(s_2^\dagger s_1^\dagger s_1 s_2)) = \mathrm{ad}_{s_1}(\mathrm{ad}_{s_2}(j(e)j(s_2^\dagger s_1^\dagger s_1 s_2)))$. An application of Lemma A.3 now shows that θ is multiplicative on $\mathcal{S}$. Hence θ is an inverse semigroup homomorphism.

If $\theta(s_1) = \theta(s_2)$, then for every $e \in \mathcal{E}(\mathcal{S})$, $\mathrm{ad}_{s_1}|_{j(e)\mathcal{D}} = \mathrm{ad}_{s_2}|_{j(e)\mathcal{D}}$, so that in particular, $s_1 e s_1^\dagger = s_2 e s_2^\dagger$ for every $e \in \mathcal{E}(\mathcal{S})$. As $\mathcal{S}$ is fundamental, $s_1 = s_2$, whence θ is one-to-one. $\square$

We now show that a regularizer may be viewed as a homomorphism of $q(R)$ into $\mathrm{Aut}(\mathcal{N})$ satisfying Fulman's conditions.

Lemma A.5. *Suppose a regularizer R exists for $(\mathcal{M}, \mathcal{N}, \mathcal{D})$. Let $\alpha : R \to \mathrm{Aut}(\mathcal{N})$ be a regularizing map. Then α induces a one-to-one group homomorphism $\tilde{\alpha}\colon q(R) \to \mathrm{Aut}(\mathcal{N}, \mathcal{D})$ such that for every $e \in \mathcal{E}(\mathcal{P})$ and $U \in R$,*

$$\tilde{\alpha}_{q(U)}(e) = \mathrm{ad}_U(e) = j(q(U))ej(q(U))^*.$$

Proof. By condition (c)(ii) of Definition 6.4, $(I, \alpha_U, I) \in \mathrm{Aut}(\mathcal{N}, \mathcal{D})$ for every $U \in R$. Applying condition (c)(i) of Definition 6.4, it follows that there exists a one-to-one group homomorphism $\tilde{\alpha}\colon q(R) \to \mathrm{Aut}(\mathcal{N}, \mathcal{D})$. If $e \in \mathcal{E}(\mathcal{P})$, and $U \in R$, then $\tilde{\alpha}_{q(U)}(e) = \alpha_U(e) = UeU^{-1}$. $\square$

Lemma A.6. *Let R be a regularizer for $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ and let $\mathcal{R} := \{q(s)e : s \in R, e \in \mathcal{E}(\mathcal{S})\}$. Then $\mathcal{R}$ is an inverse semigroup and $\mathcal{S}$ is isomorphic to the join completion of $\mathcal{R}$.*

Proof. A calculation shows $\mathcal{R}$ is an inverse semigroup, and by definition, $\mathcal{S}$ is complete. Notice that every compatible order ideal of $\mathcal{R}$ is also a compatible order ideal of $\mathcal{S}$. Thus by the proof of [21, Theorem 1.4.23], the join completion of $\mathcal{R}$ is contained in $\mathcal{S}$.

Take $s \in \mathcal{S}$ and let $t = \bigvee\{q(r) \wedge s\colon r \in R\}$. Suppose $t \neq s$. As $\{a \in \mathcal{S}\colon a \leq s\}$ is a Boolean algebra, there is a $u \in \mathcal{S}$ such that $u \vee t = s$ and $u \wedge t = 0$. There is a $w \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ such that $q(w) = u$. As R densely spans $\mathcal{M}$, there is a $U \in R$ such that $E(U^*w) \neq 0$. Hence $v = UE(U^*w) \neq 0$. Note that $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and

$$\begin{aligned} q(v) &= q(U)q(E(u^*w)) = q(U)(q(U^*)q(w) \wedge 1) \\ &= q(w) \wedge q(U) \leq s \wedge q(U). \end{aligned}$$

Hence $q(v) \leq t$. However, $q(v) \leq u$. Hence $u = 0$, and $t = s$. For every $r \in R$, $q(r) \wedge s \in \mathcal{R}$. Hence the completion of $\mathcal{R}$ is $\mathcal{S}$. $\square$

Next we show that Fulman's condition implies that there is a homomorphism of $\mathcal{S}$ into $\mathrm{pAut}(\mathcal{N}, \mathcal{D})$ which lifts the map θ described in Proposition A.4.

Lemma A.7. *Suppose $\Gamma \subseteq \mathcal{S}$ is a group (under the multiplication inherited from $\mathcal{S}$) whose unit is $1 \in \mathcal{S}$. Assume that $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{N}, \mathcal{D})$ is a one-to-one homomorphism such that for every $s \in \Gamma$, Fulman's condition (c) is satisfied for α_s, that is,*

(i) if $p \in \mathrm{proj}(\mathcal{D})$ satisfies $\alpha_s|_{p\mathcal{D}} = \mathrm{id}|_{p\mathcal{D}}$, then $\alpha_s|_{p\mathcal{N}} = \mathrm{id}|_{p\mathcal{N}}$; and
(ii) for $d \in \mathcal{D}$, $\alpha_s(d) = j(s)dj(s)^*$.

Let $\mathcal{S}_\Gamma \subseteq \mathcal{S}$ be the smallest Cartan inverse submonoid of $\mathcal{S}$ containing Γ and $\mathcal{E}(\mathcal{S})$. Then α extends uniquely to a one-to-one homomorphism $\alpha' : \mathcal{S}_\Gamma \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$. In addition $\pi \circ \alpha' = \theta|_{\mathcal{S}_\Gamma}$.

Proof. Let $\mathcal{R} := \{se : s \in \Gamma, e \in \mathcal{E}(\mathcal{S})\}$. Since Γ is a group, $\mathcal{R}$ is an inverse semigroup. As in the proof of Lemma A.5(b), $\mathcal{S}_\Gamma$ is the join completion of $\mathcal{R}$. We shall show that there is a multiplicative map of $\alpha' : \mathcal{R} \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$.

Suppose $s, t \in \Gamma$. Fulman's condition (c) applied to $s^\dagger t$ shows that if $p \in \mathrm{proj}(\mathcal{D})$ and $\alpha_s|_{p\mathcal{D}} = \alpha_t|_{p\mathcal{D}}$, then $\alpha_s|_{p\mathcal{N}} = \alpha_t|_{p\mathcal{N}}$. For $s \in \Gamma$ and $e \in \mathcal{E}(\mathcal{S})$, define

$$\alpha'(se) := (j(ses^\dagger), \alpha_s|_{j(e)\mathcal{N}}, j(e)).$$

Note that this is well-defined, for if $se = tf$ for some idempotents e, f and $t \in \Gamma$, then $\alpha_{s^\dagger t}|_{fe\mathcal{D}} = \mathrm{id}|_{fe\mathcal{D}}$, so $\alpha_s|_{ef\mathcal{N}} = \alpha_t|_{ef\mathcal{N}}$. Thus, $\alpha' : \mathcal{R} \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$ is well-defined. For $s, t \in \Gamma$ and $e, f \in \mathcal{E}(\mathcal{S})$ a calculation shows that $\alpha'((se)(tf)) = \alpha'(st)\alpha'(tf)$, so α' is a homomorphism. Also, for any $s \in \Gamma$ and $e \in \mathcal{E}(\mathcal{S})$, $\pi(\alpha'(se)) = \theta(se)$.

By [21, Theorem 1.4.24], α' extends uniquely to a join-preserving homomorphism of $\mathcal{S}_\Gamma$ into $\mathrm{pAut}(\mathcal{N}, \mathcal{D})$. Take $s \in \Gamma$. Recall $\theta(s) = [j(ss^\dagger), \mathrm{ad}_s, j(s^\dagger s)] = [j(1), \mathrm{ad}_s, j(1)]$. Since $\alpha_s(d) = j(s)dj(s)^*$ for all $d \in \mathcal{D}$, by Lemma A.3, $\pi \circ \alpha(s) = \theta(s)$. That $\pi \circ \alpha' = \theta|_{\Gamma_\mathcal{S}}$ now follows from the definition of α'. Since θ is a one-to-one map it follows that α' is one-to-one. □

We now are prepared to recast Fulman's condition as a lifting problem.

Theorem A.8. *Let $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ be a Cartan triple with associated extension $\mathcal{P} \hookrightarrow \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$. Then $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ satisfies Fulman's condition if and only if there exists a homomorphism of inverse semigroups $\alpha\colon \mathcal{S} \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$ such that $\pi \circ \alpha = \theta$.*

Proof. Suppose $(\mathcal{M}, \mathcal{N}, \mathcal{D})$ satisfies Fulman's condition. Combining Lemmas A.5 and A.7 we obtain a homomorphism $\alpha : \mathcal{S} \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$ such that $\pi \circ \alpha = \theta$.

Conversely, suppose $\alpha : \mathcal{S} \to \mathrm{pAut}(\mathcal{N}, \mathcal{D})$ is a homomorphism satisfying $\pi \circ \alpha = \theta$. Let $R := \mathcal{U}(\mathcal{M}) \cap \mathcal{G}$. Clearly $\mathcal{U}(\mathcal{N}) \subseteq R \subseteq \mathcal{GN}(\mathcal{M}, \mathcal{D})$ and $\operatorname{span} R$ is weak-$*$ dense in $\mathcal{M}$. Let $\tau := \alpha \circ q|_R$. Then $\tau\colon R \to \mathrm{Aut}(\mathcal{N})$ is a homomorphism. For $u \in R$ write τ_u instead of $\tau(u)$.

We claim that for $u \in R$ and $d \in \mathcal{D}$, $\tau_u(d) = udu^*$. Since $\pi \circ \tau = \theta$, we obtain $\pi(\tau_u) = \theta(q(u))$, that is, $[1, \tau_u, 1] = [1, \mathrm{ad}_{q(u)}, 1]$. By Lemma A.3, we obtain $\tau_u|_\mathcal{D} = \mathrm{ad}_{q(u)}|_\mathcal{D}$. But, using Proposition A.4, for every $d \in \mathcal{D}$, $\mathrm{ad}_{q(u)}(d) = udu^*$. The claim follows.

Suppose $e \in \mathrm{proj}(\mathcal{D})$ and $\tau_u|_{e\mathcal{D}} = \mathrm{id}|_{e\mathcal{D}}$. Let $s = q(ue)$ and note that $s^\dagger s = q(e)$. For $f \in \mathcal{E}(\mathcal{S})$ we have

$$sfs^\dagger = q(uej(f)pu^*) = q(\tau_u(ej(f))) = q(ej(f))) = q(e)fq(e)^\dagger.$$

Since $\mathcal{S}$ is fundamental, we obtain $s = q(e)$. So s is an idempotent. Therefore $\alpha(s) \in \mathrm{pAut}(\mathcal{N}, \mathcal{D})$ is idempotent, which is to say that $\alpha(s) = \mathrm{id}|_{e\mathcal{N}}$. Since $\alpha(s) = \tau_u|_{e\mathcal{N}}$, we find that $\tau_u|e\mathcal{N} = \mathrm{id}|_{e\mathcal{N}}$. This completes the proof. □

Remark A.9. While we do not presently have an example, it seems unlikely that for a general Cartan triple, this lifting problem will have a solution. Thus, we expect that there should be an example of a Cartan triple which is not a crossed product by an equivalence relation.

A sufficient condition for a solution to the lifting problem is if the map $j\colon \mathcal{S} \to \mathcal{G}$ can be chosen so that $j(st)^* j(s)j(t) \in \mathcal{D}$. In this case the map $\alpha\colon s \mapsto (j(ss^\dagger), \mathrm{ad}_s, j(s^\dagger s))$ can be shown to be homomorphism. Clearly $\theta = \pi \circ \alpha$.

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