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NORMING ALGEBRAS AND AUTOMATIC COMPLETE BOUNDEDNESS OF ISOMORPHISMS OF OPERATOR ALGEBRAS

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NORMING ALGEBRAS AND AUTOMATIC COMPLETE BOUNDEDNESS OF ISOMORPHISMS OF OPERATOR ALGEBRAS

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Abstract. We combine the notion of norming algebra introduced by Pop, Sinclair and Smith with a result of Pisier to show that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are operator algebras, then any bounded epimorphism of $\mathcal{A}_1$ onto $\mathcal{A}_2$ is completely bounded provided that $\mathcal{A}_2$ contains a norming C^*-subalgebra. We use this result to give some insights into Kadison's Similarity Problem: we show that every faithful bounded homomorphism of a C^*-algebra on a Hilbert space has completely bounded inverse, and show that a bounded representation of a C^*-algebra is similar to a $*$-representation precisely when the image operator algebra λ-norms itself. We give two applications to isometric isomorphisms of certain operator algebras. The first is an extension of a result of Davidson and Power on isometric isomorphisms of CSL algebras. Secondly, we show that an isometric isomorphism between subalgebras $\mathcal{A}_i$ of C^*-diagonals $(\mathcal{C}_i, \mathcal{D}_i)$ $(i = 1, 2)$ satisfying $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$ extends uniquely to a $*$-isomorphism of the C^*-algebras generated by $\mathcal{A}_1$ and $\mathcal{A}_2$; this generalizes results of Muhly-Qiu-Solel and Donsig-Pitts.

1. Introduction and Norming Algebras

Let $\mathcal{A}$ be a unital operator algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a homomorphism. If u is contractive (resp. isometric or bounded), it is not generally the case that u is completely contractive (resp. completely isometric or completely bounded). However, in some cases it is possible to conclude that if u is isometric or contractive, then u is completely isometric or completely contractive. For example, the contractive homomorphisms of a C^*-algebra are exactly the $*$ homomorphisms, and hence any contractive homomorphism of a C^*-algebra into $\mathcal{B}(\mathcal{H})$ is completely contractive. It is not known however if every bounded representation of a C^*-algebra is completely bounded; a result of Haagerup (stated as Theorem 2.1 below) shows this question is equivalent to Kadison's similarity problem.

The purpose of this note is to give a sufficient condition on an operator algebra $\mathcal{B}$ which ensures that every bounded epimorphism $u : \mathcal{A} \to \mathcal{B}$ of the operator algebra $\mathcal{A}$ onto $\mathcal{B}$ is completely bounded. We show that this condition can be used to give simple proofs of several results in the literature, and also that every bounded faithful representation of a C^*-algebra is bounded below.

Throughout, all operator algebras are norm-closed, and we typically use $\mathcal{A}$ and $\mathcal{B}$ to denote operator algebras. (There are several texts containing the background we need for operator spaces and operator algebras, see for example [1, 4, 12].) An *operator $\mathcal{A}$-$\mathcal{B}$-bimodule* is an operator space $\mathcal{M}$ which is a left-$\mathcal{A}$, right-$\mathcal{B}$ bimodule where the bimodule action is completely contractive in the sense that for any $A \in M_{np}(\mathcal{A})$, $M \in M_p(\mathcal{M})$ and $B \in M_{pn}(\mathcal{B})$,

$$\|AMB\|_{M_n(\mathcal{M})} \leq \|A\|_{M_{np}(\mathcal{A})} \|M\|_{M_p(\mathcal{M})} \|B\|_{M_{pn}(\mathcal{B})}.$$

(We shall generally not write subscripts on the norms in the sequel, unless necessary for clarity.) We will sometimes write ${}_A\mathcal{M}_\mathcal{B}$ when $\mathcal{M}$ is an A-$\mathcal{B}$-bimodule.

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For convenience, we will always assume that operator algebras are unital unless explicitly stated otherwise. However, many of the results in the sequel are valid for non-unital algebras.

Given such an $\mathcal{A}$-$\mathcal{B}$ operator bimodule $\mathcal{M}$, we may define a family of norms η_n on $M_n(\mathcal{M})$ as follows: for $X \in M_n(\mathcal{M})$,

$$\eta_n(X) := \sup\{\|RXC\| : R \in M_{1n}(\mathcal{A}), C \in M_{n1}(\mathcal{B}) \text{ and } \max\{\|C\|, \|R\|\} \leq 1\}.$$

Clearly $\eta_n(X) \leq \|X\|_{\mathcal{M}_n(\mathcal{M})}$.

Definition 1.1.([18]) Let $\lambda > 0$ be a real number. We say that ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is λ-*normed* by $\mathcal{A}$ and $\mathcal{B}$ if for every $n \in \mathbb{N}$ and $X \in M_n(\mathcal{M})$,

$$\lambda \|X\| \leq \eta_n(X).$$

When $\lambda = 1$, we say that ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is *normed* by $\mathcal{A}$-$\mathcal{B}$. When $\mathcal{A} = \mathcal{B}$, we simply say that $\mathcal{M}$ is normed by $\mathcal{A}$.

When $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, C. Pop [17] shows that $(\mathcal{M}, \{\eta_n\})$ is the smallest operator space structure on the Banach space $\mathcal{M}$ which is compatible with the module structure of $\mathcal{M}$.

We will make essential use of the following result.

Theorem 1.2 ([16, Lemma 7.7, p. 128]). *Let $\mathcal{A}$ be a C^*-algebra and suppose that $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded homomorphism. Then for any $R \in M_{1,n}(\mathcal{A})$ and $C \in M_{n,1}(\mathcal{A})$ we have*

$$\|u_{1,n}(R)(u_{1,n}(R))^*\| \leq \|u\|^4 \|RR^*\| \qquad \textit{and}$$
$$\|(u_{n,1}(C))^* u_{n,1}(C)\| \leq \|u\|^4 \|C^*C\|.$$

Remark 1.3. Actually, [16, Lemma 7.7, p. 128] gives the statement and proof for the column C. However to obtain the statement for R, one applies the statement for C to the opposite algebra $\mathcal{A}^{\mathrm{op}}$ and the homomorphism ψ of $\mathcal{A}^{\mathrm{op}}$ on the conjugate Hilbert space $\overline{\mathcal{H}}$ given by $d \in \mathcal{A} \mapsto \psi(d)$, where $\psi(d)\overline{h} = \overline{(u(d))^* h}$.

The following is our central observation. The key idea is to combine the techniques of [18, Theorem 2.10] with Theorem 1.2. (See also [20, Theorem 2.1].)

Theorem 1.4. *Let $\mathcal{B}$ be a norm-closed operator algebra which contains a C^*-algebra $\mathcal{D}$, and assume $\mathcal{D}$ norms $\mathcal{B}$. If $\mathcal{A}$ is a norm-closed operator algebra and $u : \mathcal{A} \to \mathcal{B}$ is a bounded isomorphism, then u is completely bounded and*

$$\|u\|_{cb} \leq \|u\| \left\|u^{-1}\right\|^4.$$

Remarks. If $\mathcal{D}$ only λ-norms $\mathcal{B}$, then u is completely bounded with $\|u\|_{cb} \leq \lambda^{-1} \|u\| \left\|u^{-1}\right\|^4$. Also, clearly the theorem applies when u is a bounded epimorphism: simply replace $\mathcal{A}$ with $\mathcal{A}/\ker u$ and u with the induced isomorphism of the quotient onto $\mathcal{B}$.

Proof. Let $T = (T_{ij}) \in M_n(\mathcal{A})$. Then for any $R \in M_{1n}(\mathcal{D})$ and $C \in M_{n1}(\mathcal{D})$ with $\|R\| \le 1$ and $\|C\| \le 1$ we have,

$$\begin{aligned}
\|Ru_n(T)C\| &= \left\| \sum_{i,j=1}^{n} R_i u(T_{ij}) C_j \right\| \\
&\le \|u\| \left\| \sum_{i,j=1}^{n} u^{-1}(R_i) T_{ij} u^{-1}(C_j) \right\| \\
&= \|u\| \left\| u_{1,n}^{-1}(R) T u_{n,1}^{-1}(C) \right\| \\
&\le \|u\| \left\| u_{1,n}^{-1}(R) \right\| \|T\| \left\| u_{n,1}^{-1}(C) \right\| .
\end{aligned}$$

We may assume that $\mathcal{A}$ is represented completely isometrically as operators acting on a Hilbert space $\mathcal{H}$. Thus, $u^{-1}|_{\mathcal{D}}$ is a bounded homomorphism of $\mathcal{D}$ into $\mathcal{B}(\mathcal{H})$. By Lemma 1.2, $\left\| u_{n,1}^{-1}(C) \right\| \le \left\| u^{-1} \right\|^2$ and $\left\| u_{1,n}^{-1}(R) \right\| \le \left\| u^{-1} \right\|^2$. Taking suprema over R and C gives

$$\|u_n(T)\| \le \|u\| \left\| u^{-1} \right\|^4 \|T\| ,$$

as desired. □

When the isomorphism is isometric, more can be said. For any operator algebra $\mathcal{B}$, let $C^*_{\text{env}}(B)$ be the C^*-envelope of $\mathcal{B}$.

Corollary 1.5. *For $i = 1, 2$, suppose that $\mathcal{A}_i$ are operator algebras and $\mathcal{D}_i \subseteq \mathcal{C}_i$ is a norming C^*-subalgebra of $\mathcal{A}_i$. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism $\tilde{u} : C^*_{\text{env}}(\mathcal{A}_1) \to C^*_{\text{env}}(\mathcal{A}_2)$.*

Proof. Theorem 1.4 shows that u and u^{-1} are complete contractions, so that u is a complete isometry. The result follows from the universal property of C^*-envelopes. □

2. Applications

In this section we record some consequences of Theorem 1.4. We shall require the following closely related results of Haagerup and Paulsen.

Theorem 2.1 (Haagerup [6])**.** *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely bounded homomorphism. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| \|S^{-1}\| = \|u\|_{\text{cb}}$ such that for every $a \in \mathcal{A}$,*

$$a \mapsto Su(a)S^{-1}$$

is completely contractive (and hence a $*$-representation).

Theorem 2.2 (Paulsen [13])**.** *Suppose $\mathcal{A}$ is a unital operator algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a completely bounded unital homomorphism. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| \|S^{-1}\| = \|u\|_{\text{cb}}$ such that for every $a \in \mathcal{A}$,*

$$a \mapsto Su(a)S^{-1}$$

is a completely contractive homomorphism.

2.1. **Applications to C^*-algebras and Kadison's Similarity Problem.** We begin with a new proof of a result of Gardner.

Theorem 2.3 (Gardner [5])**.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $u : \mathcal{A} \to \mathcal{B}$ is an isomorphism. Then u is (completely) bounded and there exists a $*$-isomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ and a bounded automorphism β of $\mathcal{B}$ such that $u = \beta \circ \alpha$. If $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, there exists a positive invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| \|S^{-1}\| \leq \|u\|$ so that $\beta = \operatorname{Ad} S$.*

Proof. By [18, Lemma 2.3(i)], $\mathcal{B}$ norms itself, and as observed by Gardner, u is bounded. (One can also use a result of B. Johnson [7] (see also [19]) concerning automatic continuity of a homomorphism from a Banach algebra onto a semi-simple Banach algebra.) Theorem 1.4 implies that u is completely bounded. Assume that $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$. By Theorem 2.1, there exists an invertible operator T with $\|T\| \|T^{-1}\| = \|u\|_{\text{cb}}$ such that $\operatorname{Ad} T \circ u$ is a completely contractive homomorphism, so that $\operatorname{Ad} T \circ u$ is therefore a $*$-homomorphism. Let $S = |T|$, and let U be the polar part of T, so $T = US$. Then $(\operatorname{Ad} S)(\mathcal{B})$ is a C^*-algebra, and it follows that $S^2 \mathcal{B} S^{-2} = \mathcal{B}$. By Gardner's Invariance Theorem [5, Theorem 3.5], $\beta := \operatorname{Ad} S^{-1}$ is an automorphism of $\mathcal{B}$. Now let $\alpha = \operatorname{Ad}(U^*T) \circ u$. Since $\operatorname{Ad} T \circ u$ is a $*$-isomorphism of $\mathcal{A}$ onto its range, α is a $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Then $u = \beta \circ \alpha$. ☐

Remark 2.4. Gardner's original arguments give somewhat more. In particular, he shows that if $\mathcal{A}$ and $\mathcal{B}$ are faithfully represented using the universal atomic representations, then α can be taken to have the form $\operatorname{Ad} U$ for some unitary U.

The following is an immediate corollary of Theorem 1.4.

Theorem 2.5. *Suppose $\mathcal{A}$ is an operator algebra and $\mathcal{B}$ is a C^*-algebra. If $u : \mathcal{A} \to \mathcal{B}$ is an isomorphism, then u is automatically completely bounded and $\|u\|_{cb} \leq \|u\| \|u^{-1}\|^4$.*

Proof. Continuity of u follows from Johnson's theorem [7], then complete boundedness follows from Theorem 1.4 together with the fact that a C^*-algebra norms itself [18, Lemma 2.3(i)]. ☐

We now wish to make some observations regarding Kadison's Similarity Problem. Recall that this problem asks whether every bounded representation of a C^*-algebra is similar to a $*$-representation, which as noted above, is equivalent to the question of whether bounded representations of C^*-algebras are automatically completely bounded. Theorem 2.5 can be used to prove that bounded representations of C^*-algebras are (modulo the kernel) "completely bounded below."

Theorem 2.6. *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded homomorphism. Put $\mathcal{J} = \ker u$. Then there exists a real number $k > 0$ such that for every $n \in \mathbb{N}$ and every $T \in M_n(\mathcal{A})$,*

$$k \operatorname{dist}(T, M_n(\mathcal{J})) \leq \|u_n(T)\| .$$

Proof. Since $\mathcal{A}/\mathcal{J}$ is a C^*-algebra isomorphic under the map induced by u to $u(\mathcal{A})$, without loss of generality we may assume that u is one-to-one. Thus our task is to prove that $u := u^{-1}$ is completely bounded. This will follow from Theorem 2.5 once we prove that the image $\mathcal{B} := u(\mathcal{A})$ is closed, and hence an operator algebra. We may also assume that $\mathcal{A}$ is unital and that $u(I) = I$.

Let x be a non-zero element of $\mathcal{A}$, and let $\mathcal{C}$ be the unital C^*-subalgebra of $\mathcal{A}$ generated by x^*x. Since $\mathcal{C}$ is abelian, the restriction of u to $\mathcal{C}$ is completely bounded on $\mathcal{C}$, so by the Dixmier-Day Theorem on amenable groups, there exists an invertible operator S with $\|S\| \|S^{-1}\| \leq \|u|_{\mathcal{C}}\|^2$ such that $(\operatorname{Ad} S) \circ u|_{\mathcal{C}}$ is a $*$-homomorphism. Since u is one-to-one, we have

$$\|x\|^2 = \|x^*x\| = \|Su(x^*x)S^{-1}\| \leq \|u\|^2 \|u(x^*)u(x)\| \leq \|u\|^3 \|x\| \|u(x)\| ,$$

so that $\|x\| \leq \|u\|^3 \|u(x)\|$. Thus, u is bounded below, so that the range of u is closed. ☐

Combining Theorems 2.2 and 2.6 with the structure of completely contractive representations we obtain the following corollary.

Corollary 2.7. *Suppose $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H}_u)$ is a faithful bounded representation. Then there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{K})$, an isometry $W : \mathcal{H} \to \mathcal{K}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| \|S^{-1}\| \le \|u^{-1}\| \|u\|^4$ such that for every $x \in \mathcal{A}$,*

$$x = SW^*\pi(u(x))WS^{-1}. \tag{1}$$

Proof. Let $\mathcal{B} = u(\mathcal{A})$. We may assume that $\mathcal{A}$ is unital and $u(I) = I$. Theorem 2.6 shows that u^{-1} is a completely bounded map from $\mathcal{B}$ to $\mathcal{B}(\mathcal{H})$. Therefore, by Theorem 2.2, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ with $\|S\| \|S^{-1}\| = \|u^{-1}\|_{\text{cb}}$ so that $\psi := \text{Ad}(S^{-1}) \circ u^{-1}$ is completely contractive. By Arveson's Structure Theorem for completely contractive representations of operator algebras, there exists a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{B}(\mathcal{K})$ and an isometry W so that for all $b \in \mathcal{B}$, $\psi(b) = W^*\pi(b)W$. Letting $b = u(x)$ (for $x \in \mathcal{A}$) yields (1). The estimate for the condition number of S follows from Theorem 1.4. ☐

Remark 2.8. Unfortunately, we have been unable to solve for $u(x)$ in (1); doing so would of course lead to a solution of Kadison's problem.

The range of the isometry W appearing in Corollary 1 is a semi-invariant subspace for $\pi(\mathcal{B})$. Thus, $WW^* = PQ^\perp$ for some projections $P, Q \in \text{Lat}\,(\pi(\mathcal{B}))$, with $Q \le P$. The map $x \mapsto WS^{-1}xSW^*$ is a homomorphism into the "$2,2$-diagonal piece" of $\pi(u(x))$ relative to the block decomposition of $\pi(u(x))$ according to $I = Q + PQ^\perp + Q^\perp P^\perp$.

We now show that Kadison problem is equivalent to the issue of whether the image of a C^*-algebra under a bounded homomorphism λ norms itself.

Theorem 2.9. *Suppose $\mathcal{A}$ is a C^*-algebra and $u : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a bounded representation with image $\mathcal{B} := u(\mathcal{A})$, and let $\tilde{u} : \mathcal{A}/\ker u \to \mathcal{B}$ be the induced map. If $\mathcal{B}$ λ-norms itself for some $\lambda > 0$, then u is completely bounded and $\|u\|_{cb} \le \lambda^{-1} \|u\|^9 \|\tilde{u}^{-1}\|^2$. Conversely, if u is completely bounded, then $\mathcal{B}$ λ-norms itself for any λ with $0 < \lambda \le \dfrac{1}{\|u\|_{cb} \|\tilde{u}^{-1}\| \|u\|^4}$.*

Proof. We may assume that u is a monomorphism, so Theorem 2.5 gives $\|u^{-1}\|_{cb} \le \|u\|^4 \|u^{-1}\|$. Suppose that $\mathcal{B}$ is λ-normed by itself for some $\lambda > 0$, and let $T \in M_n(\mathcal{A})$. The calculation in the proof of Theorem 1.4 shows that if $R \in M_{1,n}(\mathcal{B})$ and $C \in M_{n,1}(\mathcal{B})$ satisfy $\|R\| \le 1$ and $\|C\| \le 1$, then

$$\|Ru_n(T)C\| \le \|u\| \left\|u_{1,n}^{-1}(R)\right\| \|T\| \left\|u_{n,1}^{-1}(C)\right\| \le \|u\|^9 \left\|u^{-1}\right\|^2 \|T\|.$$

Taking suprema over all such R and C gives

$$\lambda \|u_n(T)\| \le \|u\|^9 \left\|u^{-1}\right\|^2 \|T\|,$$

so u is completely bounded with $\|u\|_{cb} \le \lambda^{-1} \|u\|^9 \left\|u^{-1}\right\|^2$.

Conversely, suppose that u is completely bounded, and view $\mathcal{B}$ as a bimodule over itself. For any $R \in \mathcal{M}_{1,n}(\mathcal{A})$, $C \in \mathcal{M}_{n,1}(\mathcal{A}))$ with $\|R\|, \|C\| \le 1$, we have,

$$\begin{aligned}
\left\|Ru_n^{-1}(T)C\right\| &\le \left\|u^{-1}\right\| \left\|u(Ru_n^{-1}(T)C)\right\| \\
&= \left\|u^{-1}\right\| \|u_{1,n}(R)\| \|u_{n,1}(C)\| \left\| \frac{u_{1,n}(R)}{\|u_{1,n}(R)\|} T \frac{u_{n,1}(C)}{\|u_{n,1}(C)\|} \right\| \\
&\le \left\|u^{-1}\right\| \|u\|^4 \eta_n(T) \quad \text{(using Lemma 1.2).}
\end{aligned}$$

Taking suprema yields $\left\|u_n^{-1}(T)\right\| \leq \left\|u^{-1}\right\| \|u\|^4 \eta_n(T)$, and hence

$$\|T\| \leq \|u_n\| \left\|u_n^{-1}(T)\right\| \leq \|u\|_{cb} \left\|u^{-1}\right\| \|u\|^4 \eta_n(T).$$

Thus $\mathcal{B}$ λ-norms itself for any $\lambda \leq \dfrac{1}{\|u\|_{cb} \|u^{-1}\| \|u\|^4}$. □

Corollary 2.10. *Suppose $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is an operator algebra which is isomorphic to a C^*-algebra $\mathcal{A}$. Then there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S\mathcal{B}S^{-1}$ is a C^*-algebra if and only if $\mathcal{B}$ λ-norms itself for some $\lambda > 0$.*

Proof. Let $u : \mathcal{A} \to \mathcal{B}$ be the isomorphism. Then Theorem 2.9 shows that if $\mathcal{B}$ λ-norms itself, then u is completely bounded, hence by Theorem 2.1, u is similar to a $*$-representation, and so $\mathcal{B}$ is similar to a C^*-algebra. Conversely, if $\mathcal{B}$ is similar to a C^*-algebra, then u is similar to a $*$-representation and an application of Theorem 2.9 completes the proof. □

The following question is thus a reformulation of Kadison's question.

Question 2.11. Suppose $\mathcal{B}$ is an operator algebra which is isomorphic to a C^*-algebra. Does $\mathcal{B}$ λ-norm itself for some $\lambda > 0$?

Remark 2.12. Haagerup [6] showed that every bounded, cyclic representation u of a C^*-algebra $\mathcal{A}$ is completely bounded with $\|u\|_{cb} \leq \|u\|^3$. So given an arbitrary representation u of $\mathcal{A}$ on $\mathcal{H}$, let $\mathcal{B} = u(\mathcal{A})$. For each unit vector $\xi \in \mathcal{H}$, let P_ξ be the projection onto the cyclic subspace $[\mathcal{B}\xi]$ and let $\mathcal{B}_\xi$ be the restriction of $\mathcal{B}$ to $\mathcal{H}_\xi := P_\xi\mathcal{H}$. The representation u_ξ of $\mathcal{A}$ given by $u_\xi(x) = u(x)P_\xi$ is thus completely bounded and $\|u_\xi\|_{cb} \leq \|u\|_{cb}$. Theorem 2.2 shows that there exists an invertible operator $S_\xi \in \mathcal{B}(\mathcal{H}_\xi)$ such that $\|u_\xi\|_{cb} = \left\|S_\xi\right\| \left\|S_\xi{}^{-1}\right\|$ and $(\operatorname{Ad} S_\xi)(\mathcal{B}_\xi)$ is a C^*-algebra, call it $\mathcal{A}_\xi$. Applying Theorem 2.9 to $\operatorname{Ad} S_\xi^{-1} : \mathcal{A}_\xi \to \mathcal{B}_\xi$, we find $\|\operatorname{Ad} S_\xi\| \leq \left\|S_\xi\right\| \left\|S_\xi{}^{-1}\right\|$, so that $\mathcal{B}_\xi$ λ-norms itself with $\lambda = \|u\|^{-18}$.

In the following example, we show that there exists an operator algebra which does not λ-norm itself. The idea for the proof is due to Ken Davidson.

Example 2.13. Let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disk and let $\mathbb{A}(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$ be the disk algebra, that is, the collection of all continuous functions on the closed unit disk which are analytic in $\mathbb{D}$. We use $\mathcal{P} \subseteq \mathbb{A}(\mathbb{D})$ to denote the collection of all polynomials.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is polynomially bounded if there exists $K > 0$ such that for every $p \in \mathcal{P}$, $\|p(T)\| \leq K\|p\|$. Polynomial boundedness of T is equivalent to the existence of a bounded homomorphism $u : \mathbb{A}(\mathbb{D}) \to \mathcal{B}(\mathcal{H})$ such that for every $p \in \mathcal{P}$, $u(p) = p(T)$. A polynomially bounded operator T is *completely polynomially bounded* if u is completely bounded. Paulsen [14] showed that T is completely polynomially bounded if and only if T is similar to a contraction. Pisier [15] showed that there exists a polynomially bounded operator $T \in \mathcal{B}(\mathcal{H})$ which is not completely polynomially bounded, so T is not similar to a contraction.

Fix a polynomially bounded operator $T \in \mathcal{B}(\mathcal{H})$. Notice that the spectrum of T is contained in $\overline{\mathbb{D}}$. If U is a unitary operator with $\sigma(U) = \mathbb{T}$, then T is completely polynomially bounded if and only if $T \oplus U$ is completely polynomially bounded. We may therefore assume that

$$\mathbb{T} \subseteq \sigma(T).$$

Let $u : \mathbb{A}(\mathbb{D}) \to \mathcal{B}(\mathcal{H})$ be the extension of the map $p \in \mathcal{P} \mapsto p(T)$ and set $\mathcal{B} = u(\mathbb{A}(\mathbb{D}))$. Since $\mathbb{T} \subseteq \sigma(T)$, we have $\|u(f)\| \geq \|f\|$ for every $f \in \mathbb{A}(\mathbb{D})$, so that $\mathcal{B}$ is closed and u^{-1} is contractive. Since $\mathbb{A}(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$, the operator space structure on $\mathbb{A}(\mathbb{D})$ is minimal among all all possible operator space structures on $\mathbb{A}(\mathbb{D})$ (see [1, Paragraph 1.2.21]), hence u^{-1} is completely contractive.

View $\mathcal{B}$ as a $\mathcal{B}$-bimodule and let η_n be the norm on $M_n(\mathcal{B})$ as in Definition 1.1. We shall show that for every n, the norm $\eta_n(u_n(\cdot))$ and the usual norm $\|\cdot\|_{M_n(\mathbb{A}(\mathbb{D}))}$ are equivalent norms on $M_n(\mathbb{A}(\mathbb{D}))$.

Choose $X \in M_n(\mathbb{A}(\mathbb{D}))$, $R \in M_{1n}(\mathbb{A}(\mathbb{D}))$ and $C \in M_{n1}(\mathbb{A}(\mathbb{D}))$. Since u is bounded, we have $\|u_{1n}(R)u_n(X)u_{n1}(C)\| = \|u(RXC)\| \leq \|u\| \, \|RXC\|$. Since $\|u_{1n}(R)\| \geq \|R\|$ and $\|u_{n1}(C)\| \geq \|C\|$, we obtain

$$\eta_n(u_n(X)) \leq \|u\| \, \|X\| \,. \tag{2}$$

On the other hand, a result of Bourgain [2] (see also [16, Theorem 9.9]) shows that there exists a constant $s > 0$ (independent of u) such that for every n, $\max\{\|u_{1n}\|, \|u_{n1}\|\} \leq s \|u\|$. Therefore,

$$\begin{aligned} \|RXC\| &\leq \|u_{1n}(R)u_n(X)u_{n1}(C)\| \\ &\leq s^2 \|u\|^2 \left\| \frac{u_{1n}(R)}{\|u_{1n}(R)\|} u_n(X) \frac{u_{n1}(C)}{\|u_{n1}(C)\|} \right\| \|R\| \, \|C\| \,. \end{aligned}$$

Since the operator space structure on $\mathbb{A}(\mathbb{D})$ is minimal, $\mathbb{A}(\mathbb{D})$ norms itself. Taking the supremum over R and C with norm one, we obtain

$$\|X\| \leq s^2 \|u\|^2 \eta_n(u_n(X)). \tag{3}$$

Combining (2) and (3) establishes the claim.

Thus, when T is chosen to be polynomially bounded but not completely polynomially bounded, we see that $\mathcal{B}$ cannot λ-norm itself.

2.2. **An Application to CSL Algebras.** Recall that a *CSL algebra* is an operator algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ which is both reflexive and such that there exists a MASA $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ with $\mathcal{D} \subseteq \mathcal{A}$. The lattice of invariant projections of a CSL algebra is a commutative family of projections, and when this lattice is completely distributive, the CSL algebra is called a *completely distributive* CSL algebra.

Davidson and Power proved that isometric isomorphisms of completely distributive CSL algebras are unitarily implemented. Their techniques involved homological ideas and were somewhat intricate. We can give a simpler proof of their result, and which also extends theirs. For convenience, we assume irreducibilty. If $\mathcal{A}$ is not irreducible, one can use direct integrals along the center of $\mathcal{A} \cap \mathcal{A}^*$, together with an appropriate hypothesis on the lattices of each term in the direct integral to obtain a more general result.

Theorem 2.14. *For $i = 1, 2$, let $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{H}_i)$ be CSL algebras such that $\mathcal{A}_i \cap \mathcal{K}_i \neq (0)$ and suppose that $\mathcal{A}_1$ is irreducible. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $u = \operatorname{Ad} U$.*

Proof. Since the isomorphism u is isometric, its restriction to $\mathcal{A}_1 \cap (\mathcal{A}_1)^*$ is a $*$-isomorphism, so in particular, $\operatorname{Lat}(\mathcal{A}_2)$ is isomorphic to $\operatorname{Lat}(\mathcal{A}_1)$. Thus, since $\mathcal{A}_1$ is irreducible, so is $\mathcal{A}_2$. Let $\mathcal{C}_i$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_i)$ generated by $\mathcal{A}_i$. Then $\mathcal{C}_i$ is an irreducible C^*-algebra containing a compact operator, hence $\mathcal{C}_i$ contains all the compact operators. There exists a $*$-epimomorphism $\pi : \mathcal{C}_i \to C^*_{\text{env}}(\mathcal{A}_i)$ such that $\pi|_{\mathcal{A}_i} = \iota$, where ι is the canonical inclusion of $\mathcal{A}_i$ into $C^*_{\text{env}}(\mathcal{A}_i)$. If $\ker \pi \neq (0)$, then since $\mathcal{C}_i$ contains the compact operators, $\ker \pi \cap \mathcal{K}_i \neq (0)$. Hence $\mathcal{K}_i \subseteq \ker \pi$, since $\ker \pi \cap \mathcal{K}_i$ is an ideal in an irreducible C^*-algebra. But this is impossible, since π is isometric on $\mathcal{A}_i \cap \mathcal{K}_i$. Therefore $\ker \pi = (0)$, so that $\mathcal{C}_i$ is the C^*-envelope of $\mathcal{A}_i$.

Theorem 2.7 of [18] shows that any MASA is norming for $\mathcal{B}(\mathcal{H})$, hence $\mathcal{A}_i$ contain norming C^*-subalgebras. Theorem 1.4 shows that u and u^{-1} are completely contractive, so that u is a complete isometry. By the universal property of C^*-envelopes (applied to u and u^{-1}), u extends to a $*$-isomorphism $\tilde{u}$ of $\mathcal{C}_1$ onto $\mathcal{C}_2$. The compact operators are the smallest closed two-sided ideal contained in $\mathcal{C}_i$, so that the restriction of $\tilde{u}$ to the compact operators is a $*$-isomorphism of $\mathcal{K}(\mathcal{H}_1)$ onto $\mathcal{K}(\mathcal{H}_2)$. Therefore, there exists a unitary operator U so that $(\operatorname{Ad} U)|_{\mathcal{K}(\mathcal{H}_1)} = \tilde{u}|_{\mathcal{K}(\mathcal{H}_1)}$. Finally, if $T \in \mathcal{C}_1$ and if $\eta \in \mathcal{H}_2$, we may find a finite rank projection P so that $2\|T\| \|P^\perp \eta\| < \varepsilon$. Then since $\tilde{u}^{-1}(P) = (\operatorname{Ad} U^*)(P)$, we have

$$\begin{aligned}\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,\eta\| &\leq \left\|\left(\tilde{u}(T\tilde{u}^{-1}(P)) - (\operatorname{Ad} U)(T(\operatorname{Ad} U^*(P)))\right)\eta\right\| \\ &\quad + \left\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,P^\perp \eta\right\| \\ &= \left\|(\tilde{u}(T) - (\operatorname{Ad} U)(T))\,P^\perp \eta\right\| < \varepsilon,\end{aligned}$$

so $\tilde{u}(T) = (\operatorname{Ad} U)(T)$. Since $\mathcal{A}_1 \subseteq \mathcal{C}_1$, the proof is complete. ∎

2.3. **Applications to Subalgebras of C^*-Diagonals.** In this subsection, we provide applications to subalgebras of certain classes of C^*-algebras.

A C^*-diagonal is a pair $(\mathcal{C}, \mathcal{D})$ of C^*-algebras such that $\mathcal{D}$ is abelian and such that

i) every pure state of $\mathcal{D}$ extends uniquely to a pure state of $\mathcal{C}$;
ii) the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ (whose existence is guaranteed by (i)) is faithful;
iii) the closed linear span of the set $\{v \in \mathcal{C} : v\mathcal{D} = \mathcal{D}v\}$ is $\mathcal{C}$.

We will assume that both $\mathcal{C}$ and $\mathcal{D}$ are unital. The extension property then implies that $\mathcal{D}$ is a MASA in $\mathcal{C}$.

Such pairs were introduced by Kumjian [8], who used slightly different, but essentially equivalent axioms (see [3] for a discussion of the equivalence). Also, C^*-diagonals and their subalgebras were further in several papers, see for example [3, 10, 11].

Our first task is to show that $\mathcal{D}$ norms $\mathcal{C}$.

Lemma 2.15. *Suppose $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal. Then $\mathcal{C}$ is normed by $\mathcal{D}$.*

Proof. Theorem 5.9 of [3] shows that there exists a faithful $*$-representation $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ such that $\pi(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H})$. It follows from [18, Lemma 2.2 and Theorem 2.7] that $\pi(\mathcal{D})$ norms $\mathcal{B}(\mathcal{H})$, hence $\pi(\mathcal{D})$ norms $\pi(\mathcal{C})$. As π is a faithful $*$-representation of a C^*-algebra, it is a complete isometry, so $\mathcal{D}$ norms $\mathcal{C}$. ∎

The following notation will be useful. When $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal and $\mathcal{A}$ is a norm closed algebra with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$, we will write $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$.

For $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be C^*-diagonals. Muhly, Qiu and Solel [11, Theorem 1.1] proved that when $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ are triangular, that is $\mathcal{A}_i \cap (\mathcal{A}_i)^* = \mathcal{D}_i$, which generate $\mathcal{C}_i$ and $(\mathcal{C}_i, \mathcal{D}_i)$ are nuclear, then an isometric isomorphism $u : \mathcal{A}_1 \to \mathcal{A}_2$ extends to a $*$-isomorphism of $\mathcal{C}_1$ onto $\mathcal{C}_2$. Later Donsig and Pitts [3, Theorem 8.9] extended this result: they showed that the hypothesis of nuclearity can be removed. To prove their result, Donsig and Pitts used showed that the isometric isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ induces isomorphism of an appropriate CSL algebras, then used the structure theory for isomorphisms of CSL algebras. The techniques used to prove [11, Theorem 1.1] and [3, Theorem 8.9] do not apply for non-triangular subalgebras. A

Donsig and Pitts [3, Theorem 4.22] showed that the C^*-envelope of any subalgebra (triangular or not) $\mathcal{A} \subseteq (\mathcal{C}, \mathcal{D})$ is the C^*-subalgebra, $C^*(\mathcal{A})$, of $\mathcal{C}$ generated by $\mathcal{A}$. In the context of both [11, Theorem 1.1] and [3, Theorem 8.9], $(C^*(\mathcal{A}_i), \mathcal{D}_i)$ are C^*-diagonals. When $\mathcal{C}$ is separable and nuclear, the Spectral Theorem for Bimodules [10] shows that $(C^*(\mathcal{A}), \mathcal{D})$ is a C^*-diagonal. In the general case however, it is not clear that the pair $(C^*(\mathcal{A}), \mathcal{D})$ is a C^*-diagonal—one needs to verify that condition (ii) of the definition of C^*-diagonal holds. The following consequence of [3, Theorem 4.22], Corollary 1.5 and Lemma 2.15 is therefore a significant extension of [11, Theorem 1.1] and [3, Theorem 8.9].

Theorem 2.16. *Let $\mathcal{A}_i \subseteq (\mathcal{C}_i, \mathcal{D}_i)$ be norm-closed subalgebras of C^*-diagonals. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism of $C^*(\mathcal{A}_1)$ onto $C^*(\mathcal{A}_2)$.*

Remark 2.17. In [9], Mercer proves a result similar to Theorem 2.16, but where the algebras $\mathcal{A}_i$ are taken to be weak-$*$ closed subalgebras of von Neumann algebras $\mathfrak{M}_i$ and there are Cartan MASAs $\mathcal{D}_i \subseteq \mathfrak{M}_i$ such that $\mathcal{D}_i \subseteq \mathcal{A}_i \subset \mathcal{M}_i$. We expect that Cartan MASAs norm their containing von Neumann algebras, and thus expect that it should be possible to give a proof of Mercer's result based on Theorem 1.4 as well.

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