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STRUCTURE FOR REGULAR INCLUSIONS

DAVID R. PITTS

Dedicated to the memory of William B. Arveson

Abstract. We study pairs $(\mathcal{C}, \mathcal{D})$ of unital C^*-algebras where $\mathcal{D}$ is an abelian C^*-subalgebra of $\mathcal{C}$ which is regular in $\mathcal{C}$ in the sense that the span of $\{v \in \mathcal{C} : v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}\}$ is dense in $\mathcal{C}$. When $\mathcal{D}$ is a MASA in $\mathcal{C}$, we prove the existence and uniqueness of a completely positive unital map E of $\mathcal{C}$ into the injective envelope $I(\mathcal{D})$ of $\mathcal{D}$ whose restriction to $\mathcal{D}$ is the identity on $\mathcal{D}$. We show that the left kernel of E, $\mathcal{L}(\mathcal{C}, \mathcal{D})$, is the unique closed two-sided ideal of $\mathcal{C}$ maximal with respect to having trivial intersection with $\mathcal{D}$. When $\mathcal{L}(\mathcal{C}, \mathcal{D}) = 0$, we show the MASA $\mathcal{D}$ norms $\mathcal{C}$ in the sense of Pop-Sinclair-Smith. We apply these results to significantly extend existing results in the literature on isometric isomorphisms of norm-closed subalgebras which lie between $\mathcal{D}$ and $\mathcal{C}$.

The map E can be used as a substitute for a conditional expectation in the construction of coordinates for $\mathcal{C}$ relative to $\mathcal{D}$. We show that coordinate constructions of Kumjian and Renault which relied upon the existence of a faithful conditional expectation may partially be extended to settings where no conditional expectation exists.

As an example, we consider the situation in which $\mathcal{C}$ is the reduced crossed product of a unital abelian C^*-algebra $\mathcal{D}$ by an arbitrary discrete group Γ acting as automorphisms of $\mathcal{D}$. We characterize when the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is abelian in terms of the dynamics of the action of Γ and show that when $\mathcal{D}^c$ is abelian, $\mathcal{L}(\mathcal{C}, \mathcal{D}^c) = (0)$. This setting produces examples where no conditional expectation of $\mathcal{C}$ onto $\mathcal{D}^c$ exists.

In general, pure states of $\mathcal{D}$ do not extend uniquely to states on $\mathcal{C}$. However, when $\mathcal{C}$ is separable, and $\mathcal{D}$ is a regular MASA in $\mathcal{C}$, we show the set of pure states on $\mathcal{D}$ with unique state extensions to $\mathcal{C}$ is dense in $\mathcal{D}$. We introduce a new class of well behaved state extensions, the compatible states; we identify compatible states when $\mathcal{D}$ is a MASA in $\mathcal{C}$ in terms of groups constructed from local dynamics near an element $\rho \in \hat{\mathcal{D}}$.

A particularly nice class of regular inclusions is the class of C^*-diagonals; each pair in this class has the extension property, and Kumjian has shown that coordinate systems for C^*-diagonals are particularly well behaved. We show that the pair $(\mathcal{C}, \mathcal{D})$ regularly embeds into a C^*-diagonal precisely when the intersection of the left kernels of the compatible states is trivial.

Contents

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1. Introduction, Background and Preliminaries

In this paper, we investigate the structure of the class of regular inclusions.

Definition 1.1. An *inclusion* is a pair $(\mathcal{C},\mathcal{D})$ of C^*-algebras with $\mathcal{D}$ abelian, and $\mathcal{D}\subseteq\mathcal{C}$. When $\mathcal{C}$ has a unit I, we always assume that $I\in\mathcal{D}$. For any inclusion, let

$$\mathcal{N}(\mathcal{C},\mathcal{D}) := \{v\in\mathcal{C} : v^*\mathcal{D}v\cup v\mathcal{D}v^*\subseteq\mathcal{D}\};$$

elements of $\mathcal{N}(\mathcal{C},\mathcal{D})$ are called *normalizers*. If $(\mathcal{C}_i,\mathcal{D}_i)$ $(i=1,2)$ are inclusions, and $\theta:\mathcal{C}_1\rightarrow\mathcal{C}_2$ is a $*$-homomorphism, we will say that θ is a *regular $*$-homomorphism* if

$$\theta(\mathcal{N}(\mathcal{C}_1,\mathcal{D}_1))\subseteq\mathcal{N}(\mathcal{C}_2,\mathcal{D}_2).$$

When θ is regular and one-to-one, we will say that $(\mathcal{C}_1,\mathcal{D}_1)$ *regularly embeds* into $(\mathcal{C}_2,\mathcal{D}_2)$.

The inclusion $(\mathcal{C},\mathcal{D})$ is a

- *regular inclusion* if $\operatorname{span}\mathcal{N}(\mathcal{C},\mathcal{D})$ is norm-dense in $\mathcal{C}$;
- *MASA inclusion* if $\mathcal{D}$ is a MASA in $\mathcal{C}$;
- *EP inclusion* if $\mathcal{D}$ has the extension property relative to $\mathcal{C}$, that is, every pure state σ on $\mathcal{D}$ has a unique extension to a state on $\mathcal{C}$ and no pure state of $\mathcal{C}$ annihilates $\mathcal{D}$;
- *Cartan inclusion* if $(\mathcal{C},\mathcal{D})$ is a regular MASA inclusion and there exists a faithful conditional expectation $E:\mathcal{C}\rightarrow\mathcal{D}$;
- *C^*-diagonal* if $(\mathcal{C},\mathcal{D})$ is a Cartan inclusion and also an EP-inclusion.

Examples of regular inclusions are commonplace in the theory of C^*-algebras: any MASA $\mathcal{D}$ in a finite dimensional C^*-algebra $\mathcal{C}$ yields a C^*-diagonal $(\mathcal{C},\mathcal{D})$; the categogy of C^*-diagonals and regular $*$-monomorphisms is closed under inductive limits [10, Theorem 4.23]; when a discrete group Γ acts topologically freely on a compact Hausdorff space X, the reduced crossed product $C(X)\rtimes_r\Gamma$ together with the canonical embedding of $C(X)$ yields a Cartan pair $(C(X)\rtimes_r\Gamma, C(X))$ [31]; when $\mathcal{C}$ is the C^*-algebra of a directed graph and $\mathcal{D}$ is the C^*-subalgebra generated by the range and source projections of the partial isometries corresponding to the edges of the graph, the pair $(\mathcal{C},\mathcal{D})$ is a regular inclusion. Other examples arise from certain constructions in the theory of groupoid C^*-algebras or from C^*-algebras constructed from combinatoral data.

In this paper, we present a number of structural results for regular inclusions. Our main results include: Theorem 3.10, which establishes the existence and uniqueness of a psuedo-expectation for a regular MASA inclusion; Theorem 3.22, which shows that the left kernel $\mathcal{L}(\mathcal{C},\mathcal{D}))$ of the pseudo-expectation is a two-sided ideal in $\mathcal{C}$ which is maximal with respect to being diagaonal disjoint; Theorem 5.9, which characterizes when a regular inclusion can be regularly embedded into a C^*-diagonal; Theorem 8.14 which shows how constructions of Kumjian and Renault may be used to produce a twist associated to a regular inclusion; Theorem 9.2, which shows that for a regular MASA inclusion with $\mathcal{L}(\mathcal{C},\mathcal{D})=(0)$, $\mathcal{D}$ norms $\mathcal{C}$ in the sense of Pop-Sinclair-Smith; and Theorem 9.4, which gives conditions on which an isometric isomorphism of a subalgebra $\mathcal{A}$ of $\mathcal{C}$ containing $\mathcal{D}$ can be extended to a $*$ isomorphism of the C^*-algebra generated by $\mathcal{A}$. We turn now to some background.

In a landmark paper, J. Feldman and C. Moore [13] considered pairs $(\mathcal{M}, \mathcal{D})$ consisting of a (separably acting) von Neumann algebra $\mathcal{M}$ containing a MASA $\mathcal{D} \simeq L^\infty(X, \mu)$ such that the set of normalizing unitaries, $\{u \in M : u \text{ is unitary and } u\mathcal{D}u^* = \mathcal{D}\}$, has σ-weakly dense span in $\mathcal{M}$ and there exists a faithful normal conditional expectation $E : \mathcal{M} \to \mathcal{D}$. In this context, Feldman and Moore showed that there is a Borel equivalence relation R on X and a cocycle c such that the pair $(M, \mathcal{D})$ can be identified with a von-Neumann algebra arising from certain Borel functions on R. In a heuristic sense, their construction may be viewed as the left regular representation of the equivalence relation where the multiplication is twisted by the cocycle c. This result may be viewed as a means of coordinatizing the von Neumann algebra along $\mathcal{D}$.

Work on a C^*-algebraic version of the Feldman-Moore result was studied by Kumjian in [22]. In that article, Kumjian introduced the notion of a C^*-diagonals as well as the notion of regularity. (We should mention that the axioms for a C^*-diagonal given in [22], while equivalent to the axioms given in Definition 1.1, do not explicitly mention the extension property. In the sequel, we will have considerable interest in the extension property or its failure, which is why we use the axioms given in Definition 1.1.) Kumjian showed that if $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal, with $\mathcal{C}$ separable and $\hat{\mathcal{D}}$ second countable, then it can be coordinatized via a twisted groupoid over a topological equivalence relation. This provided a very satisfying parallel to the von Neumann algebraic context.

The requirement of the extension property in the axioms for a C^*-diagonal is at times too stringent, which is one of the advantages to Cartan inclusions, which need not have the extension property. For example, let $\mathcal{H} = \ell^2(\mathbb{N})$, with the usual orthonormal basis $\{e_n\}$, and let S be the unilateral shift, $Se_n = e_{n+1}$. Let $\mathcal{C} := C^*(S)$ be the Toeplitz algebra, and let $\mathcal{D} = C^*(\{S^nS^{*n} : n \geq 0\})$. Routine arguments show this is a Cartan inclusion, but the state $\rho_\infty(T) = \lim_{n\to\infty} \langle Te_n, e_n\rangle$ on $\mathcal{D}$ fails to have a unique extension to a state on $\mathcal{C}$,

Cartan inclusions were introduced by Renault in [31], where he showed that if $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion (again with the separability and second countability hypotheses), then there is a satisfactory coordinatization of the pair $(\mathcal{C}, \mathcal{D})$ via a twisted groupoid. In this paper, Renault makes a very convincing case that Cartan inclusions are the appropriate analog of the Feldman-Moore setting in the C^*-context.

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion. A conditional expectation $E : \mathcal{C} \to \mathcal{D}$ gives a preferred class, $\{\rho \circ E : \rho \in \hat{\mathcal{D}}\}$, of extensions of pure states on $\mathcal{D}$ to states on $\mathcal{C}$, and when the expectation E is unique, this class may be used for construction of coordinates. Indeed, when $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion (or C^*-diagonal), elements of the twisted groupoid arise from ordered pairs $[v, \rho]$, where $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $\rho \in \hat{\mathcal{D}}$ with $\rho(v^*v) \neq 0$. Such a pair determines a linear functional of norm one on $\mathcal{C}$ by the rule,

$$[v, \rho](x) = \frac{\rho(E(v^*x))}{\rho(v^*v)^{1/2}} \qquad (x \in \mathcal{C}). \tag{1}$$

The work of Feldman-Moore also makes essential use of conditional expectations.

By [2, Theorem 3.4], any EP inclusion $(\mathcal{C}, \mathcal{D})$ has a unique conditional expectation $E : \mathcal{C} \to \mathcal{D}$. Thus for a C^*-diagonal, the extension property guarantees the the uniqueness of the expectation expectation $E : \mathcal{C} \to \mathcal{D}$. Interestingly, the extension property is not necessary to guarantee uniqueness of expectations. Indeed, Renault showed that when $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion, with expectation $E : \mathcal{C} \to \mathcal{D}$, then E is the unique conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$.

What other regular inclusions $(\mathcal{C}, \mathcal{D})$ have unique expectations of $\mathcal{C}$ onto $\mathcal{D}$? While we do not know the answer to this question in general, we give a positive result along these lines in Theorem 2.10 below: this result shows that when $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion with $\mathcal{D}$ an injective C^*-algebra, then $(\mathcal{C}, \mathcal{D})$ is an EP inclusion, and therefore has a unique expectation. Section 2 also contains results on dynamics of regular inclusions and introduces the notion of a *quasi-free action*. Theorem 2.9 characterizes the extension property for regular MASA inclusions in terms of the dynamics of regular inclusions: the regular MASA inclusion $(\mathcal{C},\mathcal{D})$ is an EP-inclusion if and only if the $*$-semigroup $\mathcal{N}(\mathcal{C},\mathcal{D})$ acts quasi-freely on $\hat{\mathcal{D}}$.

Unfortunately, conditional expectations do not always exist, even when $(\mathcal{C},\mathcal{D})$ is a regular MASA inclusion, and the C^*-algebras involved are well-behaved. Here is a simple example, which is a special case of the far more general setting considered in Section 6.

Example 1.2. Let X be a connected, compact Hausdorff space, and let $\alpha : X \to X$ be a homeomorphism such that α^2 is the identity map on X. Let F° be the interior of the set of fixed points for α; we assume that F° is neither empty nor all of X. (For a concrete example, take $X = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } \Re(z)\Im(z) = 0\}$ and let $\alpha(z) = \overline{z}$.) Define $\theta : C(X) \to C(X)$ by $\theta(f) = f \circ \alpha^{-1}$, and set

$$\mathcal{C} := \left\{ \begin{pmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{pmatrix} : f_0, f_1 \in C(X) \right\} \quad \text{and} \quad \mathcal{D} := \left\{ \begin{pmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{pmatrix} : f_0 \in C(X) \right\}.$$

Then $\mathcal{C}$ is a C^*-subalgebra of $M_2(C(X))$, and $(\mathcal{C},\mathcal{D})$ is a regular inclusion. ($\mathcal{C}$ may be regarded as $C(X) \rtimes (\mathbb{Z}/2\mathbb{Z})$.)

A calculation shows that the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is

$$\mathcal{D}^c = \left\{ \begin{pmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{pmatrix} \in \mathcal{C} : \operatorname{supp}(f_1) \subseteq F^\circ \right\}.$$

As $F^\circ \notin \{\emptyset, X\}$, we have $\mathcal{D} \subsetneq \mathcal{D}^c \subsetneq \mathcal{C}$, and another calculation shows $\mathcal{D}^c$ is abelian. Since $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{N}(\mathcal{C},\mathcal{D}^c)$, it follows that $(\mathcal{C},\mathcal{D}^c)$ is a regular MASA inclusion.

Suppose $E : \mathcal{C} \to \mathcal{D}^c$ is a conditional expectation. Then for some $f_0, f_1 \in C(X)$ with $\operatorname{supp}(f_1) \subseteq F^\circ$, we have $E\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{smallmatrix}\right)$. Notice $\theta(f_1) = f_1$ and $\left(\begin{smallmatrix} 0 & f_1 \\ f_1 & 0 \end{smallmatrix}\right) \in \mathcal{D}^c$. We have

$$\begin{pmatrix} f_1^2 & f_1\theta(f_0) \\ f_1 f_0 & f_1^2 \end{pmatrix} = \begin{pmatrix} 0 & f_1 \\ f_1 & 0 \end{pmatrix} E \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = E\left(\begin{pmatrix} 0 & f_1 \\ f_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right) = \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix},$$

so $f_1^2 = f_1$. As X is connected, this yields $f_1 = 0$ or $f_1 = I$. But $\operatorname{supp}(f_1) \subseteq F^\circ \neq X$, so $f_1 = 0$. Thus $E(\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{smallmatrix}\right)$.

Now if $g_1 \in \mathcal{D}$ is such that $\operatorname{supp}(g_1) \subseteq F^\circ$, then $\theta(g_1) = g_1$, so $\left(\begin{smallmatrix} 0 & g_1 \\ g_1 & 0 \end{smallmatrix}\right) \in \mathcal{D}^c$. Thus,

$$\begin{aligned} \begin{pmatrix} 0 & g_1 \\ g_1 & 0 \end{pmatrix} &= E\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right) = \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} E \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{pmatrix} \\ &= \begin{pmatrix} g_1 f_0 & 0 \\ 0 & \theta(g_1 f_0) \end{pmatrix}. \end{aligned}$$

Hence $g_1 = 0$ for every such g_1. This implies that $F^\circ = \emptyset$, contrary to hypothesis. Hence no conditional expectation of $\mathcal{C}$ onto $\mathcal{D}^c$ exists.

One of the goals of this paper is to show that even though conditional expectations may fail to exist for a regular MASA inclusion, there is a map which which may be used as a replacement. Here is the relevant definition.

Definition 1.3. Let $(\mathcal{C},\mathcal{D})$ be an inclusion and let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$. A *pseudo-conditional expectation for* ι, or more simply, a *pseudo-expectation for* ι, is a unital completely positive map $E : \mathcal{C} \to I(\mathcal{D})$ such that $E|_{\mathcal{D}} = \iota$. When the context is clear, we sometimes drop the reference to ι and simply call E a pseudo-expectation.

The existence of pseudo-expectations follows immediately from the injectivity of $I(\mathcal{D})$. In general, the pseudo-expectation need not be unique.

However, in Section 3 below, we show that for any regular MASA inclusion $(\mathcal{C},\mathcal{D})$, there is always a unique pseudo-expectation $E : \mathcal{C} \to I(\mathcal{D})$, see Theorem 3.10. Let $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ be the family of all states on $\mathcal{C}$ which restrict to elements of $\hat{\mathcal{D}}$. The family of states, $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) := \{\rho \circ E : \rho \in \widehat{I(\mathcal{D})}\}$ covers $\hat{\mathcal{D}}$ in the sense that the restriction map, $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) \ni \rho \mapsto \rho|_{\mathcal{D}} \in \hat{\mathcal{D}}$, is onto. Interestingly, $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is the unique minimal closed subset of $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ which covers $\hat{\mathcal{D}}$, see Theorem 3.13. We also show that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is closely related to the extension property. When $(\mathcal{C},\mathcal{D})$ is "countably generated," Theorem 3.13 also shows that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is the closure of all states in $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ whose restrictions to $\mathcal{D}$ extend uniquely to $\mathcal{C}$.

For a regular MASA inclusion, the intersection of the left kernels of the states in $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is the left kernel of the pseudo-expectation E. Let $\mathcal{L}(\mathcal{C},\mathcal{D})$ be the left kernel of E. Theorem 3.22 shows that $\mathcal{L}(\mathcal{C},\mathcal{D})$ is an ideal of $\mathcal{C}$, and moreover, is the unique ideal of $\mathcal{C}$ which is maximal with respect to the property of having trivial intersection with $\mathcal{D}$. When the pseudo-expectation takes values in $\mathcal{D}$ (rather than $I(\mathcal{D})$), the ideal $\mathcal{L}(\mathcal{C},\mathcal{D})$ may be viewed as a measure of the failure of the inclusion to be Cartan in Renault's sense.

We define a regular MASA inclusion to be a *virtual Cartan inclusion* when the pseudo-expectation is faithful, or equivalently, when $\mathcal{L}(\mathcal{C},\mathcal{D}) = 0$. The purpose of Section 6 is to give a large class of virtual Cartan inclusions. Theorem 6.9 shows that when $\mathcal{C}$ is the reduced crossed product of the abelian C^*-algebra $\mathcal{D}$ by a discrete group Γ, then, provided the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is abelian, $(\mathcal{C},\mathcal{D}^c)$ is a virtual Cartan inclusion. We characterize when $\mathcal{D}^c$ is abelian in terms of the dynamics of the action of Γ on $\hat{\mathcal{D}}$ in Theorem 6.6; this result shows that $\mathcal{D}^c$ is abelian precisely when the germ isotropy subgroup H^x of Γ is abelian for every $x \in \hat{\mathcal{D}}$. The results of Section 6 are summarized in Theorem 6.10.

One of the motivations for our study of inclusions was to provide a context for the study of certain nonselfadjoint subalgebras. If $(\mathcal{C},\mathcal{D})$ is a C^*-diagonal, there are numerous papers devoted to the study of various (usually nonselfadjoint) closed algebras $\mathcal{A}$ with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$, see [7, 9, 10, 21, 23, 24, 25] to name just a few. An often successful strategy for the analysis of the subalgebras of C^*-diagonals is to use the coordinatization of $(\mathcal{C},\mathcal{D})$ (via the twist) to impose coordinates on the subalgebras; properties of the coordinate system then reflect properties of the subalgebra.

Since many classes of regular MASA inclusions are neither C^*-diagonals nor Cartan inclusions, it is natural to wonder whether coordinate methods may be used to analyze nonselfadjoint subalgebras of regular MASA inclusions. A strategy for doing so is to try to regularly embed a given regular MASA inclusion $(\mathcal{C},\mathcal{D})$ into a C^*-diagonal $(\mathcal{C}_1,\mathcal{D}_1)$ and then to restrict the coordinates obtained from $(\mathcal{C}_1,\mathcal{D}_1)$ to the $(\mathcal{C},\mathcal{D})$ or to the given subalgebra. This leads to the following problem.

Problem 1.4. Characterize when a given regular inclusion $(\mathcal{C},\mathcal{D})$ can be regularly embedded into a C^*-diagonal.

We give a solution to Problem 1.4 in Section 5. To do this, we introduce a new family $\mathfrak{S}(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Mod}(\mathcal{C},\mathcal{D})$, which we call *compatible states*. When $(\mathcal{C},\mathcal{D})$ is a regular MASA inclusion, $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) \subseteq \mathfrak{S}(\mathcal{C},\mathcal{D})$. The intersection of the left kernels of the states in $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is an ideal of $\mathcal{C}$, $\mathrm{Rad}(\mathcal{C},\mathcal{D})$. The regular inclusion $(\mathcal{C},\mathcal{D})$ regularly embeds in a C^*-diagonal if and only if $\mathrm{Rad}(\mathcal{C},\mathcal{D}) = (0)$, see Theorem 5.9. In particular, any virtual Cartan inclusion regularly embeds into a C^*-diagonal.

The needed properties of compatible states are developed in Section 4. Compatible states can be defined for any inclusion, and we expect that they may be useful in other contexts as well. While compatible states exist in abundance for any regular MASA inclusion, Theorem 4.8 implies that compatible states need not exist for a general regular inclusion.

For a regular MASA inclusion $(\mathcal{C},\mathcal{D})$, it is always the case that $\mathrm{Rad}(\mathcal{C},\mathcal{D}) \subseteq \mathcal{L}(\mathcal{C},\mathcal{D})$. We have been unable to resolve the question of whether equality holds. We provide some insight into this question in Section 7. Given $\sigma \in \hat{\mathcal{D}}$, there is an equivalence relation R_1 on $H_\sigma := \{v \in \mathcal{N}(\mathcal{C},\mathcal{D}) : \rho(v^*dv) = \rho(d) \text{ for all } d \in \mathcal{D}\}$, and the set H_σ/R_1 of equivalence classes of this equivalence relation may be made into a $\mathbb{T}$-group. The main result of this section, Theorem 7.13, shows that there is a bijective correspondence between $\{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}) : \rho|_{\mathcal{D}} = \sigma\}$ and a certain family of pre-homomorphisms on H_σ/R_1. Theorem 7.13 thus gives a description of a certain class of state extensions of σ. Our description leads us to suspect that it is possible for $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$ to be a proper subset of $\mathcal{L}(\mathcal{C}, \mathcal{D})$.

The purpose of Section 8 is to discuss certain twisted groupoids arising from a regular MASA inclusion. The methods of this section are suitable modifications to our context of the methods used by Kumjian and Renault when coordinatizing C^*-diagonals and Cartan inclusions. We show that given any regular MASA inclusion $(\mathcal{C}, \mathcal{D})$, there is a twist associated to $(\mathcal{C}, \mathcal{D})$. We use $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ or another suitable subset F of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ as unit space of the twist, and define functionals $[v, \rho]$ on $\mathcal{C}$ much as in equation (1), except that the functional $\rho \circ E$ appearing in that formula is replaced with an element of F. The main result of this section, Theorem 8.14, shows that when $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$ is trivial, there is a regular $*$-monomorphism of $(\mathcal{C}, \mathcal{D})$ into the Cartan inclusion arising from the twist associated to $(\mathcal{C}, \mathcal{D})$. This result gives perspective to the embedding results of Section 5, and suggests that it is indeed possible to coordinatize subalgebras using this twist, as indicated prior to Problem 1.4.

In [28], we gave a method for extending an isometric isomorphism between subalgebras $\mathcal{A}_i$ of C^*-diagonals $(\mathcal{C}_i, \mathcal{D}_i)$ with $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$ to a $*$-isomorphism of the C^*-subalgebra $C^*(\mathcal{A}_1)$ of $\mathcal{C}_1$ generated by $\mathcal{A}_1$ onto the corresponding subalgebra $C^*(\mathcal{A}_2)$. The two main ingredients of this method were: a) show that $\mathcal{D}_i$ norms $\mathcal{C}_i$ in the sense of Pop-Sinclair-Smith ([29]), and b) show that the C^*-envelope of $\mathcal{A}_i$ is isometrically isomorphic to $C^*(\mathcal{A}_i)$. In Section 9, we show that if the hypothesis that $(\mathcal{C}_i, \mathcal{D}_i)$ is weakened from C^*-diagonal to virtual Cartan inclusion, then both these ingredients still hold. Theorem 9.4 generalizes [28, Theorem 2.16] to the context of virtual Cartan inclusions. This is a considerable generalization, and allows for the simplification of some arguments in the literature.

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1.1. **Preliminaries.** Given a Banach space $\mathcal{X}$, we will use $\mathcal{X}^\#$ instead of the traditional $\mathcal{X}^*$ to denote the Banach space dual. Likewise if $\alpha : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map between Banach spaces, we use $\alpha^\#$ to denote the adjoint map, $f \in \mathcal{Y}^\# \mapsto f \circ \alpha \in \mathcal{X}^\#$.

If X is a topological space and $E \subseteq X$, E° denotes the interior of E. Also, for $f : X \to \mathbb{C}$, we write $\operatorname{supp} f$ for the set $\{x \in X : f(x) \neq 0\}$.

Standing Assumption 1.5. For the remainder of this paper, all C^*-algebras will be unital, and if $\mathcal{D}$ is a sub-C^*-algebra of the C^*-algebra $\mathcal{C}$, we assume that the unit for $\mathcal{D}$ is the same as the unit for $\mathcal{C}$.

Let $\mathcal{C}$ be a C^*-algebra, and let $\mathcal{S}(\mathcal{C})$ be the state space of $\mathcal{C}$. For $\rho \in \mathcal{S}(\mathcal{C})$ let $L_\rho = \{x \in \mathcal{C} : \rho(x^*x) = 0\}$ be the left kernel of ρ, and let $(\pi_\rho, \mathcal{H}_\rho, \xi)$ be the GNS representation corresponding to ρ. We regard $\mathcal{C}/L_\rho$ as a dense subset of $\mathcal{H}_\rho$, and for $x \in \mathcal{C}$ will often write $x + L_\rho$ to denote the vector $\pi_\rho(x)\xi$. Denote the inner product on $\mathcal{H}_\rho$ by $\langle \cdot, \cdot \rangle_\rho$.

We now recall some facts about projective topological spaces, projective covers, and injective envelopes of abelian C^*-algebras. Following [17], given a compact Hausdorff space X, a pair (P, f) consisting of a compact Hausdorff space P and a continuous map $f : P \to X$ is called a *cover for X* (or simply a *cover*) if f is surjective. A cover (P, f) is *rigid* if the only continuous map $h : P \to P$ which satisfies $f \circ h = f$ is $h = \mathrm{id}_P$; the cover (P, f) is *essential* if whenever Y is a compact Hausdorff space and $h : Y \to P$ is continuous and satisfies $f \circ h$ is onto, then h is onto.

A compact Hausdorff space P is *projective* if whenever X and Y are compact Hausdorff spaces and $h : Y \to X$ and $f : P \to X$ are continuous maps with h surjective, there exists a continuous map $g : P \to Y$ with $g \circ h = f$. A Hausdorff space which is extremally disconnected (i.e. the closure of every open set is open) and compact is *Stonean*. In [15, Theorem 2.5], Gleason proved that a compact Hausdorff space P is projective if and only P is Stonean.

By [17, Proposition 2.13], if (P, f) is a cover for X with P a projective space, then (P, f) is rigid if and only if (P, f) is essential. A *projective cover* for X is a rigid cover (P, f) for X such that P is projective. A projective cover for X always exists [17, Theorem 2.16] and is unique in the sense that if (P_1, f_1) and (P_2, f_2) are projective covers for X, then there is a unique homeomorphism $h : P_1 \to P_2$ such that $f_1 = f_2 \circ h$.

The concept of an injective envelope for an abelian unital C^*-algebras is dual to the concept of a projective cover of a compact Hausdorff space: if (P, f) is a cover for X, let $\iota : C(X) \to C(P)$ be the map $d \mapsto d \circ f$; then (P, f) is a projective cover if and only if $(C(P), \iota)$ is an injective, envelope of $C(X)$ [17, Corollary 2.18]. (Injective envelopes can be also be defined for general unital C^*-algebras, not just abelian, unital C^*-algebras. Like projective covers, injective envelopes of unital C^*-algebras have a uniqueness property. If $\mathcal{A}$ is a unital C^*-algebra, and $(\mathcal{B}_1, \sigma_1)$ and $(\mathcal{B}_2, \sigma_2)$ are injective envelopes for $\mathcal{A}$, then there exists a unique $*$-isomorphism $\theta : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\theta \circ \sigma_1 = \sigma_2$ [18, Theorem 4.1].)

Recall that a unital C^*-algebra $\mathcal{C}$ is *monotone complete* if every bounded increasing net in the self-adjoint part, $\mathcal{C}_{s.a.}$, of $\mathcal{C}$ has a least upper bound in $\mathcal{C}_{s.a.}$. Hamana [20] shows that every injective C^*-algebra is monotone complete. When (x_λ) is a bounded increasing net in $\mathcal{C}_{s.a.}$, $\sup_{\mathcal{C}} x_\lambda$ means the least upper bound of (x_λ) in $\mathcal{C}$.

Theorem 6.6 of [20] implies that if $\mathcal{A}$ is a unital, abelian C^*-algebra then any injective envelope $(\mathcal{B}, \sigma)$ for $\mathcal{A}$ is *Hamana-regular* in the sense that whenever $x \in \mathcal{B}$ is self-adjoint, $x = \sup_{\mathcal{B}}\{\sigma(a) : a \in \mathcal{A}, a = a^* \text{ and } a \leq x\}$. (Since $\mathcal{A}$ is abelian, we regard $\{\sigma(a) : a \in \mathcal{A}, a = a^* \text{ and } a \leq x\}$ as a net indexed by itself.) As Hamana observes in [20], when $x \in \mathcal{A}$ is positive, then also, $x = \sup_{\mathcal{B}}\{\sigma(a) : a \in \mathcal{A} \text{ and } 0 \leq \sigma(a) \leq b\}$.

Here is a description of an injective envelope of an abelian C^*-algebra. For details, see [16, Theorem 1]. Let X be a compact Hausdorff space. Define an equivalence relation on the algebra $\mathcal{B}(X)$ of all bounded Borel complex-valued functions on X by $f \sim g$ if and only if $\{x \in X : f(x) - g(x) \neq 0\}$ is a set of first category. The equivalence class J of the zero function is an ideal in $\mathcal{B}(X)$ and the quotient $\mathcal{D}(X) := \mathcal{B}(X)/J$ is called the *Dixmier algebra*. Define $j : C(X) \to \mathcal{D}(X)$ by $j(f) = f + J$. Then $(\mathcal{D}(X), j)$ is an injective envelope for $\mathcal{D}$.

We conclude this section with a few comments regarding categories. Let $\mathfrak{C}$ be the category of unital abelian C^*-algebras with $*$-homomorphisms, and let $\mathfrak{O}$ be the category of operator systems with completely positive (unital) maps. Let $\mathcal{D}$ be a unital abelian C^*-algebra. Then $\mathcal{D}$ is an injective object in $\mathfrak{C}$ if and only if $\mathcal{D}$ is an injective object in $\mathfrak{O}$, see [17, Theorem 2.4]. (The statement of [17, Theorem 2.4], mentions the category of operator systems without explicitly giving the morphisms, but the proof makes it clear that the authors mean the category of operator systems and unital, completely positive maps.)

Hamana shows that in the category $\mathfrak{O}$, there is an injective object $I(\mathcal{D})$ and a one-to-one completely positive $\iota : \mathcal{D} \to I(\mathcal{D})$ such that the extension $(I(\mathcal{D}), \iota)$ is rigid and essential. Hamana and Hadwin-Paulsen (see [19] and [17, Corollary 2.18]) observe that $I(\mathcal{D})$ is endowed with a product which makes it into an abelian C^*-algebra (and ι a $*$-monomorphism). Set $X = \hat{\mathcal{D}}$, and let (P, f) be a projective cover for X, so that the map $\tau : \mathcal{D} \to C(P)$ given by $\tau(x) = \hat{x} \circ f$ is a one-to-one $*$-homomorphism of $\mathcal{D}$ into $C(P)$. Corollary 2.18 of [17] also shows the existence of a $*$-isomorphism $\theta : C(P) \to I(\mathcal{D})$ such that $\theta \circ \tau = \iota$.

Thus, for us an injective envelope $(I(\mathcal{D}), \iota)$ for $\mathcal{D}$ will be a rigid and essential extension of $\mathcal{D}$ in the category $\mathfrak{C}$. The comments above show that this is equivalent to saying that $(I(\mathcal{D}), \iota)$ is a rigid and essential extension for $\mathcal{D}$ in $\mathfrak{O}$.

2. Dynamics of Regular Inclusions

Given a regular inclusion $(\mathcal{C}, \mathcal{D})$, the $*$-semigroup $\mathcal{N}(\mathcal{C}, \mathcal{D})$ of normalizers acts via partial homeomorphisms on the maximal ideal space $\hat{\mathcal{D}}$ of $\mathcal{D}$. The purpose of this section is to discuss some of the features of this action. The first subsection is devoted to notation and some background facts regarding normalizers and intertwiners. The second subsection gives a characterization of the extension property in terms of the dynamics associated with the action of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ on $\hat{\mathcal{D}}$ (Theorem 2.9). An interesting consequence is that for any regular MASA inclusion $(\mathcal{C}, \mathcal{D})$ with $\mathcal{D}$ injective is an EP inclusion, see Theorem 2.10.

2.1. **Normalizers and Intertwiners.** Let $(\mathcal{C}, \mathcal{D})$ be an inclusion. Closely related to normalizers are intertwiners.

Definition 2.1. An *intertwiner* for $\mathcal{D}$ is an element $v \in \mathcal{C}$ such that $v\mathcal{D} = \mathcal{D}v$. We denote the set of all intertwiners by $\mathcal{I}(\mathcal{C}, \mathcal{D})$.

Proposition 3.3 of [10] shows that for any intertwiner v, the elements v^*v and vv^* belong to the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$. When v is a normalizer, the fact that $\mathcal{D}$ is unital shows that both v^*v and vv^* belong to $\mathcal{D}$, and it is easy to see that $\{v \in \mathcal{I}(\mathcal{C}, \mathcal{D}) : v^*v, vv^* \in \mathcal{D}\} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. In general however, there exists $v \in \mathcal{I}(\mathcal{C}, \mathcal{D})$ with $\{v^*v, vv^*\} \not\subseteq \mathcal{D}$. (For a simple example, observe that every operator in $M_2(\mathbb{C})$ is an intertwiner for the inclusion, $(M_2(\mathbb{C}), \mathbb{C}I_2)$.) Also, by [10, Proposition 3.4], $\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \overline{\mathcal{I}(\mathcal{C}, \mathcal{D})}$. However, the final paragraph of the proof of [10, Proposition 3.4], shows somewhat more than this, namely that $\overline{\{v \in \mathcal{I}(\mathcal{C}, \mathcal{D}) : v^*v, vv^* \in \mathcal{D}\}} = \mathcal{N}(\mathcal{C}, \mathcal{D})$. Summarizing, we have the following result.

Proposition 2.2 ([10, Propositions 3.3 and 3.4]). *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion. Then*

$$\overline{\{v \in \mathcal{I}(\mathcal{C}, \mathcal{D}) : v^*v, vv^* \in \mathcal{D}\}} = \mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \overline{\mathcal{I}(\mathcal{C}, \mathcal{D})}.$$

Furthermore, when $\mathcal{D}$ is a MASA in $\mathcal{C}$, $\mathcal{N}(\mathcal{C}, \mathcal{D}) = \overline{\mathcal{I}(\mathcal{C}, \mathcal{D})}$.

The following example of a regular homomorphism will be useful in the sequel.

Lemma 2.3. *Suppose $(\mathcal{C}, \mathcal{D})$ is an inclusion such that the relative commutant, $\mathcal{D}^c$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian. Then $(\mathcal{C}, \mathcal{D}^c)$ is a MASA inclusion and $\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}^c)$; in particular the identity map $id : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}^c)$ is a regular $*$-homomorphism.*

Proof. Since $\mathcal{D}$ and $\mathcal{D}^c$ are abelian, $(\mathcal{C}, \mathcal{D}^c)$ is a MASA inclusion.

To show that $\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}^c)$, suppose $v \in \mathcal{I}(\mathcal{C}, \mathcal{D})$. Fix $h \in \mathcal{D}^c$, and let $d \in \mathcal{D}$ be self-adjoint. Since v is a $\mathcal{D}$-intertwiner, we may find a self-adjoint $d' \in \mathcal{D}$ so that $dv = vd'$. Then $d(vhv^*) = vd'hv^* = vhd'v^* = (vhv^*)d$, from which it follows that $vhv^* \in \mathcal{D}^c$. Similarly, $v^*hv \in \mathcal{D}^c$. We conclude that $\mathcal{I}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}^c)$. Since $\mathcal{N}(\mathcal{C}, \mathcal{D}^c)$ is norm-closed, Proposition 2.2 yields,

$$\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \overline{\mathcal{I}(\mathcal{C}, \mathcal{D})} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}^c).$$

Thus, $(\mathcal{C}, \mathcal{D}^c)$ is a MASA inclusion and the identity mapping $\mathrm{id} : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}^c)$ is a regular $*$-homomorphism. $\square$

For any topological space X, a *partial homeomorphism* is a homeomorphism $h : S \to R$, where S and R are open subsets of X. As usual, $\mathrm{dom}(h)$ and $\mathrm{ran}(h)$ will denote the domain and range of the partial homeomorphism h. We use $\mathrm{Inv}_{\mathcal{O}}(X)$ to denote the inverse semigroup of all partial homeomorphisms of X. When $\mathcal{S}$ is a $*$-semigroup, a semigroup homomorphism $\alpha : \mathcal{S} \to \mathrm{Inv}_{\mathcal{O}}(X)$ is a *$*$-homomorphism* if for every $s \in \mathcal{S}$, $\alpha(s^*) = \alpha(s)^{-1}$. A subset $\mathcal{G}$ of $\mathrm{Inv}_{\mathcal{O}}(X)$ which is closed under composition and inverses (i.e. a sub inverse semigroup) is called a *pseudo-group on X*. Associated to any pseudo-group $\mathcal{G}$ on X is the *groupoid of germs* which is the set of equivalence classes, $\{[x, \phi, y] : \phi \in \mathcal{G}, y \in \mathrm{dom}(\phi), x = \phi(y)\}$, where $[x, \phi, y] = [x_1, \phi_1, y_1]$ if and only $y = y_1$

and there exists a neighborhood N of y such that $\phi|_N = \phi_1|_N$. Elements $[w, \phi, x]$ and $[y, \psi, z]$ are composable if $x = y$ and then $[w, \phi, x]\,[y, \psi, z] = [w, \phi\psi, z]$ and $[x, \phi, y]^{-1} = [y, \phi^{-1}, x]$. The range and source maps are $r([x, \phi, y]) = x$ and $s([x, \phi, y]) = y$. Thus X may be identified with the unit space of the groupoid of germs.

Recall (see [22, Proposition 6]) that a normalizer v determines a partial homeomorphism

$$\beta_v : \{\rho \in \hat{\mathcal{D}} : \rho(v^*v) > 0\} \to \{\rho \in \hat{\mathcal{D}} : \rho(vv^*) > 0\} \quad \text{given by} \quad \beta_v(\rho)(d) = \frac{\rho(v^*dv)}{\rho(v^*v)} \qquad (d \in \mathcal{D}).$$

Clearly $\mathcal{N}(\mathcal{C}, \mathcal{D})$ and $\mathcal{I}(\mathcal{C}, \mathcal{D})$ are $*$-semigroups under multiplication. Routine, but tedious, calculations show that the map $\mathcal{N}(\mathcal{C}, \mathcal{D}) \ni v \mapsto \beta_v$ is a $*$-semigroup homomorphism $\beta : \mathcal{N}(\mathcal{C}, \mathcal{D}) \to \mathrm{Inv}_{\mathcal{O}}(\hat{\mathcal{D}})$. We record this fact as a proposition.

Proposition 2.4 ([31, Lemma 4.10])**.** *Suppose $(\mathcal{C}, \mathcal{D})$ is an inclusion. Then the following statements hold.*

1. *Suppose that $v, w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $\rho \in \hat{\mathcal{D}}$ satisfies $\rho(w^*v^*vw) \neq 0$. Then $\rho(w^*w) \neq 0$, and $\beta_{vw}(\rho) = \beta_v(\beta_w(\rho))$.*
2. *For every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\beta_{v^*} = (\beta_v)^{-1}$.*

Following Renault [31] we will call the collection,

$$\mathcal{PG}(\mathcal{C}, \mathcal{D}) := \{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$$

the *Weyl pseudo-group* of the inclusion. Also, the groupoid of germs of $\mathcal{PG}(\mathcal{C}, \mathcal{D})$ is the *Weyl groupoid* of the inclusion, which we denote by $\mathcal{WG}(\mathcal{C}, \mathcal{D})$.

Remarks 2.5.

(1) Observe that if θ is a regular homomorphism, then $\theta(\mathcal{D}_1) \subseteq \mathcal{D}_2$: indeed, for $d \in \mathcal{D}_1$ with $d \geq 0$, $d^{1/2} \in \mathcal{N}(\mathcal{C}_1, \mathcal{D}_1)$ and so $\theta(d) = \theta(d^{1/2})1\theta(d^{1/2}) \in \mathcal{D}_2$. It follows that the dynamics of inclusions under regular homomorphisms are well-behaved in the sense that if $\theta : (\mathcal{C}_1, \mathcal{D}_1) \to (\mathcal{C}_2, \mathcal{D}_2)$ is a regular homomorphism of the inclusions $(\mathcal{C}_i, \mathcal{D}_i)$, then whenever $v \in \mathcal{N}(\mathcal{C}_1, \mathcal{D}_1) \setminus \ker\theta$, the following diagram commutes:

$$\begin{tikzcd}
\hat{\mathcal{D}}_2 \supseteq \mathrm{dom}(\beta_{\theta(v)}) \arrow[r, "\beta_{\theta(v)}"] \arrow[d, "\theta^{\#}"'] & \mathrm{range}(\beta_{\theta(v)}) \subseteq \hat{\mathcal{D}}_2 \arrow[d, "\theta^{\#}"] \\
\hat{\mathcal{D}}_1 \supseteq \mathrm{dom}(\beta_v) \arrow[r, "\beta_v"] & \mathrm{range}(\beta_v) \subseteq \hat{\mathcal{D}}_1.
\end{tikzcd}$$

(2) For some purposes, it is easier to work with intertwiners than normalizers. Thus, one might be tempted to define a regular homomorphism using intertwiners instead of normalizers, that is, by mandating $\theta(\mathcal{I}(\mathcal{C}_1, \mathcal{D}_1)) \subseteq \mathcal{I}(\mathcal{C}_2, \mathcal{D}_2)$. However, a disadvantage of doing so is that such a θ need not carry $\mathcal{D}_1$ into $\mathcal{D}_2$, which is why we use $\mathcal{N}(\mathcal{C}, \mathcal{D})$ rather than $\mathcal{I}(\mathcal{C}, \mathcal{D})$ in Definition 1.1. For an example of this, let $\mathcal{C} = C([0, 1])$ and let $\mathcal{D} = \{f \in C([0, 1]) : f(0) = f(1/2)\}$. Then $\mathcal{C} = \mathcal{I}(\mathcal{C}, \mathcal{D})$, and since every unitary in $\mathcal{C}$ normalizes $\mathcal{D}$, we see that $(\mathcal{C}, \mathcal{D})$ is regular (as in Definition 1.1). Taking $(\mathcal{C}_i, \mathcal{D}_i) = (\mathcal{C}, \mathcal{D})$, then any automorphism θ of $\mathcal{C}$ satisfies $\theta(\mathcal{I}(\mathcal{C}_1, \mathcal{D}_1)) \subseteq \mathcal{I}(\mathcal{C}_2, \mathcal{D}_2)$, yet clearly one may choose θ so that $\theta(\mathcal{D}_1) \not\subseteq \theta(\mathcal{D}_2)$.

2.2. **Quasi-Freeness and the Extension Property.** By Proposition 2.4, $\mathcal{S} := \{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ is an inverse semigroup of partial homeomorphisms of $\hat{\mathcal{D}}$. Recall that a group G of homeomorphisms of a space X acts freely if whenever $g \in G$ has a fixed point, then g is the identity. Paralleling the notion for groups, we make the following definition.

Definition 2.6. Suppose that $\mathcal{S}$ is a $*$-semigroup and X is a compact Hausdorff space, and that $\alpha : \mathcal{S} \to \mathrm{Inv}_{\mathcal{O}}(X)$ is a $*$-semigroup homomorphism. We say that $\mathcal{S}$ *acts quasi-freely* on X if whenever $s \in \mathcal{S}$, $\{x \in \mathrm{dom}(\alpha(s)) : \alpha(s)(x) = x\}$ is an open set in X.

When Γ is a group acting quasi-freely on X, this says that for each $s \in \Gamma$, the set of fixed points of $\alpha(s)$ is a clopen set; in particular, when X is a connected set, the notions of free and quasi-free actions for a group (acting as homeomorphisms) on X coincide.

In some circumstance, quasi-freeness is automatic. Recall that a Hausdorff topological space X is *extremally disconnected* if the closure of every open subset of X is open and that X is a *Stonean space* if X is compact, Hausdorff, and extremally disconnected. We shall show that if a $*$-semigroup acts on a Stonean space, then it acts quasi-freely. To do this we require the following topological proposition. The proof is a straightforward adaptation of the elegant proof by Arhangel′skii of Frolík's Theorem [14, Theorem 3.1] on fixed points of homeomorphisms of extremally disconnected spaces. We provide a sketch of the proof for the convenience of the reader.

Proposition 2.7 (Frolík's Theorem)**.** *Let X be an extremally disconnected space, let V, W be clopen subsets of X, and suppose $h : V \to W$ is a homeomorphism of V onto W. Then the set of fixed points $F := \{x \in V : h(x) = x\}$ is a clopen subset of X. Moreover, there are three disjoint clopen subsets C_1, C_2, C_3 of X such that for $i = 1, 2, 3$, $h(C_i) \cap C_i = \emptyset = C_i \cap F$ and $V = F \cup C_1 \cup C_2 \cup C_3$.*

Proof (see [4, Theorem 1]*).* Call an open subset $A \subseteq V$ *h-simple* if $h(A) \cap A = \emptyset$. By the Hausdorff maximality theorem, there exists a maximal chain $\mathcal{G}$ of h-simple sets. Put $U = \bigcup \mathcal{G}$. Then U is also a h-simple subset of V, and since $\overline{U}$ is open, maximality shows that U is in fact clopen.

Next observe that $h(U) \cap V$ and $h^{-1}(U \cap W)$ are clopen h-simple sets, and put

$$M = U \cup (h(U) \cap V) \cup h^{-1}(U \cap W).$$

Since the intersection of F with any h-simple subset of V is empty, we have $M \cap F = \emptyset$. We shall show that $F = V \setminus M$.

Suppose to the contrary, that $x \in V \setminus M$ satisfies $h(x) \neq x$. Let H be an open subset of V such that $x \in H$ and $H \cap M$ and $h(H) \cap H$ are both empty. Then H is h-simple and

$$H \cap U = H \cap (h(U) \cap V) = H \cap h^{-1}(U \cap W) = \emptyset. \tag{2}$$

But (2) implies that $H \cup U$ is a h-simple set which properly contains U, contradicting the maximality of U. So $F = V \setminus M$.

Since both V and M are clopen, so is F. Finally, to complete the proof, take $C_1 := U$, $C_2 := h(U) \cap V$, and $C_3 := h^{-1}(U \cap W) \setminus (h(U) \cap V)$. □

With this preparation, we now show that any action of a $*$-semigroup on a Stonean space is quasi-free.

Theorem 2.8. *Suppose that X is a Stonean space, $\mathcal{S}$ is a $*$-semigroup, and $\alpha : \mathcal{S} \to \mathrm{Inv}_{\mathcal{O}}(X)$ is a $*$-semigroup homomorphism. Then $\mathcal{S}$ acts quasi-freely on X.*

Proof. Fix $s \in \mathcal{S}$, and consider the open sets $G := \mathrm{dom}(\alpha(s))$ and $H := \mathrm{ran}(\alpha(s))$. Since X is compact and extremally disconnected, the Stone-Čech compactifications βG and βH are homeomorphic to $\overline{G}$ and $\overline{H}$ respectively ([34, Exercises 15G(1) and 19G(2)]). Since $\alpha(s)$ is a homeomorphism of G onto H, general properties of the Stone-Čech compactification show that $\alpha(s)$ extends to a homeomorphism h of $\overline{G}$ onto $\overline{H}$.

Let $F \subseteq \overline{G}$ be the set of fixed points for h; Proposition 2.7 shows that F is clopen in X. Therefore,

$$\{x \in \mathrm{dom}(\alpha(s)) : \alpha(s)(x) = x\} = F \cap \mathrm{dom}(\alpha(s))$$

is open in X. Thus $\mathcal{S}$ acts quasi-freely on X. □

Quasi-freeness is intimately related to the extension property. The following result is known in the case when $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion with faithful expectation, see [31, Proposition 5.11]. Parts of the proof below follow the proof of [10, Proposition 3.12], but we reproduce it here for convenience of the reader.

Theorem 2.9. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion. Then the following statements are equivalent:*

a) *$\mathcal{D}$ has the extension property in $\mathcal{C}$;*
b) *$\mathcal{D}$ is a MASA in $\mathcal{C}$ and the action $v \mapsto \beta_v$ of the semigroup $\mathcal{N}(\mathcal{C}, \mathcal{D})$ is a quasi-free action on $\hat{\mathcal{D}}$;*
c) *$\mathcal{D}$ is a MASA in $\mathcal{C}$ and for each $\sigma \in \hat{\mathcal{D}}$, the isotropy group of $\mathcal{WG}(\mathcal{C}, \mathcal{D})$ at σ is the trivial group.*

Proof. (a)⇒(b). Suppose that $\mathcal{D}$ has the extension property. Then [3, Corollary 2.7] shows that $\mathcal{D}$ is a MASA in $\mathcal{C}$, and that there exists a conditional expectation $E : \mathcal{C} \to \mathcal{D}$. Suppose that $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, and $\sigma \in \hat{\mathcal{D}}$ satisfies $\sigma(v^*v) > 0$ and $\beta_v(\sigma) = \sigma$. By [10, Proposition 3.12], we have $v^*E(v) \in \mathcal{D}$. Also, if G is the unitary group of $\mathcal{D}$, we have for $g \in G$,

$$\sigma(v^*gvg^{-1}) = \sigma(v^*gv)\sigma(g^{-1}) = \beta_v(\sigma)(g)\sigma(v^*v)\sigma(g^{-1}) = \sigma(g)\sigma(v^*v)\sigma(g^{-1}) = \sigma(v^*v).$$

The extension property and [3, Theorem 3.7] show that $E(v) \in \overline{co}\{gvg^{-1} : g \in G\}$, so that

$$\sigma(v^*E(v)) = \sigma(v^*v),$$

whence $\sigma(v^*E(v)) = \sigma(v^*v) \neq 0$.

Hence there exists an open set $U \subseteq \hat{\mathcal{D}}$ so that $\sigma \in U$ and $\tau(v^*E(v)) \neq 0$ for every $\tau \in U$. Since $v^*E(v) \in \mathcal{D}$, we have $\tau = \beta_{v^*E(v)}(\tau) = \beta_{v^*}(\beta_{E(v)}(\tau)) = \beta_{v^*}(\tau)$ for every $\tau \in U$. But $\beta_v^{-1} = \beta_{v^*}$, so $\beta_v(\tau) = \tau$ for $\tau \in U$. Thus $\{\sigma \in \hat{\mathcal{D}} : \sigma(v^*v) > 0 \text{ and } \beta_v(\sigma) = \sigma\}$ is open in $\hat{\mathcal{D}}$, so the semigroup $\{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ acts quasi-freely on $\hat{\mathcal{D}}$.

Before proving (b)⇒(a), we establish some notation. We use $[\mathcal{C}, \mathcal{D}]$ for the set $\{cd - dc : d \in \mathcal{D}, c \in \mathcal{C}\}$. Also, a normalizer $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a *free normalizer* if $v^2 = 0$. Kumjian notes in [22], that any free normalizer belongs to $\overline{\text{span}}[\mathcal{C}, \mathcal{D}]$, because $v(v^*v)^{1/n} - (v^*v)^{1/n}v = v(v^*v)^{1/n} \to v$.

We turn now to the proof of (b)⇒(a). So assume that (b) holds. We shall prove that

$$\mathcal{N}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{D} + \overline{\text{span}}[\mathcal{C}, \mathcal{D}]. \tag{3}$$

Once this inclusion is established, an application of [3, Theorem 2.4] will show that whenever ρ_1 and ρ_2 are states of $\mathcal{C}$ with $\rho_1|_\mathcal{D} = \rho_2|_\mathcal{D} \in \hat{\mathcal{D}}$, then for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\rho_1(v) = \rho_2(v)$. Regularity of $(\mathcal{C}, \mathcal{D})$ then implies that $\rho_1 = \rho_2$, so that $\mathcal{D}$ has the extension property.

To show (3), fix $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Let $F = \{\sigma \in \hat{\mathcal{D}} : \sigma(v^*v) > 0 \text{ and } \beta_v(\sigma) = \sigma\}$. By hypothesis, F is an open subset of $\hat{\mathcal{D}}$. Let $\varepsilon > 0$, and let

$$X_\varepsilon := F \cap \{\sigma \in \hat{\mathcal{D}} : \sigma(v^*v) \geq \varepsilon^2\}.$$

Then X_ε is a closed subset of $\hat{\mathcal{D}}$. Let $f_\varepsilon \in \mathcal{D}$ be such that $0 \leq f_\varepsilon \leq I$, $\hat{f}_\varepsilon|_{X_\varepsilon} \equiv 1$ and $\overline{\text{supp}}(\hat{f}_\varepsilon) \subseteq F$. Next let

$$Y_\varepsilon := F^c \cap \{\sigma \in \hat{\mathcal{D}} : \sigma(v^*v) \geq \varepsilon^2\}.$$

Clearly $Y_\varepsilon \cap \overline{\text{supp}}(\hat{f}_\varepsilon) = \emptyset$.

For $\sigma \in Y_\varepsilon$, we have $\beta_v(\sigma) \neq \sigma$, so we may find $d \in \mathcal{D}$ with $\sigma(d) = 1$, $0 \leq d \leq 1$, $\overline{\text{supp}}(\hat{d}) \cap \overline{\text{supp}}(\hat{f}_\varepsilon) = \emptyset$ and $(vd)^2 = 0$. Compactness of Y_ε and a partition of unity argument show that there exists $n \in \mathbb{N}$ and a collection of functions $\{g_j\}_{j=1}^n \subseteq \mathcal{D}$ such that, with $g_\varepsilon = \sum_{j=1}^n g_j$, we have:

$$(vg_j)^2 = 0, \quad 0 \leq g_j \leq I, \quad \overline{\text{supp}}(\hat{g}_j) \cap \overline{\text{supp}}(\hat{f}_\varepsilon) = \emptyset, \quad 0 \leq g_\varepsilon \leq I, \quad \text{and} \quad \hat{g}_\varepsilon|_{Y_\varepsilon} \equiv 1.$$

Then $g_\varepsilon f_\varepsilon = 0$, and $\|v(I - (f_\varepsilon + g_\varepsilon))\| < \varepsilon$. So $v = \lim_{\varepsilon\to 0}(vf_\varepsilon + vg_\varepsilon)$. From Kumjian's observation, $vg_\varepsilon \in \overline{\operatorname{span}}[\mathcal{C}, \mathcal{D}]$. Moreover, since the closed support of $\hat{f}_\varepsilon$ is contained in F, we find that vf_ε commutes with $\mathcal{D}$. Hence $vf_\varepsilon \in \mathcal{D}$, as $\mathcal{D}$ is a MASA in $\mathcal{C}$.

Let ε_n be a sequence of positive numbers decreasing to 0. We may choose elements $f_{\varepsilon_n} \in \mathcal{D}$ as above but with the additional condition that $f_{\varepsilon_m} \leq f_{\varepsilon_n}$ whenever $m < n$. For $n > m$ and $\sigma \in \hat{\mathcal{D}}$, we have

$$\sigma(v^*v(f_{\varepsilon_n} - f_{\varepsilon_m})^2) = \begin{cases} 0 & \text{if } \sigma(f_{\varepsilon_n} - f_{\varepsilon_m}) = 0 \\ \sigma(v^*v)\sigma(f_{\varepsilon_n} - f_{\varepsilon_m})^2 & \text{if } \sigma(f_{\varepsilon_n} - f_{\varepsilon_m}) \neq 0. \end{cases} \tag{4}$$

Notice that if $\sigma \in \overline{F} \cap F^c$, then $\sigma(v^*v) = 0$. By continuity of $\widehat{v^*v}$, given $\delta > 0$, there exists a open set $U \subseteq \hat{\mathcal{D}}$ with $U \supseteq \overline{F} \cap F^c$ and $\sigma(v^*v) < \delta$ if $\sigma \in U$. Since $\overline{\operatorname{supp}}(f_{\varepsilon_n} - f_{\varepsilon_m}) \subseteq U$ when n, m are sufficiently large, it follows from (4) that vf_{ε_n} is a Cauchy sequence in $\mathcal{D}$, and hence converges to $k \in \mathcal{D}$. Then

$$v - k = \lim_{n\to\infty}(vg_{\varepsilon_n}) \in \overline{\operatorname{span}}[\mathcal{C}, \mathcal{D}],$$

whence $v = k + (v - k) \in \mathcal{D} + \overline{\operatorname{span}}[\mathcal{C}, \mathcal{D}]$. As noted above, this is sufficient to complete the proof that $\mathcal{D}$ has the extension property. Thus (a) holds.

(b)$\Rightarrow$(c). Fix $\sigma \in \hat{\mathcal{D}}$, and suppose that $[\sigma, \phi, \sigma]$ belongs to the isotropy group of $\mathcal{WG}(\mathcal{C}, \mathcal{D})$ at σ. Then there is an open set N and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that with $\sigma \in N \subseteq \operatorname{dom}\phi \cap \operatorname{dom}\beta_v$ such that $\beta_v|_N = \phi|_N$. By quasi-freeness of the action, there exists a neighborhood N_1 of σ contained in N such that $\beta_v|_{N_1} = \operatorname{id}|_{N_1}$. Thus, $[\sigma, \phi, \sigma] = [\sigma, \operatorname{id}, \sigma]$ so that the isotropy group of $\mathcal{WG}(\mathcal{C}, \mathcal{D})$ at σ is trivial.

(c)$\Rightarrow$(b). Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and suppose that $\sigma \in \operatorname{dom}(\beta_v)$ is a fixed point for β_v. Then $[\sigma, \beta_v, \sigma]$ is in the isotropy group for $\mathcal{WG}(\mathcal{C}, \mathcal{D})$ at σ, so that $[\sigma, \beta_v, \sigma] = [\sigma, \operatorname{id}, \sigma]$. By the definition of the Weyl groupoid, σ belongs to the interior of $\{x \in \operatorname{dom}(\beta_v) : \beta_v(x) = x\}$. Hence $\{x \in \operatorname{dom}(\beta_v) : \beta_v(x) = x\}$ is an open set in X. As this holds for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, the action $v \mapsto \beta_v$ is quasi-free. $\square$

As an immediate corollary of our work, we have the following.

Theorem 2.10. *Suppose* $(\mathcal{C}, \mathcal{D})$ *is a regular MASA inclusion, with* $\mathcal{D}$ *an injective* C^**-algebra. Then* $(\mathcal{C}, \mathcal{D})$ *has the extension property.*

Proof. Results of Dixmier and Gonshor [8, 16] show that $\mathcal{D}$ is injective if and only if $\hat{\mathcal{D}}$ is a compact extremally disconnected space. Now combine Theorems 2.8 with the equivalence of (a) and (b) in Theorem 2.9. $\square$

Example 2.11. Suppose that $\mathcal{M}$ is a von Neumann algebra and $\mathcal{D} \subseteq \mathcal{M}$ is a MASA. Let $\mathcal{C}$ be the norm-closure of $\mathcal{N}(\mathcal{M}, \mathcal{D})$. Then $(\mathcal{C}, \mathcal{D})$ has the extension property. Note that in particular, when $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$ in the sense of Feldman and Moore [13], then $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal.

Remark 2.12. The regularity hypothesis in Theorem 2.10 cannot be removed. Indeed, [1, Corollary 4.7] shows that when $\mathcal{C}$ is the hyperfinite II_1 factor, and $\mathcal{D} \subseteq \mathcal{C}$ is any MASA, then $(\mathcal{C}, \mathcal{D})$ fails to have the extension property.

2.3. **$\mathcal{D}$-modular states and ideals of $(\mathcal{C}, \mathcal{D})$.** In this final subsection, we make a simple observation: the action $\mathcal{N}(\mathcal{C}, \mathcal{D}) \ni v \mapsto \beta_v \in \operatorname{Inv}_{\mathcal{O}}\hat{\mathcal{D}}$ may be regarded as an action on certain states of $\mathcal{C}$.

Definition 2.13. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion. A state ρ on $\mathcal{C}$ is $\mathcal{D}$*-modular* if for every $x \in \mathcal{C}$ and $d \in \mathcal{D}$,

$$\rho(dx) = \rho(d)\rho(x) = \rho(xd).$$

We let $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ be the collection of all $\mathcal{D}$-modular states on $\mathcal{C}$; equip $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ with the relative weak-$*$ topology. Then $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ is closed and hence is compact. Using the Cauchy-Schwartz inequality, it is easy to see that

$$\operatorname{Mod}(\mathcal{C},\mathcal{D}) = \{\rho \in \mathcal{S}(\mathcal{C}) : \rho|_{\mathcal{D}} \in \hat{\mathcal{D}}\}.$$

For $\sigma \in \hat{\mathcal{D}}$, let $\operatorname{Mod}(\mathcal{C},\mathcal{D},\sigma)$ be the set of all state extensions of σ, so

$$\operatorname{Mod}(\mathcal{C},\mathcal{D},\sigma) := \{\rho \in \mathcal{S}(\mathcal{C}) : \rho|_{\mathcal{D}} = \sigma\}.$$

The following simple observation will be useful during the sequel.

Lemma 2.14. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion, and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$. If $\sigma \in \operatorname{dom}(\beta_v)$ and $\beta_v(\sigma) \neq \sigma$, then $\rho(v) = 0$ for every $\rho \in Mod(\mathcal{C},\mathcal{D},\sigma)$.*

Proof. Let $d \in \mathcal{D}$ satisfy $\beta_v(\sigma)(d) = 0$ and $\sigma(d) = 1$. Then for $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D},\sigma)$, we have

$$\rho(v) = \frac{\rho(vv^*v)}{\rho(v^*v)} = \frac{\rho(dvv^*v)}{\sigma(v^*v)} = \frac{\rho(vv^*dv)}{\sigma(v^*v)} = \rho(v)\beta_v(\sigma)(d) = 0.$$

$\square$

When $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ satisfies $\rho(v^*v) \neq 0$, the state $\tilde{\beta}_v(\rho)$ on $\mathcal{C}$ given by

$$\tilde{\beta}_v(\rho)(x) := \frac{\rho(v^*xv)}{\rho(v^*v)}$$

again belongs to $\operatorname{Mod}(\mathcal{C},\mathcal{D})$. When there is no danger of confusion, we sometimes simplify notation and write $\beta_v(\rho)$ instead of $\tilde{\beta}_v(\rho)$. Thus $\mathcal{N}(\mathcal{C},\mathcal{D})$ also acts on $\operatorname{Mod}(\mathcal{C},\mathcal{D})$, and for every $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D})$, we have $\tilde{\beta}_v(\rho)|_{\mathcal{D}} = \beta_v(\rho|_{\mathcal{D}})$.

Definition 2.15. A subset $F \subseteq \operatorname{Mod}(\mathcal{C},\mathcal{D})$ is $\mathcal{N}(\mathcal{C},\mathcal{D})$-*invariant* if for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $\rho \in F$ with $\rho(v^*v) \neq 0$, we have $\tilde{\beta}_v(\rho) \in F$.

We record the following fact for use in the sequel.

Proposition 2.16. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion and suppose that $F \subseteq Mod(\mathcal{C},\mathcal{D})$ is $\mathcal{N}(\mathcal{C},\mathcal{D})$-invariant. Then the set*

$$\mathcal{K}_F := \{x \in \mathcal{C} : \rho(x^*x) = 0 \textit{ for all } \rho \in F\}$$

is a closed, two-sided ideal in $\mathcal{C}$. Moreover, if $\{\rho|_{\mathcal{D}} : \rho \in F\}$ is weak-$*$ dense in $\hat{\mathcal{D}}$, then $\mathcal{K}_F \cap \mathcal{D} = (0)$.

Proof. As $\mathcal{K}_F$ is the intersection of closed left-ideals, it remains only to prove that $\mathcal{K}_F$ is a right ideal. By regularity, it suffices to prove that if $x \in \mathcal{K}_F$ and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, then $xv \in \mathcal{K}_F$. Let $\rho \in F$. If $\rho(v^*v) \neq 0$, then by hypothesis, we obtain $\rho(v^*x^*xv) = \beta_v(\rho)(x^*x)\rho(v^*v) = 0$. On the other hand, if $\rho(v^*v) = 0$, then $\rho(v^*x^*xv) \leq \|x^*x\|\,\rho(v^*v) = 0$. In either case, we find $\rho(v^*x^*xv) = 0$. As this holds for every $\rho \in \mathcal{F}$, we find $xv \in \mathcal{K}_F$, as desired. The final statement is obvious. $\square$

3. Pseudo-Conditional Expectations for Regular MASA Inclusions

As noted in Example 1.2, there exist regular MASA inclusions $(\mathcal{C},\mathcal{D})$ for which no conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$ exists. The purpose of this section is to show that nevertheless, there is always a unique pseudo-expectation for a regular MASA inclusion.

Given a normalizer v, our first task is to connect the dynamics of β_v with the ideal structure of $\mathcal{D}$.

Lemma 3.1. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion and suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$. If $d \in \mathcal{D}$ and $\operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ$, then $vd = dv$. Moreover, if $(\mathcal{C},\mathcal{D})$ is a MASA inclusion, then $vd = dv \in \mathcal{D}$.*

Proof. Note that v^*dv and v^*vd both belong to $\mathcal{D}$. We first show that for every $\rho \in \hat{\mathcal{D}}$,

$$\rho(v^*dv) = \rho(v^*vd). \tag{5}$$

Let $\rho \in \hat{\mathcal{D}}$. There are three cases. First suppose $\rho(v^*v) = 0$. Then as $\rho(v^*v) = \|v\|_\rho^2$, the Cauchy-Schwartz inequality gives,

$$|\rho(v^*dv)| = |\langle dv, v\rangle_\rho| \leq \|dv\|_\rho \|v\|_\rho = 0,$$

so (5) holds when $\rho(v^*v) = 0$.

Suppose next that $\rho(v^*v) > 0$ and $\beta_v(\rho)(d) \neq 0$. Then $\beta_v(\rho) \in \operatorname{supp}(\hat{d})$, so $\beta_v(\rho) \in \operatorname{fix}\beta_v = \operatorname{fix}\beta_{v^*}$. Thus, we get $\beta_v(\rho) = \beta_{v^*}(\beta_v(\rho)) = \rho$, and hence $\rho(v^*dv) = \rho(v^*v)\rho(d) = \rho(v^*vd)$.

Finally suppose that $\rho(v^*v) > 0$ and $\beta_v(\rho)(d) = 0$. Then $\rho(v^*dv) = 0$. We shall show that $\rho(d) = 0$. If not, the hypothesis on d shows that $\rho \in \operatorname{fix}\beta_v$. Hence, $0 \neq \rho(d) = \beta_v(\rho)(d) = \frac{\rho(v^*dv)}{\rho(v^*v)} = 0$, which is absurd. So $\rho(d) = 0$, and (5) holds in this case also. Thus we have established (5) in all cases.

Thus $v^*dv = v^*vd$. So for every $n \in \mathbb{N}$,

$$0 = v^*dv - v^*vd = v^*(dv - vd) = vv^*(dv - vd) = (vv^*)^n(dv - vd).$$

It follows that for every polynomial p with $p(0) = 0$, we have, $p(vv^*)(dv - vd) = 0$. Therefore, for every $n \in \mathbb{N}$,

$$0 = (vv^*)^{1/n}(dv - vd) = d(vv^*)^{1/n}v - (vv^*)^{1/n}vd.$$

Since $\lim_{n\to\infty}(vv^*)^{1/n}v = v$, we have $vd = dv$.

Now suppose that $(\mathcal{C}, \mathcal{D})$ is a MASA inclusion. For $a \in \mathcal{D}$, we have $\operatorname{supp}(\widehat{da}) \subseteq \operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ$, so we have $v(da) = (da)v$. Since $vd = dv$, we get $(vd)a = a(vd)$. Since $\mathcal{D}$ is a MASA, $vd \in \mathcal{D}$ and the proof is complete. ☐

Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Observe that if $d \in \mathcal{D}$ and $\operatorname{supp}(\hat{d}) \subseteq \operatorname{fix}\beta_v$, then we actually have $\operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ$. Thus

$$\{d \in \mathcal{D} : \operatorname{supp}(\hat{d}) \subseteq \operatorname{fix}\beta_v\} = \{d \in \mathcal{D} : \operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ\}$$

is a closed ideal of $\mathcal{D}$ isomorphic to $C_0((\operatorname{fix}\beta_v)^\circ)$. By the Fuglede-Putnam-Rosenblum commutation theorem,

$$\{d \in \mathcal{D} : dv = vd\} = \{d \in \mathcal{D} : dv^* = v^*d\}, \tag{6}$$

and it follows that $\{d \in \mathcal{D} : dv = vd \in \mathcal{D}\}$ is a closed ideal of $\mathcal{D}$. The next proposition shows how the set $(\operatorname{fix}\beta_v)^\circ$ may be described algebraically.

Proposition 3.2. *Let $(\mathcal{C}, \mathcal{D})$ be a MASA inclusion. If $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, then*

$$\{d \in \mathcal{D} : \operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ\} = \overline{\mathcal{D}v^*v} \cap \{d \in \mathcal{D} : dv = vd \in \mathcal{D}\}.$$

Proof. Notice that $\overline{\mathcal{D}v^*v} = \{d \in \mathcal{D} : \operatorname{supp}(\hat{d}) \subseteq \operatorname{supp}(\widehat{v^*v})\}$. Since $(\operatorname{fix}\beta_v)^\circ \subseteq \operatorname{supp}(\widehat{v^*v})$, Lemma 3.1 shows that

$$\{d \in \mathcal{D} : \operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix}\beta_v)^\circ\} \subseteq \overline{\mathcal{D}v^*v} \cap \{d \in \mathcal{D} : dv = vd \in \mathcal{D}\}.$$

Now suppose that $d \in \overline{\mathcal{D}v^*v} \cap \{h \in \mathcal{D} : hv = vh \in \mathcal{D}\}$ and $d \neq 0$. Let $\rho_0 \in \operatorname{supp}(\hat{d})$ and set $r := |\rho_0(d)|$. Then $r > 0$. Put $G = \{\rho \in \hat{\mathcal{D}} : |\rho(d)| > r/2\}$. We show that $G \subseteq \operatorname{fix}\beta_v$. Fix $\rho \in G$. Since $d \in \overline{\mathcal{D}v^*v}$, we have $\operatorname{supp}(\hat{d}) \subseteq \operatorname{supp}(\widehat{(v^*v)})$, so $\rho(v^*v) \neq 0$. Since d belongs to the ideal $\{f \in \mathcal{D} : fv = vf \in \mathcal{D}\}$, we find (using (6)) that for every $a \in \mathcal{D}$,

$$\beta_v(\rho)(a) = \frac{\rho(v^*av)}{\rho(v^*v)} = \frac{\rho(v^*avd)}{\rho(v^*v)\rho(d)} = \frac{\rho(v^*(ad)v)}{\rho(v^*v)\rho(d)} = \frac{\rho(adv^*v)}{\rho(v^*v)\rho(d)} = \rho(a).$$

It follows that $G \subseteq \operatorname{fix} \beta_v$. Since G is an open subset of $\hat{\mathcal{D}}$ with $\rho_0 \in G$, we have $\rho_0 \in (\operatorname{fix} \beta_v)^\circ$. So $\operatorname{supp}(\hat{d}) \subseteq (\operatorname{fix} \beta_v)^\circ$, as desired.

$\square$

We need some notation.

Notation. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion.

a) For a closed ideal $\mathfrak{J}$ in $\mathcal{D}$, we let
$$\mathfrak{J}^\perp := \{d \in \mathcal{D} : dg = 0 \text{ for all } g \in \mathfrak{J}\}$$
denote the complement of $\mathfrak{J}$ in the lattice of closed ideals of $\mathcal{D}$.

b) Given any two closed ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$ in $\mathcal{D}$, $\mathfrak{J}_1 \vee \mathfrak{J}_2$ denotes the closed ideal generated by $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

c) For $S \subseteq \mathcal{D}$, $\langle S \rangle_{\mathcal{D}}$ denotes the closed two-sided ideal of $\mathcal{D}$ generated by the set S. When the context is clear, we drop the subscript and simply use $\langle S \rangle$.

d) For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, let

$$J_v := \{d \in \mathcal{D} : vd = dv \in \mathcal{D}\} \cap \langle v^*v \rangle \quad \text{and} \quad K_v := \langle v^*v \rangle^\perp \vee \langle \{v^*hv - hv^*v : h \in \mathcal{D}\} \rangle . \tag{7}$$

Recall that an ideal $\mathfrak{J}$ in a C^*-algebra $\mathcal{C}$ is an *essential ideal* if $\mathfrak{J} \cap L \neq (0)$ for every closed two-sided ideal $(0) \neq L \subseteq \mathcal{C}$.

We will show $J_v \vee K_v$ is an essential ideal in $\mathcal{D}$. It is easy to see that $J_v \subseteq K_v^\perp$. If equality holds, then the fact that $J_v \vee K_v$ is an essential ideal follows readily. However, we have not found a simple proof of this fact, so we proceed along different lines.

Proposition 3.3. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Then $J_v \vee K_v$ is an essential ideal in $\mathcal{D}$.*

Proof. We shall show that $J_v \vee K_v$ is essential by showing that $(J_v \vee K_v)^\perp = (0)$. Suppose that $d \in \mathcal{D}$ and $d(J_v \vee K_v) = 0$.

First we show
$$\operatorname{supp} \hat{d} \subseteq \overline{\operatorname{supp} \widehat{v^*v}}. \tag{8}$$

Indeed, if $\rho \in \hat{\mathcal{D}}$ and $\rho \notin \overline{\operatorname{supp} \widehat{v^*v}}$, then we may find $h \in \mathcal{D}$ such that $\hat{h}(\rho) = 1$ and $\hat{h}(\sigma) = 0$ for every $\sigma \in \overline{\operatorname{supp} \widehat{v^*v}}$. Then $h \in \langle v^*v \rangle^\perp \subseteq K_v$, so $dh = 0$. As $\rho(h) = 1$, this shows that $\rho \notin \operatorname{supp} \hat{d}$. Thus (8) holds.

Next, we claim that
$$\operatorname{supp} \hat{d} \cap \operatorname{supp} \widehat{v^*v} \subseteq (\operatorname{fix} \beta_v)^\circ. \tag{9}$$

Let $\rho \in \operatorname{supp} \hat{d} \cap \operatorname{supp} \widehat{v^*v}$. For every $h \in \mathcal{D}$, we have $v^*hv - v^*vh \in K_v$. Since $\rho(d) \neq 0$ and $d \in K_v^\perp$, this gives $\rho(v^*hv) = \rho(v^*v)\rho(h)$ for every $h \in \mathcal{D}$. Since $\rho(v^*v) \neq 0$ by hypothesis, we have $\rho \in \operatorname{fix} \beta_v$. As $\operatorname{supp} \hat{d} \cap \operatorname{supp} \widehat{v^*v}$ is an open subset of $\hat{\mathcal{D}}$, we obtain (9).

Suppose that $\rho \in (\operatorname{fix} \beta_v)^\circ$. By Proposition 3.2 there exists $h \in J_v$ so that $\rho(h) = 1$. Since $d \in J_v^\perp$, we obtain $\rho(d) = 0$. Hence
$$(\operatorname{fix} \beta_v)^\circ \cap \operatorname{supp} \hat{d} = \emptyset. \tag{10}$$

Combining (8), (9), and (10) we obtain,
$$\operatorname{supp} \hat{d} \subseteq \overline{\operatorname{supp} \widehat{v^*v}} \setminus \operatorname{supp} \widehat{v^*v}.$$

But $\overline{\operatorname{supp} \widehat{v^*v}} \setminus \operatorname{supp} \widehat{v^*v}$ has empty interior, so $\operatorname{supp} \hat{d} = \emptyset$. Therefore, $d = 0$ as desired. $\square$

The following is probably well known, but we include it for completeness. Recall that if $\mathcal{A}$ is a unital injective C^*-algebra, then any bounded increasing net x_λ of self-adjoint elements of $\mathcal{A}$ has a least upper bound in $\mathcal{A}$, which we denote by $\sup_{\mathcal{A}} x_\lambda$.

Lemma 3.4. *Let $\mathcal{D}$ be a unital, abelian C^*-algebra, let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$ and suppose that $J \subseteq \mathcal{D}$ is a closed ideal. Let J_1^+ be the positive part of the unit ball of J and regard J_1^+ as a net indexed by itself. Put $P := \sup_{I(\mathcal{D})} \iota(J_1^+)$. Then P is an projection in $I(\mathcal{D})$.*

If in addition, J is an essential ideal of $\mathcal{D}$, the following hold:

i) *if $a, b \in I(\mathcal{D})$ and $a\iota(h) = b\iota(h)$ for every $h \in J$, then $a = b$;*
ii) *$P = I$.*

Proof. The mapping $x \in J_1^+ \mapsto x^{1/2} \in J_1^+$ is an order isomorphism of J_1^+, so $P = \sup_{I(\mathcal{D})}\{\iota(x^{1/2}) : x \in J_1^+\}$. So P^2 is also an upper bound for $\iota(J_1^+)$. Hence $P \leq P^2$. But as $\|P\| \leq 1$, we have $P^2 \leq P$. So P is a projection.

Now assume J is an essential ideal, and suppose $a, b \in I(\mathcal{D})$ satisfy $(a-b)\iota(h) = 0$ for every $h \in J$. By Hamana-regularity of $\iota(\mathcal{D})$ in $I(\mathcal{D})$, we have $|a-b| = \sup_{I(\mathcal{D})}\{d \in \mathcal{D} : 0 \leq \iota(d) \leq |a-b|\}$. But $K := \{d \in \mathcal{D} : \iota(d) \in \langle |a-b| \rangle_{I(\mathcal{D})}\}$ is a closed ideal of $\mathcal{D}$ with $K \subseteq J^\perp$. Hence $K = 0$, so $|a-b| = 0$.

Notice that if $d \in J_1^+$, then $\iota(d)P = \iota(d)$. It follows that for every $d \in J$, $\iota(d)P = \iota(d)$. By part (i), $P = I$ when J is an essential ideal. □

The following extension of Definition 2.13 will be useful.

Definition 3.5. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and $\mathcal{B}$ an algebra.

(1) A linear map $\Delta : \mathcal{C} \to \mathcal{B}$ is $\mathcal{D}$-*modular* (or more simply *modular*) if for every $x \in \mathcal{C}$ and $d \in \mathcal{D}$,
$$\Delta(xd) = \Delta(x)\Delta(d) \quad \text{and} \quad \Delta(dx) = \Delta(d)\Delta(x).$$
(2) A homomorphism $\theta : \mathcal{D} \to \mathcal{B}$ is $\mathcal{D}$-*thick in* $\mathcal{B}$ if for every non-zero element $b \in \mathcal{B}$, the ideal $\{d \in \mathcal{D} : b\theta(d) = 0\}^\perp$ is a non-zero ideal of $\mathcal{D}$.

When $\mathcal{B}$ is abelian, notice that the restriction of a $\mathcal{D}$-modular map to $\mathcal{D}$ is a homomorphism. The next lemma gives an example which will be used in the proof of Theorem 3.10.

Lemma 3.6. *Let $\mathcal{D}$ be an abelian C^*-algebra and let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Then ι is $\mathcal{D}$-thick in $I(\mathcal{D})$.*

Proof. Suppose $b \in I(\mathcal{D})$ is non-zero. The Hamana regularity of $I(\mathcal{D})$ ensures that there exists a non-zero $h \in \mathcal{D}$ such that $0 \leq \iota(h) \leq b^*b$. If $d \in \mathcal{D}$ satisfies $\iota(d)b = 0$, then $\operatorname{supp}(\widehat{\iota(d)}) \cap \operatorname{supp}(\hat{b}) = \emptyset$. Since $\operatorname{supp}(\widehat{\iota(h)}) \subseteq \operatorname{supp}(\hat{b})$, we get $\iota(dh) = 0$, whence $h \in \{d \in \mathcal{D} : \iota(d)b = 0\}^\perp$. □

Our interest in $\mathcal{D}$-thick homomorphisms will be with the restrictions of $\mathcal{D}$-modular maps to $\mathcal{D}$. The following lemma will be useful.

Lemma 3.7. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, let $\mathcal{B}$ be a unital abelian Banach algebra and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. For $i = 1, 2$, suppose $\Delta_i : \mathcal{C} \to \mathcal{B}$ are bounded $\mathcal{D}$-modular maps such that $\Delta_1|_{\mathcal{D}} = \Delta_2|_{\mathcal{D}}$ and set $\iota := \Delta_i|_{\mathcal{D}}$. Then for every $h \in J_v \vee K_v$,*
$$\Delta_1(vh) = \Delta_2(vh). \tag{11}$$
In fact,

a) *for every $h \in K_v$, $\Delta_1(vh) = 0 = \Delta_2(vh)$;*
b) *for every $h \in J_v$, $\Delta_1(vh) = \iota(vh) = \Delta_2(vh)$.*

Moreover, if ι is also $\mathcal{D}$-thick in $\mathcal{B}$, then $\Delta_1 = \Delta_2$.

Proof. For (a), we consider two cases. First, if $h \in \langle v^*v\rangle^\perp$, then for $i = 1, 2$,

$$\Delta_i(vh) = \lim_{n\to\infty} \Delta_i(v(v^*v)^{1/n}h) = 0.$$

Second, suppose that $h \in \{v^*dv - v^*vd : d \in \mathcal{D}\}$. Then for some $d \in \mathcal{D}$,

$$\begin{aligned}\Delta_i(vh) &= \Delta_i(v(v^*dv - v^*vd)) = \Delta_i(vv^*dv) - \Delta_i(vv^*vd)\\ &= \iota(dv^*v)\Delta_i(v) - \Delta_i(vv^*v)\iota(d) = \iota(d)\Delta_i(v)\iota(v^*v) - \iota(d)\Delta_i(vv^*v) = 0.\end{aligned}$$

Thus $\Delta_i(vh) = 0$ for all h in a generating set for K_v. Since Δ_i are $\mathcal{D}$-modular and bounded, we obtain (a).

Next, suppose that $h \in J_v$. Then since $vh \in \mathcal{D}$,

$$\Delta_1(vh) = \iota(vh) = \Delta_2(vh). \tag{12}$$

This gives (b).

Parts (a) and (b) imply that $\Delta_1(vh) = \Delta_2(vh)$ for all h in a generating set for $J_v \vee K_v$, so boundedness and $\mathcal{D}$-modularity of Δ_i yields (11).

Now suppose that ι is $\mathcal{D}$-thick in $\mathcal{B}$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Let $b = \Delta_1(v) - \Delta_2(v)$ and choose any $h \in \{d \in \mathcal{D} : b\iota(d) = 0\}^\perp \cap (J_v \vee K_v)$. Since $h \in J_v \vee K_v$, we have $b\iota(h) = \Delta_1(v)\iota(h) - \Delta_2(v)\iota(h) = \Delta_1(vh) - \Delta_2(vh) = 0$. Since $h \in \{d \in \mathcal{D} : b\iota(d) = 0\}^\perp$, we get $h^2 = 0$. As $\mathcal{D}$ is abelian, $h = 0$. This shows that $\{d \in \mathcal{D} : b\iota(d) = 0\}^\perp \cap (J_v \vee K_v) = (0)$. Since $J_v \vee K_v$ is an essential ideal, $\{d \in \mathcal{D} : b\iota(d) = 0\}^\perp = (0)$. Since ι is $\mathcal{D}$-thick in $\mathcal{B}$, we see that $b = 0$. Hence $\Delta_1(v) = \Delta_2(v)$.

Since this holds for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, regularity of $(\mathcal{C}, \mathcal{D})$ yields $\Delta_1 = \Delta_2$.

□

Lemma 3.7 has an interesting consequence for uniqueness of extensions of pure states on $\mathcal{D}$ to $\mathcal{C}$, which we now present. This result generalizes a result found in [31], however, the proof is rather different. Notice that Theorem 3.8 holds when $\mathcal{C}$ is separable or when there is a countable subset $X \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ is the C^*-algebra generated by $\mathcal{D}$ and X. We shall use Theorem 3.8 in the proof of Theorem 9.2.

Theorem 3.8. *Suppose $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion and that $N \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a countable set such that the norm-closed $\mathcal{D}$-bimodule generated by N is $\mathcal{C}$. Let*

$$\mathfrak{U} := \{\sigma \in \hat{\mathcal{D}} : \sigma \text{ has a unique state extension to } \mathcal{C}\}.$$

Then $\mathfrak{U}$ is dense in $\hat{\mathcal{D}}$.

Proof. For each $v \in N$, let $G_v := \{\sigma \in \hat{\mathcal{D}} : \sigma|_{J_v \vee K_v} \neq 0\}$. Clearly G_v is open in $\hat{\mathcal{D}}$ and since $J_v \vee K_v$ is an essential ideal in $\mathcal{D}$, G_v is dense in $\hat{\mathcal{D}}$. Baire's theorem shows that

$$P := \bigcap_{v \in N} G_v$$

is dense in $\hat{\mathcal{D}}$.

Let $\sigma \in P$ and suppose for $i = 1, 2$, ρ_i are states on $\mathcal{C}$ such that $\rho_i|_{\mathcal{D}} = \sigma$. The Cauchy-Schwartz inequality shows that $\rho_i : \mathcal{C} \to \mathbb{C}$ are $\mathcal{D}$-modular maps.

Fix $v \in N$. Since $\sigma \in G_v$, we may find $h \in J_v \vee K_v$ such that $\sigma(h) = 1$. By Lemma 3.7 we have

$$\rho_1(v) = \rho_1(v)\sigma(h) = \rho_1(vh) = \rho_2(vh) = \rho_2(v)\sigma(h) = \rho_2(v).$$

Since N generates $\mathcal{C}$ as a $\mathcal{D}$-bimodule and ρ_i are $\mathcal{D}$-modular, we see that $\rho_1 = \rho_2$. Hence $P \subseteq \mathfrak{U}$, and the proof is complete. □

We now show that any regular MASA inclusion has a unique completely positive mapping $E : \mathcal{C} \to I(\mathcal{D})$ which extends the inclusion mapping of $\mathcal{D}$ into $I(\mathcal{D})$.

Lemma 3.9. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion and $(I(\mathcal{D}),\iota)$ an injective envelope of $\mathcal{D}$. Let $E:\mathcal{C}\to I(\mathcal{D})$ be a pseudo-expectation for ι. Then E is $\mathcal{D}$-modular.*

Proof. Let $\rho\in\widehat{I(\mathcal{D})}$ and put $\sigma=\rho\circ E$. Then $\sigma|_{\mathcal{D}}\in\hat{\mathcal{D}}$, so the Cauchy-Schwartz inequality implies that for every $x\in\mathcal{C}$ and $d\in\mathcal{D}$, $\sigma(xd)=\sigma(x)\sigma(d)$. Hence,

$$\rho(E(xd))=\sigma(xd)=\sigma(x)\sigma(d)=\rho(E(x))\rho(E(d))=\rho(E(x)E(d)).$$

As this holds for every $\rho\in\widehat{I(\mathcal{D})}$, we obtain $E(xd)=E(x)E(d)$. The proof that $E(dx)=E(d)E(x)$ is similar. $\square$

Theorem 3.10. *Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion, and let $(I(\mathcal{D}),\iota)$ be an injective envelope of $\mathcal{D}$. Then there exists a unique pseudo-expectation $E:\mathcal{C}\to I(\mathcal{D})$ for ι. Furthermore, suppose $v\in\mathcal{N}(\mathcal{C},\mathcal{D})$. Then*

a) *$E(vh)=\iota(vh)$ for every $h\in J_v$;*
b) *$E(vh)=0$ for every $h\in K_v$;*
c) *$|E(v)|^2=\iota(v^*v)P$, where $P:=\sup_{I(\mathcal{D})}(\iota((J_v)_1^+))$.*

Proof. The injectivity of $I(\mathcal{D})$ guarantees the existence of a completely positive unital map E : $\mathcal{C}\to I(\mathcal{D})$ such that for every $d\in\mathcal{D}$,

$$E(d)=\iota(d). \tag{13}$$

Lemma 3.7 and Lemma 3.6 imply that E is unique.

Now suppose that $v\in\mathcal{N}(\mathcal{C},\mathcal{D})$. Since $E|_{\mathcal{D}}=\iota$, parts (a) and (b) follow from Lemma 3.7.

We turn now to (c). Observe that $K_v\subseteq J_v^\perp$. Indeed $J_v\subseteq\langle v^*v\rangle$, so if $h\in\langle v^*v\rangle^\perp$ then $hJ_v=0$. On the other hand, if $d\in D$ and $h=v^*dv-v^*vd$, then for $g\in J_v$, we have $hg=(v^*dv-v^*vd)g=v^*dgv-v^*vdg=0$ because $dg\in J_v$. Thus $hJ_v=0$ for every h in a generating set for K_v, so $K_v\subseteq J_v^\perp$.

Now let $h\in K_v$. By [20, Corollary 4.10], we have

$$\iota(h)^*P\iota(h)=\iota(h)^*\left[\sup_{I(\mathcal{D})}\iota((J_v)_1^+)\right]\iota(h)=\sup_{I(\mathcal{D})}\iota(h^*(J_v)_1^+h)=0.$$

Since P is a projection, it follows that $\iota(h)P=0$. Hence $\iota(v^*v)P\iota(h^*h)=0$. Next, by part (b), we have for every $h\in K_v$, $|E(v)|^2\iota(h^*h)=\iota(h^*)E(v)^*E(v)\iota(h)=\iota(h^*)E(v)^*E(vh)=0$. Therefore, for $h\in K_v$,

$$|E(v)|^2\iota(h^*h)=\iota(v^*v)P\iota(h^*h)=0. \tag{14}$$

Since $E(vh)=\iota(vh)$ for every $h\in J_v$, we see that for $h\in J_v$,

$$E(v^*)E(v)\iota(h^*h)=E((vh)^*)E(vh)=\iota(h^*v^*)\iota(vh)=\iota(v^*v)\iota(h^*h)=\iota(v^*v)P\iota(h^*h). \tag{15}$$

Combining (14) and (15), we see that for every $h\in J_v\vee K_v$,

$$E(v)^*E(v)\iota(h^*h)=\iota(v^*v)P\iota(h^*h).$$

Then $E(v)^*E(v)=\iota(v^*v)P$ by Lemma 3.4, so we have (c). $\square$

The following "dual" to Theorem 3.10 is now easily established. Notice that in the context of Theorem 3.10, when $\rho\in\widehat{I(\mathcal{D})}$, $E^\#(\rho)=\rho\circ E\in\text{Mod}(\mathcal{C},\mathcal{D})$.

Theorem 3.11. *Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion, let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$, and let E be the pseudo-expectation for ι. The map $E^\#:\widehat{I(\mathcal{D})}\to Mod(\mathcal{C},\mathcal{D})$ is the unique continuous map of $\widehat{I(\mathcal{D})}$ into $Mod(\mathcal{C},\mathcal{D})$ such that for every $\rho\in\widehat{I(\mathcal{D})}$, $E^\#(\rho)|_{\mathcal{D}}=\rho\circ\iota$.*

Proof. Clearly $E^{\#}$ has the stated property, so we need only prove uniqueness.

Suppose that ℓ is a continuous map of $\widehat{I(\mathcal{D})}$ into $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ such that for every $\rho \in \widehat{I(\mathcal{D})}$, $\ell(\rho)|_{\mathcal{D}} = \rho \circ \iota$. For $x \in \mathcal{C}$, define a function $\phi_x : \widehat{I(\mathcal{D})} \to \mathbb{C}$ by $\phi_x(\rho) = \ell(\rho)(x)$. Since ℓ is continuous, ϕ_x is continuous. Hence there exists a unique element $E_1(x) \in I(\mathcal{D})$ such that ϕ_x is the Gelfand transform of $E_1(x)$. Using the fact that $\mathrm{Mod}(\mathcal{C},\mathcal{D}) \subseteq \mathcal{S}(\mathcal{C})$, we find E_1 is linear, bounded, unital and positive. Since $I(\mathcal{D})$ is abelian, E_1 is completely positive. For $d \in \mathcal{D}$ we have $\rho(E_1(d)) = \ell(\rho)(d) = \rho(\iota(d))$. Therefore, $E_1|_{\mathcal{D}} = \iota$, so E_1 is a pseudo-expectation for ι. By Theorem 3.10, $E_1 = E$, hence $\ell = E^{\#}$.

$\square$

Definition 3.12. Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion, let $(I(\mathcal{D}),\iota)$ be a C^*-envelope for $\mathcal{D}$, and let E be the (unique) pseudo-expectation for ι. Define

$$\mathfrak{S}_s(\mathcal{C},\mathcal{D}) := \{\rho \circ E : \rho \in \widehat{I(\mathcal{D})}\}.$$

We shall call states belonging to $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ *strongly compatible* states. Clearly, $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is a closed subset of $\mathrm{Mod}(\mathcal{C},\mathcal{D})$. Observe that $\hat{\mathcal{D}} = \{\tau|_{\mathcal{D}} : \tau \in \mathfrak{S}_s(\mathcal{C},\mathcal{D})\}$; this is because $\hat{\mathcal{D}} = \{\rho\circ\iota : \rho \in \widehat{I(\mathcal{D})}\}$.

Let $r : \mathrm{Mod}(\mathcal{C},\mathcal{D}) \to \hat{\mathcal{D}}$ be the restriction map, $r(\rho) = \rho|_{\mathcal{D}}$. We now show that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is the unique minimal closed subset of $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ for which r is onto. In a certain sense, this allows us to determine $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ without the use of the pseudo-expectation.

Theorem 3.13. *Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion and suppose $F \subseteq Mod(\mathcal{C},\mathcal{D})$ is closed and $r(F) = \hat{\mathcal{D}}$. Then $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) \subseteq F$.*

Suppose further that there exists a countable subset $N \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ such that the norm-closed $\mathcal{D}$-bimodule generated by N is $\mathcal{C}$ and set

$$\mathfrak{U}(\mathcal{C},\mathcal{D}) := \{\rho \in Mod(\mathcal{C},\mathcal{D}) : \rho|_{\mathcal{D}} \textit{ has a unique state extension to } \mathcal{C}\}.$$

Then

$$\mathfrak{S}_s(\mathcal{C},\mathcal{D}) = \overline{\mathfrak{U}(\mathcal{C},\mathcal{D})}^{w\text{-}*}.$$

Proof. Since F is closed and $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ is compact, F is compact and Hausdorff. As $\widehat{I(\mathcal{D})}$ is projective (in the category of compact Hausdorff spaces and continuous maps) and r maps F onto $\hat{\mathcal{D}}$, there exists a continuous map $\ell : \widehat{I(\mathcal{D})} \to F$ such that $\iota^{\#} = r \circ \ell$. Let $\epsilon : F \to \mathrm{Mod}(\mathcal{C},\mathcal{D})$ be the inclusion map. Then $\ell' := \epsilon \circ \ell$ is a continuous map of $\widehat{I(\mathcal{D})}$ into $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ such that $r \circ \ell' = \iota^{\#}$. Theorem 3.11 shows that $\ell' = E^{\#}$. Therefore, the range of $E^{\#}$ is contained in F, that is, $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) \subseteq F$.

Suppose now that there is a countable subset $N \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ which generates $\mathcal{C}$ as a $\mathcal{D}$-bimodule. Theorem 3.8 implies that $\hat{\mathcal{D}} = r(\overline{\mathfrak{U}(\mathcal{C},\mathcal{D})})$, so we have $\mathfrak{S}_s(\mathcal{C},\mathcal{D}) \subseteq \overline{\mathfrak{U}(\mathcal{C},\mathcal{D})}$. To complete the proof, observe that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is closed and $\mathfrak{U}(\mathcal{C},\mathcal{D}) \subseteq \mathfrak{S}_s(\mathcal{C},\mathcal{D})$. $\square$

The following result shows that, in the terminology of Section 4, each element of $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is a compatible state.

Proposition 3.14. *Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion and let $\sigma \in \mathfrak{S}_s(\mathcal{C},\mathcal{D})$. Then for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$,*

$$|\sigma(v)|^2 \in \{0, \sigma(v^*v)\}.$$

Proof. Let $\rho \in \widehat{I(\mathcal{D})}$ be such that $\sigma = \rho \circ E$ and suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ is such that $\sigma(v) \neq 0$. Then $0 \neq |\rho(E(v))|^2$, so by part (c) of Theorem 3.10, $\rho(P) \neq 0$. By Lemma 3.4, P is a projection, so $\rho(P) = 1$. Thus, $|\rho(E(v))|^2 = \rho(\iota(v^*v)) = \rho(E(v^*v)) = \sigma(v^*v)$. $\square$

Our next goal is Theorem 3.21 below, which shows that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is an $\mathcal{N}(\mathcal{C},\mathcal{D})$-invariant subset of $\operatorname{Mod}(\mathcal{C},\mathcal{D})$. In the case that $\mathcal{C}$ is countably generated as a $\mathcal{D}$-bimodule, this follows from the second part of Theorem 3.13, but we have not found a proof in the general case using Theorem 3.13. The fact that the left kernel of the pseudo-expectation is an ideal will follow easily from Theorem 3.21, see Theorem 3.22.

Our route to Theorem 3.21 involves a study of the Gelfand support of $E(v)$ for $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$; we also obtain a formula for the Gelfand transform of $E(v)$. We begin with three lemmas on properties of projective covers.

Lemma 3.15. *Let X be a compact Hausdorff space and let (P, f) be a projective cover for X.*

a) *If $G \subseteq X$ is an open set, then $(\overline{f^{-1}(G)}, f|_{\overline{f^{-1}(G)}})$ is a projective cover for $\overline{G}$.*

b) *If $Q \subseteq P$ is clopen, then $f(Q)^\circ$ is dense in $f(Q)$ and $Q = \overline{f^{-1}(f(Q)^\circ)}$.*

Proof. Before beginning the proof, observe that if $\iota : C(X) \to C(P)$ is given by $\iota(d) = d \circ f$, then $(C(P), \iota)$ is an injective envelope for $C(X)$.

a) Let $Y := \overline{f^{-1}(G)}$. Then Y is compact, so $f(Y)$ is a closed subset of X which contains G. Hence $\overline{G} \subseteq f(Y)$. On the other hand, $f^{-1}(\overline{G})$ is a closed subset of P containing $f^{-1}(G)$, so $Y \subseteq f^{-1}(\overline{G})$. Hence $f(Y) \subseteq \overline{G}$. Therefore (Y, f) is a cover for $\overline{G}$.

Since P is projective, it is Stonean, and hence Y is clopen in P. Since clopen subsets of projective spaces are projective, Y is projective. The proof of part (a) will be complete once we show that $(Y, f|_Y)$ is a rigid cover for $\overline{G}$ (see [17, Theorem 2.16]).

So suppose that $h : Y \to Y$ is continuous and $f \circ h = f|_Y$. Define $\tilde{h} : P \to P$ by

$$\tilde{h}(t) = \begin{cases} h(t) & \text{if } t \in Y \\ t & \text{if } t \notin Y. \end{cases}$$

Since Y is clopen in P, $\tilde{h}$ is continuous. Moreover, $f \circ \tilde{h} = f$, so by the rigidity of (P, f), we see that $\tilde{h}$ is the identity map on P. Therefore h is the identity map on Y, which shows that $(Y, f|_Y)$ is a rigid cover.

b) The case when $Q = \emptyset$ is trivial, so we assume Q is non-empty. Let

$$\mathcal{M} := \{d \in C(X) : 0 \le \iota(d) \le \chi_Q\} \quad \text{and set} \quad G := \bigcup_{d \in \mathcal{M}} \operatorname{supp}(d).$$

Then G is non-empty because $\chi_Q \neq 0$ and $(C(P), \iota)$ is Hamana regular. Notice that $f^{-1}(G) \subseteq Q$: indeed, if $p \in f^{-1}(G)$, then there exists $d \in \mathcal{M}$ such that $f(p) \in \operatorname{supp}(d)$, so $0 < d(f(p)) \le \chi_Q(p)$, whence $p \in Q$.

As P is extremally disconnected, $W := \overline{f^{-1}(G)}$ is a clopen subset of P. We will show that $W = Q$. Clearly $W \subseteq Q$. If $Q \setminus W \neq \emptyset$, then $Q \setminus W$ is a clopen subset of P, so we may find a non-zero $d_1 \in C(X)$ with $0 \le \iota(d_1) \le \chi_{Q\setminus W} \le \chi_Q$. But then $d_1 \in \mathcal{M}$, so $\operatorname{supp}(\iota(d_1)) \subseteq f^{-1}(G) \subseteq W$, contradicting $0 \le \iota(d_1) \le \chi_{Q\setminus W}$. Hence $W = Q$.

By part (a), $(W, f|_W)$ is a cover for $\overline{G}$. Thus, $f(W) = f(Q) = \overline{G}$. As $f^{-1}(G) \subseteq Q$, we have $G \subseteq f(Q)^\circ$, so that $f(Q)^\circ$ is dense in $f(Q)$.

Finally, put $W_1 := \overline{f^{-1}(f(Q)^\circ)}$. Since $G \subseteq f(Q)^\circ$, we have $Q = W \subseteq W_1$. Part (a) again shows that $(W_1, f|_{W_1})$ is a projective cover for $\overline{f(Q)^\circ} = f(Q)$, so in particular, this cover is essential. The inclusion map α of W into W_1 satisfies $f(\alpha(W)) = f(Q)$. Because $(W_1, f|_{W_1})$ is an essential cover of $f(Q)$, we conclude α is onto. Thus $W = W_1$, and the proof of (b) is complete. $\square$

We leave the proof of the following lemma to the reader.

Lemma 3.16. *Suppose that for $i \in \{1,2\}$, X_i is a compact Hausdorff space and that $\phi : X_1 \to X_2$ is a homeomorphism. Let (C_i, f_i) be a projective cover for X_i. Then there exists a unique homeomorphism $\Phi : C_1 \to C_2$ such that $f_2 \circ \Phi = \phi \circ f_1$.*

Our final lemma on projective covers is a strengthening of Lemma 3.16: rather than extending a homeomorphism to the injective envelope, partial homeomorphisms are extended.

Lemma 3.17. *Let (P, f) be a projective cover for the compact Hausdorff space X and suppose $h \in \mathit{Inv}_{\mathcal{O}}(X)$ is a partial homeomorphism. Then there exists a unique partial homeomorphism $I(h) \in \mathit{Inv}_{\mathcal{O}}(P)$ such that:*

a) $\mathrm{dom}(I(h)) = f^{-1}(\mathrm{dom}(h))$, $\mathrm{range}(I(h)) = f^{-1}(\mathrm{range}(h))$ *and*
b) $h \circ f = f \circ I(h)$.

Proof. Let $H_1 = \mathrm{dom}(h)$ and $H_2 = \mathrm{range}(h)$. Set

$$\mathcal{T} := \{G \subseteq H : G \text{ is open in } X \text{ and } \overline{G} \subseteq H_1\}.$$

Let $G \in \mathcal{T}$. Lemma 3.15 shows that $(\overline{f^{-1}(G)}, f)$ and $(\overline{f^{-1}(h(G))}, f)$ are projective covers for $\overline{G}$ and $\overline{h(G)}$. By Lemma 3.16, there exists a unique homeomorphism $h_G : \overline{f^{-1}(G)} \to \overline{f^{-1}(h(G))}$ such that

$$h \circ f|_{\overline{f^{-1}(G)}} = f \circ h_G.$$

We now let $I(h)$ be the inductive limit of $\{h_G\}_{G\in\mathcal{T}}$. Here is an outline.

Since

$$\bigcup_{G\in\mathcal{T}} G = H_1 = \bigcup_{G\in\mathcal{T}} \overline{G} \quad \text{we have} \quad \bigcup_{G\in\mathcal{T}} f^{-1}(G) = f^{-1}(H_1) = \bigcup_{G\in\mathcal{T}} f^{-1}(\overline{G}).$$

Let $G_1, G_2 \in \mathcal{T}$ and suppose $G_1 \cap G_2 \neq \emptyset$. Put $Q := \overline{f^{-1}(G_1)} \cap \overline{f^{-1}(G_2)}$. Then Q is a clopen set in P, so by Lemma 3.15, $Q = \overline{f^{-1}(f(Q)^\circ)}$. Note that $f(Q)^\circ \in \mathcal{T}$. Hence

$$h \circ f|_Q = h \circ f|_{\overline{f^{-1}(f(Q)^\circ)}} = f \circ h_{f(Q)^\circ}.$$

Thus, for $i = 1, 2$,

$$f \circ h_{G_i}|_Q = h \circ f|_Q.$$

This means that given $p \in P$, we may define $I(h)(p) = h_G(p)$ where G is any element of $\mathcal{T}$ containing p.

Clearly $I(h)$ satisfies (a) and (b). If H is another such map, then for every $G \in \mathcal{T}$, the restrictions of H and $I(h)$ to $\overline{f^{-1}(G)}$ are both equal to h_G; this gives uniqueness of $I(h)$. The continuity and bijectivity of $I(h)$ are left to the reader.

$\square$

When $(I(\mathcal{D}), \iota)$ is an injective envelope for $\mathcal{D}$, $(\widehat{I(\mathcal{D})}, \iota^{\#}|_{\widehat{I(\mathcal{D})}})$ is a projective cover for $\hat{\mathcal{D}}$. In the following technical result, we will simply write $\iota^{\#}$ instead of $\iota^{\#}|_{\widehat{I(\mathcal{D})}}$.

Proposition 3.18. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$, and let E be the pseudo-expectation for ι. Then*

$$(\iota^{\#})^{-1}((\mathrm{fix}\,\beta_v)^\circ) \subseteq \mathrm{supp}(\widehat{E(v)}) \subseteq (\iota^{\#})^{-1}(\mathrm{fix}\,\beta_v) \quad \text{and} \quad \overline{(\iota^{\#})^{-1}((\mathrm{fix}\,\beta_v)^\circ)} = \overline{\mathrm{supp}(\widehat{E(v)})}. \tag{16}$$

Moreover, if $\rho \in (\iota^{\#})^{-1}((\mathrm{fix}\,\beta_v)^\circ)$, then for any $d \in J_v$ with $\rho(\iota(d)) \neq 0$,

$$\rho(E(v)) = \frac{\rho(\iota(vd))}{\rho(\iota(d))}. \tag{17}$$

Proof. Suppose that $\rho \in \widehat{I(\mathcal{D})}$ and $\rho \circ \iota \in (\operatorname{fix}\beta_v)^\circ$. Then $\rho(\iota(v^*v)) \neq 0$ (as $\operatorname{fix}\beta_v \subseteq \operatorname{dom}(\beta_v) = \operatorname{supp}(\widehat{v^*v})$). By Proposition 3.2 there exists $d \in J_v$ such that $\rho(\iota(d)) \neq 0$. Since $vd \in \mathcal{D}$,

$$0 \neq \rho(\iota(d^*d))\rho(\iota(v^*v)) = \rho(\iota(d^*v^*vd)) = |\rho(\iota(vd))|^2.$$

Hence

$$\rho(E(v)) = \frac{\rho(E(v))\rho(\iota(d))}{\rho(\iota(d))} = \frac{\rho(E(vd))}{\rho(\iota(d))} = \frac{\rho(\iota(vd))}{\rho(\iota(d))} \neq 0.$$

Thus we obtain (17) and also $(\iota^\#)^{-1}((\operatorname{fix}\beta_v)^\circ) \subseteq \operatorname{supp}(\widehat{E(v)})$.

We next show

$$\operatorname{supp}(\widehat{E(v)}) \subseteq (\iota^\#)^{-1}(\operatorname{fix}\beta_v). \tag{18}$$

Suppose that $\rho \in \operatorname{supp}(\widehat{E(v)})$. By Proposition 3.14, $0 \neq |\rho(E(v))|^2 = \rho(E(v^*v)) = \rho(\iota(v^*v))$. Hence $\rho \circ \iota \in \operatorname{dom}(\beta_v)$. Let $d \in \mathcal{D}$ be such that $d \geq 0$ and $\rho(\iota(d)) \neq 0$. Then $\rho(E(d^{1/2}v)) = \rho(\iota(d))^{1/2}\rho(E(v)) \neq 0$. Then

$$\beta_v(\rho\circ\iota)(d) = \frac{\rho(\iota(v^*dv))}{\rho(\iota(v^*v))} = \frac{\rho(E(v^*dv))}{\rho(\iota(v^*v))} = \frac{\rho(E((d^{1/2}v)^*(d^{1/2}v)))}{\rho(\iota(v^*v))} = \frac{|\rho(E(d^{1/2}v))|^2}{\rho(\iota(v^*v))} \neq 0. \tag{19}$$

(The last equality in (19) follows from Proposition 3.14.) As (19) holds for every $d \in \mathcal{D}^+$ with $\rho(\iota(d)) \neq 0$, we conclude that $\beta_v(\rho\circ\iota) = \rho\circ\iota$. This gives (18).

The first paragraph of the proof gives $\overline{(\iota^\#)^{-1}((\operatorname{fix}\beta_v)^\circ)} \subseteq \overline{\operatorname{supp}(\widehat{E(v)})}$. Let

$$Q := \overline{\operatorname{supp}(\widehat{E(v)})} \setminus \overline{(\iota^\#)^{-1}((\operatorname{fix}\beta_v)^\circ)} \quad \text{and} \quad \mathfrak{I} := \{d \in \mathcal{D} : \operatorname{supp}\widehat{\iota(d)} \subseteq Q\}.$$

Then Q is a clopen set, and $\mathfrak{I}$ is a closed ideal in $\mathcal{D}$.

We claim that $\mathfrak{I} \subseteq (J_v \vee K_v)^\perp$. To see this, fix $d \in \mathfrak{I}$. Proposition 3.2 shows that for any $h \in J_v$, $\operatorname{supp}(\widehat{\iota(h)}) \subseteq (\iota^\#)^{-1}((\operatorname{fix}(\beta_v)^\circ))$; thus $dh = 0$ for any $h \in J_v$. Now we show $dK_v = 0$. Suppose $h \in K_v$. By Theorem 3.10(b), $E(v)\iota(h) = 0$, so $\widehat{\iota(h)}$ vanishes on $\operatorname{supp}(\widehat{E(v)})$. Continuity implies that $\widehat{\iota(h)}$ vanishes on Q as well. Therefore, $\iota(d)\iota(h) = 0$, so $dh = 0$. As $dh = 0$ for all h belonging to a generating set for $J_v \vee K_v$, $d \in (J_v \vee K_v)^\perp$. The claim follows.

As $J_v \vee K_v$ is an essential ideal, $(J_v \vee K_v)^\perp = (0)$, whence $\mathfrak{I} = (0)$. The Hamana regularity of $(I(\mathcal{D}), \iota)$ now implies that $Q = \emptyset$, which completes the proof. ☐

Notation 3.19. Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Define $\hat{v} : (\operatorname{fix}\beta_v)^\circ \to \mathbb{C}$ as follows. Given $\sigma \in (\operatorname{fix}\beta_v)^\circ$, choose $d \in J_v$ so that $\sigma(d) \neq 0$ and set

$$\hat{v}(\sigma) = \frac{\sigma(vd)}{\sigma(d)}.$$

Proposition 3.18 shows this is well-defined, and it is easy to show that $\hat{v}$ is a bounded continuous function on $(\operatorname{fix}\beta_v)^\circ$. Extend $\hat{v}$ to a bounded Borel function on $\hat{\mathcal{D}}$ by defining it to be zero off $(\operatorname{fix}\beta_v)^\circ$; we denote this extension by $\hat{v}$ as well.

Remark 3.20. Take $(I(\mathcal{D}), \iota)$ to be the Dixmier algebra of $\hat{\mathcal{D}}$ and ι to be the map which takes $d \in \mathcal{D}$ to the equivalence class of $\hat{d}$ in $I(\mathcal{D})$. Then $E(v)$ is the equivalence class of the bounded Borel function $\hat{v}$ in the Dixmier algebra.

Theorem 3.21. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion. Then $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ is a compact $\mathcal{N}(\mathcal{C}, \mathcal{D})$-invariant subset of Mod$(\mathcal{C}, \mathcal{D})$ and the restriction mapping $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \ni \rho \mapsto \rho|_{\mathcal{D}}$ is a continuous surjection.*

In fact, given $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $\rho \in \mathfrak{S}_s(\mathcal{C},\mathcal{D})$ such that $\rho(v^*v) \neq 0$, let $\tau \in \widehat{I(\mathcal{D})}$ satisfy $\rho = \tau \circ E$. Then $\tilde{\beta}_v(\rho) = I(\beta_v)(\tau) \circ E$.

Proof. As noted following Definition 3.12, $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is a closed subset of $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ and $r(\mathfrak{S}_s(\mathcal{C},\mathcal{D})) = \hat{\mathcal{D}}$. So $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ is compact, and the weak-$*$-weak-$*$ continuity of r is clear.

Now let $\rho \in \mathfrak{S}_s(\mathcal{C},\mathcal{D})$ and fix $\tau \in \widehat{I(\mathcal{D})}$ so that $\rho = \tau \circ E$. Let $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ be such that $\rho(v^*v) \neq 0$. Then $\tau(E(v^*v)) = \tau(\iota(v^*v)) \neq 0$, so Lemma 3.17 shows that $\tau \in \mathrm{dom}(I(\beta_v))$.

For $\lambda \in \mathrm{dom}(I(\beta_v))$, define states on $\mathcal{C}$ by

$$\mu_\lambda(x) = \frac{\lambda(E(v^*xv))}{\lambda(\iota(v^*v))} \quad \text{and} \quad \mu'_\lambda(x) = I(\beta_v)(\lambda)(E(x)).$$

(Observe that $\mu_\tau = \tilde{\beta}_v(\rho)$.) Note that

$$\mu_\lambda|_{\mathcal{D}} = \beta_v(\lambda \circ \iota) = \mu'_\lambda|_{\mathcal{D}}.$$

Hence for $d \in \mathcal{D}$ and $x \in \mathcal{C}$, we have

$$\mu_\lambda(x)\beta_v(\lambda \circ \iota)(d) = \mu_\lambda(xd) \quad \text{and} \quad \mu'_\lambda(x)\beta_v(\lambda \circ \iota)(d) = \mu'_\lambda(xd).$$

In particular, if $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $d \in J_w$ we have

$$\mu_\lambda(w)\beta_v(\lambda \circ \iota)(d) = \mu_\lambda(wd) = \beta_v(\lambda \circ \iota)(wd) = \mu'_\lambda(wd) = \mu'_\lambda(w)\beta_v(\lambda \circ \iota)(d). \tag{20}$$

To complete the proof, it suffices to show that for every $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$, $\mu_\tau(w) = \mu'_\tau(w)$. So fix $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$. We show that $\mu_\tau(w) = \mu'_\tau(w)$ by proving the following two statements:

(1) if $\mu'_\tau(w) \neq 0$, then $\mu_\tau(w) = \mu'_\tau(w)$; and
(2) if $\mu_\tau(w) \neq 0$, then $\mu'_\tau(w) = \mu_\tau(w)$.

If $\mu'_\tau(w) \neq 0$, then $I(\beta_v)(\tau) \in \mathrm{supp}(\widehat{E(w)})$, so Proposition 3.18 implies that there exists a net $\rho_\alpha \in (\iota^\#)^{-1}((\mathrm{fix}\,\beta_w)^\circ)$ such that $\rho_\alpha \to I(\beta_v)(\tau)$. As $\mathrm{range}(I(\beta_v))$ is an open subset of $\widehat{I(\mathcal{D})}$, we may assume that $\rho_\alpha \in \mathrm{range}(I(\beta_v))$ for every α. Put $\tau_\alpha = I(\beta_v)^{-1}(\rho_\alpha)$, so $(\tau_\alpha)_\alpha$ is a net in $\mathrm{dom}(I(\beta_v))$ such that $\tau_\alpha \to \tau$. Since $\rho_\alpha \circ \iota \in (\mathrm{fix}\,\beta_w)^\circ$, given α we may find $d_\alpha \in J_w$ such that $0 \neq \rho_\alpha(\iota(d_\alpha))$. But $\rho_\alpha(\iota(d_\alpha)) = I(\beta_v)(\tau_\alpha)(\iota(d_\alpha)) = \beta_v(\tau_\alpha \circ \iota)(d_\alpha)$. Taking $\lambda = \tau_\alpha$ in (20) shows that $\mu_{\tau_\alpha}(w) = \mu'_{\tau_\alpha}(w)$. Continuity of the maps $\lambda \mapsto \mu_\lambda$ and $\lambda \mapsto \mu'_\lambda$ gives $\mu_\tau(w) = \mu'_\tau(w)$.

Turning to (2), suppose $\mu_\tau(w) \neq 0$. Then $\tau \in \mathrm{supp}(\widehat{E(v^*wv)})$, so there exists a net $\tau_\alpha \in (\iota^\#)^{-1}((\mathrm{fix}\,\beta_{v^*wv})^\circ)$ with $\tau_\alpha \to \tau$. Then $\tau_\alpha \circ \iota \in \mathrm{dom}(\beta_{v^*wv}) \subseteq \mathrm{dom}\,\beta_v$. Thus, for a given α, we may find a neighborhood N of $\tau_\alpha \circ \iota$ with $N \subseteq (\mathrm{fix}\,\beta_{v^*wv})^\circ \cap \mathrm{dom}(\beta_v)$. Now for each $y \in N$, we have $\beta_{v^*wv}(y) = y$, so $\beta_w((\beta_v)(y)) = \beta_v(y)$. Hence $\beta_v(N) \subseteq \mathrm{fix}(\beta_w)$. Therefore $\beta_v(\tau_\alpha \circ \iota) \in \mathrm{fix}(\beta_w)^\circ$. So if $d \in J_w$ satisfies $\beta_v(\tau_\alpha \circ \iota)(d) \neq 0$, then (20) gives $\mu'_{\tau_\alpha}(w) = \mu_{\tau_\alpha}(w)$. Continuity again gives $\mu'_\tau(w) = \mu_\tau(w)$.

Thus both (1) and (2) hold, and the proof is complete. □

We now show the left kernel of the pseudo-expectation on a regular MASA inclusion is an ideal which is the unique ideal which is maximal with respect to being disjoint from $\mathcal{D}$.

Theorem 3.22. *Let $(\mathcal{C},\mathcal{D})$ be a regular MASA inclusion. Then the left kernel of the pseudo-expectation E,*

$$\mathcal{L}(\mathcal{C},\mathcal{D}) := \{x \in \mathcal{C} : E(x^*x) = 0\}$$

is an ideal of $\mathcal{C}$ such that $\mathcal{L}(\mathcal{C},\mathcal{D}) \cap \mathcal{D} = (0)$.

Moreover, if $\mathcal{K} \subseteq \mathcal{C}$ is an ideal such that $\mathcal{K} \cap \mathcal{D} = (0)$, then $\mathcal{K} \subseteq \mathcal{L}(\mathcal{C},\mathcal{D})$.

Proof. Theorem 3.21 shows that $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ satisfies the hypotheses of Proposition 2.16. In the notation of Proposition 2.16, we have $\mathcal{L}(\mathcal{C},\mathcal{D}) = \mathcal{K}_{\mathfrak{S}_s(\mathcal{C},\mathcal{D})}$, so $\mathcal{L}(\mathcal{C},\mathcal{D})$ is a norm-closed two-sided ideal of $\mathcal{C}$. If $x \in \mathcal{L}(\mathcal{C},\mathcal{D}) \cap \mathcal{D}$, then $0 = E(x^*x) = \iota(x^*x)$. As ι is one-to-one, $x = 0$.

Suppose now that $\mathcal{K} \subseteq \mathcal{C}$ is an ideal with $\mathcal{K} \cap \mathcal{D} = (0)$. Let $\mathcal{C}_1 = \mathcal{C}/\mathcal{K}$, and let $q : \mathcal{C} \to \mathcal{C}_1$ be the quotient map. Since $\mathcal{K} \cap \mathcal{D} = (0)$, $q|_{\mathcal{D}}$ is faithful, so we may regard $\mathcal{D}$ as a subalgebra of $\mathcal{C}_1$. Let $F := \{\rho \circ q : \rho \in \operatorname{Mod}(\mathcal{C}_1, \mathcal{D})\}$. Then F is a closed subset of $\operatorname{Mod}(\mathcal{C}, \mathcal{D})$, and the restriction map, $f \in F \mapsto f|_{\mathcal{D}}$ maps F onto $\hat{\mathcal{D}}$. By Theorem 3.13, $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq F$. Hence every element of $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ annihilates $\mathcal{K}$, so $\mathcal{K} \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D})$. ☐

The ideal $\mathcal{L}(\mathcal{C}, \mathcal{D})$ behaves reasonably well with respect to certain regular $*$-homomorphisms, as the next result shows.

Corollary 3.23. *Suppose for $i = 1, 2$ that $(\mathcal{C}_i, \mathcal{D}_i)$ are regular MASA inclusions, and that $\alpha : (\mathcal{C}_1, \mathcal{D}_1) \to (\mathcal{C}_2, \mathcal{D}_2)$ is a regular $*$-homomorphism such that $\alpha|_{\mathcal{D}}$ is one-to-one. Then*

$$\{x \in \mathcal{C}_1 : \alpha(a) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)\} \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1). \tag{21}$$

Proof. The set $\{x \in \mathcal{C}_1 : \alpha(x) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)\}$ is an ideal of $\mathcal{C}_1$ whose intersection with $\mathcal{D}_1$ is trivial. ☐

Remark 3.24. We expect that equality holds in (21) if for every $0 \leq h \in \mathcal{D}_2$, $h = \sup_{\mathcal{D}_2}\{\alpha(d) : d \in \mathcal{D}_1 \text{ and } 0 \leq \alpha(d) \leq h\}$. Also, it would not be surprising if this condition characterized equality in (21).

Since every Cartan inclusion $(\mathcal{C}, \mathcal{D})$ satisfies $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, we make the following definition.

Definition 3.25. A *virtual Cartan inclusion* is a regular MASA inclusion such that $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$.

4. Compatible States

Since the extension property does not always hold for an inclusion $(\mathcal{C}, \mathcal{D})$, we identify a useful class of states in $\operatorname{Mod}(\mathcal{C}, \mathcal{D})$, which we call $\mathcal{D}$-compatible states.

To motivate the definition, observe that when $(\mathcal{C}, \mathcal{D})$ is a regular EP inclusion, the only way to extend a pure state $\sigma \in \hat{\mathcal{D}}$ to $\mathcal{C}$ is via composition with the expectation: $\rho := \sigma \circ E$. Then the GNS representation $(\pi_\rho, \mathcal{H}_\rho)$ arising from ρ has the property that for any $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ either $I + L_\rho$ and $v + L_\rho$ are orthogonal in the Hilbert space $\mathcal{H}_\rho$, or one is a scalar multiple of the other, according to whether or not the Gelfand transform of $E(v)$ is zero in a neighborhood of σ. Furthermore, the techniques used in the proof of [10, Proposition 5.4] show that $\pi_\rho(\mathcal{D})''$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$ and also that for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$. The intersection $\mathcal{J}$ of the kernels of such representations is the left kernel of the expectation E, $\mathcal{D} \cap \mathcal{J} = (0)$, and the quotient of $(\mathcal{C}, \mathcal{D})$ by $\mathcal{J}$ yields a C^*-diagonal with the same coordinate system as $(\mathcal{C}, \mathcal{D})$, see [10, Theorem 4.8].

We shall define the set of compatible states to be those states ρ on $\mathcal{C}$ for which the vectors $\{v + L_\rho : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ form an orthogonal set of vectors. These states have many of the properties listed in the previous paragraph, but have the advantage of not needing the extension property or a conditional expectation for their definition. Here is the formal definition.

Definition 4.1. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion.

(1) A state ρ on $\mathcal{C}$ is called $\mathcal{D}$-*compatible* if for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$,
$$|\rho(v)|^2 \in \{0, \rho(v^*v)\}.$$
When the context is clear, we will simply use the term *compatible state* instead of $\mathcal{D}$-compatible state.

(2) We will use $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ to denote the set of all $\mathcal{D}$-compatible states on $\mathcal{C}$. Topologize $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ with the relative weak-$*$ topology.

(3) For $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, let $\Delta_\rho := \{v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) : \rho(v) \neq 0\}$, and $\Lambda_\rho := \{v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) : \rho(v^*v) > 0\}$. Define a relation $\sim_\rho$ on Λ_ρ by $(v, w) \in \sim_\rho$ if and only if $v^*w \in \Delta_\rho$. (We shall prove that $\sim_\rho$ is an equivalence relation momentarily, and then will simply write $v \sim_\rho w$.)

Remarks 4.2.

(1) When $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, Proposition 3.14 shows that $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq \mathfrak{S}(\mathcal{C}, \mathcal{D})$.
(2) As $|\rho(x)|^2 \leq \rho(x^*x)$ for any state $\rho \in \mathcal{C}^\#$ and any $x \in \mathcal{C}$, we see that $\mathcal{D}$-compatible states satisfy an extremal property relative to the normalizers for $\mathcal{D}$, and one might expect an inclusion relationship between compatible states and pure states. However, there is not. Example 7.17 gives an example of a Cartan inclusion $(\mathcal{C}, \mathcal{D})$ and element of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ which is not a pure state on $\mathcal{C}$, while Example 7.16 gives an example of a Cartan inclusion $(\mathcal{C}, \mathcal{D})$ and a pure state ρ on $\mathcal{C}$ such that $\rho \in \mathrm{Mod}(\mathcal{C}, \mathcal{D})$, yet $\rho \notin \mathfrak{S}(\mathcal{C}, \mathcal{D})$. As we shall see momentarily, $\mathfrak{S}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Mod}(\mathcal{C}, \mathcal{D})$. Thus no such inclusion relationship exists.
(3) For general inclusions, it is possible that $\mathfrak{S}(\mathcal{C}, \mathcal{D}) = \emptyset$ (see Theorem 4.8).
(4) The following simple observation will be useful during the sequel: for $i = 1, 2$, let $(\mathcal{C}_i, \mathcal{D}_i)$ be inclusions and suppose that $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ is a regular and unital $*$-homomorphism. Then
$$\alpha^\#(\mathfrak{S}(\mathcal{C}_2, \mathcal{D}_2)) \subseteq \mathfrak{S}(\mathcal{C}_1, \mathcal{D}_1).$$

Here are some properties of elements of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ which hold for any inclusion.

Proposition 4.3. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$. The following statements hold.*

(1) *Suppose $v \in \Delta_\rho$. Then for every $x \in \mathcal{C}$,*
$$\rho(vx) = \rho(v)\rho(x) = \rho(xv).$$
(2) *The restriction of ρ to $\mathcal{D}$ is a multiplicative linear functional on $\mathcal{D}$.*
(3) *Suppose $v \in \Delta_\rho$. Then for every $x \in \mathcal{C}$,*
$$\rho(v^*xv) = \rho(v^*v)\rho(x).$$
(4) *If $v_1, v_2 \in \Lambda_\rho$ and $(v_1, v_2) \in \sim_\rho$, then*
$$|\rho(v_1^*v_2)|^2 = \rho(v_1^*v_1)\rho(v_2^*v_2).$$
Moreover, $\sim_\rho$ is an equivalence relation on Λ_ρ.
(5) *$\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is an $\mathcal{N}(\mathcal{C}, \mathcal{D})$-invariant subset of Mod$(\mathcal{C}, \mathcal{D})$.*
(6) *If $v \in \Lambda_\rho$, then $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$; in particular, for every $d \in \mathcal{D}$,*
$$\pi_\rho(d)(v + L_\rho) = \frac{\rho(v^*dv)}{\rho(v^*v)} v + L_\rho.$$
(7) *The set $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is weak-$*$ closed in $\mathcal{C}^\#$ and the restriction mapping, $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}) \mapsto \rho|_\mathcal{D}$, is a weak-$*$–weak-$*$ continuous mapping of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ into $\hat{\mathcal{D}}$.*

Proof. Statement (1). Since $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, an easy calculation yields $v - \rho(v)I \in L_\rho$. But L_ρ is a left ideal and $L_\rho \subseteq \ker \rho$. So for $x \in \mathcal{C}$, we have $\rho(x(v - \rho(v)I)) = 0$. So $\rho(xv) = \rho(x)\rho(v)$. As $\rho(v^*) = \overline{\rho(v)} \neq 0$, a similar argument shows that $0 = \rho((v - \rho(v)I)x)$. So part (1) holds.

Statement (2). Since $\mathcal{D} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$, this follows from part (1) and continuity of ρ.

Statement (3). This follows from part (1) and the fact that Δ_ρ is closed under the adjoint operation.

Statement (4). Let $\sigma = \rho|_\mathcal{D}$ and for $i = 1, 2$ put $\sigma_i = \beta_{v_i}(\sigma)$. Then $\sigma_1 = \sigma_2$ by statement (3) and Proposition 2.4. Therefore, since $\rho(v_1^*v_2) \neq 0$, we have
$$|\rho(v_1^*v_2)|^2 = \rho(v_2^*v_1v_1^*v_2) = \sigma_2(v_1v_1^*)\sigma(v_2^*v_2) = \sigma_1(v_1v_1^*)\sigma(v_2^*v_2) = \rho(v_1^*v_1)\rho(v_2^*v_2).$$

Clearly the relation $\sim_\rho$ is reflexive and symmetric on Λ_ρ. For $i = 1, 2, 3$, suppose $v_i \in \Lambda_\rho$, $(v_1, v_2) \in\sim_\rho$ and $(v_2, v_3) \in\sim_\rho$. The equality verified in the previous paragraph shows that in $\mathcal{H}_\rho$, $|\langle v_1 + L_\rho, v_2 + L_\rho\rangle_\rho|^2 = \|v_1 + L_\rho\|_\rho^2 \|v_2 + L_\rho\|_\rho^2$. Hence there exists a non-zero scalar t such that $tv_1 + L_\rho = v_2 + L_\rho$. Similarly, there exists a non-zero scalar s such that $v_2 + L_\rho = sv_3 + L_\rho$. So $\{v_1 + L_\rho, v_3 + L_\rho\}$ is a linearly dependent set of non-zero linearly vectors in $\mathcal{H}_\rho$. Thus $\rho(v_1^* v_3) \neq 0$, whence $(v_1, v_3) \sim_\rho$.

Statement (5). Let $v \in \Lambda_\rho$. For $w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, we claim that $|\rho(w^*v)|^2 \in \{0, \rho(w^*w)\rho(v^*v)\}$. If $\rho(w^*v) \neq 0$, then as $|\rho(w^*v)|^2 \leq \rho(w^*w)\rho(v^*v)$, we find that $w \in \Lambda_\rho$ and $w \sim_\rho v$, so the claim holds by statement (4). Hence

$$|\beta_v(\rho)(w)|^2 = \frac{|\rho(v^*(wv))|^2}{\rho(v^*v)^2} \in \left\{0, \frac{\rho(v^*v)\rho(v^*w^*wv)}{\rho(v^*v)^2}\right\} = \{0, \beta_v(\rho)(w^*w)\},$$

so $\beta_v(\rho) \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

Statement (6). Suppose $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and that $\rho(v^*v) \neq 0$. For $d \in \mathcal{D}$, let $\sigma_1(d) = \dfrac{\rho(v^*dv)}{\rho(v^*v)}$. Then $\sigma_1 \in \hat{\mathcal{D}}$, and for $d \in \mathcal{D}$, we have

$$\begin{aligned}\|(\pi_\rho(d) - \sigma_1(d)I)v + L_\rho\|_\rho^2 &= \rho(v^*(d - \sigma_1(d)I)^*(d - \sigma_1(d)I)v)\\ &= \sigma_1((d - \sigma_1(d)I)^*(d - \sigma_1(d)I))\rho(v^*v) = 0.\end{aligned}$$

We conclude that $\pi_\rho(d)v + L_\rho = \sigma_1(d)v + L_\rho$, so $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$ and statement (6) holds.

Statement (7). Suppose $(\rho_\lambda)_{\lambda\in\Lambda}$ is a net in $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ and ρ_λ converges weak-$*$ to $\rho \in \mathcal{C}^\#$. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. If $\rho(v) \neq 0$, then for large enough λ, $\rho_\lambda(v) \neq 0$. Hence $|\rho(v)|^2 = \lim_\lambda |\rho_\lambda(v)|^2 = \lim_\lambda \rho_\lambda(v^*v) = \rho(v^*v)$. It follows that $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, so $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is weak-$*$ closed. The continuity of the restriction mapping is obvious.

□

Remark 4.4. Statement (1) says that if $v \in \Delta_\rho$, then $v \in \mathfrak{M}_\rho$, where $\mathfrak{M}_\rho = \{x \in \mathcal{C} : \rho(xy) = \rho(yx) = \rho(x)\rho(y)\ \forall\, y \in \mathcal{C}\}$, see [2]. Also, if $\mathcal{B}$ is the closed linear span of Δ_ρ, then $\mathcal{B}$ is a C^*-algebra because Δ_ρ is closed under multiplication. Clearly $\mathcal{D} \subseteq \mathcal{B}$, so that $(\mathcal{B}, \mathcal{D})$ is an inclusion enjoying the properties of regularity or MASA inclusion when $(\mathcal{C}, \mathcal{D})$ has the same properties.

We turn now to the issue of existence of compatible states. When $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, Theorem 3.21, shows that every $\sigma \in \hat{\mathcal{D}}$ extends to an element of $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$. Applying Proposition 3.14, we see that compatible states exist in abundance for regular MASA inclusions. We record this fact as a theorem.

Theorem 4.5. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion. If $\sigma \in \hat{\mathcal{D}}$, there exists $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that $\rho|_{\mathcal{D}} = \sigma$. Moreover, ρ may be chosen so that $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$.*

The following result summarizes what we know regarding the existence of compatible states when the hypothesis of regularity in Theorem 4.5 is weakened. Notice that in both parts of the following result, a conditional expectation is present.

Theorem 4.6. *Suppose $(\mathcal{C}, \mathcal{D})$ is an inclusion.*

a) *If $(\mathcal{C}, \mathcal{D})$ is a MASA inclusion and there exists a conditional expectation $E : \mathcal{C} \to \mathcal{D}$, then $E^\#|_{\hat{\mathcal{D}}}$ is a continuous one-to-one map of $\hat{\mathcal{D}}$ into $\mathfrak{S}(\mathcal{C}, \mathcal{D})$.*
b) *When $(\mathcal{C}, \mathcal{D})$ has the extension property (but is not necessarily regular), then $E^\#|_{\hat{\mathcal{D}}}$ is a homeomorphism of $\hat{\mathcal{D}}$ onto $\mathfrak{S}(\mathcal{C}, \mathcal{D})$.*

Proof. a) Since E is onto, $E^{\#}$ is injective and continuous. We must show that $E^{\#}$ carries $\hat{\mathcal{D}}$ into $\mathfrak{S}(\mathcal{C},\mathcal{D})$.

Let $\sigma \in \hat{\mathcal{D}}$ and set $\rho = \sigma \circ E$. Suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, and $\rho(v) \neq 0$. By the Cauchy-Schwartz inequality, $\rho(vv^*) \neq 0$. The definition of ρ shows $E(v) \neq 0$. Let $x := \dfrac{v^*E(v)}{\rho(vv^*)}$.

We claim that x commutes with $\mathcal{D}$. This is easy to see when $v \in \mathcal{I}(\mathcal{C},\mathcal{D})$. Since $\mathcal{D}$ is a MASA, Proposition 2.2 gives $\mathcal{N}(\mathcal{C},\mathcal{D}) = \overline{\mathcal{I}(\mathcal{C},\mathcal{D})}$. A continuity argument now establishes the claim.

Therefore, $x \in \mathcal{D}$. Hence $\rho(v)\rho(x) = \rho(vx) = \rho(vv^*E(v)\rho(vv^*)^{-1}) = \rho(v)$, so that $\rho(x) = 1$. Since $vx \in \mathcal{D}$ we obtain,

$$|\rho(v)|^2 = |\rho(vx)|^2 = \rho(x^*v^*vx) = \rho(x^*)\rho(v^*v)\rho(x) = \rho(v^*v).$$

We conclude that $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, as desired.

b) Now suppose that $(\mathcal{C},\mathcal{D})$ has the extension property. If $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, put $\sigma = \rho|_{\mathcal{D}}$. Then $\sigma \in \hat{\mathcal{D}}$. By the extension property, we have $\rho = \sigma \circ E$, so $\rho = E^{\#}(\sigma)$, whence $E^{\#}|_{\hat{\mathcal{D}}}$ is onto. If $E^{\#}(\sigma_1) = E^{\#}(\sigma_2)$, then the extension property yields $\sigma_1 = \sigma_2$. So $E^{\#}$ is a continuous bijection of $\hat{\mathcal{D}}$ onto $\mathfrak{S}(\mathcal{C},\mathcal{D})$. Since $\hat{D}$ and $\mathfrak{S}(\mathcal{C},\mathcal{D})$ are both compact and Hausdorff, $E^{\#}|_{\hat{\mathcal{D}}}$ is a homeomorphism. $\square$

Theorem 4.7. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion (we do not assume $\mathcal{D}$ is a MASA in $\mathcal{C}$). The following statements hold.*

i) *Suppose $(\mathcal{C}_1,\mathcal{D}_1)$ is a regular MASA inclusion and $\alpha : (\mathcal{C},\mathcal{D}) \to (\mathcal{C}_1,\mathcal{D}_1)$ is a regular and unital $*$-homomorphism. Then $\alpha^{\#}$ maps $\mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1)$ into $\mathfrak{S}(\mathcal{C},\mathcal{D})$.*

ii) *If the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is abelian, then $\mathfrak{S}_s(\mathcal{C},\mathcal{D}^c) \subseteq \mathfrak{S}(\mathcal{C},\mathcal{D})$ and the restriction map $\rho \in \mathfrak{S}_s(\mathcal{C},\mathcal{D}^c) \mapsto \rho|_{\mathcal{D}}$, is a weak-$*$–weak-$*$ continuous mapping of $\mathfrak{S}_s(\mathcal{C},\mathcal{D}^c)$ onto $\hat{\mathcal{D}}$.*

Proof. We have already observed in Remark 4.2(4) that $\alpha^{\#}$ carries $\mathfrak{S}(\mathcal{C}_1,\mathcal{D}_1)$ into $\mathfrak{S}(\mathcal{C},\mathcal{D})$. As $\mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1) \subseteq \mathfrak{S}(\mathcal{C}_1,\mathcal{D}_1)$, the first statement holds.

Now suppose that $\mathcal{D}^c$ is abelian. Lemma 2.3 shows that $(\mathcal{C},\mathcal{D}^c)$ is a regular MASA inclusion and that the identity mapping of $\mathcal{C}$ onto itself is regular. By part (i), $\mathrm{Id}^{\#}$ carries $\mathfrak{S}_s(\mathcal{C},\mathcal{D}^c)$ into $\mathfrak{S}(\mathcal{C},\mathcal{D})$; thus $\mathfrak{S}_s(\mathcal{C},\mathcal{D}^c) \subseteq \mathfrak{S}(\mathcal{C},\mathcal{D})$. As any element of $\hat{\mathcal{D}}$ can be extended to an element of $\widehat{\mathcal{D}^c}$, we see that the restriction map is onto. Part (7) of Proposition 4.3 gives the weak-$*$ continuity.

$\square$

We turn now to a result which shows that there are inclusions with few compatible states. In fact, some inclusions have no compatible states. This result applies when the relative commutant of $\mathcal{D}$ in $\mathcal{C}$ is all of $\mathcal{C}$, e.g. $(\mathcal{C},\mathbb{C}I)$. The result shows that when $\mathcal{N}(\mathcal{C},\mathcal{D})$ is too large, it may happen that $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is empty. For example, when $\mathcal{C}$ is a unital simple C^*-algebra with $\dim(\mathcal{C}) > 1$, then $\mathfrak{S}(\mathcal{C},\mathbb{C}I) = \emptyset$.

Theorem 4.8. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion and let $\mathcal{U}(\mathcal{C})$ be the unitary group of $\mathcal{C}$. Assume that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$. Then $(\mathcal{C},\mathcal{D})$ is regular and $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is the set of all multiplicative linear functionals on $\mathcal{C}$.*

Proof. Since $\mathrm{span}(\mathcal{U}(\mathcal{C})) = \mathcal{C}$, $(\mathcal{C},\mathcal{D})$ is a regular inclusion. As every multiplicative linear functional on $\mathcal{C}$ is a compatible state, we need only prove that every element of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is a multiplicative linear functional.

Fix $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$. Then for every unitary $U \in \mathcal{C}$ we have $\rho(U) \in \{0\} \cup \mathbb{T}$. Let π be a universal representation of $\mathcal{C}$, and identify $\mathcal{C}^{\#\#}$ with the von Neumann algebra $\pi(\mathcal{C})''$. Also, regard $\mathcal{C}$ as a subalgebra of $\mathcal{C}^{\#\#}$. Let $\rho^{\#\#}$ denote the normal state on $\mathcal{C}^{\#\#}$ obtained from ρ. By [33, II.4.11], every unitary in $\mathcal{C}^{\#\#}$ is the strong-$*$ limit of a net of unitaries in $\mathcal{C}$. Since $\rho^{\#\#}$ is normal, $\rho^{\#\#}(W) \in \{0\} \cup \mathbb{T}$ for every unitary $W \in \mathcal{C}^{\#\#}$.

Let P be a projection in $\mathcal{C}^{\#\#}$. We shall show that $\rho^{\#\#}(P) \in \{0, 1\}$. We argue by contradiction. Suppose that $0 < \rho^{\#\#}(P) < 1$. Then $0 < |\rho^{\#\#}(P) + i\rho^{\#\#}(I - P)| < 1$. Put $W = P + i(I - P)$. Then W is a unitary belonging to $\mathcal{C}^{\#\#}$, and therefore we may find a net U_α of unitaries in $\mathcal{C}$ so that U_α converges strong-$*$ to W. But then $|\rho(U_\alpha)| \to |\rho^{\#\#}(W)| \in (0, 1)$. This implies that there exists a unitary $U \in \mathcal{C}$ such that $|\rho(U)| \in (0, 1)$, which is a contradiction. Therefore $\rho^{\#\#}(P) \in \{0, 1\}$ for every projection $P \in \mathcal{C}^{\#\#}$.

Now let $P, Q \in \mathcal{C}^{\#\#}$ be projections. We claim that $\rho^{\#\#}(PQ) = \rho^{\#\#}(P)\rho^{\#\#}(Q)$. By the Cauchy-Schwartz inequality, $|\rho^{\#\#}(PQ)| \leq \rho^{\#\#}(P)\rho^{\#\#}(Q)$, so that $\rho^{\#\#}(PQ) = 0$ if $0 \in \{\rho^{\#\#}(P), \rho^{\#\#}(Q)\}$. Suppose then that $\rho^{\#\#}(P) = \rho^{\#\#}(Q) = 1$. Since $2P - I$ and $2Q - I$ are unitaries in $\mathcal{C}^{\#\#}$, we may find nets of unitaries u_α and v_α in $\mathcal{C}$ so that u_α and v_α converge $*$-strongly to $2P - I$ and $2Q - I$ respectively. Both $\rho(u_\alpha)$ and $\rho(v_\alpha)$ are eventually non-zero because

$$\lim \rho(u_\alpha) = \rho^{\#\#}(2P - I) = 1 = \rho^{\#\#}(2Q - I) = \lim \rho(v_\alpha).$$

As multiplication on bounded subsets of $\mathcal{C}^{\#\#}$ is jointly continuous in the strong-$*$ topology, $u_\alpha v_\alpha$ converges strongly to $(2P - I)(2Q - I)$. By Proposition 4.3(1),

$$\rho^{\#\#}((2P - I)(2Q - I)) = \lim \rho(u_\alpha v_\alpha) = \lim \rho(u_\alpha)\rho(v_\alpha) = \rho^{\#\#}(2P - I)\rho^{\#\#}(2Q - I) = 1.$$

On the other hand, a calculation shows that

$$\rho^{\#\#}((2P - I)(2Q - I)) = 4\rho^{\#\#}(PQ) - 3.$$

Combining these equalities gives $\rho^{\#\#}(PQ) = 1$, as desired. The claim follows.

Let $X = \sum_{j=1}^n \lambda_j P_j$ and $Y = \sum_{j=1}^n \mu_j Q_j$ be linear combinations of projections $\{P_j\}_{j=1}^n$ and $\{Q_j\}_{j=1}^n$ in $\mathcal{C}^{\#\#}$. It follows from the previous paragraph that $\rho^{\#\#}(XY) = \rho^{\#\#}(X)\rho^{\#\#}(Y)$. Since any von Neumann algebra is the norm closure of the span of its projections, $\rho^{\#\#}$ is multiplicative on $\mathcal{C}^{\#\#}$. It then follows that ρ is multiplicative on $\mathcal{C}$.

□

We now turn to the representations arising from states in $\mathfrak{S}(\mathcal{C}, \mathcal{D})$. We begin with a simple lemma concerning states on regular inclusions, whose proof we leave to the reader.

Lemma 4.9. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and suppose that ρ is a state on $\mathcal{C}$. Then*

$$\operatorname{span}\{v + L_\rho : v \in \mathcal{N}(\mathcal{C}, \mathcal{D}),\ \rho(v^*v) > 0\}$$

is norm-dense in $\mathcal{H}_\rho$.

Proposition 4.10. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion, let $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, and let $T \subseteq \Lambda_\rho$ be chosen so that for every $v \in T$, $\rho(v^*v) = 1$ and T contains exactly one element from each $\sim_\rho$ equivalence class. Then the following statements hold.*

1. *$\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$.*
2. *For $v \in T$, let $\mathcal{K}_v := \{\xi \in \mathcal{H}_\rho : \pi_\rho(d)\xi = \rho(v^*dv)\xi$ for all $d \in \mathcal{D}\}$ and let $\sigma = \rho|_{\mathcal{D}}$. Then $\mathcal{K}_v = \overline{\operatorname{span}}\{w + L_\rho : w \in T$ and $\beta_w(\sigma) = \beta_v(\sigma)\}$.*
3. *For $v \in T$, let P_v be the orthogonal projection of $\mathcal{H}_\rho$ onto $\mathcal{K}_v$. Then P_v is a minimal projection in $\pi_\rho(\mathcal{D})''$ and $\bigvee_{v \in T} P_v = I$.*
4. *$\pi_\rho(\mathcal{D})''$ is an abelian and atomic von Neumann algebra.*

Proof. Statement (1). If $v, w \in T$ are distinct, then $\rho(v^*w) = 0$, so that $\{v + L_\rho : v \in T\}$ is an orthonormal set. Part (4) of Proposition 4.3 and the Cauchy-Schwartz inequality show that if $v \in T$, and $w \in \Lambda_\rho$ is such that $v \sim_\rho w$, then $w + L_\rho \in \operatorname{span}\{v + L_\rho\}$. This, together with Lemma 4.9, shows that

$$\overline{\operatorname{span}}\{v + L_\rho : v \in T\} = \overline{\operatorname{span}}\{v + L_\rho : v \in \Lambda_\rho\} = \overline{\operatorname{span}}\{v + L_\rho : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\} = \mathcal{H}_\rho.$$

Thus $\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$.

Statement (2). If $\xi \in \overline{\operatorname{span}}\{w + L_\rho : \beta_w(\sigma) = \beta_v(\sigma)\}$, Part (6) of Proposition 4.3 implies that $\xi \in \mathcal{K}_v$. For the opposite inclusion, suppose $\xi \in \mathcal{K}_v$. Then for $w \in T$ and $d \in \mathcal{D}$ we have

$$\beta_v(\sigma)(d)\langle \xi, w + L_\rho\rangle = \langle \pi_\rho(d)\xi, w + L_\rho\rangle = \langle \xi, \pi_\rho(d^*)(w + L_\rho)\rangle = \beta_w(\rho)(d)\langle \xi, w + L_\rho\rangle.$$

Hence if $\langle \xi, w + L_\rho\rangle \neq 0$, then $\beta_v(\sigma) = \beta_w(\sigma)$. This yields $\xi \in \overline{\operatorname{span}}\{w + L_\rho : w \in T \text{ and } \beta_w(\sigma) = \beta_v(\sigma)\}$.

Statement (3). First note that for $v \in T$, $v + L_\rho \in \mathcal{K}_v$; thus, since $\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$, we obtain $\bigvee_{v \in T} P_v = I$.

Let $X \in \pi_\rho(\mathcal{D})'$ and $\xi \in \mathcal{K}_v$. Then for $d \in \mathcal{D}$,

$$\pi_\rho(d)X\xi = X\pi_\rho(d)\xi = \rho(v^* dv)X\xi.$$

Therefore $X\xi \in \mathcal{K}_v$, showing that $\mathcal{K}_v$ is an invariant subspace for X. As this holds for every $X \in \pi_\rho(\mathcal{D})'$, we conclude that $P_v \in \pi_\rho(\mathcal{D})''$.

Let $v \in T$ and suppose that $Q \in \pi_\rho(\mathcal{D})''$ is a projection with $0 \leq Q \leq P_v$. For all $d \in \mathcal{D}$ we have $\pi_\rho(d)P_v = \beta_v(\sigma)(d)P_v = \langle \pi_\rho(d)(v + L_\rho), v + L_\rho\rangle P_v$. The Kaplansky Density Theorem shows that for every $X \in \pi_\rho(\mathcal{D})''$ we have $XP_v = \langle X(v + L_\rho), v + L_\rho\rangle P_v \in \mathbb{C}P_v$. Since Q commutes with P_v, QP_v is a projection; hence $QP_v \in \{0, P_v\}$, so P_v is a minimal projection in $\pi_\rho(\mathcal{D})''$.

Statement (4). This follows from statement (3).

□

The following result shows that elements of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ arise from regular representations π of $(\mathcal{C}, \mathcal{D})$, which can be taken so that $\pi(\mathcal{D})''$ is atomic. For vectors h_1, h_2 in a Hilbert space $\mathcal{H}$ we use the notation $h_1 h_2^*$ for the rank-one operator $h \mapsto \langle h, h_2\rangle h_1$.

Theorem 4.11. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion. The following statements hold.*

i) *Let $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$. and let*

$$\mathcal{A}_\rho := \{(v + L_\rho)(v + L_\rho)^* : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}'' \subseteq \mathcal{B}(\mathcal{H}_\rho).$$

Then $\mathcal{A}_\rho$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$ and $\pi_\rho : (\mathcal{C}, \mathcal{D}) \to (\mathcal{B}(\mathcal{H}_\rho), \mathcal{A}_\rho)$ is a regular $*$-homomorphism.

ii) *Conversely, suppose $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ is a regular $*$-homomorphism with $\pi(\mathcal{D})''$ a (not necessarily atomic) MASA in $\mathcal{B}(\mathcal{H})$, and let $E : \mathcal{B}(\mathcal{H}) \to \pi(\mathcal{D})''$ be any conditional expectation. Then for any pure state σ of $\pi(\mathcal{D})''$, $\sigma \circ E \circ \pi \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.*

Proof. For the first statement, choose T as in the statement of Proposition 4.10. For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, we have $v + L_\rho = 0$ if $v \notin \Lambda_\rho$; and if $v + L_\rho \neq 0$, then there exists $w \in T$ such that $v \sim_\rho w$, so $(v + L_\rho)(v + L_\rho)^* \in \mathbb{C}(w + L_\rho)(w + L_\rho)^*$. Since $\mathcal{B} := \{w + L_\rho : w \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$, we see that $\mathcal{A}_\rho$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$.

We now show that π_ρ is a regular homomorphism. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and let $w \in T$. Then $\pi_\rho(v)(w + L_\rho)(w + L_\rho)^*\pi_\rho(v)^* = (vw + L_\rho)(vw + L_\rho)^* \in \mathcal{A}_\rho$. As $\operatorname{span}\{(w + L_\rho)(w + L_\rho)^* : w \in T\}$ is weakly dense in $\mathcal{A}_\rho$, we conclude that $\pi_\rho(v)\mathcal{A}_\rho\pi_\rho(v)^* \subseteq \mathcal{A}_\rho$. Similarly $\pi_\rho(v)^*\mathcal{A}_\rho\pi_\rho(v) \subseteq \mathcal{A}_\rho$. Thus π_ρ is a regular $*$-homomorphism.

For the second statement, Theorem 4.6 shows that if $\sigma \in \widehat{\pi_\rho(\mathcal{D})''}$, then $\sigma \circ E \in \mathfrak{S}(\mathcal{B}(\mathcal{H}), \pi(\mathcal{D})'')$. Remark 4.2(4) completes the proof. □

Remark 4.12. We have $\pi_\rho(\mathcal{D})'' \subseteq \mathcal{A}_\rho$ always, but in general they can be very different. Consider the state $\rho = \rho_\infty$ from Example 7.17. Then, using the notation from that example, $\{S^n + L_\rho : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{H}_\rho$ (where $S^n = S^{*|n|}$ when $n < 0$). Note that $\pi_\rho(\mathcal{D})'' = \mathbb{C}I$, while $\mathcal{A}_\rho$ is a MASA.

The following proposition characterizes when $\pi_\rho(\mathcal{D})''$ and $\mathcal{A}_\rho$ coincide. We first make a definition.

Definition 4.13. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $f \in \operatorname{Mod}(\mathcal{C}, \mathcal{D})$. The $\mathcal{D}$-*stabilizer* of f is the set,

$$\mathcal{D}\text{-stab}(f) := \{v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) : \text{for all } d \in \mathcal{D},\ f(v^*dv) = f(d)\}.$$

If for every $v \in \mathcal{D}$-stab(f) and $x \in \mathcal{C}$, we have $f(x) = f(v^*xv)$, then we call f a $\mathcal{D}$-*rigid state.*

Proposition 4.14. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion, and suppose that $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$. The following statements are equivalent.*

1. *$\pi_\rho(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\rho)$.*
2. *If $v \in \mathcal{D}$-stab(ρ), then $\rho(v) \neq 0$.*
3. *ρ is a pure and $\mathcal{D}$-rigid state.*

Proof. Throughout the proof, we let $\sigma = \rho|_\mathcal{D}$, which by Proposition 4.3(2), belongs to $\hat{\mathcal{D}}$.

Suppose $\pi_\rho(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\rho)$ and let $v \in \mathcal{D}$-stab(ρ), so $v \in \Lambda_\rho$ and $\beta_v(\sigma) = \sigma$. Then, using the notation of Proposition 4.10, we find that $P_I(v + L_\rho) = v + L_\rho$. Since $\pi_\rho(\mathcal{D})''$ is a MASA, P_I is the orthogonal projection onto $\mathbb{C}(I + L_\rho)$. We conclude that $v + L_\rho$ is a non-zero scalar multiple of $I + L_\rho$. Hence $0 \neq \langle v + L_\rho, I + L_\rho\rangle = \rho(v)$. Thus $v \in \Delta_\rho$, so statement (1) implies statement (2).

Before proving the next implication, we pause for some generalities. Suppose that f is any state on $\mathcal{C}$ with the property that $f|_\mathcal{D} = \sigma$. If $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f(v^*v) = 0$, then the Cauchy-Schwartz inequality yields $f(v) = 0$. Also note that if $v \in \Lambda_\rho$ satisfies $\beta_v(\sigma) \neq \sigma$, then $f(v) = 0$. Indeed, for such $v \in \Lambda_\rho$, choose $d \in \mathcal{D}$ so that $\beta_v(\sigma)(d) = 0$ and $\sigma(d) = 1$. Then as $v^*dv \in \mathcal{D}$,

$$0 = f(v^*dv) = f(v)f(v^*dv) = f(vv^*dv) = f(dvv^*v) = f(d)f(v)f(v^*v).$$

As $f(d)$ and $f(v^*v)$ are both non-zero, we conclude that $f(v) = 0$.

Now suppose statement (2) holds. We first prove that ρ is pure. So suppose that $t \in [0, 1]$ and that for $i = 1, 2$, ρ_i are states on $\mathcal{C}$ and $\rho = t\rho_1 + (1 - t)\rho_2$. As $\rho|_\mathcal{D}$ is a pure state on $\mathcal{D}$, we have $\rho_i|_\mathcal{D} = \sigma$. We claim that for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\rho_1(v) = \rho_2(v) = \rho(v)$. By the previous paragraph, it remains only to prove this for $v \in \Lambda_\rho$ such that $\beta_v(\sigma) = \sigma$. So suppose v has this property. By the hypothesis in statement (2), $\rho(v) \neq 0$. Clearly $|\rho_i(v)| \leq \rho_i(v^*v)^{1/2} = \rho(v^*v)^{1/2} = |\rho(v)|$. Thus we have

$$t\frac{\rho_1(v)}{\rho(v)} + (1 - t)\frac{\rho_2(v)}{\rho(v)} = 1,$$

which expresses 1 as a convex combination of elements of the closed unit disk. Hence $\rho_i(v) = \rho(v)$, establishing the claim. By regularity, we conclude that $\rho_1 = \rho_2 = \rho$, so ρ is a pure state.

Next, if $v \in \Lambda_\rho$ and $\beta_v(\sigma) = \sigma$, then by hypothesis, $\rho(v) \neq 0$. So the final part of statement (3) follows from part (1) of Proposition 4.3. Thus statement (2) implies statement (3).

Finally, suppose that statement (3) holds. Let $v, w \in \Lambda_\rho$ be such that $\beta_v(\sigma) = \beta_w(\sigma)$. We shall show that $\{v + L_\rho, w + L_\rho\}$ is a linearly dependent set, showing that $\mathcal{K}_v$ is one-dimensional. We have $\beta_{w^*v}(\sigma) = \sigma = \beta_{v^*w}(\sigma)$, so $\rho(v^*ww^*v)^{-1/2}w^*v \in \mathcal{D}$-stab$(\rho)$. By hypothesis, $\rho(x) = \dfrac{\rho(v^*wxw^*v)}{\rho(v^*ww^*v)}$ for every $x \in \mathcal{C}$. Thus if $\eta = \rho(v^*ww^*v)^{-1/2}w^*v + L_\rho$, we have $\langle \pi_\rho(x)\eta, \eta\rangle = \langle \pi_\rho(x)(I + L_\rho), I + L_\rho\rangle$ for every $x \in \mathcal{C}$. Since ρ is pure, $\pi_\rho(\mathcal{C})'' = \mathcal{B}(\mathcal{H}_\rho)$, so that for every $T \in \mathcal{B}(\mathcal{H}_\rho)$ we obtain

$$\langle T\eta, \eta\rangle = \langle T(I + L_\rho), I + L_\rho\rangle.$$

Hence $\{\eta, I + L_\rho\}$ is a linearly dependent set. Thus, $\{w^*v + L_\rho, I + L_\rho\}$ is linearly dependent. Since both vectors in this set are non-zero, we find $0 \neq \langle w^*v + L_\rho, I + L_\rho\rangle = \rho(w^*v)$. Applying part (4) of Proposition 4.3 and the Cauchy-Schwartz inequality, we obtain $\{v + L_\rho, w + L_\rho\}$ is linearly dependent, as desired.

As $\mathcal{K}_v$ is one-dimensional, Proposition 4.10 implies that $\pi_\rho(\mathcal{D})''$ is a MASA.

∎

5. The $\mathcal{D}$-Radical and Embedding Theorems

Our purpose in this section is to prove two embedding theorems. The first characterizes when a regular inclusion can be regularly embedded into a regular MASA inclusion, while the second characterizes when a regular inclusion may be regularly embedded into a C^*-diagonal.

The first of these theorems shows that the obvious necessary condition suffices.

Theorem 5.1. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion. The following statements are equivalent.*

a) *There exists a regular MASA inclusion $(\mathcal{C}_1, \mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$.*
b) *The relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is abelian.*

Proof. Suppose that $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA inclusion and $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$ is a regular $*$-monomorphism. Let $(I(\mathcal{D}_1), \iota_1)$ be an injective envelope for $\mathcal{D}_1$ and let $E_1 : \mathcal{C}_1 \to I(\mathcal{C}_1)$ be the pseudo-expectation for ι_1.

Observe that $(\mathcal{D}^c, \mathcal{D})$ is a regular inclusion, and $\mathcal{N}(\mathcal{D}^c, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. Let $\rho \in \mathfrak{S}_s(\mathcal{C}_1, \mathcal{D}_1)$. Part (i) of Theorem 4.7 shows that $\rho \circ \alpha \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, and hence $\rho \circ \alpha|_{\mathcal{D}^c} \in \mathfrak{S}(\mathcal{D}^c, \mathcal{D})$. Theorem 4.8 implies $\rho \circ \alpha|_{\mathcal{D}^c}$ is a multiplicative linear functional on $\mathcal{D}^c$. By the definition of $\mathfrak{S}_s(\mathcal{C}_1, \mathcal{D}_1)$, we see that for every $\tau \in \widehat{I(\mathcal{D}_1)}$, $\tau \circ E_1 \circ \alpha|_{\mathcal{D}^c}$ is a multiplicative linear functional on $\mathcal{D}^c$. We conclude $E_1 \circ \alpha|_{\mathcal{D}^c}$ is a $*$-homomorphism of $\mathcal{D}^c$ into $I(\mathcal{D}_1)$.

Let $u \in \mathcal{D}^c$ be a unitary element. Clearly, $u \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, so regularity of α implies that $\alpha(u) \in \mathcal{N}(\mathcal{C}_1, \mathcal{D}_1)$. Let $J_{\alpha(u)}$ be the ideal of $\mathcal{D}_1$ as defined in (7). Since $E_1(\alpha(u))$ is unitary, Theorem 3.10(c) shows $\sup_{I(\mathcal{D}_1)}(\iota(J_{\alpha(u)})_1^+)$ is the identity of $I(\mathcal{D}_1)$. Hence $J_{\alpha(u)}$ is an essential ideal in $\mathcal{D}_1$. Then $(\operatorname{fix} \beta_{\alpha(u)})^\circ$ is dense in $\hat{\mathcal{D}}_1$ by Proposition 3.2. As $\operatorname{dom}(\beta_{\alpha(u)}) = \hat{\mathcal{D}}_1$, we get $\hat{\mathcal{D}}_1 = \overline{(\operatorname{fix} \beta_{\alpha(u)})^\circ} = \operatorname{fix} \beta_{\alpha(u)}$. Therefore, $\hat{\mathcal{D}}_1 = (\operatorname{fix} \beta_{\alpha(u)})^\circ$. Another application of Proposition 3.2 gives $J_{\alpha(u)} = \mathcal{D}_1$, and hence $\alpha(u) \in \mathcal{D}_1$. This shows that the image of the unitary group of $\mathcal{D}^c$ under α is abelian. Since α is faithful, we see that the unitary group of $\mathcal{D}^c$ is abelian, and hence $\mathcal{D}^c$ is abelian.

For the converse, take α to be the identity map on $\mathcal{C}$. Lemma 2.3 shows that $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}^c)$ is regular, so statement (a) follows from statement (b).

∎

Question 5.2. *When does a regular inclusion regularly embed into a regular EP-inclusion?*

We conjecture that the conditions of Theorem 5.1 also characterize when a regular inclusion may be regularly embedded into a regular EP-inclusion. Here is an approach to this problem. Suppose $(\mathcal{C}, \mathcal{D})$ is a regular inclusion with $\mathcal{D}^c$ abelian. Let $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathcal{C}$, let $\mathcal{D}_1 = \pi(\mathcal{D}^c)''$ and let $\mathcal{C}_1$ be the (concrete) C^*-algebra generated by $\pi(\mathcal{C})$ and $\mathcal{D}_1$. Then $(\mathcal{C}_1, \mathcal{D}_1)$ is regular, and $\pi : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$ is a regular $*$-monomorphism. If $\mathcal{D}_1$ is a MASA in $\mathcal{C}_1$, then Theorem 2.10 shows that $(\mathcal{C}_1, \mathcal{D}_1)$ is an EP-inclusion. Unfortunately, we have not been able to decide whether the faithful represetation π can be chosen so that $(\mathcal{C}_1, \mathcal{D}_1)$ is a MASA inclusion.

We now define a certain ideal, the $\mathcal{D}$-radical of an inclusion, and show its relevance to embedding regular inclusions into C^*-diagonals.

Definition 5.3. For an inclusion $(\mathcal{C}, \mathcal{D})$, the *$\mathcal{D}$-radical of* $(\mathcal{C}, \mathcal{D})$ is the set

$$\operatorname{Rad}(\mathcal{C}, \mathcal{D}) := \{x \in \mathcal{C} : \|\pi_\rho(x)\| = 0 \text{ for all } \rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})\},$$

provided $\mathfrak{S}(\mathcal{C}, \mathcal{D}) \neq \emptyset$; otherwise define $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = \mathcal{C}$. (Note that $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \neq \mathcal{C}$ whenever $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion.)

When $(\mathcal{C}, \mathcal{D})$ is a regular inclusion, we have the following description of $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$.

Proposition 5.4. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a regular inclusion. Then*

$$\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : \rho(x^*x) = 0 \textit{ for all } \rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})\}.$$

Proof. Let $J := \{x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})\}$. If $x \in \operatorname{Rad}(\mathcal{C}, \mathcal{D})$ and $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, then $\rho(x^*x) = \|\pi_\rho(x)(I + L_\rho)\| = 0$, and we find that $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq J$. For the opposite inclusion, let $x \in J$. Part (5) of Proposition 4.3 and Corollary 2.16 show that J is a closed, two-sided ideal of $\mathcal{C}$. Hence for every $c \in \mathcal{C}$ and $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ we have $\rho(c^*x^*xc) = 0$, which means that $\pi_\rho(x) = 0$ for every ρ. So $x \in \operatorname{Rad}(\mathcal{C}, \mathcal{D})$, showing $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = J$. ∎

Examples 5.5. Here are some examples of the $\mathcal{D}$-radical.

1. By Proposition 5.7, $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = (0)$ for any Cartan inclusion.
2. Suppose that $(\mathcal{C}, \mathcal{D})$ is a regular EP inclusion. Then $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$ is the left kernel of the associated conditional expectation E, that is, $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : E(x^*x) = 0\}$. This follows from Propositions 5.4 and 4.6.
3. Suppose that $(\mathcal{C}, \mathcal{D})$ is an inclusion such that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. Then Theorem 4.8 shows that $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is the set of characters on $\mathcal{C}$. Since the intersection of the kernels of all characters is the commutator ideal, it follows from Proposition 5.4 that $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$ is the commutator ideal of $\mathcal{C}$.

Question 5.6. *Observe that when $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D})$, because $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq \mathfrak{S}(\mathcal{C}, \mathcal{D})$. Is it possible for the inclusion to be proper?*

Proposition 5.7. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and suppose $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA inclusion. If $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$ is a regular and unital $*$-homomorphism, then $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \alpha^{-1}(\mathcal{L}(\mathcal{C}_1, \mathcal{D}_1))$.*

Proof. By Theorem 4.7, $\alpha(\operatorname{Rad}(\mathcal{C}, \mathcal{D})) \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1)$. ∎

Notice that when $\mathcal{L}(\mathcal{C}_1, \mathcal{D}_1) = (0)$, Proposition 5.7 implies that $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \ker \alpha$. We have been unable to decide whether equality holds in general. However, the following lemma shows that one can construct a C^*-diagonal and a regular $*$-homomorphism such that equality holds.

Lemma 5.8. *Suppose that $(\mathcal{C}, \mathcal{D})$ is a regular inclusion. Then there exists a C^*-diagonal $(\mathcal{C}_1, \mathcal{D}_1)$ and a regular $*$-homomorphism $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}, \mathcal{D}_1)$ with $\ker \alpha = \operatorname{Rad}(\mathcal{C}, \mathcal{D})$.*

Proof. For each $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, let $(\pi_\rho, \mathcal{H}_\rho)$ be the GNS representation of $\mathcal{C}$ arising from ρ. Let $\mathcal{H} := \bigoplus_{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})} \mathcal{H}_\rho$ and let $\mathcal{D}_1 = \bigoplus_{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})} \mathcal{A}_\rho$, where $\mathcal{A}_\rho$ is as in the statement of Theorem 4.11. As $\mathcal{A}_\rho$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$, we see that $\mathcal{D}_1$ is an atomic MASA in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{C}_1 = \overline{\operatorname{span}} \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D}_1)$. By Theorem 2.10 and the fact that the expectation onto an atomic MASA in $\mathcal{B}(\mathcal{H})$ is faithful, $(\mathcal{C}_1, \mathcal{D}_1)$ is a C^*-diagonal.

For each $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, the regularity of π_ρ (Theorem 4.11) shows that $\bigoplus_{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})} \pi_\rho(v) \in \mathcal{N}(\mathcal{B}(\mathcal{H}), \mathcal{D}_1)$. Hence for each $x \in \mathcal{C}$, $\bigoplus_{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})} \pi_\rho(x) \in \mathcal{C}_1$. Thus if $\alpha : \mathcal{C} \to \mathcal{C}_1$ is given by $\alpha(x) = \bigoplus_{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})} \pi_\rho(x)$, then α is a regular $*$-homomorphism. By construction, $\ker \alpha = \operatorname{Rad}(\mathcal{C}, \mathcal{D})$. ∎

The following is our main embedding result.

Theorem 5.9. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion. Then there exists a C^*-diagonal $(\mathcal{C}_1, \mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$ if and only if $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = 0$.*

Proof. Suppose that $(\mathcal{C}_1, \mathcal{D}_1)$ is a C^*-diagonal and $\alpha : (\mathcal{C}, \mathcal{D}) \to (\mathcal{C}_1, \mathcal{D}_1)$ is a regular $*$-monomorphism. Since $\mathcal{L}(\mathcal{C}_1, \mathcal{D}_1) = (0)$, Proposition 5.7 gives $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \ker \alpha = (0)$.

The converse follows from Lemma 5.8. ∎

Corollary 5.10. *Suppose that $(\mathcal{C}, \mathcal{D})$ is an inclusion such that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. Then there is a regular $*$-monomorphism of $(\mathcal{C}, \mathcal{D})$ into a C^*-diagonal if and only if $\mathcal{C}$ is abelian.*

Proof. Since the commutator ideal is the intersection of the kernels of all multiplicative linear functionals, the result follows directly from Theorem 4.8 and Example 5.5(3). ∎

6. An Example: Reduced Crossed Products by Discrete Groups

In this section we consider the regular inclusion $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C} = \mathcal{D} \rtimes_r \Gamma$ is the reduced crossed product of the unital abelian C^*-algebra $\mathcal{D} = C(X)$ by a discrete group Γ of homeomorphisms of X.

The main results of this section are: Theorem 6.6, which characterizes when the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{D} \rtimes_r \Gamma$ is abelian in terms of the associated dynamical system; Theorem 6.9, which shows that when $\mathcal{D}^c$ is abelian, $\mathcal{L}(\mathcal{D} \rtimes_r \Gamma, \mathcal{D}^c) = (0)$; and a summary result, Theorem 6.10 which gives a number of characterizations for when $(\mathcal{D} \rtimes_r \Gamma, \mathcal{D})$ regularly embeds into a C^*-diagonal. By choosing the space X and group Γ appropriately, the methods in this section can be used to produce an example of a virtual Cartan inclusion $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is not nuclear, see [6, Theorem 4.4.3 or Theorem 5.1.6].

Some of the results in this section complement results from [32].

We begin by establishing our notation. This is standard material, but we include it because there are a number of variations in the literature.

Throughout, let X be a compact Hausdorff space, let Γ be a discrete group with unit element e acting on X as homeomorphisms of X. Thus there is a homomorphism Ξ of Γ into the group of homeomorphisms of X, and for $(s, x) \in \Gamma \times X$, we will write sx instead of $\Xi(s)(x)$. We will sometimes refer to the pair (X, Γ) as a *discrete dynamical system.* For $s \in \Gamma$, let $\alpha_s \in \operatorname{Aut}(C(X))$ be given by

$$(\alpha_s(f))(x) = f(s^{-1}x), \qquad f \in C(X),\ x \in X.$$

If Y is any set, and $z \in Y$, we use δ_z to denote the characteristic function of the singleton set $\{z\}$.

Let $\mathcal{D} = C(X)$, and let $C_c(\Gamma, \mathcal{D})$ be the set of all functions $a : \Gamma \to \mathcal{D}$ such that $\{s \in \Gamma : a(s) \neq 0\}$ is a finite set. We will sometimes write $a(s, x)$ for the value of $a(s)$ at $x \in X$ instead of $a(s)(x)$. Then $C_c(\Gamma, \mathcal{D})$ is a $*$-algebra under the usual twisted convolution product and adjoint operation: for $a, b \in C_c(\Gamma, \mathcal{D})$,

$$(ab)(t) = \sum_{r \in \Gamma} a(r)\alpha_r(b(r^{-1}t)) \quad \text{and} \quad (a^*)(t) = \alpha_t(a(t^{-1}))^*.$$

Let $\mathcal{C} = C(X) \rtimes_r \Gamma$ be the reduced crossed product of $C(X)$ by Γ.

The group Γ is naturally embedded into $\mathcal{C}$ via $s \mapsto w_s$, where w_s is the element of $C_c(\Gamma, \mathcal{D})$ given by $w_s(t) = \begin{cases} 0 & \text{if } t \neq s \\ I & \text{if } t = s. \end{cases}$ Also, $\mathcal{D}$ is embedded into $C_c(\Gamma, \mathcal{D})$ via the map $d \mapsto dw_e$ and we identify $\mathcal{D}$ with its image under this map. Now $w_s d w_{s^{-1}} = \alpha_s(d)$ and $\operatorname{span}\{dw_s : d \in \mathcal{D}, s \in \Gamma\}$ is norm dense in $\mathcal{C}$, so $\{w_s : s \in \Gamma\} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. Thus $(\mathcal{C}, \mathcal{D})$ is a regular inclusion.

It is well known (see for example, the discussion of crossed products in [6]) that the map $\mathbb{E} : C_c(\Gamma, \mathcal{D}) \to \mathcal{D}$ given by $\mathbb{E}(a) = a(e)$ extends to a faithful conditional expectation $\mathbb{E}$ of $\mathcal{C}$ onto $\mathcal{D}$.

Likewise, the maps $\mathbb{E}_s : C_c(\Gamma, \mathcal{D}) \to \mathcal{D}$ given by $\mathbb{E}_s(a) = a(s)$ extend to norm-one linear mappings $\mathbb{E}_s$ of $\mathcal{C}$ onto $\mathcal{D}$. Notice that for $a \in \mathcal{C}$ and $s \in \Gamma$,

$$\mathbb{E}_s(a) = \mathbb{E}(aw_{s^{-1}}).$$

The maps $\mathbb{E}_s$ allow a useful "Fourier series" viewpoint for elements of $\mathcal{C}$: $a \sim \sum_{s\in\Gamma} \mathbb{E}_s(a)w_s$.

The following is well-known. We sketch a proof for convenience of the reader.

Proposition 6.1. *If $a \in \mathcal{C}$ and $\mathbb{E}_s(a) = 0$ for every $s \in \Gamma$, then $a = 0$.*

Proof. For $a \in C_c(\Gamma, \mathcal{D})$ and $s, t \in \Gamma$, a calculation shows that

$$\mathbb{E}_s(w_t a w_{t^{-1}}) = \alpha_t(\mathbb{E}_{t^{-1}st}(a)); \tag{22}$$

a continuity argument then shows that (22) actually holds for every $a \in \mathcal{C}$.

Let $J = \{a \in \mathcal{C} : \mathbb{E}_t(a) = 0\ \forall t \in \Gamma\}$. Clearly J is closed. Then (22) shows that if $a \in J$ and $s \in \Gamma$, then $w_s a w_{s^{-1}} \in J$. Easy calculations now show that if $d \in \mathcal{D}$, $s \in \Gamma$ and $a \in J$, then $\{da, ad, w_s a, aw_s\} \subseteq J$, and by taking linear combinations and closures, we find that J is a closed two-sided ideal of $\mathcal{C}$. Thus, if $a \in J$, $a^*a \in J$, so that $\mathbb{E}_e(a^*a) = \mathbb{E}(a^*a) = 0$. Hence $a = 0$ by faithfulness of $\mathbb{E}$. This shows that $J = (0)$, completing the proof. ☐

Definition 6.2. We make the following definitions.

1. For $s \in \Gamma$, let $F_s = \{x \in X : sx = x\}$ be the set of fixed points of s.
2. For $s \in \Gamma$, let $\mathfrak{F}_s = \{f \in \mathcal{D} : \mathrm{supp}(f)) \subseteq F_s^\circ\}$. Thus $\{\mathfrak{F}_s : s \in \Gamma\}$ is a family of closed ideals in $\mathcal{D}$.
3. For $x \in X$, let $\Gamma^x := \{s \in \Gamma : sx = x\}$ be the isotropy group at x.
4. For $x \in X$, let $H^x := \{s \in \Gamma : x \in (F_s)^\circ\}$. We will call H^x the *germ isotropy group at x.*

Remarks. We chose the terminology 'germ isotropy' because $s \in H^x$ if and only if the homeomorphisms s and $\mathrm{id}|_X$ agree in a neighborhood of x, that is, they have the same germ. It is easy to see that H^x is a group; in fact, H^x is a normal subgroup of Γ^x. To see that H^x is a normal subgroup of Γ^x, fix $x \in X$ and let $s \in H^x$. Then there exists an open neighborhood V of x such that $V \subseteq F_s$. Let $t \in \Gamma^x$ and put $W = t^{-1}V$. Since $tx = x$, $x \in W$. For $y \in W$, $ty \in V$, so $sty = ty$. Hence $t^{-1}sty = y$. Therefore, $W \subseteq F_{t^{-1}st}$. As W is open and $x \in W$, we see that x belongs to the interior of $F_{t^{-1}st}$, so $t^{-1}st \in H^x$ as desired.

Simple examples show the inclusion of H^x in Γ^x can be proper.

We record a description of the relative commutant of $\mathcal{D}$ in $\mathcal{C}$.

Proposition 6.3. *We have*

$$\begin{aligned}\mathcal{D}^c &= \{a \in \mathcal{C} : \alpha_s(d)\mathbb{E}_s(a) = d\mathbb{E}_s(a) \text{ for all } d \in \mathcal{D} \text{ and all } s \in \Gamma\}\\ &= \{a \in \mathcal{C} : \mathbb{E}_s(a) \in \mathfrak{F}_s \text{ for all } s \in \Gamma\}.\end{aligned}$$

Proof. A computation shows that for $a \in \mathcal{C}$, $d \in \mathcal{D}$ and $s \in \Gamma$,

$$\mathbb{E}_s(da - ad) = (d - \alpha_s(d))\mathbb{E}_s(a). \tag{23}$$

Thus if $a \in \mathcal{D}^c$, we obtain $\alpha_s(d)\mathbb{E}_s(a) = d\mathbb{E}_s(a)$ for every $d \in \mathcal{D}$ and $s \in \Gamma$. Conversely, if $\mathbb{E}_s(a)\alpha_s(d) = d\mathbb{E}_s(a)$ for every $d \in \mathcal{D}$ and $s \in \Gamma$, Proposition 6.1 gives $a \in \mathcal{D}^c$.

For the second equality, suppose that $a \in \mathcal{C}$ and $\mathbb{E}_s(a) \in \mathfrak{F}_s$ for every $s \in \Gamma$. Since $\mathbb{E}_s(a)$ is supported in F_s°, an examination of (23) shows that $\mathbb{E}_s(da - ad) = 0$ for every $d \in \mathcal{D}$. By Proposition 6.1 again, $a \in \mathcal{D}^c$. For the reverse inclusion, suppose that $a \in \mathcal{D}^c$. Then for $d \in \mathcal{D}$ and $s \in \Gamma$, $0 = (d - \alpha_s(d))\mathbb{E}_s(a)$. Thus if $x \in X$ and $\mathbb{E}_s(a)(x) \neq 0$, we have $d(x) - d(s^{-1}x) = 0$ for every $d \in \mathcal{D}$. It follows that the support of $\mathbb{E}_s(a)$ is contained in F_s. But $\mathrm{supp}(\mathbb{E}_s(a))$ is open, so the reverse inclusion holds. ☐

We now describe a representation useful for establishing certain formulae.

The very discrete representation. Let $\mathcal{H} = \ell^2(\Gamma \times X)$. Then $\{\delta_{(t,y)} : (t,y) \in \Gamma \times X\}$ is an orthonormal basis for $\mathcal{H}$. For $f \in C(X)$, $s \in \Gamma$, and $\xi \in \mathcal{H}$, define representations π of $C(X)$ and U of Γ on $\mathcal{H}$ by

$$(\pi(f)\xi)(t,y) = f(ty)\xi(t,y) \quad \text{and} \quad (U_s\xi)(t,y) = \xi(s^{-1}t,y).$$

In particular,

$$\pi(f)\delta_{(t,y)} = f(ty)\delta_{(t,y)} \quad \text{and} \quad U_s\delta_{(t,y)} = \delta_{(st,y)}.$$

The C^*-algebra generated by the images of π and U is isometrically isomorphic to the reduced crossed product of $C(X)$ by Γ (see [6, pages 117-118]), and hence determines a faithful representation $\theta : \mathcal{C} \to \mathcal{B}(\mathcal{H})$.

A computation shows that for $a \in \mathcal{C}$, $t, r \in \Gamma$ and $x, y \in X$,

$$\left\langle \theta(a)\delta_{(t,y)}, \delta_{(r,x)} \right\rangle = \begin{cases} 0 & \text{if } x \neq y; \\ \mathbb{E}_{rt^{-1}}(a)(ry) & \text{if } x = y. \end{cases}$$

Also for $a \in \mathcal{C}$, $t \in \Gamma$ and $y \in X$, we have

$$\theta(a)\delta_{(t,y)} = \sum_{s \in \Gamma} \mathbb{E}_s(a)(sty)\delta_{(st,y)}. \tag{24}$$

We now define some notation. Let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation, and for $x \in X$, regard $\ell^2(H^x)$ as a subspace of $\ell^2(\Gamma)$. Then $C_r^*(H^x)$ is the C^*-algebra generated by $\{\lambda_s|_{\ell^2(H^x)} : s \in H^x\}$. Define $V_x : \ell^2(H^x) \to \mathcal{H}$ by

$$(V_x\eta)(s,y) = \begin{cases} 0 & \text{if } (s,y) \notin H^x \times \{x\} \\ \eta(s) & \text{if } (s,y) \in H^x \times \{x\}. \end{cases}$$

Then for $r \in H^x$, we have $V_x\delta_r = \delta_{(r,x)}$, so V_x is an isometry.

Proposition 6.4. *For $x \in X$ and $a \in \mathcal{C}$, define $\Phi_x(a) := V_x^*\theta(a)V_x$. Then Φ_x is a completely positive unital mapping of $\mathcal{C}$ onto $C_r^*(H^x)$ and $\Phi_x|_{\mathcal{D}^c}$ is a $*$-epimorphism of $\mathcal{D}^c$ onto $C_r^*(H^x)$.*

Proof. Clearly Φ_x is completely positive and unital. For $d \in \mathcal{D}$, $r \in \Gamma$ and $s, t \in H^x$ we have

$$\begin{aligned} \langle \Phi_x(dw_r)\delta_s, \delta_t \rangle &= \langle V_x^*\theta(dw_r)V_x\delta_s, \delta_t \rangle = \left\langle \pi(d)U_r\delta_{(s,x)}, \delta_{(t,x)} \right\rangle \\ &= \left\langle \pi(d)\delta_{(rs,x)}, \delta_{(t,x)} \right\rangle = d(rsx) \left\langle \delta_{(rs,x)}, \delta_{(t,x)} \right\rangle = d(rx) \left\langle \delta_{rs}, \delta_t \right\rangle \\ &= d(rx) \left\langle \lambda_r\delta_s, \delta_t \right\rangle. \end{aligned} \tag{25}$$

Hence for every $d \in \mathcal{D}$ and $r \in \Gamma$,

$$\Phi_x(dw_r) = \begin{cases} 0 & \text{if } r \notin H^x, \\ d(x)\lambda_r|_{\ell^2(H^x)} & \text{if } r \in H^x. \end{cases} \tag{26}$$

Therefore Φ_x maps a set of generators for $\mathcal{C}$ into $C_r^*(H^x)$, giving $\Phi_x(\mathcal{C}) \subseteq C_r^*(H^x)$.

To show that $\Phi_x|_{\mathcal{D}^c}$ is a $*$-homomorphism, it suffices to prove that the range of V_x is an invariant subspace for $\theta(\mathcal{D}^c)$. Note that $\operatorname{range}(V_x) = \overline{\operatorname{span}}\{\delta_{(t,x)} : t \in H^x\}$. Let $a \in \mathcal{D}^c$ and fix $t \in H^x$. We claim that if $s \in \Gamma$, $d \in \mathfrak{F}_s$ and $stx \in \operatorname{supp}(d)$, then $s \in H^x$. Indeed, suppose that $stx \in \operatorname{supp}(d)$. As $t \in H^x$, $stx = sx$. So $sx \in F_s^\circ = F_{s^{-1}}^\circ$, which yields $x \in F_s^\circ$. Thus $s \in H^x$, so the claim holds.

Next by (24) and Proposition 6.3, for $t \in H^x$, we have

$$\theta(a)\delta_{(t,x)} = \sum_{s \in \Gamma} \mathbb{E}_s(a)(stx)\delta_{(st,x)} = \sum_{s \in H^x} \mathbb{E}_s(a)(stx)\delta_{(st,x)} \in \operatorname{range}(V_x),$$

as desired. It follows that $\Phi_x|_{\mathcal{D}^c}$ is a $*$-homomorphism.

It remains to show $\Phi_x(\mathcal{D}^c) = C_r^*(H^x)$. If $s \in H^x$, let $d \in \mathfrak{F}_s$ be such that $d(x) = 1$, and put $a = dw_s$. Then (26) shows that $\Phi_x(a) = \lambda_s|_{\ell^2(H^x)}$. By Proposition 6.3, $a \in \mathcal{D}^c$, and hence $\Phi_x(\mathcal{D}^c)$ is dense in $C_r^*(H^x)$. Since $\Phi_x|_{\mathcal{D}^c}$ is a homomorphism, it has closed range. Therefore $\Phi_x(\mathcal{D}^c) = C_r^*(H^x)$.

$\square$

Let

$$\bigoplus_{x\in X} C_r^*(H^x) := \left\{ f \in \prod_{x\in X} C_r^*(H^x) : \sup_{x\in X} \|f(x)\| < \infty \right\}$$

and for $f \in \bigoplus_{x\in X} C_r^*(H^x)$, define $\|f\| = \sup_{x\in X} \|f(x)\|$. Then with product, addition, scalar multiplication and involution defined point-wise, $\bigoplus_{x\in X} C_r^*(H^x)$ is a C^*-algebra.

Corollary 6.5. *The map $\Phi : \mathcal{C} \to \bigoplus_{x\in X} C_r^*(H^x)$ given by $\Phi(a)(x) = \Phi_x(a)$ is a faithful completely positive unital mapping such that $\Phi|_{\mathcal{D}^c}$ is a $*$-monomorphism.*

Proof. It follows from the definition of Φ_x that Φ is unital and completely positive. Proposition 6.4 shows that $\Phi|_{\mathcal{D}^c}$ is a $*$-homomorphism; it remains to check that Φ is faithful.

For $x \in X$, let tr_x be the the trace on $C_r^*(H^x)$. For $d \in \mathcal{D}$ and $s \in \Gamma$ equation (26) gives,

$$\mathrm{tr}_x(\Phi_x(dw_s)) = \begin{Bmatrix} 0 & \text{if } s \neq e \\ d(x) & \text{if } s = e \end{Bmatrix} = \mathbb{E}(dw_s)(x).$$

This formula extends by linearity and continuity, so that for $a \in \mathcal{C}$, $\mathrm{tr}_x(\Phi_x(a)) = \mathbb{E}(a)(x)$. So if $a \geq 0$ belongs to $\mathcal{C}$ and $\Phi(a) = 0$, then $\mathbb{E}(a) = 0$, so $a = 0$. Thus, Φ is faithful. $\square$

Theorem 6.6. *The relative commutant, $\mathcal{D}^c$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian if and only if H^x is an abelian group for every $x \in X$.*

Proof. Corollary 6.5 shows that if H^x is abelian for every $x \in X$, then $\mathcal{D}^c$ is abelian.

For the converse, we prove the contrapositive. Suppose that H^x is non-abelian for some $x \in X$. Fix $s, t \in H^x$ so that $st \neq ts$. Then $x \in (F_s)^\circ \cap (F_t)^\circ$, so we may find $d \in \mathcal{D}$ so that $d(x) = 1$ and $\overline{\mathrm{supp}}(d) \subseteq (F_s)^\circ \cap (F_t)^\circ$. Then for $h \in \mathcal{D}$ and $z \in X$ we have (by examining the cases $z \in F_s$ and $z \notin F_s$),

$$(\alpha_s(h)(z) - h(z))d(z) = (h(s^{-1}z) - h(z))d(z) = 0.$$

Proposition 6.3 shows that $dw_s \in \mathcal{D}^c$. Likewise, $dw_t \in \mathcal{D}^c$.

Then $dw_s dw_t = d\alpha_s(d)w_{st}$. Note that by choice of d, $s\,\overline{\mathrm{supp}}(d) = \overline{\mathrm{supp}}(d)$. For $z \in X$,

$$\alpha_s(d)(z) = d(s^{-1}z) = \begin{cases} 0 & \text{if } z \notin \mathrm{supp}(d) \\ d(z) & \text{if } z \in \mathrm{supp}(d). \end{cases}$$

Thus, $\alpha_s(d) = d$, and likewise, $\alpha_t(d) = d$. Therefore,

$$(dw_s)(dw_t) = d^2 w_{st} \neq d^2 w_{ts} = (dw_t)(dw_s),$$

so $\mathcal{D}^c$ is not abelian.

$\square$

Proposition 6.7. *Let (X, Γ) be a discrete dynamical system such that for each $x \in X$, the germ isotropy group H^x is abelian. Let $\Gamma_1 \subseteq \Gamma$ be a subgroup of Γ, set*

$$\mathcal{C}_1 := \mathcal{D} \rtimes_r \Gamma_1, \quad \textit{and let} \quad \mathcal{D}_1 = \{x \in \mathcal{C}_1 : xd = dx \textit{ for all } d \in \mathcal{D}\}.$$

Then $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA inclusion and $\mathcal{C}_1 \cap \mathcal{L}(\mathcal{C}, \mathcal{D}^c) \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1)$.

Proof. As $\mathcal{D}_1 \subseteq \mathcal{D}^c$, $\mathcal{D}_1$ is abelian, and as $\mathcal{D}_1$ is the relative commutant of $\mathcal{D}$ in $\mathcal{C}_1$, $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA inclusion. Let $\epsilon : \mathcal{C}_1 \to \mathcal{C}$ be the inclusion map. Notice that each map in the diagram,

$$(\mathcal{C}_1, \mathcal{D}_1) \xrightarrow{\epsilon} (\mathcal{C}, \mathcal{D}_1) \xrightarrow{\mathrm{id}} (\mathcal{C}, \mathcal{D}^c)$$

is a regular map. The first is clearly regular, while the regularity of the second follows from the fact that the relative commutant of $\mathcal{D}_1$ in $\mathcal{C}$ is $\mathcal{D}^c$ and an application of Lemma 2.3. Therefore, $\epsilon : (\mathcal{C}_1, \mathcal{D}_1) \to (\mathcal{C}, \mathcal{D}^c)$ is a regular $*$-monomorphism. An application of Corollary 3.23 completes the proof. $\square$

Notation 6.8. When G is an abelian group with dual group $\hat{G}$, we use the notation $\langle g, \gamma \rangle$ to denote the value of $\gamma \in \hat{G}$ at $g \in G$. Also, we will identify $C^*(G)$ with $C(\hat{G})$; lastly, for $\gamma \in \hat{G}$ and $a \in C^*(G)$, we will write $\gamma(a)$ instead of $\hat{a}(\gamma)$.

Theorem 6.9. *Suppose that (X, Γ) is a discrete dynamical system such that for each $x \in X$, the germ isotropy group H^x is abelian. Then $\mathcal{L}(\mathcal{C}, \mathcal{D}^c) = (0)$.*

Proof. First assume that Γ is a countable discrete group. Let

$$P = \prod_{x \in X} \widehat{H^x}$$

be the Cartesian product of the dual groups. Denote by $p(x)$ the "x-th component" of $p \in P$. For $(x, p) \in X \times P$, define a state $\rho_{(x,p)}$ on $\mathcal{C}$ by

$$\rho_{(x,p)}(a) = p(x)\,(\Phi_x(a)) \quad \text{(here } a \in \mathcal{C}\text{)}, \quad \text{and let} \quad A := \{\rho_{(x,p)} : (x, p) \in X \times P\}.$$

Corollary 6.5 shows that the restriction of $\rho_{(x,p)}$ to $\mathcal{D}^c$ is a multiplicative linear functional, so in particular, $A \subseteq \mathrm{Mod}(\mathcal{C}, \mathcal{D}^c)$.

For each $s \in \Gamma$, let

$$X_s := (X \setminus F_s) \cup F_s^{\circ}.$$

Then X_s is a dense, open subset of X. Set

$$Y := \bigcap_{s \in \Gamma} X_s \quad \text{and} \quad B := \{\rho_{(y,p)} : (y, p) \in Y \times P\}.$$

Our goal is to show that

$$\overline{B} \subseteq \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c). \tag{27}$$

Fix $(y, p) \in Y \times P$, and suppose that $\tau \in \mathrm{Mod}(\mathcal{C}, \mathcal{D}^c)$ satisfies $\rho_{(y,p)}|_{\mathcal{D}^c} = \tau|_{\mathcal{D}^c}$. We claim that $\rho_{(y,p)} = \tau$. To see this, it suffices to show that for each $s \in \Gamma$, $\rho_{(y,p)}(w_s) = \tau(w_s)$. Given $s \in \Gamma$, if $sy \neq y$, we may choose $d \in \mathcal{D}$ so that $d(sy) = 1$ and $d(y) = 0$. Using (26),

$$\rho_{(y,p)}(d) = p(y)(\Phi_y(d)) = 0 \quad \text{and} \quad \rho_{(y,p)}(w_s^* d w_s) = \rho_{(y,p)}(\alpha_{s^{-1}}(d)) = p(y)(d(sy)I) = 1.$$

Then

$$\rho_{(y,p)}(w_s) = \rho_{(y,p)}(w_s)\rho_{(y,p)}(w_s^* d w_s) = \rho_{(y,p)}(w_s(w_s^* d w_s)) = \rho_{(y,p)}(d)\rho_{(y,p)}(w_s) = 0.$$

Likewise, $\tau(w_s) = 0$, so $\tau(w_s) = \rho_{(y,p)}(w_s) = 0$ when $y \notin F_s$.

On the other hand, if $sy = y$, then as $y \in X_s$, we have $y \in F_s^{\circ}$, so $s \in H^y$. Choose $d \in \mathcal{D}$ so that $\hat{d}(y) = 1$ and $\operatorname{supp} \hat{d} \subseteq F_s^{\circ}$. Then $d w_s \in \mathcal{D}^c$, so that

$$\rho_{(p,y)}(w_s) = \rho_{(y,p)}(d w_s) = \tau(d w_s) = \tau(w_s).$$

Therefore, $\rho_{(p,y)} = \tau$.

Let $\mathfrak{U}(\mathcal{C}, \mathcal{D}^c) = \{\tau \in \mathrm{Mod}(\mathcal{C}, \mathcal{D}^c) : \tau|_{\mathcal{D}^c} \text{ extends uniquely to } \mathcal{C}\}$. The previous paragraph shows that $B \subseteq \mathfrak{U}(\mathcal{C}, \mathcal{D}^c)$. By Theorem 3.13, $\overline{B} \subseteq \overline{\mathfrak{U}(\mathcal{C}, \mathcal{D}^c)} = \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c)$, so (27) holds.

Suppose now that $a \in \mathcal{L}(\mathcal{C}, \mathcal{D}^c)$. Then for every $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c)$, we have $\rho(a^*a) = 0$. In particular, for each $(y,p) \in Y \times P$,

$$0 = \rho_{(y,p)}(a^*a) = p(y)\left(\Phi_y(a^*a)\right).$$

Now $\widehat{H^y} = \{p(y) : p \in P\}$, so holding y fixed and varying p, yields $\Phi_y(a^*a) = 0$. Hence, we have $\mathbb{E}_e(a^*a)(y) = \mathbb{E}(a^*a)(y) = 0$ for every $y \in Y$. By Baire's theorem, Y is dense in X, so that $\mathbb{E}(a^*a) = 0$. Since $\mathbb{E}$ is faithful, $a = 0$. This gives the theorem in the case when Γ is countable.

We turn now to the general case. Let Γ be any discrete group and suppose $a \in \mathcal{L}(\mathcal{C}, \mathcal{D}^c)$. Then there exists a countable subgroup $\Gamma_1 \subseteq \Gamma$ such that $a \in \mathcal{D} \rtimes_r \Gamma_1$. Put $\mathcal{C}_1 = \mathcal{D} \rtimes_r \Gamma_1$ and let $\mathcal{D}_1 = \{x \in \mathcal{C}_1 : dx = xd \text{ for all } d \in \mathcal{D}\}$ be the relative commutant of $\mathcal{D}$ in $\mathcal{C}_1$. By Proposition 6.7, we have $a \in \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1) = (0)$. This completes the proof. $\square$

We collect the main results of this section into a main theorem.

Theorem 6.10. *Let X be a compact Hausdorff space and let Γ be a discrete group acting as homeomorphisms on X. Let $\mathcal{C} = C(X) \rtimes_r \Gamma$ and $\mathcal{D} = C(X)$. The following statements are equivalent.*

a) *For every $x \in X$, the germ isotropy group H^x is abelian;*
b) *The relative commutant, $\mathcal{D}^c$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian;*
c) $\mathcal{L}(\mathcal{C}, \mathcal{D}^c) = (0)$*;*
d) $(\mathcal{C}, \mathcal{D})$ *regularly embeds into a C^*-diagonal.*

Proof. Theorem 6.6 gives the equivalence of (a) and (b) and Theorem 6.9 shows that (a) implies (c).

Suppose (c) holds. Since $\operatorname{Rad}(\mathcal{C}, \mathcal{D}^c) \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D}^c)$, Theorem 5.9 shows that $(\mathcal{C}, \mathcal{D}^c)$ regularly embeds into a C^*-diagonal. Lemma 2.3 shows that the inclusion map of $(\mathcal{C}, \mathcal{D})$ into $(\mathcal{C}, \mathcal{D}^c)$ is a regular $*$-monomorphism. Composing the embedding of $(\mathcal{C}, \mathcal{D}^c)$ into a C^*-diagonal with the inclusion map shows that (c) implies (d).

Finally, if (d) holds, Theorem 5.1 shows that $\mathcal{D}^c$ is abelian, so (d) implies (b). $\square$

7. A Description of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ for a Regular MASA Inclusion

For a regular inclusion, $(\mathcal{C}, \mathcal{D})$, the $\mathcal{D}$-radical, $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$ is the intersection of the left kernels of compatible states, and when $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is the intersection of the left kernels of strongly compatible states. Question 5.6 asks whether it is possible for these ideals to be distinct. In order to make progress on this question, it seems likely that a description of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ will be useful. The purpose of this section is to provide this description.

The description is in terms of groups which are determined locally by the action of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ and certain positive definite forms on these groups.

We begin with some generalities on $\mathbb{T}$-groups, and describe a class of positive-definite functions on $\mathbb{T}$-groups which behave like compatible states. The following is more-or-less standard.

Definition 7.1. Let G be a locally compact group with identity element 1, and let U be the connected component of the identity. We say that G is a $\mathbb{T}$-*group* if U is clopen, isomorphic and homeomorphic to $\mathbb{T}$, and contained in the center of G. A subgroup H of G is a $\mathbb{T}$-*subgroup of* G if H contains U. When G is a $\mathbb{T}$-group, we will always identify U with $\mathbb{T}$ (and so will write $G/\mathbb{T}$ instead of G/U).

Equivalently, a $\mathbb{T}$-group is a central extension of $\mathbb{T}$ by a discrete group K,

$$1 \to \mathbb{T} \hookrightarrow G \overset{q}{\twoheadrightarrow} K \to 1.$$

If $f : G \to \mathbb{T}$ is a continuous homomorphism, we define the *index* of f to be the unique integer n for which $f(\lambda) = \lambda^n$ for every $\lambda \in \mathbb{T}$.

As a set, G may be identified with $\mathbb{T}\times K$, and the topology on G is the product of the usual topology on $\mathbb{T}$ with the discrete topology on K. Also, the Haar measure on G is the product of Haar measure on $\mathbb{T}$ with the counting measure on K.

The $\mathbb{T}$-subgroups of G are in one-to-one correspondence with the subgroups of K: if H is a $\mathbb{T}$-subgroup of G, then $q(H)$ is a subgroup of K and for any subgroup Γ of K, $q^{-1}(\Gamma)$ is a $\mathbb{T}$-subgroup of G.

We also recall that a function $f : G \to \mathbb{C}$ is *positive definite* if f is continuous, and if for every $n\in\mathbb{N}$ and $g_1,\dots,g_n\in G$, the $n\times n$ complex matrix, $A := (f(g_i^{-1}g_j))_{i,j}$ satisfies $A\geq 0$.

Proposition 7.2. *Let G be a $\mathbb{T}$-group with identity* 1.

(1) *Let $f : G\to\mathbb{C}$ be a positive-definite function such that $f(1) = 1$ and which satisfies $|f(g)|\in\{0,1\}$ for every $g\in G$. Set*
$$H := \{g\in G : f(g)\neq 0\}.$$
Then
a) *$f(g_1g_2) = f(g_1)f(g_2)$ for any $g_1,g_2\in G$ such that $H\cap\{g_1,g_2\}\neq\emptyset$; and*
b) *H is a $\mathbb{T}$-subgroup of G and $f|_H$ is a continuous homomorphism of H onto $\mathbb{T}$.*

(2) *Let $H\subseteq G$ be a $\mathbb{T}$-subgroup and suppose $\phi : H\to\mathbb{T}$ is a continuous homomorphism. Define $f : G\to\mathbb{C}$ by*
$$f(g) = \begin{cases}\phi(h) & \text{if } h\in H\\ 0 & \text{if } h\notin H.\end{cases}$$
Then f is a positive definite function on G such that $f(1) = 1$ and for every $g\in G$, $|f(g)|\in\{0,1\}$.

Proof. Suppose f satisfies the hypotheses in (1). Since $f(1) = 1$ and f is positive definite, we have $\overline{f(g)} = f(g^{-1})$ for every $g\in G$. Continuity of f and connectedness of $\mathbb{T}$ yield $\mathbb{T}\subseteq H$.

Recall that if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, $A\in\mathcal{B}(\mathcal{H}_1)$, $C\in\mathcal{B}(\mathcal{H}_2)$, and $B\in\mathcal{B}(\mathcal{H}_2,\mathcal{H}_1)$ with A invertible, then

$$\begin{pmatrix}A & B\\ B^* & C\end{pmatrix}\geq 0 \quad\text{if and only if } A\geq 0,\ C\geq 0 \text{ and } C-B^*A^{-1}B\geq 0. \tag{28}$$

Let $g_1,g_2\in G$. Then the positive definiteness of f (using the group elements $h_1 = 1, h_2 = g_1^{-1}$, and $h_3 = g_2$) implies that
$$\begin{pmatrix}1 & f(g_1^{-1}) & f(g_2)\\ f(g_1) & 1 & f(g_1g_2)\\ f(g_2^{-1}) & f(g_2^{-1}g_1^{-1}) & 1\end{pmatrix}\geq 0.$$
Applying (28) with $A = (1)$, $B = \begin{pmatrix}f(g_1^{-1}) & f(g_2)\end{pmatrix}$, and $C = \begin{pmatrix}1 & f(g_1g_2)\\ f(g_2^{-1}g_1^{-1}) & 1\end{pmatrix}$ gives
$$0\leq\begin{pmatrix}1-f(g_1)f(g_1^{-1}) & f(g_1g_2)-f(g_1)f(g_2)\\ f(g_2^{-1}g_1^{-1})-f(g_2^{-1})f(g_1^{-1}) & 1-f(g_2^{-1})f(g_2)\end{pmatrix} =: M.$$

Suppose now that $g_2\in H$, that is, $f(g_2)\neq 0$. Then $f(g_2)f(g_2^{-1}) = 1$, so that
$$M = \begin{pmatrix}1-f(g_1^{-1})f(g_1) & f(g_1g_2)-f(g_1)f(g_2)\\ f(g_2^{-1}g_1^{-1})-f(g_2^{-1})f(g_1^{-1}) & 0\end{pmatrix}.$$
But then $0\leq\det(M) = -|f(g_1g_2)-f(g_1)f(g_2)|^2$, so $f(g_1g_2) = f(g_1)f(g_2)$.

The case when $f(g_1)\neq 0$ is the same. Thus when $H\cap\{g_1,g_2\}\neq\emptyset$ we obtain,
$$f(g_1g_2) = f(g_1)f(g_2).$$

The facts that H is a $\mathbb{T}$-subgroup of G and $f|_H : H \to \mathbb{T}$ is a continuous homomorphism are now apparent.

Turning now to statement (2), let H be a $\mathbb{T}$-subgroup of G, and ϕ a continuous homomorphism of H into $\mathbb{T}$. Let $f : G \to \mathbb{T}$ be given as in the statement. The continuity of f is clear, as is the fact that $f(1) = 1$ and $|f(g)| \in \{0,1\}$ for every $g \in G$. To show f is positive definite, let $g_1, \ldots, g_n \in G$. Since $\phi(h) = \overline{\phi(h^{-1})}$, it follows that $f(g_i^{-1} g_j) = \overline{f(g_j^{-1} g_i)}$. Put $X = \{1, \ldots, n\}$. Define an equivalence relation R on X by $(i,j) \in R$ if and only if $g_i^{-1} g_j \in H$, and let X/R be the set of equivalence classes. Let $q : X \to X/R$ be the map which sends $j \in X$ to its equivalence class, and let $u : X/R \to X$ be a section for q. Let $\delta_{x,y}$ be the Kronecker delta function on X/R and for $x \in X/R$, define

$$c_x := \begin{pmatrix} f(g_{u(q(1))}^{-1} g_1)\delta_{q(1),x} & \cdots & f(g_{u(q(n))}^{-1} g_n)\delta_{q(n),x} \end{pmatrix}.$$

Then $c_x^* c_x$ is an $n \times n$ matrix whose i,j-th entry is

$$f(g_i^{-1} g_{u(q(i))}) f(g_{u(q(j))}^{-1} g_j) \delta_{q(i),x} \delta_{q(j),x} = \begin{cases} f(g_i^{-1} g_j) & \text{if } q(i) = q(j) = x \\ 0 & \text{otherwise.} \end{cases}$$

Hence the i,j-th entry of $\sum_{x \in X/R} c_x^* c_x$ is $f(g_i^{-1} g_j)$ if $(i,j) \in R$ and 0 otherwise. Therefore

$$(f(g_i^{-1} g_j))_{i,j \in X} = \sum_{x \in X/R} c_x^* c_x \geq 0,$$

as desired.

□

Corollary 7.3. *Let f be a continuous positive definite function on the $\mathbb{T}$-group G such that $|f(g)| \in \{0,1\}$ for every $g \in G$. Then there exists $p \in \mathbb{Z}$ such that for every $\lambda \in \mathbb{T}$ and $g \in G$,*

$$f(\lambda g) = \lambda^p f(g).$$

Proof. The set $\{g \in G : f(g) \neq 0\}$ contains $\mathbb{T}$, and Proposition 7.2 shows the restriction of f to $\mathbb{T}$ is a character on $\mathbb{T}$. So there exists $p \in \mathbb{Z}$ such that $f(\lambda) = \lambda^p$ for every $\lambda \in \mathbb{T}$. The corollary now follows from another application of Proposition 7.2. □

Definition 7.4. Given a $\mathbb{T}$-group G, call a positive-definite function f on G satisfying $f(1) = 1$ and $|f(g)| \in \{0,1\}$ a *pre-homomorphism.*

We will call the number p appearing in Corollary 7.3 the *index* of f, and will denote it by $\mathrm{ind}(f)$. Finally, the group $H := \{g \in G : f(g) \neq 0\}$ will be called the *supporting subgroup* for f, and will be denoted by $\mathrm{supp}(f)$.

Notation. Some notation will be useful.

(1) Let $(\mathcal{C}, \mathcal{D})$ be an inclusion. For $\sigma \in \hat{\mathcal{D}}$, let

$$\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma) := \{\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}) : \rho|_{\mathcal{D}} = \sigma\}, \qquad \text{and}$$
$$\mathfrak{S}_s(\mathcal{C}, \mathcal{D}, \sigma) := \{\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}) : \rho|_{\mathcal{D}} = \sigma\}.$$

(2) For any $\mathbb{T}$-group G, let $\mathrm{pHom}_1(G)$ denote the set of all pre-homomorphisms $f : G \to \mathbb{T} \cup \{0\}$ with $\mathrm{ind}(f) = 1$.

(3) Finally, recall the seminorms, $B_{\rho,\sigma}$ on $\mathcal{C}$ from [10, Definition 2.4]: for each $\rho, \sigma \in \hat{\mathcal{D}}$, the seminorm $B_{\rho,\sigma}$ is defined on $\mathcal{C}$ by

$$B_{\rho,\sigma}(x) := \inf\{\|dxe\| : d, e \in \mathcal{D}, \rho(d) = \sigma(e) = 1\} \qquad (x \in \mathcal{C}).$$

We shall require these seminorms for $\rho = \sigma$; however, instead of writing $B_{\sigma,\sigma}$ we shall write B_σ.

Associated to each regular inclusion $(\mathcal{C}, \mathcal{D})$ and $\sigma \in \hat{\mathcal{D}}$ is a certain $\mathbb{T}$-group, denoted H_σ/R_1, which we now construct. We produce a distinguished unitary representation T of H_σ/R_1. Our goal is to exhibit a bijection between elements of

$$\{f \in \mathrm{pHom}_1(H_\sigma/R_1) : f \text{ determines a state on the } C^*\text{-algebra generated by } T(H_\sigma/R_1)\}$$

and $\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$.

Definition 7.5. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, and let $\sigma \in \hat{\mathcal{D}}$.

(1) Define

$$H_\sigma := \{v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) : \sigma(v^*dv) = \sigma(d) \text{ for every } d \in \mathcal{D}\}.$$

We remark that H_σ is the set which arises when considering the $\mathcal{D}$-stabilizer of $\rho \in \mathrm{Mod}(\mathcal{C}, \mathcal{D})$, where $\sigma = \rho|_{\mathcal{D}}$ (see Definition 4.13). However, there our interest was in a particular extension of σ, while here we do not wish to specify the extension.

Notice that for $v \in H_\sigma$, we have $\sigma(v^*v) = 1$, and that H_σ is closed under the adjoint operation: replace d by v^*dv in the definition. Furthermore, it is easy to see that H_σ is a $*$-semigroup.

(2) Let $\Lambda \subseteq \mathbb{T}$ be a subgroup, (we write the product multiplicatively). Define

$$R_\Lambda := \{(v, w) \in H_\sigma \times H_\sigma : B_\sigma(\lambda I - w^*v) = 0 \text{ for some } \lambda \in \Lambda\}.$$

When $\Lambda = \{1\}$, we write R_1 instead of $R_{\{1\}}$.

Lemma 7.6. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, let $\sigma \in \hat{\mathcal{D}}$, and let $v, w \in H_\sigma$. Then*

$$B_\sigma(v) = 1 \quad \textit{and} \quad B_\sigma(v - w) = B_\sigma(I - v^*w) = B_\sigma(I - vw^*).$$

Proof. Note that for any $h \in \mathcal{D}$, $|\sigma(h)| = \inf\{\|d^*hd\| : d \in \mathcal{D}, \sigma(d) = 1\}$. Hence given $\varepsilon > 0$, and $e \in \mathcal{D}$ with $\sigma(e) = 1$, we may find $d \in \mathcal{D}$ with $\sigma(d) = 1$ and

$$|\|d^*v^*e^*evd\| - \sigma(v^*e^*ev)| < \varepsilon.$$

Since $\sigma(v^*e^*ev) = \sigma(e^*e) = 1$, we obtain $1 - \varepsilon < \|evd\|^2 < 1 + \varepsilon$. Hence $1 - \varepsilon < B_\sigma(v)^2 < 1 + \varepsilon$, and the fact that $B_\sigma(v) = 1$ follows.

Next, let $x, d, e \in \mathcal{D}$ with $\sigma(x) = \sigma(d) = \sigma(e) = 1$. Since $\sigma(xvdv^*) = 1$, we have

$$\begin{aligned} B_\sigma(v - w) &\le \|(xvdv^*)(v - w)e\| = \|xv(dv^*ve - dv^*we)\| \\ &\le \|xv\| \, \|dv^*ve - dv^*we\| \\ &\le \|xv\| \left[\|dv^*ve - de\| + \|d(I - v^*w)e\|\right]. \end{aligned}$$

It follows that

$$B_\sigma(v - w) \le B_\sigma(I - v^*w).$$

A similar argument using multiplication on the right by w^*ewx gives $B_\sigma(I - v^*w) \le B_\sigma(w^* - v^*)$. But $B_\sigma(v^* - w^*) = B_\sigma(v - w)$, so we obtain $B_\sigma(v - w) = B_\sigma(I - v^*w)$. Finally, $B_\sigma(v - w) = B_\sigma(v^* - w^*) = B_\sigma(I - vw^*)$.

∎

Proposition 7.7. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, $\sigma \in \hat{\mathcal{D}}$ and suppose that $v \in H_\sigma$. Let $\Lambda \subseteq \mathbb{T}$ be a subgroup. The following statements are equivalent:*

(1) *for some $\lambda \in \Lambda$, $B_\sigma(\lambda I - v) = 0$;*
(2) *there exists $\lambda \in \Lambda$ such that $f(v) = \lambda$ whenever $f \in Mod(\mathcal{C}, \mathcal{D}, \sigma)$;*
(3) *there exists $\lambda \in \Lambda$ such that $\rho(v) = \lambda$ whenever $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}, \sigma)$;*
(4) *$\sigma \in (\mathrm{fix}\, \beta_v)^\circ$ and $\hat{v}(\sigma) \in \Lambda$ (where $\hat{v}$ is as in Remark 3.19);*
(5) *there exists $\lambda \in \Lambda$ and $h, k \in \mathcal{D}$ such that $\sigma(h) = 1 = \sigma(k)$ and $vh = \lambda k$.*

Proof. (1) $\Rightarrow$ (2). If $f \in \mathrm{Mod}(\mathcal{C}, \mathcal{D}, \sigma)$ and $x \in \mathcal{C}$, then $|f(x)| = |f(dxe)| \leq \|dxe\|$ whenever $d, e \in \mathcal{D}$ and $\sigma(d) = \sigma(e) = 1$. Thus, $|f(x)| \leq B_\sigma(x)$ for every $x \in \mathcal{C}$. The implication (1) $\Rightarrow$ (2) follows.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4). Suppose $\lambda \in \Lambda$ and that $\rho(v) = \lambda$ for every $\rho \in \{f \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}) : f|_{\mathcal{D}} = \sigma\}$. By Lemma 2.14, $\sigma \in \operatorname{fix} \beta_v$. To show that $\sigma \in (\operatorname{fix} \beta_v)^\circ$, we argue by contradiction. So suppose that $\sigma \in \operatorname{fix} \beta_v \setminus (\operatorname{fix} \beta_v)^\circ$. Then every neighborhood of σ contains an element in $\hat{\mathcal{D}} \setminus \operatorname{fix} \beta_v$. Hence we may find a net (σ_s) in $\hat{\mathcal{D}}$ such that $\sigma_s \to \sigma$ and such that $\sigma_s \notin \operatorname{fix} \beta_v$. By Theorem 3.21, the restriction map $f \mapsto f|_{\mathcal{D}}$ from $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ to $\hat{\mathcal{D}}$ is onto. Thus we may choose $f_s \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ such that $f_s|_{\mathcal{D}} = \sigma_s$. By passing to a subnet if necessary, we may assume that f_s converges to a state f. Theorem 3.21 shows that $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ is closed, so $f \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$. Clearly $f|_{\mathcal{D}} = \sigma$. Lemma 2.14 gives $f_s(v) = 0$ for every s, so $0 \neq \lambda = f(v) = \lim_s f_s(v) = 0$. This is absurd, so we conclude that $\sigma \in (\operatorname{fix} \beta_v)^\circ$.

Let $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}, \sigma)$. Then for $h \in J_v$, we have

$$\hat{v}(\sigma)\sigma(h) = \sigma(vh) = \rho(vh) = \rho(v)\sigma(h). \tag{29}$$

By Proposition 3.2, $\sigma|_{J_v} \neq 0$. Statement (4) now follows from equation (29).

(4) $\Rightarrow$ (5). Let $\lambda = \hat{v}(\sigma)$ and choose $h \in J_v$ such that $\sigma(h) = 1$. Then $\sigma(vh) = \hat{v}(\sigma)$. Put $k = \overline{\hat{v}(\sigma)}vh$.

(5) $\Rightarrow$ (1). Let $h, k \in \mathcal{D}$ be chosen so that $\sigma(h) = \sigma(k) = 1$ and $vh = \lambda k$. Then

$$B_\sigma(\lambda I - v) \leq \inf_{\{d \in \mathcal{D} : \sigma(d) = 1\}} \|d(\lambda I - v)hd\| = |\sigma(\lambda h - vh)| = 0.$$

Thus (1) holds.

$\square$

We next observe that B_σ gives the quotient norm on the quotient of $\overline{\operatorname{span}} H_\sigma$ by the ideal generated by $\ker \sigma$.

Proposition 7.8. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and suppose $\sigma \in \hat{\mathcal{D}}$. Let $\mathcal{C}_\sigma = \overline{\operatorname{span}} H_\sigma$. Then $(\mathcal{C}_\sigma, \mathcal{D})$ is a regular inclusion. If $\mathfrak{J}_\sigma$ is the closed, two-sided ideal of $\mathcal{C}_\sigma$ generated by* $\ker \sigma$*, then B_σ vanishes on $\mathfrak{J}_\sigma$, and for $x \in \mathcal{C}_\sigma$,*

$$B_\sigma(x) = \inf\{\|x + j\| : j \in \mathfrak{J}_\sigma\}.$$

Moreover, the following statements hold.

1. *If ρ is a state on $\mathcal{C}_\sigma$ which annihilates $\mathfrak{J}_\sigma$, then ρ extends uniquely to a state $\tilde{\rho}$ on $\mathcal{C}$. When $\rho \in \mathfrak{S}(\mathcal{C}_\sigma, \mathcal{D})$ annihilates $\mathfrak{J}_\sigma$, $\tilde{\rho} \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.*
2. *The map $H_\sigma \ni u \mapsto u + \mathfrak{J}_\sigma \in \mathcal{C}_\sigma/\mathfrak{J}_\sigma$ is a $*$-homomorphism of the $*$-semigroup H_σ into the unitary group of $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$.*

Proof. Since H_σ is closed under multiplication and the adjoint map, we see that $\mathcal{C}_\sigma$ is a C^*-algebra.

For $d \in \mathcal{D}$, $d = \lim_{t \to 0} tI + d$. But for all sufficiently small $t \neq 0$, $(t + \sigma(d))^{-1}(tI + d) \in H_\sigma$, so d is a limit of scalar multiples of elements of H_σ. So $\mathcal{D} \subseteq \mathcal{C}_\sigma$. Therefore $(\mathcal{C}_\sigma, \mathcal{D})$ is a regular inclusion.

Next, suppose that $x \in \mathcal{C}$ and $d \in \ker \sigma$. Then $B_\sigma(xd) = B_\sigma(dx) = 0$. When $x = v \in H_\sigma$ and $y \in \mathcal{C}_\sigma$,

$$B_\sigma(vdy) = B_\sigma(vv^*vdy) = B_\sigma(vdv^*vy) = |\sigma(vdv^*)| B_\sigma(vy) = 0.$$

It follows that when $x \in \operatorname{span} H_\sigma$, we have $B_\sigma(xdy) = 0$. Taking closures we obtain $B_\sigma(xdy) = 0$ when $x, y \in \mathcal{C}_\sigma$ and $d \in \ker \sigma$. Therefore, $B_\sigma(z) = 0$ for every $z \in \mathfrak{J}_\sigma$.

This gives $B_\sigma(x) \leq B_\sigma(x + j) + B_\sigma(j) = B_\sigma(x + j) \leq \|x + j\|$ for every $x \in \mathcal{C}_\sigma$ and $j \in \mathfrak{J}_\sigma$. Thus, $B_\sigma(x) \leq \operatorname{dist}(x, \mathfrak{J}_\sigma)$.

Notice that if ρ is a state on $\mathcal{C}_\sigma$ which annihilates $\mathfrak{I}_\sigma$, then $\rho|_{\mathcal{D}}$ annihilates $\ker\sigma$, so $\rho|_{\mathcal{D}} = \sigma$. Any extension of σ to a state g on $\mathcal{C}$ belongs to $\mathrm{Mod}(\mathcal{C}, \mathcal{D})$. Hence $|g(x)| \leq B_\sigma(x)$ for every $x \in \mathcal{C}$. In particular, $|\rho(x)| \leq B_\sigma(x)$ for every $x \in \mathcal{C}_\sigma$. So if $x \in \mathcal{C}_\sigma$, we obtain,

$$\|x + \mathfrak{I}_\sigma\| = \sup\{|\rho(x)| : \rho \text{ is a state on } \mathcal{C}_\sigma \text{ and } \rho|_{\mathfrak{I}_\sigma} = 0\} \leq B_\sigma(x).$$

Hence B_σ gives the quotient norm on $\mathcal{C}_\sigma/\mathfrak{I}_\sigma$.

Turning now to statement (1), let ρ be a state on $\mathcal{C}_\sigma$ which annihilates $\mathfrak{I}_\sigma$, and suppose for $i = 1, 2$, that τ_i are states on $\mathcal{C}$ with $\tau_i|_{\mathcal{C}_\sigma} = \rho$. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) \setminus \{\lambda u : u \in H_\sigma \text{ and } \lambda \in \mathbb{C}\}$.

We claim that $\tau_i(v) = 0$. Since ρ annihilates $\mathfrak{I}_\sigma$, we have $\sigma = \rho|_{\mathcal{D}} = \tau_i|_{\mathcal{D}}$. Suppose that $\sigma(v^*v) \neq 0$. By multiplying v by a suitable scalar, we may assume that $\sigma(v^*v) = 1$. Since v is not a scalar multiple of an element of H_σ, we see that $\beta_v(\sigma) \neq \sigma$. Lemma 2.14 shows that $\tau_i(v) = 0$. On the other hand, if $\sigma(v^*v) = 0$, then $v^*v \in \mathfrak{I}_\sigma$, so $\tau_i(v) = \lim_{n\to\infty} \tau_i(v(v^*v)^{1/n}) = 0$. Thus the claim holds.

Since $\tau_1(v) = \tau_2(v)$ for every $v \in H_\sigma$, we have $\tau_1(v) = \tau_2(v)$ for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Hence $\tau_1 = \tau_2$. Thus ρ extends uniquely to a state on $\mathcal{C}$. Notice also that if $\rho \in \mathfrak{S}(\mathcal{C}_\sigma, \mathcal{D})$, then this argument shows that $\tilde{\rho} \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

To prove statement (2), use the fact that B_σ gives the quotient norm on $\mathcal{C}_\sigma/\mathfrak{I}_\sigma$ and apply Lemma 7.6 to $u \in H_\sigma$. $\square$

The following is a corollary of Proposition 7.8.

Proposition 7.9. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion, $\sigma \in \hat{\mathcal{D}}$ and let Λ be a subgroup of $\mathbb{T}$. Then R_Λ is an equivalence relation on H_σ. Denote the equivalence class of $v \in H_\sigma$ by $[v]_\Lambda$. The product $[v]_\Lambda[w]_\Lambda := [vw]_\Lambda$ is a well-defined product on H_σ/R_Λ. With this product, $[I]_\Lambda$ is the unit and for each $v \in H_\sigma$, $[v]_\Lambda^{-1} = [v^*]_\Lambda$. Thus H_σ/R_Λ is a group.*

Furthermore, the map $T_\sigma : H_\sigma/R_1 \to \mathcal{C}_\sigma/\mathfrak{I}_\sigma$ given by $T_\sigma([u]_1) = u + \mathfrak{I}_\sigma$ is a one-to-one group homomorphism of H_σ/R_1 into the unitary group of $\mathcal{C}_\sigma/\mathfrak{I}_\sigma$, and $T_\sigma(H_\sigma/\mathfrak{I}_\sigma)$ generates $\mathcal{C}_\sigma/\mathfrak{I}_\sigma$.

Proof. Let $u, v \in H_\sigma$. By Proposition 7.8 and Lemma 7.6, $(u, v) \in R_\Lambda$ if and only if there exists $\lambda \in \Lambda$ such that $u + \mathfrak{I}_\sigma = \lambda v + \mathfrak{I}_\sigma$. Routine arguments now show that H_σ/R_Λ is a group under the indicated operations. The final statement follows from Proposition 7.8(2).

$\square$

Lemma 7.10. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, $\sigma \in \hat{\mathcal{D}}$, and suppose $u, v \in H_\sigma$ are such that $(u, v) \in R_1$. Then the following statements hold.*

(1) *If $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$, then*
$$\rho(v) = \rho(u).$$
If in addition, $0 \neq \rho(v)$ then $\rho(v) \in \mathbb{T}$.

(2) *$\sigma \in (\mathrm{fix}\, \beta_v)^\circ$ if and only if $\sigma \in (\mathrm{fix}\, \beta_u)^\circ$, and when this occurs, $\hat{v}(\sigma) = \hat{u}(\sigma) \in \mathbb{T}$.*

Proof. Suppose $\rho(v) \neq 0$. Since $|\rho(x)| \leq B_\sigma(x)$ for every $x \in \mathcal{C}$ and $B_\sigma(I - u^*v) = 0$, we have $\rho(u^*v) = 1$. Therefore, by part 1 of Proposition 4.3,

$$\rho(u) = \rho(u)\rho(u^*v) = \rho(uu^*v) = \sigma(uu^*)\rho(v) = \rho(v).$$

Likewise, if $\rho(u) \neq 0$, then $\rho(u) = \rho(v)$. Thus, we have $\rho(u) = \rho(v)$ whenever $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$.

Next, when $\rho(v) \neq 0$, the fact that $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ gives

$$|\rho(v)|^2 = \rho(v^*v) = \sigma(v^*v) = 1,$$

so $\rho(v) \in \mathbb{T}$. This completes the proof of the first statement.

We now turn to the second statement. Since $(u, v) \in R_1$, Proposition 7.7 implies $\sigma \in (\mathrm{fix}\, \beta_{v^*u})^\circ$. Thus $\sigma \in (\mathrm{fix}\, \beta_v)^\circ$ if and only if $\sigma \in (\mathrm{fix}\, \beta_u)^\circ$.

Next suppose that $\sigma \in (\operatorname{fix}(\beta_v))^\circ$. Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$ and let E be the pseudo-expectation for ι. Let $h \in J_v$ satisfy $\sigma(h) = 1$ and let $\rho \in \widehat{I(\mathcal{D})}$ be such that $\rho \circ \iota = \sigma$. Then $\rho \circ E \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$, and Proposition 3.18 shows that $\rho \in \operatorname{supp}(\widehat{E(v)})$. Thus

$$\hat{v}(\sigma) = \sigma(vh) = \rho(E(v)) = \rho(E(w)) = \sigma(wh) = \hat{w}(\sigma).$$

Since $\rho(E(v)) \neq 0$, we have $\hat{v}(\sigma) \in \mathbb{T}$ by part (1). $\square$

Remark. Lemma 7.10 shows that for $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ with $\rho|_{\mathcal{D}} = \sigma$, we have a well-defined map $\tilde{\rho} : H_\sigma/R_1 \to \mathbb{T} \cup \{0\}$ given by $\tilde{\rho}([v]) = \rho(v)$.

Theorem 7.11. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion, and let $\sigma \in \hat{\mathcal{D}}$. The function*

$$d([v], [w]) := B_\sigma(v - w)$$

is a well defined metric on H_σ/R_1 and makes H_σ/R_1 into a $\mathbb{T}$-group. More specifically, the following statements hold.

(1) *Let*

$$U = \{[v] \in H_\sigma/R_1 : \sigma \in (\operatorname{fix} \beta_v)^\circ\}.$$

Then U is clopen and is the connected component of the identity in H_σ/R_1.

(2) *The subgroup U is contained in the center of H_σ/R_1.*

(3) *The map $[v] \in U \mapsto \hat{v}(\sigma)$ is an isomorphism of U onto $\mathbb{T}$.*

(4) *The quotient of H_σ/R_1 by U is $H_\sigma/R_{\mathbb{T}}$.*

Proof. Let T_σ be the isomorphism of H_σ/R_1 onto a subgroup of the unitary group of $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$ defined in Proposition 7.9. Then

$$d([v], [w]) = \|T_\sigma([v]) - T_\sigma([w])\|_{\mathcal{C}_\sigma/\mathfrak{J}_\sigma}.$$

It follows that d is a well-defined metric which makes H_σ/R_1 into a topological group.

We now show that U is an open set. Let $u \in H_\sigma$ be such that $[u] \in U$ and suppose that $v \in H_\sigma$ satisfies $d([u], [v]) < 1/2$. We will show that $\sigma \in (\operatorname{fix} \beta_v)^\circ$. To do this we modify the proof of the implication (3) $\Rightarrow$ (4) in Proposition 7.7 slightly. Since $B_\sigma(u - v) < 1/2$, for every $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$, we have $|\rho(u) - \rho(v)| < 1/2$.

Suppose, to obtain a contradiction, that $\sigma \notin (\operatorname{fix} \beta_v)^\circ$. Then we may find find a directed set S and a net $(\sigma_s)_{s \in S}$ such that $\sigma_s \notin \operatorname{fix} \beta_v$ for every s and such that $\sigma_s \to \sigma$. As usual, let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$ and let E be the pseudo-expectation for ι. For each s, choose $\tau_s \in \widehat{I(\mathcal{D})}$ such that $\tau_s \circ \iota = \sigma_s$. Passing to a subnet if necessary, we may assume that τ_s converges to $\tau \in \widehat{I(\mathcal{D})}$. Then $\tau \circ E \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ and $\tau \circ E|_{\mathcal{D}} = \sigma$. Notice that $\tau(E(v)) \neq 0$ because $|\tau(E(v)) - \tau(E(u))| < 1/2$ and $\tau(E(u)) \in \mathbb{T}$. Since $\tau_s \circ E|_{\mathcal{D}} = \sigma_s \notin \operatorname{fix} \beta_v$, Lemma 2.14 shows that $\tau_s(E(v)) = 0$ for every $s \in S$. Then

$$\tau(E(v)) = \lim_s \tau_s(E(v)) = 0,$$

contradicting the fact that $\tau(E(v)) \neq 0$. Thus $\sigma \in (\operatorname{fix} \beta_v)^\circ$. Therefore $[v] \in U$, so U is an open subset of H_σ/R_1.

Similarly, the complement of U is open in H_σ/R_1, so U is also closed.

Let $\gamma : U \to \mathbb{T}$ be the map $\gamma([u]) = \hat{u}(\sigma)$. Suppose that $[u], [v] \in U$ and $\hat{u}(\sigma) = \hat{v}(\sigma)$. Then

$$d([u], [v]) = B_\sigma(u - v) \leq B_\sigma(u - \hat{u}(\sigma)I) + B_\sigma(\hat{v}(\sigma)I - v) = 0,$$

so γ is one-to-one. Since $\gamma([\lambda u]) = \lambda\gamma([u])$ for any $\lambda \in \mathbb{T}$, γ is onto.

Let $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$. Since

$$|\tilde{\rho}([v]) - \tilde{\rho}([w])| = |\rho(v - w)| \leq B_\sigma(v - w) = d([v], [w]),$$

we see that $\tilde{\rho}$ is a continuous map on H_σ/R_1. By Lemma 7.10, for $[v] \in U$, $\tilde{\rho}([v]) = \gamma([v])$. So γ is also continuous. The map $\lambda \in \mathbb{T} \mapsto [\lambda I]$ is the inverse of γ, and we see that γ is a homeomorphism. In particular, U is connected, and hence U is the connected component of the identity in H_σ/R_1.

To see that U is contained in the center of H_σ/R_1, observe that for $[u] \in U$, we have $[u] = [\gamma([u])I]$, which evidently belongs to the center of H_σ/R_1.

Since the connected component of the identity is compact, H_σ/R_1 is a locally compact group. Finally, $[v] = [w] \mod U$ if and only if $[v^*w] \in U$. Therefore, $[v] = [w] \mod U$ if and only if $B_\sigma(\gamma([v^*w])I - v^*w) = 0$. Hence the quotient of H_σ/R_1 by U is $H_\sigma/R_{\mathbb{T}}$.

□

We now are prepared to exhibit a bijection between $\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$ and a class of pre-homomorphisms on H_σ/R_1. We pause for some notation.

Let $q : \mathcal{C}_\sigma \to \mathcal{C}_\sigma/\mathfrak{J}_\sigma$ be the quotient map. Define a $*$-homomorphism $\theta : C_c(H_\sigma/R_1) \to \mathcal{C}_\sigma/\mathfrak{J}_\sigma$ by

$$\theta(\phi) = \int_{H_\sigma/R_1} \phi(s) T_\sigma(s)\, ds,$$

where ds is Haar measure on H_σ/R_1. Then the image of θ is dense in $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$.

Definition 7.12. We will say that a positive definite function f on H_σ/R_1 is *dominated by B_σ* if for every $\phi \in C_c(G)$,

$$\left| \int_{H_\sigma/R_1} \phi(t) f(t)\, dt \right| \le \left\| \int_{H_\sigma/R_1} \phi(t) T_\sigma(t)\, dt \right\|_{\mathcal{C}_\sigma/\mathfrak{J}_\sigma}.$$

Theorem 7.13. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion and let $\sigma \in \hat{\mathcal{D}}$. Let*

$$M_\sigma := \{ f \in pHom_1(H_\sigma/R_1) : f \text{ is dominated by } B_\sigma \}.$$

For $\tau \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$ the map $\tilde{\tau} : H_\sigma/R_1 \to \mathbb{C}$ given by

$$\tilde{\tau}([v]_1) = \tau(v)$$

is well-defined, and $\tilde{\tau} \in M_\sigma$. Moreover, the map $\tau \mapsto \tilde{\tau}$ is a bijection between $\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$ and M_σ.

Proof. If $\tau \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$, then $|\tau(x)| \le B_\sigma(x)$. Hence τ annihilates $\mathfrak{J}_\sigma$, so that τ determines a state τ' on $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$ such that

$$\tau|_{\mathcal{C}_\sigma} = \tau' \circ q.$$

Then $\tilde{\tau} = \tau' \circ T_\sigma$, so $\tilde{\tau}$ is well-defined.

Set

$$G := \{ [v] \in H_\sigma/R_1 : \tilde{\tau}([v]) \neq 0 \}.$$

Proposition 4.3(1) implies that G is closed under products. For $v \in H_\sigma$ with $[v] \in G$, we have $\tau(v) \in \mathbb{T}$, and as τ is a state, $\tau(v^*) = \overline{\tau(v)}$. Therefore, $[v]^{-1} = [v^*] \in G$, so G is closed under inverses. It follows that G is a $\mathbb{T}$-subgroup of H_σ/R_1. Since $\tau \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, $|\tau([v])| \in \{0, 1\}$ for every $[v] \in H_\sigma/R_1$. Proposition 7.2 implies that $\tilde{\tau}$ is a pre-homomorphism on H_σ/R_1. Since τ is linear, for $\lambda \in \mathbb{T}$ and $v \in H_\sigma$, we have $\tilde{\tau}([\lambda v]_1) = \tau(\lambda v) = \lambda \tilde{\tau}([v]_1)$. So $\operatorname{ind}(\tilde{\tau}) = 1$.

For $\phi \in C_c(H_\sigma/R_1)$ we have

$$\left| \int_{H_\sigma/R_1} \phi(s) \tilde{\tau}(s)\, ds \right| = \left| \tau' \left(\int_{H_\sigma/R_1} \phi(s) T_\sigma(s)\, ds \right) \right| \le \left\| \int_{H_\sigma/R_1} \phi(s) T_\sigma(s)\, ds \right\|_{\mathcal{C}_\sigma/\mathfrak{J}_\sigma}.$$

Thus, $\tilde{\tau}$ is dominated by B_σ. Therefore, $\tilde{\tau} \in M_\sigma$.

Next we show that the map $\tau \mapsto \tilde{\tau}$ is surjective. So suppose that $f : H_\sigma/R_1 \to \{0\} \cup \mathbb{T}$ is a pre-homomorphism dominated by B_σ and $\operatorname{ind}(f) = 1$. Then the map $F_0 : \theta(C_c(H_\sigma/R_1)) \to \mathbb{C}$ given by

$$F_0\left(\int_{H_\sigma/R_1} \phi(t)T_\sigma(t)\,dt\right) = \int_{H_\sigma/R_1} \phi(t)f(t)\,dt \qquad (\phi \in C_c(H_\sigma/R_1))$$

extends by continuity to a bounded linear functional F on $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$. Since θ is a $*$-homomorphism and f is a positive definite function, F is a positive linear functional. Clearly $\|F\| \leq 1$.

We next show that $F(v+\mathfrak{J}_\sigma) = f([v])$ for every $v \in H_\sigma$. Since H_σ/R_1 is a $\mathbb{T}$-group, the connected component of the identity is $\mathbb{T}$. Since f is continuous, $\operatorname{ind}(f) = 1$ and $f(1) = 1$, we have $f(\lambda) = \lambda$ for every $\lambda \in \mathbb{T}$. Given $v \in H_\sigma$, let $\phi \in C_c(H_\sigma/R_1)$ be the function given by

$$\phi(t) = \begin{cases} \overline{\lambda} & \text{if } t = [\lambda v]_1 \text{ for some } \lambda \in \mathbb{T} \\ 0 & \text{otherwise.} \end{cases}$$

Now for $t \in \{[\lambda v]_1 : \lambda \in \mathbb{T}\} \subseteq H_\sigma/R_1$, we have $T_\sigma(t) = \lambda q(v)$, so

$$\theta(\phi) = \int_{H_\sigma/R_1} \phi(t)T_\sigma(t)\,dt = \int_{\mathbb{T}} \overline{\lambda} q(\lambda v)dt = q(v).$$

Thus,

$$F(q(v)) = \int_{H_\sigma/R_1} \phi(t)f(t)\,dt = \int_{\mathbb{T}} \overline{\lambda} f([\lambda v]_1)\,dt = f([v]_1).$$

It follows that $\|F\| = 1$, so F is a state on $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$.

As $\mathcal{N}(\mathcal{C}_\sigma, \mathcal{D}) \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$, we see that if $w \in \mathcal{N}(\mathcal{C}_\sigma, \mathcal{D})$, then w is a scalar multiple of an element of H_σ. Since $|F \circ q(v)| = |f([v]_1)| \in \{0, 1\}$ for each $v \in H_\sigma$, we find $F \circ q \in \mathfrak{S}(\mathcal{C}_\sigma, \mathcal{D})$. Proposition 7.8 shows that $F \circ q$ extends uniquely to an element $\tau \in \mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$. As $\tilde{\tau} = f$, we find that the map $\tau \mapsto \tilde{\tau}$ is onto.

To show that $\tau \mapsto \tilde{\tau}$ is one-to-one, suppose τ and τ_1 belong to $\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma)$ and $\tilde{\tau} = \tilde{\tau_1}$. Then $\tau(v) = \tau_1(v)$ for every $v \in H_\sigma$, so that $\tau|_{\mathcal{C}_\sigma} = \tau_1|_{\mathcal{C}_\sigma}$. Proposition 7.8 then shows $\tau = \tau_1$.

Thus the mapping $\tau \mapsto \tilde{\tau}$ is indeed a bijection. ∎

We conclude this section with a pair of very closely related questions and two examples.

Notation 7.14. Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and let

$$\mathfrak{B}(\mathcal{C}, \mathcal{D}) := \{x \in \mathcal{C} : B_\sigma(x) = 0 \text{ for all } \sigma \in \hat{\mathcal{D}}\}.$$

By Proposition 7.8, $\mathfrak{B}(\mathcal{C}, \mathcal{D}) = \{x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in \operatorname{Mod}(\mathcal{C}, \mathcal{D})\}$, and by Proposition 2.16, $\mathfrak{B}(\mathcal{C}, \mathcal{D})$ is an ideal of $\mathcal{C}$ which satisfies

$$\mathfrak{B}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Rad}(\mathcal{C}, \mathcal{D}). \tag{30}$$

Question 7.15. *Let $(\mathcal{C}, \mathcal{D})$ be a regular MASA inclusion.*

(1) *Let $\sigma \in \hat{\mathcal{D}}$. For $x \in \mathcal{C}$, is $B_\sigma(x) = \sup_{\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})} \rho(x^*x)^{1/2}$?*
(2) *Is $\mathfrak{B}(\mathcal{C}, \mathcal{D}) = \operatorname{Rad}(\mathcal{C}, \mathcal{D})$?*

When $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = (0)$, the answer to both questions is yes.

Example 7.16. This example applies the previous results to a special case of reduced crossed products to produce a Cartan inclusion $(\mathcal{C}, \mathcal{D})$ and a pure state τ on $\mathcal{C}$ such that $\tau|_{\mathcal{D}} \in \hat{\mathcal{D}}$, yet $\tau \notin \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

Let Γ be an infinite discrete group, let $X := \Gamma \cup \{\infty\}$ be the one-point compactification of Γ and let Γ act on X by extending the left regular representation to X: for $s \in \Gamma$ and $x \in X$, let

$$sx := \begin{cases} sx & \text{if } x \in \Gamma; \\ x & \text{if } x = \infty. \end{cases}$$

We now use the notation from Section 6: let $\mathcal{C} = C(X) \rtimes_r \Gamma$ and $\mathcal{D}$ be the cannonical image of $C(X)$ in $\mathcal{C}$. We identify $\hat{\mathcal{D}}$ with X. The action of Γ on X is topologically free, so $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion. Moreover, the conditional expectation $\mathbb{E} : \mathcal{C} \to \mathcal{D}$ is the pseudo-expectation.

Let $\sigma \in \hat{D}$ be the map $\sigma(f) = f(\infty)$. We claim that $H_\sigma/R_{\mathbb{T}} = \Gamma$ and $H_\sigma/R_1 = \mathbb{T} \times \Gamma$. Observe first that $\mathfrak{S}(\mathcal{C}, \mathcal{D}, \sigma) = \{\sigma \circ \mathbb{E}\}$ and the map $\theta : \Gamma \to H_\sigma/R_{\mathbb{T}}$ given by $\theta(s) = [w_s]_{\mathbb{T}}$ is a group homomorphism. Next, suppose $v \in H_\sigma$. Then there exists $t \in \Gamma$, so that $\mathbb{E}_t(v) = \mathbb{E}(vw_{t^{-1}}) \neq 0$. Since both v and $w_{t^{-1}} \in H_\sigma$ we have $vw_{t^{-1}} \in H_\sigma$, so $\sigma(\mathbb{E}(vw_{t^{-1}})) \neq 0$. By Proposition 7.7, $(v, w_t) \in R_{\mathbb{T}}$. Thus θ is surjective. Proposition 7.7 also implies θ is one-to-one: if $[w_s]_{\mathbb{T}} = [w_t]_{\mathbb{T}}$ then $\sigma(\mathbb{E}(w_{st^{-1}})) \neq 0$, so that $\mathbb{E}(w_{st^{-1}}) \neq 0$. Thus $s = t$. Therefore θ is an isomorphism of $H_\sigma/R_{\mathbb{T}}$ onto Γ, and we use θ to identify Γ with $H_\sigma/R_{\mathbb{T}}$. The map $\Gamma \ni s \mapsto [w_s]_1 \in H_\sigma/R_1$ is a group homomorphism and also a section for the quotient map of H_σ/R_1 onto $H_\sigma/R_{\mathbb{T}}$. It follows that H_σ/R_1 is isomorphic to $\mathbb{T} \times \Gamma$.

Let $\rho = \sigma \circ \mathbb{E}$ and let $(\pi_\rho, \mathcal{H}_\rho)$ be the GNS representation of $\mathcal{C}$ associated to ρ. Proposition 7.8 implies that $\mathfrak{J}_\sigma \subseteq \ker \pi_\rho$, so π_ρ induces a representation, again denoted π_ρ, of $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$ on $\mathcal{H}_\sigma$.

We shall show that the image of $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$ under π_ρ is isomorphic to $C^*_r(\Gamma)$. To do this, first observe that for $v, w \in H_\sigma$, $v + L_\rho = w + L_\rho$ if and only if $(v, w) \in R_1$. Indeed, since $v, w \in H_\sigma$, $\rho(w^*w) = \rho(v^*v) = 1$, so $\rho((v-w)^*(v-w)) = 2 - 2\Re(\rho(v^*w))$. Since $|\rho(v^*w)| \in \{0, 1\}$, we get $v + L_\rho = w + L_\rho$ if and only if $\rho(v^*w) = 1$, which by Proposition 7.7 gives the observation. This observation and regularity of $\mathcal{C}_\sigma$ implies that

$$\{w_s + L_\rho : s \in \Gamma\}$$

is an orthonormal basis for $\mathcal{H}_\rho$. Thus there is a unitary operator $U : \ell^2(\Gamma)$ onto $\mathcal{H}_\rho$ which carries the basis element $\delta_s \in \ell^2(\Gamma)$ to $w_s + L_\rho \in \mathcal{H}_\rho$.

Now let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation. For $s, t \in \Gamma$ we have $U\lambda(s)\delta_t = w_{st} + L_\rho = \pi_\rho(w_s)U\delta_t$, so

$$U\lambda(s) = \pi_\rho(w_s)U. \tag{31}$$

It follows from Proposition 7.9 that the set $\{w_s + \mathfrak{J}_\sigma : s \in \Gamma\}$ generates $\mathcal{C}_\sigma/\mathfrak{J}_\sigma$, so (31) shows that $\pi_\rho(\mathcal{C}_\sigma/\mathfrak{J}_\sigma)$ is isomorphic to $C^*_r(\Gamma)$.

Thus there is a surjective $*$-homomorphism $\Psi : \mathcal{C}_\sigma \to C^*_r(\Gamma)$ which annihilates $\mathfrak{J}_\sigma$. The composition of Ψ with any pure state f on $C^*_r(\Gamma)$ yields a pure state on $\mathcal{C}_\sigma$, which in turn may be extended to a pure state $\tau \in \mathrm{Mod}(\mathcal{C}, \mathcal{D}, \sigma)$. Apply this process when $\Gamma = \mathbb{F}^2$ is the free group on 2-generators u_1 and u_2. By [26, Theorem 2.6 and Remark 3.4], there exists a pure state f on $C^*_r(\mathbb{F}^2)$ such that $|f(u_1)| \notin \{0, 1\}$. It follows that there exists a pure state τ on $\mathcal{C}$ such that $\tau|_{\mathcal{D}} = \sigma$, yet $\tau \notin \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

Example 7.17. Denote by $\{e_n\}_{n\in\mathbb{N}}$ the standard orthonormal basis for $\mathcal{H} := \ell^2(\mathbb{N})$. Consider the inclusion $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ is the Toeplitz algebra (the C^*-algebra generated by the unilateral shift S acting on $\mathcal{H}$) and $\mathcal{D} \subseteq \mathcal{B}(\mathcal{H})$ is the C^*-algebra generated by $\{S^k S^{*k} : k \geq 0\}$. Then $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion and $\hat{\mathcal{D}}$ is homeomorphic to the one-point compactification of $\mathbb{N}$, $\mathbb{N} \cup \{\infty\}$. We identify $\hat{\mathcal{D}}$ with this space.

Here the pseudo-expectation is the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ which takes $T \in \mathcal{C}$ to the operator $E(T)$ which acts on basis elements via $E(T)e_n = \langle Te_n, e_n \rangle e_n$.

We shall do the following:

(1) give a description of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$;

(2) show that not every element of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is a pure state of $\mathcal{C}$ and identify the pure states in $\mathfrak{S}(\mathcal{C},\mathcal{D})$.

The strongly compatible states are easy to identify. Let ρ_n and ρ_∞ be the states on $\mathcal{C}$ given by $\rho_n(X) = \langle Xe_n, e_n\rangle$ and $\rho_\infty(X) = \lim_{n\to\infty} \rho_n(X)$. Then

$$\mathfrak{S}_s(\mathcal{C},\mathcal{D}) = \{\rho_n : n \in \mathbb{N}\cup\{\infty\}\} = \{\sigma\circ E : \sigma \in \hat{\mathcal{D}}\}.$$

For each $n \in \mathbb{N}$, the set $\{n\}$ is clopen in $\hat{\mathcal{D}}$, so ρ_n is the unique extension of $\rho_n|_{\mathcal{D}}$ to a state on $\mathcal{C}$. (This can be proved directly or viewed as a consequence of Theorem 3.8.)

Thus, to complete a description of $\mathfrak{S}(\mathcal{C},\mathcal{D})$, we need only describe $\mathfrak{S}(\mathcal{C},\mathcal{D},\sigma_\infty)$, where $\sigma_\infty = \rho_\infty|_{\mathcal{D}}$.

To do this, let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the compact operators and let $q : \mathcal{C} \to \mathcal{C}/K = C(\mathbb{T})$ be the quotient map. Given $z \in \mathbb{T}$, we write τ_z for the state on $\mathcal{C}$ given by $\tau_z(T) = q(T)(z)$. Also, for $z \in \mathbb{T}$, we let α_z be the gauge automorphism on $\mathcal{C}$ determined by $\alpha_z(S) = zS$. For each $N \in \mathbb{N}$, let $\lambda(N) = \exp(2\pi i/N)$ and define $\Phi_N : \mathcal{C} \to \mathcal{C}$ by

$$\Phi_N(T) := \frac{1}{N}\sum_{k=0}^{N-1} \alpha^k_{\lambda(N)}(T).$$

(Note that if $T = \sum_{k=-p}^{p} a_k S^k$ is a "trigonometric polynomial," then $\Phi_N(T) = \sum_{k\in N\mathbb{Z}} a_k S^k$.)

We claim that

$$\mathfrak{S}(\mathcal{C},\mathcal{D},\sigma_\infty) = \{\rho_\infty\} \cup \{\tau_z \circ \Phi_N : z \in \mathbb{T}, N \in \mathbb{N}\}. \tag{32}$$

Each state of the form $\tau_z \circ \Phi_1 = \tau_z$ is multiplicative on $\mathcal{C}$. Therefore, $\{\tau_z \circ \Phi_1 : z \in \mathbb{T}\}$ is a set of pure states and is a subset of $\mathfrak{S}(\mathcal{C},\mathcal{D})$. Also, we have

$$\rho_\infty = \int_{\mathbb{T}} \tau_z \, dz \quad \text{and for } N \geq 1, \quad \tau_z \circ \Phi_N = \frac{1}{N}\sum_{k=0}^{N-1} \tau_{\lambda(N)^k z},$$

so the only pure states on the right hand side of (32) are those of the form τ_z. We will show that states of the form $\tau_z \circ \Phi_N$ are compatible states. We proceed by first identifying the elements of $\mathcal{N}(\mathcal{C},\mathcal{D})$ with $\sigma_\infty(v^*v) \neq 0$.

Suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ satisfies $\sigma_\infty(v^*v) > 0$. Put $A_v = \{n \in \mathbb{N} : \langle v^*ve_n, e_n\rangle \neq 0\}$ and $B_v = \{n \in \mathbb{N} : \langle vv^*e_n, e_n\rangle \neq 0\}$. Then β_v induces a bijection $f : A_v \to B_v$, and there exist scalars c_j so that

$$v_{ij} := \langle ve_j, e_i\rangle = \begin{cases} c_j & \text{if } i = f(j); \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that $q(v) = cq(S)^m$ for some $c \in \mathbb{C}$ and $m \in \mathbb{Z}$. Since $\sigma_\infty(v^*v) \neq 0$, $c \neq 0$, so v is a Fredholm operator. Let m be the Fredholm index of v. Then $q(v) = \rho_\infty(vS^{-m})q(S)^m$. Thus $v = S^m d$ for some $d \in \mathcal{D}$ with $\sigma_\infty(d) \neq 0$. The fact that each $\tau_z \circ \Phi_N \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, now follows. It remains to show that we have found all elements of $\mathfrak{S}(\mathcal{C},\mathcal{D},\sigma_\infty)$.

We will write H_∞, B_∞, C_∞, and $\mathfrak{J}_\infty$ rather than the more cumbersome H_{σ_∞}, B_{σ_∞}, $\mathcal{C}_{\sigma_\infty}$, and $\mathfrak{J}_{\sigma_\infty}$. Then

$$H_\infty = \{S^m d : m \in \mathbb{Z} \text{ and } d \in \mathcal{D}, \sigma_\infty(d) \in \mathbb{T}\}.$$

Thus, $\mathfrak{J}_\infty = \mathcal{K}$ and $\mathcal{C}_\infty = \mathcal{C}$. By Proposition 7.8, B_∞ gives the quotient norm on $\mathcal{C}_\infty/\mathfrak{J}_\infty = C(\mathbb{T})$.

Next, for $v, w \in H_\infty$, we have $B_\infty(v - w) = 0$ if and only if $\rho_\infty(w^*v) = 1$. In other words, $(v, w) \in R_1$ if and only if $q(v) = q(w)$. Therefore, H_∞/R_1 is isomorphic to the direct product $\mathbb{T}\times\mathbb{Z}$. Observe that $H_\infty/R_{\mathbb{T}}$ is isomorphic to $\mathbb{Z}$, and we obtain the trivial $\mathbb{T}$-group extension,

$$1 \to \mathbb{T} \to \mathbb{T}\times\mathbb{Z} \to \mathbb{Z} \to 1.$$

The generators of the subgroups of $\mathbb{Z}$ are the non-negative integers, so the $\mathbb{T}$-subgroups of $\mathbb{T} \times \mathbb{Z}$ are

$$\{\mathbb{T} \times n\mathbb{Z} : n \geq 0\}.$$

Let f be a pre-homomorphism of index 1 on H_∞/R_1. Then there exists a non-negative integer N such that f is a character on $\mathbb{T} \times N\mathbb{Z}$. Since f has index 1, there exists $\lambda \in \mathbb{Z}$ such that for $(z,n) \in \mathbb{T} \times \mathbb{Z} = H_\infty/R_1$,

$$f(z,n) = \begin{cases} z\lambda^n & \text{if } n \in N\mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}$$

As B_∞ is the quotient norm on $\mathcal{C}_\infty$, f is dominated by B_∞. Let $\tau = \tau_z \circ \Phi_N$. With the notation of Theorem 7.13, we get $\tilde{\tau} = f$, so by Theorem 7.13, the compatible state corresponding to f is $\tau_z \circ \Phi_N$. This completes the proof of (32).

We now identify the topology on $\mathfrak{S}(\mathcal{C}, \mathcal{D})$. Notice that if M, N are positive integers with $N \notin M\mathbb{Z}$, then for any $z \in \mathbb{T}$, $\tau_z(\Phi_N(S^N)) = \tau_z(S^N) = z^N \neq 0 = \tau_z(\Phi_M(S^N))$. Given distinct positive integers N, M, either $N \notin M\mathbb{Z}$ or $M \notin N\mathbb{Z}$. Thus for $N > 0$, $\{\phi_z \circ \Phi_N : z \in \mathbb{T}\}$ is a connected component of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ and is homeomorphic to $\mathbb{T}$.

We next show that if G is a weak-$*$ open neighborhood of ρ_∞, then there exists $N \in \mathbb{N}$ such that $\mathfrak{T}_n := \{\tau_z \circ \Phi_n : z \in \mathbb{Z}\} \subseteq G$ for every $n \geq N$. To do this, it suffices to show that for every $a \in \mathcal{C}$ and $\varepsilon > 0$, the set $G_{a,\varepsilon} := \{\phi \in \mathfrak{S}(\mathcal{C}, \mathcal{D}) : |\phi(a) - \rho_\infty(a)| < \varepsilon\}$ contains $\mathfrak{T}_n$ for all sufficiently large n, and this is what we shall do. First observe that for every $b \in \mathcal{C}$,

$$\lim_{n \to \infty} \|\Phi_n(b) - E(b)\| = 0. \tag{33}$$

(This is clear for "trigonometric polynomials" in S, approximate b in norm with a trigonometric polynomial to obtain (33).) Fix $N \in \mathbb{N}$ so that $\|\Phi_n(a) - E(a)\| < \varepsilon$ for every $n \geq N$. For any $z \in \mathbb{T}$, we have $\tau_z(E(a)) = \rho_\infty(a)$, so we find

$$|\tau_z(\Phi_n(a)) - \rho_\infty(a)| < \varepsilon.$$

So $\mathfrak{T}_n \subseteq G_{a,\varepsilon}$ for all $n \geq N$.

We conclude that

$$\mathfrak{S}(\mathcal{C}, \mathcal{D}) = \mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \cup \{\tau_z \circ \Phi_N : z \in \mathbb{T}, N \in \mathbb{N}\},$$

which may be viewed as the one-point compactification of the space $\mathbb{N} \cup \left(\bigcup_{N \in \mathbb{N}} \mathbb{T} \times \{N\}\right)$, with ρ_∞ corresponding to the point at infinity.

8. The Twist of a Regular Inclusion

Throughout this section, we fix, once and for all, a regular inclusion $(\mathcal{C}, \mathcal{D})$ and a closed $\mathcal{N}(\mathcal{C}, \mathcal{D})$-invariant subset $F \subseteq \mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that the restriction map, $f \in F \mapsto f|_{\mathcal{D}}$, is a surjection of F onto $\hat{\mathcal{D}}$. When $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, Theorem 3.13 shows $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq F$. For this reason, we have in mind taking $F = \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$, though other choices (e.g. $F = \mathfrak{S}(\mathcal{C}, \mathcal{D})$) may be useful for some purposes.

In this section, we show that associated to this data, there is a twist (Σ, G), which, when $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal (in which case F is necessarily $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$) or a Cartan pair (with $F = \mathfrak{S}_s(\mathcal{C}, \mathcal{D})$) gives the twist of the pair as defined by Kumjian [22] for C^*-diagonals, or for Cartan pairs given by Renault in [31]. The set F will be used as the unit space for the étale groupoid G associated to the twist (Σ, G).

Our construction parallels the constructions by Kumjian and Renault, but with several differences. First, in the Renault and Kumjian contexts, a conditional expectation $E : \mathcal{C} \to \mathcal{D}$ is present, and since $\{\rho \circ E : \rho \in \hat{\mathcal{D}}\}$ is homeomorphic to $\hat{\mathcal{D}}$, Renault and Kumjian use $\hat{\mathcal{D}}$ as the unit space for the twists they construct. In our context, we need not have a conditional expectation, so we use the set F as a replacement for $\hat{\mathcal{D}}$. Next, as in the constructions of Kumjian and Renault, we construct a regular $*$-homomorphism $\theta : (\mathcal{C}, \mathcal{D}) \to (C_r^*(\Sigma, G), C(G^{(\circ)}))$, however, $\theta(\mathcal{C})$ need not equal $C_r^*(\Sigma, G)$, and the kernel of θ is not trivial unless the ideal $\mathcal{K}_F = \{x \in \mathcal{C} : f(x^*x) = 0 \text{ for all } f \in F\} = (0)$. We note however, that this ideal is trivial in the cases considered by Kumjian and Renault.

8.1. **Twists and their C^*-algebras.** Before proceeding, it is helpful to recall some generalities on twists and the (reduced) C^*-algebras associated to them.

Definition 8.1. The pair (Σ, G) is a *twist* if Σ and G are Hausdorff locally compact topological groupoids, G is an étale groupoid and the following hold:

(1) there is a free action of $\mathbb{T}$ by homeomorphisms of Σ such that whenever $(\sigma_1, \sigma_2) \in \Sigma^{(2)}$ and $z_1, z_2 \in \mathbb{T}$, we have $(z_1\sigma_1, z_2\sigma_2) \in \Sigma^{(2)}$ and $(z_1\sigma_1)(z_2\sigma_2) = (z_1z_2)(\sigma_1\sigma_2)$;
(2) there is a continuous surjective groupoid homomorphism $\gamma : \Sigma \twoheadrightarrow G$ such that for every $\sigma \in \Sigma$, $\gamma^{-1}(\gamma(\sigma)) = \{z\sigma : z \in \mathbb{T}\}$;
(3) the bundle (Σ, G, γ) is locally trivial.

Notice that $\gamma|_{\Sigma^{(\circ)}} : \Sigma^{(\circ)} \to G^{(\circ)}$ is a homeomorphism of the unit space of Σ onto the unit space of G. We will usually use this map to identify $\Sigma^{(\circ)}$ and $G^{(\circ)}$.

Recall that given a twist Σ over the étale topological groupoid G, one can form the twisted groupoid C^*-algebra of the pair (Σ, G). We summarize the construction in our context, for details, see [31, Section 4] and [22, Section 2]. We note that in both [31] and [22], there is a blanket assumption that the étale groupoid G is second countable. However, for what we require here, this hypothesis is not used. The reader may also wish to consult Section 3 of [12].

Let $C_c(\Sigma, G)$ be the family of all compactly supported continuous complex valued functions f on Σ which are *equivariant*, that is, which satisfy $f(z\sigma) = zf(\sigma)$ for all $\sigma \in \Sigma, z \in \mathbb{T}$. Given $f, g \in C_c(\Sigma, G)$, notice whenever $\tau, \sigma \in \Sigma$ with $s(\tau) = s(\sigma)$ and $z \in \mathbb{T}$, we have $f(\sigma\tau^{-1})g(\tau) = f(\sigma(z\tau)^{-1})g(z\tau)$. For $x \in G$ and $\sigma \in \Sigma$ with $s(x) = s(\sigma)$, let $\tau \in \gamma^{-1}(x)$. Then

$$(f \circledast g)(\sigma, x) := f(\sigma\tau^{-1})g(\tau)$$

does not depend on the choice of $\tau \in \gamma^{-1}(x)$. The product of f with g is defined by

$$(f \star g)(\sigma) = \sum_{\substack{x \in G \\ s(x)=s(\sigma)}} (f \circledast g)(\sigma, x),$$

and the adjoint operation is defined by

$$f^*(\sigma) = \overline{f(\sigma^{-1})}.$$

These operations make $C_c(\Sigma, G)$ into a $*$-algebra.

Similarly, notice that for $f \in C_c(\Sigma, G)$, $z \in \mathbb{T}$ and $\sigma \in \Sigma$, $|f(\sigma)| = |f(z\sigma)|$, so for $x \in G$, we denote by $|f(x)|$ the number $|f(\sigma)|$, where $\sigma \in \gamma^{-1}(x)$. In particular, $|f|$ may be viewed as a function on G.

One may norm $C_c(\Sigma, G)$ as in [31] or [22]. For convenience, we provide a sketch of an equivalent, but slightly different method. Given $x \in G^{(\circ)}$, let $\eta_x : C_c(\Sigma, G) \to \mathbb{C}$ by $\eta_x(f) = f(x)$. Then η_x is a positive linear functional in the sense that for each $f \in C_c(\Sigma, G)$, $\eta_x(f^* \star f) \geq 0$. Let $\mathcal{N}_x := \{f \in C_c(\Sigma, G) : \eta_x(f^* \star f) = 0\}$ and let $\mathcal{H}_x$ be the completion of $C_c(\Sigma, G)/\mathcal{N}_x$ with respect to the inner product $\langle f + N_x, g + N_x\rangle = \eta_x(g^* \star f)$.

Recall that a *slice* of G is an open set $U \subseteq G$ so that $r|_U$ and $s|_U$ are one-to-one. By [12, Proposition 3.10], given $f \in C_c(\Sigma, G)$, there exist $n \in \mathbb{N}$ and slices $U_1, \dots, U_n$ of G such that the support of $|f|$ is contained in $\bigcup_{k=1}^n U_k$. Let $n(f) \geq 0$ be the smallest integer such that there exist slices $U_1, \dots U_{n(f)}$ with the support of $|f|$ is contained in $\bigcup_{k=1}^{n(f)} U_k$.

For $f, g \in C_c(\Sigma, G)$, a calculation shows that

$$\|(f \star g) + \mathcal{N}_x\| \leq n(f)\, \|f\|_\infty\, \|g + \mathcal{N}_x\|_{\mathcal{H}_x}\,.$$

Therefore, the map $g + \mathcal{N}_x \mapsto (f \star g) + \mathcal{N}_x$ extends to a bounded linear operator $\pi_x(f)$ on $\mathcal{H}_x$. It is easy to see that π_x is a $*$-representation of $C_c(\Sigma, G)$. Also, if $\pi_x(f) = 0$ for every $x \in G^{(\circ)}$, then $\|f + \mathcal{N}_x\|_{\mathcal{H}_x} = 0$ for each $x \in G^{(\circ)}$. A calculation then gives $f = 0$. Thus,

$$\|f\| := \sup_x \|\pi_x(f)\|\,.$$

defines a norm on $C_c(\Sigma, G)$. The *(reduced) twisted* C^*-algebra, $C^*(\Sigma, G)$, is the completion of $C_c(\Sigma, G)$ relative to this norm. Clearly, the representation π_x extend by continuity to a representation, again called π_x, of $C^*(\Sigma, G)$.

As observed in the remarks following [31, Proposition 4.1], elements of $C^*(\Sigma, G)$ may be regarded as equivariant continuous functions on Σ, and the formulas defining the product and involution on $C_c(\Sigma, G)$ remain valid for elements of $C^*(\Sigma, G)$. Also, as in [31, Proposition 4.1], for $\sigma \in \Sigma$, and $f \in C^*(\Sigma, G)$,

$$|f(\sigma)| \leq \|f\|\,.$$

Definition 8.2. We shall call the smallest topology on $C^*(\Sigma, G)$ such that for every $\sigma \in \Sigma$, the point evaluation functional, $C^*(\Sigma, G) \ni f \mapsto f(\sigma)$ is continuous, the $G^{(\circ)}$*-compatible topology* on $C^*(\Sigma, G)$. Clearly this topology is Hausdorff.

The open support of $f \in C^*(\Sigma, G)$ is $\operatorname{supp}(f) = \{\sigma \in \Sigma : f(\sigma) \neq 0\}$. Then $C_0(G^{(\circ)})$ may be identified with

$$\{f \in C^*(\Sigma, G) : \operatorname{supp}(f) \subseteq G^{(\circ)}\}.$$

In order to remain within the unital context, we now assume that the unit space of G is compact. In this case $C^*(\Sigma, G)$ is unital, and $C(G^{(\circ)}) \subseteq C^*(\Sigma, G)$, so that $(C^*(\Sigma, G), C(G^{(\circ)}))$ is an inclusion. We wish to show that it is a regular inclusion.

Recall (see [12, Section 3]) that a *slice* (or G-set) of G is an open subset $S \subseteq G$ such that the restrictions of the range and source maps to S are one-to-one. We will say that an element $f \in C^*(\Sigma, G)$ is *supported in the slice* S if $\gamma(\operatorname{supp}(f)) \subseteq S$.

If $f \in C^*(\Sigma, G)$ is supported in a slice U, then a computation (see [31, Proposition 4.8]) shows that $f \in \mathcal{N}(C^*(\Sigma, G), C(G^{(\circ)}))$, and because the collection of slices forms a basis for the topology of G ([12, Proposition 3.5]), it follows (as in [31, Corollary 4.9]) that $(C^*(\Sigma, G), C(G^{(\circ)}))$ is a regular inclusion.

Proposition 8.3. *Let Σ be a twist over the Hausdorff étale groupoid G. Assume that the unit space X of G is compact. Then there is a faithful conditional expectation $E : C^*(\Sigma, G) \to C(G^{(\circ)})$, the inclusion $(C^*(\Sigma, G), C(G^{(\circ)}))$ is regular. If in addition, is $C(G^{(\circ)})$ is a MASA, then $\operatorname{Rad}(C^*(\Sigma, G), C(G^{(\circ)})) = (0)$.*

Remark 8.4. The condition that $C(G^{(\circ)})$ is a MASA is satisfied when $G^{(\circ)}$ is second countable and G is essentially principal, that is, when the interior of the isotropy bundle for G is $G^{(\circ)}$, see [31, Proposition 4.2]. We expect that it is possible to remove the hypothesis of second countability here, but we have not verified this.

Proof of Proposition 8.3. The existence of the conditional expectation is proved as in [31, Proposition 4.3] or [30, Proposition II.4.8], and we have already observed that the inclusion is regular.

When $C(G^{(\circ)})$ is a MASA in $C^*(\Sigma, G)$, the triviality of the radical follows from Proposition 5.7. $\square$

8.2. **Compatible Eigenfunctionals and the Twist for** $(\mathcal{C}, \mathcal{D})$**.** We turn next to a discussion of eigenfunctionals, for a certain class of eigenfunctionals will yield our twist. Recall (see [10]) that an *eigenfunctional* is a non-zero element $\phi \in \mathcal{C}^\#$ which is an eigenvector for both the left and right actions of $\mathcal{D}$ on $\mathcal{C}^\#$; when this occurs, there exist unique elements $\rho, \sigma \in \hat{\mathcal{D}}$ so that whenever $d_1, d_2 \in \mathcal{D}$ and $x \in \mathcal{C}$, we have $\phi(d_1 x d_2) = \rho(d_1)\phi(x)\sigma(d_2)$. We write

$$s(\phi) := \sigma \quad \text{and} \quad r(\phi) := \rho.$$

Definition 8.5. A *compatible eigenfunctional* is a eigenfunctional ϕ such that for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$,

$$|\phi(v)|^2 \in \{0, s(\phi)(v^*v)\}. \tag{34}$$

Let $\mathcal{E}_c(\mathcal{C}, \mathcal{D})$ denote the set consisting of the zero functional together with the set of all compatible eigenfunctionals, and let $\mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$ be the set of compatible eigenfunctionals which have unit norm. Equip both $\mathcal{E}_c(\mathcal{C}, \mathcal{D})$ and $\mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$ with the relative $\sigma(\mathcal{C}^\#, \mathcal{C})$ topology.

Remark. Notice that when ϕ is an eigenfunctional and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is such that $\phi(v) \neq 0$, then for every $d \in \mathcal{D}$,

$$\frac{s(\phi)(v^*dv)}{s(\phi)(v^*v)} = r(\phi)(d). \tag{35}$$

Indeed, $\phi(v)s(\phi)(v^*dv) = \phi(vv^*dv) = \phi(dvv^*v) = r(\phi)(d)\phi(v)s(\phi)(v^*v)$. Thus, taking $d = 1$, the condition in (34) is equivalent to

$$|\phi(v)|^2 \in \{0, r(\phi)(vv^*)\}. \tag{36}$$

We now show that associated with each $\phi \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$ is a pair $f, g \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ which extend $r(\phi)$ and $s(\phi)$. Note that regularity of the inclusion $(\mathcal{C}, \mathcal{D})$ ensures the existence of $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that $\phi(v) > 0$.

Proposition 8.6. *Let $\phi \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$, and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ satisfy $\phi(v) > 0$. Define elements $f, g \in \mathcal{C}^\#$ by*

$$f(x) = \frac{\phi(xv)}{\phi(v)} \quad \text{and} \quad g(x) = \frac{\phi(vx)}{\phi(v)}.$$

Then the following statements hold.

i) *$f, g \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, $r(\phi) = f|_{\mathcal{D}}$ and $s(\phi) = g|_{\mathcal{D}}$.*

ii) *For every $x \in \mathcal{C}$,*

$$\phi(x) = \frac{g(v^*x)}{g(v^*v)^{1/2}} = \frac{f(xv^*)}{f(vv^*)^{1/2}}.$$

iii) *For every $x \in \mathcal{C}$, $g(v^*xv) = g(v^*v)f(x)$ and $f(vxv^*) = f(vv^*)g(x)$.*

Proof. The definitions show $f|_{\mathcal{D}} = r(\phi)$ and $g|_{\mathcal{D}} = s(\phi)$. We next claim that $\|f\| = \|g\| = 1$. For any $d \in \mathcal{D}$ with $s(\phi)(d) = 1$, replacing v by vd in the definition of f does not change f. Thus, if $x \in \mathcal{C}$ and $\|x\| \leq 1$, we have $|f(x)| \leq \inf\{\frac{\|vd\|}{\phi(v)} : d \in \mathcal{D}, s(\phi)(d) = 1\} = 1$ (because d may be chosen so that $\|vd\| = \|d^*v^*vd\|^{1/2}$ is as close to $s(\phi)(v^*v)^{1/2}$ as desired). This shows $\|f\| = 1$. Likewise $\|g\| = 1$. As $f(1) = g(1) = 1$, both f and g are states on $\mathcal{C}$.

If $w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f(w) \neq 0$, we have (using (35))

$$|f(w)|^2 = \left|\frac{\phi(wv)^2}{\phi(v)}\right|^2 = \frac{s(\phi)(v^*w^*wv)}{s(\phi)(v^*v)} = r(\phi)(w^*w) = f(w^*w),$$

and it follows that $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$. Likewise, $g \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

Statements (ii) and (iii) are calculations using (34) and (36) whose verification is left to the reader. $\square$

Notation. For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that $f(v^*v) > 0$, let $[v, f] \in \mathcal{C}^\#$ be defined by

$$[v, f](x) := \frac{f(v^*x)}{f(v^*v)^{1/2}} = \left\langle x + L_f, \frac{v + L_f}{\|v + L_f\|_{\mathcal{H}_f}} \right\rangle_{\mathcal{H}_f}.$$

(This notation is borrowed from Kumjian [22]. There, Kumjian works in the context of C^*-diagonals and uses states on $\mathcal{C}$ of the form $\sigma \circ E$ with $\sigma \in \hat{\mathcal{D}}$. As we assume no conditional expectation here, we replace functionals of the form $\sigma \circ E$, with elements from $\mathfrak{S}(\mathcal{C}, \mathcal{D})$. See also [31].)

We have the following.

Lemma 8.7. *If $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ with $f(v^*v) > 0$, then $[v, f] \in \mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$ and the following statements hold.*

i) *$s([v, f]) = f|_{\mathcal{D}}$ and $r([v, f]) = \beta_v(f|_{\mathcal{D}})$.*
ii) *$[v, f] = [w, g]$ if and only if $f = g$ and $f(v^*w) > 0$.*

Proof. Suppose that $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f(v^*v) \neq 0$. Let $\phi = [v, f]$. A calculation shows that ϕ is a norm-one eigenfunctional and that statement (i) holds.

If $w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $\phi(w) \neq 0$, then

$$|\phi(w)|^2 = \frac{|f(v^*w)|^2}{f(v^*v)} = \frac{f(v^*ww^*v)}{f(v^*v)} = \beta_v(s(\phi))(ww^*) = r(\phi)(ww^*),$$

so ϕ belongs to $\mathcal{E}_c^1(\mathcal{C}, \mathcal{D})$ by (36).

Turning now to part (ii), suppose that $\phi = [v, f] = [w, g]$. Then we have $s(\phi) = f|_{\mathcal{D}} = g|_{\mathcal{D}}$. For every $x \in \mathcal{C}$, Proposition 8.6 gives

$$f(x) = \frac{\phi(vx)}{\phi(v)} \quad \text{and} \quad g(x) = \frac{\phi(wx)}{\phi(w)}.$$

Since $\frac{g(w^*v)}{g(w^*w)^{1/2}} = \phi(v) = f(v^*v)^{1/2}$, we obtain

$$g(w^*v) = f(v^*v)^{1/2} g(w^*w)^{1/2} > 0.$$

Likewise, $f(v^*w) > 0$. Also,

$$f(x) = \frac{\phi(vx)}{\phi(v)} = \frac{[w, g](vx)}{[v, f](v)} = \frac{g(w^*vx)}{f(v^*v)^{1/2} g(w^*w)^{1/2}} = \frac{g(w^*v)g(x)}{g(w^*v)} = g(x),$$

where the fourth equality follows from Proposition 4.3.

Conversely, if $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ and $v, w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ with $f(v^*w) > 0$, Proposition 4.3 shows that $f(v^*w)^2 = f(w^*w)f(v^*v)$, so that in the GNS Hilbert space $\mathcal{H}_f$, we have $\langle v + L_f, w + L_f \rangle = \|v + L_f\| \, \|w + L_f\|$. By the Cauchy-Schwartz inequality, there exists a positive real number t so that $v + L_f = tw + L_f$. But then for any $x \in \mathcal{C}$,

$$[v, f](x) = \frac{\langle x + L_f, v + L_f \rangle}{\|v + L_f\|} = \frac{\langle x + L_f, tw + L_f \rangle}{\|tw + L_f\|} = [w, f](x).$$

$\square$

Combining Proposition 8.6 and Lemma 8.7 we obtain the following.

Theorem 8.8. *If $\phi \in \mathcal{E}^1_c(\mathcal{C}, \mathcal{D})$, then there exist unique elements $\mathfrak{s}(\phi), \mathfrak{r}(\phi) \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that whenever $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ satisfies $\phi(v) \neq 0$ and $x \in \mathcal{C}$,*

$$\phi(vx) = \phi(v)\,\mathfrak{s}(\phi)(x) \quad \textit{and} \quad \phi(xv) = \mathfrak{r}(\phi)(x)\,\phi(v).$$

If $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ satisfies $\phi(v) > 0$, then $\phi = [v, \mathfrak{s}(\phi)]$. Moreover,

$$\mathcal{E}^1_c(\mathcal{C}, \mathcal{D}) = \{[v, f] : v \in \mathcal{N}(\mathcal{C}, \mathcal{D}), f \in \mathfrak{S}(\mathcal{C}, \mathcal{D}) \textit{ and } f(v^*v) \neq 0\}.$$

Proof. Suppose $v, w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ are such that $\phi(v) > 0$ and $\phi(w) > 0$. For $x \in \mathcal{C}$, set

$$f(x) := \frac{\phi(vx)}{\phi(v)} \quad \text{and} \quad g(x) := \frac{\phi(wx)}{\phi(w)}.$$

Proposition 8.6 shows that $\phi = [v, f] = [w, g]$. Lemma 8.7 yields $f = g$. Another application of Proposition 8.6 shows that for any $x \in \mathcal{C}$,

$$\frac{\phi(xv)}{\phi(v)} = \frac{\phi(xw)}{\phi(w)}.$$

Then taking $\mathfrak{s}(\phi) = f$, and $\mathfrak{r}(\phi) = \frac{\phi(xv)}{\phi(v)}$, we obtain the result. ☐

Notice that for $\phi \in \mathcal{E}^1_c(\mathcal{C}, \mathcal{D})$, we have $\mathfrak{s}(\phi) \in F$ if and only if $\mathfrak{r}(\phi) \in F$.

Definition 8.9. Let $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D}) := \{\phi \in \mathcal{E}^1_c(\mathcal{C}, \mathcal{D}) : \mathfrak{s}(\phi) \in F\}$. We shall call $\phi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ an *F-compatible eigenfunctional*. Notice that

$$\mathcal{E}^1_F(\mathcal{C}, \mathcal{D}) = \{[v, f] : f \in F \text{ and } f(v^*v) \neq 0\}.$$

With these preparations in hand, we can show that $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ forms a topological groupoid. The topology has already been defined, so we need to define the source and range maps, composition and inverses.

Definition 8.10. Given $\phi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$, let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ be such that $\phi(v) > 0$. We make the following definitions.

(1) We say that $\mathfrak{s}(\phi)$ and $\mathfrak{r}(\phi)$ are the *source* and *range* of ϕ respectively.
(2) Define the *inverse*, ϕ^{-1} by the formula,

$$\phi^{-1}(x) := \overline{\phi(x^*)}.$$

If $\phi \in \mathcal{E}^1_c(\mathcal{C}, \mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is such that $\phi(v) > 0$, (so that $\phi = [v, \mathfrak{s}(\phi)]$), then a calculation shows that $\phi^{-1} = [v^*, \mathfrak{r}(\phi)]$. The fact that F is $\mathcal{N}(\mathcal{C}, \mathcal{D})$-invariant ensures that $\phi^{-1} \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$. Thus, our definition of ϕ^{-1} is consistent with the definition of inverse in the definition of the twist of a C^*-diagonal arising in [22] and the twist of a Cartan MASA from [31].

(3) For $i = 1, 2$, let $\phi_i \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$. We say that the pair (ϕ_1, ϕ_2) is a *composable pair* if $\mathfrak{s}(\phi_2) = \mathfrak{r}(\phi_1)$. As is customary, we write $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})^{(2)}$ for the set of composable pairs. To define the composition, choose $v_i \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ with $\phi_i(v_i) > 0$, so that $\phi_i = [v_i, \mathfrak{s}(\phi_i)]$. By Proposition 8.6(iii), we have

$$\mathfrak{s}(\phi_2)(v_2^*v_1^*v_1v_2) = \mathfrak{r}(\phi_2)(v_1^*v_1)\mathfrak{s}(\phi_2)(v_2^*v_2) = \mathfrak{s}(\phi_1)(v_1^*v_1)\mathfrak{s}(\phi_2)(v_2^*v_2) > 0,$$

so that $[v_1v_2, \mathfrak{s}(\phi_2)]$ is defined. The product is then defined to be $\phi_1\phi_2 := [v_1v_2, \mathfrak{s}(\phi_2)]$.

We show now that this product is well defined. Suppose that $(\phi_1, \phi_2) \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})^{(2)}$, $f = \mathfrak{s}(\phi_2)$, $\mathfrak{r}(\phi_2) = g = \mathfrak{s}(\phi_1)$, and that for $i = 1, 2$, $v_i, w_i \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ are such that $\phi_1 = [v_1, g] = [w_1, g]$ and $[v_2, f] = [w_2, f]$. Then using Lemma 8.7, we have $g(w_1^*v_1) > 0$ and

$f(v_2^* w_2) > 0$, so, as $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, there exists a positive scalar t such that $v_2 + L_f = tw_2 + L_f$. Hence,

$$\begin{aligned} f((w_1w_2)^*(v_1v_2)) &= \langle \pi_f(v_1)(v_2 + L_f), \pi_f(w_1)(w_2 + L_f)\rangle \\ &= t\,\langle \pi_f(v_1)(w_2 + L_f), \pi_f(w_1)(w_2 + L_f)\rangle \\ &= tf(w_2^*(w_1^*v_1)w_2) \\ &= tf(w_2^*w_2)\mathfrak{r}(\phi_2)(w_1^*v_1) \\ &= tf(w_2^*w_2)\mathfrak{s}(\phi_1)(w_1^*v_1) \\ &= tf(w_2^*w_2)g(w_1^*v_1) > 0. \end{aligned}$$

By Lemma 8.7, $[v_1v_2, f] = [w_1w_2, f]$, so that the product is well defined.

(4) For $\phi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$, denote the map $\mathcal{C} \ni x \mapsto |\phi(x)|$ by $|\phi|$. Observe that for $\phi, \psi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$, $|\phi| = |\psi|$ if and only if there exists $z \in \mathbb{T}$ such that $\phi = z\psi$. Let $\mathfrak{R}_F(\mathcal{C}, \mathcal{D}) := \{|\phi| : \phi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})\}$. We now define source and range maps, along with inverse and product maps on $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$.

Since a state on $\mathcal{C}$ is determined by its values on the positive elements of $\mathcal{C}$, we identify $f \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ with $|f| \in \mathfrak{R}_F(\mathcal{C}, \mathcal{D})$. Define $\mathfrak{s}(|\phi|) = \mathfrak{s}(\phi)$ and $\mathfrak{r}(|\phi|) = \mathfrak{r}(\phi)$. Next we define inversion in $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ by $|\phi|^{-1} = |\phi^{-1}|$, and composable pairs by $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})^{(2)} := \{(|\phi|, |\psi|) : (\phi, \psi) \in \mathcal{E}^1_c(\mathcal{C}, \mathcal{D})^{(2)}\}$, and the product by $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})^{(2)} \ni (|\phi|, |\psi|) \mapsto |\phi\psi|$. Topologize $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ with the topology of point-wise convergence: $|\phi_\lambda| \to |\phi|$ if and only if $|\phi_\lambda|(x) \to |\phi|(x)$ for every $x \in \mathcal{C}$. We call $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ the *spectral groupoid over F* of $(\mathcal{C}, \mathcal{D})$.

(5) Define an action of $\mathbb{T}$ on $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ by $\mathbb{T} \times \mathcal{E}^1_F(\mathcal{C}, \mathcal{D}) \ni (z, \phi) \mapsto z\phi$, where $(z\phi)(x) = \phi(zx)$. Notice that if ϕ is written as $\phi = [v, f]$, where $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f \in F$, then $z\phi = [\overline{z}v, f]$.

We have the following fact, whose proof is essentially the same as that of [10, Proposition 2.3] (the continuity of the range and source maps follows from their definition).

Proposition 8.11. *The set $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D}) \cup \{0\}$ is a weak-$*$ compact subset of $\mathcal{C}^\#$, and the maps $\mathfrak{s}, \mathfrak{r} : \mathcal{E}^1_F(\mathcal{C}, \mathcal{D}) \to \mathfrak{S}(\mathcal{C}, \mathcal{D})$ are weak-$*$–weak-$*$ continuous.*

Theorem 8.12. *Let $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ and $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ be as above. Then $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ and $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ are locally compact Hausdorff topological groupoids and $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$ is an étale groupoid. Their unit spaces are $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})^{(\circ)} = \mathfrak{R}_F(\mathcal{C}, \mathcal{D})^{(\circ)} = F$. Moreover, $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ is a locally trivial topological twist over $\mathfrak{R}_F(\mathcal{C}, \mathcal{D})$.*

Proof. That inversion on $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ is continuous follows readily from the definition of inverse map and the weak-$*$ topology. Suppose $(\phi_\lambda)_{\lambda\in\Lambda}$ and $(\psi_\lambda)_{\lambda\in\Lambda}$ are nets in $\mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ converging to $\phi, \psi \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})$ respectively, and such that $(\phi_\lambda, \psi_\lambda) \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})^{(2)}$ for all λ. Since the source and range maps are continuous, we find that $\mathfrak{s}(\phi) = \lim_\lambda \mathfrak{s}(\phi_\lambda) = \lim_\lambda \mathfrak{r}(\psi_\lambda) = \mathfrak{r}(\psi)$, so $(\phi, \psi) \in \mathcal{E}^1_F(\mathcal{C}, \mathcal{D})^{(2)}$. Let $v, w \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ be such that $\phi(v) > 0$ and $\psi(w) > 0$. There exists λ_0, so that $\lambda \geq \lambda_0$ implies $\phi_\lambda(v)$ and $\psi_\lambda(w)$ are non-zero. For each $\lambda \geq \lambda_0$, there exists scalars $\xi_\lambda, \eta_\lambda \in \mathbb{T}$ such that $\phi_\lambda(v) = \xi_\lambda[v, \mathfrak{s}(\phi_\lambda)]$ and $\psi_\lambda = \eta_\lambda[v, \mathfrak{s}(\psi_\lambda)]$. Since

$$\lim_\lambda \phi_\lambda(v) = \phi(v) = \lim_\lambda [v, \mathfrak{s}(\phi_\lambda)](v) \quad \text{and} \quad \lim_\lambda \psi_\lambda(v) = \psi(v) = \lim_\lambda [v, \mathfrak{s}(\psi_\lambda)](v),$$

we conclude that $\lim \eta_\lambda = 1 = \lim \xi_\lambda$. So for any $x \in \mathcal{C}$,

$$\begin{aligned}(\phi\psi)(x) = \frac{\mathfrak{s}(\psi)((vw)^*x)}{(\mathfrak{s}(\psi)((vw)^*(vw)))^{1/2}} &= \lim_\lambda \frac{\mathfrak{s}(\psi_\lambda)((vw)^*x)}{(\mathfrak{s}(\psi_\lambda)((vw)^*(vw)))^{1/2}} = \lim_\lambda [v, \mathfrak{s}(\phi_\lambda)][w, \mathfrak{s}(\psi_\lambda)] \\ &= \lim_\lambda (\phi_\lambda\psi_\lambda)(x),\end{aligned}$$

giving continuity of multiplication. Notice that for $\phi \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, $\mathfrak{s}(\phi) = \phi^{-1}\phi$ and $\mathfrak{r}(\phi) = \phi\phi^{-1}$, and $F \subseteq \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$. Thus, $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$ is a locally compact Hausdorff topological groupoid with unit space F.

The definitions show that $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ is a groupoid. By construction, the map q defined by $\phi \mapsto |\phi|$ is continuous and is a surjective groupoid homomorphism. The topology on $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ is clearly Hausdorff. If $\phi \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ is such that $\phi(v) \neq 0$, then $W := \{\alpha \in \mathfrak{R}_F(\mathcal{C},\mathcal{D}) : \alpha(v) > |\phi(v)|/2\}$ has compact closure so $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ is locally compact. Also, if $\alpha_1, \alpha_2 \in W$ and $\mathfrak{r}(\alpha_1) = \mathfrak{r}(\alpha_2) = f$, then writing $\alpha_i = |\psi_i|$ for $\psi_i \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, we see that $\psi_i(v) \neq 0$, so there exist $z_1, z_2 \in \mathbb{T}$ so that for $i = 1, 2$ and every $x \in \mathcal{C}$, $\psi_i(x) = z_i f(xv^*)f(v)^{-1}$. Hence $\alpha_1 = \alpha_2$ showing that the range map is locally injective. We already know that the range map is continuous, so by local compactness, the range map is a local homeomorphism.

Note that convergent nets in $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ can be lifted to convergent nets in $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$. Indeed, if $q(\phi_\lambda) \to q(\phi)$ for some net (ϕ_λ) and ϕ in $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, choose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ so that $\phi(v) > 0$. Then for large enough λ, $\phi_\lambda(v) \neq 0$, and we have $[\phi_\lambda, \mathfrak{s}(\phi_\lambda)] \to \phi$. Also, $|[\phi_\lambda, \mathfrak{s}(\phi_\lambda)]| \to |\phi|$. The fact that the groupoid operations on $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ are continuous now follows easily from the continuity of the groupoid operations on $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$. Thus $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$ is a locally compact Hausdorff étale groupoid.

Finally, $q(\phi_1) = q(\phi_2)$ if and only if there exist $z \in \mathbb{T}$ so that $\phi_1 = z\phi_2$. Moreover, for each $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, the map $f \mapsto [v, f]$, where $f \in \{g \in \mathfrak{S}(\mathcal{C},\mathcal{D}) : g(v^*v) > 0\}$ is a continuous section for q. Also, the action of $\mathbb{T}$ on $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$. given above makes $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$ into a $\mathbb{T}$-groupoid. So $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$ is a twist over $\mathfrak{R}_F(\mathcal{C},\mathcal{D})$.

□

Notation 8.13. We now let

$$\Sigma = \mathcal{E}^1_F(\mathcal{C},\mathcal{D}) \text{ and } G = \mathfrak{R}_F(\mathcal{C},\mathcal{D}), \quad \text{so that} \quad G^{(\circ)} = F.$$

For $a \in \mathcal{C}$, define $\hat{a} : \mathcal{E}^1_F(\mathcal{C},\mathcal{D}) \to \mathbb{C}$ to be the 'Gelfand' map: for $\phi \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, $\hat{a}(\phi) = \phi(a)$. Then $\hat{a}$ is a continuous equivariant function on $\mathcal{E}^1_F(\mathcal{C},\mathcal{D})$.

Note that if $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$, then $\hat{w}$ is compactly supported. Indeed, for $\phi = [v, f] \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$, $\phi \in \operatorname{supp}(\hat{w})$ if and only if $[v, f](w) \neq 0$, which occurs exactly when $f(v^*w) \neq 0$. Proposition 4.3 shows this occurs precisely when $f(w^*w) \neq 0$. Hence,

$$\operatorname{supp} \hat{w} = \{\phi \in \mathcal{E}^1_F(\mathcal{C},\mathcal{D}) : \mathfrak{s}(\phi)(w^*w) \neq 0\},$$

and it follows that $\hat{w}$ has compact support. Moreover, $\hat{w}$ is supported on a slice, so that we find $\hat{w} \in \mathcal{N}(C^*(\Sigma, G), C(G^{(\circ)}))$.

Before stating the main result of this section, recall that Proposition 2.16 shows that $\mathcal{K}_F = \{x \in \mathcal{C} : f(x^*x) = 0 \text{ for all } f \in F\}$ is an ideal of $\mathcal{C}$ whose intersection with $\mathcal{D}$ is trivial.

Theorem 8.14. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion, and let $G := \mathfrak{R}_F(\mathcal{C},\mathcal{D})$ and $\Sigma := \mathcal{E}^1_F(\mathcal{C},\mathcal{D})$. The map sending $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$ to $\hat{w} \in C^*(\Sigma, G)$ extends uniquely to a regular $*$-homomorphism $\theta : (\mathcal{C},\mathcal{D}) \to (C^*(\Sigma, G), C(G^{(\circ)}))$ with $\ker\theta = \mathcal{K}_F$. Furthermore, $\theta(\mathcal{C})$ is dense in $C^*(\Sigma, G)$ in the $G^{(\circ)}$-compatible topology.*

Remark. In general, $\theta(\mathcal{C})$ and $\theta(\mathcal{D})$ may be proper subsets of $C^*(\Sigma, G)$ and $C(G^{(\circ)})$ respectively.

Proof. The point is that the norms on $\mathcal{C}/\mathcal{K}_F$ and $C^*(\Sigma, G)$ both arise from the left regular representation on appropriate spaces. Here are the details.

We have already observed that the map $w \mapsto \hat{w}$ sends normalizers to normalizers. Let $\mathcal{C}_0 = \operatorname{span}\mathcal{N}(\mathcal{C},\mathcal{D})$. Then for any $a \in \mathcal{C}_0$, $\hat{a} \in C_c(\Sigma, G)$.

Let $f \in F = G^{(\circ)}$. Then f can be regarded as either a state on $\mathcal{C}$ or as determining a state on $C^*(\Sigma, G)$ via evaluation at f. We write $f_{\mathcal{C}}$ when viewing f as a state on $\mathcal{C}$, and f_Σ when viewing f as a state on $C^*(\Sigma, G)$.

Let $(\pi_{\mathcal{C},f}, \mathcal{H}_{\mathcal{C},f})$ be the GNS representation of $\mathcal{C}$ arising from $f_{\mathcal{C}}$, and let $(\pi_{\Sigma,f}, \mathcal{H}_{\Sigma,f})$ be the GNS representation of $C^*(\Sigma, G)$ determined by f_Σ. (Writing $x = f$, the restriction of $\pi_{\Sigma,f}$ to $C_c(\Sigma, G)$ is the representation π_x discussed above when defining the norm on $\mathcal{C}_c(\Sigma, G)$.)

For typographical reasons, when the particular f is understood, we will drop the extra f in the notation: write $\pi_{\mathcal{C}}$ or π_Σ instead of $\pi_{\mathcal{C},f}$ or $\pi_{\Sigma,f}$.

Now fix $f \in G^{(\circ)}$. For $a \in \mathcal{C}_0$, we claim that

$$\|\hat{a} + \mathcal{N}_f\|_{\mathcal{H}_\Sigma} = \|a + L_f\|_{\mathcal{H}_{\mathcal{C}}}. \tag{37}$$

Let $T \subseteq \Lambda_f$ be chosen so that $f(w^*w) = 1$ for every $w \in T$ and so that T contains exactly one element from each $\sim_f$ equivalence class. Proposition 8.7 shows that when $w_1, w_2 \in \Lambda_f$, we have $|[w_1, f]| = |[w_2, f]|$ if and only if $w_1 \sim_f w_2$. Proposition 4.10 shows that $\{w + L_f : w \in T\}$ is an orthonormal basis for $\mathcal{H}_{\mathcal{C}}$. Writing $\phi = [w, f]$, we have,

$$\begin{aligned}\|\hat{a} + \mathcal{N}_f\|^2_{\mathcal{H}_\Sigma} &= \sum_{\substack{|\phi| \in G \\ \mathfrak{s}(|\phi|) = f}} |\hat{a}(|\phi|)|^2 = \sum_{w \in T} |[w, f](a)|^2 = \sum_{w \in T} |f(w^*a)|^2 \\ &= \sum_{w \in T} |\langle a + L_f, w + L_f \rangle|^2 = \|a + L_f\|^2_{\mathcal{H}_{\mathcal{C}}}.\end{aligned}$$

It follows that the map $a + L_f \mapsto \hat{a} + \mathcal{N}_f$ extends to an isometry $W_f : \mathcal{H}_{\mathcal{C}} \to \mathcal{H}_\Sigma$.

To see that W_f is a unitary operator, fix $\xi \in C_c(\Sigma, G)$. Since ξ is compactly supported, the set

$$S_\xi := \{|\phi| \in G : \mathfrak{s}(|\phi|) = f \text{ and } \xi(|\phi|) \neq 0\}$$

is a finite set. Let $|\phi_1|, \dots, |\phi_n|$ be the elements of S_ξ. For $1 \leq j \leq n$, we may find $v_j \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that $|\phi_j| = |[v_j, f]|$. By Proposition 8.7, we may assume each v_j belongs to the set T. Let $z_j = \xi(\phi_j)$ and set $a = \sum_{j=1}^n z_j v_j$. Clearly $a \in \mathcal{C}_0$. Using the fact that $f(w^*w) = 1$ for each $w \in T$ and the fact that $f(w_1^* w_2) = 0$ for distinct elements $w_1, w_2 \in T$, we find that for $1 \leq k \leq n$,

$$[v_k, f](a) = z_k.$$

Then

$$\begin{aligned}\|(\hat{a} - \xi) + \mathcal{N}_f\|^2_{\mathcal{H}_\Sigma} &= f_\Sigma\left((\hat{a} - \xi)^* \star (\hat{a} - \xi)\right) = \sum_{j=1}^n |(\hat{a} - \xi)(|\phi_j|)|^2 \\ &= \sum_{j=1}^n \overline{(\hat{a}(\phi_j) - \xi(\phi_j))}(\hat{a}(\phi_j) - \xi(\phi_j)) = \sum_{j=1}^n |([v_j, f](a) - \xi(\phi_j))|^2 \\ &= \sum_{j=1}^n (z_j - \xi(\phi_j))^2 = 0.\end{aligned}$$

Therefore, $\{\hat{a} + \mathcal{N}_f : a \in \mathcal{C}_0\} = \{\xi + \mathcal{N}_f : \xi \in C_c(\Sigma, G)\}$. As this set is dense in $\mathcal{H}_\Sigma$, W_f is a unitary operator.

Next, we show that for $a \in \mathcal{C}_0$, we have

$$\pi_\Sigma(\hat{a}) W_f = W_f \pi_{\mathcal{C}}(a). \tag{38}$$

To do this, it suffices to show that for each $v \in T$,

$$\pi_\Sigma(\hat{a}) W_f(v + L_f) = W_f \pi_{\mathcal{C}}(a)(v + L_f).$$

Letting $\phi = [w_1, f]$ and $y = [w_2, f]$ we find $\phi y^{-1} = [w_1 w_2, \beta_{w_2}(f)]$, and a computation yields,

$$\hat{a}(\phi y^{-1})\hat{v}(y) = \frac{f(w_1^* a w_2) f(w_2^* v)}{f(w_1 w_1)^{1/2} f(w_2^* w_2)} = \begin{cases} 0 & \text{if } w_2 \not\sim_f v \\ \dfrac{f(w_1^* a v)}{f(w_1^* w_1)^{1/2}} & \text{if } w_2 \sim_f v \end{cases} = \begin{cases} 0 & \text{if } w_2 \not\sim_f v \\ \widehat{av}(\phi) & \text{if } w_2 \sim_f v. \end{cases}$$

Therefore, when $\mathfrak{s}(\phi) = f$, we have

$$(\hat{a} \star \hat{v})(\phi) = \sum_{\substack{|y| \in G \\ \mathfrak{s}(|y|) = f}} (\hat{a} \circledast \hat{v})(\phi, |y|) = \widehat{av}(\phi).$$

So for $|y| \in G$ with $\mathfrak{s}(|y|) = f$, $|(\hat{a} \star \hat{v} - \widehat{av})(|y|)| = 0$. Then,

$$\hat{a} \star \hat{v} + \mathcal{N}_f = \widehat{av} + \mathcal{N}_f$$

because

$$\|((\hat{a} \star \hat{v}) - \widehat{av}) + \mathcal{N}_f\|^2_{\mathcal{H}_\sigma} = \sum_{\substack{|y| \in G \\ s(|y|) = f}} |(\hat{a} \star \hat{v} - \widehat{av})(|y|)|^2 = 0.$$

Hence,

$$\pi_\Sigma(\hat{a}) W_f(v + L_f) = \pi_\Sigma(\hat{a})(\hat{v} + \mathcal{N}_f) = \widehat{av} + \mathcal{N}_f = W_f(\pi_{\mathcal{C}}(a))(v + L_f),$$

which gives (38).

The definition of the norm on $C^*(\Sigma, G)$ and (38) imply that for $a, b \in \mathcal{C}_0$, $\left\|\widehat{ab} - \hat{a}\hat{b}\right\|_{C^*(\Sigma,G)} = 0$. Therefore, the map $a \in \mathcal{C}_0 \mapsto \hat{a}$ is multiplicative and

$$\|\hat{a}\|_{C^*(\Sigma,G)} = \sup_{f \in \mathfrak{S}(\mathcal{C},\mathcal{D})} \left\|\pi_{\mathcal{C},f}(a)\right\|.$$

The existence of θ now follows from continuity, and the fact that $\ker\theta = \mathcal{K}_F$ follows from Proposition 5.4. For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\theta(v) = \hat{v}$ belongs to $\mathcal{N}(C^*(\Sigma, G), C(G^{(\circ)}))$, so θ is a regular $*$-homomorphism.

Finally, we turn to showing the $G^{(\circ)}$-compatible density of $\theta(\mathcal{C})$ in $C^*(\Sigma, G)$. Let $\mathcal{M} \subseteq C^*(\Sigma, G)^\#$ be the linear span of the evaluation functionals $\xi \mapsto \xi(\sigma)$ where $\xi \in C^*(\Sigma, G)$ and $\sigma \in \Sigma$. Suppose $\mu \in \mathcal{M}$ annihilates $\theta(\mathcal{C})$. Then there exists $n \in \mathbb{N}$, scalars $\lambda_1, \dots, \lambda_n$, elements $v_1, \dots, v_n \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and $f_1, \dots, f_n \in F$ such that for any $\xi \in C^*(\Sigma, G)$,

$$\mu(\xi) = \sum_{k=1}^n \lambda_k \xi([v_k, f_k]).$$

Without loss of generality we may assume that $[v_i, f_i] \neq [v_j, f_j]$ if $i \neq j$. Since μ annilhlates $\theta(\mathcal{C})$, for every $a \in \mathcal{C}_0$,

$$0 = \mu(\hat{a}) = \sum_{k=1}^n \lambda_k [v_k, f_k](a).$$

Fix $1 \leq j \leq n$, and let $d, e \in \mathcal{D}$ be such that $\beta_{v_j}(f_j)(d) = f_j(e) = 1$. For $i \neq j$, since $[v_i, f_i] \neq [v_j, f_j]$, either $f_j \neq f_i$ or $\beta_{v_i}(f_i) \neq \beta_{v_j}(f_j)$. Hence we assume that d and e have been chosen so that if $i \neq j$, then $[v_i, f_i](dv_j e) = 0$. Then

$$\mu(\hat{v_j}) = \lambda_j f_j(v_j^* v_j)^{1/2} = 0.$$

As $f_j(v_j^* v_j) \neq 0$, we obtain $\lambda_j = 0$. It follows that $\mu = 0$. Since the dual of $C^*(\Sigma, G)$ equipped with the $G^{(\circ)}$-compatible topology is $\mathcal{M}$, we conclude that $\theta(\mathcal{C})$ is dense in the $G^{(\circ)}$-compatible topology on $C^*(\Sigma, G)$. This completes the proof.

$\square$

9. Applications

In this section we give some applications which apply to regular MASA inclusions with $\mathcal{L}(\mathcal{C},\mathcal{D}) = (0)$. Theorem 6.10 gives a very large class of such inclusions.

Here is an application of our work to norming algebras. We begin with a definition. Recall that $\mathcal{N}(\mathcal{C},\mathcal{D})$ is a closed $*$-semigroup containing $\mathcal{D}$.

Definition 9.1. A $*$-subsemigroup $\mathcal{F} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{F}$ is *countably generated over* $\mathcal{D}$ if there exists a countable set $F \subseteq \mathcal{F}$ so that the smallest $*$-subsemigroup of $\mathcal{N}(\mathcal{C},\mathcal{D})$ containing $F \cup \mathcal{D}$ is $\mathcal{F}$. The set F will be called a *generating set* for $\mathcal{F}$.

We will say that the inclusion $(\mathcal{C},\mathcal{D})$ is *countably regular* if there exists a $*$-subsemigroup $\mathcal{F} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ such that $\mathcal{F}$ is countably generated over $\mathcal{D}$ and $\mathcal{C} = \overline{\operatorname{span}}(\mathcal{F})$.

The following result generalizes [28, Lemma 2.15] and gives a large class of norming algebras. In particular, notice that the result holds for Cartan inclusions.

Theorem 9.2. *Suppose $(\mathcal{C},\mathcal{D})$ is a regular MASA inclusion such that $\mathcal{L}(\mathcal{C},\mathcal{D}) = (0)$. Then $\mathcal{D}$ norms $\mathcal{C}$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ be a $*$-subsemigroup which is countably generated over $\mathcal{D}$ by the (countable) set F. Let $\mathcal{C}_\mathcal{F} \subseteq \mathcal{C}$ be the C^*-subalgebra generated by $\mathcal{F}$. (Notice that $\mathcal{C}_\mathcal{F}$ is simply the closed linear span of $\mathcal{F}$.) Then $(\mathcal{C}_\mathcal{F},\mathcal{D})$ is a countably regular MASA inclusion.

We will show that $\mathcal{D}$ norms $\mathcal{C}_\mathcal{F}$. Let

$$Y := \{\sigma \in \hat{\mathcal{D}} : \sigma \text{ has a unique state extention to } \mathcal{C}_\mathcal{F}\}.$$

Theorem 3.8 shows that Y is dense in $\hat{\mathcal{D}}$. For each element $\sigma \in Y$, let σ' denote the unique extension of σ to all of $\mathcal{C}_\mathcal{F}$. Notice that if $\rho \in \widehat{I(\mathcal{D})}$ and $\rho \circ \iota = \sigma$, then $\sigma' = \rho \circ E$ because $\sigma'|_\mathcal{D} = \sigma = \rho \circ E|_\mathcal{D}$.

For $\sigma \in Y$, let π_σ be the GNS representation for σ'. Proposition 4.14 shows that $\pi_\sigma(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\sigma)$.

Define an equivalence relation R on Y by $\sigma_1 \sim \sigma_2$ if and only if there exists $v \in \mathcal{F}$ such that $\sigma_2 = \beta_v(\sigma_1)$. (Since $\mathcal{F}$ is a $*$-semigroup, this is an equivalence relation.)

We claim that if π_{σ_1} is unitarily equivalent to π_{σ_2}, then $\sigma_1 \sim \sigma_2$. To see this, we use a modification of the argument in [10, Lemma 5.8]. Let $U \in \mathcal{B}(\mathcal{H}_{\sigma_2},\mathcal{H}_{\sigma_1})$ be a unitary operator such that

$$U^*\pi_{\sigma_1}U = \pi_{\sigma_2}.$$

Let L_{σ_i} be the left kernel of σ_i'. Since π_{σ_i} is irreducible, $\mathcal{C}/L_{\sigma_i} = \mathcal{H}_{\sigma_i}$. Hence we may find $X \in \mathcal{C}$ such that $U(I + L_{\sigma_2}) = X + L_{\sigma_1}$. Then for every $x \in \mathcal{C}$,

$$\sigma_2'(x) = \langle \pi_{\sigma_2}(x)(I + L_{\sigma_2}), (I + L_{\sigma_2})\rangle = \sigma_1'(X^*xX).$$

Fix $\rho_i \in \widehat{I(\mathcal{D})}$ such that $\rho_i \circ \iota = \sigma_i$.

The map $\mathcal{C} \ni x \mapsto \sigma_1'(X^*x)$ is a non-zero linear bounded linear functional on $\mathcal{C}$. Since $\operatorname{span}(\mathcal{F})$ is dense in $\mathcal{C}$, there exists $v \in \mathcal{F}$ so that $\sigma_1'(X^*v) \neq 0$. The Cauchy-Schwartz inequality for completely positive maps shows that for any $d \in \mathcal{D}$,

$$|\sigma_1'(X^*vd)|^2 = \rho_1(E(X^*vd)E(d^*v^*X)) \leq \rho_1(E(X^*vdd^*v^*X)) = \sigma_1'(X^*vdd^*v^*X) = \sigma_2(vdd^*v^*).$$

When $d \in \mathcal{D}$ and $\sigma_1(d) \neq 0$, we have $\sigma_1'(X^*vd) = \sigma_1'(X^*v)\sigma_1(d) \neq 0$. Therefore, when $d \in \mathcal{D}$ satisfies $\sigma_1(d) \neq 0$,

$$0 < \sigma_2(vdd^*v^*). \tag{39}$$

In particular, $\sigma_2(vv^*) \neq 0$. For any $d \in \mathcal{D}$, we have

$$\beta_{v^*}(\sigma_2)(d) = \frac{\sigma_2(vdv^*)}{\sigma_2(vv^*)}.$$

If $\beta_{v^*}(\sigma_2) \neq \sigma_1$, then there exists $d \in \mathcal{D}$ with $\sigma_1(dd^*) \neq 0$ and $\beta_{v^*}(dd^*) = 0$. But this is impossible by (39). So $\beta_{v^*}(\sigma_2) = \sigma_1$. Hence $\sigma_1 \sim \sigma_2$ as claimed.

Thus, if $\sigma_1 \not\sim \sigma_2$, then π_{σ_1} and π_{σ_2} are disjoint representations (as they are both irreducible).

Let $\mathcal{Y} \subseteq Y$ be chosen so that Y contains exactly one element from each equivalence class of Y. Put

$$\pi = \bigoplus_{\sigma \in \mathcal{Y}} \pi_\sigma.$$

Then

$$\ker \pi = \bigcap_{\sigma \in \mathcal{Y}} \ker \pi_\sigma = \bigcap_{\sigma \in Y} \ker \pi_\sigma = \{x \in \mathcal{C}_{\mathcal{F}} : \sigma'(z^* x^* x z) = 0 \text{ for all } \sigma \in Y \text{ and all } z \in \mathcal{C}_{\mathcal{F}}\}. \tag{40}$$

We next prove that

$$\mathcal{L}(\mathcal{C}_{\mathcal{F}}, \mathcal{D}) \supseteq \ker \pi. \tag{41}$$

Suppose to obtain a contradiction, that $x \in \ker \pi$ and that $E(x^*x)$ is a non-zero element of $I(\mathcal{D})$. Let

$$L := \overline{\{\rho \in \widehat{I(\mathcal{D})} : \rho(E(x^*x)) > \|E(x^*x)\|/2\}}.$$

Then L is a clopen set. By Lemma 3.15, $\iota^*(L) = \{\sigma \in \hat{\mathcal{D}} : \sigma = \rho \circ \iota \text{ for some } \rho \in L\}$ has non-empty interior. Since Y is dense in $\hat{\mathcal{D}}$, we may find $\sigma \in Y$ and $\rho \in L$ such that $\rho \circ \iota = \sigma$. Then $\sigma'(x^*x) = \rho(E(x^*x)) \neq 0$, so $x \notin \ker \pi_\rho$, contradicting (40). Hence (41) holds.

Since $\mathcal{L}(\mathcal{C}, \mathcal{D}) \supseteq \mathcal{L}(\mathcal{C}_{\mathcal{F}}, \mathcal{D}) \supseteq \ker \pi$, we see that π is a faithful representation of $\mathcal{C}_{\mathcal{F}}$.

Since the representations in the definition of π are disjoint and each $\pi_\sigma(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\sigma)$, $\pi(\mathcal{D})''$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\pi)$. Therefore, $\pi(\mathcal{D})''$ is locally cyclic (see [29, p. 173]) for $\mathcal{B}(\mathcal{H}_\pi)$. By [29, Theorem 2.7 and Lemma 2.3] $\pi(\mathcal{D})$ norms $\mathcal{B}(\mathcal{H}_\pi)$. But then $\pi(\mathcal{D})$ norms $\pi(\mathcal{C}_{\mathcal{F}})$. Since π is faithful, $\mathcal{D}$ norms $\mathcal{C}_{\mathcal{F}}$.

Finally, suppose that $k \in \mathbb{N}$ and that $x = (x_{ij}) \in M_n(\mathcal{C})$. For each $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, k\}$, we may find a finite set $F_{n,i,j} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ so that $\left\| x_{ij} - \sum_{v \in F_{n,i,j}} v \right\| < 1/n$. Let

$$F = \cup\{F_{n,i,j} : n \in \mathbb{N}, i, j \in \{1, \ldots, k\}\}.$$

Then F is countable. Let $\mathcal{F}$ be the closed $*$-subsemigroup of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ generated by F and $\mathcal{D}$. Then for $i, j \in \{1, \ldots, k\}$, $x_{ij} \in \mathcal{C}_{\mathcal{F}}$. Since $\mathcal{D}$ norms $\mathcal{C}_{\mathcal{F}}$, we conclude that

$$\|x\|_{M_k(\mathcal{C})} = \|x\|_{M_k(\mathcal{C}_{\mathcal{F}})} = \sup\{\|RxC\| : R \in M_{1,n}(\mathcal{D}), C \in M_{n,1}(\mathcal{D}), \|R\| \leq 1, \|C\| \leq 1\}.$$

Hence $\mathcal{D}$ norms $\mathcal{C}$.

□

For any norm closed subalgebra $\mathcal{A}$ of the C^*-algebra $\mathcal{C}$, let $C^*(\mathcal{A})$ be the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$, and let $C^*_e(\mathcal{A})$ be the C^*-envelope of $\mathcal{A}$. (There are a number of references which discuss C^*-envelopes; see [5, 11, 27].)

The following result is a significant generalization of [10, Theorem 4.21]. Theorem 9.3 was observed by Vrej Zarikian, who has kindly consented to its inclusion here.

Theorem 9.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras, with $\mathcal{D} \subseteq \mathcal{C}$ ($\mathcal{D}$ is not assumed abelian). Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Suppose there exists a unique unital completely positive map $\Phi : \mathcal{C} \to I(\mathcal{D})$ such that $\Phi|_{\mathcal{D}} = \iota$, and assume also that Φ is faithful. Let $\mathcal{A}$ be a norm-closed (not necessarily self-adjoint) subalgebra of $\mathcal{C}$ such that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$. Then the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$ is the C^*-envelope of $\mathcal{A}$.*

Proof. Let $\theta : \mathcal{A} \to C^*_e(\mathcal{A})$ be a unital completely isometric (unital) homomorphism such that the C^*-algebra generated by the image of θ is $C^*_e(\mathcal{A})$. Then there exists a unique $*$-epimorphism $q : C^*(\mathcal{A}) \to C^*_e(\mathcal{A})$ such that $q|_{\mathcal{A}} = \theta$. Our task is to show that q is one-to-one.

Since $I(\mathcal{D})$ is injective in the category of operator systems and completely contractive maps, there exists a unital completely contractive map $\Phi_e : C^*_e(\mathcal{A}) \to I(\mathcal{D})$ such that $\Phi_e \circ \theta = \iota$. Also, there exists a unital completely contractive map $\Delta : \mathcal{C} \to I(\mathcal{D})$ so that $\Delta|_{C^*(\mathcal{A})} = \Phi_e \circ q$. Then for $d \in \mathcal{D}$, we have $\theta(d) = q(d)$, so $\iota(d) = \Phi_e(\theta(d)) = \Phi_e(q(d)) = \Delta(d)$. The uniqueness of Φ gives $\Delta = \Phi$. Then if $x \in C^*(\mathcal{A})$ and $q(x) = 0$, we have $\Phi(q(x^*x)) = 0$, so $q(x^*x) = 0$ by the faithfulness of Φ. Thus q is one-to-one, and the proof is complete. ☐

We now obtain the following generalization of [28, Theorem 2.16]. While the outline of the proof is the same as the proof of [28, Theorem 2.16], the details in obtaining norming subalgebras are different.

Theorem 9.4. *For $i = 1, 2$, suppose that $(\mathcal{C}_i, \mathcal{D}_i)$ are regular MASA inclusions such that $\mathcal{L}(\mathcal{C}_i, \mathcal{D}_i) = (0)$ and that $\mathcal{A}_i \subseteq \mathcal{C}_i$ are norm closed subalgebras such that $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$. Let $C^*(\mathcal{A}_i)$ be the C^*-subalgebra of $\mathcal{C}_i$ generated by $\mathcal{A}_i$. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism of $C^*(\mathcal{A}_1)$ onto $C^*(\mathcal{A}_2)$.*

Proof. Theorem 9.2 implies that $\mathcal{D}_i$ norms $\mathcal{C}_i$. Taken together, Theorem 3.10 and Theorem 9.3 imply that $C^*(\mathcal{A}_i)$ is the C^*-envelope of $\mathcal{A}_i$. Finally, an application of [28, Corollary 1.5] completes the proof. ☐

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