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Brian Harbourne

Craig Hunkeke

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BRIAN HARBOURNE & CRAIG HUNEKE

ABSTRACT. Searching for structural reasons behind old results and conjectures of Chudnovksy regarding the least degree of a nonzero form in an ideal of fat points in $\mathbf{P}^N$, we make conjectures which explain them, and we prove the conjectures in certain cases, including the case of general points in $\mathbf{P}^2$. Our conjectures were also partly motivated by the Eisenbud-Mazur Conjecture on evolutions, which concerns symbolic squares of prime ideals in local rings, but in contrast we consider higher symbolic powers of homogeneous ideals in polynomial rings.

1. INTRODUCTION

Both authors of this paper have been interested for many years in the behavior of symbolic powers of ideals in regular rings, especially in polynomial rings. In the case of defining ideals of points, symbolic powers are special cases of ideals of so-called fat points, and their study provides a meeting ground of geometry and algebra. Hereafter R will denote the ring $K[x_0, \ldots, x_N] = K[\mathbf{P}^N]$ where K is a field and $I \subseteq R$ will be a homogeneous ideal, with $M = (x_0, \ldots, x_N)$ being the maximal homogeneous ideal in R. Many of our arguments hold for arbitrary fields, but in some cases the field K must be infinite. By symbolic power we mean $I^{(m)} = R \cap (\cap_P (I^m)_P)$ where the intersections take place in the field of fractions of $K[\mathbf{P}^N]$, and the second intersection is over all associated primes P of I. An important special case is that of ideals of fat points; i.e., $I = \cap_i I(p_i)^{m_i}$ for non-negative integers m_i and a finite set of distinct points $p_i \in \mathbf{P}^N$, where $I(p_i)$ is the ideal generated by all forms that vanish at p_i. In this case $I^{(m)}$ is just $\cap_i I(p_i)^{mm_i}$. When $\mathbf{P}^N$ is clear, we denote the subscheme defined by I by $Z = m_1 p_1 + \cdots + m_n p_n$, and denote its ideal I by $I(Z)$. In this case, $I^{(m)}$ becomes $I(mZ)$.

If J is a homogeneous ideal, we let $\alpha(J)$ be the least degree of a polynomial in J. Let I be the radical ideal of a finite set of points in $\mathbf{P}^N$. Using complex analytic techniques, Waldschmidt and Skoda [W1, Sk] showed that

$$\alpha(I^{(m)})/m \geq \alpha(I)/N \qquad (\circ)$$

for every $m > 0$. Interestingly, this result also follows for any homogeneous ideal $I \subseteq R = K[\mathbf{P}^N]$ because $I^{(Nm)} \subseteq I^m$ by [ELS], [HH1] and clearly $(I^{(m)})^N \subseteq I^{(mN)}$. These containments imply that $N\alpha(I^{(m)}) = \alpha((I^{(m)})^N) \geq \alpha(I^{(Nm)}) \geq \alpha(I^m) = m\alpha(I)$, so $\alpha(I^{(m)})/m \geq \alpha(I^{(Nm)})/(Nm) \geq \alpha(I)/N$.

When $N = 2$, Chudnovsky [Ch] improved the bound of Waldschmidt and Skoda [W1, Sk]. Since only a sketch of Chudnovksy's proof is given in [Ch], we later give a proof (see Proposition 3.1). Chudnovsky's improvement is the following: Let $p_1, \ldots, p_n \in \mathbf{P}^2$ be distinct points. Let $I = \cap_i I(p_i) \subset K[\mathbf{P}^2]$. Then $\alpha(I^{(m)})/m \geq (\alpha(I) + 1)/2$ for all $m > 0$.

Because the result of Waldschmidt and Skoda can be explained by a general property of symbolic powers, that $I^{(Nm)} \subseteq I^m$ for all m, it is natural to speculate whether or not there is a similar property which might underlie Chudnovsky's improved bound. This led us to our first conjecture, which gives a structural reason for the result of Chudnovsky. Namely, we conjecture that $I^{(2r)} \subseteq M^r I^r$ for an ideal I of points in $\mathbf{P}^2$. This conjecture, which we prove for general points (see Proposition 3.10), easily implies the result of Chudnovsky. We generalize this conjecture in a natural way to arbitrary dimension. As it turns out, a positive answer to our more general conjecture also gives a positive answer to a conjecture of Chudnovsky, that if I is the ideal of a finite set of points in $K[\mathbf{P}^N]$, then $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$.

Our conjecture relates to evolutions. Evolutions are certain kinds of ring homomorphisms that arose in proving Fermat's Last Theorem [F, TW, Wi]; see [B] for an exposition. An important step in the proof was to show in certain cases only trivial evolutions occurred. Eisenbud and Mazur [EM] showed the question of triviality (which for the work of Wiles was in mixed characteristic) could be translated into a statement involving symbolic powers. They then made the following conjecture in characteristic 0:

Conjecture 1.1 (Eisenbud-Mazur). *Let $P \subset \mathbf{C}[[x_1, \ldots, x_d]]$ be a prime ideal. Then $P^{(2)} \subseteq MP$, where $M = (x_1, \ldots, x_d)$.*

Our main conjecture can at least heuristically be thought of as a generalization of the conjecture of Eisenbud-Mazur to higher symbolic powers. The homogeneous version of Conjecture 1.1 for symbolic squares is easy to verify:

Fact 1.2. *Let $I \subseteq K[x_0, \ldots, x_N]$ be a proper homogeneous ideal where $\mathrm{char}(K) = 0$. Then $I^{(2)} \subseteq MI$.*

Proof. For any $F \in I^{(2)}$ we have $\partial F/\partial x_i \in I$; if $\mathrm{char}(K) = 0$, then by the Euler identity we have $\deg(F)F = \sum_i x_i \partial F/\partial x_i \in MI$, so $I^{(2)} \subseteq MI$. $\qquad\square$

The general question we wish to raise is:

Question 1.3. *Let $I \subset R$ be a homogeneous ideal. For which m, i and j do we have $I^{(m)} \subseteq M^j I^i$?*

A complete answer will typically depend on I, but it is also of interest to ask what holds for all I, knowing only N. Since $M^j I^i \subseteq I^i$, we see that whenever $I^{(m)} \subseteq M^j I^i$ is true we also have $I^{(m)} \subseteq I^i$. It is known that $I^{(m)} \subseteq I^i$ holds whenever $m/i \geq N$ [ELS], also [HH1], and that whenever $m/i < N$, there exist ideals I for which $I^{(mt)} \subseteq I^{it}$ fails for $t \gg 0$ [BH1].

Thus an interesting starting point is:

Question 1.4. *Let $I \subset R$ be a homogeneous ideal. For which j does $I^{(rN)} \subseteq M^j I^r$ hold for all I and all r?*

The best known general results concerning this question are found in [HH2] and [TY].

In Section 2 we state our main conjecture, and prove it in a very special case. Section 3 gives the proof of the result of Chudnovsky, which we then slightly generalize and relate to the Noetherian property for symbolic power algebras. We prove our conjecture for general points in $\mathbf{P}^2$. The last section relates our work to a conjecture of the first author, and has further speculations and examples.

2. An optimistic conjecture

Note that $I^{(rN)} \subseteq M^j I^r$ fails in general if $j > r(N - 1)$, even for $I = M$. So the best we can hope for is:

Conjecture 2.1. *Let $I = \cap_i I(p_i)^{m_i} \subset K[\mathbf{P}^N]$ be any fat points ideal. Then $I^{(rN)} \subseteq M^{r(N-1)}I^r$ holds for all $r > 0$.*

In fact, as far we know, there is no reason not to raise this question for arbitrary homogeneous ideals in a polynomial ring, or even for arbitrary ideals in a regular local ring. In this paper, all our main arguments are for the case of points, so we have chosen not to make the general conjecture unless more evidence can be found that supports it.

In fact, Proposition 3.3 shows that the conjecture is true for $N = 2$ for fat point ideals arising as symbolic powers of radical ideals generated in a single degree. Examples of point sets whose ideals are generated in a single degree include star configurations (see Definition 3.8) and any set of $\binom{s}{2}$ general points in $\mathbf{P}^2$.

Most of our progress on this conjecture relates to the minimal degrees of elements in symbolic powers. We need the following definition.

Definition 2.2. *Let R be a polynomial ring over a field, and let J be a homogeneous ideal. We set $\alpha(J)$ equal to the smallest integer l such that $J_l \neq 0$, and set $\beta(J)$ equal to the smallest integer n such that J_n contains a regular sequence of length two.*

We first note that the conjecture holds provided we know some information about the symbolic powers.

Proposition 2.3. *Let $J = \cap_i I(p_i) \subset R = K[\mathbf{P}^N]$ and let $I = J^{(m)} \subset R$ be a fat points ideal. If for some s, J is generated by a set of homogeneous elements each having degree at most s, and if $\alpha(I^{(Nr)}) \geq rms + rm(N - 1)$, then $I^{(Nr)} \subseteq M^{r(N-1)}I^r$.*

Proof. We use that $J^{(Nmr)} \subseteq J^{mr}$ by [ELS] (also see [HH1]). Since J is generated in degree s and less, so J^{mr} is generated in degree at most mrs, and hence $J_t^{mr} = (M^{mr(N-1)}J^{mr})_t$ for $t \geq mrs + mr(N - 1)$. Of course, $0 = J_t^{(Nmr)} \subseteq (M^{mr(N-1)}J^{mr})_t$ for $t < \alpha(J^{(Nmr)})$, while for $t \geq \alpha(J^{(Nmr)}) = \alpha(I^{(Nr)}) \geq rms + rm(N-1)$ we have $(J^{(Nmr)})_t \subseteq (J^{mr})_t = (M^{mr(N-1)}J^{mr})_t$ and so $I^{(Nr)} = J^{(Nmr)} \subseteq M^{mr(N-1)}J^{mr} \subseteq M^{r(N-1)}I^r$. $\qquad\square$

3. Points in projective space

As discussed in the introduction, if I is the radical ideal of a finite set of points in $\mathbf{P}^N$, Waldschmidt and Skoda [W1, Sk] showed that $\alpha(I^{(m)})/m \geq \alpha(I)/N$ for every $m > 0$, using complex analysis, but it also follows for any homogeneous ideal $I \subseteq R = K[\mathbf{P}^N]$ using the result $I^{(Nm)} \subseteq I^m$ of [ELS], [HH1]. Among other things, the algebraic argument suggests that Conjecture 2.1 is closely related to the study of $\alpha(I^{(m)})$.

When $N = 2$, Chudnovsky improved the bound of Waldschmidt and Skoda [W1, Sk]. Since only a sketch of Chudnovksy's proof is given in [Ch], we give both the statement and proof here. We will give a small improvement of this result in Proposition 3.5.

Proposition 3.1 (Chudnovsky). *Let $p_1, \ldots, p_n \in \mathbf{P}^2$ be distinct points. Let $I = \cap_i I(p_i) \subset K[\mathbf{P}^2]$. Then*

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + 1}{2}$$

for all $m > 0$.

Proof. Let $b = \alpha(I)$. Choose distinct points $q_1, \ldots, q_t \in \{p_1, \ldots, p_n\}$ with t as small as possible such that $\alpha(J) = b$, where $J = \cap_i I(q_i)$. By minimality, the points q_i impose independent conditions in degree $b - 1$, hence $t = \binom{b+1}{2}$ (since this is the dimension of the space of all forms of degree $b - 1$) and $\alpha(J) = \text{reg}(J)$. Thus J is generated in degree b and hence the only base points of J_b are the points q_i; in particular, J_b is fixed component free (i.e., there is no nonconstant common factor for the homogeneous elements of J of degree b). Now let A be a nonzero form in $I_a^{(m)}$, where

$a = \alpha(I^{(m)})$. Since J_b is fixed component free, we can choose an element $B \in J_b$ with no factor in common with A. By Bézout's Theorem, it follows that $ab = \deg(A)\deg(B) \geq mt = m\binom{b+1}{2}$, and hence that $\alpha(I^{(m)}) = a \geq (b+1)/2 = (\alpha(I)+1)/2$. □

Our original motivation for Conjecture 2.1 was exactly this result of Chudnovsky. Just as the containment result $I^{(rN)} \subseteq I^r$ of [HH1] implies the bound $\alpha(I^{(m)})/m \geq \alpha(I)/N$ of Waldschmidt and Skoda, we looked for a new containment which in a similar way would imply the bound in Proposition 3.1. In addition to the result of Proposition 3.1, Chudnovsky [Ch] has conjectured for $N > 2$ that $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$ (actually his conjecture was stated for $K = \mathbf{C}$ for points in affine N-space). We now show that Conjecture 2.1 implies not only Proposition 3.1 but also $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$.

Lemma 3.2. *Let $p_1, \ldots, p_n \in \mathbf{P}^N$ be distinct points, and let $I = \cap_i I(p_i) \subset K[\mathbf{P}^N]$. If $I^{(Nr)} \subseteq M^{r(N-1)}I^r$ holds for all $r \geq 1$, then $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$ holds for all $m \geq 1$.*

We pause to recall a numerical quantity introduced by Waldschmidt [W1] for sets of points, but which extends to homogeneous ideals $0 \neq I \subset K[\mathbf{P}^N]$. Define

$$\gamma(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

The limit exists and satisfies $\gamma(I) \leq \frac{\alpha(I^{(m)})}{m}$ for all $m \geq 1$; see [BH1, Lemma 2.3.1] and its proof.
We now prove Lemma 3.2.

Proof. Since $I^{(Nr)} \subseteq M^{r(N-1)}I^r$, we have $\alpha(I^{(Nr)}) \geq \alpha(M^{r(N-1)}I^r) = r\alpha(I) + r(N-1)$. Now divide by rN and take limits as $r \to \infty$ to get $\alpha(I^{(m)})/m \geq \gamma(I) = (\alpha(I) + N - 1)/N$. □

Conversely, we can also use Proposition 3.1 to prove certain cases of Conjecture 2.1.

Proposition 3.3. *Let $J = \cap_i I(p_i) \subset K[\mathbf{P}^N]$ and let $I = J^{(m)} \subset R$ be a fat points ideal. If $N = 2$ and if J is generated in degree $\alpha(J)$, then $I^{(Nr)} \subseteq M^{r(N-1)}I^r$ for all r.*

Proof. Let $s = \alpha(J)$; then $\alpha(J^{(Nmr)}) \geq mrs + mr(N-1)$ by Proposition 3.1. Now apply Proposition 2.3. □

Remark 3.4. Given $I = \cap_i I(p_i)$ for any distinct points $p_1, \ldots, p_n \in \mathbf{P}^N$ for $N > 2$, if Chudnovsky's conjecture [Ch] $\alpha(I^{(m)})/m \geq (\alpha(I) + N - 1)/N$ holds, then the proof of Proposition 3.3 would work for any N, not just $N = 2$.

We next refine Proposition 3.1 by bringing into play $\beta(I^{(m)})$.

Proposition 3.5. *Let Let $p_1, \ldots, p_n \in \mathbf{P}^2$ be distinct points. Let $I = \cap_i I(p_i) \subset K[\mathbf{P}^2] = R$. Set $\alpha_m = \alpha(I^{(m)})$, and $\beta_m = \beta(I^{(m)})$. Then we have:*

(i) $\alpha_m \beta_m \geq m^2 n$, and

(ii) $\dfrac{\alpha_m}{m} \geq \left(\dfrac{\alpha_1 + 1}{2}\right)\left(\dfrac{m\alpha_1}{\beta_m}\right).$

Moreover, if $\alpha_m \beta_m = m^2 n$, then $(I^{(m)})^k = I^{(mk)}$ for all $k \geq 1$.

Proof. We first prove that $\alpha_m \beta_m \geq m^2 n$, which is basically by Bézout's theorem. Let l be a general linear form. Choose f of degree α_m and g of degree β_m in $I^{(m)}$ which form a regular sequence. Then $\alpha_m \beta_m = \ell(R/(f, g, l)) = \sum_P \ell(_P/(f, g)_P)\ell(R/(P, l))$, where the sum is over all prime ideals minimal over the ideal (f, g). Since such P include all the ideals corresponding to the points $p_1, \ldots, p_n$, we can restrict the sum to $P_1, \ldots, P_n$, where $P_i = I(p_i)$. In this case, $\ell((R/(P_i, l)) = 1$

for each i, and $\ell(R_{P_i}/(f,g)_{P_i}) \geq m^2$ since R_{P_i} is a regular local ring and by assumption, the images of f and g are in $(P_i^m)_{P_i}$[1] . This gives the first inequality.

To prove the second inequality, we use the idea from the proof of Proposition 3.1. As in that proof, choose distinct points $q_1, \ldots, q_t \in \{p_1, \ldots, p_n\}$ with t as small as possible such that $\alpha(J) = \alpha(I)$, where $J = \cap_i I(q_i)$. As above, $t = \binom{\alpha(I)+1}{2}$, and $\alpha(J) = \operatorname{reg}(J)$. Thus J is generated in degree $\alpha(I)$. Note that $I^{(m)} \subset J^{(m)}$, and so $\alpha_m \beta_m \geq \alpha(J^{(m)}))\beta(J^{(m)}) \geq m^2\binom{\alpha(I)+1}{2}$, using the first inequality of this theorem. This proves the second inequality.

Finally we prove the last part. Suppose that $\alpha_m \beta_m = m^2 n$. Choose f and g as in the first part of this theorem. We use the ideas of Theorem 3.1 of [Hu], which cannot be used directly since it deals with the local case of a prime ideal.

To show $I^{(mk)} = (I^{(m)})^k$ for all $k \geq 1$, since $(f,g)^{k-1}I^{(m)} \subseteq (I^{(m)})^k \subseteq I^{(mk)}$, it's enough to show $I^{(mk)} = (f,g)^{k-1}I^{(m)}$. To prove this note that $(f,g)^{k-1}I^{(m)} \subset I^{(mk)}$, so it suffices to prove the equality locally at each associated prime of $(f,g)^{k-1}I^{(m)}$. We prove that the associated primes of this ideal are exactly the primes ideals $P_i = I_{p_i}$. Clearly each of these are associated since they are minimal over the ideal. To prove they are all the associated primes, we use induction on k. If $k = 0$, this is clear by the definition of symbolic powers. For $k \geq 1$, $(f,g)^{k-1}/(f,g)^{k-1}I^{(m)} \cong (f,g)^{k-1}/(f,g)^k \otimes R/I^{(m)}$ since $(f,g)^k \subset (f,g)^{k-1}I^{(m)}$, and the tensor product is isomorphic to a free $R/I^{(m)}$-module as $(f,g)^{k-1}/(f,g)^k$ is a free $R/(f,g)$-module because f, g form a regular sequence. Hence the only associated primes of $(f,g)^{k-1}/(f,g)^{k-1}I^{(m)}$ are $P_1, \ldots, P_n$. Now the exact sequence

$$0 \to (f,g)^{k-1}/(f,g)^{k-1}I^{(m)} \to R/(f,g)^{k-1}I^{(m)} \to R/(f,g)^{k-1} \to 0$$

shows that the associated primes of $R/(f,g)^{k-1}I^{(m)}$ are contained in the union of the associated primes of $(f,g)^{k-1}/(f,g)^{k-1}I^{(m)}$ together with the associated primes of $R/(f,g)^{k-1}$. Since the associated primes of $R/(f,g)^{k-1}$ are exactly those of $R/(f,g)$, to finish the proof of our claim we need to prove that all the associated primes of (f,g) are $P_1, \ldots, P_n$. As this ideal is unmixed, this is equivalent to proving that the only points both f and g vanish at are $P_1, \ldots, P_n$. Recall from above that $\alpha_m \beta_m = \ell(R/(f,g,l)) = \sum_P \ell(R_P/(f,g)_P)\ell(R/(P,l))$, where the sum is over all prime ideals minimal over the ideal (f,g). Since we are assuming that $\alpha_m \beta_m = m^2 n$, and since $\sum_{P_i} \ell(R_{P_i}/(f,g)_{P_i})\ell(R/(P_i,l)) \geq m^2 n$, we see that the only primes minimal over (f,g) are $P_1, \ldots, P_n$.

To finish the proof, we need to prove that $(I^{(mk)})_{P_i} = ((f,g)^{k-1}I^{(m)})_{P_i}$ for every i. We know from the fact equality holds that $\ell(R_{P_i}/(f,g)_{P_i}) = m^2$. However the multiplicity of $(P_i)_{P_i}^m$ is exactly m^2, and since $(f,g)_{P_i} \subset (P_i)_{P_i}^m$, it follows from Rees's theorem [SH, Theorem 11.3.1] that $(f,g)_{P_i}$ is a minimal reduction of $(P_i^m)_{P_i}$. Moreover, $(I^{(m)})_{P_i} = (P_i^m)_{P_i}$ is integrally closed. By the result of Lipman and Teissier [LT], the result follows. □

Remark 3.6. The second inequality of Proposition 3.5 can be thought of as an improvement of Proposition 3.1 in the case in which $\beta(I^{(m)}) < \alpha(I)m$. However, the proof actually shows that

$$\frac{\alpha(I^{(m)})}{m} \geq \left(\frac{\alpha(I)+1}{2}\right)\left(\frac{\alpha(I)m}{\beta(J^{(m)})}\right), \tag{$*$}$$

where J is as in the proof. Moreover, $\beta(J^{(m)}) \leq \alpha(I)m$, since J is generated in degree $\alpha(J) = \alpha(I)$, and $J^m \subset J^{(m)}$. Thus $(*)$ does represent a small improvement on the orginal result of Chudnovsky. In fact, suppose that equality occurs in Chudnovksy's bound, so that $\frac{\alpha(I^{(m)})}{m} = \frac{\alpha(I)+1}{2}$. Then

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[1]In general, if (R, M) is a regular local ring and if $x_1, \ldots, x_d \in M^n$ form a maximal regular sequence, then $\ell(R/(x_1, \ldots, x_d)) \geq n^d$. This can be seen, for example, as follows. If I denotes the integral closure of the ideal $(x_1, \ldots, x_d)$, then $I \subset M^n$ as the latter ideal is integrally closed. Hence the multiplicity of I, which is $\ell(R/(x_1, \ldots, x_d))$, is at least the multiplicity of M^n, which is n^d.

necessarily $\alpha(I)m = \beta(J^{(m)})$ by (*). But $\alpha(I^{(m)}) \geq \alpha(J^{(m)}) \geq (\frac{\alpha(J)+1}{2})m$, so that equality must hold and we obtain that $\alpha(J^{(m)})\beta(J^{(m)}) = m^2\binom{\alpha(I)+1}{2}$. Therefore by Proposition 3.5, it follows that $(J^{(m)})^k = J^{(mk)}$ for all $k \geq 1$.

Remark 3.7. The last conclusion in Theorem 3.5 implies that the symbolic power algebra, $\oplus I^{(n)}$ is a Noetherian ring. (This is a homogeneous version of [Sc, Theorem 1.3].)

Definition 3.8. Let $H_1, \ldots, H_s \in \mathbf{P}^N$ be $s \geq N$ hyperplanes such that no $N + 1$ meet at a single point, and let $p_1, \ldots, p_n$ be the $n = \binom{s}{N}$ points such that each point is the intersection of a subset of N of the s hyperplanes; following a suggestion of Geramita, we refer to such a set of points p_i as a *star configuration* for s hyperplanes in $\mathbf{P}^N$ (since 5 general lines in the plane if drawn appropriately give a 5 pointed star).

We now show that Conjecture 2.1 and the conjecture of Chudnovsky mentioned in Remark 3.4 both hold for star configurations. In fact, we show more.

Corollary 3.9. *Let* $I = \cap_i I(p_i)$ *where the* $n = \binom{s}{N}$ *points* $p_i \in \mathbf{P}^N$ *give a star configuration coming from* $s \geq N$ *hyperplanes in* $\mathbf{P}^N$. *Then* $\frac{\alpha(I^{(r)})}{r} \geq \frac{\alpha(I)+N-1}{N}$ *and* $I^{(Nr)} \subseteq M^{r(N-1)}I^r$ *hold for all* $r \geq 1$, *with equality in the former when* r *is a multiple of* N. *If moreover* $N = 2$ *and* $m > 0$ *is an even integer, then: equality holds in* (*) *of Remark 3.6;* $\alpha(I^{(m)})\beta(I^{(m)}) = m^2 n$; *and* $\alpha(I)k = \beta(I^{(k)})$ *and* $(I^{(m)})^k = I^{(mk)}$ *hold for all* $k \geq 1$.

Proof. We have $\alpha(I^{(Nr)}) = sr$ by [BH1, Lemma 2.4.1], so $\gamma(I) = \lim_{r\to\infty} \alpha(I^{(rN)})/(rN) = s/N$, but we also have $\operatorname{reg}(J) = \alpha(I) = s - N + 1$ by [BH1, Lemma 2.4.2], so we have the equality $\alpha(I^{(Nr)})/(rN) = \gamma(I) = s/N = (\alpha(I) + N - 1)/N$, as claimed. Since as pointed out above $\alpha(I^{(r)})/r \geq \gamma(I)$ holds for all r, we also have $\frac{\alpha(I^{(r)})}{r} \geq \frac{\alpha(I)+N-1}{N}$.

Since $\alpha(I) = \operatorname{reg}(I)$, we see I is generated in degree $\alpha(I) = s - (N - 1)$ and since $\alpha(I^{(mrN)}) = mrs \geq mr(s - (N - 1)) + mr(N - 1)$, it follows by Proposition 2.3 that $I^{(Nr)} \subseteq M^{r(N-1)}I^r$.

Now assume $N = 2$. Then we have $\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)+1}{2}$ from above, so (*) will be an equality if we verify that $\beta(I^{(m)}) = m\alpha(I)$. Note that each of the s lines defining the star configuration contain exactly $s - 1$ of the $n = \binom{s}{2}$ points p_i. Let L be the linear form defining one of these lines. If $F \in (I^{(k)})_t$ is a form of degree $t < k(s - 1)$, then L divides F by Bézout's Theorem. Thus $k(s - 1) \leq \beta(I^{(k)})$, but $\beta(I^{(k)}) \leq \operatorname{reg}(I^{(k)})$, and, by [GGP, Theorem 1.1], $\operatorname{reg}(I^{(k)}) \leq k\operatorname{reg}(I)$. Since $\operatorname{reg}(I) = s - 1$, we see $\beta(I^{(k)}) = k(s - 1) = k\alpha(I)$, as we wanted to show. Moreover, $\alpha(I^{(m)})\beta(I^{(m)}) = (sm/2)m(s - 1) = m^2 n$, so $(I^{(m)})^k = I^{(mk)}$ holds by Proposition 3.5. $\qquad\square$

Of course, star configurations are very special sets of points, but if one takes the ideal J of a general set of $\binom{s}{2}$ points of $\mathbf{P}^2$, then $\alpha(J) = \operatorname{reg}(J)$ so J is generated in degree $\alpha(J)$ and Conjecture 2.1 holds for $I = J^{(m)}$ by Proposition 3.3. More generally, we now show that Conjecture 2.1 holds for the radical ideal I of any set of n general points of $\mathbf{P}^2$.

Proposition 3.10. *Let* $I = \cap_i I(p_i) \subset R$ *for* n *general points* $p_i \in \mathbf{P}^2$. *Then* $I^{(2r)} \subseteq M^r I^r$ *holds for all* r.

Proof. For $n = 1, 3, 6$, n is a binomial coefficient, so $I^{(2r)} \subseteq M^r I^r$ holds as we observed immediately above. For $n = 2, 4$, the points are a complete intersection, and so $I^{(2r)} = I^{2r} \subseteq M^r I^r$ holds. By Proposition 2.3 it is enough to show that I is generated in degrees s and less for some s such that $\alpha(I^{(2r)})/(2r) \geq (s + 1)/2$ for all r. Thus it is enough to show that $\gamma(I) \geq (s + 1)/2$.

Consider the case $n = 5$. By [BH2, Lemma 3.1], $\alpha(I^{(2r)}) = 4r$ and hence $\gamma(I) = 2$. Since the 5 points impose independent conditions on forms of degree s for any $s \geq 2$, we see $\operatorname{reg}(I) = 3$ (so I is generated in degree $s = 3$ and less). Thus $\gamma(I) = 2 \geq (s + 1)/2$ as we wanted to show.

For $n = 7$, I is generated in degrees 3 and less [Ha1] and $\gamma(I) = 21/8$ (see the proof of [BH2, Proposition 4.3]); for $n = 8, 9$, I is generated in degrees 4 and less since $\operatorname{reg}(I) = 4$, $\gamma(I) = 48/17$ (see the proof of [BH2, Proposition 4.4]) when $n = 8$ and (it is easy to see) $\gamma(I) = 3$ when $n = 9$. Thus $I^{(2r)} \subseteq M^r I^r$ holds for $7 \leq n \leq 9$. Now say $n > 9$. If n is a binomial coefficient $\binom{s}{2}$ we saw above that $I^{(2r)} \subseteq M^r I^r$ holds, so assume that $\binom{s}{2} < n < \binom{s+1}{2}$ for some $s \geq 5$. It is known that $\gamma(I) \geq \sqrt{n-1}$ (see [PSC, Remark 8.3.5] and the proof of [BH1, Theorem 4.2]) and that I is generated in degree at most s (since $\operatorname{reg}(I) = s$). So we want to check that $\sqrt{n-1} \geq (s+1)/2$, or that $n - 1 \geq (s+1)^2/4$, but $n - 1 \geq \binom{s}{2}$ and $\binom{s}{2} > (s+1)^2/4$ for $s \geq 5$. $\qquad \square$

For later use we have the following results regarding the ideal of 5 general points of $\mathbf{P}^2$.

Lemma 3.11. *Let I be the ideal of 5 general points of $\mathbf{P}^2$. Then $I^{(2r)} = (I^{(2)})^r$ and $I^{(2r+1)} = I^{(2r)}I$ for all $r \geq 1$.*

Proof. We saw in the proof of Proposition 3.10 that $\alpha(I^{(m)}) = 2m$. Also, by Bézout's Theorem we must have $2\beta(I^{(m)}) \geq 5m$, and by [Ha2, Remark I.5.5] (or by [Ha3, Theorem III.1(a)]), $(I^{(m)})_t$ has no non-constant common factors for $2t \geq 5m$. Thus $\beta(I^{(m)}) = \lceil \frac{5m}{2} \rceil$. In particular, $\alpha(I^{(2r)})\beta(I^{(2r)}) = 5(2r)^2$, so $I^{(2r)} = (I^{(2)})^r$ by Proposition 3.5.

Now consider $I^{(2r+1)}$. Clearly, $I^{(2r)}I = I^{2r}I = I^{2r+1} \subseteq I^{(2r+1)}$, so consider the reverse inclusion. By Bézout's Theorem, since $\beta(I^{(2)}) = 5$, we have $5\alpha(I^{(2r+1)}) \geq 2*5(2r+1)$, and hence $\alpha(I^{(2r+1)}) \geq 2(2r+1)$, but $\alpha(I^{(2r+1)}) \leq \alpha(I^{2r+1}) = 2(2r+1)$, so we have $\alpha(I^{(2r+1)}) = 2(2r+1)$. Thus $0 = (I^{(2r+1)})_t \subseteq (I^{(2r)}I)_t$ for $t < 2(2r+1)$. If $2(2r+1) \leq t < 5(2r+1)/2$, then Q is a common factor for $(I^{(2r+1)})_t$, by Bézout's Theorem, where Q is a homogeneous form of degree 2 defining the unique conic through the five general points. Thus $(I^{(2r+1)})_t = Q(I^{(2r)})_{t-2}$, but $Q \in I_2$, so $(I^{(2r+1)})_t \subseteq (I^{(2r)})_{t-2}I_2 \subseteq ((I^{(2r)})I)_t$. Finally, assume $t \geq 5(2r+1)/2 = 5r + 3$. Then $(I^{(2r+1)})_t = (I^{(2r)})_{5r}I_{3+(t-5r)} \subseteq ((I^{(2r)})I)_t$ by [BH2, Proposition 2.4], and hence $I^{(2r+1)} \subseteq I^{(2r)}I$. $\qquad \square$

Remark 3.12. We close this section with a discussion of issues raised by Proposition 3.5. For example, the bound in Proposition 3.5(ii) improves on Chudnovsky's bound in Proposition 3.1 only when $m\alpha(I) > \beta(I^{(m)})$, but it certainly can happen that $m\alpha(I) < \beta(I^{(m)})$. For example, consider the ideal I of $n > d^2$ points on a smooth plane curve of degree d. Then (by Bézout's Theorem) we have both $\alpha(I) = d$ and $\beta(I^{(m)})d \geq nm$, so $\beta(I^{(m)}) \geq nm/d > \alpha(I)m$. There also are cases with $m\alpha(I) > \beta(I^{(m)})$. For example, given $m > 0$, let $a = (2m+1)^2 - 1$ and let $n = 2(2m+1)$. It is easy now to check that $\binom{a+1}{2} = n^2\binom{m+1}{2}$. Let I be the ideal of the union Z of n^2 general points in $\mathbf{P}^2$. By [CM, E, R], the fat point scheme iZ imposes independent conditions on forms of degree t for any t such that $(I^{(i)})_t \neq 0$. In particular, it follows that $\binom{\alpha(I)+2}{2} > n^2$, so $m\alpha(I) > m(\sqrt{2}n - 2)$. Since $\binom{a+1}{2} = n^2\binom{m+1}{2}$, it also follows that $\beta(I^{(m)}) \leq \operatorname{reg}(I^{(m)}) = a = (2m+1)^2 - 1$, hence for $m \gg 0$ we have $\beta(I^{(m)}) < (2m+1)^2 < 4\sqrt{2}m^2 < m(\sqrt{2}n - 2) < m\alpha(I)$.

Another issue raised by Proposition 3.5 is whether there are cases of n points in the plane and integers $m > 0$ where the ideal I of the points satisfies $\alpha(I^{(m)})\beta(I^{(m)}) = m^2n$. This holds for the ideal I of a star configuration by Corollary 3.9, hence $I^{(2k)} = (I^{(2)})^k$ for all $k \geq 1$. For additional examples, consider the scheme Z consisting of n general points of the plane. For $n \leq 8$ points, we can assume the points lie on a smooth cubic curve, in which case the results of [Ha3, GuH, GHM] can be used to determine $\alpha(I^{(m)})$ and $\beta(I^{(m)})$. For $n = 1$, $\alpha(I^{(m)}) = \beta(I^{(m)}) = m$ for all $m \geq 1$, hence by Proposition 3.5 we have $I^{(k)} = I^k$ for all $k \geq 1$. Of course, in this case Z is a complete intersection. For $n = 2$, $\alpha(I^{(m)}) = m$ and $\beta(I^{(m)}) = 2m$ for all $m \geq 1$, hence we have $I^{(k)} = I^k$ for all $k \geq 1$. Of course, in this case Z is again a complete intersection. For $n = 3$, $\alpha(I^{(m)}) = \lceil 3m/2 \rceil$ and $\beta(I^{(m)}) = 2m$ for all $m \geq 1$, hence we have $I^{(2k)} = (I^{(2)})^k$ for all $k \geq 1$. In this case Z is a star configuration. For $n = 4$, $\alpha(I^{(m)}) = \beta(I^{(m)}) = 2m$ for all $m \geq 1$, hence we have $I^{(k)} = I^k$ for all $k \geq 1$. Of course, in this case Z is yet again a complete intersection. For $n = 5$, we

have $I^{(2k)} = (I^{(2)})^k$ for all $k \geq 1$; see Lemma 3.11 and its proof. For $n = 6$, $\alpha(I^{(m)}) = \lceil 12m/5 \rceil$ and $\beta(I^{(m)}) = \lceil 5m/2 \rceil$ for all $m \geq 1$, hence we have $I^{(10k)} = (I^{(10)})^k$ for all $k \geq 1$. For $n = 7$, $\alpha(I^{(m)}) = \lceil 21m/8 \rceil$ and $\beta(I^{(m)}) = \lceil 8m/3 \rceil$ for all $m \geq 1$, hence we have $I^{(24k)} = (I^{(24)})^k$ for all $k \geq 1$. For $n = 8$, $\alpha(I^{(m)}) = \lceil 48m/17 \rceil$ and $\beta(I^{(m)}) = \lceil 17m/6 \rceil$ for all $m \geq 1$, hence we have $I^{(102k)} = (I^{(102)})^k$ for all $k \geq 1$.

Again applying the results of [Ha3], for $n = 9$ general points (which we may assume therefore lie on a smooth cubic curve), we have $\alpha(I^{(m)}) = 3m$ and $\beta(I^{(m)}) = 3m + 1$, so we never have $\alpha(I^{(m)})\beta(I^{(m)}) = m^2 n$, but if the 9 points p_i are chosen to be points on a smooth plane cubic curve such that $p_1 + \cdots + p_9 - p$ has order r in the divisor class group of the cubic (where p is a flex point), then $\alpha(I^{(m)}) = 3m$ and $\beta(I^{(m)})$ is $3m$ if $r|m$, and $3m + 1$ if r does not divide m. Thus $\alpha(I^{(r)})\beta(I^{(r)}) = 9r^2$, so $I^{(rk)} = (I^{(r)})^k$ for all $k \geq 0$.

Remark 3.13. Finally, we raise two questions. Given distinct points p_i in the plane and positive integers m_i, consider the ideal $I = \cap_i I(p_i)^{m_i}$. Is it true that $I^{(mk)} = (I^{(m)})^k$ for $k \geq 1$ (and hence that the symbolic Rees algebra $\oplus I^{(n)}$ is Noetherian) if $\alpha(I^{(m)})\beta(I^{(m)}) = m^2 \sum_i m_i^2$? See [BH2, Example 5.1] for examples with $\alpha(I)\beta(I) = \sum_i m_i^2$ for which $I^{(k)} = I^k$ for all $k \geq 1$. Conversely, if $\oplus I^{(n)}$ is Noetherian, must $\alpha(I^{(m)})\beta(I^{(m)}) = m^2 \sum_i m_i^2$ hold for some m?

4. ADDITIONAL QUESTIONS AND CONJECTURES

If for some ideal I there is a d such that one has $I^{(m)} \subseteq I^r$ for all $m \geq dr$, one can next ask for what constants c does $m \geq dr - c$ guarantee $I^{(m)} \subseteq I^r$. We discuss questions and conjectures related to this problem of subtracting a constant in Section 4.1. In Section 4.2 we discuss questions and conjectures arising out of refinements of the Waldschmidt-Skoda bound ($\circ$) of the Introduction.

4.1. **Subtracting a constant.** The second named author has raised the question whether a radical ideal I of a finite set of points in $\mathbf{P}^2$ always satisfies $I^{(3)} \subseteq I^2$. Examples suggested to the first named author the following conjectural generalization [PSC, Conjecture 8.4.2]:

Conjecture 4.1.1. *Let $I \subseteq K[\mathbf{P}^N]$ be a homogeneous ideal. Then $I^{(rN-(N-1))} \subseteq I^r$ holds for all r.*

Conjecture 4.1.1 holds for star configurations [PSC, Example 8.4.8]; examples of star configurations also show $I^{(rN-N)} \subseteq I^r$ fails in general. When $N = 2$, Conjecture 4.1.1 is true for any finite set of general points in $\mathbf{P}^2$ by [BH1, Remark 4.3]. Thus, while Conjecture 4.1.1 is open in general, it is plausible (at least for ideals of points). In fact, the second named author observed that the principle underlying the main result of [HH1] shows that Conjecture 4.1.1 is true for radical ideals of any finite set of points if $\mathrm{char}(K) = p > 0$ when r is a power of p [PSC, Remark 8.4.4]. The same basic argument also verifies Conjecture 4.1.1 for monomial ideals (with no restriction on the characteristic) [PSC, Remark 8.4.5].

The key in both cases is the use of nice behavior of Frobenius powers. If $I \subseteq R$ is an ideal, define its qth *Frobenius power* $I^{[q]}$ to be the ideal generated by all v^q for $v \in I$. If I is a monomial ideal, then $I^{[q]}$ is generated by the qth powers of any set of monomial generators of I. And if $\mathrm{char}(K) = p > 0$ and q is a power of p, then $I^{[q]}$ is generated by qth powers of any set of generators of I. A fundamental fact for ideals $J_1, \ldots, J_s \subseteq R$ is that $(\cap_i J_i)^{[q]} = \cap_i (J_i^{[q]})$ if either J_i is monomial for each i (see [PSC, Remark 8.4.5]) or if $p = \mathrm{char}(K) > 0$ and q is a power of p (by flatness of Frobenius; see [SH, Lemma 13.1.3, p. 247] and [K]). As a direct consequence we obtain:

Lemma 4.1.2. *Let $g > 0$ be an integer and let $J_1, \ldots, J_s \subseteq R$ be ideals, each generated by at most g elements. Assume either that each J_i is monomial, or that $p = \mathrm{char}(K) > 0$ and r is a power of p. Then $\cap_i(J_i^m) \subseteq (\cap_i J_i)^r$ as long as $m \geq gr - g + 1$.*

Proof. Because J_i has at most g generators, any product of a choice of $gr - g + 1$ of these generators is divisible by the rth power of one of the generators. Thus $J_i^m \subseteq J_i^{gr-g+1} \subseteq J_i^{[r]}$, so we have $\cap_i(J_i^m) \subseteq \cap_i(J_i^{[r]}) = (\cap_i J_i)^{[r]} \subseteq (\cap_i J_i)^r$. $\qquad \square$

Remark 4.1.3. We now see that Conjecture 4.1.1 holds for any monomial ideal $I \subset K[\mathbf{P}^N]$. If I is not saturated, then $I^{(rN-(N-1))} = I^{rN-(N-1)} \subseteq I^r$, and if I is saturated then there is a (not necessarily irredundant) primary decomposition $I = \cap_i J_i$ where each J_i is monomial, primary and generated by positive powers of the variables in some proper subset (depending on i) of the $N + 1$ variables (and hence J_i has at most N generators). Moreover, it is not hard to show that $I^{(m)} \subseteq \cap_i(J_i^m)$ (see [PSC, Remark 8.4.5]). Lemma 4.1.2 now applies, and we have $I^{(m)} \subseteq I^r$ whenever $m \geq rN - N + 1$. Similarly, if $p = \mathrm{char}(K) > 0$ and r is a power of p, then for the ideal $I = \cap_i I(p_i)$ of any finite set of distinct points $p_i \in \mathbf{P}^N$, by Lemma 4.1.2 we have $I^{(m)} = \cap_i(I(p_i)^m) \subseteq (\cap_i I(p_i))^r = I^r$ as long as $m \geq rN - N + 1$.

When $N = 2$ Chudnovsky's bound $\gamma(I) \geq (\alpha(I) + 1)/2$ suggests another conjecture (in the perhaps weak sense that Conjecture 4.1.4 implies the bound):

Conjecture 4.1.4. *Let $I \subseteq K[\mathbf{P}^2]$ be the homogeneous radical ideal of a finite set of points. Then $I^{(m)} \subseteq I^r$ holds whenever $m/r \geq 2\alpha(I)/(\alpha(I) + 1)$.*

This conjecture does indeed imply Chudnovsky's bound $\gamma(I) \geq (\alpha(I) + 1)/2$. To see this, let $m = 2\alpha(I)t$ and let $r = (\alpha(I) + 1)t$ for integers $t \geq 1$. Conjecture 4.1.4 then implies that $I^{(m)} \subseteq I^r$ which in turn implies $\alpha(I^{(m)}) \geq \alpha(I^r)$ and hence $\alpha(I^{(m)})/m \geq (r/m)\alpha(I)$. Taking limits as $t \to \infty$ gives $\gamma(I) \geq (\alpha(I) + 1)/2$.

Conjecture 4.1.4 is asymptotically stronger than Conjecture 4.1.1 since $2r - 1 > 2r\alpha(I)/(\alpha(I)+1)$ for $r \gg 0$. But as with Conjecture 4.1.1, Conjecture 4.1.4 holds for star configurations: if I is the ideal of a star configuration, then $\alpha(I) = \mathrm{reg}(I)$ and $\gamma(I) = (\alpha(I) + 1)/2$, hence we have $I^{(m)} \subseteq I^r$ by [BH1, Theorem 1.2.1(b)] (since $m/r \geq 2\alpha(I)/(\alpha(I) + 1) = \mathrm{reg}(I)/\gamma(I)$).

It is easy to see that Conjecture 4.1.4 holds if I is a complete intersection, since then $I^{(m)} = I^m$. Conjecture 4.1.4 also holds for general sets of n points (use [BH1, Remark 4.3] in case $n \geq 6$, since then $\alpha(I) \geq 3$ and so $2\alpha(I)/(\alpha(I) + 1) \geq 3/2$, use [BH2, Theorem 3.4(b)] for $n = 5$, and for $n < 5$ note that n general points give either a star or a complete intersection or both).

Giving Conjecture 4.1.1 an evolutionary twist, we obtain another possibility:

Conjecture 4.1.5. *Let $I \subseteq K[\mathbf{P}^N]$ be the ideal of a finite set of points $p_i \in \mathbf{P}^N$. Then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)}I^r$ holds for all $r \geq 1$.*

Lemma 4.1.6. *Let $I \subseteq K[\mathbf{P}^N]$ be the ideal of a finite set of points $p_i \in \mathbf{P}^N$. If $I^{(Nr-(N-1))} \subseteq I^r$ and $\alpha(I^{(Nr-(N-1))}) \geq rs + (r-1)(N-1)$ for some s such that I is generated in degrees s and less, then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)}I^r$.*

Proof. The same argument (with $m = 1$) used in the proof of Proposition 2.3 works here. $\qquad \square$

Corollary 4.1.7. *Let $I \subseteq K[\mathbf{P}^N]$ be the ideal of a finite set of points $p_i \in \mathbf{P}^N$, comprising either a star configuration or a complete intersection. Then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)}I^r$ holds for all $r \geq 1$.*

Proof. Since for star configurations $\alpha(I^{(Nr-(N-1))}) = r\mathrm{reg}(I) + (r-1)(N-1)$ by [PSC, Lemma 8.4.7], Lemma 4.1.6 applies with $s = \mathrm{reg}(I)$ both conditions of Lemma 4.1.6, so the result follows. For complete intersections we have $I^{(rN-(N-1))} = I^{rN-(N-1)} = I^{rN-r-(N-1)}I^r = I^{(r-1)(N-1)}I^r \subseteq M^{(r-1)(N-1)}I^r$. $\qquad \square$

Of course, if Conjecture 4.1.5 is true, then so must be the following:

Conjecture 4.1.8. *Let $I \subseteq K[\mathbf{P}^N]$ be the ideal of a finite set of points $p_i \in \mathbf{P}^N$. Then*

$$\alpha(I^{(rN-(N-1))}) \geq r\alpha(I) + (r-1)(N-1)$$

for every $r > 0$.

Remark 4.1.9. Conjecture 4.1.8 is clearly true if $\alpha(I) = 1$ or if $r = 1$. It also holds if $\gamma(I) \geq (r\alpha(I) + (r-1)(N-1))/(rN - (N-1))$, since $\alpha(I^{(rN-(N-1))})/(rN - (N-1)) \geq \gamma(I)$. But if $\alpha(I) > 1$, then $(r\alpha(I) + (r-1)(N-1))/(rN - (N-1))$ is biggest when r is least, so Conjecture 4.1.8 will hold for all r if $\gamma(I) \geq (r\alpha(I) + (r-1)(N-1))/(rN - (N-1))$ holds for $r = 2$.

Remark 4.1.10. As mentioned in Remark 3.4, if I is the radical ideal of a finite set of points $p_i \in \mathbf{P}^N$, Chudnovsky conjectured that $\gamma(I) \geq (\alpha(I) + N - 1)/N$. It seems plausible in fact that $\gamma(I) > (\alpha(I) + N - 1)/N$ unless either the points lie on a hyperplane or give a star configuration. This is however open, even for $N = 2$. The best current result is: if $N = 2$ and if $\alpha(I^{(2)})/2 = (\alpha(I)+1)/2$, then the points p_i either lie on a line or give a star configuration [BC]. If it were true that $\gamma(I) > (\alpha(I) + N - 1)/N$ unless either the points were contained in a hyperplane or formed a star configuration, then Conjecture 4.1.8 would at least hold for all $r \gg 0$, as the next result shows.

Corollary 4.1.11. *For the radical ideal I of a finite set of points $p_1, \ldots, p_n \in \mathbf{P}^N$ we have*

$$\alpha(I^{(Nr-N+1)}) \geq r\alpha(I) + (r-1)(N-1)$$

for all $r \gg 0$ if either $\alpha(I) = 1$ or the points p_i form a star configuration or $\gamma(I) > (\alpha(I)+N-1)/N$.

Proof. If $\alpha(I) = 1$, then $\alpha(I^{(rN-(N-1))}) = rN - (N-1) = r\alpha(I) + (r-1)(N-1)$. By [PSC, Lemma 8.4.7], if the points form a star configuration on s lines, then $\alpha(I) = s - N + 1$ and $\alpha(I^{(rN-(N-1))}) = (r-1)s + s - N + 1 = r\alpha(I) + (r-1)(N-1)$. If $\gamma(I) > (\alpha(I) + N - 1)/N$, then $\gamma(I) = \delta + (g + N - 1)/N$ for some $\delta > 0$ where $g = \alpha(I)$. Hence for $r \gg 0$ we have $\alpha(I^{(Nr-(N-1))}) \geq rg + (r-1)(N-1)$, since $\alpha(I^{(Nr-(N-1))}) \geq (Nr - (N-1))\gamma(I) = (Nr - (N-1))\delta + (Nr - (N-1))(g+N-1)/N = rg + (r-1)(N-1) + (N(Nr-(N-1))\delta - (g-1)(N-1))/N$ and the last term is positive for $r \gg 0$. $\qquad\square$

Example 4.1.12. We now check that Conjecture 4.1.8 holds for every set of $n \leq 8$ points of $\mathbf{P}^2$. By Remark 4.1.9, it is enough to check $\gamma(I) \geq (2\alpha(I) + 1)/3$. When $N = 2$, $\gamma(I)$ can be found for each set of $n \leq 8$ points of $\mathbf{P}^2$ using the results of [GuH, GHM] (see http://www.math.unl.edu/~bharbourne1/GammaFile.html). It turns out that $\gamma(I) \geq (2\alpha(I) + 1)/3$ holds for every configuration of $n \leq 8$ points except for four cases: the 3 points coming from the star for 3 lines; the 6 points coming from the star for 4 lines; 6 points where 3 of them are a star for 3 lines and an additional point is chosen on each of those 3 lines but such that these three additional points are not collinear; and 7 points where 6 of them form the star on 4 lines and an additional point is placed on one of those 4 lines. In each of these cases except for the star configuration of 6 points, $\gamma(I) \geq (r\alpha(I) + r - 1)/(2r - 1)$ holds for $r = 3$ (and hence by Remark 4.1.9 for $r \geq 3$) and $\alpha(I^{(2r-1)}) \geq (r\alpha(I) + r - 1)/(2r - 1)$ holds for $r = 2$. This leaves the 6 point star configuration, but Corollary 4.1.11 shows that Conjecture 4.1.8 holds for stars.

Now we show Conjecture 4.1.5 holds in the case of n general points when $N = 2$. (We use the characteristic 0 hypothesis only for some values of r and n where we apply [D]; see the last paragraph of the proof.)

Corollary 4.1.13. *Let I be the ideal of n general points of $\mathbf{P}^2$; assume the ground field K has $\mathrm{char}(K) = 0$. Then $I^{(2r-1)}I \subseteq M^{r-1}I^r$.*

Proof. As noted in Corollary 4.1.7, $I^{(2r-1)}I \subseteq M^{r-1}I^r$ holds for complete intersections and for star configurations. Thus it holds for $n = 1, 2, 4$ since in these cases we have a complete intersection, and it holds for $n = 3$ since this is a star configuration. It holds for $n = 5$ by Proposition 3.10

and Lemma 3.11, since $I^{(2r-1)} = I^{(2(r-1))}I \subseteq M^{r-1}I^{r-1}I = M^{r-1}I^r$. For larger n, note that $I^{(2r-1)} \subseteq I^r$ holds for general points by [PSC, Example 8.4.9], so $I^{(2r-1)}I \subseteq M^{r-1}I^r$ also holds (by Lemma 4.1.6) if $\gamma(I) \geq (r\operatorname{reg}(I) + r - 1)/(2r - 1)$. Therefore $I^{(2r-1)}I \subseteq M^{r-1}I^r$ holds for $n = 6$ since then $\alpha(I) = \operatorname{reg}(I)$ and in Example 4.1.12 we verified that $\gamma(I) \geq (r\alpha(I) + r - 1)/(2r - 1)$ holds for $r \geq 2$ for every configuration of $3 \neq n \leq 8$ general points. Similarly, $I^{(2r-1)}I \subseteq M^{r-1}I^r$ holds for $n = 7$ since (as noted in the proof of Proposition 3.10) I is generated in degree $\alpha(I)$; now argue as in the case $n = 6$. For $n = 8$, $\operatorname{reg}(I) = 4$ and (as in the proof of Proposition 3.10) we have $\gamma(I) = 48/17 \geq (r\operatorname{reg}(I) + r - 1)/(2r - 1)$ for $r = 3$ (and hence for all $r \geq 3$), while for $r = 2$, we have $\alpha(I^{(2r-1)}) = 9 = r\operatorname{reg}(I) + r - 1$, so $I^{(2r-1)}I \subseteq M^{r-1}I^r$ holds for $n = 8$. And for $n = 9$ general points of $\mathbf{P}^2$, $\gamma(I) = 3 = (r\operatorname{reg}(I) + r - 1)/(2r - 1)$ for $r = 2$, so again $I^{(2r-1)}I \subseteq M^{r-1}I^r$ holds.

Now let $n \geq 10$; as in the proof of Proposition 3.10, $\gamma(I) \geq \sqrt{n-1}$. If $n = \binom{s+1}{2}$ for some $s \geq 4$, then $\operatorname{reg}(I) = \alpha(I) = s$, and so we have $\sqrt{n-1} \geq (r\operatorname{reg}(I) + r - 1)/(2r - 1)$ for $r = 2$, hence for all r. So now assume $\binom{s}{2} < n < \binom{s+1}{2}$ for some $s \geq 5$. Then $\operatorname{reg}(I) = s$, so we want to check that $\sqrt{n-1} \geq (2s + 1)/3$, or that $n - 1 \geq (2s + 1)^2/9$, but $n - 1 \geq \binom{s}{2}$ and $\binom{s}{2} \geq (2s + 1)^2/9$ for $s \geq 18$. So we need to check $s \leq 17$; i.e., $10 < n < 153$. By direct check we have $\sqrt{n-1} \geq (r\operatorname{reg}(I) + r - 1)/(2r - 1)$ for $r = 7$ for $10 < n < 153$. So now we just need to check that $\alpha(I^{(2r-1)}) \geq r\alpha(I) + (r - 1)$ holds for $2 \leq r \leq 6$ for $10 < n < 153$. We verified this using [D] to determine $\alpha(I^{(2r-1)})$; note that [D] assumes characteristic 0. $\qquad\square$

Remark 4.1.14. It may be worthwhile to consider the maximum height of the associated primes. If e is the maximum of the heights of the associated primes of a given homogeneous ideal I, then from [HH1] we know $I^{(re)} \subseteq I^r$, and it is conjectured in [PSC] that $I^{(re-(e-1))} \subseteq I^r$. This raises the question of whether $I^{(re)} \subseteq M^{r(e-1)}I^r$ and $I^{(re-(e-1))} \subseteq M^{(r-1)(e-1)}I^r$ are also true.

Remark 4.1.15. As a minor remark, we show how classical methods can be used to show $I^{(2m)} \subseteq I^m$ in some cases of ideals of points $p_1, \ldots, p_n \in \mathbf{P}^2$. Assume I is a radical ideal for a finite set of points in $\mathbf{P}^2$. Assume the characteristic is 0, and that $\alpha(I) = \operatorname{reg}(I)$. Using the Euler identity as in the proof of Fact 1.2, we have $I^{(2m)} \subseteq M^m I^{(m)}$. But $\alpha(I) = \operatorname{reg}(I)$ implies that $I^m = M^j \cap I^{(m)}$ for $j = m\alpha(I) = \alpha(I^m)$, since the saturation degree of I^m is bounded above by $m\operatorname{reg}(I)$ which by our hypothesis is equal to $\alpha(I^m)$, so truncating $I^{(m)}$ at degree $\alpha(I^m)$ gives I^m (see [BH2, Remark 4.2]).

Since $M^m I^{(m)} \subseteq I^{(m)}$, we now see that $M^m I^{(m)} \subseteq M^j \cap I^{(m)} = I^m$ if $M^m I^{(m)} \subseteq M^j$; i.e., if $\alpha(M^m I^{(m)}) \geq j$. But $\alpha(M^m I^{(m)}) = \alpha(I^{(m)}) + m \geq m\gamma(I) + m$ so $\alpha(M^m I^{(m)}) \geq j$ holds if $m\gamma(I) + m \geq j = m\alpha(I)$; i.e., if $\gamma(I) \geq \alpha(I) - 1$. This holds if $(\alpha(I) + 1)/2 \geq \alpha(I) - 1$ (i.e., if $\alpha(I) \leq 3$) since $\gamma(I) \geq (\alpha(I) + 1)/2$.

This shows that $I^{(2m)} \subseteq I^m$ holds in characteristic 0 if $\alpha(I) = \operatorname{reg}(I)$ and $3 \geq \alpha(I)$. We note that $\alpha(I) = \operatorname{reg}(I)$ implies that n is a binomial coefficient, so we have either $n = 6$ and $p_1, \ldots, p_6$ lie on a cubic but not on a conic (there are five essentially different such configurations [GuH]), or $n = 3$ and p_1, p_2, p_3 lie on a conic but not on a line, or $n = 1$.

Additional cases follow from $\gamma(I) \geq \alpha(I) - 1$. For example, for any 10 points which do not lie on a cubic, we have $\alpha(I) = \operatorname{reg}(I)$. If some 9 of the points lie on a smooth cubic, then $3\alpha(I^{(m)}) \geq 9m$ holds by Bézout's Theorem, so we have $\gamma(I) \geq 3 = \alpha(I) - 1$ and thus $I^{(2m)} \subseteq I^m$.

4.2. **Further refinements.** A refinement of (°) of the Introduction is given in [W2, Lemme 7.5.2]. In our terms, this refinement is that

$$\frac{\alpha(I^{(m)})}{m + N - 1} \leq \gamma(I). \tag{**}$$

for the radical ideal I for a finite set of points in the complex projective space $\mathbf{P}^N$. The proof given in [W2] uses complex analytic techniques; for an easy proof using multiplier ideals, see [La, Proposition 10.1.1 and Example 10.1.3].

In fact, by a variation of the proof given in the introduction for the case $m = 1$, $(^{**})$ holds for any homogeneous ideal $0 \neq I \subseteq K[\mathbf{P}^N]$ over any field K. In particular, we have $I^{(t(m+N-1))} \subseteq (I^{(m)})^t$ for $m \geq 1$ by [HH2], so $t\alpha(I^{(m)}) \leq \alpha(I^{(t(m+N-1))})$. Dividing by $t(m+N-1)$ and taking limits as $t \to \infty$ gives $(^{**})$.

Again let I be the radical ideal I for a finite set of points in the projective space $\mathbf{P}^N$ but over the complex numbers. A further refinement, proved using complex projective techniques, is given in [EV] for the case that $N \geq 2$:

$$\frac{\alpha(I^{(m)}) + 1}{m + N - 1} \leq \gamma(I). \tag{***}$$

This is just Proposition 3.1 when $m = 1$ and $N = 2$. Comparing $(^{***})$ with Chudnovsky's conjecture $(\alpha(I) + N - 1)/N \leq \gamma(I)$ raises the question:

Question 4.2.1. *Let I be the radical ideal I for a finite set of points in $\mathbf{P}^N$. Is it true for all $m \geq 1$ that*

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \gamma(I)?$$

Using the fact that for the ideal I of a star configuration defined by s hyperplanes in $\mathbf{P}^N$ with $m = Ni + j$ for $0 \leq i$ and $0 < i \leq N$ we have $\alpha(I^{(m)}) = (i+1)s - N + j$, one can check that Question 4.2.1 has an affirmative answer for star configurations.

These speculations and observations raise two additional questions for the radical ideal I of a finite set of points in $\mathbf{P}^N$:

Question 4.2.2. *Is it true for all positive integers m and t that $I^{(t(m+N-1))} \subseteq M^t(I^{(m)})^t$, where $M \subset K[\mathbf{P}^N]$ is the ideal generated by the variables?*

If Question 4.2.2 has an affirmative answer, then we obtain an alternate proof of $(^{***})$ in the usual way.

Finally we ask:

Question 4.2.3. *Is it true for all positive integers m and t that $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$?*

If so, then Question 4.2.1 must also have an affirmative answer.

REFERENCES

[BC]   C. Bocci and L. Chiantini. *The effect of points fattening on postulation*, J. Pure Appl. Alg. 215 (2011) 89–98.

[BH1]   C. Bocci and B. Harbourne. *Comparing Powers and Symbolic Powers of Ideals*, J. Algebraic Geometry, 19 (2010) 399–417.

[BH2]   C. Bocci and B. Harbourne. *The resurgence of ideals of points and the containment problem*, Proc. Amer. Math. Soc., Volume 138, Number 4, (2010) 1175–1190.

[B]   A. Boocher. *The Module of Differentials, Evolutions, and the Eisenbud-Mazur Conjecture*, senior thesis, Notre Dame University, 2008.

[Ch]   G. V. Chudnovsky. *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*, Séminaire de Théorie des Nombres, Paris 1979–80, Séminaire Delange-Pisot-Poitou, Progress in Math vol. 12, M-J Bertin, editor, Birkhäuser, Boston-Basel-Stutgart (1981).

[CM]   C. Ciliberto and R. Miranda. *Nagata's Conjecture for a Square Number of Points*, Ricerche di Matematica, 55 (2006), 71–78.

[D]   M. Dumnicki. *Reduction method for linear systems of plane curves with base fat points*, preprint, 2006 (arXiv: math.AG/0606716).

[ELS]   L. Ein, R. Lazarsfeld, and K.E. Smith. *Uniform behavior of symbolic powers of ideals*, Invent. Math., 144 (2001), 241–252.

[EM]   D. Eisenbud and B. Mazur. *Evolutions, symbolic squares, and Fitting ideals*, J. Reine Angew. Math., 488 (1997) 189–201.

[EV]   Hd. Esnault and E. Viehweg. *Sur une minoration du degré d'hypersurfaces s'annulant en certains points*, Math. Ann. 263 (1983), no. 1, 75–86.

[E]   L. Evain. *Computing limit linear series with infinitesimal methods*, preprint (2004), http://xxx.lanl.gov/abs/math/0407143.

[F]   M. Flach. *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math., 109(2) (1992) 307–327.

[GGP]   A. V. Geramita, A. Gimigliano and Y. Pitteloud. *Graded Betti numbers of some embedded rational n-folds*, Math. Annalen 301 (1995), 363–380. MR1314592 (96f:13022)

[GHM]   A. Geramita, B. Harbourne and J. Migliore. *Classifying Hilbert functions of fat point subschemes in* $\mathbf{P}^2$, Collect. Math. 60, 2 (2009), 159–192.

[GuH]   E. Guardo and B. Harbourne. *Resolutions of Ideals of Any Six Fat Points in* $\mathbf{P}^2$, J. Algebra 318 (2), 619–640 (2007).

[Ha1]   B. Harbourne. *An Algorithm for Fat Points on* $\mathbf{P}^2$, Can. J. Math. 52 (2000), 123–140.

[Ha2]   B. Harbourne. *The geometry of rational surfaces and Hilbert functions of points in the plane*, Canadian Mathematical Society Conference Proceedings 6 , 95–111 (1986).

[Ha3]   B. Harbourne. *Anticanonical rational surfaces*, Trans. Amer. Math. Soc. 349, 1191–1208 (1997).

[HH1]   M. Hochster and C. Huneke. *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. **147** (2002), no. 2, 349–369.

[HH2]   M. Hochster and C. Huneke. *Fine behavior of symbolic powers of ideals*, Illinois J. Math. 51 (2007), 171–183.

[Hu]   C. Huneke. *Hilbert functions and symbolic powers*, Mich. Math. J. 34 (1987), 293–318.

[K]   E. Kunz. *Characterizations of regular local rings of characteristic p*, Amer. J. Math. 91 (1969), 772–784.

[La]   R. Lazarsfeld. Positivity in Algebraic Geometry I.-II. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 48-49., Springer Verlag, Berlin, 2004.

[LT]   J. Lipman and B. Teissier. *Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals*, Mich. Math. J. 28 (1981), 97–116.

[PSC]   T. Bauer, S. Di Rocco, B. Harbourne, M. Kapustka, A. Knutsen, W. Syzdek and T. Szemberg. *A primer on Seshadri constants*, pp. 33–70, in: Interactions of Classical and Numerical Algebraic Geometry, Proceedings of a conference in honor of A. J. Sommese, held at Notre Dame, May 22–24 2008. Contemporary Mathematics vol. 496, 2009, eds. D. J. Bates, G-M. Besana, S. Di Rocco, and C. W. Wampler, 362 pp. (arXiv:0810.0728).

[R]   J. Roé. *Limit linear systems and applications*, arXiv:math/0602213, preprint, 2006.

[Sc]   P. Schenzel. *Filtrations and Noetherian symbolic blow-up rings*, Proc. Amer. Math. Soc. 102 (1988), no. 4, 817–822.

[Sk]   H. Skoda, *Estimations L^2 pour l'opérateur $\hat{\partial}$ et applications arithmétiques*, in: Séminaire P. Lelong (Analyse), 1975/76, Lecture Notes in Mathematics 578, Springer, 1977, 314–323.

[SH]   I. Swanson and C. Huneke. Integral Closure of Ideals, Rings, and Modules. London Math. Soc. Lecture Note Series**336** (2006), Cambridge University Press, pp. 421.

[TY]   S. Takagi and K. Yoshida. *Generalized test ideals and symbolic powers*, Michigan Math. J. 57 (2008), 711–724.

[TW]   R. Taylor and A. Wiles. *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2), 141(3), (1995) 553–572.

[W1]   M. Waldschmidt. *Propriétés arithmétiques de fonctions de plusieurs variables II*, Séminaire P. Lelong (Analyse), 1975–76, Lecture Notes Math. 578, Springer-Verlag, 1977, 108–135.

[W2]   M. Waldschmidt. Nombres transcendants et groupes algébriques, Astérisque 69/70, Socéte Mathématiqué de France, 1979.

[Wi]   A. Wiles. *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2), 141(3) (1995) 443–551.

BRIAN HARBOURNE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEBRASKA, LINCOLN, NE 68588-0130, USA
    E-mail address: bharbour@math.unl.edu

CRAIG HUNEKE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KS 66045-7523, USA
    E-mail address: huneke@math.ukans.edu