A Dilemma for Mathematical Constructivism

In this paper I argue that constructivism in mathematics faces a dilemma. In particular, I maintain that constructivism is unable to explain (i) the application of mathematics to nature and (ii) the intersubjectivity of mathematics unless (iii) it is conjoined with two theses that reduce it to a form of mathematical Platonism.

The paper is divided into five sections. In the first section of the paper, I explain the difference between mathematical constructivism and mathematical Platonism and I outline my argument. In the second, I argue that the best explanation of how mathematics applies to nature for a constructivist is a thesis I call Copernicanism. In the third, I argue that the best explanation of how mathematics can be intersubjective for a constructivist is a thesis I call Ideality. In the fourth, I argue that once constructivism is conjoined with these two theses, it collapses into a form of mathematical Platonism. In the fifth, I confront some objections.

1 Introduction

I take mathematical constructivism to be defined by the following thesis:

> **MC** Mathematical objects are constructed by and thus known through creative acts of the mind.

I take mathematical Platonism to be defined by the following thesis:

> **MP** Mathematical objects are mind-independent and non-physical.

MC and MP are generally contrasted with mathematical nominalism, which I take to be defined by the following thesis:

> **MN** There are no *sui generis* mathematical objects.

MC, MP, and MN are *prima facie* pairwise contraries: although they all might be false (e.g., if there are *sui generis* but physical mathematical objects), the conjunction of any two would yield a contradiction. MC and MP both entail that there are *sui generis* mathematical objects (contrary to MN) whereas MC entails that mathematical objects are mind-dependent (contrary to MP).

The difference between these three positions has important theoretical and practical implications. One is that MC and MN do not have the kind of epistemological problems that MP has.[1] MP, unlike MC and MN, seems to require that we have a special faculty of rational intuition that gives us immediate access to the realm of mathematical objects.[2]

Another is that MC, unlike MP, requires rejecting the unrestricted law of excluded middle (uLEM). The uLEM presupposes that the domain of quantification exists independently of construction. But rejecting it requires retooling many theorems from classical mathematics: some are demonstrably false in MC whereas others are true but significantly more difficult to prove (and the status of others is simply unknown).[3]

However, in this paper I am going to argue that, in light of the problem of application and the problem of intersubjectivity, MC reduces to MP. Here are the three main premises of my argument:

1. The best solution to the problem of application for MC is the thesis of Copernicanism.

2. The best solution to the problem of intersubjectivity for MC is the thesis of Ideality.

3. The conjunction of MC, Copernicanism, and Ideality is a form of MP.

[1]The classic exposition of this problem for MP is (Benacerraf, 1965).

[2]However, MC does presuppose some special faculty of the mind that is capable of engaging in synthetic, constructive procedures, and some assert that this is simply substituting one mysterious faculty for another. I am sympathetic to such worries, but I shelve them for now: they will become more pressing in section 4.

[3]This can be illustrated by means of a well-known example. Suppose you want to demonstrate that there are two irrational numbers, a and b, such that a^b is rational. You might begin by observing that (PREMISE) either $\sqrt{2}^{\sqrt{2}}$ is rational or it is not. If it is rational, then let $a = b = \sqrt{2}$ and you are done. If it is not, then let $a = \sqrt{2}^{\sqrt{2}}$ and let $b = \sqrt{2}$ and you are done. Because these two cases are exhaustive, you might take the proof to be complete even if we do not know which case is correct. But constructivists would not accept it. The problem is that the most obvious way to prove PREMISE is by appeal to the principle that all real numbers are rational or not (and the claim that $\sqrt{2}^{\sqrt{2}}$ is a real number). But the reals are an infinite set, so if this is how one went about proving PREMISE, it would involve an appeal to the uLEM.

Two things are worth pointing out here. One is that if it could be proved that $\sqrt{2}^{\sqrt{2}}$ belongs to a finite set (not necessarily of cardinality 1) each of whose members is either rational or irrational, this would suffice for MC (*pace* Bridges and Palmgren who assert that MC requires us to "decide whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational" (Bridges and Palmgren, 2018, section 1)). The other is that the Gelfond-Schneider theorem, proved in the first half of the 20^{th} century, shows that $\sqrt{2}^{\sqrt{2}}$ is transcendental and, thus, irrational.

The reduction of MC to MP should be surprising, not only because in discussions of the philosophy of mathematics MC is generally taken to be an attractive alternative to MP and MN, but also because, as I myself defined them above (and as I took care to emphasize), these positions are contraries. What is going on?

The answer to this question, and the key step of my argument, will come in section four of this paper, when I defend premise 3. I maintain that the conjunction of Copernicanism and Ideality requires that the mind-dependence in MC be understood in a way that entails the existence of mind-independent but non-physical mathematical objects as in MP. That is, I argue that a proponent of MC can retain their position only by equivocation on 'mind'.

2 The problem of application

In this section I am going to defend premise 1 of my argument above: the best solution to the problem of application for MC is the thesis of Copernicanism. The problem of application is this:

> **PoA** How can mathematics be applied physical objects?[4]

That mathematics is applied to nature is undeniable.[5] For example, consider the explanation of the appearance of certain kinds of cicadas in prime number periods or of the hexagonal structure of honeycomb. The former appeals to the fact that prime number periods will have a lower incidence of recurrence in years with periodic predators and competitors. The latter appeals to the fact that hexagonal structures make a more efficient use of resources than other shapes. Both explanations appeal to ideas about evolution and various mathematical truths, presupposing a correspondence between mathematical and physical objects.[6] But if mathematical objects are constructions in the mind, it is entirely unclear how or why physical objects in the world would correspond to them.

Now the PoA is partly what motivated Kant, who is often taken to be the father of MC, to develop his theory of transcendental idealism in the *Critique of Pure Reason*. Transcendental

[4]The PoA also is raised for MP. However, the PoA is not a problem for MN. For example, on Mill's famous account, addition is merely the abstract function of gathering objects together. MN, by way of contrast with both MC and MP, faces the problem of universality (PoU): if mathematical truths are generalized from experience, how can they have strict (rather than merely inductive) universality? The PoU also brings with it another problem, one related to the PoA, the problem of projectability (PoP): how can the projection of mathematics onto previously unobserved objects be explained? However, such issues are beyond the scope of the present investigation.

[5]Steiner recently has argued that the application of mathematics in modern physics is especially problematic. His argument is based on the fact that modern physics frequently moves immediately from mathematical possibility to physical reality, presupposing what Steiner calls a Pythagorean (and I would call a Platonic) faith in mathematical formalism (Steiner, 1998).

[6]Mancosu, 1998, section 1.

idealism says that space, time, and spatiotemporal objects are empirically real but transcendentally ideal. That is, they are real in relation to us and our experience, but considered as things in themselves (not in relation to us), they are nothing. According to Kant, the only way to explain the application of geometry and arithmetic to nature is to take space and time to be constructions of the mind and to take the laws of geometry and arithmetic to be the laws that govern the mind's constructive procedures; spatiotemporal physical objects are then also constructed in accordance with these laws.[7] Kant then had further arguments to show that space, time, and spatiotemporal physical objects are *merely* constructions of the mind and do not have any independent existence.[8]

That is, Kant solved the PoA by appeal to a thesis I shall call Copernicanism:

Copernicanism Physical objects are constructed by creative acts of the mind.[9]

Note that Copernicanism does not say that physical objects are entirely constructed by creative acts of the mind. Thus, Copernicanism does not run the risk of erasing the boundary between imagination and reality. Although Copernicanism is consistent with an idealism which says that nature is entirely mind-dependent, it also is consistent with a weaker idea, the idea that sense data is processed by the mind in accordance with various rules in order to generate the world we experience. These rules include mathematics as we know it, and that is why mathematics may be applied to experience.

To show that Copernicanism is the best solution to the PoA for MC, I want to consider two other possible solutions:

Pre-established Harmony A benevolent God established a fit between the mental rules of mathematics and nature.

Evolved Harmony The fit between the mental rules of mathematics and nature is fitness-enhancing or is a byproduct of some fitness-enhancing trait.

[7] See (Friedman, 1992).

[8] In brief, Kant argued that (1) because mathematical truths are (a) universal, (b) necessary, and (c) non-analytic, they must be conditions of the possibility of experience; (2) conditions of the possibility of experience must be understood as laws that govern the functioning of the mind; therefore (3) mathematical truths must be understood as laws that govern the functioning of the mind. He then further argued that (4) because of (a), (b), and (c), mathematical truths cannot apply to mind-independent things as they are in themselves; (5) if mathematical truths cannot apply to mind-independent things as they are in themselves, then space and time are *only* mental constructs (i.e., they cannot be both mental constructs *and* things in themselves); therefore (6) space and time are only mental constructs. See (Guyer, 1987).

[9] The name 'Copernicanism' is taken from Kant's characterization of his idea that the only way to explain our knowledge of objects of experience is by means of a "Copernicanism revolution" in which we try out the hypothesis that objects conform to the mind (rather than the other way around) and see how far we get.

I think there are at least two problems with both of these solutions. First, neither one offers a genuine solution to the PoA. Pre-established Harmony and Evolved Harmony merely point in the direction of solutions to the PoA. To fill out Pre-established Harmony, more would need to be said about why a fit between the mental rules of mathematics and nature would be benevolent, especially given the fact that our mathematical aptitude so frequently has been used to generate weapons of mass destruction. Similarly, to fill out Evolved Harmony, more would need to be said about why such a fit would be fitness-enhancing (or a byproduct of something fitness-enhancing), especially in the wake of the many and conspicuous failures to answer Platinga's evolutionary argument against naturalism.[10]

Copernicanism might require explanation itself, and that explanation might be theological or naturalistic. Copernicanism also might require further discussion of how these constructive procedures take place, and that explanation also might be theological or naturalistic. But unlike both Harmony solutions, Copernicanism provides an immediate explanation of how to solve the PoA, and that, I think, is a decisive point in its favor.

The other problem with the Harmony solutions is that both of them threaten to collapse into MP directly. That is, if there is a harmony between constructed mathematical objects and objects in the world, then mathematical truths are true independently of any construction procedure. Because the rules that make them so are non-physical on any account (MN goes hand in hand with fictionalism), it is hard to see how the Harmony solutions can avoid the conclusion that there are two kinds of mathematical objects and two kinds of mathematical truths, constructed ones and Platonistic ones. I shall return to this point in section 5 of this paper. But for now the point is that, although I shall argue in section 4 of this paper that the conjunction of Copernicanism, Ideality, and MC reduces to MP, if Ideality is not included in this conjunction, the reduction does not hold: Copernicanism and MC does not reduce to MP.

This concludes my argument that Copernicanism is the best solution to the PoA for MC.

3 The Problem of Intersubjectivity

In this section I am going to defend premise 2 of my argument above: the best solution to the problem of intersubjectivity for MC is the thesis of Ideality. The problem of intersubjectivity is this:

PoI How can mathematics be intersubjective?

[10] Bielby, 2002.

As with the PoA, there is no question but that mathematics is intersubjective. But if mathematical objects are constructed by *individual* minds, then mathematical truths will be relative to the individual. If any individual has not gone through the corresponding construction procedure him/herself, the mathematical objects will not exist for him/her and any statements about them will be non-referring (and, thus, will not be true for him/her).

For example, a child who is determined to give his/her playmates time to hide thereby might make it true *for him/her* that 201 comes after 200 even though it is not true (yet) for his/her less patient peers. But this cannot be right. The constructivist furtively scribbling down sums of very large numbers does not thereby make various propositions true for him/herself and him/herself alone.[11] So MC requires some solution to the PoI.

Brouwer, who formalized the MC program in the 20^{th} century and is probably more responsible than anyone else for the recognition of MC as a genuine alternative to MP and MC, grappled with the PoI. Most discussions of MC take him to have solved the PoI by appeal to the thesis of Ideality:

> **Ideality** The mind referred to in MC is an ideal mind, not the mind of any particular individual.

To show that Ideality is the best solution to the PoI for MC, I am going to consider three other possible solutions:

> **B1** The mind referred to in MC is the mind of God.
>
> **B2** The creative acts referred to in MC are counterfactual.
>
> **T** Mathematical truths are tensed (in accordance with the creative acts referred to in MC).

I am going to attack these in order.

The B1 solution is based on the work of Berkeley, widely considered the father of idealism.

Berkeley struggled to explain the persistence of objects in the external world in light of his thesis that the external world is not material but ideal, constituted by a series of perceptions. The problem Berkeley faced was that tables and chairs do not seem to go out of existence when we stop perceiving them. One solution Berkeley explored was that, even when the ideas that constitute the external world are not in any particular individual's mind, they are in the mind of God. Thus, Berkeley argues that a table does not cease to exist when I close my eyes because God continues to see it.

[11] The Pythagoreans guarded epistemic access to a truth about $\sqrt{2}$, but I doubt that any of them would have conflated Kp with p in the way suggested in the sentence to which this note is appended.

Along the same lines, a proponent of MC might solve the PoI by maintaining that the creative acts (constructions) that generate mathematical objects are carried out by the mind of God. You or I might carry out similar creative acts when learning a geometric proof or something. But *our* creative acts are not the ones that create the objects that really matter any more than my particular viewing of a movie is constitutive of the movie itself (had I not seen the movie, it would have existed all the same).

The problem with B1 is that it reduces immediately to MP: the mind of God is non-physical, and thus the objects of mathematics will be non-physical, *sui generis*, and independent of any individual's mind. Note that this does not entail that B1 is false: the problem I am raising is not intended to be a problem for B1 in itself. Rather, the problem is that B1 strips MC of its distinctive content and so it is a problem for the conjunction of B1 and MC. Perhaps the easiest way to see this is that problems distinctive to MP, like the epistemic problem gestured to in section 1 of this paper, now arise for MC.

The B2 solution is also based on the work of Berkeley. In addition to appealing to the mind of God to explain object permanence, Berkeley appealed to coutnerfactuals: he argued that to say that an object persists when I am not perceiving it is to say that *if* I were to perceive it, *then* it would have properties X, Y, and Z (i.e., I would have an idea of such and such a form).[12] Along the same lines, some might maintain that the creative acts in MC are counterfactual and, thus, that 201 comes after 200 for all of the hide-and-seek players above because all of them *could* engage in the relevant constructive act even if they actually have not.

The problem with B2 is that it is incomplete: it is unclear which mind is doing these counterfactual constructions. If it is an individual's mind, as suggested in the previous paragraph, then the PoI arises once again because different individuals have different levels of mathematical acuity. So B2 requires supplementation, and this supplementation is likely to come from Ideality.

The T solution is inspired from recent revisionary interpretations of Brouwer.[13] To make sense of T, consider Andrew Wiles' proof of Fermat's last theorem. Suppose that this proof were in accordance with MC (and the rejection of the uLEM, as noted in section 1). According to T, Fermat's last theorem, $\forall a\,\forall b\,\forall c\,\forall n\,\Big((a,b,c>1\wedge n>2)\rightarrow a^n+b^n\neq c^n\Big)$, would be false for anybody unlucky enough to have died before 1995 but true for those who lived to see the discovery of a constructive procedure for the proof of the theorem.

However, T, like B2, is incomplete. Again it is unclear which mind is doing the constructions.

[12]There are difficult textual debates about whether the God solution or the counterfactual solution is logically prior on Berkeley's account. I cannot pursue such debates here.

[13]Niekus, 2010.

That is, it is unclear which mind serves as index for the tenses. If, for example, it is an individual's mind, then the absurd consequences of the hide-and-seek example stand. Thus, T requires supplementation in order to solve the PoI, and again this supplementation is likely to come from Ideality.

To put the point another way, the problem with B2 and T is that, although they might seem to offer a solution to the PoI (in the way that the Harmony theses seemed to offer a solution to the PoA), this is only because they enable us to smuggle in a thesis like Ideality without being explicit about our assumptions. Openly confronting the PoI makes clear that these are not genuine alternatives to Ideality.

Now the thesis of Ideality raises some difficult questions of its own. For example, one might wonder whether there is one ideal mind for all rational beings in the universe or whether there is a plurality of ideal minds. Perhaps the members of an Amazonian tribe as yet uncontacted by the "outside world" would not work off the same ideal mind as you and I. After all, what if we are the interstellar analog of this tribe, as yet uncontacted by the galactic empire of aliens much more mathematically sophisticated than we?[14] Added layers of complication arise when it is the constructive proof (rather than the individual who discovers it) that is isolated: if Fermat really did discover some simple and elegant proof of his theorem but, Kafka-esque, had it consigned to the flames posthumously rather than share it—what then?

Answering these questions is only the beginning: in addition to delimiting the boundaries of our constructivist ideal mind(s), we need to figure out how this mind maintains identity through time. For example, can there be fusion and fission events when communities intermingle? However, I am not able to engage these complications here. For now I turn to the third premise of my main line of argument.

4 MC $\wedge$ Copernicanism $\wedge$ Ideality $\Rightarrow$ MP

The conjunction of questions about application and intersubjectivity is no accident: together, these are often taken to be individually necessary and jointly sufficient for objectivity. And mathematics, like logic, is nothing if not objective. But I now want to argue that the solution of the PoA I advocated in section 2, when conjoined with the solution of the PoI I advocated in section 3, reduces MC to a form of MP.

[14] These kinds of questions are familiar: they come up in metaethics in discussions of cultural relativism, and they also come up in social epistemology in discussions of communities of knowers and their respective epistemic authorities.

From Ideality, we get the conclusion that different members of a given mathematical community will rest the truth of their mathematical propositions on numerically identical objects constructed in the ledger of their shared ideal mind. This is similar to MP (and different from MN) in having *sui generis*, unique, and non-physical mathematical objects. Nonetheless it involves a subtle shift in the notion of mind dependence appealed to in MC, for the mind in question has no real existence (that is the point of calling it ideal).

From Copernicanism we get the conclusion that nature and the physical objects that comprise it are mind-dependent in the same way that mathematical objects are. But calling something a mental construct is to appeal to a mind/nature dualism. So although it might be true that physical objects are mental constructs, the thesis of Copernicanism also involves a subtle shift in the conventional notion of mind dependence appealed to in MC.

These two shifts in the notion of mind dependence are jointly but not individually sufficient to reduce MC to a form of MP.

To see that they are not individually sufficient, note that (i) the proponent of the conjunction of Ideality and MC might maintain mind dependence by pointing out that dependence on an ideal mind is nonetheless mind dependence, not something instantiated in nature, and (ii) the proponent of the conjunction of Copernicanism and MC might maintain mind/nature dualism by pointing out that the ontology of mathematical objects is different from that of physical objects because mathematical objects, unlike physical objects, do not involve sense data and thus are entirely mind dependent (i.e., physical objects are the result of mental activity acting on sense data; mathematical objects are the result of pure mental activity).

To see that they are jointly sufficient to reduce MC to a form of MP, note that (i) and (ii) are no longer available once the two independent shifts in mind dependence are added together. This is because, even if mathematical objects are entirely ideal mind dependent whereas physical objects are only partly so, the constructive procedures themselves, the procedures that are taken to encapsulate the ideal mind, cannot themselves be ideal mind dependent. They cannot be so for they are constitutive of the ideal mind. It follows that these rules are taken to be ontologically on a par with the physical data itself (or whatever produces this physical data) that distinguishes mind from nature.

MC seems to offer a novel solution to traditional problems in the philosophy of mathematics. But I maintain that, once these novel solutions are put together, MC collapses under critical scrutiny into a form of MP.

5 Some Objections

I want to conclude this paper by considering two objections to my argument.

First, some might object that my argument is too far reaching. Constructivism appears in many other areas of philosophy. For example, Rawls advocated a constructivist theory of value, tracing its provenance (like that of MC) to Kant. But if my argument works against MC, then it seems that it should work against other forms of constructivism too. But surely it cannot be the case that all of these forms of constructivism reduce to forms of Platonism. This suggests that something in my argument has gone awry.

I would like to say two things in response to this objection. One is that if my argument does apply to other forms of constructivism, not merely to MC, that is not, I think, in itself a problem. At most it might motivate someone to *look* for a problem with my argument. But if my argument applies to these other forms of constructivism and there is no independent reason to doubt my argument, then that, I think, is simply so much the worse for these other forms of constructivism.

The other thing I would like to say in response to this objection is that I do not think that my argument does apply to other forms of constructivism. Constructivism about values, for example, has no need of Copernicanism: moral truths are not taken to apply to or be instantiated in nature in the same way that mathematical truths are, so it is open to the metaethical constructivist to maintain an unadulterated mind dependent intersubjectivity when it comes to ethical truths. Thus, if constructivism in other domains reduces to Platonism, it will not be on account of anything said here.

The second objection I want to consider has to do with the distinction between pure and applied mathematics. Some proponents of MC might argue that MC is about pure mathematics. But they might maintain that applied mathematics is entirely different from pure mathematics. The difference between pure and applied mathematics can be seen even with basic truths of mathematics: in pure mathematics "1+1=2," but in applied mathematics this is always an open question (if two molecules are added together, they might form a bond, in which case "1+1=1"). And if this account of the distinction between pure and applied mathematics is accepted, then MC has no need of Copernicanism. So even if it is true that the conjunction of Copernicanism, Ideality, and MC reduces to MP, it is irrelevant.

The problem with this is that whether this is the correct account of the way mathematics is applied to nature is an empirical question and, importantly, recent work in the philosophy of mathematics suggests that natural scientists would reject it: their practice suggests that applied mathematics tends toward MP rather than MN.[15] To put the point more forcefully: many mathematicians already object to MC on the grounds that it is too revisionary and requires the rejection of too much of classical mathematics; if MC also requires the rejection of much of natural science, the view will become, for most, entirely unsustainable.[16]

6 Bibliography

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Steiner, Mark. 1998. *The Application of Mathematics as a Philosophical Problem.* Harvard University Press.

[15] Steiner, 1998).

[16] Bangu, 2012. Further problems arise, I think, if we begin to explore some of the issues highlighted at the end of section 2 above, the nature of the mind, or even the nature of logical truths and logical rules. However, I cannot pursue these issues here.