The ontology of number

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ABSTRACT

What is a number? Answering this will answer questions about its philosophical foundations - rational numbers, the complex numbers, imaginary numbers. If we are to write or talk about something, it is helpful to know whether it exists, how it exists, and why it exists, just from a common-sense point of view [Quine, 1948, p. 6]. Generally, there does not seem to be any disagreement among mathematicians, scientists, and logicians about numbers existing in some way, but currently, in the mainstream arena only definitions, descriptions of properties, and effects are presented as evidence. Enough historical description of numbers in history provides an empirical basis of number, although a case can be made that numbers do not exist by themselves empirically. Correspondingly, numbers exist as abstractions. All the while, though, these "descriptions" beg the question of what numbers are ontologically. Advocates for numbers being the ultimate reality have the problem of wrestling with the nature of reality. I start on the road to discovering the ontology of number by looking at where people have talked about numbers as already existing: history. Of course, we need to know not only what ontology is but the problems of identifying one, leading to the selection between metaphysics and provisional approaches. While we seem to be dimensionally limited, at least we can identify a more suitable bootstrapping ontology than mere definitions, leading us to the unity of opposites. The rest of the paper details how this is done and modifies Peano's Postulates.

BACKGROUND

History of number

I start the history of number with the 37,000 year-old Lebombo bone [2020] with 29 notches found in Swaziland's Lebombo mountains that has gained currency in the mathematical and scientific world as evidence of counting. Following this is the 20-35,000 year-old Ishango stick [2020] from Central Africa with its parallel and grouped tally marks archaeologists have identified as placeholders.

Ancient peoples observed and placed objects in groups, abstracting by applying "... one and the same word … to any aggregate of objects ... [Kramer, 1981, p. 5]". Repeated and consistent application is the function of unit. Speculation can abound on how early peoples came by number systems, but they all depend upon the ability to group things, "sets" [Ibid., p. 4] in modern parlance. Groupings, or placeholders for numbers, might be found in how many bowls of grain might be stored in a building and how many buildings of grain would be needed to feed a village for a winter. It is the ability to group that establishes counting. In other words, practically, one cannot have endless counting without an idea of a base (2, 10, 12, 60, etc.) on which to establish numeracy. Our ten-place counting system can be attributed to the Chinese Shang Dynasty in the14th century BCE [Temple, 1986]. In 499 CE during the Indian Gupta Period, Aryabhata in his now-famous *Aryabhatiya* used the 10-place/decimal system [Chakrabarti, 1996, p. 203;]. How the system came about in India, at least originated from "...from an age of mystic intuition in remote antiquity" [Dutta, 2015, p. 2]. From this point on, quantities took on the symbols we recognize today as the numerals 1, 2, 3, etc.

Elsewhere in history, we read of base 10, 12, and even 60, dating back thousands of years [Kramer, pp. 8-12]. An interesting discussion emerges about how thousands of years ago tokens indicating quantities formed the basis of number systems [Schmandt-Besserat, 1982]. The Mayans around 36 b.c.e used a vigesimal (base-20) counting system with horizontal bars and dots above them, each bar as a placeholder for five units [Maya numerals, 2020].

We will see later modern research showing that numeracy and grouping are hardwired into our brains (and those of other animals). More will be said later ("The display of numbers") on places in number systems.

What is said specifically about number

At the core of the debate about the existence of numbers is whether they are mere ideas (abstractions) or exist because of physical entities (empirically-based). Plato in his *Republic* said numbers are independent of concrete things: "... the odd and the even … one can see in no other way than with thought" [Plato, 1968, 510c-e]. Aristotle [1924] in his *Metaphysics* said not that numbers don't exist but by "... the mode of their having being" [1076a] … the objects of mathematics are not substances to a greater degree than bodies nor prior in being to perceptible things" [Ibid., 1077b].

I present now some representatives of this divergence of abstract from empirical just to give a flavor of the issues involved.

Max Tegmark argues "a sufficiently broad definition of mathematics" [Tegmark, 2007, p. 1] makes the universe "a mathematical structure ... an abstract, immutable entity existing outside of space and time" [Ibid., p. 3]. He bootstraps, i.e., "... we will assume that the ERH [external reality hypothesis] is correct ..." [Ibid., p. 1].

Euclid [2008, p. 194] gives definitions:

> 1. A unit is (that) according to which each existing (thing) is said (to be) one. 2. And a number (is) a multitude composed of units.

Leibniz says, "The concept of unity is abstracted from the concept of one being, and the whole itself, abstracted from unities, or the totality, is called number. Quantity is therefore the number of parts" (Russell, 1958, L 76/GP IV 35 - The Art of Combination].

Frege [1953, p. 15] says "... numbers...are not in space or time." They are spacetime independent [Ibid., p. 13]. "... psychology should not imagine that it can contribute anything whatever to the foundation of arithmetic" [Ibid., p. viiii]. Then, "Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one" [Ibid., p.99].

If numbers were empirically based, then what of zero [Ibid., p. 11]? Also, a problem exists with application, as in 5+1=6 not yielding the same when applied to various situations [Ibid., p. 13], such as adding numbers of events or volumes of liquid not being physically the same.

Frege seeks to "… complete the Leibnizian definitions of the individual numbers ...." [Ibid., 67], not draw any metaphysical conclusions. For him, "Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one." and by definitions [Ibid., p. 91, et seq.].

Dedekind [2007] is not concerned with ontology, because for him, "I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought." [Ibid., p. 14], that "The development of the arithmetic of rational numbers is here presupposed … we are led to relate things to things and thus to use that faculty of the mind on which the creation of numbers depends; … numbers are free creations of the human mind;" [Ibid., p. 15- Preface]. Indeed, the first part of his Essays sets forth "Properties of rational numbers" [Ibid., p. 2]. Indeed, Dedekind "... takes the informal notion of arbitrary set of natural numbers for granted, and in those terms the axioms are categorical and hence complete" [Feferman, p. 3].

Peano said,

> ... there is no need for the teacher to give any definition of number, … Therefore, number cannot be defined, since it is evident that however these words are combined among themselves, we can never have an expression equivalent to number. If number cannot be defined, however, we can still state those properties from which the many other well known properties of the numbers are derived. … The concepts, then, that we do not define are those of a number a, of number N, of one 1, and of successor

which we indicate for the moment by a+. ... Here, number is not defined, but its principal properties are stated. Instead, Dedekind defines number as precisely that which satisfies the preceding conditions. [Kennedy, 1974, p. 406.]

A standard rendition of his postulates is:

1. 0 is a number.
2. If *a* is a number, the successor of *a* is a number.
3. If two numbers have the same successor, the two numbers are identical.
4. O is not the successor of any number.
5. If *s* be a class to which belongs 0 and also the successor of every number belonging to *s*, then every number belongs to *s*.

The last of these propositions is the principle of mathematical induction. [Russell, 1938, p. 125]

What is lacking in Peano's Postulates are concepts of magnitude, what can be enumerated (existence postulate), a concept of unit, and a theorem that maps the concept of magnitude to units. Peano cared nothing about existence [Bidiou, §5.5, p. 48].

For Russell [1919, p. 7, 10], Peano's use of "number" is ambiguous:

> ... Peano's three primitive ideas namely, "0," number," and " successor" - are capable of an infinite number of different interpretations, all of which will satisfy the five primitive propositions.[7] … Our numbers should have a definite meaning, not merely that they should have certain formal properties. [10]

However, Russell falls into the same trap as Peano did and for which the former criticizes the latter: "Number is what is characteristic of numbers, as man is characteristic of men."; we know – like Dedekind – what a number is by its property.

In fact, it can be argued that Peano never really exposits the concept of number because he said it was indefinable [Segre, 1994, p. 287].

Observations of the above

We need to focus (but hold the thought for later) on the above saying that from logic (the "laws of thought" – after George Boole) comes mathematics. The common theme is that mathematics stems from logic. It will be demonstrated later than both math and logic emanate from something more fundamental – number. If we not only establish the existence of number but understand truly what it is, we then can see why they produce logic and math.

All of the above are representatives of set theorists (Frege being a nascent set theorist), along with many more, such as Cantor, Zermelo/Fraenkel, and von Neumann, but set theory, in general (formal set theory, in particular), is predicated upon definitions (the axiomatic versions being formalizations).

As if to summarize the problems of identifying what numbers really are, Benacerraf aptly addresses both the issue of Platonism and numbers being sets. In an argument against Platonism, that there is no "THE number", Benacerraf demonstrates that different methods can reduce natural numbers to different sets. What, then, is the "meta-set" that equates the two methods? "Benacerraf concludes that they [numbers], too, are not sets at all" [Horsten, 2019].

Somewhat of a sidebar discussion is that of Benacerraf's identification problem [2029] that arithmetic just describes structural relations (as in the structure of the natural numbers), not objects. We know that structures can be objects and vice versa, depending upon level, more familiar to us as object – meta-object. For example, there is the word actually used and then our talking about the word (spelling, etymology, meaning, etc.). There are objects and sets of them, each set, in turn becoming an object at a higher (meta) level. Yet, at the very core is the question about what the most fundamental element is. If a collection of objects is set, what about the objects within it? It is Russell's "set of all sets" in the opposite direction, and it is not clear if a type theory or empty sets are any resolution. Later, we will encounter the problem in identifying the smallest of the small. Structures also can be processes, as the discussion about binary logical space below will show.

Skirting the discussion by describing what are thought to be effects, properties, or by positing definitions fail to locate essence [Kramer, 1982; Bidiou, 2008]. The idea of numbers and counting, as will be explained documented later, are so fundamental as to enable instinctual behavior. For example, whether to avoid a situation can depend upon the number of nearby animals as potential predators. Research indicates that mammals can count [Rumbaugh et al., 1987; Range et al, 2014; Vonk, J. and Beran, M.J., 2012] and even bees (up to four) [Gross et al., 2009]. Such says nothing about how critical numeracy is to science and resulting technology.

In summary, numbers are said to exist in three different ways:

- realists - numbers as abstractions actually exist coequally with physical objects;
- conceptualists - numbers are universals (things that exist and can be given as examples, or instantiated) and abstractions still are only in our minds;
- nominalists - denying universals (as in numbers, their relations, or structures) or claiming they are not abstractions [Barker -Philosophy of Mathematics, p. 68 et seq.].

It is not as easy as one might think to say why numbers exist [Russell, loc. cit. Ibid.]. Too, it is by no means clear what "abstraction", "reality", or even "empiricism" mean. This is why we cannot garner much substance from views like Tegmark's, only Platonism.

All these arguments throughout the ages about what a number is raise core philosophical problems like (and among others):

- discrete (integers) and irrationals (continuum);
- number being grounded on physical entities (as in zero not referring to anything but, nevertheless, still being a number);
- the nature of unity;
- sets and individuals (an individual can be a set and vice versa) and classes skirting the paradox of the set of all sets).

We are left, then, with widely divergent views of numbers being the ultimate reality through abstractions, and their being based on definitions, none of these seeming to bring us any closer to a systematic description of an ontology, let alone how to identify one.

The way we discuss the foundation of numbers beyond mere definitions needs to change if we are going to establish their ontology. This paper will say what a number is, how we know, and why we know. First, we have to confront the problems in establishing what exists and how we know, i.e., ontology and epistemology. The answers here will explain the rationale for the approach I take.

ABOUT ONTOLOGY AND EPISTEMOLOGY

At the risk of being tutorial, I still think it necessary to disclose a foundation in confronting those who slough off philosophy and prefer to concentrate on the "practical" or applicative aspect of numbers. Too, I feel I need to address those who are satisfied with merely a definitions-based ontology. I recall one professor answering a student's question about "why" by "don't worry about that now, just learn the concept, and it will come to you later."

As a logician, I sympathize with the frustration of not being able to "know" ultimate reality and wanting to defer merely to properties, definitions, axioms, and so forth, being content that all of mathematics is our creation. That there is so much turmoil, contradiction, and lack of understanding in this universe, all the while it finding ability for its continued existence, lends sympathy to Platonism. Too, Platonism does seem to describe conceptually statistics (the "shadows" being samples of something we never will be able to know completely) and even Kant's appearance (instances) and reality (complete "life" of something). Yet, just bear with me, as I propose to describe a richer basis on which we can gain acceptance of the existence of numbers. Simplistically stated, there are two approaches to ontology – metaphysical (external – "absolute" reality) and conditional (internal – we determine what exists).

Metaphysical ontology

The forms

The Stanford Encyclopedia of Philosophy says, "As a first approximation, ontology is the study of what there is" [Hofweber, 2018]. Often, there are excursions into "meaning", "reality", there being a god, and so forth. Those like Tegmark avow an external reality of mathematics, a counterpoint being the "Copenhagen interpretation" whereby reality is created by the observer. Modern literature even suggests the possibility of our being a simulation (Bostrom, 2011; Canuretto, 2007), but leaving for speculation even the very existence (and why) of a programmer creating us. Plato's [1968] classic allegory of the cave in his Book 7 of the *Republic* articulates the dilemma. There is the reality of the "forms", something we may never see; we observe analogously only shadows of those forms. For example, the form is of a table, but we know only of instances of the table. In logical parlance, we know only with a degree of probability what a table really is by sampling the tables around us, but since it is impossible to sample all the tables that have come before and will in the future, we never can see all the tables to know what the form is. How, though, do we know without question if such is the case? Just as problematical, how can this be known exactly in the same way by each of us? We see things through our own eyes. Other problems about existence seem to defy resolution, such as the following.

Superposition

How could instances and the whole of something exist at the same time? Quantum physicists refer to "superposition", where two different states of something exist simultaneously, illustrated by the Schroedinger thought experiment of a cat being both alive and dead at the same time [Schroedinger]. A similar view is by Kant (1787/1929) in his discussion of appearance and reality in his *Critique of Pure Reason*:

> All our representations are, it is true, referred by the understanding to some object; and since appearances are nothing but representations, the understanding refers them to a something, as the object of sensible intuition. But this something, thus conceived, is only the transcendental object; and by that is meant a something = X, of which we know, and with the present constitution of our understanding can know, nothing whatsoever, but which, as a correlate of the unity of apperception, can serve only for the unity of the manifold in sensible intuition. By means of this unity the understanding combines the manifold into the concept of an object. This transcendental object cannot be separated from the sense data, for nothing is then left through which it might be thought. Consequently it is not in itself an object of knowledge, but only the representation of appearances under the concept of an object in general a concept which is determinable through the manifold of these appearances" (Ibid., A250, A 251 p. 268)

We experience an object in one moment after another (appearing, or appearance), but the reality is that the object persists, manifesting itself through these appearances.

> When, therefore, we say that the senses represent objects as they appear, and the understanding objects as they are, the latter statement is to be taken, not in the transcendental, but in the merely empirical meaning of the terms, namely as meaning that the objects must be represented as objects of experience, that is, as appearances in thoroughgoing interconnection with one another, and not as they may be apart from their relation to possible experience (and consequently to any senses), as objects of the pure understanding (Ibid., A258 p. 274).

The instance (appearance) exists because of the totality of instances (reality), but we are no closer to understanding why we cannot perceive each by itself.

Material and immaterial

Even broader in scope than differentiating abstraction from the concrete is the tension between material and immaterial, brain and mind, tangible and intangible, and so forth, all stemming from the centuries-old debate over the mind-body distinction, Rene Descartes coming to mind. Using his method, " ... to divide each of the difficulties under examination into as many parts as possible, and as might be necessary for its adequate solution. "... by showing that we cannot conceive body unless as divisible" (Descartes, 1641, p. 122), we arrive at Planck scale (and even smaller), where entities (called by physicists "particles") appear and disappear (Hawking). Beyond the observable is what we take to be the abstraction occurring in our brains and what I generally call "mentation". We seem to know what these are by the effects, not by what they may be, just like some people say about numbers.

I pose a "replacement exercise", where ongoing efforts are to create a human. It starts with mechanical computation – the abacus, adding machines, computers, and so forth, all designed replicate human thinking, robotics, androids, transhumanism, and, finally, the artificial human, which, incidentally, does not have to be hydrocarbon-based. Now, it is supercomputers and quantum computers that are seeking to replicate human thinking. Start replacing human body parts (transhumanism) and neurons (as in neurons on chips), and ultimately we arrive at a "crossover point" that reputedly gives rise to consciousness, thinking, and all that. In the background are the Turing tests (or their analogs) that are to tell us that what is created is indiscernible from the "natural" human. The problem is the physical aspects required to support the mental. Presented another way, let us say we can replace atom for atom in the human with a non-hydrocarbon substance. Would that structure be capable of supporting "consciousness"? Here, the "mind" meets brain, the "immaterial" the "material".

It is beyond the scope to identify the criteria for separating the material from the immaterial in the macro world [Rosen, 2018], but much can be said about their converging and divergence. The former occurs in repeated Cartesian subdivision, a subdivision that is computationally infinite and resulting in a fuzzy boundary that may disappear altogether and not perceivable in our dimension. The immaterial blends with the material. Rightfully, these two words should be in quotes, as it by no means is established what either of these is, just because of the boundary problem, which I will get to in a moment. The reason why extends back to when the Universe was formed when everything was bound up with the singularity. Speculation about the nature of it abounds, as in it being dimensionless, a geometric point, but incorporating all that we have now, including that "immaterial", ideas, or however one may call it. Conversations quickly turn into debates about whether there is something called creation and the existence of something – or in terms of – nothing. Our knowledge of what there actually is seems to be dimensionally limited, scientists considering that there is no such thing as creation, as Hawking [2017] argued or time "is redundant" [Barbour, 2008] time, the latter as a "timeless law that governs change" [Ibid.]. By now, it should be clear that we experience things with no explanation, but why?

In this mix is object and process, and in the subquantum world, these may actually merge. Aristotle was looking for the substratum. Aristotle hints,

> Pairs of opposites which fall under the category of relation are explained by reference of the one to the other, the reference being indicated by the preposition "of" or by some other preposition. Thus, 'double' is a relative term, for that which is double is explained as the 'double of something'" [Aristotle,1984, *Categories* 10 – 11b22-33; 192a25-192a34, p. 455/18].

Such is "... the underlying nature to substance, i.e. the 'this' or existent". (Ibid., 191a9-191a12, p. 453/15).

Hegel said "But we can say, too, that it has been the conviction of every age that what is substantial [substratum] is only reached through the reworking of the immediate by our thinking about it." (Hegel, 1830/2001, §22 , p.54). At the core of experience is how we see ourselves and our environment through ourselves.

Point, continuum, and boundary

Coupled with material-immaterial is what really is the smallest of the smallest. Two philosophers generate this question: Descartes and Leibniz. Leibniz stated that no two points are identical in space; each is unique. [Forest, 2016]. Descartes just gets us down to this level.

Digital physicists see spacetime as discrete, a collection of points, but against what background do these points find themselves? For Hawking, it is those "particles" flickering in and out of existence [Hawking, 2017].

Geometers for millennia were confronted with finding the area under a curve. This problem as well as Zeno's Paradox of the rabbit never catching up with the hare were resolved by the calculus, but such has never given us an answer about where to draw boundaries definitively or absolutely. Space not having continuity at Planck scale complicates the problem even more. "Space itself turns out to have a discrete and combinatorial character" [Rovelli, 2004, p. 21]. #.

Another stark example of the boundary problem underpinning the above is exemplified by our determining how to measure anything, such as a length or weight. Just how small are you to draw those demarcation lines separating one value from the next? When does 5 meters, four centimeters, two millimeters, for example, become 5 meters, four centimeters, three millimeters? It seems that we never can "really" measure anything. I see this as another type of problem addressed by calculus: WE determine the limits. The problem is very similar to the Copenhagen interpretation and Heisenberg, that we inherently determine the observation, or bias it.

Equals and equivalence

The difference between = and ≡ evidences the existence of space-time. A curious distinction arises between equals and equivalence, the former often assuming a dual-use with respect to how an entity is placed. With current practice "x = x" can mean either an entity is itself or has the same value as something else. [Feibleman, 1979]. Regardless, the equals symbol also designates the value of one entity being the same as another, as in 7 = 5+2. The value is apart from what it designates, an abstraction. In Platonic terms, 7, is the form; what it refers to is the instance (or shadow of it). This is not unlike the form of a table, the table in front of me the instance, or sample (as is the case in statistics). The "=" jumping outside of itself to connect another entity is not the same as the "=" saying something is itself. Instead, the equivalence (≡) accomplishes the former. Logicians proclaim that the equivalence signifies that the truth value of one entity is the same as another, but there is nothing to say that "truth value" cannot refer to magnitude, though we do not normally see the equivalence used this way in mathematics. We will discover later that the convergence of logic and mathematics establishes – no, even mandates – the use of equal and equivalence in their proper domains in space-time.

The unity of opposites

All of the above problem areas (and there are many more, such as motion and stasis - there being no absolute reference frame - and time, itself - what is it?) describe our being able to know something only because of what it is not, what philosophically is known as the dialectic, or, as ancient Eastern philosopher say, the "unity of opposites". Look at these pairs:

- left – right
- up – down
- induction – deduction
- yes – no
- analytic - synthetic
- … and so forth

A number of words describe the situation: contrast, difference, contradiction, opposites, and context. We also experience the effects, our thinking, minds, consciousness, or however one may term it, analogous to the effects of electromagnetic radiation, and, yes, even numbers. How or why this is the case eludes us in the same manner that consciousness, ideas, mind, ideas, and such "immaterial" things do. So, hold these thoughts, as they will be incorporated into our description of how numbers will be regarded.

The limitation of dimension

Let's bring all of what has been said to focus on ascertaining when to call something left or right, this measurement or that one, or one or other of any pair: boundary. Arguably, the "doorway" to describing our problems with understanding is Hawking's discovery of those "particles" flickering in and out of our existence, or more stridently put, our dimension. Take the "particle", itself, where repeated subdivision can occur forever (even given that something called "time" exists). If the unity of opposites is the ontology, our perceptions of phenomena as "paradoxes" turn into understanding of them by seeing that which creates them in another world, or dimension. That what we think of as a "particle" (or discrete) occurs as what it is not – a continuum. Infinite becomes infinitesimal; static becomes dynamic, life is death; and so forth. What is becomes its opposite in another dimension. Something becoming its opposite is inconceivable to us. Such is all speculation perhaps, save for the possible testability of the theory, the hallmark of scientific discovery. Mine is not the expertise of the quantum physicist but a logician calling for the mathematical physicists to explain the consequences of repeated subdivision of particles in our world and description of Hawking's findings. While conventional quantum physics suggests that something cannot be subdivided below Planck scale, recent research suggests otherwise [Oliva and Steuernagel, 2017; Ghosh et al., 2009; Kumar and Lee, 2017; Zurek, 2001; Jafarov and Jeugt, 2010]. A close reading of this evidence will bring into focus the ancient question of movement in the form of oscillation. We ask "a back and forth between what and what", the "what" raising the question of stasis. Together with the boundary problem discussed above is that often our ability to detect physical phenomena is limited "simply" by instrumentation precision. The question remains, then about limitation … if any at all. Aside from all this, sub-Planck structures seem to exist.

It may be that an answer would be similar to the answer to the question, "what is the nature of superposition?"

Ascertaining what exists, including number, may be constrained by inherent inability to apprehend in other, dimensions. Such appears to be straying far afield from establishing the ontology of number, but the following should elucidate.

In 1884 Edwin A. Abbott (1884), his *nom de plume* as “A. Square”, described in his *Flatland* novelette a two-dimensional world, where inhabitants perceived objects existing only in a plane. A sphere descending on Flatland would appear to Flatland inhabitants first as a dot on the ground (or horizon), then progressively become larger in diameter (or as a line on the horizon) as the sphere passed through the planar landscape, and lastly be a dot (or line on the horizon) again. One dimensional persons live only in a line and can see only forward or backward (if they have eyes in the backs of their heads). Abbott's lesson is that our knowledge and experience of higher dimensions is required to explain occurrences fully in the lower ones. A two-dimensional person detects falling objects by observing effects on the plane, detecting pressure on the head, or being displaced but having no way of explaining why, because of not benefiting from the third dimension. We experience all the metaphysical problems, as they emanate from a dimension "above" or containing ours.

How and why unity of opposites "works" (and other metaphysical problems) may be unexplainable because of dimensional limitation. Aside from what we perceive in four dimensions, including space-time, how do we see things? How then do we know that the problem of mentation (including “consciousness”) may not be a dimensional problem? After all, there have been many other problems throughout history that were just as problematic as mentioned above. We are being impacted by something we cannot explain, such as the Heisenberg Uncertainty Principle, various paradoxes (e.g., the set of all sets), what happens after life, wave function collapse, our comprehending how the dialectic operates to allow us the ability to apprehend anything, our inability to apprehend anything except through ourselves (problem of "objectivity"), emergence, and the problem of mentation (as in what it means to be “aware of ourselves”), to cite several examples. In all of these, one has to be in two places or mindsets at once (superposition), something we clearly cannot do.

Homo sapiens sapiens, then, is in the same situation as Abbot's two-dimensional persons. They see things in terms of what they know, unaware of what “lies behind” the phenomenon but thinking something is not right in the environment. “Hey, did you feel that? Something just hit my head, and I can't tell from where it came, because everywhere I look in my two-dimensional space, there is nothing.” Many such objects are falling in our world, such as boundary problems, life/death, and consciousness. Like Abbott's people, we know something is happening but are unable to explain how or why. My approach to understanding explains how an ontology may be provided to these things, as well as number.

Internal ontology

To summarize the five (among other metaphysical problems) reasons for opting for internal ideology are:

- the boundary problem;
- failure to understand why the unity of opposites is necessary for apprehension – as in wave-particle, material-immaterial, and so forth;
- lack of ability to identify absolute reference frame (general and special theory of relativity);
- inability to explain mentation (ideas, consciousness, thinking, etc.) –except through the effects;
- inability to identify the substratum.

Such says nothing about larger problems, like Gödel's incompleteness theorem (undecidability of consistency in any system giving rise to mathematics, including arithmetic).

Perforce, we must resort to a bootstrap ontology, but of a specialized type. My approach is similar to how calculus resolves the boundary problem – by our setting the limit, and because of the Copenhagen and Heisenberg problems, perforce, individual human bias is unavoidable, hence internal ontology.

All these arguments throughout the ages about what a number is center on

- discrete (integers) and irrationals (continuum);
- number being grounded on physical entities (as in zero not referring to anything but, nevertheless, still being a number;
- the nature of unity;
- sets and individuals (an individual can be a set and vice versa) and classes, skirting the paradox of the set of all sets.

Quantum ontology

How could it be that numbers by themselves could only be abstractions but also located in space-time? We have both the abstraction – the "immaterial" and a physical presence in a special way to give us numbers. If something does exist because of what it is not, both it and its negation existing in the same time and space, how do these opposites "work" to allow apprehension? Quantum physics allows for this with superposition, but we do not apprehend this phenomenologically. Instead of either-or, there exists a "process", the nature of which is obscured arguably because of our dimensional limitations. Hence, we revert to a Flatland method, bootstrapping.

The unity of opposites ontology

We start understanding by experience, but such need reason, or as Kant said, "the categories [pure concept of understanding – something apart from experience, or how it may exist] have meaning only in relation to the unity of intuition in space and time;" (Ibid., B308, p. 269), a key word here being "unity", as in "unity of opposites". Upon closer reading, we may see that the empirical is in the physical, or material domain, and ideas, intuition, concepts are in the immaterial or mental domain. We will return to this bipolarity later in establishing the foundation of number.

Recall the section above, "the unity of opposites". Our universe exists as sets of polarities and as one aspect of a polarity, itself – that which the Universe is not. It came into being from a single entity, the singularity, and it will dissipate in a heat death, all energy and movement ending with an equal distribution of that energy. Creation (order) exists because of destruction (entropy). From where the singularity came is unknown. Conversations quickly turn into debates about whether there is something called creation and the existence of something – or in terms of – nothing. Whatever exists and how we see it, including "consciousness" was included in that singularity will have the same fate. If there is a beginning, there is an end. The debate about existence lies in between.

In terms of our range of “vision” one starts with the infinitesimal (singularity) and extends to how large the universe may be (infinite). Our knowledge of its size is bounded by the speed of light, giving us an estimate literally of 13.8 billion light-years when the universe emerged in the form of rapid inflation from a single point, which was chaos. It very well may be that the universe is larger, only we cannot see it, hence our allowing for the possibility of infinite (never-ending) size, albeit maybe in some form other than that circumscribed by electromagnetic radiation.

What of its opposite, the infinitesimal? Too, what is it “made of”? Is it something also that may not be electromagnetically bound? Similar to peering out towards the infinite, how can we go the other way towards the infinitesimal? Recall the discussion above about Descartes.

We seem to apprehend either one element or its opposite, but a paradox emerges, the question being whether we apprehend both at once. How can something and its opposite be the same thing at once? Can up also be down, left right, or hot cold? Relativistically, such would make sense, each extreme taking its opposite place according to an observer, but both by the same person at the same time? Yet, to be able to do so might provide a foundational understanding of something's essence. Such may be the barrier to our obtaining absolute knowledge or even "simpler" things like knowing what "life" is. That is, to know anything, we need to know what it is not. More starkly, something exists because of what it is not. Exploring superposition may be another road to discovery., more pointedly the nature of quantum collapse. At this point, whether I am not uncomfortable with quantum/unity of opposites ontology matters little as an inhabitant of some version of Flatland.

We need to go back to Abbott and realize that just because we do not understand the reasons for an effect (like the raindrop falling on our head), that it does not mean there is nothing producing the effect. As in accepting the existence of any other natural phenomenon, like gravity, we realize that it is impossible empirically or cognitively to apprehend anything except by what it is not. In any event, accept it or not, such is the "bootstrap" ontology I adopt in this paper, a bootstrap more substantial than definitions. We may not know why this unity "works", but work it does. It may be that the unity of opposites is another facet of quantum mechanics, i.e., leading us to quantum ontology. For now, though, I will conserve these as separate categories of discussion.

THE ONTOLOGY OF NUMBER - FOUNDATION

Because of the inherent difficulties in arriving at a metaphysical ontology, the alternative internal, or provisional ontology will be used here, but more specifically, adopting the unity of opposites, or dialectics. This does not mean, however, that we are forced to use only definitions, axioms, postulates, and rules. These emerge from what we observe about us through our senses (again, sympathizing with Quine [p. 6]). Yet, data make no sense unless "filtered", and such is the role of mentation, as in "rationalism", a word that also needs explication. We should be reminded that these are two main epistemologies – ways of knowing – among others (tradition, history, science, and even intuition).

We have seen that many mathematicians say that numbers are mere abstractions and have no physical basis. The following explains why what they say is not the case. Upfront, there are two pillars on which the ontology of number rests: abstraction and empiricism.

Untangling abstraction

Abstractionists say that numbers exist irrespective of space-time. Gödel says "Unlike physical objects and properties, mathematical objects do not exist in space and time, and mathematical concepts are not instantiated in space or time" [Horsten, Ibid.].

It is by no means clear what is an abstraction, something that needs clarification if we are to think about the role it plays in providing an existence status for number. At the outset, an abstraction is intangible (not capable of being detected by our senses). Abstraction is "opposite" the "concrete", also known as "material", "physical", "solid", and so forth. It is on par with "idea", "concept", "mind", "intelligence", "consciousness", "thinking", "immaterial", and related notions, all of which are unclear to us, if the proceeding of the Towards a Science of Consciousness [www.consciousness.arizona.edu] conferences are any indication. Bear in mind also that not only do we not have a solid foundation for gaging the material and immaterial, but the boundary issue also remains. Before launching into these in contrast with what we detect with our senses, let's observe a bit more about abstraction and its role in numeracy.

Searching for phrases like "the ontology of abstraction" yields works by computer scientists about their specialized languages in their field, nothing metaphysical. In essence, we are talking about elaborate dictionaries.

A divergence in abstraction occurs when we can apply an idea to what is observed and those ideas remaining only as ideas with no correspondence to our observations. Classical examples are gods, flying horses, unicorns, and so forth. Quine's [p. 6] "On what there is", may have a satisfactory answer to resolving existence problems with logic, but I am more interested here in the nature of abstraction and how it is used. Focusing on how and why abstraction occurs will partially help answer why numeracy occurs. (I get to the other part – empiricism – in a while.) Again and contrary to existing "explanations" of number, explaining the effects (as in properties) of numeracy is not sufficient for understanding what numeracy is, any more than measuring the effects of heat tells what gives rise to it and why or the metrics of electricity (volts, ohms, amps, etc.) describe the electron. To the point in the present dilemma, how do we account for abstraction in numeracy?

While abstraction is not necessarily restricted to a specific time or space, it may be associated with the same (or another) [Abstraction, 2020]. Now, what explains abstraction? Neurocorrelates do, if we pay attention to Churchland, Koch, and other neuroanatomists. Below, I refer to recent work supporting the physical correlates to numeracy, hence explaining what abstraction is.

Untangling empiricism

Empiricism, itself, epistemologically is knowing through the senses – sight, taste, touch, hearing, and smell. Yet, these by themselves are insufficient, as the very neuroanatomy of nerves and the brain interpret them. Missing is our understanding of consciousness, cognition, mind, and so forth. Even Kant in his day of limited knowledge realized the problem.

Kant said,

> There can be no doubt that all our knowledge begins with experience. For *how should our faculty of knowledge be awakened into action did not objects affecting our senses partly of themselves produce representations*, partly arouse the activity of our understanding to compare these representations, and, by combining or separating them, work up the *raw material* of the sensible impressions into that knowledge of objects which is entitled experience?"
>
> *Reason is never in immediate relation to an object, but only to the understanding*; and it is only through the understanding that it has its own [specific] empirical employment. It does not, therefore, create concepts (of objects) but only orders them, and gives them that unity which they can have only if they be employed in their widest possible application, that is, with a view to obtaining totality in the various series. The understanding does not concern itself with this totality [of reason], but only with that connection through which, in accordance with concepts, such a series of conditions come into being. Reason has, therefore, as its sole object, the understanding and its effective application. Just as the understanding unifies the manifold in the object by means of concepts, so reason unifies the manifold of concepts by means of ideas, positing a certain collective unity as the goal of the activities of the understanding. [Ibid., B1: and B 671-672: (emphasis added]

We now should hold these thoughts for a while, returning to them to provide our new way of apprehending number.

DESCRIBING NUMBER

As with apprehending anything else, we sense magnitude biologically and then "store" the experience as memory, but when we recall it, we do so as an abstraction. Such is a main property of rationalism.

The biological basis of magnitude

We biologically acquire a sense of magnitude through our senses, or empirically. Magnitude is hard-wired into our brains. For example, the following speaks for itself about "state of the art" scientific discoveries in this domain:

> ... at the beginning of postnatal life, 0- to 3-[day]-old neonates reacted to a simultaneous increase (or decrease) in spatial extent and in duration or numerical quantity, but they did not react when the magnitudes varied in opposite directions. The findings provide evidence that representations of space, time, and number are systematically interrelated at the start of postnatal life, before acquisition of language and cultural metaphors, and before extensive experience with the natural correlations between these dimensions. [de Hevia et al., 2014]

Descriptions of tantalizing discoveries in the physical foundation of number are in *Space, Time and Number in the Brain: Searching for the Foundations of Mathematical Thought* [Montemayor and Winther, 2015]. Substantial other research supports magnitude being biologically-based [Bueti and Walsh, 2009; Vicario, 2013; Leibovich, Katzin, Harel, and Henik, 2017; Walsh, 2003; Lambrechts, Walsh, and van Wassenhove, 2013]

This must be refined or qualified by the Weber-Fechner law. The Weber part [Weber–Fechner law, 2020] says that one's perception of change in a stimulus is proportional to the original stimulus that produced the perception. The Fechner part [1860/1912] says each individual's sensitivity to certain stimuli is different. The U.S. National Institutes of Mental Health is taking the grounding of mental states in biology seriously enough to be sponsoring Research Domain Criteria (RDOC), as a reading of its website, https://www.nimh.nih.gov/research/research-funded-by-nimh/rdoc/index.shtml, will confirm.

As a final note to this section, we should not have to be reminded about animal cognition and non-human ability to apprehend magnitude.

Geometry and number

Another example of invoking space-time in considering number is geometric representation of numbers. For example, ask, "What are the numbers in the physical world? We will see that the answer to our question "What are the numbers in physical theory?" may depend on the dimension of spacetime [Volovich, 2010; Andréka et al., 2012, p. 2].

Whether spacetime embodies abstractions (without our representing them) is another question. A sidebar controversy of time even existing colors the durability of abstractions. This is consistent with abstraction as rationalism and succession of abstractions as application as empiricism.

The display of numbers

Polya [1945, p. 225] argued persuasively in his "working backwards" that we should look at the outcome of a problem and trace back how it came to be, as in looking at the exit of a maze and following it back through the passageways to find the entrance. Discovering the ontology of counting is no different. In a backhanded way, I am studying the effect of number to trace back to origin, rather than simply stopping at accepting the effect or property as an explanation. In particular, I now examine the display of numbers.

We have seen how thinkers like Frege and Russell revert to logic (as in the laws of thought – after George Boole) to explain the origin of mathematics. Yet, knowing about how numbers exist implies our knowing how both logic and mathematics exist. In fact, both have the same origin, logic neither the source of math nor vice versa. Three steps show how and why:

- modulo and number places, counting, display of modulo 2, and identification of the relationships between two variables;
- our binary world based on the unity of opposites;
- all possible arrangements of the two poles of the binary world.

Modulo

Peano and Dedekind especially made succession as one of the pillars of number properties. We must describe how the characteristics of counting as a type of succession came to be – by classifying accumulations, or modulos. This establishes counting, a type of succession more specific than Peano described [Russell, 1936; Kennedy, 1974; Bidiou, 2008].

The Lebombo and Ishango bones contain tally marks as a basis for placeholders, the essence of modulo (grouping of magnitudes), radix, or bases in mathematics, i.e., an early foundation of counting. How exactly our canonization of number places and the origin of modulo came about is speculative, but examples abound [cf: Boyer - *A history of mathematics*]. The following is a hypothetical scenario explaining how empirically-based counting may have originated. While I describe base 10 (as in people counting with their 10 digits) and base 5, the same principle applies for every base.

A space holds so many bags. This space becomes a measurement unit. When that space is filled, the bags are moved to a building that can hold 10 spaces. The original space is refilled, and its contents are transferred to a second space in the building. On an on this happens until we have 10 spaces filled. Hence the building is filled to capacity. Thus, the contents have to be transferred to a larger building that can hold ten of the smaller buildings. By inspection, the places become affixed by observing their regular sizes.

For modulo 5, we have:

1 5 A bags into 1 bag – Bag B
1. 5 B bags into Bag C.
2. 5 C bags into Bag D
3. etc.

[Kramer, p. 8]

Our binary world and the unity of opposites

It is somewhat absurd that the utilitarian filling of bags would have resulted in the binary method, simply because of its inefficiency. Instead, we look to how the binary system appeared in modern history, and Leibniz's work is to whom historians of mathematics turn [Leibniz, 1703] for describing the transition from base 10 to binary [Lande, 2014, p. 525]

Aside from it being the basis of our digital age, the display of binary counting has philosophical significance, although not like Leibniz had in mind with his reference to the binary aspects of the ancient Chinese I-Ching. A word of caution is avoiding numerology, reading into the arrangement of numbers a significance that is not there. We look to how well the concepts attributed to the arrangement can predict, something the I-Ching could not do.

However, it is by Leibniz [Ibid.] observing

> ... j'ai employe despuis plusieurs annees la progression la plus simple de toutes, qui ca de deux en deux; ... je n'y employe pour d'autres caracteres que 0 & 1, & puis allant a deux, je recommence." (After several years I have used the simplest progression of all – two by two. (I have only used two characters – 0 and 1, and thus using the two, proceed.)

In seeing that 2 is the maximum value, Leibniz equates it in concept to the highest value in the base 10 system, 10. "C'est pourquoi deux s'ecrit ici par 10, & deux fois deux ou quatre par 100; & deux fois quatre ou huit par 1000; …" (It is because two is written here [base 10] by 10 and two times two or four for 100; and two times four or eight for 1000.) [Ibid., p. 2]. With this method, he constructs his binary table of numbers. Notice the words, "la plus simple de toutes" (the simplest of all), meaning that you cannot use less than two, or only one, character to create a progression that describes each member as different from the other. Leibniz said,

> Mais le calcul par deux, c'est-a-dire par 0 & par 1, en recompense de longeur, est le plus fondamental pour science, & donne de nouveles descouvres, que se trouvent utiles ensuite, meme pour la practique des nombres, & surtout pour la Geometrie; dont la raison est, que les nombres enant reduits aux plus simples principles, comme 0 & 1, il paroit partout un ordre merveilleux." [Ibid., p. 87]
>
> "Because calculating by two, that is to say by 0 and 1, as a tradeoff for its length, is the most fundamental of all science, and gives us new discoveries which have been found useful, the same for the practice of numbers, and above-all, geometry; the reason being that numbers are reduced to the most fundamental principles, as 0 and 1, they appear everywhere in a marvelous order." [Ibid., p. 87]

I also focus on his words "nouveles descouvres", some of which will be discussed shortly, especially pertaining to the relationship of logic to mathematics.

To appreciate the philosophy, we return not only to the unity of opposites but order, itself. After all, one of the foundations of counting is order and succession. James K. Feibleman said that logic is the theory of order [Feibleman, p. 89]. "Order" is arrangement, the simplest form (and as Leibniz observed) binary. After all, you have to have something to arrange; something cannot arrange itself. Too, the very essence of our world is that things exist for what they are not, the unity of opposites.

Permutation displaying the laws of thought

Permutation is arrangement, and there are only so many ways you can arrange two things in a plane:

p	**q**
0	0
0	1
1	0
1	1

… just as Leibniz said. Here, it is worthy to note two things. First, the very nature of single dimensionality (a line) physically allows only for so many arrangements. Second, the manner of listing them could be different, as in:

p	**q**
0	1
0	0
1	1
1	0

However, the former reflects a consistent method, as in starting with nothing and ending with a full complement of something. Much more could be said about this, but suffice it to say (and argued elsewhere [Horne, 2017]) that the first is a more methodical treatment of the unity of opposites. Philosophers would also refer to Hegel's *Phenomenology of Mind* in the description of something, its "other" and full development of the opposite.

The following table of functional completeness (ToFC) tracks Leibniz, and logicians (Copi, Church) have been keenly aware of it, although without its implications in originating both logic and math.

p	q	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		$\sim f_{15}$	$\sim f_{14}$	$\sim f_{13}$	$\sim f_{12}$	$\sim f_{11}$	$\sim f_{10}$	$\sim f_9$	$\sim f_8$	$\sim f_7$	$\sim f_6$	$\sim f_5$	$\sim f_4$	$\sim f_3$	$\sim f_2$	$\sim f_1$	$\sim f_0$

We should now be able to see contained within this logical space the complete logical functions:

f_0	X – Contradiction
f_1	&, and, conjunction: p & q
f_2	$\sim (p \supset q)$
f_3	1>, 1 precedes, or, simply "p"
f_4	$\sim (q \supset p)$
f_5	>1, 1 follows (or simply "q")
f_6	≠, p or q is true (1) but not both (XOR); exclusive "or": $\sim (p \equiv q)$
f_7	∨, p or q is true or both are true; inclusive "or", disjunction
f_8	NOR, neither p nor q or both is/are true: ~ (p v q)
f_9	≡, p is equivalent to q in truth value
f_{10}	>0, 0 follows (or simply "not q")
f_{11}	$(q \supset p)$, $(p \subset q)$
f_{12}	0>, 0 precedes (or simply "not p")
f_{13}	$(p \supset q)$ ⊃, or → p contains q (often called "IMP") – **defines deduction**
f_{14}	NAND, not both p and q are true ~ (p & q)
f_{15}	T, tautology

I will return to these numerical expressions of functions shortly, but observe that they occur, or embedded, in counting in base 2, displayed from left to right.

A separate discussion exists about our being prejudiced by the zeros and ones and the necessity of showing how the display should progress using other symbols, as in:

p	q	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
*	*	*	*	*	*	*	*	*	*	&	&	&	&	&	&	&	&
*	&	*	*	*	*	&	&	&	&	*	*	*	*	&	&	&	&
&	*	*	*	&	&	*	*	&	&	*	*	&	&	*	*	&	&
&	&	*	&	*	&	*	&	*	&	*	&	*	&	*	&	*	&
		$\sim f_{15}$	$\sim f_{14}$	$\sim f_{13}$	$\sim f_{12}$	$\sim f_{11}$	$\sim f_{10}$	$\sim f_9$	$\sim f_8$	$\sim f_7$	$\sim f_6$	$\sim f_5$	$\sim f_4$	$\sim f_3$	$\sim f_2$	$\sim f_1$	$\sim f_0$

The way the ampersand (&) "invades" the spaces occupied by the asterisk (*), as in the bags filling buildings is the same for the left (blue) as for the right (yellow) until the ultimate column is completely occupied. We are witnessing how "places" in a counting system develop, as in units, tens, hundreds, and so forth in the base 10 system. In the end, bear in mind we can do such tables based on other modulos, but such physically requires a large space to display all the permutations. Note that the right half is the negative (negation) of the left half. Replacement, more than a mathematical construct, in essence, is the foundation of counting.

This is to be developed later, it being sufficient to say that single dimensionality (the simplest) constrains the number of permutations possible and that the emergence of patterns reflects our emerging grasp of numeracy (as in modulos). As to our counting with any symbols, the obvious answer is "yes", merely by substituting a symbol uniformly for 0 and a different one uniformly for 1. That is, there is something very basic underpinning counting – not the symbol of course, but what they represent. I see Quine's [Ibid., p. 4, et seq.] distinction between naming and meaning (Quine – On what there is) apropos here, that both the zeros/ones and asterisks/ampersands being the meaning and that to which they refer – the name.

Another observation is that Peano's five postulates will not create this or any other grid of permutations but only reflect some of the properties. His is no algorithm to create a grid that displays counting, but grids as discrete displays are not necessarily indicative of or a substitution for Peano's description of mathematical induction, a valuable addition to number theory. Here, we have it, a non-definitional revision of Peano and Dedekind, combining a description of number properties as resulting from permutations, or simple arrangements, with the ontology giving rise to the numbers, themselves.

HOW NUMBERS EXIST - THE UNITY OF OPPOSITES

What, then, do we experience, and then, what do we do with those experiences ("process" it)?

One of the epistemologies mentioned above – how we know – is history. We know number exists by its history, as in the Ishango stick. The single physical dimensional limitation – known at least by the senses – tells us the minimal requirements to have ordering: two elements, something being in front of another or it being in the back [Horne, 2017]. There also is the neuroanatomical correlation with our perception of magnitude and number, indicated by the research described above.

Once we are aware of the empirical and historical aspects of numeracy they have to be persistent in order to have utility. As Kramer and others have stated, we transfer our observations (as in matching objects with symbols) and applying them to other situations through abstraction. This requires memory. Abstraction through memory is the central aspect of reason. More controversial are the "laws" of reason and their origin or even existence. However, by inspection (empiricism), we can see that in the display of binary space above relationships such as:

p	q
0	0
0	1
1	0
1	1

That f_9 (equivalence) describes in the first row that 0 indeed is 0, 0 is not 1 in the second row, and so forth. For f_1 1, does not exist in the first row but does in the succeeding ones. Each of the columns reflects a physical description of the relationship of the first to the second column in some way (Horne, 2017). These can be said to be the "laws of thought", but they are also embedded in counting (modulo 2). Note that the table could be written as:

p	q
^	^
^	%
%	^
%	%

Any symbols or objects (of any of the senses – sight, sound, touch, etc.) could be used to instantiate the variables, of course, but the empirical comparing of the two still would yield the same judgements. If we relied on memory (as in abstraction), the same still would apply.

Notice the shift of epistemologies – from the physical (organically-based) empiricism used for apprehension to the intangibility of rationalism to invoke memory. More explicit, receptors accept stimuli and neurons transmit electrical signals to the brain, where quite an unknown entity – "mind", "consciousness" or whatever we choose to call "brain processing" produces the best we can call as "ideas". Immediately after the creation of these "ideas", because of the nature of what we perceive as the passage of time, we rely upon recall, that is, memory. Memory is used to establish uniformity, or the unit number. Inasmuch as the present contains the past, recollection is deduction. One extrapolates from the past – recalling through memory the initial apprehension and projects to the future via induction to describe the future. (Notice the unity of opposites with respect to the relationship of deduction and induction, one existing because of the other.) Rationalism is memory operating in this fashion. A person recalls the apprehension (as in the size or weight of something) and in physical application matches it to a physical object. This object becomes the unit that is physically applied repeatedly. An objection, given the previous remarks about boundary problems, may be raised about the precision of each application, as in using the object to calibrate. Here, successive observations, perforce, are subjective, the boundaries being determined by us, the same approach used in calculus. In the case of abstraction, the application and succession also are abstract.

Otherwise stated, remembering is deduction; projecting is induction (not the same as mathematical induction, a proof procedure). Hence, we have demarcated a unit (establishing the characteristic of a space) and projected it as a process through induction using memory (rationalism). Be aware that any apprehension can serve as a unit – meter stick, shoe, refrigerator, table, or even an idea. Dimensions, weights, waiting periods (as in establishing units of time), or any singly observed property – or anything sensed or experienced (empiricism, again) will serve as a unit. As to various bases, one merely need to determine the instances of applications of an apprehension before re-starting the cycle. Repetitions of this are successions.

As can be seen, number and systems of numbers do not establish themselves except as through this unity of opposites of characterizing space by the empiricism of apprehension and rationalizing by abstraction with memory through time by repeated applications of apprehension. Both the deduction of recollection and the induction of applying the abstraction of memory must exist for there to be numbers and their properties.

Notice all the while that the above is a bootstrapping, inherently subject to human bias. It does not appeal to any Platonic form (which admittedly may even exist, but which we cannot demonstrate without question – external metaphysics). Because of our dimensional limitation we do not seem to be able to explain the working of this unity of opposites but, like four-dimensional Flatlandlanders do realize its effects.

We have established the ontology of number but not of counting. Succession is not necessarily counting, the latter not only referring to succession but a following having specific properties. We need that rationalism/memory aspect, as well as empiricism. Peano did establish those properties but not their ontology.

THE MISSING POSTULATES OF PEANO

Within the framework of the unity of opposites, the abstraction is the immaterial, the empirical the material, both needed to achieve the ontology of number. In a reverse sort of way, the immaterial is the object (theory) and the empirical (application) the process. One would think that the empirical would be material and how could material be process? We hark back to the subquantum world, where the material becomes the immaterial and vice versa. Our confusion stems from the dimensional limitation.

1 Space-time is the environment in which the following events occur.

1. Rationality depends upon memory and is abstraction.
2. An apprehension may be empirical or rational.
3. Recollection of an apprehension is through memory and is a deductive process, and such establishes a unit characterizing a space.
4. Application of this apprehension – extrapolation through memory and projecting it to the future as an inductive process – describes succession and characterizes time.
5. Magnitude is the accumulation of successions.
6. For there to be a number, both recollection (space) and application (time) must exist, one existing because of the other.
7. The absence of occupation of space – determined by apprehension – establishes zero as a number serving as a placeholder in permuted space and signifying accumulated magnitude.
8. Every number is a successor, except zero, which is not a successor of any number.
9. The principle of mathematical induction holds, that any property that holds at zero, and is preserved under successor, holds for all natural numbers.
10. No two numbers can have the same successor.
11. The process of succession is endless in the abstract but in reality limited to quantum cosmological physics (mathematical induction).

SUMMARY AND CONCLUSIONS

The existence of numbers typically is described by their effects, our attributing to those effects characteristics, or properties. The framework for doing so is by bootstrapping, not unlike logicians doing deductive proofs, with definitions, rules, axioms, and postulates. If you accept these, then you have to accept the conclusion. Yet, this method does not advance us any closer to the foundation of number, their existence. While we do not seem to be able to arrive at the "absolute" or "truth", there is a method, a process that allows us to place the argument for the existence of number on equal footing for arguing the existence of anything else.

Numbers are abstractions, as well as representations of objects, both material and non-material. A number is a thing, an object (a spatial aspect), but it is a process (a temporal aspect), as well. Both object and process co-exist as a unity of opposites, just as space and time do to give us our dimensional environment. The unity of opposites allows a number to exist. Without both elements, number does not exist.

Our justification of the ontological description calls upon the epistemologies of empiricism and rationalism, the former referring to our sense and experiential world in which numbers affirm themselves as representations of objects and the latter in which number affirms its quality of abstraction.

There, you have it, the ontology of number.

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Quantum ontology – see also https://en.wikipedia.org/wiki/Quantum_superposition

See Julian Barbour - https://www.popsci.com/science/article/2012-09/book-excerpt-there-no-such-thing-time/ Similar to superposition and

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