A compatibility relation for sets

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Abstract

In this paper, I generalise the logical concept of compatibility into a set theoretical one. The basic idea is that two sets are incompatible if they *produce* at least one pair of opposite or contrary objects. In section 1, I formalise opposition as an operation $' : \mathbf{E} \longrightarrow \mathbf{E}$, where $\mathbf{E}$ is the set of opposable elements of our universe $\mathbf{U}$, and I propose some models in section 2. From this, I define in section 3 a relation $\mathsf{C} : \wp\mathbf{U} \times \wp\mathbf{U} \times \wp\mathbf{U}^{\wp\mathbf{U}}$, which has logical compatibility as its more natural interpretation. Other models are sketched in section 4 and, in the final section, I discuss some limitations of this proposal.
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1 Opposition

That some set is compatible with another depends on whether they *produce* a pair of *opposite* or *contrary* objects. In order for a function $' : \mathbf{E} \longrightarrow \mathbf{E}$ to be an operation of opposition it needs to be involutory:

Axiom 1. $x = x''$.

From where we obtain the following corollaries:

Corollary 2. $x' = y' \Rightarrow x = y$.

Corollary 3. $x' = y \Rightarrow x = y'$.

The first means that $'$ is *injective* and, since $'$ is its own inverse function, we can interpret as saying that $'$ is *surjective*. This establishes that $'$ is a *bijective* function.

Intuitively, x' denotes the opposite or contrary of x; in other words, an operation of opposition transforms an element of $\mathbf{E}$ into its opposite. Since it is not necessary that all elements of our domain have an opposite, the domain of this operation is restricted to $\mathbf{E}$, which is the set of *opposable* elements of our universe $\mathbf{U}$.

These properties, however, are not sufficient to completely characterise the concept of opposition. Some additional properties depend on the introduction of other operations. In fact, there is room for debating whether some of these properties are adequate or not. For example, we might say that the white bishop in `h1` (see

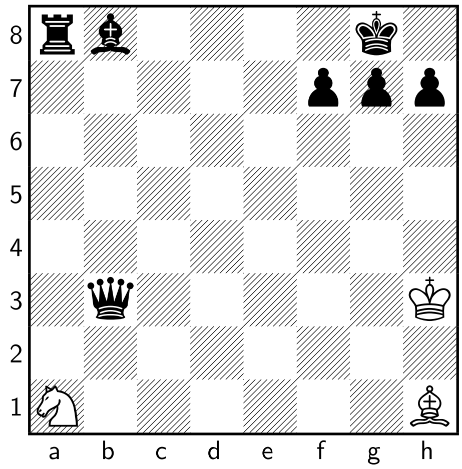

Figure 1: The chess table.

figure 1) is opposed to the black rook in a8. But it may also be said that this rook is opposed to the white knight in a1 (which is opposed to the black queen in b3?).

Now, this may be fairly taken as a silly consideration, but it shows to some extent that the properties of our formalisation depends on what notion of opposition are we trying to capture. Since I do not pretend to be exhaustive, I will restrict this concept even more by requiring that that $'$ be irreflexive.

Axiom 4. $x \neq x'$.

This axiom prevents any object from being its own opposite.

2 Interpretations of $'$

Several mathematical functions are relations of opposition. For example, *classical negation* satisfies all the previous properties. Furthermore, if we take our domain $\mathbf{U}$ to the set of wffs of a language, we have that $\mathbf{E} = \mathbf{U}$. If p and q are wffs, we may take $p = q$ to mean that p and q have the same logical value.

Also, the *inverse operation* of group theory is also an operation of opposition. This implies that the *additive inverse* ($x' \mapsto -x$) and the *multiplicative inverse* ($x' \mapsto 1/x$) are also operations of opposition in the relevant domains. This obvious holds also for the inverse operation of any model of group theory.

A remarkable interpretation is the *absolute complement operation* ($\mathbf{A}' \mapsto \mathbf{A}^C$) of set theory. This means that we can also have properties in the domain of $'$. For example, if P is in the domain of $'$, we may define P$'$ as the predicate in whose extension are all x such that $\neg$Px; that is, the extension of P$'$ is the absolute complement of the set corresponding to the extension of P. We can alternatively take P$'$ as the one antonymous property of P, which is the one antonymous property of P$'$.

In the first approach, if P stands for 'x is transparent', P$'$ would stand for 'x is opaque', since all non transparent things are opaque. (We are restricting our domain to normal sized physical objects.) In the second approach, if P stands for

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'x is white', P$'$ could stand for 'x is black'. Notice that this works only if we take blackness and whiteness as opposite properties in the sense previously defined.

3 The relation C

I define *classical compatibility* as a three-place relation $\mathsf{C} : \wp\mathbf{U} \times \wp\mathbf{U} \times \wp\mathbf{U}^{\wp\mathbf{U}}$ satisfying:[1]

Definition 5. $\mathsf{C}(\mathbf{A}, \mathbf{B})^* \Leftrightarrow x, x' \in (\mathbf{A} \cup \mathbf{B})^*$*, for no* x*.*

The expression $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$ is read "$\mathbf{A}$ is $*$-*compatible* with $\mathbf{B}$," where $\mathbf{A}, \mathbf{B} \in \wp\mathbf{U}$ and $* \in \wp\mathbf{U}^{\wp\mathbf{U}}$. Accordingly, we say that $\mathbf{A}$ is $*$-*incompatible* with $\mathbf{B}$ iff not $\mathsf{C}(\mathbf{A}, \mathbf{B})^*$. The relation C is *symmetric* in the sense that:

Theorem 6. $\mathsf{C}(\mathbf{A}, \mathbf{B})^* \Leftrightarrow \mathsf{C}(\mathbf{B}, \mathbf{A})^*$*, for all* $\mathbf{A}, \mathbf{B}$ *and* $*$*.*

However, C is no equivalence relation since it is neither *reflexive* nor *transitive.* Those properties clearly do not hold when $*$ is *increasing* and *monotone* (which are properties of closure operations). In that case, it cannot be reflexive since $\mathsf{C}(\{x, x'\}, \{x, x'\})^*$ never holds. Nor can it be transitive because even in the hypothesis that $\mathsf{C}(\{x\}, \mathbf{B})^*$ and $\mathsf{C}(\mathbf{B}, \{x'\})^*$, it still does not hold that $\mathsf{C}(\{x\}, \{x'\})^*$.

The most natural interpretation of C is the relation of logical compatibility between sets of sentences with respect to a relation of consequence. Such interpretation depends on defining *consistency.*

Definition 7. $\mathbf{A}$ *is* consistent $\Leftrightarrow \alpha, \neg\alpha \in \mathbf{A}$*, for no* α*, and* inconsistent *otherwise.*

For our interpretation we take $\mathbf{U}$ to be the set of statements or propositions of a formal language, $'$ the operation of negation, and $*$ a relation of logical consequence ($\vdash$: $\wp\mathbf{U} \longrightarrow \mathbf{U}$).[2] From this, it follows that two sets of sentences $\mathbf{A}$ and $\mathbf{B}$ are compatible iff there is no α such that $\mathbf{A} \cup \mathbf{B} \vdash \alpha$ and $\mathbf{A} \cup \mathbf{B} \vdash \neg\alpha$. That is, in order for two sets of sentences to be consistent, it is necessary that the set of their logical consequences be consistent too.

Let us compare this definition with that of Batens and Meheus (2000). Although they initially define compatibility as a relation between sentences and sets of sentences, they clarify in their footnote 1 that it is a symmetric relation. The following definition is enough, for my purposes, to capture their syntactic definition of compatibility.

Definition 8. $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash} \Leftrightarrow (\mathbf{A} \vdash \alpha \Rightarrow \mathbf{B} \nvdash \neg\alpha)$*, for all* α*.*

Assuming $\vdash$ is increasing, any pair of sets in the extension of C is also in the extension of D.

Theorem 9. $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash} \Rightarrow \mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$*.*

[1] Remember that Y^X is the set of all functions from X to Y. Hence, $\wp\mathbf{U}^{\wp\mathbf{U}}$ is the set of all functions from and to sets of $\mathbf{U}$ (cf. Halmos, 1974, 30).

[2] Remember that $\alpha \in \mathbf{A}^{\vdash}$ iff $\mathbf{A} \vdash \alpha$.

Proof. Assume $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$ and let $\mathbf{A} \vdash \alpha$. Since $\vdash$ is increasing, this means that $\mathbf{A} \cup \mathbf{B} \vdash \alpha$. For the same property it would follow from $\mathbf{B} \vdash \neg\alpha$ that $\mathbf{A} \cup \mathbf{B} \vdash \neg\alpha$, which is forbidden by $\mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$. Hence, $\mathbf{B} \nvdash \neg\alpha$. ❧

The converse follows if $\vdash$ is classic, in which case it means that it is increasing and satisfies the *compactness theorem*.

Theorem 10. $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash} \Rightarrow \mathsf{C}(\mathbf{A}, \mathbf{B})^{\vdash}$.

Proof. We assume $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$ and suppose for reductio that $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ is inconsistent. In that case, the compactness theorem guarantees that $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ has an inconsistent subset. Since $\vdash$ is classical, $\mathbf{A}^{\vdash}$ is consistent, otherwise $\mathbf{A}$ would imply all formulae, including the negations of tautologies, and since $\mathbf{B}$ implies all tautologies, this would mean that not $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$. We can prove that $\mathbf{B}$ is consistent in a similar way. Hence, in order for $(\mathbf{A} \cup \mathbf{B})^{\vdash}$ to be inconsistent it is necessary that some α be such that $\mathbf{A} \vdash \alpha$ and $\mathbf{B} \vdash \neg\alpha$, which is forbidden by $\mathsf{D}(\mathbf{A}, \mathbf{B})^{\vdash}$. ❧

This proves that (classical) logical compatibility, as defined by Batens and Meheus, is a model of C. Let us now turn to other interpretations.

4 Other (informal) interpretations of C

We may speak about incompatible sets of entities when $\mathbf{U}$ is interpreted as the set of all (conceivable) entities. One way to show this is through answering the question triggering the *irresistible force paradox*, i.e. *what happens when an unstoppable force meets an immovable object*? Let M stand for the relation 'x can move y'. We have then that an immovable object is any y satisfying $\forall x(\neg x\mathrm{M}y)$. An unstoppable force is instead an object that can move any object that encounters; that is, an x satisfying $\forall y(x\mathrm{M}y)$.

Now, is it possible that an unstoppable force and an immovable object thus defined can exist in the same possible world? Unless we dismiss the principle of non contradiction, the answer is clearly no. Otherwise, if there where an object, say a, such that $\forall y(a\mathrm{M}y)$, and another b such that $\forall x(\neg x\mathrm{M}b)$, it would follow that $a\mathrm{M}b$ and $\neg a\mathrm{M}b$. Informally, we can say that the opposite of an immovable object is an unstoppable force, which makes incompatible any set containing both. [3]

In another interpretation, it is possible to state that two sets of properties are compatible or incompatible for a given entity. In order to do this we can treat entities as sets of properties: the properties that those entities have. This treatment corresponds to Russell's conception of proper names, for whom "what would commonly be called a 'thing' is nothing but a *bundle of coexisting qualities* such as redness, hardness, etc." (1995, 97, my emphasis). For example, if we let B stand for 'x is single' and M for 'x is married', we may say that $\mathrm{B}' = \mathrm{M}$. The properties of being single and being married are in this sense incompatible, since all non married persons are single.

[3] This solution was well presented by Isaac Asimov in his *Book of Facts*: "The rules of the game of reason say the question is meaningless and requires no answer. The question: 'What would happen if an irresistible force met an immovable body?' In a universe where one of the above conditions exists, by definition the other cannot exist." (Asimov, 1979, 281)

5 Limitations of classical compatibility

It does not seem right that C not be reflexive. After all, how can a set be incompatible with itself? Nevertheless, $\mathsf{C}(\mathbf{A}, \mathbf{A})^*$ only fails for those sets such that $x, x' \in \mathbf{A}^*$, for some x.

Corollary 11. $\mathsf{C}(\mathbf{A}, \mathbf{A})^* \Leftrightarrow x, x' \in \mathbf{A}^*$*, for no* x*.*

In this framework we can state that all sets that are incompatible with themselves are (logically) unacceptable or inconceivable, depending on the kind of incompatibility we are talking about.

The problem with this approach is that it would make it impossible to analyse (in)compatibility between inconsistent sets. For example, if we had an inconsistent (though non trivial) theory $\mathbf{T}$, we would have to conclude that all sets of observation statements (consistent or otherwise) are incompatible with $\mathbf{T}$. This would result in $\mathbf{T}$ being a priori false instead of falsifiable.

As it happens, this situation can be corrected if we stick Batens' and Meheus' definition. In such case, though, compatibility could not be a symmetric relation, as they want. One such theory of *para-compatibility* is a topic for another paper.

References

Asimov, I. (1979). *Isaac Asimov's Book of Facts.* Wings Books, New York.

Batens, D. and Meheus, J. (2000). The adaptive logic of compatibility. *Studia Logica*, 66(3):327–48.

Halmos, P. R. (1974). *Naive Set Theory.* Dover Books on Mathematics. Springer, New York.

Russell, B. (1995). *An Inquiry into Meaning and Truth.* Routledge, London.