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Absolutely Convergent Fourier Series of Functions over Homogeneous Spaces of Compact Groups

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Abstract. This paper presents a systematic study for classical aspects of functions with absolutely convergent Fourier series over homogeneous spaces of compact groups. Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the left coset space G/H associated with Weil's formula with respect to the probability measures of G and H. We introduce the abstract notion of functions with absolutely convergent Fourier series in the Banach function space $L^1(G/H, \mu)$. We then present some analytic characterizations for the linear space consisting of functions with absolutely convergent Fourier series over the compact homogeneous space G/H.

1. Introduction

The classical aspect of harmonic analysis over homogeneous spaces of compact non-Abelian groups, more precisely, left (resp. right) coset spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis, constructive approximation, and particle physics, see [3; 2; 5; 15; 16] and the references therein. Over the last decades, abstract and computational aspects of Fourier transforms, Fourier series, and Plancherel formulas over symmetric spaces have achieved significant popularity in abstract harmonic analysis, geometric analysis, mathematical physics, and scientific computing, see [1; 4; 6; 14; 8; 17; 19; 18] and the references therein.

Let G be a compact group and H be a closed subgroup of G. Then the left coset space G/H is a compact homogeneous space, where G acts on it via the left action. Let μ be the normalized G-invariant measure on the compact homogeneous space G/H associated with Weil's formula with respect to the probability measures of G and H. This paper consists of theoretical aspects of a unified approach to the nature of absolutely convergent Fourier series in the Banach function space $L^1(G/H, \mu)$. We aim to further develop the abstract notion of absolutely convergent Fourier series of functions over homogeneous spaces of compact groups, which has not been studied when compared to the absolutely convergent Fourier series of functions over compact groups [16].

This paper, which contains five sections, is organized as follows. Section 2 is devoted to fixing notation and preliminaries including a summary on operator

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theory, abstract non-Abelian Fourier analysis over compact groups, and classical results on abstract harmonic analysis over homogeneous spaces of compact groups. In Section 3, we present some abstract aspects of harmonic analysis on the Banach function spaces over the compact homogeneous space G/H. We then briefly introduce the abstract notion of dual space $\widehat{G/H}$ for the homogeneous space G/H, and also we present a summary including the theoretical definition of the abstract operator-valued Fourier transform over the compact homogeneous space G/H and its basic properties in Section 4. Finally, we introduce the abstract notion of functions with absolutely convergent Fourier series over the compact homogeneous space G/H, and then we study basic properties of the linear space consisting of all functions with absolutely convergent Fourier series over the compact homogeneous space G/H in the framework of abstract harmonic analysis. We shall also present some analytic characterizations for these spaces.

2. Preliminaries and Notations

Throughout this section we present preliminaries and basic notations.

2.1. Operator Theory

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a Hilbert–Schmidt operator if for one, hence for any orthonormal basis $\{e_k\}$ of $\mathcal{H}$, we have $\sum_k \|Te_k\|^2 < \infty$. The set of all Hilbert–Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$, and for $T \in \mathrm{HS}(\mathcal{H})$, the Hilbert–Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2 = \sum_k \|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self-adjoint two-sided ideal in $\mathcal{B}(\mathcal{H})$, and if $\mathcal{H}$ is finite-dimensional, we have $\mathrm{HS}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is trace-class if $\|T\|_{\mathrm{tr}} = \mathrm{tr}[|T|] < \infty$, where $\mathrm{tr}[T] = \sum_k \langle Te_k, e_k\rangle$ and $|T| = (TT^*)^{1/2}$ [16; 20].

Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension d and $1 \le p \le \infty$. For a (bounded) linear operator $T \in \mathcal{B}(\mathcal{H})$, the Schatten operator p-norm of T, denoted by $\|T\|_p$, is defined as [16]

$$\|T\|_p := \left(\sum_{j=1}^{d} |s_j(T)|^p\right)^{1/p},$$

where $\{s_j(T) : 1 \le j \le d\}$ is the sequence of singular values of T, that is, the sequence of eigenvalues of the positive-definite operator $|T|$. It is a well-known result that $\|T\|_p^p = \mathrm{tr}[|T|^p]$. Thus, $\|T\|_1 = \|T\|_{\mathrm{tr}}$ and $\|T\|_2 = \|T\|_{\mathrm{HS}}$.

2.2. Abstract Fourier Analysis over Compact Groups

Let G be a compact group with the probability Haar measure dx. Then each irreducible representation of G is finite-dimensional, and every unitary representation of G is a direct sum of irreducible representations, see [5; 15; 16]. The set of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, see [5; 15]. For a subset $\Omega \subseteq \widehat{G}$, the $*$-algebra $\prod_{\pi\in\Omega} \mathcal{B}(\mathcal{H}_\pi)$

is denoted by $\mathfrak{C}(\Omega)$, where scalar multiplication, addition multiplication, and the adjoint of an element are defined coordinatewise. For $1 \le p \le \infty$, $\mathfrak{C}^p(\Omega)$ is defined by

$$\mathfrak{C}^p(\Omega) = \left\{ \mathbf{T} = (T_\pi)_{\pi \in \Omega} \in \mathfrak{C}(\Omega) \,\Big|\, \|\mathbf{T}\|^p_{\mathfrak{C}^p(\Omega)} := \sum_{[\pi] \in \Omega} d_\pi \|T_\pi\|_p^p < \infty \right\} \tag{2.1}$$

and

$$\mathfrak{C}^\infty(\Omega) = \left\{ \mathbf{T} = (T_\pi)_{\pi \in \Omega} \in \mathfrak{C}(\Omega) \,\Big|\, \|\mathbf{T}\|_{\mathfrak{C}^\infty(\Omega)} := \sup_{[\pi] \in \Omega} \|T_\pi\|_\infty < \infty \right\},$$

where $\|T_\pi\|_p$ with $\pi \in \widehat{\Omega}$ are the Schatten operator p-norms. The set of all $\mathbf{T} \in \mathfrak{C}(\Omega)$ such that $\{\pi \in \Omega : T_\pi \neq 0\}$ is finite, is denoted by $\mathfrak{C}_{00}(\Omega)$, and $\mathfrak{C}_0(\Omega)$ is defined as the set of all $\mathbf{T} \in \mathfrak{C}(\Omega)$ such that $\mathbf{T}_\epsilon := \{\pi \in \Omega : \|T_\pi\|_\infty \ge \epsilon\}$ is finite for all $\epsilon > 0$, see [16, Section 28.24 and Section 28.34] and [20].

If π is any unitary representation of G, for $\zeta, \xi \in \mathcal{H}_\pi$, the functions $\pi_{\zeta,\xi}(x) = \langle \pi(x)\zeta, \xi \rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The Fourier transform of $f \in L^1(G)$ at $\pi \in \widehat{G}$ is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* \, dx. \tag{2.2}$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x)) \in \mathbb{C}^{d_\pi \times d_\pi}$, then $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ is the matrix with entries given by $\widehat{f}(\pi)_{ij} = d_\pi^{-1} c^\pi_{ji}(f)$ satisfying

$$\sum_{i,j=1}^{d_\pi} c^\pi_{ij}(f)\pi_{ij}(x) = d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ji}\pi_{ij}(x) = d_\pi \operatorname{tr}[\widehat{f}(\pi)\pi(x)],$$

where $c^\pi_{i,j}(f) = d_\pi \langle f, \pi_{ij} \rangle_{L^2(G)}$. Then the Peter–Weyl theorem implies

$$\|f\|^2_{L^2(G)} = \sum_{[\pi] \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_2^2 = \sum_{[\pi] \in \widehat{G}} \sum_{j,k=1}^{d_\pi} |\langle f, d_\pi^{1/2} \pi_{jk} \rangle_{L^2(G)}|^2 \tag{2.3}$$

and

$$f(x) = \sum_{[\pi] \in \widehat{G}} d_\pi \operatorname{tr}[\widehat{f}(\pi)\pi(x)] \quad \text{for a.e. } x \in G. \tag{2.4}$$

Thus, the abstract Fourier transform $\mathcal{F}_G : L^2(G) \to \mathfrak{C}^2(\widehat{G})$, given by $f \mapsto \mathcal{F}_G(f) = (\widehat{f}(\pi))_{\pi \in \widehat{G}}$, is a unitary transform, see [16, Section 27.40 and Section 28.43] and [5].

A function $f \in L^1(G)$ is said to have an absolutely convergent Fourier series if

$$\sum_{[\pi] \in \widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1 < \infty. \tag{2.5}$$

The linear space consisting of all functions with absolutely convergent Fourier series over the compact group G is denoted by $\mathcal{R}(G)$, and we have $\mathcal{R}(G) = \{f \in$

$L^1(G) : \widehat{f} \in \mathfrak{C}^1(\widehat{G})\}$. Then

$$f \mapsto \|f\|_{(1)} := \|\widehat{f}\|_{\mathfrak{C}^1(\widehat{G})} = \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1$$

defines a norm on the linear space $\mathcal{R}(G)$, which makes $\mathcal{R}(G)$ into a Banach space, see [16, Section 34].

2.3. *Classical Harmonic Analysis over Homogeneous Spaces of Compact Groups*

Let H be a closed subgroup of G with the probability Haar measure dh. The left coset space G/H is considered as a locally compact homogeneous space that G acts on it from the left, and $q : G \to G/H$ given by $x \mapsto q(x) := xH$ is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces are quite well studied by several authors, see [5; 15; 16; 21] and the references therein. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}(G)$ and

$$T_H(f)(xH) = \int_H f(xh)\, dh. \tag{2.6}$$

Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$ for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$ for all $x \in G$. The compact homogeneous space G/H has a normalized G-invariant measure μ associated with the following Weil formula:

$$\int_{G/H} T_H(f)(xH)\, d\mu(xH) = \int_G f(x)\, dx, \tag{2.7}$$

and hence the linear map T_H is norm-decreasing, that is,

$$\|T_H(f)\|_{L^1(G/H,\mu)} \le \|f\|_{L^1(G)}$$

for all $f \in L^1(G)$, see [5; 15; 21].

3. Abstract Harmonic Analysis of Functions over Homogeneous Spaces of Compact Groups

Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and μ is the normalized G-invariant measure on the compact homogeneous space G/H associated with Weil's formula (2.7) with respect to the probability measures of G and H. From now on, we say that μ is the normalized G-invariant measure over the compact homogeneous space G/H, at times.

This section is devoted to review some classical aspects of abstract harmonic analysis over the Banach function spaces $L^p(G/H, \mu)$ with $p \ge 1$. For more details, we refere the readers to [8; 9] and references therein.

PROPOSITION 3.1. *Let H be a closed subgroup of a compact group G. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.*

THEOREM 3.2. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ has a unique extension to a norm-decreasing linear map from $L^p(G)$ onto $L^p(G/H, \mu)$.*

COROLLARY 3.3. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. Let $\varphi \in L^p(G/H, \mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^p(G)$ with*

$$\|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \tag{3.1}$$

Let $\mathcal{J}^2(G, H) := \{f \in L^2(G) : T_H(f) = 0\}$ and $\mathcal{J}^2(G, H)^\perp$ be the orthogonal completion of the closed subspace $\mathcal{J}^2(G, H)$ in $L^2(G)$.

PROPOSITION 3.4. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then $T_H : L^2(G) \to L^2(G/H, \mu)$ is a partial isometric linear map.*

COROLLARY 3.5. *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G, H)$ and $\mathcal{J}^2(G, H)^\perp$ respectively. Then, for $f \in L^2(G)$ and $x \in G$, we have*

$$\begin{aligned} P_{\mathcal{J}^2(G,H)^\perp}(f)(x) &= T_H(f)(xH), \\ P_{\mathcal{J}^2(G,H)}(f)(x) &= f(x) - T_H(f)(xH). \end{aligned} \tag{3.2}$$

COROLLARY 3.6. *Let H be a compact subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*:

1. $\mathcal{J}^2(G, H)^\perp = \{\psi_q = \psi \circ q : \psi \in L^2(G/H, \mu)\}$.
2. *For $f \in \mathcal{J}^2(G, H)^\perp$ and $h \in H$, we have $R_h f = f$.*
3. *For $\psi \in L^2(G/H, \mu)$, we have $\|\psi_q\|_{L^2(G)} = \|\psi\|_{L^2(G/H,\mu)}$.*
4. *For $f, g \in \mathcal{J}^2(G, H)^\perp$, we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

REMARK 3.7. Invoking Corollary 3.6, one can regard the Hilbert space $L^2(G/H, \mu)$ as a closed subspace of $L^2(G)$, that is, the closed subspace consists of all $f \in L^2(G)$, which satisfies $R_h f = f$ for all $h \in H$. Then Theorem 3.2 and Proposition 3.4 guarantee that the linear map

$$T_H : L^2(G) \to L^2(G/H, \mu) \subset L^2(G)$$

is an orthogonal projection onto $L^2(G/H, \mu)$.

The following results present L^∞-properties of the linear map T_H.

THEOREM 3.8. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ has a unique extension to a norm-decreasing linear map from $L^\infty(G)$ onto $L^\infty(G/H, \mu)$.*

Proof. Let $f \in \mathcal{C}(G)$. Then it is easy to see that

$$|T_H(f)(xH)| \leq \|f\|_\infty$$

for μ-almost $xH \in G/H$. Thus, we deduce that $\|T_H(f)\|_\infty \leq \|f\|_\infty$. Hence, we can extend T_H to a norm-decreasing linear operator from $L^\infty(G)$ onto $L^\infty(G/H, \mu)$, which still will be denoted by T_H. □

COROLLARY 3.9. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Let $\varphi \in L^\infty(G/H, \mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^\infty(G)$ with*

$$\|\varphi_q\|_{L^\infty(G)} = \|\varphi\|_{L^\infty(G/H,\mu)}. \tag{3.3}$$

4. Abstract Fourier Transforms over Homogeneous Spaces of Compact Groups

Throughout this section, we briefly present the abstract notion of operator-valued Fourier transforms over homogeneous spaces of compact groups, for proofs and more details we refer the readers to [7; 10; 11; 12; 13]. It is still assumed that H is a closed subgroup of a compact group G and μ is the normalized G-invariant measure on the compact homogeneous space G/H.

For a closed subgroup H of G, define

$$H^\perp := \{[\pi] \in \widehat{G} : \pi(h) = I \text{ for all } h \in H\}. \tag{4.1}$$

If G is Abelian, each closed subgroup H of G is normal, and the locally compact group G/H is Abelian, then the character group $\widehat{G/H}$ is the set of all characters (one-dimensional irreducible representations) of G, which are constant on H, that is, precisely $H^\perp$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$, which is the set of all unitary equivalence classes of unitary representations of G/H, has meaning and it is well-defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi : \widehat{G/H} \to H^\perp$ defined by $\sigma \mapsto \Phi(\sigma) := \sigma \circ q$ is a Borel isomorphism and $\widehat{G/H} = H^\perp$, see [5; 15]. Thus, if H is normal, $H^\perp$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

For a closed subgroup H of G and a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^\pi := \int_H \pi(h)\, dh, \tag{4.2}$$

where the operator valued integral (4.2) is considered in the weak sense.

In other words,

$$\langle T_H^\pi \zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle\, dh \quad \text{for } \zeta, \xi \in \mathcal{H}_\pi. \tag{4.3}$$

The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Therefore, T_H^π is a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^\pi\| \leq 1$.

The following proposition states basic properties of the linear operator T_H^π.

PROPOSITION 4.1. *Let H be a closed subgroup of a compact group G and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Then the linear operator T_H^π is an orthogonal projection.*

The next definition presents the abstract notion of dual homogeneous space $\widehat{G/H}$.

DEFINITION 4.2. Let H be a closed subgroup of a compact group G. Then we define the dual space of G/H as the subset of $\widehat{G}$, which is given by

$$\widehat{G/H} := \{[\pi] \in \widehat{G} : T_H^\pi \neq 0\} = \left\{[\pi] \in \widehat{G} : \int_H \pi(h)\, dh \neq 0\right\}. \tag{4.4}$$

Evidently, any closed subgroup H of G satisfies

$$H^\perp \subseteq \widehat{G/H}. \tag{4.5}$$

Let $\varphi \in L^1(G/H, \mu)$ and $[\pi] \in \widehat{G/H}$. The Fourier transform of φ at $[\pi]$ is defined as the operator

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \widehat{\varphi}(\pi) := \int_{G/H} \varphi(xH)\Gamma_\pi(xH)^*\, d\mu(xH) \tag{4.6}$$

on the Hilbert space $\mathcal{H}_\pi$, where for $xH \in G/H$ the notation $\Gamma_\pi(xH)$ stands for the bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\langle \zeta, \Gamma_\pi(xH)\xi\rangle = \langle \zeta, \pi(x)T_H^\pi \xi\rangle \tag{4.7}$$

for all $\zeta, \xi \in \mathcal{H}_\pi$.

The operator-valued integral (4.6) is also considered in the weak sense. That is,

$$\langle \widehat{\varphi}(\pi)\zeta, \xi\rangle = \int_{G/H} \varphi(xH)\langle \Gamma_\pi(xH)^*\zeta, \xi\rangle\, d\mu(xH) \tag{4.8}$$

for all $\zeta, \xi \in \mathcal{H}_\pi$.

In other words, for $[\pi] \in \widehat{G/H}$ and $\zeta, \xi \in \mathcal{H}_\pi$, we have

$$\langle \widehat{\varphi}(\pi)\zeta, \xi\rangle = \int_{G/H} \varphi(xH)\langle \zeta, \pi(x)T_H^\pi \xi\rangle\, d\mu(xH). \tag{4.9}$$

Therefore, $\widehat{\varphi}(\pi)$ is a bounded linear operator on $\mathcal{H}_\pi$ satisfying

$$\|\widehat{\varphi}(\pi)\| \leq \|\varphi\|_{L^1(G/H,\mu)}. \tag{4.10}$$

From now on, we may use $\widehat{\varphi}(\pi)$ or $\mathcal{F}_{G/H}(\varphi)(\pi)$ at times.

The following propositions give us the connection of the Fourier transform over the homogeneous space G/H with the Fourier transform on the group G.

PROPOSITION 4.3. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Let $\varphi \in L^1(G/H, \mu)$ and $[\pi] \in \widehat{G/H}$. Then*

$$\mathcal{F}_{G/H}(\varphi)(\pi) = \mathcal{F}_G(\varphi_q)(\pi). \tag{4.11}$$

PROPOSITION 4.4. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*:

1. *For $f \in L^1(G)$ and $[\pi] \in \widehat{G/H}$, we have* $\widehat{T_H(f)}(\pi) = T_H^\pi \widehat{f}(\pi)$.
2. *For $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$, we have* $T_H^\pi \widehat{\varphi}(\pi) = \widehat{\varphi}(\pi)$.
3. *For $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$, we have* $\mathrm{tr}[\widehat{\varphi}(\pi)T_H^\pi] = \mathrm{tr}[\widehat{\varphi}(\pi)]$.

DEFINITION 4.5. Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The linear map $\mathcal{F}_{G/H}$: $L^1(G/H,\mu) \to \mathfrak{C}(\widehat{G/H})$ is called as *abstract operator-valued Fourier transform* over the compact homogeneous space G/H.

THEOREM 4.6. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Fourier transform*

$$\mathcal{F}_{G/H} : L^1(G/H,\mu) \to \mathfrak{C}_0(\widehat{G/H})$$

is a norm-decreasing homomorphism onto a subalgebra of $\mathfrak{C}_0(\widehat{G/H})$.

PROPOSITION 4.7. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Let $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G}$ be such that $[\pi] \notin \widehat{G/H}$. Then we have*

$$\widehat{\varphi_q}(\pi) = 0. \tag{4.12}$$

Proof. Let $\varphi \in L^1(G/H,\mu)$ be given. Then we have $\varphi_q \in L^1(G)$. If $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$, then we have $T_H^\pi = 0$. Hence, for $\zeta, \xi \in \mathcal{H}_\pi$, we have $T_H(\pi_{\xi,\zeta}) = 0$. Therefore, using Weil's formula, for $\zeta, \xi \in \mathcal{H}_\pi$, we get

$$\langle \widehat{\varphi_q}(\pi)\zeta, \xi\rangle = \int_{G/H} \varphi(xH)\overline{T_H(\pi_{\xi,\zeta})(xH)}\, d\mu(xH) = 0,$$

which implies (4.12). □

Since $L^2(G/H,\mu) \subset L^1(G/H,\mu)$, the Fourier transform defined in (4.6) is well defined for L^2-functions over the homogeneous space G/H.

The next result shows that the Fourier transform defined in (4.6) satisfies a generalized version of the Plancherel formula in L^2-sense.

THEOREM 4.8. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Each $\varphi \in L^2(G/H,\mu)$ satisfies the Plancherel formula*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_2^2 = \|\varphi\|^2_{L^2(G/H,\mu)}. \tag{4.13}$$

COROLLARY 4.9. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then*:

1. *For $\varphi, \psi \in L^2(G/H,\mu)$, we have*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi\, \mathrm{tr}[\widehat{\varphi}(\pi)\widehat{\psi}(\pi)^*] = \langle \varphi, \psi\rangle_{L^2(G/H,\mu)}. \tag{4.14}$$

2. *For $f, g \in \mathcal{J}^2(G,H)^{\perp}$, we have*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{f}(\pi)\widehat{g}(\pi)^*] = \langle f, g\rangle_{L^2(G)}. \tag{4.15}$$

THEOREM 4.10. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The range of the Fourier transform $\mathcal{F}_{G/H} : L^2(G/H,\mu) \to \mathfrak{C}^2(\widehat{G/H})$ is the closed subspace*

$$\mathfrak{R}^2(G/H) := \{\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^2(\widehat{G/H}) | T_H^\pi T_\pi = T_\pi \text{ for all } \pi \in \widehat{G/H}\}.$$

COROLLARY 4.11. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The abstract Fourier transform $\mathcal{F}_{G/H} : L^2(G/H,\mu) \to \mathfrak{R}^2(G/H)$ is a unitary isomorphism of Hilbert spaces.*

COROLLARY 4.12. *Let H be a closed subgroup of a compact group G. Then $\widehat{T_H}$ maps $\mathfrak{C}^2(\widehat{G})$ onto $\mathfrak{R}^2(G/H)$.*

DEFINITION 4.13. The inverse Fourier transform $\mathbf{T} \to \check{\mathbf{T}}$ from $\mathfrak{R}^2(G/H)$ onto $L^2(G/H,\mu)$ is defined as the inverse of the Fourier transform $\varphi \to \widehat{\varphi}$, which maps $L^2(G/H,\mu)$ isometrically onto $\mathfrak{R}^2(G/H)$.

Plainly $\mathbf{T} \to \check{\mathbf{T}}$ is a unitary isomorphism of Hilbert spaces.

The following theorem shows that the abstract Fourier transform

$$\mathcal{F}_{G/H} : L^2(G/H,\mu) \to \mathfrak{C}^2(\widehat{G/H})$$

is surjective only when H is normal in G.

THEOREM 4.14. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The abstract Fourier transform $\mathcal{F}_{G/H}$ maps $L^2(G/H,\mu)$ onto $\mathfrak{C}^2(\widehat{G/H})$ if and only if H is normal in G.*

REMARK 4.15. Theorem 4.14 can be considered as an abstract characterization of the algebraic structure of the homogeneous space G/H via analytic and topological aspects.

We hereby finish the section by the following useful results concerning abstract harmonic analysis over the dual homogeneous space $\widehat{G/H}$.

For $\mathbf{T} = (T_\pi)_{[\pi]\in\widehat{G}} \in \mathfrak{C}(\widehat{G})$, let $\mathbf{T}_H := (T_H^\pi T_\pi)_{[\pi]\in\widehat{G/H}} \in \mathfrak{C}(\widehat{G/H})$. Then

$$\widehat{T_H} : \mathfrak{C}(\widehat{G}) \to \mathfrak{C}(\widehat{G/H})$$

given by $\mathbf{T} \mapsto \mathbf{T}_H$ is a well-defined linear operator.

The abstract operator-valued Fourier transform over G/H satisfies

$$\mathcal{F}_{G/H} \circ T_H = \widehat{T_H} \circ \mathcal{F}_G. \tag{4.16}$$

Next we show that the linear map $\widehat{T_H} : \mathfrak{C}^p(\widehat{G}) \to \mathfrak{C}^p(\widehat{G/H})$ is norm-decreasing for all $1 \le p \le \infty$.

THEOREM 4.16. *Let H be a closed subgroup of a compact group G and $p \geq 1$. Then*

$$\widehat{T_H} : \mathfrak{C}^p(\widehat{G}) \to \mathfrak{C}^p(\widehat{G/H})$$

is a norm-decreasing linear operator.

Proof. Let $\mathbf{T} = (T_\pi)_{[\pi]\in\widehat{G}} \in \mathfrak{C}^p(\widehat{G})$. We then have

$$\begin{aligned}\|\widehat{T_H}(\mathbf{T})\|^p_{\mathfrak{C}^p(\widehat{G/H})} &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|T_H^\pi T_\pi\|_p^p \\ &\leq \sum_{[\pi]\in\widehat{G/H}} d_\pi \|T_\pi\|_p^p \\ &\leq \sum_{[\pi]\in\widehat{G}} d_\pi \|T_\pi\|_p^p = \|\mathbf{T}\|^p_{\mathfrak{C}^p(\widehat{G})}.\end{aligned}$$

□

THEOREM 4.17. *Let H be a closed subgroup of a compact group G. Then*

$$\widehat{T_H} : \mathfrak{C}^\infty(\widehat{G}) \to \mathfrak{C}^\infty(\widehat{G/H})$$

is a norm-decreasing linear operator.

Proof. Let $\mathbf{T} = (T_\pi)_{[\pi]\in\widehat{G}} \in \mathfrak{C}^\infty(\widehat{G})$. Then we get

$$\begin{aligned}\|\widehat{T_H}(\mathbf{T})\|_{\mathfrak{C}^\infty(\widehat{G/H})} &= \sup_{[\pi]\in\widehat{G/H}} \|T_H^\pi T_\pi\|_\infty \\ &\leq \sup_{[\pi]\in\widehat{G/H}} \|T_H^\pi\|_\infty \|T_\pi\|_\infty \\ &\leq \sup_{[\pi]\in\widehat{G/H}} \|T_\pi\|_\infty \\ &\leq \sup_{[\pi]\in\widehat{G}} \|T_\pi\|_\infty = \|\mathbf{T}\|_{\mathfrak{C}^\infty(\widehat{G})}.\end{aligned}$$

□

The following results present other interesting properties of the linear map $\widehat{T_H} : \mathfrak{C}^\infty(\widehat{G}) \to \mathfrak{C}^\infty(\widehat{G/H})$.

THEOREM 4.18. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then, for $f \in L^1(G)$, we have*

$$\|\widehat{T_H(f)}\|_{\mathfrak{C}^\infty(\widehat{G/H})} \leq \|\widehat{f}\|_{\mathfrak{C}^\infty(\widehat{G})}.$$

Proof. Let $f \in L^1(G)$ and $[\pi] \in \widehat{G/H}$. Then we have

$$\begin{aligned}\|\widehat{T_H(f)}(\pi)\| &= \|T_H^\pi \widehat{f}(\pi)\| \\ &\leq \|T_H^\pi\| \|\widehat{f}(\pi)\| \\ &\leq \|\widehat{f}(\pi)\| \\ &\leq \sup_{[\sigma]\in\widehat{G}} \|\widehat{f}(\sigma)\| = \|\widehat{f}\|_{\mathfrak{C}^\infty(\widehat{G})}.\end{aligned}$$

Thus, we deduce that

$$\|\widehat{T_H(f)}\|_{\mathfrak{C}^\infty(\widehat{G/H})} = \sup_{[\pi]\in\widehat{G/H}} \|\widehat{T_H(f)}(\pi)\| \leq \|\widehat{f}\|_{\mathfrak{C}^\infty(\widehat{G})}.$$

□

COROLLARY 4.19. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then, for $\varphi \in L^1(G/H,\mu)$, we have*

$$\|\widehat{\varphi}\|_{\mathfrak{C}^\infty(\widehat{G/H})} = \|\widehat{\varphi_q}\|_{\mathfrak{C}^\infty(\widehat{G})}.$$

Proof. Let $\varphi \in L^1(G/H,\mu)$. Then, using Proposition 4.3 and also (4.12), we get

$$\begin{aligned}
\|\widehat{\varphi_q}\|_{\mathfrak{C}^\infty(\widehat{G})} &= \sup_{[\pi]\in\widehat{G}} \|\widehat{\varphi_q}(\pi)\| \\
&= \sup_{[\pi]\in\widehat{G/H}} \|\widehat{\varphi_q}(\pi)\| \\
&= \sup_{[\pi]\in\widehat{G/H}} \|\widehat{\varphi}(\pi)\| = \|\widehat{\varphi}\|_{\mathfrak{C}^\infty(\widehat{G/H})}.
\end{aligned}$$

□

5. Absolutely Convergent Fourier Series of Functions over Homogeneous Spaces of Compact Groups

Throughout this section we study analytic aspects of absolutely convergent Fourier series of functions over homogeneous spaces of compact groups. We first introduce the abstract notion of the linear subspace of all functions with absolutely convergent Fourier series over homogeneous spaces of compact groups, and then we study basic properties of these spaces in the framework of abstract harmonic analysis. We shall also present some characterizations for these spaces.

It is still assumed that H is a closed subgroup of a compact group G and μ is the normalized G-invariant measure on the compact homogeneous space G/H.

A function $\varphi \in L^1(G/H,\mu)$ is said to have an absolutely convergent Fourier series if

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_1 < \infty. \tag{5.1}$$

The linear space consisting of all functions with absolutely convergent Fourier series over the homogeneous spaces G/H is denoted by $\mathcal{R}(G/H)$. Thus, it is easy to see that

$$\mathcal{R}(G/H) = \{\varphi \in L^1(G/H,\mu) : \widehat{\varphi} \in \mathfrak{C}^1(\widehat{G/H})\}.$$

Also, it can be readily checked that

$$\varphi \mapsto \|\varphi\|_{(1)} := \|\widehat{\varphi}\|_{\mathfrak{C}^1(\widehat{G/H})} = \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_1$$

defines a norm on the linear space $\mathcal{R}(G/H)$.

We then have the following interesting observations.

PROPOSITION 5.1. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. The abstract Fourier transform $\varphi \mapsto \widehat{\varphi}$ is a linear isometric isomorphism of $\mathcal{R}(G/H)$ into $\mathfrak{C}^1(\widehat{G/H})$.*

COROLLARY 5.2. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. The linear space $\mathcal{R}(G/H)$ with respect to $\|\cdot\|_{(1)}$ is a Banach space.*

The next theorem shows that the linear operator T_H maps $\mathcal{R}(G)$ into $\mathcal{R}(G/H)$.

THEOREM 5.3. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. The linear operator T_H maps $\mathcal{R}(G)$ into $\mathcal{R}(G/H)$.*

Proof. Let $f \in \mathcal{R}(G)$. Using Proposition 4.4, we have

$$\widehat{T_H(f)}(\pi) = T_H^\pi \widehat{f}(\pi),$$

for all $[\pi] \in \widehat{G/H}$. Since T_H^π is an orthogonal projection, we get

$$\begin{aligned}\|\widehat{T_H(f)}(\pi)\|_1 &= \|T_H^\pi \widehat{f}(\pi)\|_1 \\ &\leq \|T_H^\pi\| \|\widehat{f}(\pi)\|_1 \leq \|\widehat{f}(\pi)\|_1.\end{aligned}$$

Let Ω be a finite subset of the dual space $\widehat{G/H}$. Then we get

$$\sum_{[\pi]\in\Omega} d_\pi \|\widehat{T_H(f)}(\pi)\|_1 \leq \sum_{[\pi]\in\Omega} d_\pi \|\widehat{f}(\pi)\|_1.$$

Since each finite subset of $\widehat{G/H}$ is a finite subset of $\widehat{G}$ as well, we have

$$\sum_{[\pi]\in\Omega} d_\pi \|\widehat{f}(\pi)\|_1 \leq \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_1 = \|f\|_{(1)}. \tag{5.2}$$

Invoking the fact that Ω was an arbitrary finite subset of $\widehat{G/H}$, we deduce that the following series

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{T_H(f)}(\pi)\|_1$$

converges, which implies that $T_H(f) \in \mathcal{R}(G/H)$. □

Then we deduce the following consequences.

COROLLARY 5.4. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Then, for $f \in \mathcal{R}(G)$, we have*

$$\|T_H(f)\|_{(1)} \leq \|f\|_{(1)}.$$

Proof. Let $f \in \mathcal{R}(G)$. Then, using (5.2), we have

$$\|T_H(f)\|_{(1)} = \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{T_H(f)}(\pi)\|_1$$
$$= \sup_{\substack{\Omega\subset\widehat{G/H}\\ \text{finite}}} \sum_{[\pi]\in\Omega} d_\pi \|\widehat{T_H(f)}(\pi)\|_1 \leq \|f\|_{(1)}.$$

□

COROLLARY 5.5. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H and $\varphi \in \mathcal{R}(G/H)$. Then $\varphi_q \in \mathcal{R}(G)$ and*

$$\|\varphi\|_{(1)} = \|\varphi_q\|_{(1)}. \tag{5.3}$$

Proof. Let $\varphi \in \mathcal{R}(G/H)$. Since $\varphi \in L^1(G/H, \mu)$, we have $\varphi_q \in L^1(G)$. Let Ω be a finite subset of the dual space $\widehat{G}$. Then, using (4.12), we have

$$\sum_{[\pi]\in\Omega} d_\pi \|\widehat{\varphi_q}(\pi)\|_1 = \sum_{[\pi]\in\Omega\cap\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_1 + \sum_{[\pi]\in\Omega-\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_1$$
$$= \sum_{[\pi]\in\Omega\cap\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_1$$
$$= \sum_{[\pi]\in\Omega\cap\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_1.$$

Since $\Omega \cap \widehat{G/H}$ is a finite subset of $\widehat{G/H}$, we get

$$\sum_{[\pi]\in\Omega\cap\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_1 \leq \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_1 = \|\varphi\|_{(1)}.$$

Thus, we conclude that

$$\sum_{[\pi]\in\Omega} d_\pi \|\widehat{\varphi_q}(\pi)\|_1 \leq \|\varphi\|_{(1)}.$$

Invoking the fact that Ω was an arbitrary finite subset of $\widehat{G}$, we deduce that the series

$$\sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{\varphi_q}(\pi)\|_1$$

converges and hence $\varphi_q \in \mathcal{R}(G)$ with

$$\|\varphi_q\|_{(1)} = \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{\varphi_q}(\pi)\|_1 \leq \|\varphi\|_{(1)}. \tag{5.4}$$

Using Corollary 5.4 and since $T_H(\varphi_q) = \varphi$, we achieve

$$\|\varphi\|_{(1)} = \|T_H(\varphi_q)\|_{(1)} \leq \|\varphi_q\|_{(1)}. \tag{5.5}$$

Then (5.4) and (5.5) guarantee (5.3). □

The next result summarizes basic properties of the linear map $T_H : \mathcal{R}(G) \to \mathcal{R}(G/H)$.

COROLLARY 5.6. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Then $T_H : \mathcal{R}(G) \to \mathcal{R}(G/H)$ is a norm-decreasing surjective linear operator.*

In the following we present some characterizations for elements of the linear space $\mathcal{R}(G/H)$.

THEOREM 5.7. *Let G be a compact group and H be a closed subgroup of G. Let μ be the normalized G-invariant measure over the homogeneous space G/H and $\varphi \in L^1(G/H, \mu)$. Then $\varphi \in \mathcal{R}(G/H)$ if and only if there exists a constant $B > 0$ such that*

$$\left| \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi) T_\pi^*] \right| \le B \|\mathbf{T}\|_{\mathfrak{C}^\infty(\widehat{G/H})} \tag{5.6}$$

for all $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}_{00}(\widehat{G/H})$. In this case, we have $\|\varphi\|_{(1)} \le B$.

Proof. Let $\varphi \in \mathcal{R}(G/H)$. Then we have $\varphi_q \in \mathcal{R}(G)$. Thus, using Theorem 34.29 of [16], there exists a constant $B > 0$ such that

$$\left| \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi) S_\pi^*] \right| \le B \|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})} \tag{5.7}$$

for all $\mathbf{S} = (S_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}_{00}(\widehat{G})$ and $\|\varphi_q\|_{(1)} \le B$. Now let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}_{00}(\widehat{G/H})$ be given. Define $\mathbf{S} := (S_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}_{00}(\widehat{G})$ such that $S_\pi = 0$ for $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$ and $S_\pi = T_\pi$ for all $[\pi] \in \widehat{G/H}$. Then it can be readily checked that $\|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})} = \|\mathbf{T}\|_{\mathfrak{C}^\infty(\widehat{G/H})}$. Applying (5.7), we achieve

$$\left| \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi) S_\pi^*] \right| \le B \|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})} = B \|\mathbf{T}\|_{\mathfrak{C}^\infty(\widehat{G/H})}. \tag{5.8}$$

Thus, using (5.8), we can write

$$\begin{aligned}\left| \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi) T_\pi^*] \right| &= \left| \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi) T_\pi^*] \right| \\ &= \left| \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi) T_\pi^*] \right| \le B \|\mathbf{T}\|_{\mathfrak{C}^\infty(\widehat{G/H})},\end{aligned}$$

which implies (5.6). Also, using (5.3), we deduce that

$$\|\varphi\|_{(1)} = \|\varphi_q\|_{(1)} \le B.$$

Conversely, suppose that there exists a constant $B > 0$ such that inequality (5.6) holds for all $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}_{00}(\widehat{G/H})$. Now we claim that $\varphi_q \in \mathcal{R}(G)$. To this end, let $\mathbf{S} = (S_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}_{00}(\widehat{G})$ be given. Then $\mathbf{S}_H := (T_H^\pi S_\pi)_{[\pi]\in\widehat{G/H}} \in$

$\mathfrak{C}_{00}(\widehat{G/H})$ with $\|\mathbf{S}_H\|_{\mathfrak{C}^\infty(\widehat{G/H})} \leq \|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})}$. Applying (5.6) to $\mathbf{S}_H$, we get

$$\left|\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)S_\pi^* T_H^\pi]\right| \leq B\|\mathbf{S}_H\|_{\mathfrak{C}^\infty(\widehat{G/H})} \leq B\|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})}. \tag{5.9}$$

Then, using (5.9), we have

$$\begin{aligned}
\left|\sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)S_\pi^*]\right| &= \left|\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)S_\pi^*]\right| \\
&= \left|\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)S_\pi^*]\right| \\
&= \left|\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_H^\pi\widehat{\varphi}(\pi)S_\pi^*]\right| \\
&= \left|\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)S_\pi^* T_H^\pi]\right| \leq B\|\mathbf{S}\|_{\mathfrak{C}^\infty(\widehat{G})}.
\end{aligned}$$

Since $\mathbf{S} = (S_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}_{00}(\widehat{G})$ was arbitrary, Theorem 34.29 of [16] implies that $\varphi_q \in \mathcal{R}(G)$ with $\|\varphi_q\|_{(1)} \leq B$. Then Theorem 5.3 and equation (3.3) guarantee that $\varphi = T_H(\varphi_q) \in \mathcal{R}(G/H)$ with

$$\|\varphi\|_{(1)} = \|\varphi\|_{(1)} \leq B. \qquad \square$$

We then deduce the following consequence.

Corollary 5.8. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\varphi \in L^1(G/H,\mu)$ and $\varphi(xH) = \lim_n \varphi_n(xH)$ for μ-almost all $xH \in G/H$, where each $\varphi_n \in \mathcal{R}(G/H)$ with*

$$B := \sup_n \|\varphi\|_{(1)} < \infty.$$

Then $\varphi \in \mathcal{R}(G/H)$ with $\|\varphi\|_{(1)} \leq B$.

We can also present the following characterization.

Proposition 5.9. *Let G be a compact group and H be a closed subgroup of G. Let μ be the normalized G-invariant measure over the homogeneous space G/H and $\varphi \in L^\infty(G/H,\mu)$. Then $\varphi \in \mathcal{R}(G/H)$ if and only if there exists a constant $B > 0$ such that*

$$\left|\int_{G/H} \varphi(xH)\overline{\psi(xH)}\,d\mu(xH)\right| \leq B\|\widehat{\psi}\|_{\mathfrak{C}^\infty(\widehat{G/H})} \tag{5.10}$$

for all $\psi \in L^1(G/H,\mu)$. In this case, we have $\|\varphi\|_{(1)} \leq B$.

Proof. Let $\varphi \in \mathcal{R}(G/H)$. Then we have $\varphi_q \in \mathcal{R}(G)$. Using Corollary 34.31 of [16], we deduce that

$$\left|\int_G \varphi_q(x)\overline{g(x)}\,dx\right| \le \|\varphi_q\|_{(1)}\|\widehat{g}\|_{\mathfrak{C}^\infty(\widehat{G})} \tag{5.11}$$

for all $g \in L^1(G)$. Now let $\psi \in L^1(G/H,\mu)$. Then, using Weil's formula, Corollary 4.19, (5.3), and (5.11), we get

$$\begin{aligned}
\left|\int_{G/H} \varphi(xH)\overline{\psi(xH)}\,d\mu(xH)\right| &= \left|\int_{G/H} \varphi(xH)\overline{T_H(\psi_q)(xH)}\,d\mu(xH)\right| \\
&= \left|\int_{G/H} \overline{T_H(\overline{\varphi_q}\cdot\psi_q)(xH)}\,d\mu(xH)\right| \\
&= \left|\int_{G/H} T_H(\varphi_q\cdot\overline{\psi_q})(xH)\,d\mu(xH)\right| \\
&= \left|\int_G \varphi_q(x)\cdot\overline{\psi_q(x)}\,dx\right| \\
&\le \|\varphi_q\|_{(1)}\|\widehat{\psi_q}\|_{\mathfrak{C}^\infty(\widehat{G})} \\
&= \|\varphi_q\|_{(1)}\|\widehat{\psi}\|_{\mathfrak{C}^\infty(\widehat{G/H})} = \|\varphi\|_{(1)}\|\widehat{\psi}\|_{\mathfrak{C}^\infty(\widehat{G/H})},
\end{aligned}$$

which implies (5.10). Conversely, assume that there exists a constant $B > 0$ such that (5.10) holds for all $\psi \in L^1(G/H,\mu)$. Now we claim that $\varphi_q \in \mathcal{R}(G)$. To this end, let $g \in L^1(G)$. Then, using Weil's formula and Proposition 4.18, we have

$$\begin{aligned}
\left|\int_G \varphi_q(x)\overline{g(x)}\,dx\right| &= \left|\int_{G/H} T_H(\varphi_q\cdot\overline{g})(xH)\,d\mu(xH)\right| \\
&= \left|\int_{G/H} \varphi(xH)T_H(\overline{g})(xH)\,d\mu(xH)\right| \\
&= \left|\int_{G/H} \varphi(xH)\overline{T_H(g)(xH)}\,d\mu(xH)\right| \\
&\le B\|\widehat{T_H(g)}\|_{\mathfrak{C}^\infty(\widehat{G/H})} \le B\|\widehat{g}\|_{\mathfrak{C}^\infty(\widehat{G/H})}.
\end{aligned}$$

Thus, using Corollary 34.31 of [16], we get $\varphi_q \in \mathcal{R}(G)$ with $\|\varphi_q\|_{(1)} \le B$. Therefore, Theorem 5.3 implies that $\varphi = T_H(\varphi_q) \in \mathcal{R}(G/H)$ with

$$\|\varphi\|_{(1)} = \|\varphi_q\|_{(1)} \le B. \qquad \square$$

Next we study analytic aspects of general Fourier series over the homogeneous space G/H.

THEOREM 5.10. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H and* $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^1(\widehat{G/H})$.

1. *The nonnegative real-valued series*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi |\operatorname{tr}[T_\pi \pi(x) T_H^\pi]|$$

converges uniformly on the homogeneous space G/H.

2. *The series*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x) T_H^\pi]$$

converges uniformly on the homogeneous spaces G/H to a continuous function.

Proof. Let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^1(G/H)$ be given.

(1) Let $[\pi] \in \widehat{G/H}$ and $x \in G$. Then we can write

$$|\operatorname{tr}[T_\pi \pi(x) T_H^\pi]| \leq \|T_\pi\|_1 \|\pi(x) T_H^\pi\| = \|T_\pi\|_1 \|T_H^\pi\| \leq \|T_\pi\|_1. \tag{5.12}$$

Now let Ω be a finite subset of the dual space $\widehat{G/H}$. Using (5.12), for $x \in G$, we get

$$\begin{aligned}\sum_{[\pi]\in\Omega} d_\pi |\operatorname{tr}[T_\pi \pi(x) T_H^\pi]| &\leq \sum_{[\pi]\in\Omega} d_\pi \|T_\pi\|_1 \\ &\leq \sum_{[\pi]\in\widehat{G/H}} d_\pi \|T_\pi\|_1 = \|\mathbf{T}\|_{\mathfrak{C}^1(\widehat{G/H})}.\end{aligned}$$

Thus, we deduce that the nonnegative real-valued series

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi |\operatorname{tr}[T_\pi \pi(x) T_H^\pi]|$$

converges uniformly on the homogeneous space G/H, and we have

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi |\operatorname{tr}[T_\pi \pi(x) T_H^\pi]| \leq \|\mathbf{T}\|_{\mathfrak{C}^1(\widehat{G/H})} \tag{5.13}$$

for all $x \in G$.

(2) Let $\widetilde{\mathbf{T}} = (\widetilde{T}_\pi)_{\pi\in\widehat{G}} \in \mathfrak{C}^1(\widehat{G})$ such that $\widetilde{T}_\pi = 0$ for $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$ and $\widetilde{T}_\pi = T_\pi$ for all $[\pi] \in \widehat{G/H}$. Then, using Theorem 34.5 of [16], the nonnegative real-valued series

$$\sum_{[\pi]\in\widehat{G}} d_\pi |\operatorname{tr}[\widetilde{T}_\pi \pi(x)]| = \sum_{[\pi]\in\widehat{G/H}} d_\pi |\operatorname{tr}[T_\pi \pi(x)]|$$

converges uniformly on G, and also the series

$$\sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widetilde{T}_\pi \pi(x)] = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x)]$$

converges uniformly on G to a continuous function, namely $f \in \mathcal{C}(G)$. Thus, we have

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widetilde{T}_\pi \pi(x)] = \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x)] \tag{5.14}$$

for all $x \in G$. Let $\varphi := T_H(f) \in \mathcal{C}(G/H)$. Then, for $x \in G$, we have

$$\begin{aligned}
T_H(f)(xH) &= \int_H f(xh)\,dh \\
&= \int_H \Bigg(\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(xh)] \Bigg) dh \\
&= \int_H \Bigg(\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x)\pi(h)] \Bigg) dh \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \left(\int_H \operatorname{tr}[T_\pi \pi(x)\pi(h)]\,dh \right) \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \left(\operatorname{tr}\left[\int_H T_\pi \pi(x)\pi(h)\,dh \right] \right) \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \left(\operatorname{tr}\left[T_\pi \pi(x) \left(\int_H \pi(h)\,dh \right) \right] \right) \\
&= \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x) T_H^\pi],
\end{aligned}$$

which implies that the series

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x) T_H^\pi]$$

converges uniformly on the homogeneous spaces G/H to the continuous function φ. □

We then conclude the following corollaries.

COROLLARY 5.11. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^1(\widehat{G/H})$ and*

$$\varphi(xH) := \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi \pi(x) T_H^\pi], \quad \forall xH \in G/H.$$

Then we have $\varphi \in \mathcal{R}(G/H)$.

Proof. Let $f \in \mathcal{C}(G)$ be the function given by equation (5.14). Then, by Theorem 34.5 of [16], we have $f \in \mathcal{R}(G)$. Thus, using Theorem 5.3, we get $\varphi = T_H(f) \in \mathcal{R}(G/H)$. □

COROLLARY 5.12. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\varphi \in \mathcal{R}(G/H)$. Then φ is equal μ-almost everywhere on G/H to the continuous function given by*

$$xH \mapsto \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)\pi(x)].$$

Let $\mathfrak{R}^1(G/H)$ be the linear subspace of $\mathfrak{C}^1(\widehat{G/H})$ given by

$$\mathfrak{R}^1(G/H) := \{\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{C}^1(\widehat{G/H}) | T_H^\pi T_\pi = T_\pi \text{ for all } \pi \in \widehat{G/H}\}.$$

PROPOSITION 5.13. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{R}^1(\widehat{G/H})$. Then the nonnegative real-valued series*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi |\operatorname{tr}[T_\pi\pi(x)]|$$

converges uniformly on the homogeneous space G/H. Hence, the series

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi\pi(x)]$$

converges uniformly on the homogeneous spaces G/H to a continuous function.

Then we conclude the following corollary.

COROLLARY 5.14. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\mathbf{T} = (T_\pi)_{\pi\in\widehat{G/H}} \in \mathfrak{R}^1(\widehat{G/H})$ and*

$$\varphi(xH) := \sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[T_\pi\pi(x)], \quad \forall xH \in G/H.$$

Then $\varphi \in \mathcal{R}(G/H)$, and we have

$$\|\varphi\|_{\sup} \le \|\varphi\|_{(1)}.$$

Then we can present the following result.

THEOREM 5.15. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. The inverse Fourier transform $\mathbf{T} \mapsto \check{\mathbf{T}}$ given by (4.13) is a linear isometric isomorphism of $\mathfrak{R}^1(\widehat{G/H})$ onto $\mathcal{R}(G/H)$.*

Proof. Let $\mathbf{T} = (T_\pi)_{\pi \in \widehat{G/H}} \in \mathfrak{R}^1(G/H)$. Since $\mathfrak{R}^1(\widehat{G/H}) \subseteq \mathfrak{R}^2(\widehat{G/H})$, we have $\check{\mathbf{T}} \in L^2(G/H, \mu)$ with $\widehat{\check{\mathbf{T}}} = \mathbf{T}$. Since $L^2(G/H, \mu) \subseteq L^1(G/H, \mu)$, we get $\check{\mathbf{T}} \in L^1(G/H, \mu)$ as well. Thus, we can write

$$\sum_{[\pi] \in \widehat{G/H}} d_\pi \|\widehat{\check{\mathbf{T}}}(\pi)\|_1 = \sum_{[\pi] \in \widehat{G/H}} d_\pi \|T_\pi\|_1 = \|\mathbf{T}\|_{\mathfrak{C}^1(\widehat{G/H})},$$

which implies that $\check{\mathbf{T}} \in \mathcal{R}(G/H)$ and also $\|\mathbf{T}\|_{\mathfrak{C}^1(\widehat{G/H})} = \|\check{\mathbf{T}}\|_{(1)}$. Let $\varphi \in \mathcal{R}(G/H)$. Then we have $\widehat{\varphi} \in \mathfrak{C}^1(\widehat{G/H})$. By Proposition 4.4 we achieve that $\widehat{\varphi} \in \mathfrak{R}^1(G/H)$. Then it is straightforward to see that $\check{\widehat{\varphi}} = \varphi$. $\square$

COROLLARY 5.16. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. The inverse Fourier transform $\mathbf{T} \mapsto \check{\mathbf{T}}$ is a linear isometric isomorphism of $\mathfrak{R}^1(\widehat{G/H})$ onto $\mathcal{R}(G/H)$.*

We finish the paper by presenting the following result.

THEOREM 5.17. *Let G be a compact group, H be a closed subgroup of G, and μ be the normalized G-invariant measure over the homogeneous space G/H. Let $\varphi \in L^\infty(G/H, \mu)$ with $\mathrm{tr}[\widehat{\varphi}(\pi)] \geq 0$ for all $[\pi] \in \widehat{G/H}$. Then the series*

$$\sum_{[\pi] \in \widehat{G/H}} d_\pi \, \mathrm{tr}[\widehat{\varphi}(\pi)]$$

converges, and we have

$$\sum_{[\pi] \in \widehat{G/H}} d_\pi \, \mathrm{tr}[\widehat{\varphi}(\pi)] \leq \|\varphi\|_{L^\infty(G/H,\mu)}.$$

Proof. Let $\varphi \in L^\infty(G/H, \mu)$ with $\mathrm{tr}[\widehat{\varphi}(\pi)] \geq 0$ for all $[\pi] \in \widehat{G/H}$. Then we have $\varphi_q \in L^\infty(G)$ and also

$$\mathrm{tr}[\widehat{\varphi_q}(\pi)] = \mathrm{tr}[\widehat{\varphi}(\pi)] \geq 0$$

for all $[\pi] \in \widehat{G/H}$. Thus, using Proposition 4.3, we deduce that $\mathrm{tr}[\widehat{\varphi_q}(\pi)] \geq 0$ for all $[\pi] \in \widehat{G}$. Then Theorem 34.9 of [16], guarantees that the series $\sum_{[\pi] \in \widehat{G}} d_\pi \, \mathrm{tr}[\widehat{\varphi_q}(\pi)]$ converges and satisfies

$$\sum_{[\pi] \in \widehat{G}} d_\pi \, \mathrm{tr}[\widehat{\varphi_q}(\pi)] \leq \|\varphi_q\|_{L^\infty(G)}.$$

Let Ω be a finite subset of the dual space $\widehat{G/H}$. Then Ω is a finite subset of $\widehat{G}$ as well. Thus, we can write

$$\sum_{[\pi] \in \Omega} d_\pi \, \mathrm{tr}[\widehat{\varphi}(\pi)] = \sum_{[\pi] \in \Omega} d_\pi \, \mathrm{tr}[\widehat{\varphi_q}(\pi)] \leq \sum_{[\pi] \in \widehat{G}} d_\pi \, \mathrm{tr}[\widehat{\varphi_q}(\pi)].$$

Since Ω was arbitrary, we conclude that the series $\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)]$ converges. Then we get

$$\begin{aligned}\sum_{[\pi]\in\widehat{G/H}} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)] &= \sup_{\substack{\Omega\subset\widehat{G/H}\\ \text{finite}}} \sum_{[\pi]\in\Omega} d_\pi \operatorname{tr}[\widehat{\varphi}(\pi)] \\ &\leq \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{tr}[\widehat{\varphi_q}(\pi)] \\ &\leq \|\varphi_q\|_{L^\infty(G)} = \|\varphi\|_{L^\infty(G/H,\mu)},\end{aligned}$$

which completes the proof. □

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