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Square-integrability of multivariate metaplectic wave-packet representations

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Abstract
This paper presents a systematic study for harmonic analysis of metaplectic wave-packet representations on the Hilbert function space $L^2(\mathbb{R}^d)$. The abstract notions of symplectic wave-packet groups and metaplectic wave-packet representations will be introduced. We then present an admissibility condition on closed subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$, which guarantees the square-integrability of the associated metaplectic wave-packet representation on $L^2(\mathbb{R}^d)$.

Keywords: symplectic group, multivariate metaplectic wave-packet representations, symplectic wave-packet group, metaplectic wave-packet transform, square-integrable representations

1. Introduction

Many intresting applications of mathematical analysis in theoretical physics (e.g. paraxial optic, quantum mechanics, etc) prompt particular forms of multivariate metaplectic (Shale-Weyl) representation [14–16, 25, 41] under various names; quadratic-phase transforms, linear canonical transforms [10, 36], Fresnel transforms, fractional Fourier transforms [54], Gaussian integral [51]. In the following article, we shall approache the topic from the classical theory of coherent state transforms [3].

The abstract theory of covariant/coherent state transforms is the mathematical basis of modern high frequency approximation techniques and time-frequency (resp. time-scale) analysis [37, 44, 48, 49]. Over the last decades, abstract and computational aspects of covariant/coherent state transforms have achieved significant popularity in mathematical and theoretical physics, see [3, 5, 37, 47] and references therein. Coherent state transforms are classically obtained by a given coherent function systems. Then admissibility conditions on the coherent system imply analyzing of functions with respect to the system by the inner product evaluation

[22, 23, 35]. From harmonic and functional analysis aspects such coherent structures are classically originated from squar-integrable representations of locally compact groups, see [33, 46, 50, 59] and references therein. Commonly used coherent states transforms in theoretical physics, computational science and engineering are wavelet transform [49], Gabor transform [37], wave-packet transform [27–30, 32].

The mathematical theory of Gabor analysis is based on the coherent state generated by modulations and translations of a given window function [4, 6, 31, 34]. Wavelet analysis is a time-scale analysis which is based on the continuous affine group as the group of dilations and translations [9]. Abstract harmonic analysis extensions of wavelet analysis are studied in [7, 49]. The theory of wave packet transform over the real line has been extended for higher dimensions by several authors, see [11]. The mathematical theory of classical wave-packet analysis on the real line is originated from classical dilations, translations, and modulations of a given window function. The mathematical theory of wave-packet analysis as a coherent state analysis has been recently abstracted in the setting of locally compact Abelian groups in [28]. In a nutshell, wave-packet analysis which is also well-known as Gabor-wavelet analysis is a shrewd extensions of the two most prominent coherent states analysis, namely Gabor and wavelet analysis.

The following paper consists of abstract aspects of nature of metaplectic wave-packet transforms over $L^2(\mathbb{R}^d)$. This paper aims to introduce the notion of metaplectic wave-packet transform over the Hilbert function space $L^2(\mathbb{R}^d)$. We shall address analytic aspects of metaplectic wave-packet transforms over $L^2(\mathbb{R}^d)$ using tools from representation theory of locally compact groups and abstract harmonic analysis.

This article contains 6 sections. Section 2 is devoted to fix notations and a summary of classical Fourier analysis on $\mathbb{R}^d$ and classical harmonic analysis on projective representations and square-integrable representations over locally compact groups. In section 3 we present a brief study of harmonic analysis over the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$. We introduce the abstract notion of symplectic wave-packet groups associated to closed subgroups of $\mathrm{Sp}(\mathbb{R}^d)$. We shall also show that the group structure of symplectic wave-packet groups canonically determines an irreducible projective (unitary) group representation of the group, which is called as metaplectic wave-packet representation. We then present an admissibility criterion on closed subgroups of $\mathrm{Sp}(\mathbb{R}^d)$ to guarantee the square-integrability of the associated metaplectic wave-packet representation on $L^2(\mathbb{R}^d)$. As an application of our results we study analytic aspects of metaplectic wave-packet transforms associated to closed subgroups of the real symplectic goup $\mathrm{Sp}(\mathbb{R}^d)$. It is also shown that, if $\mathbb{H}$ is a compact subgroup of $\mathrm{Sp}(\mathbb{R}^d)$, for all non-zero window functions we can continuously reconstruct any L^2-function from metaplectic wave-packet coefficients. Finally, we will illustrate application of these techniques in the case of well-known compact subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$.

2. Preliminaries and notations

Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. Let $\mathcal{U}(\mathcal{H})$ be the multiplicative group of all unitary operators on $\mathcal{H}$. A projective group representation of G on $\mathcal{H}$ is a mapping $\Gamma : G \to \mathcal{U}(\mathcal{H})$ which satisfies

$$\Gamma(gg') = z(g, g')\Gamma(g)\Gamma(g') \quad \text{for all } g, g' \in G$$

where $z(g,g')$ are unimodular numbers. The projective group representation Γ is called irreducible on $\mathcal{H}$, if $\{0\}$ and $\mathcal{H}$ are the only closed Γ-invariant subspaces of $\mathcal{H}$.

A projective group representation $(\Gamma,\mathcal{H})$ is called left square integrable if there exists a non-zero vector $\zeta\in\mathcal{H}$ such that

$$\int_G |\langle\zeta,\Gamma(g)\zeta\rangle|^2\mathrm{d}m_G(g)<\infty,$$

for some left Haar measure m_G of G. Similarly, it is called right square integrable if there exists a non-zero vector $\zeta\in\mathcal{H}$ such that

$$\int_G |\langle\zeta,\Gamma(g)\zeta\rangle|^2\mathrm{d}n_G(g)<\infty,$$

for some right Haar measure n_G of G.

Since $\mathbb{R}^d$ is an LCA (locally compact Abelian) group, according to the Schur's lemma, all irreducible representations of $\mathbb{R}^d$ are one-dimensional. Thus any irreducible unitary representation $(\pi,\mathcal{H}_\pi)$ of $\mathbb{R}^d$ satisfies $\mathcal{H}_\pi=\mathbb{C}$ and hence there exists a continuous homomorphism ω of $\mathbb{R}^d$ into the circle group $\mathbb{T}$, such that for each $x=(x_1,...,x_d)\in\mathbb{R}^d$ and $z\in\mathbb{C}$ we have $\pi(x)(z)=\omega(x)z$. Such homomorphisms are called characters of $\mathbb{R}^d$ and the set of all such characters of $\mathbb{R}^d$ is denoted by $\widehat{\mathbb{R}^d}$. If $\widehat{\mathbb{R}^d}$ equipped with the topology of compact convergence on $\mathbb{R}^d$ which coincides with the w^*-topology that $\widehat{\mathbb{R}^d}$ inherits as a subset of $L^\infty(\mathbb{R}^d)$, then $\widehat{\mathbb{R}^d}$ with respect to the product of characters is an LCA group which is called the dual (character) group of $\mathbb{R}^d$. The character group $\widehat{\mathbb{R}^d}$, that is the multiplicative group of all continuous additive homomorphisms of $\mathbb{R}^d$ into the circle group $\mathbb{T}$, can be parametrizes by $\mathbb{R}^d$ via the following duality notation $\widehat{\mathbb{R}^d}$ with $\mathbb{R}^d$ via

$$\omega(x)=\langle x,\omega\rangle=\mathrm{e}^{2\pi\mathrm{i}\omega^T\cdot x}$$

for each $\omega\in\widehat{\mathbb{R}^d}$. The linear map $\mathcal{F}_{\mathbb{R}^d}:L^1(\mathbb{R}^d)\to\mathcal{C}(\widehat{\mathbb{R}^d})$ defined by $f\mapsto\mathcal{F}_{\mathbb{R}^d}(f)=\widehat{f}$ via

$$\mathcal{F}_{\mathbb{R}^d}(f)(\omega)=\widehat{f}(\omega)=\int_{\mathbb{R}^d}f(s)\overline{\omega(s)}\mathrm{d}m_{\mathbb{R}^d}(s),\tag{2.1}$$

is called the Fourier transform on $\mathbb{R}^d$. It is a norm-decreasing $*$-homomorphism from $L^1(\mathbb{R}^d)$ into $\mathcal{C}_0(\widehat{\mathbb{R}^d})$ with a uniformly dense range in $\mathcal{C}_0(\widehat{\mathbb{R}^d})$. If a Haar measure $m_{\mathbb{R}^d}$ on $\mathbb{R}^d$ is given and fixed then there is a Haar measure $m_{\widehat{\mathbb{R}^d}}$ on $\widehat{\mathbb{R}^d}$, which is called the normalized Plancherel measure associated to $m_{\mathbb{R}^d}$, such that the Fourier transform (2.1) is an isometric transform on $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and hence it can be extended uniquely to a unitary isomorphism from $L^2(\mathbb{R}^d)$ onto $L^2(\widehat{\mathbb{R}^d})$, see [24]. Then each $f\in L^1(\mathbb{R}^d)$ with $\widehat{f}\in L^1(\widehat{\mathbb{R}^d})$ satisfies the following Fourier inversion formula

$$f(s)=\int_{\widehat{\mathbb{R}^d}}\widehat{f}(\omega)\omega(s)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega)\text{ for a.e. }s\in\mathbb{R}^d.\tag{2.2}$$

For $x\in\mathbb{R}^d$ and $f\in L^2(\mathbb{R}^d)$, the translation of f by x is defined by $T_xf(y)=f(y-x)$ for $y\in\mathbb{R}^d$. The translation $T_x:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is a unitary operator. For $\omega\in\widehat{\mathbb{R}^d}$ and $f\in L^2(\mathbb{R}^d)$, the modulation of f by ω is defined by $M_\omega f(y)=\overline{\omega(y)}f(y)$ for $s\in\mathbb{R}^d$. The modulation operator $M_\omega:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is unitary as well. The modulation and translation operators are connected via the Fourier transform by

$$\widehat{M_\omega f} = T_{-\omega}\widehat{f}, \qquad \widehat{T_k f} = M_k\widehat{f}, \tag{2.3}$$

for all $f\in L^2(\mathbb{R}^d)$, $\omega\in\widehat{\mathbb{R}^d}$, and $k\in\mathbb{R}^d$, see [24, 38, 52].

From now on and in this article, for a fixed Haar (Lebesgue) measure $m_{\mathbb{R}^d}$ on $\mathbb{R}^d$, by $\mu_{\mathbb{R}^{2d}}$ or $\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}$ we mean the induced product measure on $\mathbb{R}^{2d}=\mathbb{R}^d\times\widehat{\mathbb{R}^d}$, that is $\mathrm{d}\mu_{\mathbb{R}^{2d}}(x,\omega)=\mathrm{d}m_{\mathbb{R}^d}(x)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega)$, where $m_{\widehat{\mathbb{R}^d}}$ is the normalized Plancherel measure associated to $m_{\mathbb{R}^d}$.

For $\lambda=(x,\omega)\in\mathbb{R}^{2d}=\mathbb{R}^d\times\widehat{\mathbb{R}^d}$, the time-frequency shift operator $\pi(\lambda):L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ is defined by $\pi(\lambda)=M_\omega T_x$. Then, it is well-known as the Moyal's formula, that

$$\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\langle f,\pi(\lambda)g\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)=\|f\|^2_{L^2(\mathbb{R}^d)}\|g\|^2_{L^2(\mathbb{R}^d)}, \tag{2.4}$$

for all $f,g\in L^2(\mathbb{R}^d)$, see [37] and classical references therein.

3. Harmonic analysis over symplectic groups

Throughout this section, we briefly present basics of classical harmonic analysis over the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$, for a complete picture of this matrix group we referee the readers to [18–20, 44–46] and the comprehensive list of classical references therein.

For $d\geqslant 1$, let $\Omega: M_{d\times d}(\mathbb{C})\to M_{2d\times 2d}(\mathbb{R})$ be the linear map given by

$$\Omega(A+\mathrm{i}B):=\begin{pmatrix}A & -B\\ B & A\end{pmatrix},$$

for all $A,B\in M_{d\times d}(\mathbb{R})$.

A matrix $S\in M_{2d\times 2d}(\mathbb{R})$ is called symplectic if and only if $S^TJS=SJS^T=J$, with $J=\begin{pmatrix}0 & I_{d\times d}\\ -I_{d\times d} & 0\end{pmatrix}$, where $I_{d\times d}$ is $d\times d$ identity matrix.. The group consists of all symplectic matrices is called the (real) symplectic group which is denoted by $\mathrm{Sp}(\mathbb{R}^d)$. It is a simple non-compact finite-dimensional real Lie group. In block-matrix notation, the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ consists of all real $2d\times 2d$ matrices in block form

$$S=\begin{pmatrix}A & B\\ C & D\end{pmatrix}, \qquad A,B,C,D\in M_{d\times d}(\mathbb{R}),$$

such that $A^TC=C^TA$, $B^TD=D^TB$, and $A^TD-C^TB=I_{d\times d}$.

The real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ satisfies the following decomposition, namely Iwasawa (Gram-Schmidt) decomposition, $\mathrm{Sp}(\mathbb{R}^d)=\mathcal{KAN}$ where [55, 56]

$$\mathcal{K}_d:=\left\{\Omega(A+\mathrm{i}B)=\begin{pmatrix}A & -B\\ B & A\end{pmatrix}: A+\mathrm{i}B\in\mathrm{U}(d,\mathbb{C})\right\}, \tag{3.1}$$

$$\mathcal{A}:=\{\mathrm{diag}(h_1,...,h_d,h_1^{-1},...,h_d^{-1}): h_1,...,h_d>0\}, \tag{3.2}$$

and

$$\mathcal{N}:=\left\{\begin{pmatrix}A & B\\ 0 & (A^{-1})^T\end{pmatrix}: A \text{ is unit upper triangular, } AB^T=BA^T\right\}, \tag{3.3}$$

If we regard elements of $\mathrm{Sp}(\mathbb{R}^d)$ as linear transformations over the vector space (time-frequency phase space) $\mathbb{R}^{2d} = \mathbb{R}^d \times \widehat{\mathbb{R}^d}$, then the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ is precisely the group of all linear automorphisms of $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ which preserve the canonical (symplectic) form. Also, it is easy to check that, if $\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}$ is the Lebesgue measure on $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, then

$$\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(S \cdot \lambda) = \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda), \tag{3.4}$$

for all $S \in \mathrm{Sp}(\mathbb{R}^d)$.

A metaplectic operator on $L^2(\mathbb{R}^d)$ is a unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ which satisfies the following intertwining identity

$$U\pi(\lambda)U^{-1} = \alpha(\lambda)\pi(S \cdot \lambda), \qquad (\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}) \tag{3.5}$$

for some $S \in \mathrm{Sp}(\mathbb{R}^d)$ and a second degree character $\alpha : \mathbb{R}^d \times \widehat{\mathbb{R}^d} \to \mathbb{T}$.

In coordinate terms, a metaplectic operator on $L^2(\mathbb{R}^d)$ is a unitary operator $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ which satisfies the following intertwining identity

$$UM_\omega T_x U^{-1} = \alpha(x, \omega) M_{C\cdot x + D\cdot\omega} T_{A\cdot x + B\cdot\omega}, \qquad ((x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d})$$

for some $S \in \mathrm{Sp}(\mathbb{R}^d)$ and a second degree character $\alpha : \mathbb{R}^d \times \widehat{\mathbb{R}^d} \to \mathbb{T}$. In this case, the operator U is called as the metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix S.

For $H \in \mathrm{GL}(d, \mathbb{R})$, the dilation operator $D_H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is given by

$$D_H f(t) := |\det H|^{-1/2} f(H^{-1} \cdot t),$$

for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

For $C \in M_{d\times d}(\mathbb{R})$ with $C = C^T$, the chrip multiplication operator $E_C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by

$$E_C f(t) := \exp(\pi \mathrm{i} \cdot t^T C t) f(t),$$

for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

The following proposition [43] shows that the Fourier transform, dilations, and chrip multiplications can be considered as metaplectic operators.

Proposition 3.1. *Let $H \in \mathrm{GL}(d, \mathbb{R})$ and $C \in M_{d\times d}(\mathbb{R})$ with $C^T = C$. Then*

(1) *The Fourier transform $\mathcal{F}_{\mathbb{R}^d} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and satisfies the following intertwining identity*

$$\mathcal{F}_{\mathbb{R}^d}\pi(x, \omega)\mathcal{F}_{\mathbb{R}^d}^{-1} = \mathrm{e}^{2\pi \mathrm{i}\omega^T \cdot x}\pi(\omega, -x)$$

(2) *The dilation operator $D_H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} H & 0 \\ 0 & (H^T)^{-1} \end{pmatrix}$ and satisfies the following intertwining identity*

$$D_H \pi(x, \omega) D_H^{-1} = \pi(H \cdot x, (H^T)^{-1} \cdot \omega)$$

(3) *The chrip multiplication operator $E_C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a metaplectic operator on $L^2(\mathbb{R}^d)$ associated to the symplectic matrix $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ and satisfies the following intertwining identity*

$$E_C\pi(x,\omega)E_C^{-1} = \mathrm{e}^{-\pi \mathrm{i} x^T\cdot C\cdot x}\pi(x, C\cdot x+\omega)$$

Then the following [43] result gives us a unified and also explicit construction of metaplectic operators on $L^2(\mathbb{R}^d)$ by splitting them into simple operators given in proposition 3.1.

Theorem 3.2. *Let* $S=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in \mathrm{Sp}(\mathbb{R}^d)$ *be given. Let* $\mathbb{I}_A\subseteq\mathbb{N}_d$ *be such that the columns of A indexed by* $\mathbb{I}_A$ *form a basis for* $\mathcal{R}(A)$ *and* $\Lambda\in M_{d\times d}(\mathbb{Z})$ *be the diagonal matrix whose diagonal is 0 at* $\mathbb{I}_A$ *and 1 at the complementary set* $\mathbb{N}_d\setminus\mathbb{I}_A$. *Let* $H:=A+B\Lambda$ *and* $Q:=C+D\Lambda$. *Then* $H\in\mathrm{GL}(d,\mathbb{R})$ *and the unitary operator*

$$U_S := E_{QH^{-1}}D_H\mathcal{F}_{\mathbb{R}^d}^{-1}E_{-H^{-1}B}\mathcal{F}_{\mathbb{R}^d}E_{-\Lambda} \tag{3.6}$$

is the metaplectic operator associated to the symplectic matrix S.

4. Multivariate metaplectic wave packet representations

In this section we present the abstract structure of multivariate symplectic wave-packet groups associated to closed subgroups of the real symplectice group $\mathrm{Sp}(\mathbb{R}^d)$. Then we introduce the associated multivariate metaplectic wave-packet representation. We shall also study classical properties of these representations.

For a closed subgroup $\mathbb{H}$ of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$, the underlying manifold

$$\mathbb{G}(d,\mathbb{H}) := \mathbb{H}\times\mathbb{R}^d\times\widehat{\mathbb{R}^d} = \mathbb{H}\times\mathbb{R}^d\times\mathbb{R}^d,$$

equipped with operations given by

$$(S,\lambda)\rtimes(S',\lambda') := (SS', S'^{-1}\cdot\lambda+\lambda'), \tag{4.1}$$

$$(S,\lambda)^{-1} := (S^{-1}, -S\cdot\lambda), \tag{4.2}$$

is a group with the identity element $(\mathbf{1},0,0)$.

We call this group as *symplectic wave-packet group* associated to the subgroup $\mathbb{H}$ over $\mathbb{R}^d$. For simplicity, we may use $\mathbb{G}(\mathbb{H})$ instead of $\mathbb{G}(d,\mathbb{H})$, at times. The groups $\mathbb{H}$ and $\mathbb{R}^d\times\widehat{\mathbb{R}^d}$ can be considered as closed subgroups of $\mathbb{G}(\mathbb{H})$.

Then we present the following theorem concerning basic properties of the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ in the framework of harmonic analysis.

Theorem 4.1. *Let* $\mathbb{H}$ *be a closed subgroup of the symplectic group* $\mathrm{Sp}(\mathbb{R}^d)$ *with the modular function* $\Delta_{\mathbb{H}}$ *and* $m_{\mathbb{H}}$ *(resp.* $n_{\mathbb{H}}$*) be a left (resp. right) Haar measure of* $\mathbb{H}$. *Then,* $\mathbb{G}(\mathbb{H})$ *is a locally compact group with a left Haar measure given by* $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S,\lambda) := \mathrm{d}m_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$, *and a right Haar measure given by* $\mathrm{d}n_{\mathbb{G}(\mathbb{H})}(S,\lambda) := \mathrm{d}n_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$.

Proof. It can readily be checked that the mapping $\tau:\mathbb{H}\times\mathbb{R}^d\times\widehat{\mathbb{R}^d}\to\mathbb{R}^d\times\widehat{\mathbb{R}^d}$ given by $(S,\lambda)\to S\cdot\lambda$ is continuous. This automatically implies that the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ is a locally compact group. Let $F\in\mathcal{C}_c(\mathbb{G}(\mathbb{H}))$ and $\mathbf{g}=(S,\lambda)\in\mathbb{G}(\mathbb{H})$. Since the Lebesgue measure $\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}$ is translation invariant and also $m_{\mathbb{H}}$ is a left Haar measure on $\mathbb{H}$, we have

$$
\begin{aligned}
\int_{\mathbb{G}(\mathbb{H})} F(\mathbf{g}\cdot\mathbf{g}')\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\mathbf{g}') &= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((S,\lambda)\rtimes(S',\lambda'))\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((SS',S'^{-1}\cdot\lambda+\lambda'))\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((SS',S'^{-1}\cdot\lambda+\lambda'))\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}m_{\mathbb{H}}(S') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(SS',\lambda')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}m_{\mathbb{H}}(S') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(SS',\lambda')\mathrm{d}m_{\mathbb{H}}(S')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(S',\lambda')\mathrm{d}m_{\mathbb{H}}(S')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S',\lambda')\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') = \int_{\mathbb{G}(\mathbb{H})} F(\mathbf{g}')\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\mathbf{g}'),
\end{aligned}
$$

which implies that $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S,\lambda) := \mathrm{d}m_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$ is a left Haar measure for $\mathbb{G}(\mathbb{H})$. Similarly, using (3.4), Fubini's theorem and also since the Lebesgue measure $\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}$ is translation invariant, we get

$$
\begin{aligned}
\int_{\mathbb{G}(\mathbb{H})} F(\mathbf{g}'\cdot\mathbf{g})\mathrm{d}n_{\mathbb{G}(\mathbb{H})}(\mathbf{g}') &= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((S',\lambda')\rtimes(S,\lambda))\mathrm{d}n_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S'S,S^{-1}\cdot\lambda'+\lambda)\mathrm{d}n_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S'S,S^{-1}\cdot\lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}n_{\mathbb{H}}(S') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S'S,\lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(S\cdot\lambda')\right)\mathrm{d}n_{\mathbb{H}}(S') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S'S,\lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}n_{\mathbb{H}}(S') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S'S,\lambda')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}n_{\mathbb{H}}(S') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(S'S,\lambda')\mathrm{d}n_{\mathbb{H}}(S')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(S',\lambda')\mathrm{d}n_{\mathbb{H}}(S')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(S',\lambda')\mathrm{d}n_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') = \int_{\mathbb{G}(\mathbb{H})} F(\mathbf{g}')\mathrm{d}n_{\mathbb{G}(\mathbb{H})}(\mathbf{g}'),
\end{aligned}
$$

implying that $\mathrm{d}n_{\mathbb{G}(\mathbb{H})}(S,\lambda) := \mathrm{d}n_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^{2d}}(\lambda)$ is a right Haar measure for $\mathbb{G}(\mathbb{H})$. □

Next we deduce the following consequences.

Corollary 4.2. *Let $\mathbb{H}$ be a closed subgroup of the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ with the modular function $\Delta_{\mathbb{H}}$ and $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$) be a left (resp. right) Haar measure of $\mathbb{H}$. Then*

(1) *The modular function $\Delta_{\mathbb{G}(\mathbb{H})} : \mathbb{G}(\mathbb{H}) \to (0, \infty)$ is given by $\Delta_{\mathbb{G}(\mathbb{H})}(S, \lambda) := \Delta_{\mathbb{H}}(S)$. In particular, the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ is unimodular if and only if $\mathbb{H}$ is unimodular.*
(2) *The closed subgroup $\mathbb{H}$ is normal in $\mathbb{G}(\mathbb{H})$ if and only if $\mathbb{H} = \{\mathbf{I}\}$.*
(3) *The closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is a normal Abelian subgroup of $\mathbb{G}(\mathbb{H})$.*

Proof.

(1) Let $F \in \mathcal{C}_c(\mathbb{G}(\mathbb{H}))$ be a non-zero and positive function. Also, let $(S, \lambda) \in \mathbb{G}(\mathbb{H})$. Then we can write

$$\begin{aligned}
\Delta_{\mathbb{G}(\mathbb{H})}(S,\lambda)^{-1} \cdot \int_{\mathbb{G}(\mathbb{H})} F(S',\lambda')\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S',\lambda') &= \int_{\mathbb{G}(\mathbb{H})} F((S',\lambda') \rtimes (S,\lambda))\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S',\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(S',\lambda') \rtimes (S,\lambda)\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(S'S, S^{-1}\cdot\lambda' + \lambda)\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(S'S, \lambda' + \lambda)\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(S\cdot\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(S'S, \lambda + \lambda')\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(S'S, \lambda')\mathrm{d}m_{\mathbb{H}}(S')\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(S'S, \lambda')\mathrm{d}m_{\mathbb{H}}(S') \right)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \Delta_{\mathbb{H}}(S)^{-1} \cdot \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(S', \lambda')\mathrm{d}m_{\mathbb{H}}(S') \right)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \Delta_{\mathbb{H}}(S)^{-1} \cdot \int_{\mathbb{G}(\mathbb{H})} F(S',\lambda')\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S',\lambda'),
\end{aligned}$$

implying that $\Delta_{\mathbb{G}(\mathbb{H})}(S, \lambda) = \Delta_{\mathbb{H}}(S)$ for all $(S, \lambda) \in \mathbb{G}(\mathbb{H})$.
(2) and (3) are straightforward from structure of the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$. □

Remark 4.3. From now on, once the left (resp. right) Haar measure $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$) over $\mathbb{H}$ is fixed, we call the associated left (resp. right) Haar measure on the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$, which is constructed via theorem 4.1, as left (resp. right) Haar measure induced by $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$).

For $\mathbf{g} = (S, \lambda) = (A, x, \omega) \in \mathbb{G}(\mathbb{H})$, define the linear operator $\Gamma_{\mathbb{H}}(\mathbf{g}) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$\Gamma_{\mathbb{H}}(\mathbf{g}) := U_S\pi(\lambda) = U_S T_x M_\omega. \tag{4.3}$$

The following theorem shows that $\mathbf{g} \mapsto \Gamma_{\mathbb{H}}(\mathbf{g})$ given by (4.3), defines an irreducible projective group representation of the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ on the Hilbert function space $L^2(\mathbb{R}^d)$.

Theorem 4.4. *Let $\mathbb{H}$ be a closed subgroup of the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ and $\mathbb{G}(\mathbb{H})$ be the associated symplectic wave-packet group. Then $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ given by $\mathbf{g} \mapsto \Gamma_{\mathbb{H}}(\mathbf{g})$ is an irreducible projective group representation of the locally compact group $\mathbb{G}(\mathbb{H})$ on the Hilbert function space $L^2(\mathbb{R}^d)$.*

Proof. Plainly, we have $\Gamma_{\mathbb{H}}(\mathbf{1}, 0, 0) = I_{L^2(\mathbb{R}^d)}$, where $I : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the identity operator. Let $(S, \lambda), (S', \lambda') \in \mathbb{G}(\mathbb{H})$. Invoking definition of $\Gamma_{\mathbb{H}}(S, \lambda)$, it is evident to check that $\Gamma_{\mathbb{H}}(S, \lambda)$ is a unitary operator, because it is the composition of two unitary operators, namely U_S and $\pi(\lambda)$. Let $\beta : \mathbb{R}^d \times \widehat{\mathbb{R}^d} \to \mathbb{T}$ be a second degree character such that the intertwining identity (3.5) holds for S'. Hence, we get

$$\begin{aligned} U_{S'}\pi(S'^{-1} \cdot \lambda) &= \beta(S'^{-1} \cdot \lambda)\pi(S' \cdot (S'^{-1} \cdot \lambda))U_{S'} \\ &= \beta(S'^{-1} \cdot \lambda)\pi(\lambda)U_{S'}. \end{aligned}$$

Also, the operator $U_S U_{S'}$ is a metaplectic operator associated to SS'. Thus, there exists a complex number $z(S, S') \in \mathbb{T}$ such that $U_{SS'} = z(S, S')U_S U_{S'}$. Then we can write

$$\begin{aligned} U_{SS'}\pi(S'^{-1} \cdot \lambda + \lambda') &= z(S, S')U_S U_{S'}\pi(S'^{-1} \cdot \lambda + \lambda') \\ &= z(S, S')U_S U_{S'}\pi(S'^{-1} \cdot \lambda)\pi(\lambda') = z(S, S')\beta(S'^{-1} \cdot \lambda)U_S\pi(\lambda)U_{S'}\pi(\lambda'). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \Gamma_{\mathbb{H}}((S, \lambda) \rtimes (S', \lambda')) &= \Gamma_{\mathbb{H}}(SS', S'^{-1} \cdot \lambda + \lambda') \\ &= U_{SS'}\pi(S'^{-1} \cdot \lambda + \lambda') \\ &= z(S, S')\beta(S'^{-1} \cdot \lambda)U_S\pi(\lambda)U_{S'}\pi(\lambda') = z(S, S')\beta(S'^{-1} \cdot \lambda) \\ &\quad \Gamma_{\mathbb{H}}(S, \lambda)\Gamma_{\mathbb{H}}(S', \lambda'), \end{aligned}$$

which implies that $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is a projective group representation of the locally compact group $\mathbb{G}(\mathbb{H})$ on the Hilbert function space $L^2(\mathbb{R}^d)$. Since restriction of $\Gamma_{\mathbb{H}}$ to the closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is equivalent with the projective Shrödinger representation of the subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ on $L^2(\mathbb{R}^d)$, we deduce that $\Gamma_{\mathbb{H}}$ is irreducible on $L^2(\mathbb{R}^d)$ as well. $\square$

Remark 4.5.

(i) The restriction of the metaplectic wave-packet representation to the closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is unitarily equivalent to the projective Schrödinger representation of $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ on $L^2(\mathbb{R}^d)$, see [37] and references therein.

(ii) Let $\mathbb{H}$ be a closed subgroup of the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ which contains $\mathrm{GL}(d, \mathbb{R})$. Then the restriction of the metaplectic wave-packet representation to the closed subgroup $\mathrm{GL}(d, \mathbb{R}) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is unitarily equivalent to the classic wave-packet representation associated to the action of the multiplicative matrix group $\mathrm{GL}(d, \mathbb{R})$ on the time-frequency plan $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, see [28, 42, 57, 58] and the comprehensive list of references therein.

5. Square-integrability of multivariate metaplectic wave-packet representations

Throughout this section, we study the square-integrability of multivariate metaplectic wave-packet representations. We still assume that $\mathbb{H}$ is a closed subgroup of the symplectic group $\mathrm{Sp}(\mathbb{R}^d)$.

It should be mentioned that in the framework of classical voice/coherent state transforms [59], the problem of admissibility conditions for subgroups of the symplectic group studied from an algebraic perspective in [1, 2, 12, 13, 17, 21].

Let $\psi \in L^2(\mathbb{R}^d)$ be a window function. The metaplectic wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ is given by the voice transform associated to the metaplectic wave-packet representation, that is

$$\mathcal{V}_\psi f(S,x,\omega) := \langle f, \Gamma_{\mathbb{H}}(S,x,\omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, U_S T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)}, \tag{5.1}$$

for $(S,x,\omega) \in \mathbb{H} \times \mathbb{R}^d \times \widehat{\mathbb{R}^d}$.

Remark 5.1.

(i) The restriction of the metaplectic wave-packet transform to the closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is the continuous Gabor (short-time Fourier) transform over $L^2(\mathbb{R}^d)$, see [37] and references therein.

(ii) Let $\mathbb{H}$ be a closed subgroup of $\mathrm{Sp}(\mathbb{R}^d)$ which contains $\mathrm{GL}(d,\mathbb{R})$. Then the restriction of the metaplectic wave-packet transform to the closed subgroup $\mathrm{GL}(d,\mathbb{R}) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is the classic wave-packet transform induced by the action of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$ on the time-frequency plan $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, see [28] and the comprehensive list of references therein.

The following theorem can be considered as a constructive topological criterion on the closed subgroup $\mathbb{H}$, which guarantees the square-integrability of the associated metaplectic wave-packet representation $\Gamma_{\mathbb{H}}$ on the Hilbert function space $L^2(\mathbb{R}^d)$.

Theorem 5.2. *Let $\mathbb{H}$ be a closed subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ and $\mathbb{G}(\mathbb{H})$ be the associated symplectic wave-packet group. Then, the metaplectic wave-packet representation $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is left (resp. right) square-integrable over the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ if and only if $\mathbb{H}$ is compact. In this case, all non-zero functions in the Hilbert function space $L^2(\mathbb{R}^d)$ are square-integrable over $\mathbb{G}(\mathbb{H})$ with respect to $\Gamma_{\mathbb{H}}$.*

Proof. Let $m_{\mathbb{H}}$ be a left Haar measure for $\mathbb{H}$. Then by theorem 4.1, the positive Radon measure $m_{\mathbb{G}(\mathbb{H})}$ given by $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S,\lambda) = \mathrm{d}m_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a left Haar measure for the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$. Now, suppose that the metaplectic wave-packet representation $\Gamma_{\mathbb{H}}$ be left square-integrable over $\mathbb{G}(\mathbb{H})$. Then, there exists a non-zero function $\psi \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{G}(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\mathbf{g}) < \infty.$$

Then, using Fubini's theorem and also the Moyal's formula (2.4), we get

$$
\begin{aligned}
\int_{\mathbb{G}(\mathbb{H})}|\langle\psi,\Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\mathbf{g}) &= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\langle\psi,\Gamma_{\mathbb{H}}(S,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}m_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\langle\psi,\Gamma_{\mathbb{H}}(S,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\\
&\quad \mathrm{d}m_{\mathbb{H}}(S)\\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\langle\psi,U_S\pi(\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\mathrm{d}m_{\mathbb{H}}(S)\\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}|\langle U_S^*\psi,\pi(\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\mathrm{d}m_{\mathbb{H}}(S)\\
&= \int_{\mathbb{H}}(\|U_S^*\psi\|^2_{L^2(\mathbb{R}^d)}\|\psi\|^2_{L^2(\mathbb{R}^d)})\mathrm{d}m_{\mathbb{H}}(S)\\
&= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|U_S^*\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(S)\right).
\end{aligned}
$$

Since metaplectice operators are unitary on $L^2(\mathbb{R}^d)$, we deduce that

$$
\begin{aligned}
\|\psi\|^4_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\mathrm{d}m_{\mathbb{H}}\right) &= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(S)\right)\\
&= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|U_S^*\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(S)\right)\\
&= \int_{\mathbb{G}(\mathbb{H})}|\langle\psi,\Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\mathbf{g}) < \infty.
\end{aligned}
$$

Thus $m_{\mathbb{H}}(\mathbb{H}) < \infty$ and hence $\mathbb{H}$ is compact. Conversely, let $\mathbb{H}$ be a compact subgroup of $\mathrm{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_{\mathbb{H}}$, that is the unique positive Radon measure $\sigma_{\mathbb{H}}$ which is both left and right Haar measure of $\mathbb{H}$ with $\sigma_{\mathbb{H}}(\mathbb{H}) = 1$. Then, each non-zero function $\psi \in L^2(\mathbb{R}^d)$ satisfies

$$
\int_{\mathbb{G}(\mathbb{H})}|\langle\psi,\Gamma_{\mathbb{H}}(S,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2\,\mathrm{d}\sigma_{\mathbb{H}}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^4_{L^2(\mathbb{R}^d)}, \tag{5.2}
$$

which implies the square-integrability of the metaplectic wave-packet representation $\Gamma_{\mathbb{H}}$ over the symplectic wave-packet group $\mathbb{G}(\mathbb{H})$. □

As a consequence of theorem 5.2, we deduce the following orthogonality relation concerning the metaplectic wave-packet transforms.

Corollary 5.3. *Let $\mathbb{H}$ be a compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{G}(\mathbb{H})$ be the associated metaplectic wave-packet group with the induced Haar measure $m_{\mathbb{G}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Also, let $\psi,\varphi \in L^2(\mathbb{R}^d)$ be non-zero window functions and $f, g \in L^2(\mathbb{R}^d)$. Then, we have*

$$
\langle\mathcal{V}_\psi f,\mathcal{V}_\varphi g\rangle_{L^2(\mathbb{G}(\mathbb{H}),m_{\mathbb{G}(\mathbb{H})})} = \langle\varphi,\psi\rangle_{L^2(\mathbb{R}^d)}\langle f,g\rangle_{L^2(\mathbb{R}^d)}. \tag{5.3}
$$

Proof. The same argument used in theorem 5.2 implies that

$$
\|\mathcal{V}_\psi f\|^2_{L^2(\mathbb{G}(\mathbb{H}),m_{\mathbb{G}(\mathbb{H})})} = \|\psi\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{5.4}
$$

Then (5.4) and also twice applying the Polarization identity guarantees (5.3). □

Next result is an inversion (reconstruction) formula for the metaplectic wave-packet transform defined by (5.1).

Theorem 5.4. *Let $\mathbb{H}$ be a compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{G}(\mathbb{H})$ be the associated symplectic wave-packet group with the induced Haar measure $m_{\mathbb{G}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Also, let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then, each function $f \in L^2(\mathbb{R}^d)$ can be recovered continuously in the weak sense of the Hilbert function space $L^2(\mathbb{R}^d)$, from metaplectic wave-packet coefficients generated by ψ, via the following resolution of the identity formula;*

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \cdot \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(S, \lambda) \Gamma_{\mathbb{H}}(S, \lambda)\psi \, \mathrm{d}\sigma_{\mathbb{H}}(S) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda). \tag{5.5}$$

Proof. Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. For $f \in L^2(\mathbb{R}^d)$, define

$$f_{(\psi)} := \int_{\mathbb{H}} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(S, \lambda) \Gamma_{\mathbb{H}}(S, \lambda)\psi \, \mathrm{d}\sigma_{\mathbb{H}}(S) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda),$$

in the weak sense of the Hilbert function space $L^2(\mathbb{R}^d)$. Using (5.3), for all $g \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
\langle f_{(\psi)}, g\rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(S, \lambda) \langle \Gamma_{\mathbb{H}}(S, \lambda)\psi, g\rangle_{L^2(\mathbb{R}^d)} \, \mathrm{d}\sigma_{\mathbb{H}}(S) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(S, \lambda) \overline{\langle g, \Gamma_{\mathbb{H}}(S, \lambda)\psi\rangle_{L^2(\mathbb{R}^d)}} \, \mathrm{d}\sigma_{\mathbb{H}}(S) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(S, \lambda) \overline{\mathcal{V}_\psi g(S, \lambda)} \, \mathrm{d}\sigma_{\mathbb{H}}(S) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) \\
&= \langle \mathcal{V}_\psi f, \mathcal{V}_\psi g\rangle_{L^2(\mathbb{G}(\mathbb{H}), m_{\mathbb{G}(\mathbb{H})})} = \|\psi\|^2_{L^2(\mathbb{R}^d)} \langle f, g\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Then $f_{(\psi)} \in L^2(\mathbb{R}^d)$ and $f_{(\psi)} = \|\psi\|^2_{L^2(\mathbb{R}^d)} f$ in $L^2(\mathbb{R}^d)$, which equivalently implies the reconstruction formula (5.5) in the weak sens of the Hilbert function space $L^2(\mathbb{R}^d)$. □

Then we can present the following reproducing property for the metaplectic wave-packet representations.

Corollary 5.5. *Let $\mathbb{H}$ be a compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ with the probability Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{G}(\mathbb{H})$ be the associated symplectic wave-packet group with the induced Haar measure $m_{\mathbb{G}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function and $\mathcal{H}_\psi$ be range of the metaplectic wave-packet transform $\mathcal{V}_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{G}(\mathbb{H}), m_{\mathbb{G}(\mathbb{H})})$. Then*

(1) *$\mathcal{H}_\psi$ is a closed subspace of $L^2(\mathbb{G}(\mathbb{H}), m_{\mathbb{G}(\mathbb{H})})$.*
(2) *$\mathcal{H}_\psi$ is the unique reproducing kernel Hilbert space (RKHS) over $\mathbb{G}(\mathbb{H})$ associated to the positive definite kernel $K_\psi : \mathbb{G}(\mathbb{H}) \times \mathbb{G}(\mathbb{H}) \to \mathbb{C}$ given by*

$$K_\psi[(S, \lambda), (S', \lambda')] := \langle U_S \pi(\lambda)\psi, U_{S'}\pi(\lambda')\psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(S, \lambda), (S', \lambda') \in \mathbb{G}(\mathbb{H})$.

Next corollary summarizes our recent results in terms of continuous frame theory [8, 53].

Corollary 5.6. *Let $\mathbb{H}$ be a compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ and $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then the multivariate wave-packet system*

$$\mathfrak{A}(\mathbb{H},\psi) := \{\Gamma_{\mathbb{H}}(S,\lambda)\psi : (S,\lambda) \in \mathbb{G}(\mathbb{H})\},$$

is a continuous tight frame for the Hilbert space $L^2(\mathbb{R}^d)$*.*

6. Analysis of multivariate metaplectic wave-packet representations over compact subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$

Throughout this section, we study analytic aspects of compact subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ in the framework of coherent state metaplectic wave-packet analysis.

As it is proved in theorem 5.2, just compact subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ are interesting from the L^2-theory and reproducing property of metaplectic wave-packet representations. Roughly speaking, only compact subgroups of $\mathrm{Sp}(\mathbb{R}^d)$ are highly important in the framework of coherent state metaplectic wave-packet analysis over the Hilbert function space $L^2(\mathbb{R}^d)$, since they guarantee that the associated metaplectic wave-packet transforms over $L^2(\mathbb{R}^d)$ satisfy resolution of the identity formulas which are valid in the weak sense of the Hilbert function space $L^2(\mathbb{R}^d)$.

6.1. The case $d = 1$

In this case [26], the real symplectic group $\mathrm{Sp}(\mathbb{R})$ is precisely the special linear group $\mathrm{SL}(2,\mathbb{R})$, that is the the multiplicative matrix group, consists of all real 2×2 matrices with determinant one. That is,

$$\mathrm{SL}(2,\mathbb{R}) := \left\{S = \begin{pmatrix} a & b \\ c & d\end{pmatrix} : a,b,c,d \in \mathbb{R} \quad \text{and} \quad ad - bc = 1\right\}.$$

It is a simple real 3-dimensional Lie group. The special linear group $\mathrm{SL}(2,\mathbb{R})$ satisfies the following decomposition, namely Iwasawa (Gram-Schmidt) decomposition, $\mathrm{SL}(2,\mathbb{R}) = \mathcal{KAN}$ where $\mathcal{K} = \mathrm{SO}(2)$ is the special orthogonal group consists of all 2×2-orthogonal matrices with real entries and the subgroups $\mathcal{A},\mathcal{N}$ are given by

$$\mathcal{A} = \left\{\mathbf{D}(h) := \begin{pmatrix} h & 0 \\ 0 & h^{-1}\end{pmatrix}\middle|\, h > 0\right\}, \qquad \mathcal{N} = \left\{\mathbf{N}(x) := \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\middle|\, x \in \mathbb{R}\right\}.$$

The group $\mathrm{SL}(2,\mathbb{R})$ is non-compact but unimodular. A Haar measure of $\mathrm{SL}(2,\mathbb{R})$ is given by

$$\phi \mapsto \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \phi\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y}\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\right)\mathrm{d}\theta y^{-2}\mathrm{d}y\mathrm{d}x,$$

for all $\phi \in \mathcal{C}_c(\mathrm{SL}(2,\mathbb{R}))$.

6.1.1. Continuous compact subgroups of $\mathrm{SL}(2,\mathbb{R})$*.* The subgroup $\mathbb{H} = \mathrm{SO}(2)$ is the most significant compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. The compact subgroup $\mathrm{SO}(2)$ is the multiplicative matrix group consists of all 2×2-orthogonal matrices with unit determinant. That is, $\mathrm{SO}(2) = \{\mathbf{H}(\theta) : 0 < \theta \leqslant 2\pi\}$, where

$$\mathbf{H}(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}.$$

The subgroup SO(2) is isomorphic, as a real Lie group, to the circle group, also known as $\mathbb{T} = \mathrm{U}(1)$, via the canonical Lie group isomorphism which sends the complex number $\mathrm{e}^{\mathrm{i}\theta}$ of absolute value 1, to the special orthogonal matrix $\mathbf{H}(\theta)$. From now on, we may call SO(2) as the circle group, at times. It can be readily checked that, any closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ conjugated to SO(2) is also compact in $\mathrm{SL}(2,\mathbb{R})$. In addition, the circle group SO(2) is a maximal compact subgroup of the multiplicative matrix Lie group $\mathrm{SL}(2,\mathbb{R})$, which means that SO(2) is a compact subgroup and it is maximal among such subgroups as well. Thus, any continuous (non-discrete) and compact subgroup is one-dimensional. Then by proposition 3.2 of [45], it is conjugated to the compact subgroup SO(2).

(i) The circle group. By the above argument and theoretical motivation, first we shall focus on analytic and constructive analysis of metaplectic wave-packet representations over the compact subgroup SO(2).

The normalized Haar measure $\sigma_{\mathrm{SO}(2)}$ of the circle group SO(2) is given by

$$\int_{\mathrm{SO}(2)} \phi(S)\mathrm{d}\sigma_{\mathrm{SO}(2)}(S) = (2\pi)^{-1}\int_0^{2\pi} \phi(\mathbf{H}(\theta))\mathrm{d}\theta, \tag{6.1}$$

for all $\phi \in \mathcal{C}(\mathrm{SO}(2))$.

The following theorem characterizes analytic aspects of the metaplectic wave-packet representation associated to the compact subgroup SO(2).

Theorem 6.1. *Let $0 < \theta \leqslant 2\pi$ and $U_\theta := U_{\mathbf{H}(\theta)}$ be the associated metaplectic operator to $\mathbf{H}(\theta)$.*

(1) *For $\theta \neq \pi/2, 3\pi/2$, we have $U_\theta = E_{-\tan\theta}D_{\cos\theta}\mathcal{F}_{\mathbb{R}}^{-1}E_{\tan\theta}\mathcal{F}_{\mathbb{R}}$.*
(2) *For $\theta = \pi/2$, we have $U_{\pi/2} = E_{-1}\mathcal{F}_{\mathbb{R}}^{-1}E_{-1}\mathcal{F}_{\mathbb{R}}E_{-1}$.*
(3) *For $\theta = 3\pi/2$, we have $U_{3\pi/2} = E_{-1}D_{-1}\mathcal{F}_{\mathbb{R}}^{-1}E_{-1}\mathcal{F}_{\mathbb{R}}E_{-1}$.*

Proof.

(1) Let $0 < \theta \leqslant 2\pi$ with $\theta \neq \pi/2, 3\pi/2$. Then $a := \cos\theta \neq 0$. Hence, using theorem 3.2 with $a = d$ and $b := \sin\theta = -c$, we get

$$U_\theta = E_{ca^{-1}}D_a\mathcal{F}_{\mathbb{R}}^{-1}E_{-a^{-1}b}\mathcal{F}_{\mathbb{R}} = E_{-\tan\theta}D_{\cos\theta}\mathcal{F}_{\mathbb{R}}^{-1}E_{\tan\theta}\mathcal{F}_{\mathbb{R}}.$$

(2) and (3) are straightforward from theorem 3.2. □

Also, we can deduce the following result.

Proposition 6.2. *$\mathbb{G}(\mathrm{SO}(2))$ is a non-Abelian, non-compact, and unimodular group with a Haar measure given by*

$$\int_{\mathbb{G}(\mathrm{SO}(2))} F(S,\lambda)\mathrm{d}m_{\mathbb{G}(\mathrm{SO}(2))}(S,\lambda) = (2\pi)^{-1}\int_0^{2\pi}\int_{\mathbb{R}\times\widehat{\mathbb{R}}} F(\mathbf{H}(\theta),\lambda)\mathrm{d}\theta\mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(\lambda),$$

for all $F \in \mathcal{C}_c(\mathbb{G}(\mathrm{SO}(2)))$.

Let $\psi \in L^2(\mathbb{R})$ be a non-zero window function. The metaplectic wave-packet transform can be regarded as $\mathcal{V}_\psi : L^2(\mathbb{R}) \to L^2((0,2\pi]\times\mathbb{R}\times\widehat{\mathbb{R}})$ given by $f \mapsto \mathcal{V}_\psi f$, where

$$\mathcal{V}_\psi f(\theta,x,\omega) := \langle f, U_\theta M_\omega T_x\psi\rangle_{L^2(\mathbb{R})}, \tag{6.2}$$

for all $(\theta,x,\omega) \in (0,2\pi]\times\mathbb{R}\times\widehat{\mathbb{R}}$.

The Plancherel formula for (6.2) reads as follows;

$$\int_0^{2\pi}\int_{\mathbb{R}\times\widehat{\mathbb{R}}}|\langle f, U_\theta M_\omega T_x\psi\rangle_{L^2(\mathbb{R})}|^2 \mathrm{d}\theta \mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(x,\omega) = (2\pi)\cdot\|f\|^2_{L^2(\mathbb{R})}\cdot\|\psi\|^2_{L^2(\mathbb{R})}. \tag{6.3}$$

Then (6.3) guarantees the following reconstruction formula;

$$f = (2\pi)^{-1}\cdot\|\psi\|^{-2}_{L^2(\mathbb{R})}\cdot\int_0^{2\pi}\int_{\mathbb{R}\times\widehat{\mathbb{R}}}\mathcal{V}_\psi f(\theta,x,\omega)U_\theta M_\omega T_x\psi\ \mathrm{d}\theta \mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(x,\omega). \tag{6.4}$$

6.1.2. Finite subgroups of $\mathrm{SL}(2,\mathbb{R})$*.* Since every subgroup of the circle group is either dense or finite, we deduce that any closed proper subgroup of the circle group is finite.

Let $N\in\mathbb{N}$ be a positive integer and $\mathbb{T}_N := \{z\in\mathbb{T} : z^N = 1\}$. Then $\mathbb{T}_N$ is a finite subgroup of $\mathbb{T}$ of order N. One can also check that, $\mathrm{SO}_N(2) := \{\mathbf{H}(2\pi k/N) :\ k = 0, ..., N-1\}$, is a finite subgroup of SO(2) of order N. Also, it is easy to check that any finite subgroup of $\mathrm{SL}(2,\mathbb{R})$ of order N, is conjugated to $\mathrm{SO}_N(2)$.

(i) Finite circle groups Let $N\in\mathbb{N}$ be a positive integer. The normalized Haar measure of $\mathrm{SO}_N(2)$ is given by

$$\int_{\mathrm{SO}_N(2)}\phi(S)\mathrm{d}\sigma_{\mathrm{SO}_N(2)}(S) := \frac{1}{N}\sum_{k=0}^{N-1}\phi(\mathbf{H}(2\pi k/N)),$$

for all $\phi : \mathrm{SO}_N(2)\to\mathbb{C}$.

Proposition 6.3. *Let $N\in\mathbb{N}$ be a positive integer. Then $\mathbb{G}(\mathrm{SO}_N(2))$ is a non-Abelian, non-compact, and unimodular group with a Haar measure given by*

$$\int_{\mathbb{G}(\mathrm{SO}_N(2))}F(S,\lambda)\mathrm{d}m_{\mathbb{G}(\mathrm{SO}(2))}(S,\lambda) = \frac{1}{N}\sum_{k=0}^{N-1}\int_{\mathbb{R}\times\widehat{\mathbb{R}}}F(\mathbf{H}(2\pi k/N),\lambda)\mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(\lambda),$$

for all $F\in\mathcal{C}_c(\mathbb{G}(\mathrm{SO}_N(2)))$.

Let $\psi\in L^2(\mathbb{R})$ be a non-zero window function. The metaplectic wave-packet transform can be regarded as $\mathcal{V}_\psi : L^2(\mathbb{R})\to L^2(\mathbb{Z}_N\times\mathbb{R}\times\widehat{\mathbb{R}})$ given by $f\mapsto\mathcal{V}_\psi f$, where

$$\mathcal{V}_\psi f(k,x,\omega) := \langle f, U_{2\pi k/N}M_\omega T_x\psi\rangle_{L^2(\mathbb{R})}, \tag{6.5}$$

for all $(k,x,\omega)\in\mathbb{Z}_N\times\mathbb{R}\times\widehat{\mathbb{R}}$.

The Plancherel formula for (6.5) reads as follows;

$$\sum_{k=0}^{N-1}\int_{\mathbb{R}\times\widehat{\mathbb{R}}}|\langle f, U_{2\pi k/N}M_\omega T_x\psi\rangle_{L^2(\mathbb{R})}|^2\mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(x,\omega) = N\cdot\|f\|^2_{L^2(\mathbb{R})}\cdot\|\psi\|^2_{L^2(\mathbb{R})}. \tag{6.6}$$

Then (6.6) guarantees the following reconstruction formula;

$$f = N^{-1}\cdot\|\psi\|^{-2}_{L^2(\mathbb{R})}\cdot\sum_{k=0}^{N-1}\int_{\mathbb{R}\times\widehat{\mathbb{R}}}\mathcal{V}_\psi f(k,x,\omega)U_{2\pi k/N}M_\omega T_x\psi\ \mathrm{d}\mu_{\mathbb{R}\times\widehat{\mathbb{R}}}(x,\omega). \tag{6.7}$$

6.2. The case $d>1$

It is well-known that $\mathcal{K}_d$ is the maximal compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$, see [18–20, 45] and the classical list of references therein. Also, it can readily be check that

$$\mathcal{K}_d = \mathrm{Sp}(\mathbb{R}^d) \cap \mathrm{O}(2d, \mathbb{R}).$$

The following theorem presents an explicit construction for metaplectic operators associated to the maximal compact subgroup $\mathcal{K}_d$.

Theorem 6.4. *Let $S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathcal{K}_d$ be given. Let $\mathbb{I}_A \subseteq \mathbb{N}_d$ be such that the columns of A indexed by $\mathbb{I}_A$ form a basis for $\mathcal{R}(A)$ and $\Lambda \in M_{d\times d}(\mathbb{Z})$ be the diagonal matrix whose diagonal is 0 at $\mathbb{I}_A$ and 1 at the complementary set $\mathbb{N}_d \setminus \mathbb{I}_A$. Let $H := A - B\Lambda$ and $Q := B + A\Lambda$. Then $H \in \mathrm{GL}(d, \mathbb{R})$ and the unitary operator*

$$U_S := E_{QH^{-1}} D_H \mathcal{F}_{\mathbb{R}^d}^{-1} E_{-H^{-1}B} \mathcal{F}_{\mathbb{R}^d} E_{-\Lambda} \tag{6.8}$$

is the metaplectic operator associated to the symplectic matrix S.

Next we can also present the following characterizations.

Corollary 6.5. *Let $d > 1$ and $S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathcal{K}_d$.*

(1) *If $A \in \mathrm{GL}(d, \mathbb{R})$ we have $U_S = E_{BA^{-1}} D_A \mathcal{F}_{\mathbb{R}^d}^{-1} E_{A^{-1}B} \mathcal{F}_{\mathbb{R}^d}$.*
(2) *If $A = 0$, then $B \in \mathrm{O}(d, \mathbb{R})$ and we have $U_S = E_I D_B \mathcal{F}_{\mathbb{R}^d}^{-1} E_{-I} \mathcal{F}_{\mathbb{R}^d} E_{-I}$.*
(3) *If $B = 0$, then $A \in \mathrm{O}(d, \mathbb{R})$ and we have $U_S = D_A$.*

Proof. Let $d > 1$ and $S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathcal{K}_d$.

(1) Let $A \in \mathrm{GL}(d, \mathbb{R})$. Then, $\Lambda = 0$ and hence $H = A$ and $Q = B$. Thus, using theorem 6.4, we deduce that

$$U_S = E_{QH^{-1}} D_H \mathcal{F}_{\mathbb{R}^d}^{-1} E_{-H^{-1}B} \mathcal{F}_{\mathbb{R}^d} E_{-\Lambda} = E_{BA^{-1}} D_A \mathcal{F}_{\mathbb{R}^d}^{-1} E_{A^{-1}B} \mathcal{F}_{\mathbb{R}^d}.$$

(2) Let $A = 0$. Then $\Lambda = I$. Also, since $AA^T + BB^T = I$ and $A^TA + B^TB = I$, we get $B^TB = BB^T = I$. Hence, $B \in \mathrm{O}(d, \mathbb{R})$ and $-H = Q = B$. Thus, using theorem 6.4, we deduce that

$$U_S = E_{QH^{-1}} D_H \mathcal{F}_{\mathbb{R}^d}^{-1} E_{-H^{-1}B} \mathcal{F}_{\mathbb{R}^d} E_{-\Lambda} = E_{-I} D_{-B} \mathcal{F}_{\mathbb{R}^d}^{-1} E_I \mathcal{F}_{\mathbb{R}^d} E_{-I}.$$

(3) Let $B = 0$. Since $AA^T + BB^T = I$ and $A^TA + B^TB = I$, we get $A^TA = AA^T = I$. Therefore, $A \in \mathrm{O}(d, \mathbb{R})$ and hence $\Lambda = 0$. Then, $H = A$ and $Q = 0$. Thus, using theorem 6.4, we deduce that

$$U_S = E_{QH^{-1}} D_H \mathcal{F}_{\mathbb{R}^d}^{-1} E_{-H^{-1}B} \mathcal{F}_{\mathbb{R}^d} E_{-\Lambda} = D_A.$$

□

6.2.1. The maximal compact subgroup $\mathcal{K}_d$. Let $\mathbb{H} = \mathcal{K}_d$ be the maximal compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ and $\sigma_{\mathcal{K}_d}$ be the probability measure over the compact group $\mathcal{K}_d$. In this case, the associated multivariate symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ is the underlying manifold $\mathcal{K}_d \times \mathbb{R}^d \times \widehat{\mathbb{R}^d}$, equipped with the following group law

$$(S, \lambda) \rtimes (S', \lambda') = (SS', S'^{-1}\lambda + \lambda'),$$

for all $(S, \lambda), (S', \lambda') \in \mathbb{G}(\mathbb{H})$. Then $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(S, \lambda) = \mathrm{d}\sigma_{\mathrm{O}(d)}(S)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the symplectice wave-packet group $\mathbb{G}(\mathbb{H})$. The multivariate symplectic wave-packet representation $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is given by $\Gamma_{\mathbb{H}}(S, \lambda) = U_S\pi(\lambda)$ for all $(S, \lambda) \in \mathbb{G}(\mathbb{H})$.

The multivariate metaplectic wave-packet transform of $f\in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(S,\lambda)=\langle f,\Gamma_{\mathbb{H}}(S,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}=\langle f,U_S\pi(\lambda)\psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(S,\lambda)\in\mathbb{G}(\mathbb{H})$. Then, corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathcal{K}_d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}}|\langle f,\Gamma_{\mathbb{H}}(S,\lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2\,\mathrm{d}\sigma_{\mathcal{K}_d}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)=\|\psi\|^2_{L^2(\mathbb{R}^d)}\cdot\|f\|^2_{L^2(\mathbb{R}^d)},$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f=\|\psi\|^{-2}_{L^2(\mathbb{R}^d)}\cdot\int_{\mathcal{K}_d}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}}\mathcal{V}_\psi f(S,\lambda)\Gamma_{\mathbb{H}}(S,\lambda)\psi\;\mathrm{d}\sigma_{\mathcal{K}_d}(S)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda).$$

6.2.2. Compact subgroups of $\mathcal{K}_d$ generated by compact subgroups of $\mathrm{GL}(d,\mathbb{R})$. Let $\mathbb{K}$ be a compact subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$. Then

$$\mathbb{H}:=\left\{\widetilde{H}:=\begin{pmatrix}H & 0\\ 0 & (H^T)^{-1}\end{pmatrix}:H\in\mathbb{K}\right\},$$

is a compact subgroup of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$. Also, it is easy to check that $U_{\widetilde{H}}=D_H$ for all $H\in\mathbb{K}$, see [27].

The subgroup $\mathbb{K}=\mathrm{O}(d,\mathbb{R})$ is the most significant compact subgroup of $\mathrm{GL}(d,\mathbb{R})$. The compact subgroup $\mathrm{O}(d,\mathbb{R})$, or simply just $\mathrm{O}(d)$, is the multiplicative matrix group consists of all $d\times d$-orthogonal matrices. That is,

$$\mathrm{O}(d,\mathbb{R}):=\{A\in M_{d\times d}(\mathbb{R}):A^TA=I_{d\times d}\}.$$

The compact group $\mathrm{O}(d)$ is a $\frac{d(d-1)}{2}$-dimensional real Lie group and it is non-connected. The probability (normalized Haar) measure over $\mathrm{O}(d)$ is given by

$$\int_{\mathrm{O}(d)}\phi(H)\mathrm{d}\sigma_{\mathrm{O}(d)}(H)=\int_{\mathbb{S}^{d-1}}\widetilde{\phi}(y)\mathrm{d}\nu_{d-1}(y),$$

where ν_{d-1} is the normalized surface measure on $\mathbb{S}^{d-1}$, that is the standard unit sphere in $\mathbb{R}^d$, and the function $\widetilde{\phi}:\mathbb{S}^{d-1}\to\mathbb{C}$ is given by $\widetilde{\phi}(Hx):=\phi(H)$ for all $A\in\mathrm{O}(d)$ and a fixed point $x\in\mathbb{S}^{d-1}$.

Let $\mathbb{K}$ be a compact subgroup of $\mathrm{GL}(d,\mathbb{R})$ with the probability Haar measure σ_{K}. Then $\langle.,.\rangle_{\mathbb{K}}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ given by

$$(x,y)\mapsto\langle x,y\rangle_{\mathbb{K}}:=\int_{\mathbb{K}}\langle Hx,Hy\rangle\mathrm{d}\sigma_{\mathbb{K}}(H),$$

for all $x,y\in\mathbb{R}^d$, is a positive and symmetric bilinear from on $\mathbb{R}^d$. Also, it is a $\mathbb{K}$-invariant form, that is

$$\langle Hx,Hy\rangle_{\mathbb{K}}=\langle x,y\rangle_{\mathbb{K}},$$

for all $x,y\in\mathbb{R}^d$ and $H\in\mathbb{K}$. Thus, there exists a positive definite matrix $\mathbf{D}\in M_{d\times d}(\mathbb{R})$ such that

$$\langle x,y\rangle_{\mathbb{K}}=\langle x,\mathbf{D}y\rangle,\ \forall\, x,y\in\mathbb{R}^d.$$

Let $\mathbf{D} = B^T B$ be the Cholesky factorization of D with B invertible. Then we deduce that $B\mathbb{K}B^{-1} \subset \mathrm{O}(d)$, or equivalently $\mathbb{K} \subset B^{-1}\mathrm{O}(d)B$. This implies that, up to conjugation, $\mathrm{O}(d)$ is the maximal compact subgroup of $\mathrm{GL}(d, \mathbb{R})$.

(i) The orthogonal group. By the above argument and theoretical motivation, first we shall focus on analytic and constructive analysis of multivariate metaplectic wave-packet representations over the block diagonal compact subgroups of $\mathcal{K}_d$ generated by $\mathbb{K} = \mathrm{O}(d)$.

In this case, the associated multivariate symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ is isomorphic with the underlying manifold $\mathrm{O}(d) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{O}(d) \times \mathbb{R}^d \times \mathbb{R}^d$, equipped with the following group law

$$(H, x, \omega) \rtimes (H', x', \omega') = (HH', H'^{-1}x + x', H'\omega + \omega'),$$

for all $(H, x, \omega), (H', x', \omega') \in \mathrm{O}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\widetilde{H}, \lambda) = \mathrm{d}\sigma_{\mathrm{O}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the symplectice wave-packet group $\mathbb{G}(\mathbb{H})$. The multivariate symplectic wave-packet representation $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is given by $\Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega) = D_H T_x M_\omega$ for all $(\widetilde{H}, x, \omega) \in \mathbb{G}(\mathbb{H})$.

The multivariate metaplectic wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(\widetilde{H}, x, \omega) \in \mathbb{G}(\mathbb{H})$.

Then, corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{O}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2 \mathrm{d}\sigma_{\mathrm{O}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)},$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{O}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(\widetilde{H}, \lambda)\Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi \; \mathrm{d}\sigma_{\mathrm{O}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda).$$

(ii) The special orthogonal group. For $d > 2$, the special orthogonal $\mathbb{K} := \mathrm{SO}(d, \mathbb{R})$ or just $\mathrm{SO}(d)$ is given by

$$\mathrm{SO}(d) := \{A \in \mathrm{O}(d) : \det A = 1\}.$$

It is a connected and compact real Lie group.

In this case, the associated multivariate symplectic wave-packet group $\mathbb{G}(\mathbb{H})$ is isomorphic with the underlying manifold $\mathrm{SO}(d) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{SO}(d) \times \mathbb{R}^d \times \mathbb{R}^d$, which is equipped with the following group law

$$(H, x, \omega) \rtimes (H', x', \omega') = (HH', H'^{-1}x + x', H'\omega + \omega'),$$

for all $(H, x, \omega), (H', x', \omega') \in \mathrm{SO}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\widetilde{H}, \lambda) = \mathrm{d}\sigma_{\mathrm{SO}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the multivariate symplectic wave-packet group $\mathbb{G}(\mathbb{H})$. The metaplectic wave-packet representation $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is given by $\Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega) = D_H T_x M_\omega$ for all $(H, x, \omega) \in \mathbb{G}(\mathbb{H})$.

The multivariate metaplectic wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(\widetilde{H}, x, \omega) \in \mathbb{G}(\mathbb{H})$.

Then, corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2 \, \mathrm{d}\sigma_{\mathrm{SO}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)},$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(\widetilde{H}, \lambda)\Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi \, \mathrm{d}\sigma_{\mathrm{SO}(d)}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda).$$

(iii) The maximal tori. A circle group is a linear (matrix) group isomorphic to $\mathbb{S}^1$. A torus (tori) is a direct sum of circle groups. Thus any torus is a compact connected Abelian Lie group. A maximal torus (tori) is a torus in a linear (matrix) group which is not contained in any other torus. The rank of a maximal tori T is the number r such that $\mathrm{T} = \oplus_{j=1}^r \mathbb{S}^1$.

The following proposition [39, 40] characterizes structure of a maximal tori of the special orthogonal group $\mathrm{SO}(d)$.

Proposition 6.6. *Let $d > 2$ and T be a maximal tori of $\mathrm{SO}(d)$. Then,*

(1) *if $d = 2r$ with $r \in \mathbb{N}$, then $\mathrm{T} = \oplus_{j=1}^r \mathrm{SO}(2)$.*
(2) *if $d = 2r + 1$ with $r \in \mathbb{N}$, then $\mathrm{T} = (\oplus_{j=1}^r \mathrm{SO}(2)) \oplus \{1\}$.*

In this case, the associated multivariate symplectic wave-packet group $\mathbb{G}(\mathrm{T})$ is isomorphic with the underlying manifold $\mathrm{T} \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{T} \times \mathbb{R}^d \times \mathbb{R}^d$, which is equipped with the following group law

$$(H, x, \omega) \rtimes (H', x', \omega') = (HH', H'^{-1}x + x', H'\omega + \omega'),$$

for all $(H, x, \omega), (H', x', \omega') \in \mathrm{T} \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{G}(\mathbb{H})}(\widetilde{H}, \lambda) = \mathrm{d}\sigma_{\mathrm{T}}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the multivariate symplectic wave-packet group $\mathbb{G}(\mathbb{H})$. The multivariate metaplectic wave-packet representation $\Gamma_{\mathbb{H}} : \mathbb{G}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ is given by $\Gamma_{\mathbb{H}}(\widetilde{H}, x, \omega) = D_H T_x M_\omega$ for all $(\widetilde{H}, x, \omega) \in \mathbb{G}(\mathrm{T})$.

The multivariate metaplectic wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(\widetilde{H}, x, \omega) = \langle f, \Gamma_{\mathrm{T}}(\widetilde{H}, x, \omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_H T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(\widetilde{H}, x, \omega) \in \mathbb{G}(\mathrm{T})$.

Then, corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{T}} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2 \, \mathrm{d}\sigma_{\mathrm{T}}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)},$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{T}} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(\widetilde{H}, \lambda)\Gamma_{\mathbb{H}}(\widetilde{H}, \lambda)\psi \, \mathrm{d}\sigma_{\mathrm{T}}(H)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda).$$

Concluding Remarks. The main purpose of this article was dedicated to presenting a constructive admissibility criterion on closed subgroups of the real symplectic group $\mathrm{Sp}(\mathbb{R}^d)$ which guarantees square integrability of the associated multivariate metaplectic wave-packet representations and hence a valid resolution of the identity operator in the sense of the Hilbert function space $L^2(\mathbb{R}^d)$.

Invoking topological and geometric structure of the real Lie group $\mathrm{Sp}(\mathbb{R}^d)$, there is a high degree of freedom in selecting an admissible subgroup $\mathbb{H}$ of $\mathrm{Sp}(\mathbb{R}^d)$. Among all closed subgroups of $\mathrm{Sp}(\mathbb{R}^d)$, just compact ones are admissible and hence they guarantee a square-integrable multivariate metaplectic wave-packet representation and valid reconstruction formula.

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