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A Switching Approach to Event-Triggered Control

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***Abstract*— Event-trigger is used to obtain the measurements transmission instants in networked control systems. Under continuous measurements it can generate an infinite number of events in finite time (Zeno phenomenon) what makes it inapplicable to the real world systems. Periodic event-trigger avoids this behavior but does not use all the available information. In the present paper we aim to exploit the advantage of the continuous-time measurements and guarantee positive lower bound on the inter-event times. Our approach is based on a switching between periodic sampling and continuous event-trigger. It is applicable to the systems with polytopic-type uncertainties and assures the Input-to-State Stability in the presence of external disturbances and measurement noise. By an example we demonstrate that the switching approach to event-triggered control can reduce the network workload compared to periodic event-trigger.**

I. Introduction

Networked control systems (NCS), that are comprised of sensors, actuators, and controllers connected through a communication network, have been recently extensively studied by researchers from a variety of disciplines [**?**], [**?**], [**?**], [**?**]. One of the main challenges in such systems is that only sampled in time measurements can be transmitted through a communication network. There are different ways of obtaining the sequence of sampling instants that preserve the stability. The simplest approach is to send measurements periodically. However, under periodic sampling the measurements are sent even when the output fluctuation is small and does not significantly change the control signal. To avoid these "redundant" packets one can use *continuous event-trigger* [**?**] that sends measurements only when the relative change of the output is large enough. As it has been shown in [**?**] in case of a static output-feedback execution times, implicitly defined by continuous event-trigger, can have a finite limit, i.e. an infinite number of sampling instants is generated in finite time. To avoid this Zeno phenomenon one can use *periodic event-trigger* [**?**], [**?**], [**?**], [**?**] where event-trigger condition is checked periodically in discrete time instants. This approach guarantees a positive lower bound for the inter-event times and fits the case where the sensor measures only sampled in time outputs.

However, when the continuous measurements are available one can use this additional information to improve the control strategy. A way to do this is to wait for some fixed time after the measurement has been sent. Then start to continuously check the event-trigger condition and send the next measurement when it is violated. This natural idea has been implemented in, e.g., [**?**], [**?**], where a dynamic output controller has been studied and event-trigger condition was a function of the system output, state estimate, and the error due to triggering. To obtain the value of a waiting time that preserves the stability following [**?**], [**?**] one needs to solve some special scalar differential equations. In [**?**] some qualitative results concerning practical stability have been obtained for event-trigger with waiting time.

In this work we propose a new approach to event-triggered control that is based on a switching between periodic sampling and continuous event-trigger. In this approach an appropriate waiting time is found from LMI-based conditions. We extend our results to the systems with external disturbances and measurement noise (Section III). In Section IV by the example brought from [**?**] we demonstrate that our method can essentially reduce a workload of the network compared to periodic sampling and periodic event-trigger.

II. A switching approach to event-trigger

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

with a state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^l$. Assume that there exists $K \in \mathbb{R}^{m\times l}$ such that the control signal $u(t) = -Ky(t)$ stabilizes the system (1) and the measurements are sent at time instants

$$0 = t_0 < t_1 < t_2 < \dots, \quad \lim_{k\to\infty} t_k = \infty. \tag{2}$$

Then the closed-loop system has the form

$$\dot{x}(t) = Ax(t) - BKCx(t_k), \quad t \in [t_k, t_{k+1}),\, k \in \mathbb{N}_0, \tag{3}$$

where $\mathbb{N}_0$ is the set of nonnegative integers. According to [**?**] the closed-loop system (3) under periodic sampling $t_k = kh$ can be presented in the form

$$\dot{x}(t) = (A - BKC)x(t) + BKC \int_{t-\tau(t)}^{t} \dot{x}(s)\, ds, \tag{4}$$

where $\tau(t) = t - t_k$ for $t \in [t_k, t_{k+1})$. The system (3) under continuous event-trigger

$$\begin{aligned} t_{k+1} = \min\{t > t_k \,|\, &(y(t) - y(t_k))^T \Omega (y(t) - y(t_k)) \\ &\geq \varepsilon y^T(t) \Omega y(t)\} \end{aligned} \tag{5}$$

with a matrix $\Omega \geq 0$ and a scalar $\varepsilon \geq 0$ can be rewritten as

$$\dot{x}(t) = (A - BKC)x(t) - BKe(t) \tag{6}$$

with $e(t) = y(t_k) - y(t)$ for $t \in [t_k, t_{k+1})$.

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As we mentioned before, under periodic sampling (leading to (4)) "redundant" packets can be sent while continuous event-trigger (that leads to (6)) can cause Zeno phenomenon. To avoid the above drawbacks periodic event-trigger can be used. In this case the sampling instants are given by

$$t_{k+1} = \min\{t_k + ih \,|\, i \in \mathbb{N},\, (y(t_k + ih) - y(t_k))^T \Omega \times (y(t_k + ih) - y(t_k)) > \varepsilon y^T(t_k + ih)\Omega y(t_k + ih)\} \quad (7)$$

with a matrix $\Omega \geq 0$ and a scalar $\varepsilon \geq 0$. The system (3) under (7) can be written as

$$\dot{x}(t) = (A - BKC)x(t) + BKC\int_{t-\tau(t)}^{t} \dot{x}(s)\,ds - BKe(t) \quad (8)$$

with

$$\tau(t) = t - t_k - ih \leq h, \quad e(t) = y(t_k) - y(t_k + ih)$$

for $t \in [t_k + ih, t_k + (i+1)h)$, $i \in \mathbb{N}_0$ such that $t_k + (i+1)h \leq t_{k+1}$. As one can see, the error due to sampling that appears in (4) (the integral term) and the error $e(t)$ due to triggering from (6) are both presented in (8) what makes it more difficult to ensure the stability of (8).

We propose an event-trigger that allows to *separate these errors* by considering the switching between periodic sampling and continuous event-trigger. Namely, after the measurement has been sent, the sensor waits for at least h seconds. During this time the system is described by (4). Then the sensor begins to continuously check the event-trigger condition and sends the measurement when it is violated. During this time the system is described by (6). This leads to the following choice of sampling instants:

$$t_{k+1} = \min\{t \geq t_k + h \,|\, (y(t) - y(t_k))^T\Omega(y(t) - y(t_k)) \geq \varepsilon y^T(t)\Omega y(t)\} \quad (9)$$

with a matrix $\Omega \geq 0$ and scalars $\varepsilon \geq 0$, $h > 0$, where the inter-event times are not less than h. The system (3), (9) can be presented as a switching between (4) and (6):

$$\dot{x}(t) = (A - BKC)x(t) + BKC\int_{t-\tau(t)}^{t} \dot{x}(s)\,ds, \quad t \in [t_k, t_k + h), \quad (10)$$

$$\dot{x}(t) = (A - BKC)x(t) - BKe(t),\ t \in [t_k + h, t_{k+1}), \quad (11)$$

where

$$\begin{aligned} \tau(t) &= t - t_k \leq h, & t \in [t_k, t_k + h), \\ e(t) &= y(t_k) - y(t), & t \in [t_k + h, t_{k+1}). \end{aligned} \quad (12)$$

To obtain the stability conditions for the switched system (10), (11) we use different Lyapunov functions: for (11) we consider

$$V_P(x) = x^T(t)Px(t), \quad P > 0, \quad (13)$$

for (10) we apply the functional from [**?**]:

$$V(t, x_t, \dot{x}_t) = V_P(x(t)) + V_U(t, \dot{x}_t) + V_X(t, x_t), \quad (14)$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h, 0]$, V_P is given by (13),

$$V_U(t, \dot{x}_t) = (h - \tau(t))\int_{t_k}^{t} e^{2\delta(s-t)}\dot{x}^T(s)U\dot{x}(s)\,ds,\ U > 0,$$

$$V_X(t, x_t) = (h - \tau(t))\begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \times \begin{bmatrix} \frac{X+X^T}{2} & -X+X_1 \\ * & -X_1 - X_1^T + \frac{X+X^T}{2} \end{bmatrix}\begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}.$$

Note that the values of V and V_P coincide at the switching instants t_k and $t_k + h$.

Proposition 1: For given scalars $h > 0$, $\varepsilon \geq 0$, $\delta > 0$ let there exist $n \times n$ matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, Y_1, Y_2, Y_3 and $l \times l$ matrix $\Omega \geq 0$ such that

$$\Xi > 0, \quad \Psi_0 \leq 0, \quad \Psi_1 \leq 0, \quad \Phi \leq 0, \quad (15)$$

where

$$\Xi = \begin{bmatrix} P + h\frac{X+X^T}{2} & hX_1 - hX \\ * & -hX_1 - hX_1^T + h\frac{X+X^T}{2} \end{bmatrix},$$

$$\Psi_0 = \begin{bmatrix} \Psi_{11} - X_\delta & \Psi_{12} + h\frac{X+X^T}{2} & \Psi_{13} + X_{1\delta} \\ * & \Psi_{22} + hU & \Psi_{23} - h(X - X_1) \\ * & * & \Psi_{33} - X_{2\delta}|_{\tau=0} \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} \Psi_{11} - \frac{X+X^T}{2} & \Psi_{12} & \Psi_{13} + X - X_1 & hY_1^T \\ * & \Psi_{22} & \Psi_{23} & hY_2^T \\ * & * & \Psi_{33} - X_{2\delta}|_{\tau=h} & hY_3^T \\ * & * & * & -hUe^{-2\delta h} \end{bmatrix},$$

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & -P_2^TBK \\ * & -P_3^T - P_3 & -P_3^TBK \\ * & * & -\Omega \end{bmatrix},$$

$$\begin{aligned}
\Phi_{11} &= P_2^T(A - BKC) + (A - BKC)^TP_2 + \varepsilon C^T\Omega C + 2\delta P, \\
\Phi_{12} &= P + (A - BKC)^TP_3 - P_2^T, \\
\Psi_{11} &= A^TP_2 + P_2^TA + 2\delta P - Y_1 - Y_1^T, \\
\Psi_{12} &= P - P_2^T + A^TP_3 - Y_2, \\
\Psi_{13} &= Y_1^T - P_2^TBKC - Y_3, \\
\Psi_{22} &= -P_3 - P_3^T, \\
\Psi_{23} &= Y_2^T - P_3^TBKC, \\
\Psi_{33} &= Y_3 + Y_3^T, \\
X_\delta &= (1/2 - \delta h)(X + X^T), \\
X_{1\delta} &= (1 - 2\delta h)(X - X_1), \\
X_{2\delta} &= (1/2 - \delta(h - \tau))(X + X^T - 2X_1 - 2X_1^T).
\end{aligned}$$

Then the system (3) under the event-trigger (9) is exponentially stable with a decay rate δ.

Proof. The system (3), (9) is presented in the form of the switched system (10), (11). According to [**?**] the conditions $\Xi > 0$, $\Psi_0 \leq 0$, $\Psi_1 \leq 0$ imply $V \geq \alpha|x(t)|^2$ and $\dot{V} \leq -2\delta V$ for the system (10). Consider (11). Since for $t \in [t_k + h, t_{k+1})$ the relation (9) implies

$$0 \leq \varepsilon x^T(t)C^T\Omega Cx(t) - e^T(t)\Omega e(t), \quad (16)$$

we add (16) to $\dot{V}_P$ to compensate the cross term with $e(t)$. We have

$$\dot{V}_P + 2\delta V_P \le 2x^T P\dot{x} + 2\delta x^T Px + 2[x^T P_2^T + \dot{x}^T P_3^T]\times [(A - BKC)x - BKe - \dot{x}] + [\varepsilon x^T C^T \Omega Cx - e^T \Omega e] = \varphi^T \Phi\varphi \le 0,$$

where $\varphi = \mathrm{col}\{x(t), \dot{x}(t), e(t)\}$. Thus, $\dot{V}_P \le -2\delta V_P$.

The stability of the switched system (10), (11) follows from the fact that at the switching instants t_k and $t_k + h$ the values of V and V_P coincide.

■

By extending the proof from [**?**] we obtain the stability conditions for the system (3), (7) presented in the form (8):

Proposition 2: For given scalars $h > 0$, $\varepsilon \ge 0$, $\delta > 0$ let there exist $n \times n$ matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, Y_1, Y_2, Y_3 and $l \times l$ matrix $\Omega \ge 0$ such that

$$\Xi > 0, \quad \Sigma_0 \le 0, \quad \Sigma_1 \le 0, \tag{17}$$

where

$$\Sigma_0 = \left[\begin{array}{c|c} \overline{\Psi}_0 & \begin{matrix} -P_2^T BK \\ -P_3^T BK \\ 0 \end{matrix} \\ \hline * & -\Omega \end{array}\right], \quad \Sigma_1 = \left[\begin{array}{c|c} \overline{\Psi}_1 & \begin{matrix} -P_2^T BK \\ -P_3^T BK \\ 0 \end{matrix} \\ \hline * & -\Omega \end{array}\right],$$

and $\overline{\Psi}_i = \Psi_i + \varepsilon[I_n\, 0]^T C^T \Omega C[I_n\, 0]$, $i = 0, 1$. Then the system (3) with t_k given by (7) is exponentially stable with a decay rate δ.

Remark 1: The feasibility of (17) implies the feasibility of (15). Therefore, the stability of (3) under (9) can be guaranteed for larger h and ε compared to (7). This allows to reduce the amount of sent measurements (see the example in Section IV). Note that for the fixed h, ε, and Ω under periodic event-trigger (7) the amount of sent measurements is deliberately less than under (9). Indeed, if the measurement is sent at t_k and the event-trigger rule is satisfied at $t_k + h$, according to (7) the sensor will wait till at least $t_k + 2h$ before sending the next measurement, while according to (9) the next measurement can be sent before $t_k + 2h$.

III. L_2-GAIN ANALYSIS AND INPUT-TO-STATE STABILITY

The switching approach can be easily extended to systems with disturbances. Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \\ z(t) &= C_1 x(t) + D_1 u(t), \\ y(t) &= C_2 x(t) + D_2 v(t) \end{aligned} \tag{18}$$

with the state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, controlled output $z \in \mathbb{R}^{n_z}$, measurements $y \in \mathbb{R}^l$, disturbances $w \in \mathbb{R}^{n_w}$, $v \in \mathbb{R}^{n_v}$, and constant matrices of appropriate dimensions. We study the system (18) under the static output-feedback

$$u(t) = Ky(t_k), \quad t \in [t_k, t_{k+1}), \tag{19}$$

where t_k is the sequence of sampling instants given by (9). The control input can be presented in the form

$$\begin{aligned} u(t) &= K[C_2 x(t - \tau(t)) + D_2 v(t - \tau(t))],\ t \in [t_k, t_k + h), \\ u(t) &= K[e(t) + C_2 x(t) + D_2 v(t)],\ t \in [t_k + h, t_{k+1}) \end{aligned}$$

with $e(t)$ and $\tau(t)$ given by (12). Then the closed-loop system has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 KC_2 x(t - \tau(t)) \\ &\qquad + B_2 KD_2 v(t - \tau(t)), \\ z(t) &= C_1 x(t) + D_1 K[C_2 x(t - \tau(t)) + D_2 v(t - \tau(t))], \\ &\qquad t \in [t_k, t_k + h), \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{x}(t) &= (A + B_2 KC_2)x(t) + B_1 w(t) + B_2 Ke(t) \\ &\qquad + B_2 KD_2 v(t), \\ z(t) &= (C_1 + D_1 KC_2)x(t) + D_1 K(e(t) + D_2 v(t)), \\ &\qquad t \in [t_k + h, t_{k+1}). \end{aligned} \tag{21}$$

Define $\tau(t) = 0$ for $t \in [t_k + h, t_{k+1})$. We say that the system (9), (18), (19) has L_2-gain less than γ if for the zero initial condition $x(0) = 0$ and all $w, v \in L_2[0, \infty)$ such that $w^T(t)w(t) + v^T(t - \tau(t))v(t - \tau(t)) \not\equiv 0$ the following relation holds on the trajectories of (9), (18), (19):

$$J = \int_0^\infty \Big\{ z^T(t)z(t) - \gamma^2[w^T(t)w(t) + v^T(t - \tau(t))v(t - \tau(t))] \Big\}\, dt < 0. \tag{22}$$

Proposition 3: For given scalars $\gamma > 0$, $\delta > 0$, $h > 0$, $\varepsilon \ge 0$ let there exist $n \times n$ matrices $P > 0$, $U > 0$, X, X_1, P_2, P_3, Y_1, Y_2, Y_3 and $l \times l$ matrix $\Omega \ge 0$ such that

$$\Xi > 0, \quad F \le 0, \quad G \le 0, \quad H \le 0, \tag{23}$$

where Ξ is given in Proposition 1,

$$F = \begin{bmatrix} F_{11} & F_{12} & F_{13} & P_2^T B_1 & F_{15} \\ * & F_{22} & F_{23} & P_3^T B_1 & F_{25} \\ * & * & F_{33} & 0 & F_{35} \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & F_{55} \end{bmatrix},$$

$$G = \begin{bmatrix} G_{11} & G_{12} & G_{13} & hY_1^T & P_2^T B_1 & G_{16} \\ * & G_{22} & G_{23} & hY_2^T & P_3^T B_1 & G_{26} \\ * & * & G_{33} & hY_3^T & 0 & G_{36} \\ * & * & * & G_{44} & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & G_{66} \end{bmatrix},$$

$$H = \begin{bmatrix} H_{11} & H_{12} & P_2^T B_2 K & P_2^T B_1 & H_{15} & H_{16} \\ * & H_{22} & H_{23} & P_3^T B_1 & H_{25} & 0 \\ * & * & -\Omega & 0 & 0 & (D_1 K)^T \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & H_{55} & H_{56} \\ * & * & * & * & * & -I \end{bmatrix},$$

$$\begin{aligned} G_{11} &= -(X + X^T)/2 + 2\delta P + C_1^T C_1 - Y_1 - Y_1^T \\ &\quad + P_2^T A + A^T P_2, \\ G_{12} &= P - Y_2 + A^T P_3 - P_2^T, \\ G_{13} &= X - X_1 + (C_1^T D_1 + P_2^T B_2)KC_2 - Y_3 + Y_1^T, \\ G_{16} &= F_{15} = (C_1^T D_1 + P_2^T B_2)KD_2, \\ G_{22} &= H_{22} = -P_3^T - P_3, \\ G_{23} &= Y_2^T + P_3^T B_2 KC_2, \\ G_{26} &= F_{25} = P_3^T B_2 KD_2, \end{aligned}$$

$$
\begin{aligned}
G_{33} &= -(X+X^T)/2 + X_1 + X_1^T \\
&\quad + (D_1KC_2)^T D_1KD_2 + Y_3 + Y_3^T, \\
G_{36} &= F_{35} = (D_1KC_2)^T D_1KD_2, \\
G_{44} &= -he^{-2\delta h}U, \\
G_{66} &= F_{55} = -\gamma^2 I + (D_1KC_2)^T D_1KD_2, \\
F_{11} &= G_{11} + \delta h(X+X^T), \\
F_{12} &= G_{12} + h(X+X^T)/2, \\
F_{13} &= G_{13} + 2\delta h(X_1 - X), \\
F_{22} &= G_{22} + hU, \\
F_{23} &= G_{23} + h(X_1 - X), \\
F_{33} &= G_{33} + \delta h(X + X^T - 2X_1 - 2X_1^T), \\
H_{11} &= P_2^T(A + B_2KC_2) + (A + B_2KC_2)^T P_2 \\
&\quad + \varepsilon C_2^T \Omega C_2 + 2\delta P, \\
H_{12} &= P + (A + B_2KC_2)^T P_3 - P_2^T, \\
H_{15} &= P_2^T B_2KD_2 + \varepsilon C_2^T \Omega D_2, \\
H_{16} &= (C_1 + D_1KC_2)^T, \\
H_{23} &= P_3^T B_2K, \\
H_{25} &= P_3^T B_2KD_2, \\
H_{55} &= -\gamma^2 I + \varepsilon D_2^T \Omega D_2, \\
H_{56} &= (D_1KD_2)^T.
\end{aligned}
$$

Then the system (18), (19) under the event-trigger (9) is internally exponentially stable with the decay rate δ and has L_2-gain less than γ.

Proof. The system (9), (18), (19) is presented in the form (20), (21). For the system (20) consider the functional (14). Calculating the derivatives we obtain:

$$\dot{V}_P = 2x^T P\dot{x}, \tag{24}$$

$$
\begin{aligned}
\dot{V}_X = &-\begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}^T \begin{bmatrix} \frac{X+X^T}{2} & X_1 - X \\ * & \frac{X+X^T}{2} - X_1 - X_1^T \end{bmatrix} \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix} \\
&+ (h-\tau)\dot{x}\left[(X+X^T)x + 2(X_1 - X)x(t_k)\right],
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\dot{V}_U \le -2\delta V_U - e^{2\delta h}\int_{t-\tau(t)}^{t} \dot{x}^T(s)U\dot{x}(s)\,ds \\
+ (h-\tau)\dot{x}^T U\dot{x}.
\end{aligned} \tag{26}
$$

Define

$$f(t) = \frac{1}{\tau(t)}\int_{t-\tau(t)}^{t} \dot{x}(s)\,ds.$$

Then Jensen's inequality [**?**] implies

$$-e^{2\delta h}\int_{t-\tau(t)}^{t} \dot{x}^T(s)U\dot{x}(s)\,ds \le -\tau(t)e^{2\delta h}f^T(t)Uf(t). \tag{27}$$

Similar to [**?**] we will add the following expressions to $\dot{V}$:

$$
\begin{aligned}
0 = 2[x^T P_2^T + \dot{x}^T P_3^T][Ax + B_1 w \\
+ B_2K(C_2x(t-\tau) + D_2v(t-\tau)) - \dot{x}],
\end{aligned} \tag{28}
$$

$$0 = 2[x^T Y_1^T + \dot{x}^T Y_2^T + x^T(t-\tau)Y_3^T][-x + x(t_k) + \tau f]. \tag{29}$$

By summing up (24), (25), (26), (28), (29), using (27), and substituting z from (20) we find that

$$
\begin{aligned}
\dot{V} + 2\delta V + z^T z - \gamma^2[w^T w + v^T(t-\tau(t))v(t-\tau(t))] \\
\le \eta^T N(\tau)\eta,
\end{aligned}
$$

where $\eta = \mathrm{col}\{x, \dot{x}, x(t-\tau(t)), f, w, v(t-\tau(t))\}$ and the matrix-function $N(\tau)$ is affine in τ. The condition $F \le 0$ implies $N(0) \le 0$ and $G \le 0$ implies $N(h) \le 0$. Therefore, $N(\tau) \le 0$ for any $\tau \in [0, h]$.

Now consider the system (21). Event-trigger (9) implies

$$0 \le -e^T\Omega e + \varepsilon[C_2x + D_2v]^T\Omega[C_2x + D_2v]. \tag{30}$$

By summing up (24), (30), and

$$
\begin{aligned}
0 = 2[x^T P_2^T + \dot{x}^T P_3^T][(A + B_2KC_2)x + B_1w \\
+ B_2K(e + D_2v) - \dot{x}]
\end{aligned} \tag{31}
$$

and using Schur complement [**?**] for $z^T z$ with z given in (21) we find that

$$\dot{V}_P + 2\delta V_P + z^T z - \gamma^2[w^T w + v^T v] \le \nu^T H\nu \le 0,$$

where $\nu = \mathrm{col}\{x, \dot{x}, e, w, v\}$.

Define

$$\overline{V} = \begin{cases} V, & t \in [t_k, t_k + h), \\ V_P, & t \in [t_k + h, t_{k+1}). \end{cases}$$

This function is continuous since $V = V_P$ at time instants t_k and $t_k + h$, and

$$\dot{\overline{V}} + 2\delta\overline{V} + z^T z - \gamma^2[w^T w + v^T(t-\tau(t))v(t-\tau(t))] \le 0. \tag{32}$$

Note that $\tau(t) = 0$ for $t \in [t_k + h, t_{k+1})$. For $w \equiv 0$, $v \equiv 0$ (32) implies

$$\dot{\overline{V}} \le -2\delta\overline{V}.$$

Therefore, the system (18), (19) is internally exponentially stable with the decay rate δ. By integrating (32) from 0 to ∞ with $x(0) = 0$ we obtain (22).

■

Corollary 1: If relations in (23) are valid with $C_1 = 0$, $D_1 = 0$ then the system (18), (19) under the sampling (9) is Input-to-State Stable with respect to the disturbance $\bar{w}(t) = \mathrm{col}\{w(t), v(t-\tau(t))\}$.

*Proof.*If the function $\bar{w}^T(t)\bar{w}(t)$ is bounded by Δ^2 then (32) with $C_1 = 0$, $D_1 = 0$ transforms to

$$\dot{\overline{V}} \le -2\delta\overline{V} + \gamma^2\Delta^2.$$

The assertion of the corollary follows from the comparison principle.

■

Remark 2: The switching approach proposed in this paper can be extended to the systems with network-induced delays [**?**].

Remark 3: Differently from periodic systems approach considered in [**?**] our method is applicable to linear systems

TABLE I

AVERAGE AMOUNT OF SENT MEASUREMENTS (SM)

	ε	h	SM
Periodic sampling	—	1.173	18
Event-trigger (7)	4.6×10^{-3}	1.115	17.47
Event-trigger (7)	0.555	0.344	24.8
Switching approach (9)	0.555	0.899	11.13

with polytopic-type uncertainties. Indeed, LMIs of Propositions 1, 2, and 3 are affine in A, B, B_1, and B_2. Therefore, in the case of system matrices from the uncertain time-varying polytope

$$\mathcal{X} = \sum_{j=1}^{M} \mu_j(t)\mathcal{X}_j, \quad 0 \le \mu_j(t) \le 1, \quad \sum_{j=1}^{M} \mu_j(t) = 1,$$

where $\mathcal{X}_j = [A^{(j)}\, B^{(j)}]$ for Propositions 1, 2 and $\mathcal{X}_j = [A^{(j)}\, B_1^{(j)}\, B_2^{(j)}]$ for Proposition 3, to guarantee the robust stability of the system one needs to solve these LMIs simultaneously for all the M vertices $\mathcal{X}_j$ applying the same decision matrices.

IV. EXAMPLES

Example 1 [**?**]. Consider the system (3) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = I, \quad K = \begin{bmatrix} -1 & 4 \end{bmatrix}.$$

For $\varepsilon = 0$ (9) transforms into periodic sampling, therefore, Proposition 1 can be used to obtain the maximum period h. Under periodic sampling the amount of sent measurements is $\left[\frac{T_f}{h}\right] + 1$, where T_f is the time of simulation and $[\cdot]$ is the integer part of a given number. To obtain the amount of sent measurements for t_k given by (7) (or (9)), for each $\varepsilon = i \times 10^{-4}$ $(i = 0, \dots, 10^4)$ we find the maximum h that satisfies Proposition 2 (or Proposition 1) and for each pair of (ε, h) we perform numerical simulations with $T_f = 20$ for several initial conditions given by

$$(x_1(0), x_2(0)) = \left(10\cos\left(\frac{2\pi}{30}k\right), 10\sin\left(\frac{2\pi}{30}k\right)\right)$$

with $k = 1, \dots, 30$. Then we choose the pair (ε, h) that ensures the minimum average amount of sent measurements. In this example the best result was achieved under periodic sampling ($\varepsilon = 0$). Proposition 1 gives $h = 0.356$ for $\delta = 0.24$ and $h = 0.424$ for $\delta = 0.001$. Both event-triggers (7) and (9) did not succeed in reducing the network workload.

Example 2 [**?**]. Consider the system (3) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad K = 3. \tag{33}$$

We obtained the amount of sent measurements as described in Example 1 (taking $\delta = 0.24$, $T_f = 20$). As one can see from Table I periodic event-triggered (7) does not give any significant improvement compared to periodic sampling, while the event-trigger (9) allows to reduce the average amount of sent measurements by almost 40%. In Figs. 1 and 2 one can see the results of numerical simulations for the event-triggers (7) and (9). The vertical lines correspond to the time instants when the measurements are sent. The event-trigger (7) allows to skip the sending of two measurements (after t_4 and t_{10}), while (9) results in large inter sampling times $[t_2, t_3]$, $[t_4, t_5]$, etc. This allows to significantly reduce the network workload while the decay rate of convergence is preserved.

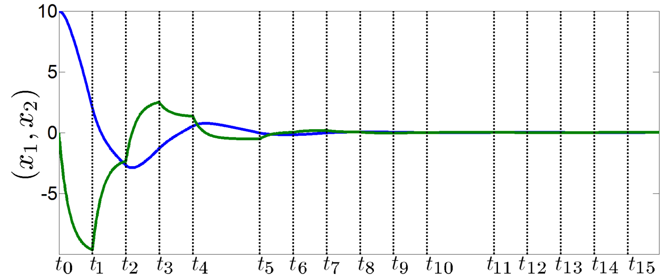

Fig. 1. Event-trigger (7): simulation of the system (3), (33), where $\varepsilon = 4.6 \times 10^{-3}$, $h = 1.115$, $[x_1(0), x_2(0)] = [10, 0]$.

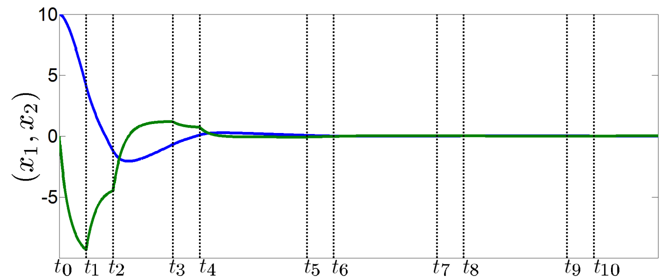

Fig. 2. Event-trigger (9): simulation of the system (3), (33), where $\varepsilon = 0.555$, $h = 0.899$, $[x_1(0), x_2(0)] = [10, 0]$.

V. CONCLUSIONS

We proposed a new approach to event-triggered control under the continuous-time measurements that ensures the given lower bound for inter-event times and can significantly reduce the workload of the network. Our idea is based on a switching between periodic sampling and continuous event-trigger. We extended this approach to L_2-gain and ISS analysis of perturbed system. Our method is applicable to the linear systems with polytopic-type uncertainties.