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The Navier–Stokes regularity problem

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There is currently no proof guaranteeing that, given a smooth initial condition, the three-dimensional Navier–Stokes equations have a unique solution that exists for all positive times. This paper reviews the key rigorous results concerning the existence and uniqueness of solutions for this model. In particular, the link between the regularity of solutions and their uniqueness is highlighted.

1. The Clay Millennium Problem

The 3D Navier–Stokes equations (NSE) that model the flow of a viscous, incompressible fluid, are

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0,$$

subject to an initial condition $u(x,0) = u_0(x)$. Here $u(x,t)$ denotes the time-dependent three-component fluid velocity, $\nu > 0$ is the kinematic viscosity, and p is the scalar pressure. Throughout the rest of this paper, for notational simplicity ν is set to be equal to 1, which can be achieved by the rescaling $u_\nu(x,t) = \nu u(x, \nu t)$.

Usually the problem is posed on one of the following three domains:

- a bounded domain Ω with Dirichlet boundary conditions $u|_{\partial \Omega} = 0$ (the 'most physical' case);
- the whole space $\mathbb{R}^3$, requiring 'sufficient decay' at infinity (this makes the problem amenable to techniques from harmonic analysis, e.g. use of the Fourier transform); or
- the three-dimensional torus $\mathbb{T}^3$, i.e. $[0, 2\pi)^3$ with periodic boundary conditions, and the convenient zero-momentum constraint $\int_{\mathbb{T}^3} u = 0$ (an artificial but useful setting, since the domain here is both bounded and boundaryless).

Here we discuss the 'regularity problem' for these equations, which can be formulated as follows.

Navier–Stokes Regularity Problem If u_0 is smooth, do the equations have a (unique) smooth solution that exists for all $t > 0$?

This is one of the seven Clay Millennium Prize Problems, the solution of which (either positive or negative) will be awarded with a prize of one million dollars. (Note, however, that the formal Clay Prize statement of the problem asks only about the domains without boundary, i.e. $\mathbb{R}^3$ and $\mathbb{T}^3$, see Fefferman [9]).

This paper will show that much of our current understanding of the problem is based on simple bounds that can be derived for smooth solutions. These employ only techniques that are likely to have been available to Stokes, although turning these into rigorous proofs required 20th-Century mathematics. It will also be demonstrated how the issue of uniqueness is closely connected to the smoothness of solutions.

For a more detailed, but nevertheless accessible, discussion of these topics see Doering & Gibbon [6]; for a more complete mathematical treatment see the review paper by Galdi [13] or the books by Constantin & Foias [10], Robinson, Rodrigo, & Sadowksi [28], or Temam [35]. This paper is not intended as an in-depth review of the mathematical state-of-the-art for this model, and experts are quite likely to find that their favourite recent result is not here; the book by Lemarié-Rieusset [20] would be a good place to start for something in this direction.

2. The significance of rigorous existence and uniqueness results

The Navier–Stokes equations are well established as the canonical model of the flow of viscous incompressible fluids, and the starting point for countless theoretical and numerical investigations of fluid behaviour. As such, it is natural to ask what would be gained from a *theorem* guaranteeing the existence of unique solutions to these equations. There are a number of possible answers, some of which I hope are more convincing to those naturally sceptical about such things than the vague reply that it is unsatisfying not to have a rigorous proof.

- Lack of uniqueness would mean a lack of predictive power, so uniqueness of solutions is crucial for a 'good model'. However, it is also of interest to have a way to check that a particular solution is unique, even in the absence of a general uniqueness result, and there are ways to do this (see Sections 5(b) and 6).

- Without an existence result, what do numerical calculations approximate? Put more abstractly, how can one prove the convergence of a numerical scheme without being able to guarantee the existence of an underlying solution towards which to converge? (This is in fact possible under the *assumption* that a sufficiently smooth solution does exist, see Chernyshenko et al. [3].)

- One could hope that a better mathematical understanding of the equations will lead to an enhanced physical understanding of the phenomena they model. (It is probably more reasonable to expect that a better mathematical understanding will *require* an enhanced physical understanding.)

- Some very natural quantitative bounds on solutions are lacking, e.g. how to estimate the maximum value of the velocity in the future given the initial velocity profile. (In fact, bounds of this sort would be sufficient to solve the regularity problem, see Section 6.)

- Theoretical methods to treat this problem should also apply to other PDE models. This is not entirely clear, however. There are artificial models that share many of the features of the NSE that have regular solutions (e.g. the dissipative Burgers equation [24]) and other such models that have solutions that blow up in finite time [22,36]. It may be that an eventual proof/disproof of the regularity of solutions of the NSE will involve such particular features of the equations that it says little about other models.

- Any such long-standing open mathematical question – particularly one so relevant to an important problem in physics – becomes something of a 'grand challenge', not to mention the financial incentive provided by the Clay Foundation.

3. Brief overview of the contents of this paper

The most important results on existence and uniqueness go back to Leray's work from 1934 [19], who treated the equations on the whole space (for a nice review of his argument in modern notation see Ożański & Pooley [23]). In 1951 Hopf [14] obtained similar results for the equations on a bounded domain. While we now know more about the properties of the class of solutions treated by Leray and Hopf, it seems not unfair to say that the resolution of the Regularity Problem is not significantly closer now than it was 85 years ago.

Both Leray and Hopf proved the existence for all time of finite-energy 'weak solutions', i.e. solutions with $\int_\Omega |u|^2 < \infty$. However, these *are not known to be unique*. We discuss this class of solutions in Section 4. They also proved the existence *for short times* of smoother 'strong solutions' with finite 'enstrophy' (square integral of the vorticity: $\int_\Omega |\omega|^2 < \infty$). While they exist these strong solutions are both smooth and unique; we cover these in Section 5.

Indeed, we will see that uniqueness of solutions is closely connected to their smoothness. It is therefore interesting to have results that guarantee the smoothness of a given solution when it satisfies some additional assumption. There are many such 'conditional regularity results'; one simple illustrative example is given in Section 6. Finally there is a class of (usually fairly sophisticated) results that, while not proving smoothness of solutions, serve to limit the size of the set of possible singularities of any weak solution (i.e. the Leray–Hopf solutions that are known to exist for all positive times). The first such results were proved by Scheffer [30,31], and then refined by a number of authors, in particular Caffarelli, Kohn, & Nirenberg [2]. These 'partial regularity' results probably give the most information that we currently have about solutions of the Navier–Stokes equations, and are briefly discussed in Section 7.

4. Weak solutions (finite kinetic energy): existence for all $t \geq 0$

First we give an indication of why we should expect the existence of *weak solutions*, which are solutions with finite kinetic energy that satisfy the equations in an averaged sense.

(a) Evolution of the kinetic energy

Suppose that u is a smooth solution of the equations, posed on a bounded domain Ω with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$, that exists for all $t \geq 0$. If we want to follow the evolution of the kinetic energy then we can dot the equations with u and integrate in space:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 + \int_\Omega [(u \cdot \nabla)u] \cdot u + \int_\Omega u \cdot \nabla p = 0,$$

where here we have integrated by parts in the linear term and by $|\nabla u|^2$ we mean $\sum_{i,j=1}^3 |\partial_i u_j|^2$.

Integrating by parts the nonlinear and pressure terms vanish[1]

$$\int_\Omega [(u \cdot \nabla)u] \cdot u = \int_\Omega u_j(\partial_j u_i)u_i = -\int_\Omega (\partial_j u_j)|u|^2 + u_j u_i(\partial_j u_i) = -\int_\Omega u_j u_i(\partial_j u_i)$$

(since $u = 0$ on $\partial\Omega$ and $\partial_j u_j = \nabla \cdot u = 0$) and

$$\int_\Omega u \cdot \nabla p = \int_\Omega u_i \partial_i p = -\int_\Omega (\partial_i u_i)p = -\int_\Omega (\nabla \cdot u)p = 0 \tag{4.1}$$

(again using the fact that u is divergence free and zero on the boundary).

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[1]In these calculations we use the Einstein summation convention and sum over repeated indices. Note that very similar manipulations show that $\int_\Omega [(u \cdot \nabla)u] \cdot v = -\int_\Omega [(u \cdot \nabla)v] \cdot u$ whenever u is divergence free and u or v are zero on $\partial\Omega$.

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|u|^2 + \int_{\Omega}|\nabla u|^2 = 0.$$

Integrating in time from 0 to t yields

$$\frac{1}{2}\int_{\Omega}|u(x,t)|^2\,\mathrm{d}x + \int_0^t\int_{\Omega}|\nabla u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_{\Omega}|u_0(x)|^2\,\mathrm{d}x,$$

which we can write more neatly as

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t\|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s = \frac{1}{2}\|u_0\|_{L^2}^2, \tag{4.2}$$

adopting the notation

$$\|u\|_{L^2}^2 = \int_{\Omega}|u(x)|^2\,\mathrm{d}x.$$

Such 'formal' computations as these (i.e. an argument in which we assume that the solutions are smooth, so that all the manipulations we make are rigorously allowed) lie at the basis of the proof of the existence of weak solutions.

(b) Definition of weak solutions and their existence

A function u is a weak solution of the Navier–Stokes equations if it satisfies

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t\|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s < \infty \qquad \text{for all } t \geq 0 \tag{4.3}$$

(cf. (4.2)) and solves the Navier–Stokes equations in an averaged sense. To obtain this formulation we dot the equations with some smooth divergence-free function φ and integrate in space and time to obtain[2]

$$\int_{\Omega}u(t)\cdot\varphi - \int_{\Omega}u_0\cdot\varphi + \int_0^t\int_{\Omega}\nabla u : \nabla\varphi + \int_0^t\int_{\Omega}(u\cdot\nabla)u\cdot\varphi = 0. \tag{4.4}$$

For a weak solution we require this 'averaged equation' to hold for every smooth divergence-free function φ that is zero outside some compact subset of Ω. Since u satisfies the integrability conditions in (4.3) all the terms in (4.4) make sense. [There are many equivalent definitions of a weak solution, but this is probably the simplest. For example, one can take φ to be a function that also depends on time, or alternatively a finite linear combination of eigenfunctions of the linear Stokes operator, see [13] or Chapter 3 of [28], for details. In order to be meaningful all the various formulations require u to satisfy (4.3).]

Note that, because we have integrated once by parts in the linear term, *a weak solution only needs to be smooth enough to make sense of one derivative, and then only in an integrated sense* (the required smoothness is precisely that in (4.3)). The fact that the formulation involves integration is significant: functions can be very bad and still integrable. For example, if $(q_n)_{n=1}^{\infty}$ is an enumeration of the rationals between 0 and 1 and

$$f(x) := \sum_{j=1}^{\infty}2^{-j}|x - q_j|^{-1/2} \quad \text{then} \quad \int_0^1 f(x)\,\mathrm{d}x = \sum_{j=1}^{\infty}2^{-j}\int_{-q_j}^{1-q_j}|y|^{-1/2} < \infty;$$

while f is integrable (and so finite almost everywhere) it is unbounded in a neighbourhood of every rational.

To prove rigorously the existence of (weak) solutions of any partial differential equation there are two well established methods. One is to use an explicit representation formula (e.g. using the a Green's function to write down the solution of a linear PDE), and the other is to proceed via approximations, usually reducing the original PDE to a system of ODEs and then taking limits.

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[2]The colon denotes the pairing of corresponding derivatives and components, i.e. $\nabla u : \nabla\varphi = \sum_{i,j=1}^3(\partial_i u_j)(\partial_i\varphi_j)$. The pressure term vanishes since φ is chosen to be divergence-free, cf. (4.1).

(Leray's proof on the whole space uses both an explicit representation formulation for solutions of the Stokes equation, and a limiting procedure.)

For example, on a bounded domain one can make the formal estimates of Section 4(a) the basis of a rigorous existence proof as follows. First find the eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the Stokes equation,

$$-\Delta\psi + \nabla p = \lambda\psi, \qquad \nabla\cdot\psi = 0, \qquad \psi|_{\partial\Omega} = 0, \tag{4.5}$$

which form a complete orthonormal basis for $L^2(\Omega)$. Then for each n look for an approximate solution expanded in terms of the $\{\psi_j\}_{j=1}^n$,

$$u_n(x,t) = \sum_{k=1}^{n} \hat{u}_{n,k}(t)\psi_k(x), \tag{4.6}$$

where the coefficients $\{\hat{u}_{n,k}\}_{k=1}^n$ satisfy the truncated equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n,k} + \lambda_k\hat{u}_{n,k} + \sum_{i,j=1}^{n} \underbrace{\left(\int_{\Omega}[(\psi_i\cdot\nabla)\psi_j]\cdot\psi_k\right)}_{c_{ijk}} \hat{u}_{n,i}\hat{u}_{n,j} = 0, \qquad k = 1,\ldots,n. \tag{4.7}$$

This is a finite system of locally Lipschitz ODEs (note that the c_{ijk} are entirely determined by the Stokes eigenfunctions), so has a unique solution (at least while the solution remains bounded).

The function u_n reconstructed from the coefficients in (4.6) satisfies similar estimates to those in Section 4(a): uniformly in n we have

$$\frac{1}{2}\|u_n(t)\|_{L^2}^2 + \int_0^t \|\nabla u_n(s)\|^2\,\mathrm{d}s = \frac{1}{2}\|u_n(0)\|_{L^2}^2 \leq \frac{1}{2}\|u_0\|_{L^2}^2.$$

These uniform bounds (along with uniform bounds on $\mathrm{d}u_n/\mathrm{d}t$) allow us to use compactness results to find a subsequence u_{n_j} that converges to a function u that one can then check really is a weak solution. All the more sophisticated mathematical argumentation is involved in what is quickly described in the previous sentence; with the details filled in, one can then guarantee the existence of at least one weak solution that exists for all $t \geq 0$.

(c) The problem of uniqueness for weak solutions

Given the existence of a weak solution we now want to consider its uniqueness. Again, we give some 'formal' calculations to illustrate the problem that arises when trying to prove such uniqueness.

Suppose that u and v are two weak solutions that arise from initial conditions u_0 and v_0. The equation for their difference $\mathfrak{z} := u - v$ is

$$\partial_t\mathfrak{z} - \Delta\mathfrak{z} + (\mathfrak{z}\cdot\nabla)u + (v\cdot\nabla)\mathfrak{z} + \nabla q = 0, \qquad \nabla\cdot\mathfrak{z} = 0, \qquad \mathfrak{z}(0) = \mathfrak{z}_0 := u_0 - v_0.$$

If we dot with $\mathfrak{z}$ and integrate in space then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}|\mathfrak{z}|^2 + \int_{\Omega}|\nabla\mathfrak{z}|^2 = -\int_{\Omega}[(\mathfrak{z}\cdot\nabla)u]\cdot\mathfrak{z} \leq \int|\nabla u||\mathfrak{z}|^2 \leq \left(\int_{\Omega}|\nabla u|^2\right)^{1/2}\left(\int_{\Omega}|\mathfrak{z}|^4\right)^{1/2},$$

where the second nonlinear term and the pressure term vanish (see footnote 1 and (4.1)), and we have used the Cauchy–Schwarz inequality

$$\int|fg| \leq \left(\int|f|^2\right)^{1/2}\left(\int|g|^2\right)^{1/2}.$$

Now we use another integral inequality due[3] to Ladyzhenskaya [17],

$$\int_{\Omega} |f|^4 \leq c \left[\int_{\Omega} |f|^2 \right]^{1/2} \left[\int_{\Omega} |\nabla f|^2 \right]^{3/2}, \tag{4.8}$$

to give[4]

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |\mathfrak{z}|^2 + \int_{\Omega} |\nabla \mathfrak{z}|^2 \leq c \left(\int_{\Omega} |\nabla u|^2 \right)^{1/2} \left(\int_{\Omega} |\mathfrak{z}|^2 \right)^{1/4} \left(\int_{\Omega} |\nabla \mathfrak{z}|^2 \right)^{3/4}.$$

If we split the terms on the right-hand side using the inequality[5] $ab \leq a^4 + b^{4/3}$ and absorb the resulting term involving $\nabla \mathfrak{z}$ in the corresponding term on the left-hand side we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{\Omega} |\mathfrak{z}|^2 \right] \leq c \|\nabla u\|_{L^2}^4 \left[\int_{\Omega} |\mathfrak{z}|^2 \right],$$

which implies that

$$\int_{\Omega} |\mathfrak{z}(t)|^2 \leq \exp \left\{ c \int_0^t \|\nabla u\|_{L^2}^4 \right\} \int_{\Omega} |\mathfrak{z}_0|^2. \tag{4.9}$$

The key question for uniqueness of weak solutions (if we follow this analysis) is therefore whether

$$\int_0^t \|\nabla u\|_{L^2}^4 < \infty;$$

if this quantity is finite then the estimate in (4.9) guarantees that $\mathfrak{z}(t) = 0$ whenever $\mathfrak{z}_0 = 0$. However, we do not know that this quantity is finite for an arbitrary weak solution: our requirement on a weak solution u in (4.3) is only that $\int_0^t \|\nabla u\|_{L^2}^2 < \infty$, and it is easy to find a function whose square is integrable but not its fourth power, e.g. suppose that $\|\nabla u(s)\|_{L^2}^2 \sim (t-s)^{-1/2}$. So *weak solutions do not have sufficient smoothness to guarantee their uniqueness.*

(d) Evolution of the kinetic energy for weak solutions

We saw in Section 4(a) that smooth solutions of the NSE satisfy the energy equality (4.2). Leray showed that for any choice of (finite-energy) initial condition there exists at least one weak solution that satisfies the strong energy inequality

$$\frac{1}{2} \|u(t)\|^2 + \int_s^t \|\nabla u(\tau)\|^2 \, \mathrm{d}\tau \leq \frac{1}{2} \|u(s)\|^2 \qquad \text{for all } t \geq s \tag{4.10}$$

for almost every s, including $s = 0$ (this is the class of 'Leray–Hopf weak solutions'). [The proof of this is quite involved when taking the equations on the whole space: in some ways Leray treated the hardest case first.] However, it is unknown whether *all* weak solutions must satisfy (4.10), *even just for the choice* $s = 0$. In other words, the existence of weak solutions in which the kinetic energy instantaneously increases cannot currently be ruled out. Of course, this is not a claim that a physical fluid system might spontaneously generate energy; rather, that with our current techniques it is not possible to rule this out for the usual mathematical model. A proof of the uniqueness of weak solutions would deal with this problem.

(e) Non-uniqueness of 'very weak' solutions

Non-uniqueness *does* occur for solutions of the Navier–Stokes equations, as Buckmaster & Vicol (2017) have recently shown. However, their proof treats 'very weak' solutions rather than

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[3] The proof is not difficult: the trick is to prove first a similar inequality in two variables, $\int |f|^4 \leq \int |f|^2 \int |\nabla f|^2$, starting from the identity $|v(x_1, x_2)|^2 = 2 \int_{-\infty}^{x_k} u \partial_k u \, \mathrm{d}x_k$ for $k = 1, 2$, and then deduce the three-dimensional version.

[4] In what follows c denotes an absolute constant that may change from line to line.

[5] Young's inequality $ab \leq a^p/p + b^q/q$ is valid for any $a, b \geq 0$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$. Since $x \mapsto \mathrm{e}^x$ is a convex function of x, we have $ab = \exp(\frac{1}{p} \log a^p + \frac{1}{q} \log b^q) \leq \frac{1}{p} \exp(\log a^p) + \frac{1}{q} \exp(\log b^q) = a^p/p + b^q/q$.

weak solutions: these are solutions with finite energy that satisfy the equation after additional integrations by parts in the third and fourth terms of (4.4), i.e.

$$\int_\Omega u(t) \cdot \varphi - \int_\Omega u_0 \cdot \varphi - \int_0^t \int_\Omega u \cdot \Delta\varphi - \int_0^t \int_\Omega (u \cdot \nabla)\varphi \cdot u = 0.$$

In this formulation we no longer require any derivatives of u for the equation to make sense.

Buckmaster & Vicol's proof of non-uniqueness adapts the method of 'convex integration', first exploited in the context of fluid dynamics by De Lellis & Székelyhidi [5], to prove non-uniqueness of weak solutions for the Euler equations. Starting with an approximate solution the method is iterative, adding finer and finer oscillations that are controlled to produce a solution satisfying any given energy profile. (As such this method necessarily produces solutions with very little smoothness.)

There is some suggestion of non-uniqueness for weak solutions (not only these 'very weak solutions') in work of Jia & Šverák: they show that non-uniqueness would follow from certain spectral properties of a particular linear operator, and present some numerical calculations that seem to indicate that these may indeed hold in [15].

5. Strong solutions (finite enstrophy)

We now show how a little smoothness of solutions is sufficient to guarantee their uniqueness. However, there is currently no proof of the long-time existence of solutions with this degree of regularity for large initial conditions.

(a) Evolution of the enstrophy

'Enstrophy' is the square integral of the vorticity $\omega := \nabla \wedge u$,

$$\int_\Omega |\omega|^2 = \|\omega\|_{L^2}^2.$$

Strong solutions are (roughly speaking) weak solutions for which the enstrophy remains bounded. We again make some 'formal' calculations to give an idea of the expected evolution of the enstrophy for smooth solutions, just as we did for the kinetic energy in Section 4(a).

We first note that controlling the enstrophy is enough is to control the square integral of all the first derivatives, since

$$\|\omega\|_{L^2}^2 = \int_\Omega |\omega|^2 = \int_\Omega [\epsilon_{ijk}\partial_j u_k][\epsilon_{ilm}\partial_l u_m] = \int_\Omega (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})(\partial_j u_k)(\partial_l u_m)$$

$$= \int_\Omega (\partial_j u_k)(\partial_j u_k) - (\partial_j u_k)(\partial_k u_j) = \int_\Omega |\nabla u|^2 - \int_\Omega (\partial_k u_k)(\partial_j u_j) = \int_\Omega |\nabla u|^2 = \|\nabla u\|_{L^2}^2.$$

By taking the curl of the equations we obtain[6] an evolution equation for ω,

$$\partial_t \omega - \Delta\omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u. \tag{5.1}$$

If we dot with ω and integrate over space then

$$\frac{1}{2}\int_\Omega |\omega|^2 + \int_\Omega |\nabla\omega|^2 = \int_\Omega [(\omega \cdot \nabla)u] \cdot \omega \le \left(\int_\Omega |\omega|^4\right)^{1/2} \left(\int_\Omega |\nabla u|^2\right)^{1/2}$$

$$\le c \left(\int_\Omega |\omega|^2\right)^{3/4} \left(\int_\Omega |\nabla\omega|^2\right)^{3/4};$$

the nonlinear term on the left-hand side vanishes (see footnote 1) and we have used Ladyzhenskaya's inequality (4.8) on the right-hand side along with the identity $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$

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[6]'By taking the curl of the equations' is somewhat misleading: the manipulations used to obtain (5.1) require the two vector identities $(u \cdot \nabla)u = \frac{1}{2}\nabla(|u|^2) - u \wedge \omega$ and $\mathrm{curl}(A \wedge B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$.

proved above. Using the inequality $ab \le a^4 + b^{4/3}$ on the right-hand side we can absorb the term involving $|\nabla\omega|^2$ in the corresponding term on the left-hand side, and so obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 \le c\|\omega\|_{L^2}^6. \tag{5.2}$$

Dropping the second term on the left-hand side it follows that the quantity $X(t) := \|\omega(t)\|_{L^2}^2$ satisfies the differential inequality $\dot{X} \le cX^3$. It is straightforward to integrate this to obtain the upper bound

$$\|\omega(t)\|_{L^2}^2 \le \frac{\|\omega_0\|_{L^2}^2}{\sqrt{1 - 2ct\|\omega_0\|_{L^2}^4}}. \tag{5.3}$$

This shows that if the initial enstrophy is bounded (equivalently if $\|\omega_0\|_{L^2} = \|\nabla u_0\|_{L^2}$ is bounded) then there is a time T (take any $T < 1/2c\|\omega_0\|_{L^2}^4$) such that the enstrophy $\|\omega(t)\|_{L^2}^2$ remains bounded for all $t \in [0, T]$. We can therefore expect *'short time' (local in time) existence of finite-enstrophy solutions.*

While this argument uses the inequality in (5.3) to tell us something about the short-time existence of solutions, it also has implications for solutions whose existence cannot be extended to all positive times. Suppose that $\|\omega(t)\|_{L^2} \to \infty$ as $t \to T$; then starting the NSE from the initial condition $\omega_0 = \omega(s)$ and using (5.3) yields

$$\|\omega(t)\|_{L^2}^2 \le \frac{\|\omega(s)\|_{L^2}^2}{\sqrt{1 - 2c(t-s)\|\omega(s)\|_{L^2}^4}} \qquad \Rightarrow \qquad \|\omega(s)\|_{L^2}^2 \ge \frac{\|\omega(t)\|_{L^2}^2}{\sqrt{1 + 2c(t-s)\|\omega(t)\|_{L^2}^4}};$$

letting $t \to T$ now yields the lower bound

$$\|\omega(s)\|_{L^2}^2 \ge c(T-s)^{-1/2}, \qquad 0 \le s \le T. \tag{5.4}$$

So the enstrophy of solutions that develop singularities must blow up at least this fast.

(b) Strong solutions: local existence and uniqueness

A weak solution is *strong on the time interval* $[0, T]$ if for some $M > 0$

$$\|\omega(t)\|_{L^2} \le M \qquad \text{for every } t \in [0, T].$$

To prove rigorously the existence of strong solutions we proceed as with weak solutions, obtaining estimates for the solutions u_n of the approximating ODEs from Section 4(b) that parallel those in the previous section (5(a)) and then showing that these persist in the limit as $n \to \infty$, i.e. for the resulting solution of the PDE. It follows that given ω_0 with $\|\omega_0\|_{L^2} < \infty$, there exists a time $T = T(\|\omega_0\|_{L^2})$ such that the equations have a strong solution on $[0, T]$.

Most importantly, strong solutions are smooth enough that we can guarantee their uniqueness. Recall from equation (4.9) in Section 4(c) that if u and v are two solutions then $\mathfrak{z} = u - v$ satisfies

$$\int_\Omega |\mathfrak{z}(t)|^2 \le \exp\left\{c\int_0^t \|\nabla u\|_{L^2}^4\right\}\int_\Omega |\mathfrak{z}_0|^2. \tag{5.5}$$

For a strong solution u the integral in the exponent is finite, since $\|\nabla u\|_{L^2} = \|\omega\|_{L^2} \le M$ for all $t \in [0, T]$; it follows that

$$\int_\Omega |\mathfrak{z}(t)|^2 \le e^{ctM^4}\int_\Omega |\mathfrak{z}_0|^2 \qquad \text{for } t \in [0, T]$$

and so, since $\mathfrak{z}_0 = 0$, we have $\mathfrak{z}(t) = 0$ for all $t \in [0, T]$, i.e. $u(t) = v(t)$ for all $t \in [0, T]$. So strong solutions[7] are unique.

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[7]Note that the solution v does not occur explicitly in the right-hand side of (5.5); this suggests that in fact strong solutions are unique in the larger class of all weak solutions. There is currently no proof of the result in this generality; rather, it is known that strong solutions are unique in the class of all weak solutions that satisfy the 'energy inequality', which is (4.10) allowing only $s = 0$ (this result is due to Serrin [33], see also [13] or Section 6.3 in [28]).

However, while strong solutions are unique, we cannot in general guarantee that they exist for all time, only for a short time that depends on the size of the initial enstrophy. Nevertheless, if the initial enstrophy is sufficiently small then we can prove the existence of strong solutions for all $t \geq 0$. If we return to (5.2) and use the Poincaré inequality (cf. Rayleigh–Ritz principle)

$$\lambda_1 \leq \frac{\int_\Omega |\nabla \omega|^2}{\int_\Omega |\omega|^2} = \frac{\|\nabla \omega\|_{L^2}^2}{\|\omega\|_{L^2}^2}$$

then we obtain[8]

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \lambda_1 \|\omega\|_{L^2}^2 \leq c\|\omega\|_{L^2}^6.$$

Once more if we set $X(t) = \|\omega(t)\|_{L^2}^2$ then this yields the differential inequality

$$\dot{X} + \lambda_1 X \leq cX^3 \qquad \Rightarrow \qquad \dot{X} \leq X(cX^2 - \lambda_1),$$

which guarantees that if $cX_0^2 < \lambda_1$ then $cX(t)^2 < \lambda_1$ for all $t \geq 0$. This means that the enstrophy of solutions starting with $\|\omega_0\|_{L^2}^2 < \lambda_1/c$ remains finite for all $t \geq 0$, i.e. for 'small initial data' a unique strong solution will exist for all time.

[If $u(x,t)$ is a solution of the equations on $\mathbb{R}^3$ with $u(x,0) = u_0(x)$ then so is the rescaled solution $u^\alpha(x,t) := \alpha u(\alpha x, \alpha^2 t)$, which arises from the rescaled initial condition $u_0^\alpha := \alpha u_0(\alpha x)$. Since the solution u exists for all $t \geq 0$ if and only if the solution u^α does, if one is not to gain 'something for nothing', the measure of 'smallness' of the initial data should be in terms of some quantity that does not change under this rescaling. Spaces in which the natural norm have this scaling-invariance property are termed 'critical', and much of the recent progress in the analysis of the NSE has been in extending arguments similar to those in this section (short-time existence when the initial condition is 'large' and existence for all $t \geq 0$ when the initial condition is 'small') to families of spaces of this type, see the book by Lemarié-Rieusset [20], for example.]

6. Conditional regularity

Strong solutions have many good properties in addition to their uniqueness. For example, if a solution is strong on $[0, T]$, i.e. once we know that the enstrophy is finite for all $t \in [0, T]$, then the solution is actually smooth for all $t \in (0, T)$, so for all $0 < t \leq T$ the equations are satisfied in the classical sense (see Theorem 10.6 in [10], Chapter 7 of [28], or Section 6 in [13]). In fact, in the periodic case, such a solution is even analytic, see [12].

This motivates the search for other 'conditional' regularity results: given a weak solution u starting from smooth initial data, what additional properties will ensure that it is in fact a smooth solution? The statement that a strong solution is smooth can be interpreted in this light, where the 'additional property' of the weak solution is that its enstrophy is bounded. Here we give another simple example of a conditional regularity result: we show that the assumption that

$$\int_0^T \|u(s)\|_\infty^2 \, \mathrm{d}s < \infty, \qquad \text{where } \|u(t)\|_\infty = \sup_{x \in \Omega} |u(x,t)|, \tag{6.1}$$

is sufficient to ensure that u is a strong solution on $[0, T]$, and hence smooth (and unique). Note that (6.1) certainly holds if for some $K > 0$, $\|u(t)\|_\infty \leq K$ for all $t \in [0, T]$, i.e. if the velocity is bounded on $[0, T]$.

As ever, we proceed with a formal calculation, taking the inner product of the equations with $-\Delta u$; then we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 = \int [(u \cdot \nabla)u] \cdot \Delta u \leq \int |u||\nabla u||\Delta u| \leq \|u\|_\infty \|\nabla u\|_{L^2}\|\Delta u\|_{L^2}$$

$$\leq \|\Delta u\|_{L^2}^2 + \frac{1}{4}\|u\|_\infty^2 \|\nabla u\|_{L^2}^2.$$

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[8]There are velocity for fields for which this estimate holds as an equality *instantaneously*, see Lu & Doering [21], but not on a time interval (Kang, Yun, & Protas [16]).

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 \leq \frac{1}{2}\|u\|_\infty^2 \|\nabla u\|_{L^2}^2;$$

recalling that $\|\omega\|_{L^2} = \|\nabla u\|_{L^2}$ this implies that

$$\|\omega(t)\|_{L^2}^2 \leq \exp\left(\frac{1}{2}\int_0^t \|u(s)\|_\infty^2 \,\mathrm{d}s\right) \|\omega_0\|_{L^2}^2.$$

Given our assumption in (6.1) the argument of the exponential is uniformly bounded over all $t \in [0, T]$ and so the solution is strong on $[0, T]$.

This is one 'endpoint' ($r = 2$, $s = \infty$) of a family of conditional regularity/uniqueness results, due severally to Ladzyhendskaya [18], Prodi [25], and Serrin [32,33]: in particular, Ladyzhenskaya showed that if u is a weak solution that in addition satisfies the Ladyzhenskaya–Prodi–Serrin condition $u \in L^r(0, T; L^s)$, i.e. $\int_0^T \left(\int_\Omega |u(x,t)|^s \,\mathrm{d}x\right)^{r/s} \,\mathrm{d}t < \infty$, with $2/r + 3/s = 1$ then u is smooth.[9] The proof of this result follows similar lines to that above, only using some more refined inequalities along the way. The proof of the other endpoint case ($r = \infty$, $s = 3$), when $\int |u|^3$ is bounded in time, is very involved and was only given in 2003 by Escauriaza, Seregin, & Šverák [7].

7. Partial regularity

Finally, we discuss 'partial regularity results', which limit the size of the set of possible singularities of weak solutions (or a subclass of weak solutions).

(a) The set of singular times

Scheffer [30] showed, using a result of Leray (!), that the set $\mathcal{T}$ of singular times has Hausdorff dimension no larger than $1/2$, $\dim_H(\mathcal{T}) \leq 1/2$. We give the proof of a similar result for the box-counting dimension here, since it is very simple but nevertheless provides a model of the proof of a much more powerful result about space-time singularities that we discuss shortly. It combines the fact that near any 'singular time' T the solution must blow up at the particular rate in (5.4) with the bound $\int_0^\infty \|\nabla u(s)\|^2 < \infty$, valid for any weak solution. The finiteness of the integral means that there cannot be too many times T near which the lower bound holds.

There are number of equivalent definitions of the (upper) box-counting dimension (see [9]), but the most convenient to use here is

$$\dim_B(A) = \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon},$$

where $N(A, \varepsilon)$ denotes the maximal number of ε-separated points in A [so $N(A, \varepsilon) \sim \varepsilon^{-\dim_B(A)}$].

Let $\mathcal{T}$ be the set of singular times, and suppose that $\dim_B(\mathcal{T}) = d > 1/2$; then taking δ with $1/2 < \delta < d$ there is a sequence $\varepsilon_j \to 0$ such that $N_j := N(A, \varepsilon_j) > \varepsilon_j^{-\delta}$. It follows that

$$\int_0^\infty \|\nabla u(s)\|^2 \,\mathrm{d}s \geq \sum_{i=1}^{N_j} \int_{T_i - \varepsilon_j}^{T_i} \|\nabla u(s)\|^2 \,\mathrm{d}s \geq \sum_{i=1}^{N_j} \int_{T_i - \varepsilon_j}^{T_i} c(T_i - s)^{-1/2} \,\mathrm{d}s = cN_j \varepsilon_j^{1/2} = c\varepsilon_j^{1/2-\delta};$$

this leads to a contradiction as $j \to \infty$ since the integral on the left-hand side is finite.

(b) The set of space-time singularities

Scheffer [31] also proved the first partial regularity result for the set $\mathcal{S}$ of all space-time singularities, i.e. the set of all 'singular points' (x, t) for which $u(x, t)$ is not bounded in an open

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[9]For a solution u with $\int_0^T \int_\Omega |u(x,t)|^4 < \infty$, Prodi [25] proved uniqueness under this condition; in [33] Serrin showed that under this condition alone a solution u is unique in the class of all weak solutions that satisfy the energy inequality. In [32] Serrin proved a local conditional regularity result: if $u \in L^r(t_1, t_2; L^s(U))$ for some $2/r + 3/s < 1$ then u is smooth in space in the space-time domain $(t_1, t_2) \times U$; the proof of this result is much harder (and that in the case $2/r + 3/s = 1$, due to Fabes, Jones, & Rivière [8] and Struwe [34], harder still).

set containing (x, t): he showed that $\dim_H(\mathcal{S}) \leq 2$. Following on from his work, Caffarelli, Kohn, & Nirenberg [2] improved this to show that $\dim_H(\mathcal{S}) \leq 1$. These results are valid for a weak solutions that also satisfy a localised version of the energy inequality, termed 'suitable weak solutions' in [2]. The argument there shows that if (x, t) is a singular point then the solution u must satisfy certain lower bounds on its rate of blow-up near (x, t); as in the previous section this can be coupled with the fact that certain integrals of the solution are finite to obtain bounds on the dimension of $\mathcal{S}$. (For a simplified version of their argument, neglecting the pressure, see [29].)

Robinson & Sadowksi showed in [27] that it is an easy corollary of the results in [2] that the box-counting dimension of the singular set is no larger than[10] $5/3$, and as a consequence were able to prove that for any particular suitable weak solution there is a unique particle trajectory for almost every initial condition: even when the solution is only 'weak', the Lagrangian formulation of the dynamics still makes sense. (The proof combines an argument due to Foias, Guillopé, & Temam [11] that produces a volume-preserving particle flow for any weak solution, with a dynamical result that for almost any initial condition such a flow must avoid the singular set. The almost-everywhere uniqueness follows relatively easily.)

8. Conclusion

We end with a quick summary of the above discussion. The key fact to take away is that proving the uniqueness of solutions depends upon being able to show that they are sufficiently smooth. There is therefore a natural hierarchy of solutions, whose properties improve with their regularity.

- *Very weak solutions* are not unique.
- *Weak* (finite-energy) *solutions* may not be unique, but exist for all time. They enjoy various 'partial regularity' properties, which are, for example, sufficient to show that 'particle paths' exist for almost all initial conditions, so that a Lagrangian approach is valid even for weak solutions. This is an avenue that deserves to be explored further.
- *Strong* (finite-enstrophy) *solutions* are unique but can be guaranteed to exist for only a short time, unless the initial enstrophy is sufficiently small.

As mentioned in Section 2, given that there are models that share many features in common with the Navier–Stokes equations, some of which form singularities in a finite time and others for which there is global existence of regular solutions, one can expect that a solution of the Regularity Problem will require a better understanding of the fine structure of interactions between modes, as embodied in the equation in Fourier space,

$$\frac{\mathrm{d}\hat{u}_k}{\mathrm{d}t} + \lambda_k \hat{u}_k + \sum_{i,j=1}^{\infty} c_{ijk} \hat{u}_i \hat{u}_j = 0, \quad k \in \mathbb{N}, \qquad \text{where} \quad c_{ijk} = \int_{\Omega} [(\psi_i \cdot \nabla)\psi_j] \cdot \psi_k,$$

(cf. (4.7)); if this understanding reflects some physical insight so much the better. Note that this structure is encoded in the eigenvalues λ_j and the coefficients c_{ijk}, which are entirely determined by the spectral properties of the linear Stokes problem in (4.5).

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[10]The gap between this and the upper bound of 1 on the Hausdorff dimension has been slowly closed by various authors, with the best current upper bound being $7/6$ due to Wang & Yang [37].

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