ON FINITELY PRESENTED FUNCTORS,
AUSLANDER ALGEBRAS, AND ALMOST SPLIT SEQUENCES

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Summary

This thesis consists of two parts:

In Part A we study the category of finitely presented functors and use it to determine the representation type of the Auslander Algebra of A_q = K-algebra $<z:z^q = 0>$, denoted R_q (K is a field). This is possible because the category of finitely generated modules over R_q, mod R_q, is equivalent to the category of finitely presented functors from $(\text{mod } A_q)^{op}$ to Mod k. Part A finishes with the construction of the Auslander-Reiten quiver of R_q in case $q = 3$.

Part B deals with the construction of almost split sequences in the category $\text{mod}^o\Lambda$ of lattices over an R-order Λ, where R is a complete discrete rank 1 valuation ring.

In the first chapter of part B we give a description of some unpublished work by J.A. Green who permitted me to include it in this thesis. This work contains a method to construct a short exact sequence $0 \to N \to E \to S \to 0$ in a way which gives an explicit expression for the subfunctor $\text{Im}(,g)$ of $(,S)$, and shows that the construction of almost split sequences can be viewed as a particular case of this problem.

In the second chapter of part B we continue this work by deducing a "trace formula" which provides a practical way of dealing with a certain step of the construction of almost split sequences in $\text{mod}^o\Lambda$. Then we consider the particular case where Λ is the group ring.

PART A.

Chapter 0 : Introduction

Let k be a field and A a finite dimensional k-algebra.

The purpose of this first part is to study the category of the finitely presented functors from $(\text{mod } A)^{op}$ to $\text{Mod } k$, denoted $\text{mmod } A$, and apply this to the particular case where A is the finite cyclic k-algebra of order q, i.e., $A = A_q = k\text{-alg}\langle z : z^q = 0\rangle$ in order to determine the representation type of its Auslander Algebra, which we shall denote R_q.

In fact the category of finitely generated modules over R_q is equivalent to $\text{mmod } A_q$; and this category can be approached by considering the elements of $D(\text{Hom}_{A_q}(W,U))$, where $W,U \in \text{mod } A_q$.

This work will be organized as follows:

In Chapter I we shall develop a matricial technique to find certain elements in $D(\text{Hom}_{A_q}(W,U))$ $(W,U \in \text{mod } A_q)$, that, later, will be called "indecomposable".

In Chapter II we consider an arbitrary finite dimensional k-algebra, A, and study the relation between finitely presented functors $F \in \text{mmod } A$ and elements of $D(W,U)$, $W,U \in \text{mod } A$, using an important result of Auslander and Reiten ([AR], pg. 318, 319) and some ideas given by J.A. Green.

In Chapter III we use the results of the previous chapter to deduce the representation type of R_q and we construct its Auslander-Reiten quiver in case $q = 3$, using a method by J.A. Green (see [Gr 2]).

But we must start by defining some of the concepts that occur, giving the required notation and stating some of the basic results we need.

§1. About categories

We begin with a few generalities about categories taken from [AI] pgs. 179 to 183:

(0.1) Let $\mathcal{C}, \mathcal{D}$ be categories and $F: \mathcal{C} \to \mathcal{D}$ be a functor. F is said to be dense if given $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ such that $F(C) \cong D$.

Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. $\mathcal{C}'$ is dense in $\mathcal{C}$ if the inclusion functor is dense, i.e., if for each $C \in \mathcal{C}$, there exists a $C' \in \mathcal{C}'$ such that $C' \cong C$.

A category $\mathcal{C}$ is skeletally small if it has a small dense subcategory $\mathcal{C}'$, i.e., a dense subcategory $\mathcal{C}'$ such that its collection of objects is a set.

Remark: All the categories that we shall consider are skeletally small.

(0.2) If $\mathcal{C}$ and $\mathcal{D}$ are categories and F is a (covariant) functor $F:\mathcal{C} \to \mathcal{D}$, then F is said to be an <u>equivalence</u> of categories if:

(1) F is dense

(2) $F:(C_1,C_2)_{\mathcal{C}} \to (F(C_1),F(C_2))_{\mathcal{D}}$ is an isomorphism, $\forall\, C_1,C_2 \in \mathcal{C}$

(0.3) A category $\mathcal{C}$ is <u>pre-additive</u> if for each $A,B \in \operatorname{Obj} \mathcal{C}$ the set of morphisms from A to B, $(A,B)_{\mathcal{C}}$ is an abelian group and the multiplication of morphisms is bilinear.

In fact most of the categories that we shall mention are k-categories (for some field k) :

(0.4) A category $\mathcal{C}$ a <u>k-category</u> if it is pre-additive and for each pair $A,B \in \mathcal{C}$, $(A,B)_{\mathcal{C}}$ is a k-space. (See [AR], pg. 309.)

(0.5) A pre-additive category where every finite family of objects has a direct sum is an <u>additive</u> category.

In any pre-additive category $\mathcal{C}$, we have the following

(0.6) Let $A,A_i \in \mathcal{C}$, $i = 1,\dots,n$. Then $A \cong A_1 \amalg A_2 \amalg \dots \amalg A_n$ iff there are morphisms $A_i \underset{\pi_i}{\overset{\mu_i}{\rightleftarrows}} A$ $(i = 1,\dots,n)$ such that

$$\pi_j \mu_i = \delta_{ij} 1_{A_i} \ (i,j = 1,\dots,n) \text{ and } \sum_{i=1}^{n} \mu_i \pi_i = 1_A \ .$$

(0.7) An object $B \in \mathcal{C}$ is said to be <u>indecomposable</u> if:

(i) B is not the zero object (i.e. End $B \neq 0$) .

(ii) If $B \cong B_1 \amalg B_2$, with $B_1, B_2 \in \mathcal{B}$ then either B_1 or B_2 is the zero object in $\mathcal{C}$.

(0.8) An endomorphism e of $A \in \mathcal{C}$ is <u>idempotent</u> if $e^2 = e$.

Notice that if $A \cong A_1 \amalg A_2$ (with $A_1, A_2 \neq 0$) , then if we take $e_1 = \mu_1\pi_1$, $e_2 = \mu_2\pi_2 \in$ End A (see 0.6), then these are idempotents and $e_1e_2 = e_2.e_1 = 0$, $e_1 + e_2 = 1_A$. Moreover $e_1, e_2 \neq 0$ (if $e_1 = 0$, $0 = \pi_1\mu_1\pi_1\mu_1 = 1^2_{A_1} = 1_{A_1}$ and this is a contradiction). Thus:

(0.9) In any pre-additive category $\mathcal{C}$, if $A = A_1 \amalg A_2$ with $A_1, A_2 \neq 0$, there exists an idempotent endomorphism $e \neq 0,1$ in End A .

(0.10) An idempotent endomorphism $e \neq 0$ of $A \in \mathcal{C}$ is said to <u>split</u> if it has a kernel in $\mathcal{C}$.

(0.11) If $\mathcal{C}$ is a category in which idempotents split then if $e \neq 0,1$ is an idempotent of A , then $A = \ker e \amalg \ker (1-e)$ and $\ker e$, $\ker (1-e) \neq 0$, so A is decomposable ([AI] pg. 188).

We also need the following concepts (see [AR] §1):

(0.12) Suppose $\mathcal{C}$ is a k-category. An <u>ideal</u> $\mathcal{J}$ of $\mathcal{C}$ is defined by giving, for each pair $A,B \in$ Obj $\mathcal{C}$, a k-subspace $\mathcal{J}(A,B)$ of $(A,B)_{\mathcal{C}}$ such that:

If $f \in \mathcal{J}(A,B)$, then for each $C \in$ Obj $\mathcal{C}$, $g \in (B,C)_{\mathcal{C}}$, one has $gf \in \mathcal{J}(A,C)$, and for each $D \in \operatorname{Obj} \mathcal{C}$, $h \in (D,A)_{\mathcal{C}}$, one has $fh \in \mathcal{J}(D,B)$.

(0.13) If $\mathcal{J}$ is an ideal of the category $\mathcal{C}$, then one can define the <u>quotient category</u> $\mathcal{C}/\mathcal{J}$ such that:

(i) The objects of $\mathcal{C}/\mathcal{J}$ and $\mathcal{C}$ are the same.

(ii) $(A,B)_{\mathcal{C}/\mathcal{J}} := (A,B)_{\mathcal{C}}/\mathcal{J}(A,B)$.

By (0.12) multiplication of morphisms $\bar{f} = f + \mathcal{J}(A,B)$, $\bar{g} = g + \mathcal{J}(B,C)$ is well defined by the rule

$$\bar{g}\bar{f} = gf + \mathcal{J}(A,C) \ (= \overline{gf}) \ .$$

§2. <u>Some categories</u>

If k is a field and A a finite dimensional k-algebra, denote by Mod A (Mod' A) the category of the left (right) A-modules.

If $M,N \in \operatorname{Mod} A$, then $\operatorname{Hom}_A(M,N)$ will be denoted simply by $(M,N)_A$ or (M,N) .

mod A (mod' A) is the full subcategory of Mod A (Mod' A) whose objects are the A-modules which are finitely generated as k-modules.

Mmod A is the category whose objects are the k-linear contravariant functors $F: \operatorname{mod} A \to \operatorname{Mod} k$ and whose morphisms are the natural transformations. (In [AI] this category is denoted Mod (mod A)).

M'mod A is the category of the k-linear covariant functors $F:\text{mod } A \to \text{Mod } k$ and the natural transformations.

Most concepts that exist in Mod A have an analogous in Mmod A (and in M'mod A), such as subfunctor, quotient functor, sums and intersections of subfunctors, direct sums, kernel and image of a morphism, exact sequences, projective and injective functors, indecomposable functors, radical of a functor. (See [F], [AI] §2 and also [Gr 1] §1.)

We state without proof some of the results we will use later (we refer to the books and papers already mentioned and also [M]).

(0.14) <u>Proposition</u>: If $0 \neq F \in \text{Mmod } A$, then F is indecomposable if and only if $\text{End } F = (F,F)_{\text{Mmod } A}$ has no idempotents except 1_F, 0_F. □

(0.15) <u>Yoneda's Lemma</u>: If $U \in \text{mod } A$ and $F \in \text{Mmod } A$, then the map:

$$p : ((\ ,U),F)_{\text{Mmod } A} \to F(U)$$

given by

$$p(\alpha) = \alpha(U)(1_U)$$

is a k-linear isomorphism. □

<u>Notation</u>: If U is any set, 1_U denotes the identity map on U.

(0.16) <u>Remark</u>: This result is also true if $W \in \text{mod } A$, $F \in \text{M'mod } A$ and $p:((W,\ ),F) \to F(W)$.

(0.17) Corollary: If $U,W \in \text{mod } A$ and $\alpha:(\ ,U) \to (\ ,W)$ is a morphism in Mmod A , then there exists a unique A-map $h:U \to W$ such that $\alpha = (\ ,h)$. □

(0.18) Proposition: For every $U \in \text{mod } A$, the functor $(\ ,U)$ is a projective object in Mmod A and the functor $D(U,\ )$ is an injective object in Mmod A . □

We also need the next definition ([AI], pg.204):

(0.19) Definition: $F \in$ Mmod A is finitely presented if there exists an exact sequence

$$(\ ,E) \xrightarrow{\beta} (\ ,V) \xrightarrow{\alpha} F \to 0$$

with $E,V \in \text{mod } A$.

This exact sequence is called a projective presentation for F .

If $\ker \alpha \leq \text{rad } (\ ,V)$ and $\ker \beta \leq \text{rad } (\ ,E)$, this presentation is called minimal.

(0.20) It can be shown that a minimal projective presentation is unique up to isomoprhism (see [AI], §4).

The full subcategory of Mmod A , whose objects are the finitely presented functors is denoted mmod A .

(0.21) Remark: One could give a definition similar to (0.19) for $F \in M'\mathrm{mod}\, A$. Then the full subcategory of $M'\mathrm{mod}\, A$ with these objects is denoted $m'\mathrm{mod}\, A$.

§3. Some functors

Besides those functors already mentioned we need to consider a few more:

The usual duality $D = \mathrm{Hom}_k(\ ,k)$, may be considered a functor : $\mathrm{mod}\, A \to \mathrm{mod}'A$ (or $\mathrm{mod}'A \to \mathrm{mod}\, A$) , with the rule:

If $X \in \mathrm{mod}\, A$ ($\mathrm{mod}'A$) then $DX \in \mathrm{mod}'A$ ($\mathrm{mod}\, A$) as follows:

(0.22) Definition ([CR] pg.410) : $(\phi a)(x) = \phi(ax)$, $((a\phi)(x) = \phi(xa))$ $\forall\ \phi \in DX$, $a \in A$, $x \in X$.

$d = \mathrm{Hom}_A(\ ,{}_AA) : \mathrm{mod}\, A \to \mathrm{mod}'A$ is a k-linear contravariant functor as follows:

If $X \in \mathrm{mod}\, A$, dX is a right A-module with:

(0.23) Definition ([CR], pg.399) $(fa)(x) = f(x)a$, $\forall\ f \in dX$, $a \in A$, $x \in X$.

We may similarly define the functor $\mathrm{Hom}_A(\ ,A_A) : \mathrm{mod}'A \to \mathrm{mod}\, A$, which is also denoted d .

d is left exact, turns projectives into projectives and $d(Ae) \cong eA$, $d(eA) \cong Ae$, where e is an idempotent of A .

(0.24) Definition ([Ga] pg.10) : $N = Dd : \text{mod } A \to \text{mod } A$

$\text{mod}' A \to \text{mod}' A$

is the Nakayama functor.

(0.25) Definition: $M = dD : \text{mod } A \to \text{mod } A$

$\text{mod}' A \to \text{mod}' A$.

§4. Some topics of Auslander-Reiten theory

In this section we look into some aspects of the Auslander-Reiten theory that will be used mainly in Chapter III. We refer to [AR III] and [AR IV].

(0.26) Definition ([AR IV] pg.456) Let $U, W \in \text{mod } A$; then $f \in (U,W)$ is irreducible if:

(i) f is neither a split monomorphism nor a split epimorphism.

(ii) If $f = hg$ where $g \in (U,X)$, $h \in (X,W)$ for some $X \in \text{mod } A$, then g is a split monomorphism or h is a split epimorphism.

Given a finite dimensional algebra A one can construct a directed graph, called Auslander-Reiten quiver, defined as follows:

(0.27) Definition: The Auslander-Reiten quiver of A is the directed graph whose vertices are the isomorphism classes $[V]$ of indecomposable A-modules and such that there is an arrow $[V] \to [V']$ if and only if there exists an irreducible map $V \to V'$.

We also need the

(0.28) <u>Definition</u> ([AR IV] pg.443): Let $E : 0 \to U \xrightarrow{f} E \xrightarrow{g} V \to 0$ be a short exact sequence in mod A . Then E is <u>almost split</u> if

(1) E is not split

(2) U, V are indecomposable modules.

(3) If $X \in$ mod A , $h \in (X,V)$ is not split epimorphism then there exists $h' \in (X,E)$ such that $h = gh'$.

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f"] & E \arrow[r, "g"] & V \arrow[r] & 0 \\ & & & X \arrow[ul, dashed, "h'"] \arrow[u, "h"'] & \end{tikzcd}$$

(0.29) <u>Remark</u>: It can be proved that (3) can be replaced by:

(3') If $Y \in$ mod A , $t \in (U,Y)$ is not split monomorphism, then there exists $t' \in (E,Y)$ such that $t = t'f$

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f"] \arrow[dr, "t"'] & E \arrow[r, "g"] \arrow[d, dashed, "t'"] & V \arrow[r] & 0 \\ & & Y & & \end{tikzcd}$$

The next theorem tells us that almost split sequences exist:

(0.30) <u>Theorem</u> (Auslander-Reiten) ([AR III] pg.263) Given any non-projective indecomposable $V \in \text{mod } A$, there exists an almost split sequence $\mathcal{E}$ ending with V. $\mathcal{E}$ is determined by V uniquely up to isomorphism of short exact sequences.

The following fact gives the connection between irreducible maps and almost split sequences:

(0.31) <u>Proposition</u>: Let $\mathcal{E}$ be an almost split sequence. Let $X,Y \in \text{mod } A$, $h \in (X,V)$, $t \in (U,Y)$; then

(i) h is irreducible iff there is a split monomorphism $h' \in (X,E)$ such that $h = gh'$, i.e. $X|E$ (X is a direct summand of E).

(ii) t is irreducible iff there is a split epimorphism $t' \in (E,Y)$ such that $t = t'f$, i.e. $Y|E$

$$\begin{tikzcd}
\mathcal{E} : 0 \arrow[r] & U \arrow[rr, "f"] \arrow[dr, "t"'] & & E \arrow[rr, "g"] \arrow[dl, dashed, "t'"] & & V \arrow[r] & 0 \\
& & Y & & X \arrow[ul, dashed, "h'"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

Thus, if $\{X_n,\dots,X_n\}$ is a full set of non isomorphic indecomposable direct summands of E,

$$\begin{tikzcd}
& {[X_1]} \arrow[dr] & \\
& {[X_2]} \arrow[r] & {[V]} \\
{[U]} \arrow[uur] \arrow[ur] \arrow[dr] & \vdots & \\
& {[X_n]} \arrow[uur] &
\end{tikzcd}$$

is a subquiver of the Auslander-Reiten quiver of A . This subquiver is called a mesh.

§5. Method to construct almost split sequences

In this section we look at P. Gabriel's version of Auslander-Reiten's construction of almost split sequences [Ga].

One can describe this method in successive steps. For details we refer to Green's paper ([Gr 2]).

We will consider right A-modules, for convenience.

Given $V \in \text{mod}'A$ such that V is indecomposable and non-projective, to construct the almost split sequence that ends in this module we proceed as follows:

(1) Construct a 2-step minimal projective resolution of V i.e. an exact sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} V \longrightarrow 0$$

such that P_1, P_0 are projective modules and $\ker p_i \le \text{rad}\ P_i, i = 0,1$.

(2) Apply the functor $d = (\ ,A_A)$ which is left exact contravariant $\text{mod}'A \to \text{mod}\ A$ (see §3)

$$dP_1 \xleftarrow{dp_1} dP_0 \ (\xleftarrow{dp_0} dV \longleftarrow 0) \ .$$

Let $\mathrm{Tr}\,V := \mathrm{coker}\,dp_1 = dP_1/\mathrm{Im}\;dp_1$ (see [AR III].§2)

Then

$$0 \longleftarrow \mathrm{Tr}\,V \xleftarrow{\text{nat}} dP_1 \xleftarrow{dp_1} dP_0$$

is exact in mod A .

(3) Apply $D : \mathrm{mod}\,A \to \mathrm{mod}'A$ which is exact

$$\text{(a)}\quad 0 \longrightarrow D\mathrm{Tr}\,V \xrightarrow{D\text{nat}} \mathcal{N}P_1 \xrightarrow{\mathcal{N}p_1} \mathcal{N}P_0$$

where $\mathcal{N} = Dd$ (1.24).

(4) Since P_0 is projective it can be written as $P_0 = \coprod_{v=1}^{n} e_v A$ where e_v are idempotents of A . Thus $dP_0 = \coprod_{v=1}^{n} Az_v$ where $z_v \in (P_0,A)$ is such that $z_v(\sum_{j=1}^{n} e_j a_j) = e_v a_v$.

If $\theta \in (V,DdP_0)$ let $t_v \in DV$ be defined by

$$t_v(s) = \theta(s)(z_v)\ ,\quad \forall\, s \in V$$

$v = 1,\ldots,n$.

Let $T_\theta \in D(V,V)$ be the element defined as follows:

$$\forall\, h \in (V,V)\ ,\ T_\theta(h) = \sum_{v=1}^{n} t_v(hp_0(e_v)) = \sum_{v=1}^{n} \theta(hp_0(e_v))(z_v)\quad .$$

Choose θ such that

$$(0.32)\qquad \begin{aligned} &T_\theta \neq 0 \\ &T_\theta(J(\mathrm{End}\ V)) = 0\ . \end{aligned}$$

(5) Make sequence (b) by "pull-back" ([Ro] pg.51)

$$\begin{array}{lccccccccc}
(a) & 0 & \longrightarrow & DTrV & \xrightarrow{Dnat} & NP_1 & \xrightarrow{Np_1} & NP_0 & & \\
 & & & \| \big\uparrow & & \big\uparrow \ell & & \big\uparrow \theta & & \\
(b) & 0 & \longrightarrow & DTrV & \xrightarrow[f]{} & E(\theta) & \xrightarrow[g]{} & V & \longrightarrow & 0
\end{array}$$

i.e.

$E(\theta) = \{(x,y) \in N\ P_1 \amalg V : N\ p_1(x) = \theta(y)\}$

$f(u) = (u,0),\ \forall\ u \in DTr\ V$

$g(x,y) = y \quad \forall (x,y) \in E(\theta)$.

Then

(0.33) $\quad 0 \to DTr\ V \xrightarrow{f} E(\theta) \xrightarrow{g} V \to 0$ is almost split sequence in mod'A .

One can change slightly this method in order to get one almost split sequence that starts with a given non-injective indecomposable module. Now we consider left A-modules:

Let $U \in \text{mod } A$ be non-injective indecomposable. Take its dual $DU = V \in \text{mod}'A$ which is indecomposable and non-projective.

(1') Construct a 2-step minimal projective resolution of V, in $\text{mod}'A$.

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} V \to 0$$

(2') Apply d (left exact) and finish sequence with coker $dp_1 = \text{Tr } V$.

Thus

$$\text{(a')} \quad 0 \leftarrow \text{Tr } V \xleftarrow{\text{nat}} dP_1 \xleftarrow{dp_1} dP_0 \; (\leftarrow dV \leftarrow 0)$$

is exact in $\text{mod } A$.

(3') Let $P_0 \cong \coprod_{v=1}^{n} e_v A$, then $dP_0 \cong \coprod_{v=1}^{n} Az_v$ where $z_v(\sum_{j=1}^{n} e_j a_j) = e_v a_v$. Choose $\Psi : dP_0 \to DV$ such that

$$T_\psi \in D(V,V) \text{ defined by}$$

$$T_\psi(h) = \sum_{v=1}^{n} \psi(z_v)\,(hp_0(e_v))$$

satisfy conditions:

(0.34) $\quad T_\psi \neq 0 \quad T_\psi(J(\text{End } V)) = 0$.

(0.35) Remark: Using previous method at this stage we should apply D and then choose $\theta : V \to DdP_0$ subject to certain conditions.

$\theta : V \to DdP_0$ defines and is defined by a bilinear form

$$\beta : V \times dP_0 \to k$$
$$(x,\ell) \to \theta(x)(\ell) \quad .$$

But we may also use this form to define a map:

$$\psi : dP_0 \to DV$$
$$\ell \to \psi(\ell) : \psi(\ell)(x) = \beta(x,\ell) = \theta(x)(\ell)$$
$$\forall\ \ell \in dP_0 \ ,\ x \in V \quad .$$

So $T_\theta(h) = \sum_{v=1}^{n} \theta(h\, p_0(e_v))(z_v) = \sum_{v=1}^{n} \psi(z_v)(h p_0(e_v)) = T_\psi(h)$.

So conditions in (3') are equivalent to conditions in (4).

(4') Make sequence (b') by "push-out" (see [Ro] pg.41)

$$\begin{array}{lccccc}
(a') & 0 \leftarrow \mathrm{Tr}\,V & \xleftarrow{\ nat\ } & dP_1 & \xleftarrow{\ dp_1\ } & dP_0 \\
 & \| & & \downarrow \ell' & & \downarrow \psi \\
(b') & 0 \leftarrow \mathrm{Tr}\,V & \xleftarrow[f']{} & F(\psi) & \xleftarrow[g']{} & DV \cong U \leftarrow 0
\end{array}$$

i.e. $$F(\psi) = \frac{U \amalg dP_1}{\{(\psi(x),\ -dp_1(x)) : x \in dP_0\}}$$

Denoting the elements of this module by $[u,y] : u \in U$, $y \in dP_1$,

$$\ell'(y) = [0,y]$$

$$g'(u) = [u,0]$$

$$f'[u,y] = \mathrm{nat}(y) .$$

Then

$$(0.36) \qquad 0 \leftarrow \mathrm{Tr}\, V \xleftarrow[f']{} F(\psi) \xleftarrow[g']{} U \leftarrow 0$$

is an almost split sequence in mod A .

(0.37) Remark: It is clear that this is dual of (0.33) (if we suppose that the module V is the same).

Chapter I : Matricial Techniques

§1. Auslander algebra of $A = k\text{-alg} \langle z : z^q = 0 \rangle$

Let k be a field and $A = A_q = k\text{-alg} \langle z : z^q = 0 \rangle$, the k-algebra generated by a single element z , subject to the relation $z^q = 0$ for some $q \in \mathbb{Z}$, $q \geq 1$.

A is commutative and every element $a \in A$ has a unique form $a = \lambda_0 1 + \lambda_1 z + .. + \lambda_{q-1} z^{q-1}$ with $\lambda_0, \lambda_1, \ldots \lambda_{q-1} \in k$.

It is well known that

$$\{V_i = A/Az^i : i = 1, \ldots, q\}$$

is a full set of indecomposable objects in mod A .

Let $C = V_1 \amalg V_2 \amalg \ldots \amalg V_q$ and $R = R_q = \mathrm{End}_A C$, the endomorphism algebra of C . R is the <u>Auslander Algebra</u> of A ([Rt 2], pg.450).

Each $f \in R$ can be given by a matrix $(f_{ij})_{i,j=1,..,q}$ where $f_{ij} = \pi_i f \mu_j \in (V_j, V_i)$, the π_i and μ_j being the projections and injections associated with C .

$$\begin{tikzcd} V_j \arrow[r, "\mu_j"] \arrow[d, "f_{ij}"'] & C \arrow[d, "f"] \\ V_i & C \arrow[l, "\pi_i"] \end{tikzcd}$$

In particular, the elements $e_i = \begin{pmatrix} 0 & \ddots & & & & 0 \\ & & 0 & 1_{V_i} & 0 & \ldots \\ 0 & & & & & 0 \end{pmatrix}$

$i = 1,\ldots,q$ are a set of primitive orthogonal idempotents of R.

Conversely every matrix $(f_{ij})_{i,j=1,\ldots,q}$ with coefficients $f_{ij} \in (V_j,V_i)$ is the matrix of a unique element $f = \sum_{i,j=1}^{q} u_i f_{ij} \pi_j \in R$. Thus the map $f \to (f_{ij})_{i,j=1,\ldots,q}$ is a k-algebra isomorphism.

§2. The A-module (V_j,V_i)

Given any two indecomposable modules $V_j,V_i \in \text{mod } A$, we can regard $(V_j,V_i) = \text{Hom}_A(V_j,V_i)$ as a (left) A-module with the rule:

$$(a\theta)(u) = \theta(au) \quad \forall\, a \in A\,,\ \theta \in (V_j,V_i)\,,\ u \in V_j$$

because A is commutative.

(1.1) Notation: Let $i,j \in \mathbb{N}_0$. Then $i \sim j$ is the element of $\mathbb{N}_0$, given by

$$i \sim j = \begin{cases} 0 & \text{if } j \geq i \\ i-j & \text{if } j < i \end{cases}$$

Remark: Observe that $\min(i,j) = i-(i\sim j)$.

(1.2) Proposition:

(a) $(V_j,V_i) \underset{A}{\cong} z^{i\sim j}\, V_i$.

(b) Let $c \geq i \sim j$. Then

$$M_{ij}(c) = \{f_{ij} \in (V_j,V_i) : f_{ij}(V_j) \subseteq z^c V_i\}$$

is the A-submodule of (V_j,V_i) generated by the element $u_{ij}(c) \in (V_j,V_i)$ such that:

$$1 + Az^j \longrightarrow z^c(1 + Az^i) .$$

(c) Each A-submodule of (V_j,V_i) is a member of the chain:

$$(V_j,V_i) = M_{ij}(i\sim j) > M_{ij}((i\sim j)+1) > \ldots > M_{ij}(i) = 0 \quad .$$

Also $M_{ij}(c) = 0 , \forall\, c \geq i$.

Proof: (a) Consider the map

$$\Lambda : (V_j,V_i) \rightarrow V_i$$

such that $\theta \rightarrow \theta(1 + Az^j)$.

Clearly it is an A-map, and if $\theta(1 + Az^j) = 0$, then $\theta(a+Az^j) = a\theta(1+Az^j) = 0$, so Λ is injective.

Also,

$z^j\Lambda(\theta) = z^j\theta(1+Az^j) = \theta(0) = 0$ and $z^i\Lambda(\theta) = 0$, because $\Lambda(\theta) \in V_i$.

Therefore,

$z^{\min(i,j)}\ \Lambda(\theta) = 0$, and, since $\min(i,j) = i-(i\sim j)$, this means that $\Lambda(\theta) \in z^{i\sim j}V_i$. Hence $\operatorname{Im}\Lambda \subseteq z^{i\sim j}V_i$.

Conversely if $r \in z^{i\sim j}V_i$, then $z^j r \in z^{j+(i\sim j)}\ V_i =$

$$= \begin{cases} z^i V_i = 0 & \text{if } j < i \\ z^j V_i \leq z^i V_i = 0 & \text{if } j \geq i \end{cases}$$

. Thus $z^j r = 0$ and so $Az^j \leq \ker\phi$ where ϕ is the A-map $A \to V_i$ such that $1 \to r$.

Therefore ϕ induces a map $\theta : A/Az^j = V_j \to V_i$ such that

$\theta \circ n = \phi$ where n is the natural epimorphism $A \to V_j$.

Thus $(\theta \circ n)(1) = \theta(1 + Az^j) = \phi(1) = r$ and so $r \in \operatorname{Im}\Lambda$.

Hence $z^{i\sim j}V_i \subseteq \operatorname{Im}\Lambda$.

(b) If $c \geq i\sim j$, $z^c(1+Az^i) \in z^{i\sim j}V_i$, and so by (a), there is some $\theta \in (V_j,V_i)$ such that $\theta(1+Az^j) = z^c(1+Az^i)$.

Call this map $u_{ij}(c)$.

Thus $u_{ij}(c) \in M_{ij}(c)$, and so $Au_{ij}(c) \leq M_{ij}(c)$ because $M_{ij}(c)$ is an A-submodule of (V_j,V_i) .

Conversely if $\theta \in M_{ij}(c)$, then $\theta(1+Az^j) \in z^c V_i$ and so $\theta(1+Az^j) = z^c\ a(1+Az^i)$ for some $a \in A$.

Then $\theta = a\, u_{ij}(c) \in A\, u_{ij}(c)$.

So $M_{ij}(c) = A\, u_{ij}(c)$.

(c) The isomorphism $\Lambda:(V_j,V_i) \to z^{i\sim j}V_i$ of (a) is such that $\Lambda(M_{ij}(c)) = z^c\, V_i$.

Since $z^{i\sim j}V_i$ is uniserial, with composition series:

$$z^{i\sim j}V_i > z^{i\sim j+1}V_i > \ldots > z^i V_i = 0$$

also (V_j,V_i) is uniserial with composition series:

$$(V_j,V_i) = M_{ij}(i\sim j) > M_{ij}(i\sim j+1) > \ldots > M_{ij}(i) = 0 \ . \qquad \square$$

(1.3) <u>Corollary</u>: $u_{ij}(i\sim j)$ generates (V_j,V_i) as an A-module.

(1.4) <u>Notation</u>: We shall denote this element by u_{ij} . Then $u_{ij}(c) = z^{c-(i\sim j)}u_{ij}$, $c \geq i\sim j$.

(1.5) <u>Corollary</u>: The elements $u_{ij}, zu_{ij}, \ldots, z^{n-1}u_{ij}$ where $n = \min(i,j)$, form a k-basis of (V_j,V_i) . Hence every element $f \in (V_j,V_i)$ has a unique expression

(1.6) $\quad f = \alpha(f)u_{ij}$

where $\alpha(f)$ is a polynomial in $k[z]$ with degree $< n$.

<u>Proof</u>: $\Lambda : (V_j, V_i) \to z^{i\sim j} V_i$ is an isomorphism of k-spaces such that

$$u_{ij} \longrightarrow u_{ij}(1+Az^j) = z^{i\sim j}+Az^i$$

$$zu_{ij} \longrightarrow z^{i\sim j+1} + Az^i$$

$$\cdots$$

$$z^{n-1}u_{ij} \longrightarrow z^{i-1} + Az^i .$$

Since $z^{i\sim j} + Az^i$, $z^{i\sim j+1} + Az^i$,... , $z^{i-1} + Az^i$ form a k-basis for $z^{i\sim j}V_i$, then u_{ij} , $zu_{ij},\dots,z^{n-1}u_{ij}$ with $n = \min(i,j)$ is a k-basis of (V_j,V_i) . □

(1.7) <u>Proposition</u>: Let V_j,V_h,V_i , be some of the indecomposable modules in mod A , and u_{hj}, u_{ih}, u_{ij} maps as in (1.4). Then

$$\text{(1.8)} \quad u_{ih}.u_{hj} = z^{(i\sim h)+(h\sim j)-(i\sim j)} . u_{ij} .$$

<u>Proof</u>: $u_{ih}.u_{hj}\,(1 + Az^j) = z^{(i\sim h)+(h\sim j)} + Az^i$.

One can easily check that $(i\sim h) + (h\sim j) \geq i \sim j$.

Thus

$$(i\sim h) + (h\sim j) = (i\sim j) + w(i,h,j) \quad \text{where} \quad w(i,h,j) \geq 0 .$$

Therefore

$$u_{ih}.u_{hj}(1+Az^j) = z^{(i\sim j)+w(i,h,j)} + Az^i =$$

$$= z^{w(i,h,j)}.z^{i\sim j} + Az^i = z^{w(i,h,j)}u_{ij}(1 + Az^j) \quad \text{and}$$

$$w(i,h,j) = (i\sim h) + (h\sim j) - (i\sim j) . \qquad \square$$

(1.9) <u>Corollary</u>: Let $f \in (V_h,V_i)$, $g \in (V_j,V_h)$. Then

$$\alpha(fg) \equiv z^{(i\sim h)+(h\sim j)-(i\sim j)}.\alpha(f).\alpha(g) \pmod{Az^n}$$

with $n = \min(i,j)$.

<u>Proof</u>: Clear by (1.8), (1.6). $\square$

(1.10) <u>Remarks</u>:

(i) If $f \in \mathrm{End}_A V_i$, $g \in (V_j,V_i)$, $h \in \mathrm{End}_A V_j$, then
$\alpha(fgh) \equiv \alpha(f)\,\alpha(g)\,\alpha(h) \pmod{Az^n}$

(ii) Since $1_{V_i} = u_{ii}$, $\alpha(1_{V_i}) = 1 \in A$. Thus by (1.9), $f \in \mathrm{Aut}\, V_i$ if and only if $\alpha(f)$ is a unit in A , i.e. $\alpha(f) = \lambda_0 + \lambda_1 z +\ldots+ \lambda_{i-1}z^{i-1}$ with $\lambda_0 \neq 0$.

(iii) If $f \in (V_j,V_i)$ is such that $\alpha(f) = \lambda_k z^k +\ldots+ \lambda_{n-1}z^{n-1}$ with $k > 0$, $\lambda_k \neq 0$, $n = \min(i,j)$, then there exists $g \in \mathrm{Aut}\, V_i$, such that $\alpha(gf) = z^k$, by (1.9) and (1.10)(ii).

§3. The A-module $D(V_j,V_i)$

Now we shall consider $D(V_j,V_i) = \mathrm{Hom}_k((V_j,V_i),k)$ which is a (right) A-module with the rule:

If $T \in D(V_j,V_i)$, $a \in A$ then Ta is such that

(1.11) $(Ta)(f) = T(af)$, $\forall\ f \in (V_j,V_i)$.

Remark: Since A is commutative we may write aT instead of Ta , when convenient.

It is well known that $\dim_k D(V_j,V_i) = \dim_k (V_j,V_i)$, and, since $\{u_{ij}, zu_{ij}, \ldots, z^{n-1}u_{ij}\}$ with $n = \min(i,j)$ is a basis for (V_j,V_i) , the k-sapce $D(V_j,V_i)$ has a basis $\{\pi_0, \pi_1, \ldots, \pi_{n-1}\}$ such that

(1.12) $\pi_k(z^\ell u_{ij}) = \delta_{k\ell} \quad k,\ell = 0,\ldots,n-1$.

(1.13) Lemma: π_{n-1} is an A-generator of $D(V_j,V_i)$.

Proof: Let $f = \alpha(f)u_{ij} \in (V_j,V_i)$ (1.6), be such that $\alpha(f) = \lambda_0 + \lambda_1 z + \ldots + \lambda_{n-1} z^{n-1}$, $n = \min(i,j)$.

Then

$$(\pi_{n-1}z^h)(f) = \pi_{n-1}(z^h f) = \pi_{n-1}(z^h(\lambda_0 + \ldots + \lambda_{n-1}z^{n-1})u_{ij}) =$$

$= \pi_{n-1}(\lambda_0 z^h u_{ij} + \lambda_1 z^{h+1} u_{ij} + \ldots + \lambda_{n-1} z^{h+n-1} u_{ij}) = \lambda_k$ such that $n-1 = h+k$, so $k = n-h-1$.

Thus $(\pi_{n-1} z^h)(f) = \lambda_{n-h-1}$, and so

$$\pi_{n-1} z^h = \pi_{n-h-1}$$

for $h = 0,\ldots,n-1$.

(1.14) <u>Notation</u>: We shall denote this element $\pi_{n-1} \in D(V_j,V_i)$ by π_{ji} .

Thus (1.12) becomes:

(1.15) $(\pi_{ji} z^r)(z^{n-s-1}.u_{ij}) = \delta_{rs}$, $r,s = 0,1,\ldots,n-1;\ n = \min(i,j)$

and the k-basis $\{\pi_0,\ldots,\pi_{n-1}\}$ of $D(V_j,V_i)$, becomes

$$\{\pi_{ji} z^{n-1},\ldots,\ \pi_{ji} z,\pi_{ji}\} \quad (n = \min(i,j)) \ .$$

Therefore, every element $T \in D(V_j,V_i)$ has a unique expression

(1.16) $T = \pi_{ji}\ \beta(T)$

where $\beta(T)$ is a polynomial in $k[z]$ with degree $< n = \min(i,j)$.

Now we consider the following:

(1.17) <u>Definition</u>: Let θ,ρ be A-maps such that

$$V \xrightarrow{\rho} W \xrightarrow{\theta} U$$

where $V,W,U \in \text{mod } A$ and $T \in D(V,U)$.

Then $T*\theta \in D(V,W)$ is defined by

$$(T*\theta)(f) = T(\theta f) \ , \quad \forall\, f \in (V,W)$$

$\rho*T \in D(W,U)$ is defined by

$$(\rho*T)(g) = T(g\rho) \ , \quad \forall\, g \in (W,U) \ .$$

The following are some of the properties of $*$:

(1.18) <u>Proposition</u>: (1) Let θ, θ', ρ,ρ' be A-maps such that

$$V \xrightarrow{\rho} V' \xrightarrow{\rho'} W \xrightarrow{\theta'} U' \xrightarrow{\theta} U$$

and $T \in D(V,U)$. Then:

(i) $(T*\theta)*\theta' = T*\theta\theta' \in D(V,W)$

(ii) $\rho'*(\rho*T) = \rho'\rho*T \in D(W,U)$

(2) Let θ,ρ be A-maps such that $V \xrightarrow{\rho} W \xrightarrow{\theta} U$ and $a \in A$.
Then

(i) $Ta*\theta = (T*\theta)a = T*a\theta \in D(V,W)$

(ii) $\rho * Ta = (\rho * T)a = a\rho * T \in D(W,U)$

(iii) $(\rho * T) * \theta = \rho * (T * \theta) \in D(W,W)$.

Proof: (1) (i)(ii) trivial.

(2) (i) $(Ta*\theta)(f) = Ta(\theta f) = T(a(\theta f)) = (T*a\theta)(f)$.

Also $T(a\theta f) = T(\theta(af)) = [(T*\theta)a](f)$.

(ii) similar (iii) trivial. □

(1.19) Proposition:

Let $i,j,h \in \{1,2,\dots q\}$ and $u_{hj} \in (V_j,V_h)$ $u_{ih} \in (V_h,V_i)$ and $\pi_{ji} \in D(V_j,V_i)$ be defined as in (1.4), (1.14). Then:

(i) $\pi_{ji} * u_{ih} = \pi_{jh} z^{h \sim i} \in D(V_j,V_h)$

(ii) $u_{hj} * \pi_{ji} = \pi_{hi} z^{h \sim j} \in D(V_h,V_i)$.

Proof: (i) It is enough to prove that

$$(\pi_{ji} * u_{ih})(u_{hj}) = \pi_{jh} z^{h \sim i}(u_{hj}) .$$

We have:

$$(\pi_{ji} * u_{ih})(u_{hj}) = \pi_{ji}(u_{ih} \cdot u_{hj}) = \pi_{ji}(z^{(i \sim h)+(h \sim j)-(i \sim j)} \cdot u_{ij}) =$$

$$= \begin{cases} 1 & \text{if } (i \sim h) + (h \sim j) - (i \sim j) = \min(i,j) - 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_{jh} z^{h\sim i}(u_{hj}) = \pi_{jh}(z^{h\sim i}u_{hj}) = \begin{cases} 1 & \text{if } h\sim i = \min(h,j)-1 \\ 0 & \text{otherwise.} \end{cases}$$

So we must prove that:

$$(i\sim h) + (h\sim j) - (i\sim j) = \min(i,j)-1 \quad \text{iff} \quad h\sim i = \min(h,j)-1 \ .$$

Writing $\min(i,j) = i-(i\sim j)$ and $\min(h,j) = j-(j\sim h)$, this is equivalent to proving that

$$(i\sim h) + (h\sim j) = i-1 \quad \text{iff} \quad (h\sim i) + (j\sim h) = j-1$$

and this can easily be checked considering all possible cases.

(ii) Similar. □

(1.20) <u>Corollary</u>: Let $a,b \in A$ and u_{hj}, u_{ih}, π_{ji} be as in (1.19).

Then

(i) $\pi_{ji}a * bu_{ih} = \pi_{jh} z^{h\sim i}ab$

(ii) $au_{hj} * \pi_{ji}b = \pi_{hi} z^{h\sim j}ab$.

<u>Proof</u>: This is clear by (1.19) and (1.18)(2)i,ii, using the commutativity of A . □

Now we have the following:

(1.21) Definition: Let $U, W \in \text{mod } A$ and $T, T' \in D(W,U)$.

Then we say that T is equivalent to T' and write $T \sim T'$ if there is a $\rho \in \text{Aut } W$ and a $\sigma \in \text{Aut } U$ such that

$$T' = \rho * T * \sigma \quad .$$

This is clearly an equivalence relation in $D(W,U)$.

We have the following:

(1.22) Corollary: If $T \in D(V_j, V_i)$, $\rho \in \text{End } V_j$, $\sigma \in \text{End } V_i$ then

$$\beta(\rho * T * \sigma) \equiv \alpha(\rho)\beta(T)\alpha(\sigma) \pmod{z^n A} \quad , \quad n = \min(i,j)$$

where α, β are defined in (1.6), (1.16).

Proof: $\pi_{ji}\, \beta(\rho * T * \sigma) = \rho * T * \sigma = \alpha(\rho) u_{jj} * \pi_{ji} \beta(T) * \alpha(\sigma) u_{ii} =$
$= \pi_{ji}\, \alpha(\rho)\beta(T)\alpha(\sigma)$ by (1.20).

Thus

$$\beta(\rho * T * \sigma) \equiv \alpha(\rho)\beta(T)\alpha(\sigma) \quad (\text{mod}(z^n A)) \; . \qquad \square$$

Therefore, if $T, T' \in D(V_j, V_i)$

$T \sim T'$ iff there exists $\rho \in \text{Aut } V_j$, $\sigma \in \text{Aut } V_i$, such that

$$\beta(T') \equiv \alpha(\rho)\beta(T)\alpha(\sigma) \pmod{z^nA}$$

and using (1.10)(ii) and (iii) one sees that

$$T \sim T' \text{ iff } \beta(T) = \lambda_k z^k + \ldots + \lambda_{n-1} z^{n-1} \text{ and}$$
$$\beta(T') = \mu_t z^t + \ldots + \mu_{n-1} z^{n-1}$$

with $\lambda_k, \mu_t \neq 0$ and $k = t$.

Thus every class of equivalence of $D(V_j, V_i)$ has one and only one representative with the form $\pi_{ji} z^k$, $k = 0, \ldots, n-1$ and so $D(V_j, V_i)/\sim$ has n elements.

§4. <u>The A-modules</u> (W,U) <u>and</u> $D(W,U)$

Suppose

(1.23) $W = \coprod_{j \in J} W_j$, $U = \coprod_{i \in I} U_i$ with $W_j, U_i \in \{V_1, \ldots, V_q\}$, $j \in J$, $i \in I$ and J, I are some finite sets.

By (0.6) these decompositions are associated with morphisms

$$(1.24) \quad \begin{array}{ll} m_i \in (U_i, U) & n_j \in (W_j, W) \\ p_i \in (U, U_i) & q_j \in (W, W_j) \end{array} \qquad i \in I, \; j \in J$$

such that

$$\beta(T') \equiv \alpha(\rho)\beta(T)\alpha(\sigma) \pmod{z^n A}$$

and using (1.10)(ii) and (iii) one sees that

$$T \sim T' \quad \text{iff} \quad \beta(T) = \lambda_k z^k + \ldots + \lambda_{n-1} z^{n-1} \quad \text{and}$$
$$\beta(T') = \mu_t z^t + \ldots + \mu_{n-1} z^{n-1}$$

with $\lambda_k, \mu_t \neq 0$ and $k = t$.

Thus every class of equivalence of $D(V_j,V_i)$ has one and only one representative with the form $\pi_{ji} z^k$, $k = 0,\ldots,n-1$ and so $D(V_j,V_i)/\sim$ has n elements.

§4. The A-modules (W,U) and $D(W,U)$

Suppose

(1.23) $W = \coprod_{j \in J} W_j$, $U = \coprod_{i \in I} U_i$ with $W_j, U_i \in \{V_1,\ldots,V_q\}$, $j \in J$, $i \in I$ and J, I are some finite sets.

By (0.6) these decompositions are associated with morphisms

$$(1.24) \qquad \begin{array}{ll} m_i \in (U_i,U) & n_j \in (W_j,W) \\ p_i \in (U,U_i) & q_j \in (W,W_j) \end{array} \qquad i \in I \text{ , } j \in J$$

such that

$$p_i m_t = \delta_{it}\, 1_{U_i} \quad , \quad q_j n_\ell = \delta_{j\ell}\, 1_{W_j} \qquad i,t \in I \ , \quad j,\ell \in J$$

$$\sum_{i\in I} m_i p_i = 1_U \quad , \quad \sum_{j\in J} n_j q_j = 1_W \quad .$$

Let:

$$\gamma : I \longrightarrow \{1,\dots,q\}$$
$$i \longrightarrow \gamma(i) \ \text{ such that } \ U_i = V_{\gamma(i)}$$

(1.25) $\delta : J \longrightarrow \{1,\dots,q\}$

$$j \longrightarrow \delta(j) \ \text{ such that } \ W_j = V_{\delta(j)} \quad .$$

Let $I_t = \{i \in I : \gamma(i) = t\}$, $J_\ell = \{j \in J : \delta(j) = \ell\}$. Then $I = \dot{\bigcup}_{t=1}^{q} I_t$, and $m_t = |I_t|$ is the multiplicity of V_t in the decomposition of U ; also $J = \dot{\bigcup}_{\ell=1}^{q} J_\ell$ and $n_\ell = |J_\ell|$ is the multiplicity of V_ℓ in the decomposition of W .

One has

$$(W,U) = (\coprod_{j\in J} W_j, \coprod_{i\in I} U_i) \cong \coprod_{\substack{i\in I\\ j\in J}} (W_j, U_i) = \coprod_{\substack{i\in I\\ j\in J}} (V_{\delta(j)}, V_{\gamma(i)}) \quad .$$

Let $f \in (W,U)$. Considering the maps associated with decompositions (1.23), we have the diagram:

$$\begin{array}{ccc} U_i = V_{\gamma(i)} & \underset{m_i}{\overset{p_i}{\rightleftarrows}} & U \\ & & \uparrow f \\ W_j = V_{\delta(j)} & \underset{n_j}{\overset{q_j}{\rightleftarrows}} & W \end{array}$$

Then f can be given by a matrix

(1.26) $$F = (f_{ij})_{\substack{i\in I \\ j\in J}} = \begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1q} \\ F_{21} & F_{22} & \cdots & F_{2q} \\ F_{q1} & F_{q2} & \cdots & F_{qq} \end{pmatrix}$$

where $f_{ij} = p_i.f.n_j$, and each $F_{t\ell}$ is an $m_t \times n_\ell$ matrix, i.e. gives a map $V_\ell^{n_\ell} \to V_t^{m_t}$.

We also can write

(1.27) $$F = (\alpha_{ij}(f)u_{\gamma(i)\delta(j)})_{\substack{i\in I \\ j\in J}} \quad \text{with} \quad \alpha_{ij}(f) \in A .$$

We can assume that $\alpha_{ij}(f)$ is a polynomial in z with degree < min $(\gamma(i),\delta(j))$ i.e. $\alpha_{ij}(f) = \alpha(f_{ij})$. (See (1.6).)

(1.28) <u>Definition</u>: Let $T \in D(W,U)$. Define $T_{ji} \in D(W_j,U_i)$ as follows:

$$T_{ji} = q_j{*}T{*}m_i$$

(where q_j, m_i are as in (1.24)).

We have $D(W,U) \cong \coprod_{\substack{i\in I\\ j\in J}} D(W_j U_i)$, thus $T \in D(W,U)$ can be given by a matrix:

$$(1.29) \quad \mathcal{T} = (T_{ji})_{\substack{i\in I\\ j\in J}} = (\pi_{\delta(j)\gamma(i)}\, \beta_{ji}(T))_{\substack{j\in J\\ i\in I}}$$

with $\beta_{ji}(T) \in A$, and we can always assume that $\beta_{ji}(T)$ is a polynomial in z with degree $< \min(\gamma(i),\delta(j))$ i.e. $\beta_{ji}(T) = \beta(T_{ji})$ (see (1.16)).

(1.30) <u>Remark</u>: If $|I| = n$ $|J| = m$ we see that the matrix of $f \in (W,U)$ is

$$\begin{array}{cl} W_j & \\ \left(\begin{array}{ccc} & \vdots & \\ & f_{ij} & \cdots\cdots \end{array}\right)_{n\times m} & U_i \end{array} \quad \text{with } f_{ij} \in (W_j,U_i)$$

and the matrix of $T \in D(W,U)$ is $\begin{array}{cl} i & \\ \left(\begin{array}{cc} \vdots & \\ T_{ji} & \cdots\cdots \end{array}\right)_{m\times n} & j \end{array}$ with $T_{ji} \in D(W_j,U_i)$

Given $f \in (W,U)$, then $f = \sum_{\substack{i\in I\\ j\in J}} m_i p_i\, f\, n_j q_j$ (by 1.24)).

Thus

$$T(f) = \sum_{\substack{i\in I\\ j\in J}} T(m_i p_i f n_j q_j) = \sum_{\substack{i\in I\\ j\in J}} (q_j{*}T{*}m_i)(p_i f n_j) = \sum_{\substack{i\in I\\ j\in J}} T_{ji}(f_{ij}) .$$

Therefore:

(1.31) If $T \in D(W,U)$, is given by the matrix $\mathcal{T} =$ $= (T_{ji})_{\substack{j\in J\\ i\in I}}$, $f \in (W,U)$ is given by the matrix $F = (f_{ij})_{\substack{i\in I\\ j\in J}}$, then

$$T(f) = \sum_{\substack{j\in J\\ i\in I}} T_{ji}(f_{ij}) .$$

Now we want to describe the equivalence classes for $\sim$ in $D(W,U)$ (see 1.21). We need the following:

(1.32) <u>Proposition</u>: If $T \in D(W,U)$ has matrix $\mathcal{T} = (T_{ji})_{\substack{j\in J\\ i\in I}}$ with $T_{ji} \in D(W_j,U_i)$, and $g \in \text{End } W$, $h \in \text{End } U$ have matrices $A = (g_{jk})_{j,k\in J}$, $B = (h_{\ell i})_{\ell,i\in I}$ respectively (with $g_{jk} \in (W_k,W_j)$, $h_{\ell i} \in (U_i,U_\ell)$) , then $g{*}T{*}h \in D(W,U)$ has matrix

$$A{*}\mathcal{T}{*}B = \Big(\sum_{\substack{k\in J\\ \ell\in I}} g_{jk}{*}T_{k\ell}{*}h_{\ell i}\Big)_{\substack{j\in J\\ i\in I}}$$

<u>Proof</u>: $(g^*T^*h)(f) = T(hfg) = \sum_{\substack{k\in J\\ \ell\in I}} T_{k\ell}(hfg)_{\ell k}$ (by (1.31))

$$= \sum_{\substack{k\in J\\ \ell\in I}} T_{k\ell}\Big(\sum_{\substack{j\in J\\ i\in I}} h_{\ell i} f_{ij} g_{jk}\Big) =$$

$$= \sum_{\substack{k,j\in J\\ \ell,i\in I}} T_k\,(h_{\ell i}\ f_{ij}\ g_{jk}) = \sum_{\substack{k,j\in J\\ \ell,i\in I}} (g_{jk}{}^*T_{k\ell}{}^*h_{\ell i})(f_{ij}) =$$

$$= \sum_{\substack{j\in J\\ i\in I}} \Big(\sum_{\substack{k\in J\\ \ell\in I}} (g_{jk}{}^*T_k\ {}^*h_{\ell i})\Big)\ (f_{ij})\ .$$

Then by (1.31)

$$\sum_{\substack{j\in J\\ i\in I}} \Big(\sum_{\substack{k\in J\\ \ell\in I}} (g_{jk}{}^*T_{k\ell}{}^*h_{\ell i})\Big)(f_{ij}) = \sum_{\substack{j\in J\\ i\in I}} (g^*T^*h)_{ji}(f_{ij}) \quad , \forall\ f \in (W,U)\ .$$

In particular, if f is such that its matrix is of the form $\begin{pmatrix} & \vdots & \\ & 0 & \\ ..0 & f_{ij} & 0.. \\ & 0 & \\ & \vdots & \end{pmatrix}$ we have:

$$(g^*T^*h)_{ji}\ (f_{ij}) = \Big(\sum_{\substack{k\in J\\ \ell\in I}} (g_{jk}{}^*T_{k\ell}{}^*h_{\ell i})\Big)(f_{ij})$$

and the proposition is proved. □

(1.33) Corollary: With the conditions of (1.32),

$$A * \mathcal{T} * B = \Big(\sum_{\substack{k \in J \\ \ell \in I}} z^{(j \sim k)+(i \sim \ell)} \alpha_{jk}(g) \cdot \beta_{k\ell}(T) \cdot \alpha_{\ell i}(h) \Big)_{\substack{i \in I \\ j \in J}} .$$

Proof: Clear by (1.32) and (1.19). □

§5. Automorphisms

Let $W = \coprod_{j \in J} W_j$ (1.23). Let $f \in \operatorname{End} W$ be given by the matrix

$$(1.34) \qquad F = \begin{pmatrix} F_{11} & F_{12} \cdots F_{1q} \\ F_{21} & F_{22} \cdots F_{2q} \\ & \\ F_{q1} & F_{q2} \cdots F_{qq} \end{pmatrix} \qquad \text{(as in (1.26))}$$

Using Fitting's theorem ([CRM], pg. 462) we have:

f is automorphism iff $F_{11}, F_{22}, \ldots, F_{qq}$ are non-singular.

In particular we may consider the automorphisms whose matrices belong to the following types:

(1.35)

$$E_1 = \begin{array}{r} \\ \\ \\ k\cdots \\ \\ \ell\cdots \\ \\ \\ \\ \end{array}\begin{pmatrix} Iu_{11} & & & & & & \\ & \ddots & & & & & \\ & & u_{ss} & & & & \\ & & & 0 & \cdots & u_{ss} & & \\ & & & \vdots & & \vdots & & \\ & & & u_{ss} & \cdots & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots & \\ & & & & & & & & Iu_{qq} \end{pmatrix}$$

$\delta(k) = s = \delta(\ell)$

(see 1.23, 1.25)

$$E_2 = \begin{array}{r} \\ \\ \\ \\ \\ k\cdots \\ \\ \\ \end{array}\begin{pmatrix} Iu_{11} & & & & & & \\ & Iu_{22} & & & & & \\ & & \ddots & & & & \\ & & & u_{ss} & & & \\ & & & & \ddots & & \\ & & & & & au_{ss} & \\ & & & & & & \ddots \\ & & & & & & & Iu_{qq} \end{pmatrix}$$

(a is a unit in A)

$$E_3 = \begin{array}{r} \\ \\ r\cdots \\ \\ k\cdots \\ \\ \end{array}\begin{pmatrix} u_{11} & & & & \\ & \ddots & & & \\ & & u_{tt} & & \\ & & \vdots & & \\ & & bu_{st} \cdots & u_{ss} & \\ & & & & \ddots \\ & & & & & u_{qq} \end{pmatrix}$$

$(b \in A)$

E_1, E_2, E_3 will be called <u>elementary</u> matrices.

If we multiply any matrix F (1.34) on the <u>left</u> by these matrices we get the following results:

(i) Multiplication by E_1 corresponds to interchanging rows k and ℓ.

(ii) Multiplication by E_2 corresponds to substituting row k by its product by the unit $a \in A$.

(iii) Multiplication by E_3 corresponds to substituting row k by its sum with the product of row r by bu_{st} (this product is calculated using rule (1.8)).

We will call these operations on F , <u>elementary operations</u> of types E_1, E_2, E_3 respectively.

Observe that using E_1 we can only interchange the rows k, such that $\delta(k) = \delta(\ell)$. (Strictly speaking E_1 should not be considered an elementary operation since it can be obtained by a number of operations E_2, E_3 in rows k,ℓ such that $\delta(k) = \delta(\ell)$).

However we can interchange any two rows, provided that we realize that this means a reordering of the decomposition of W when considered as the range of f . For example

$$\begin{pmatrix} u_{11} & 0 & 0 \\ 0 & 0 & u_{22} \\ 0 & u_{11} & 0 \end{pmatrix}$$

gives the identity map 1_W of $W = V_1^2 \amalg V_2$ but it is considered as a map

$$V_1^2 \amalg V_2 \to V_1 \amalg V_2 \amalg V_1 .$$

The multiplication of a matrix F (1.34) on the <u>right</u> by matrices of types E_1, E_2, E_3 gives similar results for columns.

And interchanging any two columns, means the reordering of the decomposition of W , when considered as the domain of f .

It is clear that the inverses of the automorphisms of types E_1, E_2, E_3 are given by matrices of the same type.

Let $f \in \operatorname{Aut} W$ be given by (1.34).

Since F_{11} is non-singular it is possible to find matrices

$A_1,\ldots,A_t$, $B_1,\ldots B_\ell$ of type E_1 , E_2 or E_3 such that

$$A_t \ldots A_1 F B_1 \ldots B_\ell = \begin{pmatrix} Iu_{11} & F'_{12} & \cdots & \\ F'_{21} & F'_{22} & & \\ \vdots & & & \\ & & & F_{qq} \end{pmatrix}$$

Then using matrices $A_p,\ldots,A_{t+1},B_{\ell+1},\ldots,B_k$ of type E_3 we can get

$$A_p \cdot \ldots \cdot A_{t+1} \cdot A_t \cdots A_1 F B_1 \cdots B_\ell B_{\ell+1} \cdots B_k = \begin{pmatrix} Iu_{11} & 0 & 0 & \cdots & 0 \\ 0 & F''_{22} & F''_{23} & & \\ 0 & F''_{32} & F_{33} & & \\ \vdots & & & & \\ 0 & & & & F_{qq} \end{pmatrix}$$

(because $u_{j1}u_{11} = u_{j1}$ (1.8), and more generally $u_{ji}u_{ii} = u_{ji}$)

Repeating the process we obtain

$$\begin{pmatrix} Iu_{11} & 0 & 0 & \cdots & 0 \\ 0 & Iu_{22} & 0 & \cdots & 0 \\ 0 & 0 & F'''_{33} & & \\ \vdots & \vdots & & & \\ 0 & 0 & & & F_{qq} \end{pmatrix}$$

... etc.

and finally:

$$\begin{pmatrix} Iu_{11} & & 0 \\ & Iu_{22} & \\ 0 & & I_{qq} \end{pmatrix} = I .$$

Thus there are matrices $A_1 \ldots A_s$, $B_1, \ldots B_r$ of types E_1, E_2, E_3 such that:

$$A_s \ldots A_1 \, F \, B_1 \ldots B_r = I$$

and so

$$F = A_1^{-1} \ldots A_s^{-1} B_r^{-1} \ldots B_1^{-1} \quad .$$

Therefore:

(1.36) If $f \in \mathrm{Aut}\, W$, the matrix F of f is a product of elementary matrices.

§6. <u>Elementary operations on the matrix of $T \in D(W,U)$</u>

Let $T \in D(W,U)$ be given by a matrix

(1.37) $$\mathcal{T} = (T_{ji})_{\substack{i \in I \\ j \in J}} = (\pi_{\delta(j)\gamma(i)} \beta_{ji}(T))_{\substack{j \in J \\ i \in I}}$$

as in (1.29).

We know that $\mathrm{End}_A W$ and $\mathrm{End}_A U$ act on $D(W,U)$ on left and right respectively as follows (see 1.17):

$$\forall f \in (W,U) \quad \begin{cases} (\rho * T)(f) = T(f\rho) & \forall \rho \in \mathrm{End}_A W \\ (T * \theta)(f) = T(\theta f) & \forall \theta \in \mathrm{End}_A U \quad . \end{cases}$$

Proposition (1.32) tells us that this action can be given by matrices.

In particular we are interested now on the action $*$ of matrices of types E_1, E_2, E_3 (1.35) on the matrix T (1.37) on left and right. These actions will be called <u>elementary operations</u> on T.

Using (1.19)(ii) one sees that:

(1.38) (i) $E_1 * T$ is the matrix obtained from T by interchanging rows k,ℓ (with $\delta(k) = \delta(\ell)$).

(ii) $E_2 * T$ is the matrix obtained from T by multiplying row k by $a \in A$ (a is a unit).

(iii) $E_3 * T$ is the matrix obtained from T by adding to row k the row r multiplied by bu_{st} (here, by "multiplication" we mean the action $*$).

Now we look with more detail, into case (iii).

Let $\delta(r) = t$, $\delta(k) = s$. Suppose $\delta(r) < \delta(k)$.

$$\begin{array}{c} r \\ \\ k \end{array}\begin{pmatrix} \cdots & u_{\delta(r)\delta(r)} & & \\ & \vdots & & \\ \cdots & bu_{\delta(k)\delta(r)} & \cdots & u_{\delta(k)\delta(k)} \cdots \end{pmatrix} * \begin{pmatrix} \cdots \pi_{\delta(r)\gamma(r)}\beta_{rr}(T) & \cdots & \pi_{\delta(r)\gamma(k)}\beta_{rk}(T) \cdots \\ \\ \cdots \pi_{\delta(k)\gamma(r)}\beta_{kr}(T) & \cdots & \pi_{\delta(k)\gamma(k)}\beta_{kk}(T) \cdots \end{pmatrix} =$$

$$= \begin{array}{c} k \end{array}\begin{pmatrix} \cdots \pi_{\delta(k)\gamma(r)}(\beta_{kr}(T) + bz^{\delta(k)-\delta(r)}\beta_{rr}(T)) \cdots \pi_{\delta(k)\gamma(k)}(\beta_{kk}(T) + bz^{\delta(k)-\delta(r)}\beta_{rk}(T)) \cdots \end{pmatrix}$$

If $\delta(k) \leq \delta(r)$

$$\begin{pmatrix} u_{\delta(k)\delta(k)} & \cdots & bu_{\delta(k)\delta(r)} \cdots \\ & & u_{\delta(r)\delta(r)} \cdots \end{pmatrix} * \begin{pmatrix} \cdots \pi_{\delta(k)\gamma(k)}\beta_{kk}(T) \cdots \pi_{\delta(k)\gamma(r)}\beta_{kr}(T) \cdots \\ \cdots \pi_{\delta(r)\gamma(k)}\beta_{rk}(T) \cdots \pi_{\delta(r)\gamma(r)}\beta_{rr}(T) \cdots \end{pmatrix} =$$

$$= \begin{array}{c} \\ k \\ \\ \end{array} \begin{pmatrix} \\ \cdots \pi_{\delta(k)\gamma(k)}(\beta_{kk}(T)+\beta_{rk}(T).b) \quad \cdots \quad \pi_{\delta(k)\gamma(r)}(\beta_{kr}(T) + \beta_{rr}(T).b) \quad \cdots \\ \\ \end{pmatrix}$$

Thus

(1.39) If $\delta(r) < \delta(k)$, $E_3 * T$ is the matrix obtained from T by adding to row k , the row r multiplied by $z^{\delta(k)-\delta(r)}.b$.

If $\delta(r) \geq \delta(k)$, $E_3 * T$ is the matrix obtained from T by adding to row k , the row r multiplied by b .

<u>Remark</u>: If $\delta(r) \geq \delta(k)$ and $\beta_{rr}(T)$ is a unit we may use an elementary operation of type E_3 to "annihilate" $\pi_{\delta(k)\gamma(r)}\beta_{kr}(T)$.

However, in case $\delta(r) < \delta(k)$, this may not be possible...

The action * of matrices of types E_1, E_2, E_3 on the right is similar to what has just been described except for the fact that it affects columns and not rows.

Because of (1.32), (1.36), this can be used to calculate $D(W,U)/\sim$ where "$\sim$" is given by (1.21).

§7. Some particular cases

In the following we will use a simplified notation:

We will write

$$T = \begin{array}{cc} \gamma(i) & \\ \begin{pmatrix} \dots & \beta_{ji}(T) & \dots \end{pmatrix} & \delta(j) \end{array}$$

instead of

$$(\dots \ \pi_{\delta(j)\gamma(i)} \ \beta_{ji}(T) \ \dots)$$

and often β_{ji} instead of $\beta_{ji}(T)$. We shall also call "multiplication *" to the action * .

Now we consider some special cases:

(a) Let $T \in D(V_t^s, V_k^m)$ $1 \le t, k \le q$, $m,s \in \mathbb{N}$.

Thus

$$T = \begin{array}{cc} \begin{matrix} k & k & \dots & k \end{matrix} & \\ \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & & \beta_{2m} \\ \vdots & \vdots & & \vdots \\ \beta_{s1} & \beta_{s2} & & \beta_{sm} \end{pmatrix} & \begin{matrix} t \\ t \\ \vdots \\ t \end{matrix} \end{array}$$

If there exists a β_{ji} that is a unit we may use operations of types E_1 and E_2 to place 1 in entrie (1,1). Then multiplying*the first row by $-\beta_{21}u_{tt}$ and adding to the second row we annihilate entrie (2,1). Using right multiplication * we also annihilate $\beta_{12},\ldots$. Observe that in this case $\delta(j)$ is always t , so this is possible.

After a number of steps we obtain:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \beta'_{rr} & \cdots & \beta'_{rm} \\ & & & & & & \\ & & & & \beta'_{sr} & \cdots & \beta'_{sm} \end{pmatrix}$$

where all β'_{ji} are divisible by z .

If no β_{ji} is a unit we may consider that T is this matrix with $r = 1$.

If there is $\beta'_{ji} = z.\text{unit}$, then using operations of type E_1 and E_2 we may assume that $\beta'_{rr} = z$ and then annihilate all entries (i,r) and (r,j) (because these are "divisible" by z).

Thus

$$\cdots \to \begin{array}{c} \begin{array}{ccccccc} k & \cdots & & & & & k \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & z & & & & \\ & & & z & & & \\ & & & & \beta''_{\ell\ell} & \cdots & \beta''_{\ell m} \\ & & & & & & \\ & & & & \beta''_{s\ell} & \cdots & \beta''_{sm} \end{pmatrix} \begin{array}{c} t \\ \\ \\ \\ \\ \\ t \end{array} \end{array}$$

with all β''_{ji} divisible by z^2 .

If no β'_{ji} has the form z.unit then consider $\ell = r$, etc. After a finite number of steps, we have:

$$(1.40) \qquad \begin{array}{c c} \begin{array}{ccccc} k & \cdots & & & k \end{array} & \\ \left(\begin{array}{ccccc} I & & & & \\ & zI & & & \\ & & z^2 I & & \\ & & & \ddots & \\ & & & & z^{n-1}I \\ & & & & 0 \end{array}\right) & \begin{array}{c} t \\ \\ \\ \vdots \\ \\ t \end{array} \end{array}$$

Thus T can be transformed into a direct sum of (1), (z),... ... (z^{n-1}) with $n = \min(t,k)$.

<u>Remark</u>: The reason why this may be called direct sum will be explained later (see (2.30)).

(b) Let $T \in D(V_i^{m_i} \amalg V_j^{m_j},\ V_i^{n_i} \amalg V_j^{n_j})$ with $i = j+1$. Using (a) it is clear that:

$$T \to \begin{array}{c c} \begin{array}{ccccccc} & & i & & & & j \end{array} & \\ \left(\begin{array}{cccccc:c} I & & & & & & B_0 \\ \hdashline & zI & & & & & B_1 \\ \hdashline & & z^2 I & & & & B_2 \\ \hdashline & & & \ddots & & & \cdots \\ \hdashline & & & & z^{i-1}I & & B_{i-1} \\ \hdashline & & & & & 0 & B_i \\ \hdashline C_0 & C_1 & C_2 & \cdots & C_{i-1} & C_i & A \end{array}\right) & \begin{array}{c} \\ \\ i \\ \\ \\ \\ j \end{array} \end{array}$$

We may also assume that B_0, $C_0 = 0$ and B_k, C_k have entries which are polynomials of degree $< k,j$, $\forall\, k = 1,\dots\, i$. This is so, since otherwise it was always possible to use z^k in the same row (column) to annihilate the terms of degree $\geq k$.

We can go further: if $0 \neq \beta^*_{sr} \in B_k$ then $\beta^*_{sr} = \pi_{ij}.z^{\ell}b$ where $b \in A$ is a unit and $\ell < k$. Thus $\pi_{ij}\, z^{\ell}.b*z^{k-\ell-1}b^{-1}u_{ji} =$ $= \pi_{ii}\, z^k$; so it is possible to annihilate $\pi_{ii}z^k$ in the same row. This may affect a column in C_k . Repeating this whenever necessary and interchanging columns and rows we have

$$\dots \to \left(\begin{array}{cccccc:c}
I & & & & & & \\
 & zI & & & & & \\
 & & \ddots & & & & \\
 & & & z^{i-1}I & & & \\
\cdashline{5-7}
 & & & & \multicolumn{1}{:c}{0} & & B_i' \\
\hdashline
C_0' & \multicolumn{1}{:c}{C_1'} & \multicolumn{1}{:c}{\dots} & \multicolumn{1}{:c}{C_{i-1}'} & \multicolumn{1}{:c}{C_i'} & & A
\end{array}\right)$$

where the columns affected by above operation are now in $\begin{pmatrix} 0 \\ C_i' \end{pmatrix}$.

Now we may proceed similarly for the C_k' $k = 0,\dots,i$. After a certain number of steps we have the following matrix.

$$\begin{array}{cc}
\begin{array}{ccccc:c}
 & & i & & & j
\end{array} & \\
\left(\begin{array}{ccccc:c}
I & & & & & \\
 & zI & & & & \\
 & & z^2I & & & \\
 & & & \ddots & & \\
 & & & & z^{i-1}I & \\
 & & & & 0 & B^* \\
\hdashline
 & & & & C^* & A
\end{array}\right) & \begin{array}{c} \\ \\ i \\ \\ \\ \\ j \end{array}
\end{array}$$

(c) Suppose in particular that $T \in D(V_2^{m_2} \amalg V_1^{m_1}, V_2^{n_2} \amalg V_1^{n_1})$.

Then using (b),

$$T \to \begin{array}{c@{}c} \begin{matrix} \qquad\qquad 2 & \qquad\quad 1 \end{matrix} & \\ \left(\begin{array}{ccc|c} I & & & 0 \\ & zI & & 0 \\ & & 0 & B \\ \hline 0 & 0 & C & A \end{array}\right) & \begin{matrix} \\ 2 \\ \\ 1 \end{matrix} \end{array}$$

Then,

$$\to \left(\begin{array}{cc:c|cc} I & & & & \\ & zI & & & \\ \hdashline & & 0 & B' & \\ \hline & & C' & I & \\ & & & & 0 \end{array}\right) \to \left(\begin{array}{cc:c|c:c} I & & & & \\ & zI & & & \\ \hdashline & & 0 & B'' & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \\ \hline & & C'' & I & \\ \hdashline & & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} & & 0 \end{array}\right) \to \begin{array}{c@{}c} \begin{matrix} \qquad\qquad\qquad 2 & \qquad\quad 1 \end{matrix} & \\ \left(\begin{array}{cc:c:c|c:c} I & & & & & \\ & zI & & & & \\ \hdashline & & 0 & 0 & I & 0 \\ & & 0 & 0 & B^* & 0 \quad 0 \\ \hline & & & C^* & I & 0 \\ & & I & 0 & & \\ & & 0 & 0 & 0 & 0 \end{array}\right) & \begin{matrix} \\ \\ 2 \\ \\ \\ 1 \\ \\ \end{matrix} \end{array}$$

Consider $\begin{array}{c@{}c} \begin{matrix} 2 & \qquad 1 \end{matrix} & \\ \begin{pmatrix} 0 & B^* \\ C^* & I \end{pmatrix} & \begin{matrix} 2 \\ 1 \end{matrix} \end{array}$. If there was a unit in B^*, then the $\pi_{11} \cdot 1$ in the same column could be annihilated.

Thus:

$$\ldots \to \begin{array}{c@{}c} \begin{matrix} 2 & \qquad 1 \end{matrix} & \\ \left(\begin{array}{c:c:c} 0 & B^* & 0 \\ \hline C^{**} & 0 & 0 \\ \hdashline 0 & 0 & I \end{array}\right) & \begin{matrix} 2 \\ 1 \\ \\ \end{matrix} \end{array} \to \left(\begin{array}{c:c:c} 0 & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \\ \hline 0 & I & 0 \\ \hdashline \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} & 0 & 0 \end{array}\right)$$

using (a) and interchanging rows and interchanging columns.

Substituting above we have:

$$\begin{array}{cc} \begin{array}{c:c} \quad 2 \quad & \quad 1 \quad \end{array} & \begin{array}{c:c} \quad 2 \quad & \quad 1 \quad \end{array} \\ \left(\begin{array}{cccc:cccc} I & & & & & & & \\ & zI & & & & & & \\ & & 0 & 0 & 0 & & I & 0 \\ & & 0 & 0 & 0 & \begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix} & 0 & \\ \hdashline & & 0 & 0 & I & 0 & 0 & \\ & & & \begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix} & 0 & 0 & & \\ & & \begin{smallmatrix} I \\ 0 \end{smallmatrix} & 0 & 0 & & 0 & \end{array}\right) = & \left(\begin{array}{ccc:cc} I & & & & \\ & zI & & & \\ & & 0 & \begin{smallmatrix} I \\ 0 \end{smallmatrix} & \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \\ \hdashline & & \begin{smallmatrix} I & 0 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 0 & 0 \end{smallmatrix} & & \begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix} \end{array}\right) \begin{array}{c} \\ \\ 2 \\ \\ 1 \end{array} = \end{array}$$

$$= \begin{array}{c} \begin{array}{cccccc} 2 & 2 & 1 & 1 & 2 & \end{array} \\ \left(\begin{array}{cccccc} I & & & & & \\ & I & & & & \\ & & I & & & \\ & & & I & & \\ & & & & zI & \\ & & & & & 0 \end{array}\right) \end{array} \begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ \\ \end{array} \qquad (1.41)$$

(d) $T \in D(V_3^{m_3} \amalg V_2^{m_2}, V_3^{s_3} \amalg V_2^{s_2})$. Using the same method we can see that:

$$T \to \begin{array}{c} \begin{array}{cccccccccc} 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 & 3 & \end{array} \\ \left(\begin{array}{cccccccccc} I & & & & & & & & & \\ & I & & & & & & & & \\ & & I & & & & & & & \\ & & & I & & & & & & \\ & & & & zI & & & & & \\ & & & & & zI & & & & \\ & & & & & & zI & & & \\ & & & & & & & zI & & \\ & & & & & & & & z^2I & \\ & & & & & & & & & 0 \end{array}\right) \end{array} \begin{array}{c} 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ \\ \end{array}$$

(e) $T \in D(V_3^{m_3} \amalg V_1^{m_1},\ V_3^{s_3} \amalg V_1^{s_1})$. In this case we use the following:

<u>Notation</u>: Denote by I_ℓ^* (I_ℓ^{**}) a matrix such that when the null columns (rows) are removed, it becomes I_ℓ.

$$
\begin{array}{c}
\begin{array}{ccccc} & & 3 & & 1 \end{array} \\
T \to \left(\begin{array}{cccc:c}
I & & & & 0 \\ \hdashline
 & zI & & & C_1 \\ \hdashline
 & & z^2I & & C_2 \\ \hdashline
 & & & 0 & C_3 \\ \hdashline
0 & B_1 & B_2 & B_3 & D
\end{array}\right)\begin{array}{c} \\ 3 \\ \\ \\ 1 \end{array}
\end{array}
\to
\left(\begin{array}{cccc:c}
I & & & & 0 \\ \hdashline
 & zI & & & C_1' \\ \hdashline
 & & z^2I & & 0 \\ \hdashline
 & & & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \\ \hdashline
0 & B_1' & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} & D'
\end{array}\right)
$$

because if C_2 had an entry $\neq 0$ it would be possible to annihilate $\pi_{33}\, z^2$ in the same row ...

$$
(1.42) \qquad \longrightarrow \left(\begin{array}{cccc:cc}
I & & & & \multicolumn{2}{c}{0} \\ \hdashline
 & zI & & & 0 & C'' \\ \hdashline
 & & z^2I & & \multicolumn{2}{c}{0} \\ \hdashline
 & & & 0 & \begin{matrix} I \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hdashline
0 & \begin{matrix} 0 \\ B'' \end{matrix} & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ D'' \end{matrix}
\end{array}\right)
$$

Now consider

$$
\left(\begin{array}{c:c}
zI & C'' \\ \hdashline
B'' & D''
\end{array}\right)
\to
\left(\begin{array}{cc:cc}
 & zA & I_t & 0 \\
 & & 0 & 0 \\ \hdashline
I_\ell & 0 & \multicolumn{2}{c}{\multirow{2}{*}{D'''}} \\
0 & 0 & &
\end{array}\right)
\to
\left(\begin{array}{cc:cc}
 & zA & I_t & 0 \\
 & & 0 & 0 \\ \hdashline
I_\ell & 0 & 0 & 0 \\
0 & 0 & 0 & D^*
\end{array}\right)
\to \cdots
$$

The last step is valid because the use of E_3 on rows will not affect $\begin{pmatrix} I_\ell & 0 \\ 0 & 0 \end{pmatrix}$, since given $\pi_{33}z\,a \in zA$, then $\forall\, b \in A$, $bu_{13}.\pi_{33}za = \pi_{13}zab = 0$; and the same holds for columns.

Then

$$\cdots \to \left(\begin{array}{cc|c:c} \multicolumn{2}{c|}{zA} & \begin{matrix} I_t \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline I_\ell & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \end{array}\right) \to \left(\begin{array}{c|c:c} zI_k & C & 0 \\ \hline B & 0 & 0 \\ \hdashline 0 & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \end{array}\right)$$

Substituting on (1.42) we have:

$$(1.43) \qquad \left(\begin{array}{cccc|c:c:c} I & & & & \multicolumn{3}{c}{0} \\ \hdashline & zI & & & 0 & C & \\ \hdashline & & z^2I & & \multicolumn{3}{c}{0} \\ \hdashline & & & 0 & \begin{matrix} I \\ 0 \end{matrix} & \multicolumn{2}{c}{\begin{matrix} 0 \\ 0 \end{matrix}} \\ \hline & 0 & & I \;\; 0 & 0 & \multicolumn{2}{c}{0} \\ \hdashline 0 & B & 0 & & & 0 & 0 \\ \hdashline & 0 & & 0 \;\; 0 & 0 & 0 & \begin{matrix} I & 0 \\ 0 & 0 \end{matrix} \end{array}\right)$$

Now we must consider the matrix $X = \left(\begin{array}{c|c} zI_k & C \\ \hline B & 0 \end{array}\right)$.

Observe that B and C are matrices of elements on k , with rank ℓ and t respectively.

Let $B = (b_{ij})_{\substack{i=1,\dots,\ell \\ j=1,\dots k}}$. If $b_{i1} \neq 0$, for some i, then by interchanging rows and multiplying by b_{i1}^{-1} we can assume that $b_{11} = 1$. Then using operations on rows only, we can annihilate all $b_{i1} \neq 0$ $(i > 0)$.

If there is no $b_{i1} \neq 0$, the first column is null and we consider the second.

$$\text{Suppose} \quad B \to \begin{pmatrix} 1 & b'_{12} & \cdots \\ 0 & b'_{22} & \cdots \\ & & \\ 0 & b'_{\ell 2} & \cdots \end{pmatrix}$$

If some b'_{t2} is $\neq 0$, then:

(i) If $t \neq 1$ we may suppose that $b'_{22} = 1$ and using only operations on rows, we can annihilate all $b'_{i2} \neq b'_{22}$.

(ii) If only $b'_{12} = b \neq 0$ then

$$X = \left(\begin{array}{cccc:c} z & & & & \\ & z & & & \\ & & \ddots & & C \\ & & & z & \\ \hdashline 1 & b & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & \end{array}\right) \to \left(\begin{array}{cccc:c} z & -bz & & & \\ & z & & & \\ & & \ddots & & C \\ & & & z & \\ \hdashline 1 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & \end{array}\right) \to \left(\begin{array}{cccc:c} z & & & & \\ & z & & & \\ & & \ddots & & C_1 \\ & & & z & \\ \hdashline 1 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & & 0 \\ 0 & 0 & & & \end{array}\right)$$

If all $b'_{t2} = 0$, the matrix has already this form.

Proceeding this way we see that

$$X \to \left(\begin{array}{c|c} zI & C' \\ \hline I^*_\ell & 0 \end{array}\right)$$

Then we use a similar method on C' . One can see that I^*_ℓ does not change, as follows:

Suppose we have the following case (which is the only that could affect I^*_ℓ):

$$\left(\begin{array}{cc|c} z & \cdots & \cdots\ 1 \\ & \ddots & \vdots \\ & z & \cdots\ c \\ \hline \multicolumn{2}{c|}{I^*_\ell} & 0 \end{array}\right)$$

where c is the only element $\neq 0$ in its row.

Then we have one of the following cases:

(i)
$$\left(\begin{array}{ccc|c} z & \cdots & & \cdots\ 1 \\ \vdots & \ddots & & \vdots \\ & & z\ \cdots & \cdots\ c \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} z & \cdots & & \cdots\ 1 \\ & \ddots & & \vdots \\ -cz & \cdots & z\ \cdots & \cdots\ 0 \\ \hline 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} z & \cdots & & \cdots\ 1 \\ & \ddots & & \vdots \\ & & z\ \cdots & \cdots\ 0 \\ \hline 1 & & & \\ \vdots & \ddots & & \\ -c & \cdots & 1 & 0 \end{array}\right) \longrightarrow$$

$$\longrightarrow \left(\begin{array}{ccc|c} z & \cdots & & \cdots\ 1 \\ & \ddots & & \vdots \\ & & z\ \cdots & \cdots\ 0 \\ \hline 1 & \ddots & & \\ 0 & \cdots & 1 & 0 \end{array}\right)$$

(ii)
$$\left(\begin{array}{cc|c} z & \cdots & \cdots\ 1 \\ \ddots & z\ \cdots & \cdots\ c \\ \hline 1 & & \\ & \ddots\ 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} z & \cdots & \cdots\ 1 \\ \vdots\ \ddots & & \vdots \\ -cz\ \cdots & z\ \cdots & \cdots\ 0 \\ \hline 1 & & \\ & \ddots\ 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} z & \cdots & \cdots\ 1 \\ \ddots & z\ \cdots & \cdots\ 0 \\ \hline 1 & & \\ & \ddots\ 0 & 0 \end{array}\right)$$

$$\text{(iii)}\quad \left(\begin{array}{c|c} \begin{matrix} z & & \\ & \ddots & \\ & & z \end{matrix} & \begin{matrix} 1 \\ \vdots \\ c \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{array}\right) \to \left(\begin{array}{c|c} \begin{matrix} z & \cdots & -c^{-1}z \\ & \ddots & \vdots \\ & & z \end{matrix} & \begin{matrix} 0 \\ \vdots \\ c \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{array}\right) \to \left(\begin{array}{c|c} \begin{matrix} z & & \\ & \ddots & \\ & & z \end{matrix} & \begin{matrix} 0 \\ \vdots \\ c \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{array}\right) \to$$

$$\to \left(\begin{array}{c|c} \begin{matrix} z & & \\ & \ddots & \\ & & z \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \end{array}\right)$$

$$\text{(iv)}\quad \left(\begin{array}{c|c} \begin{matrix} z & & \\ & \ddots & \\ & & z \end{matrix} & \begin{matrix} 1 \\ \vdots \\ c \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} & 0 \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{c|c} \begin{matrix} z & & \\ & \ddots & \\ & & z \end{matrix} & \begin{matrix} 1 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} & 0 \end{array}\right)$$

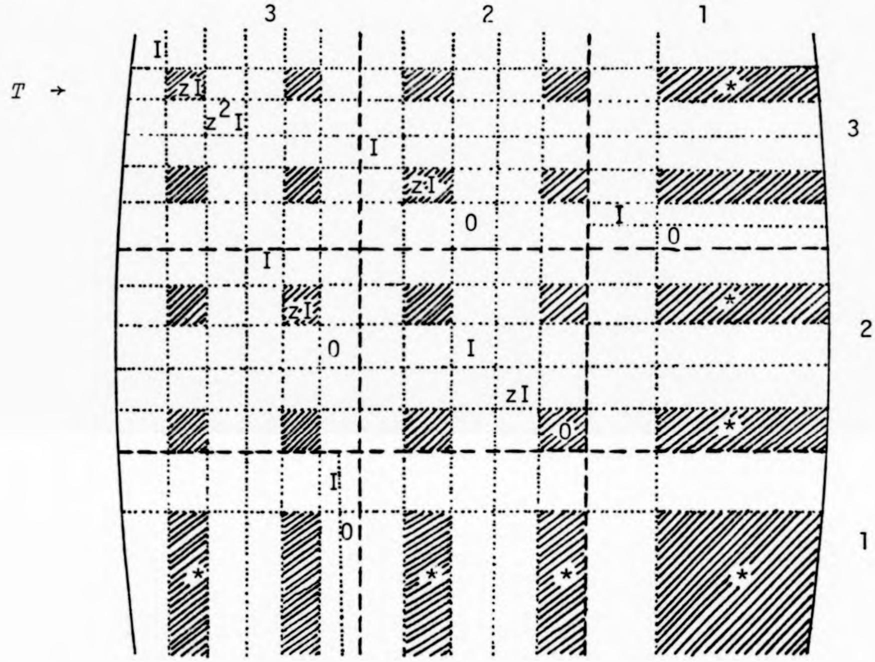
Finally we have:

$$X \to \left(\begin{array}{c|c} zI_k & I_t^{**} \\ \hline I_\ell^{*} & 0 \end{array}\right)$$

Substituting in (1.43) and interchanging conveniently rows and columns, we get:

$$(1.44)\quad \begin{array}{cc}
\begin{array}{ccccccccc} 3 & 3 & 31 & 31 & 3 & 3 & 3 & 1 & 1 \end{array} & \\
\left(\begin{array}{cccccccccc}
\boxed{z^2} & & & & & & & & & \\
 & \boxed{z} & & & & & & & & \\
 & & \boxed{\begin{matrix} z & 1 \\ 1 & 0 \end{matrix}} & & & & & & & \\
 & & & \boxed{z\ 1} & & & & & & \\
 & & & & \boxed{\begin{matrix} z \\ 1 \end{matrix}} & & & & & \\
 & & & & & \boxed{1} & & & & \\
 & & & & & & \boxed{1} & & & \\
 & & & & & & & \boxed{1} & & \\
 & & & & & & & & \boxed{1} & \\
 & & & & & & & & & 0
\end{array}\right) & \begin{array}{c} 3 \\ 3 \\ 3 \\ 1 \\ 3 \\ 3 \\ 1 \\ 3 \\ 1 \\ 3 \\ 1 \end{array}
\end{array}$$

(f) Let $T \in D(V_3^{m_3} \amalg V_2^{m_2} \amalg V_1^{m_1} \; , \; V_3^{s_3} \amalg V_2^{s_2} \amalg V_1^{s_1})$



The blocks with * are not necessarily zero; those without any symbol inside, are null.

Considering the matrix formed by the shaded blocks and applying the methods of previous examples, we obtain:

$$
\begin{array}{c}
\begin{array}{ccccccc} 3 & & 2 & & 1 & & \end{array} \\
\left(\begin{array}{cc|cc|ccc}
zI & 0 & 0 & 0 & I_m^{**} & 0 & 0 \\
0 & 0 & zI & 0 & 0 & 0 & 0 \\
\hline
0 & zI & 0 & 0 & 0 & I^{**} & 0 \\
0 & 0 & 0 & 0 & I_t^{*} & 0 & 0 \\
\hline
I_n^{*} & 0 & 0 & I_k^{**} & 0 & 0 & 0 \\
0 & 0 & I_\ell & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}
\end{array}\right)
\end{array}
\begin{array}{c} 3 \\ \\ 2 \\ \\ 1 \end{array}
=
\begin{array}{c}
\begin{array}{cccccc} 3 & 2 & 1 & 3 & 1 & 2 \end{array} \\
\left(\begin{array}{ccc|ccc}
zI & 0 & I_m^{**} & & & \\
0 & 0 & I_t^{*} & & 0 & \\
I_n^{*} & I_k^{**} & 0 & & & \\
\hline
 & & & zI & I_r^{**} & 0 \\
 & 0 & & 0 & 0 & zI \\
 & & & 0 & 0 & I_\ell^{*}
\end{array}\right)
\end{array}
\begin{array}{c} 3 \\ 2 \\ 1 \\ 2 \\ 3 \\ 1 \end{array}
$$

Now we consider the matrix

$$(1.45)\qquad \begin{array}{cccc} 3 & 2 & 1 & \\ \left(\begin{array}{c} zI \\ 0 \\ I_n^* \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ I_k^{**} \end{array} & \left.\begin{array}{c} I_m^{**} \\ I_t^* \\ 0 \end{array}\right) & \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \end{array}$$

If I has order ≥ 2 this is a decomposable matrix because in each row and each column of the I_j^{**}, I_i^* $(j = m,k,\ i = n,t)$, there is at most one 1, and the blocks $(2,3)$ and $(3,2)$ are 0.

<u>Example</u>:

$$\begin{array}{ccc} 3 & 2 & 1 \\ \left(\begin{array}{cc|cc|cc} z & 0 & 0 & 0 & 1 & 0 \\ 0 & z & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & & & 0 & 1 \\ 0 & 0 & \multicolumn{2}{c|}{0} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} 3 \\ \\ 2 \\ \\ 1 \end{array} \rightarrow \begin{array}{cccccc|c} 3 & 2 & 1 & 3 & 2 & & \\ \end{array} \left(\begin{array}{ccc|ccc} z & 0 & 1 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 1 & 0 & & & \\ \hline & & & z & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 1 & 1 & 0 \end{array}\right) \begin{array}{c} 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{array}$$

The method used can be generalized to any matrix of the type (1.45). And the matrices into which it decomposes still belong to this type.

Thus we must consider all possible 3×3 matrices with the form (1.45). Using (1.19) we can transform them as follows:

$$(1)\quad A_1 = \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{ccc} z & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{cc|c} z & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 1 \\ 2 \end{array}$$

A_1^T is similar

$$(2)\quad A_2 = \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{ccc} z & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{ccc} z & -z & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{ccc} z & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \left(\begin{array}{cc|c} z & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 1 \\ 2 \end{array}$$

A_2^T is similar.

(3)

$$A_3 = \begin{array}{cc} \begin{matrix}3&2&1\end{matrix} & \\ \begin{pmatrix} z&0&0\\ 0&0&0\\ 1&1&0 \end{pmatrix} & \begin{matrix}3\\2\\1\end{matrix} \end{array} \to \begin{array}{cc} \begin{matrix}3&2&1\end{matrix} & \\ \begin{pmatrix} z&0&0\\ 1&1&0\\ 0&0&0 \end{pmatrix} & \begin{matrix}3\\1\\2\end{matrix} \end{array}$$

A_3^T is similar.

(4)

$$A_4 = \begin{array}{cc} \begin{matrix}3&2&1\end{matrix} & \\ \begin{pmatrix} z&0&1\\ 0&0&0\\ 0&1&0 \end{pmatrix} & \begin{matrix}3\\2\\1\end{matrix} \end{array} \to \begin{array}{cc} \begin{matrix}3&1&2\end{matrix} & \\ \begin{pmatrix} z&1&0\\ 0&0&0\\ 0&0&1 \end{pmatrix} & \begin{matrix}3\\2\\1\end{matrix} \end{array}$$

A_4^T is similar.

(5)

$$A_5 = \begin{pmatrix} z&0&0\\ 0&0&1\\ 0&0&0 \end{pmatrix} \to \begin{array}{cc} \begin{matrix}3&1&2\end{matrix} & \\ \begin{pmatrix} z&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix} & \begin{matrix}3\\2\\1\end{matrix} \end{array}$$

A_5^T is similar.

(6)

$$A_6 = \begin{pmatrix} z&0&1\\ 0&0&0\\ 0&0&0 \end{pmatrix} \to \begin{array}{cc} \begin{matrix}3&1&2\end{matrix} & \\ \begin{pmatrix} z&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix} & \begin{matrix}3\\2\\1\end{matrix} \end{array}$$

A_6^T is similar.

(7)

$$A_7 = \begin{pmatrix} z&0&1\\ 0&0&1\\ 1&1&0 \end{pmatrix} \to \begin{pmatrix} z&-z&1\\ 0&0&1\\ 1&0&0 \end{pmatrix} \to \begin{pmatrix} z&-z&1\\ -z&z&0\\ 1&0&0 \end{pmatrix} \to \begin{pmatrix} z&0&1\\ 0&z&0\\ 1&0&0 \end{pmatrix} \to \begin{array}{cc} \begin{matrix}3&1&2\end{matrix} & \\ \begin{pmatrix} z&1&0\\ 1&0&0\\ 0&0&z \end{pmatrix} & \begin{matrix}3\\1\\2\end{matrix} \end{array}$$

(8)

$$A_8 = \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \left(\begin{array}{c|cc} z & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array}$$

(9)

$$A_9 = \begin{pmatrix} z & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \left(\begin{array}{cc|c} z & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 1 \\ 2 \end{array}$$

(10)

$$A_{10} = \begin{pmatrix} z & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \left(\begin{array}{c|c|c} z & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 2 \\ 1 \end{array}$$

Considering $\begin{array}{c} \begin{array}{ccc} 3 & 1 & 2 \end{array} \\ \begin{pmatrix} zI & I_r^{**} & 0 \\ 0 & 0 & zI \\ 0 & 0 & I_\ell^{*} \end{pmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array}$ one can easily see that it decomposes into the direct sum of the

following matrices: $\begin{array}{cc} 3 & 1 \\ (z & 1) \end{array} 2$ $\begin{array}{c} 2 \\ \begin{pmatrix} z \\ 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 3 \\ 1 \end{array}$, $\overset{3}{(z)}2$, $\overset{2}{(z)}3$, $\overset{2}{(1)}1$, $\overset{1}{(1)}2$.

Collecting all these matrices, we can say that:

(1.46) T can be transformed into a matrix that is the direct sum of matrices taken from the set of 21 matrices (1.47).

It will be shown later (see Chapter II, §5) that these matrices are indecomposable i.e. that they cannot be written as the direct sum of two matrices different from zero.

It will also be shown that no two of these matrices correspond to equivalent elements (as defined in (1.21)).

$$\begin{pmatrix} 0 & \pi_{13} \\ \pi_{31} & \pi_{33}z \end{pmatrix} \begin{pmatrix} \pi_{12} & \pi_{13} \\ 0 & \pi_{33}z \end{pmatrix} \quad \begin{pmatrix} \pi_{21} & 0 \\ \pi_{31} & \pi_{33}z \end{pmatrix} \begin{pmatrix} \pi_{31} & \pi_{33}z \end{pmatrix}$$

$$(1.47) \begin{pmatrix} \pi_{13} \\ \pi_{33}z \end{pmatrix} \quad \begin{pmatrix} \pi_{21} & \pi_{23}z \end{pmatrix} \quad \begin{pmatrix} \pi_{12} \\ \pi_{32}z \end{pmatrix} \begin{pmatrix} \pi_{33}z^2 \end{pmatrix} \quad \begin{pmatrix} \pi_{33}z \end{pmatrix}$$

$$(\pi_{32}z) \quad (\pi_{23}z) \quad (\pi_{22}z) \quad (\pi_{33}) \quad (\pi_{23}) \quad (\pi_{32}) \quad (\pi_{13}) \quad (\pi_{31})$$

$$(\pi_{22}) \quad (\pi_{12}) \quad (\pi_{21}) \quad (\pi_{11}) \; .$$

Chapter II : Finitely presented functors

§1. A characterization of finitely presented functors

Let k be a field and A any finitely dimensional k-algebra.

In this chapter we make use of the following important characterization of finitely presented functors ([AR] pg.318, 319):

(2.1) Theorem (Auslander-Reiten): A functor $F \in \mathrm{Mmod}\, A$ is finitely presented if and only if there exist $U,W \in \mathrm{mod}\, A$ and $\alpha:(\ ,U) \to D(W,\ )$ such that $\mathrm{Im}\,\alpha = F$. □

(2.2) Remark: In [AR] prop. 3.1. pg. 318, it is proved that $F \in \mathrm{mmod}\, A$, iff F and DF are finitely generated i.e. there are $V,W \in \mathrm{mod}\, A : (\ ,V) \to F \to 0$, $(W,\ ) \to DF \to 0$ are exact. This is obviously equivalent to (2.1) above.

Before we go further we give a more constructive proof of (2.1) than the one given in [AR]. In fact this is equivalent to answering the questions:

(1) If $F \cong \mathrm{Im}\,\alpha$ with $\alpha:(\ ,V) \to D(W,\ )$, $V,W \in \mathrm{mod}\, A$, describe $V_1 \in \mathrm{mod}\, A$ so that $(\ ,V_1) \to (\ ,V) \to F \to 0$ is exact.

(2) Conversely given $F \in \mathrm{mmod}\, A$ and an exact sequence

$$(\ ,V_1) \to (\ ,V_0) \to F \to 0$$

describe W and $\alpha:(\ ,V_0) \to D(W,\ )$ so that $F \cong \mathrm{Im}\,\alpha$.

Green answers the first question in [Gr 2] §2, by constructing what he calls the Auslander-Reiten-Gabriel (A-R-G) diagram.

For convenience we write here the main steps of this construction: Let

(2.3) $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} W \to 0$ be a projective resolution of W.

Then $D(P_1,\ ) \xrightarrow[D(p_1,\ )]{} D(P_0,\ ) \xrightarrow[D(p_0,\ )]{} D(W,\ ) \to 0$ is exact.

Applying d (see Chapter 0; §3), to (2.3), and considering $C = \operatorname{Coker} dp_1$, the sequence

$$dP_0 \xrightarrow{dp_1} dP_1 \longrightarrow C \to 0$$

is exact.

So

$$0 \to (\ ,DC) \xrightarrow{(\ ,\,\mathrm{inc})} (\ ,NP_1) \xrightarrow{(\ ,Np_1)} (\ ,NP_0)$$

is exact, where N is the Nakayama functor (0.24).

There exists a k-map

(2.4) $\alpha_Y : D(Y,\ ) \to (\ ,NY)$

which is isomorphism if Y is projective (see [Gr 2] pg. 17).

With $b = D(p_0,\ )\,\alpha_{p_0}^{-1}$, the following diagram commutes:

$$\begin{array}{ccccccc}
 & & D(P_1,\ ) & \xrightarrow{D(p_1,)} & D(P_0,\ ) & \xrightarrow{D(p_0,\ )} & D(W,\ ) \to 0 \\
 & & \downarrow {\scriptstyle \alpha_{P_1}} & & \downarrow {\scriptstyle \alpha_{P_0}} & & \downarrow {\scriptstyle 1_{D(W,\ )}} \\
0 \to (\ ,DC) & \longrightarrow & (\ ,NP_1) & \xrightarrow{(\ ,Np_1)} & (\ ,NP_0) & \xrightarrow[b]{} & D(W,\ ) \to 0
\end{array}$$

Since b is epimorphism, there is $\theta \in (V,NP_0)$ such that

$$b(V)(\theta) = \alpha(V)(1_V) \ .$$

Let

$E(\theta) = \{(u,v) \in NP_1 \amalg V : Np_1(u) = \theta(v)\}$, the pull-back over Np_1 and θ .

Then

$$0 \to DC \xrightarrow{f} E(\theta) \xrightarrow{g} V$$

with $f(u) = (u,0)$, $g(u,v) = v$, is exact

Let $\ell : E(\theta) \to NP_1$ be such that $(u,v) \to u$.

Then one can complete the commutative A-R-G diagram, where the rows are exact:

$$
\begin{tikzcd}
 & & D(P_1, \ ) \arrow[r] \arrow[d, "\alpha_{P_1}"] & D(P_0, \ ) \arrow[r] \arrow[d, "\alpha_{P_0}"] & D(W, \ ) \to 0 \arrow[d, "1_{D(W, \ )}"] \\
(2.5) & 0 \to (\ ,DC) \arrow[r] \arrow[d, "1_{(\ ,DC)}"'] & (\ ,NP_1) \arrow[r, "(\ , p_1)"] \arrow[d, "(\ ,\ell)"] & (\ ,NP_0) \arrow[r, "b"] \arrow[d, "(\ ,\theta)"] & D(W, \ ) \to 0 \arrow[d, "1_{D(W, \ )}"] \\
 & 0 \to (\ ,DC) \arrow[r, "(\ ,f)"'] & (\ ,E(\theta)) \arrow[r, "(\ ,g)"'] & (\ ,V) \arrow[r, "\alpha"'] & D(W, \ )
\end{tikzcd}
$$

It commutes because $\alpha(1_V) = b(V)\ (V,\theta)(1_V) \Rightarrow \alpha = b(\ ,\theta)$ by Yoneda's Lemma (0.15).

So

$$0 \to (\ ,DC) \to (\ ,E(\theta)) \xrightarrow{(\ ,g)} (\ ,V) \to F = \operatorname{Im} \alpha \to 0$$

is a projective resolution of F and one can take $V_1 = E(\theta)$.

An answer to the second question is given by:

(2.6) Proposition: Let

$$(\ ,V_1) \xrightarrow[(\ ,g)]{} (\ ,V) \to F \to 0$$

be exact.

Let I_0, I_1 be injective modules, and λ, θ, i_1, maps such that there exists an exact sequence

$$(2.7) \quad 0 \to V_1 \xrightarrow[\binom{\lambda}{g}]{} I_1 \amalg V \xrightarrow[(i_1, -\theta)]{} I_0 \quad .$$

Let

$$W = \text{Coker } Mi_1 \quad (\text{with } M = dD \ (\text{see } (0.25))$$

and

$$\alpha = D(n\ ,)\ \alpha^{-1}_{MI_0}\ (\ ,\theta) : (\ ,V) \to D(W,\ )$$

where

n is the natural map : $MI_0 \to \text{Coker } Mi_1$, and α_{MI_0} is given by (2.4).

Then $F \cong \text{Im}\ \alpha$.

(2.8) Remark: Condition that there is an exact sequence (2.7) is clearly equivalent to:

I_0, I_1 are injective modules such that:

(1) There is a map $\lambda : V_1 \to I_1$ such that $\lambda|_{\ker g}$ is injective.

(2) I_0 contains a module X , such that (X, i_1, θ) is a push-out of λ, g .

Such I_0, I_1, λ, θ, i_1 always exist.

Proof: Let $MI_1 = P_1$, $MI_0 = P_0$; these are projective modules (see Chapter 0, §3).

Since $i_1 : I_1 \to I_0$, then $Mi_1 : P_1 \to P_0$.

Taking $W = P_0/\mathrm{Im}\, Mi_1 = \mathrm{coker}\, Mi_1$, clearly

$$P_1 \xrightarrow{Mi_0} P_0 \xrightarrow{n} W \to 0$$

is a projective resolution of W .

So we can construct the A-R-G diagram as described above:

(2.9)

$$\begin{tikzcd}
 & D(P_1,\ ) \arrow[r, "D(Mi_1,)"] \arrow[d, "\alpha_{P_1}"] & D(P_0,\ ) \arrow[r, "D(n,\ )"] \arrow[d, "\alpha_{P_0}"] & D(W,\ ) \to 0 \arrow[d, "1_{D(W,\ )}"] \\
0 \to (\ ,A) \arrow[r, "(\ ,inc)"] \arrow[d, "1_{(\ ,A)}"] & (\ ,I_1) \arrow[r, "(\ ,i_1)"] \arrow[d, "(\ ,\lambda)"] & (\ ,I_0) \arrow[r, "b"] \arrow[d, "(\ ,\theta)"] & D(W,\ ) \to 0 \arrow[d, "1_{D(W,\ )}"] \\
0 \to (\ ,A) \arrow[r, "(\ ,f)"'] & (\ ,V_1) \arrow[r, "(\ ,g)"'] & (\ ,V) \arrow[r, "\alpha"'] & D(W,\ )
\end{tikzcd}$$

with

$$A = \ker i_1 \quad , \quad b = D(n,\ )\alpha_{P_0}^{-1} \quad , \quad \alpha = b(\ ,\theta)$$

Exactness of (2.7) implies that $\binom{\lambda}{g}$ is a monomorphism so $V_1 \cong \mathrm{Im}\,\binom{\lambda}{g} = \{(\lambda(v_1), g(v_1)) : v_1 \in V_1\}$.

Also

$$\mathrm{Im}\binom{\lambda}{g} = \ker(i_1, -\theta) = \{(u,v) \in I_1 \amalg V : i_1(u) = \theta(v)\} .$$

So

V_1 is isomorphic to the pullback over i_1, θ .

Note that if $u \in A = \text{Ker } i_1$, then $i_1(u) = 0 = \theta(0)$ so $(u,0) \in \text{Im}\binom{\lambda}{g} \cong V_1$.

Thus we can describe f as the map $u \to (u,0)$.

Also, g can be identified with the map

$$\{(\lambda(v_1),g(v_1)) : v_1 \in V\} \to V$$

such that $(\lambda(v_1),g(v_1)) \to g(v_1)$, $v_1 \in V_1$.

It is clear that

$$0 \to A \underset{f}{\to} V_1 \underset{g}{\to} V$$

is exact.

Therefore to prove that the last row of diagram (2.9) is exact one must show that $\text{Im}(\ ,g) = \ker \alpha$:

If $\rho \in \text{Im}(X,g)$ i.e. $\rho = (X,g)(\sigma)$ for $\sigma \in (X,V_1)$ then $(X,i_1)\ (X,\lambda)(\sigma) = (X,\theta)(X,g)(\sigma) \in \text{Im}(X,i_1) = \text{Ker } b(X)$.

Thus $0 = b(X)(X,\theta)(X,g)(\sigma) = \alpha(X)(X,g)(\sigma) = \alpha(X)(\rho)$ i.e. $\rho \in \text{Ker } \alpha(X)$.

Conversely, let $\rho \in \ker \alpha(X)$ i.e. $0 = \alpha(X)(\rho) = b(X)(X,\theta)(\rho)$. Thus $(X,\theta)(\rho) \in \ker b(X) = \text{Im}(X,i_1)$.

Therefore there exists $\delta \in (X,I_1)$ such that $i_1(\delta(x)) = \theta(\rho(x))$ $\forall x \in X$. So $(\delta(x),\rho(x)) \in \ker (i_1,-\theta) \cong V_1$.

Considering $(\delta,\rho) : X \to V_1$

$$x \to (\delta(x),\rho(x))$$

then

$$(X,g)(\delta,\rho) = g(\delta,\rho) : x \to g(\delta(x),\rho(x)) = \rho(x) .$$

So $(X,g)(\delta,\rho) = \rho$ i.e. $\rho \in \mathrm{Im}(X,g)$.

Therefore

$$\mathrm{Im}\,\alpha \cong (\ ,V)/\ker\alpha = (\ ,V)/\mathrm{Im}(\ ,g) \cong F . \qquad \square$$

§2. <u>mmod A and D(W,U)</u> $(W,U \in \text{mod } A)$

Yoneda's lemma (0.15) tells us that the map α in (2.1) is completely determined by the element $T = \alpha(U)\,(1_U) \in D(W,U)$, and conversely, given an element in $D(W,U)$, it determines a map $\alpha:(\ ,U) \to D(W,\ )$ and therefore a finitely presented functor.

So an element $F \in$ mmod A is completely determined by a triple T, W, U with $W,U \in$ mod A and $T \in D(W,U)$.

(2.10) <u>Notation</u>: In this case, write $F = H(T;W,U)$.

Before we can go further we must make some considerations about the map α of (2.1):

Naturality of α gives the commutative diagram:

$$\begin{tikzcd}
(U,U) \arrow[rrr, "\alpha(U)"] \arrow[ddd, "{(f,U)}"'] & & & D(W,U) \arrow[ddd, "{D(W,f)}"] \\
& 1_U \arrow[r] \arrow[d] & T \arrow[d] & \\
& f \arrow[r] & \alpha(X)(f) & \\
(X,U) \arrow[rrr, "\alpha(X)"'] & & & D(W,X)
\end{tikzcd} \tag{2.11}$$

where $f: X \to U$, $T = \alpha(U)(1_U)$.

Then:

$$\alpha(X)(f) = D(W,f)(\alpha(U))(1_U)) = D(W,f)(T) \in D(W,X) \quad .$$

If $\psi \in (W,X)$, then:

$$D(W,f)(T)(\psi) = T(f\psi) = (T*f)(\psi) \quad \text{using (1.17)} .$$

Thus

$$\alpha(X)(f) = T*f \tag{2.12}$$

Considering the covariant case, the map:

$$\begin{array}{ccc} ((W,\ ),D(\ ,U)) & \longrightarrow & D(W,U) \\ \beta & \longrightarrow & \beta(W)(1_W) \end{array}$$

is a k-linear isomorphism. (See (0.16)).

Then naturality gives the commutative diagram:

(2.13)
$$\begin{tikzcd}
(W,W) \arrow[rrr, "\beta(W)"] \arrow[ddd, "{(W,h)}"'] & & & D(W,U) \arrow[ddd, "{D(h,U)}"] \\
& 1_W \arrow[r] \arrow[d] & T' \arrow[d] & \\
& h \arrow[r] & \beta(X)(h) & \\
(W,X) \arrow[rrr, "\beta(X)"'] & & & D(X,U)
\end{tikzcd}$$

where $h:W \to X$, $T' = \beta(W)(1_W)$

$$\beta(X)(h) = D(h,U)(\beta(W)(1_W)) = D(h,U)(T') \in D(X,U) .$$

If $\phi \in (X,U)$ then:

$$D(h,U)(T')(\phi) = T'(\phi h) = (h^*T')(\phi) \quad \text{using (1.17)} .$$

Thus

(2.14) $\quad \beta(X)(h) = h^*T'$.

The next theorem, due to J.A. Green, tells us how to "describe" a morphism between functors when these are given in the form (2.10):

(2.15) Theorem (Green) : Let $F,F' \in$ mmod A , be such that $F = H(T;W,U)$, $F' = H(T';W',U')$.

Then

(i) Given a morphism $\phi:F \to F'$, there exist A-maps $f:U \to U'$, $h:W \to W'$ such that:

(2.16) $\quad T'^*f = h^*T$.

(ii) Given maps $f:U \to U'$, $h:W \to W'$ such that (2.16) holds, there is a unique morphism $\phi:F \to F'$ such that the following diagram commutes:

$$\text{(2.17)} \qquad \begin{tikzcd} (\ ,U) \arrow[r, two heads, "\alpha"] \arrow[d, "(\ ,f)"'] & F \arrow[r, hook, "inc"] \arrow[d, "\phi"] & D(W,\ ) \arrow[d, "D(h,\ )"] \\ (\ ,U') \arrow[r, two heads, "\alpha'"'] & F' \arrow[r, hook, "inc'"'] & D(W',\ ) \end{tikzcd}$$

Proof: (i) Given ϕ , then,since $(\ ,U)$ is projective, there exists $f^*:(\ ,U) \to (\ ,U')$ such that $\phi\alpha = \alpha' f^*$. And $f^* = (\ ,f)$ for some $f:U \to U'$ (0.17). Thus $\phi\alpha = \alpha'(\ ,f)$.

Also, since $D(W',\ )$ is injective, there exists $h^* = D(h,\ )$ with $h:W \to W'$ such that

$$D(h,\ )\, inc = inc'\phi$$

Thus

$$D(h,\ )\alpha = \phi\alpha = \alpha'(\ ,f)$$

and

$$D(h,U)\alpha(U)(1_U) = \alpha'(U)\ (U,f)(1_U) \Rightarrow D(h,U)(T) = \alpha'(U)(f) \Rightarrow$$
$$\Rightarrow h^*T = T'^*f \text{ by (2.12), (2.14).}$$

(ii) If f,h are such that (2.16) holds, then, since $D(h,U)\alpha(U)(1_U) = h^*T$ and $\alpha'(U)(U,f)(1_U) = T'^*f$, by Yoneda's Lemma (1.15), we have

$$D(h,\ )\alpha = \alpha'(\ ,f) \quad .$$

Now define $\phi:F \to F'$ as follows:

$$\phi = D(h,\ )\big|_F \quad \text{(restriction of } D(h,\ ) \text{ to } F) \quad .$$

We must show that $\operatorname{Im}\phi(X) \leq F'(X)$, for all $X \in \operatorname{mod} A$:

Let $S \in F(X)$; then $D(h,X)(S) = S\,(h,X) = h^*S$ (because $(S.(h,X))(t) = S(th) = (h^*S)(t)$, $\forall\, t \in (W',X))$.

Since $S \in F(X) = \operatorname{Im}\alpha(X)$, then $S = \alpha(X)(v)$ for some $v \in (X,U)$; so $S = T^*v$.

Thus $D(h,X)(S) = h^*S = h^*(T^*v) = (h^*T)^*v = (T'^*f)^*v = T'^*fv =$
$= \alpha'(X)(fv) \in \operatorname{Im}\alpha'(X) = F'(X)$.

Clearly $\phi\alpha = \alpha'(\ ,f)$ and $D(h,\ )\,\mathrm{inc} = \mathrm{inc}'\phi$ and ϕ is uniquely determined by these expressions.

Now one has to prove that $\phi(X)$ is natural in X :

Let $g:X \to Y$. We must prove that the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "\phi(X)"] & F'(X) \\
F(Y) \arrow[u, "F(g)"] \arrow[r, "\phi(Y)"'] & F'(Y) \arrow[u, "F'(g)"']
\end{tikzcd}$$

commutes.

Consider the diagram:

$$\begin{tikzcd}
(X,U) \arrow[r, "\alpha(X)"] & F(X) \arrow[r, "\phi(X)"] & F'(X) \arrow[r, "inc'(X)"] & D(W',X) \\
(Y,U) \arrow[r, "\alpha(Y)"'] \arrow[u, "{(g,U)}"] \arrow[ur, phantom, "(a)"] & F(Y) \arrow[r, "\phi(Y)"'] \arrow[u, "F(g)"'] \arrow[ur, phantom, "(b)"] & F'(Y) \arrow[r, "inc'(Y)"'] \arrow[u, "F'(g)"] \arrow[ur, phantom, "(c)"] & D(W',Y) \arrow[u, "{D(W',g)}"]
\end{tikzcd}$$

(a) and (c) commute because α and inc' are natural.

Let $R \in F(Y)$. Then $R = \alpha(Y)(v) = T^*v$ with $v \in (Y,U)$.

$\phi(X)F(g)(R) = \phi(X)F(g)\alpha(Y)(v) = \phi(X)\alpha(X)(g,U)(v) = \phi(X)\alpha(X)(vg) =$
$= \phi(X)(T^*vg) = h^*T^*vg = h^*(T^*v)^*g = h^*R^*g = D(W',g)(h^*R) =$
$= D(W',g)\phi(Y)(R) = F'(g)\phi(Y)(R)$, using commutativity of (c).

Thus $\phi(X)F(g) = F'(g)\phi(Y)$. $\square$

(2.18) <u>Remark</u>: An equivalent definition for ϕ is:

If $S \in F(X)$, $X \in \text{mod } A$,

$$\phi(X)(S) = T'^*fv$$

where $v \in (X,U)$ is such that $\alpha(X)(v) = T^*v = S$. (v exists because $\alpha:(\ ,U) \to F$ is epimorphism, and ϕ is well-defined since if v' is such that $T^*v' = T^*v$ then $T^*(v-v') = 0$; so $T'^*f(v-v') =$ $= h^*T^*(v-v') = h^*(T^*(v-v')) = 0$, i.e. $T'^*fv = T'^*fv'$.)

(2.19) <u>Corollary</u>: If there exists isomorphisms $f:U \to U'$, $h:W \to W'$ such that $T'^*f = h^*T$, then $F = H(T;W,U) \cong F' = H(T';W',U')$.

Proof: $T'*f = h*T \Rightarrow h^{-1}*T' = T*f^{-1}$. Thus by (2.15)(ii) $\exists\psi : F \to F'$ such that

$$\begin{tikzcd}
(\ ,U) \arrow[r, "\alpha"] & F \arrow[r] & D(W, \ ) \\
(\ ,U') \arrow[r, "\alpha'"] \arrow[u, "(\ ,f^{-1})"] & F' \arrow[r] \arrow[u, "\psi"] & D(W', \ ) \arrow[u, "D(h^{-1}, \ )"']
\end{tikzcd}$$

commutes.

Considering this diagram and (2.17) we have: $(\psi\phi)\alpha = \psi(\phi\alpha) = \psi\alpha'(\ ,f) = \alpha(\ ,f^{-1})(\ ,f) = \alpha = 1_F\alpha$. Since α is epimorphism, $\psi\phi = 1_F$. Similarly $\phi\psi = 1_{F'}$. Hence ϕ,ψ are isomorphisms. □

In particular, using the equivalence relation (1.21)

(2.20) Corollary: Let $U,W \in \text{mod } A$ and $T,T' \in D(W,U)$ be such that $T \sim T'$. Then

$$F = H(T;W,U) \cong F' = H(T';W,U) .$$ □

§3. The category $\mathcal{T}$

Let A be any finite dimensional k-algebra.

(2.21) Definition: Denote by $\mathcal{T}$ the following category:

$\text{Obj } \mathcal{T} = \{(T;W,U) : W,U \in \text{mod } A , T \in D(W,U)\}$

$((T;W,U),(T';W',U'))_{\mathcal{T}} = \{(f,h) : f \in (U,U')_A , h \in (W,W')_A$ and $T'*f = h*T\}$.

The "composition law" is

$$((T;W,U),(T';W',U'))\times((T';W',U'),(T'';W'',U'')) \to ((T;W,U),(T'',W',U''))$$
$$((f,h)\ ,\ (f',h')) \longrightarrow (f'f,h'h)$$

and $1_{(T;W,U)} = (1_U,\ 1_W)$.

$\mathcal{T}$ is a k-category (see (0.4)).

Theorems (2.1) and (2.15) give a k-linear covariant functor

$$H : \mathcal{T} \longrightarrow \text{mmod } A$$

such that

(2.22)

$$(T;W,U) \longrightarrow F = H(T;W,U) \quad \text{i.e.} \quad F = \operatorname{Im}\alpha$$

where $\alpha:(\ ,U) \to D(W,\ )$ is such that $\alpha(U)(1_U) = T$

$$(f,h) \longrightarrow \phi \quad \text{given by (2.15)(ii)}$$

(2.23) <u>Remark</u>: Since H is k-linear, it commutes with direct sums i.e. if $(T;W,U) \cong (T_1;W_1,U_1) \amalg (T_2;W_2,U_2)$ in $\mathcal{T}$ then $H(T;W,U) \cong H(T_1;W_1,U_1) \amalg H(T_2;W_2,U_2)$ in mmod A .

Let $\mathcal{J}$ be such that, for given objects $(T;W,U),(T';W',U')$ in $\mathcal{T}$, one has:

(2.24) $\mathcal{J}((T;W,U),(T';W',U')) = \{(f,h):f \in (U,U')_A\ ,\ h \in (W,W')_A$ and $T'*f = 0 = h*T\}$.

This is clearly an ideal in the category $\mathcal{T}$ (see (0.12)).

For simplicity denote by $\overline{(f,h)}$ the element $(f,h) + \mathcal{J}((T;W,U)$, $(T':W',U')) \in ((T;W,U)$, $(T';W',U'))_{\mathcal{T}/\mathcal{J}}$.

Then

(2.25) Lemma: The following is an equivalence of categories:

$$\mathcal{H} : \mathcal{T}/\mathcal{J} \longrightarrow \text{mmod } A$$

such that

$$(T;W,U) \longrightarrow H(T;W,U)$$
$$\overline{(f,h)} \longrightarrow H(f,h)$$

where H is the functor given in (2.22).

Proof: We use definition (0.2) of equivalence of categories:

$\mathcal{H}$ is dense by (2.1);

And $\mathcal{H}:((T;W,U),(T';W',U'))_{\mathcal{T}/\mathcal{J}} \to (H(T;,W,U),H(T';W',U'))_{\text{mmod } A}$

$$\overline{(f,h)} \longrightarrow H(f,h)$$

is an isomorphism:

In fact $\overline{(f,h)} = \overline{(f',h')} \Rightarrow T'*(f-f') = 0 = (h-h')*T \Rightarrow T'*f = T'*f' \Rightarrow$ $\Rightarrow T'*fv = T'*f'v$, $\forall\, v \in (X,U)$, $\forall\, X \in \text{mod } A$. Then by (2.18), $H(f,h) = H(f',h')$.

Also $H(f,h) = 0 \Rightarrow H(f,h)(U)(T\ ) = 0 \Rightarrow T'*f = 0 = h*T$.

Thus $\mathcal{H}$ is a monomorphism.

And given $\phi \in (H(T;W,U), H(T';W',U'))_{\text{mmod } A}$, by (2.15)(i), there are $f:U \to U'$, $h : W \to W'$ such that $T'*f = h*T$.

These are maps such that

$$D(h,\ )\alpha = \phi\alpha = \alpha'(\ ,f)$$

where α,α' are as in (2.17).

But this means that $D(h,)|_{H(T;W,U)} = \phi|_{H(T;W,U)} = \phi$

i.e. $H(f,h) = \phi$. Thus $\mathcal{H}$ is epimorphic. $\square$

Given $(T;W,U) \in \mathcal{T}$, define:

(2.26) $I = \mathcal{J}((T;W,U),(T;W,U)) =$

$= \{(f,h): f \in \text{End}_A U\ ,\ h \in \text{End}_A W \text{ and } T*f = 0 = h*T\}$.

I is an ideal of the k-algebra $\text{End}_{\mathcal{T}}(T;W,U)$.

And using lemma (2.25) we see that

(2.27) $\dfrac{\text{End}_{\mathcal{T}}(T;W,U)}{I} \cong \text{End } F$ where $F = H(T;W,U) \in \text{mmod } A$.

§4. Decomposability in $\mathcal{T}$ and mmod A

We start this section by generalizing some of the facts referred to in Chapter I, §4, to a category mod A where A is any finite-dimensional k-algebra.

Namely we have the following:

By (0.6),

$W = \coprod_{j=1}^{m} W_j$, $U = \coprod_{i=1}^{n} U_i$ in mod A iff there are morphisms

$m_i \in (U_i,U)$, $p_i \in (U,U_i)$, $n_j \in (W_j,W)$, $q_j \in (W,W_j)$, such that:

$$\begin{array}{lll} p_i m_t = \delta_{it} 1_{U_i} & q_j n_\ell = \delta_{j\ell} 1_{W_j} & i,t = 1,\dots,n \\ \sum_{i=1}^{n} m_i p_i = 1_U & \sum_{j=1}^{m} n_j q_j = 1_W & j,\ell = 1,\dots,m \end{array}$$

Then, if $T \in D(W,U)$, the matrix of T with respect to the above decompositions of W,U is:

$$(2.28) \qquad \mathcal{T} = (T_{ji})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$$

where T_{ji} is defined by the expression:

$$(2.29) \qquad T_{ji} = q_j{}^*T^*m_i \in D(W_j,U_i) \quad .$$

Remark: If the decompositions of W,U are assumed to be known, we sometimes use (abusively) the same symbol for the element $T \in D(W,U)$ and its matrix with respect to the given decompositions, writing, for example, expressions such as $T = (T_{ji})_{i,j} \in D(W,U)$.

(2.30) Proposition: $(T;W,U) \in \mathcal{T}$ is decomposable iff:

(i) There are $W_1,W_2,U_1,U_2 \neq 0$ in mod A such that: $W = W_1 \amalg W_2$, $U = U_1 \amalg U_2$.

(ii) There are elements $T_1 \in D(W_1,U_1)$, $T_2 \in D(W_2,U_2)$ such that the matrix $\mathcal{T}$ of T with respect to the decompositions of W,U given in (i), is

$$\mathcal{T} = \left(\begin{array}{c|c} T_1 & 0 \\ \hline 0 & T_2 \end{array}\right)$$

Proof: Using (0.6) and the definition of morphism in $\mathcal{T}$ we have:

$(T;W,U) \cong (T_1;W_1,U_1) \amalg (T_2;W_2,U_2)$ <=> there are morphisms $\mu_i = (m_i,n_i)$, $\pi_i = (p_i,q_i)$, with:

$$\text{(2.31)} \qquad \begin{array}{lll} m_i \in (U_i,U) & n_i \in (W_i,W) & T_i{*}p_i = q_i{*}T \\ & & \text{and} \qquad\qquad i=1,2 \\ p_i \in (U,U_i) & q_i \in (W,W_i) & T{*}m_i = n_i{*}T_i \end{array}$$

such that

$$\pi_j\mu_i = \delta_{ij}\ 1_{(T_i;W_i,U_i)} \quad (i = 1,2) \quad \text{and} \quad \sum_{i=1}^{2} \mu_i\pi_i = 1_{(T;W,U)} \iff \text{there}$$

exist morphisms m_i, n_i, p_i, q_i in mod A such that (2.31) holds and

$$\text{(2.32)} \quad \begin{array}{lcl} p_j m_i = \delta_{ij}\ 1_{U_i} & , & q_j n_i = \delta_{ij}\ 1_{W_i} \\ \sum_{i=1}^{2} m_i p_i = 1_U & , & \sum_{i=1}^{2} n_i q_i = 1_W . \end{array}$$

If there are morphisms m_i, n_i, p_i, q_i $(i = 1,2)$ such that (2.31) and (2.32) are verified then by (0.6):

$W \cong W_1 \amalg W_2$, $U \cong U_1 \amalg U_2$ and with respect to these decompositions (see (2.28), (2.29)):

$$\mathcal{T} = \begin{pmatrix} q_1{*}T{*}m_1 & q_1{*}T{*}m_2 \\ q_2{*}T{*}m_1 & q_2{*}T{*}m_2 \end{pmatrix} = \begin{pmatrix} T_1{*}p_1m_1 & T_1{*}p_1m_2 \\ T_2{*}p_2m_1 & T_2{*}p_2m_2 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

Conversley assume that (i), (ii) are verified. Then there are morphisms $U_i \underset{m_i}{\overset{p_i}{\rightleftarrows}} U$, $W_i \underset{n_i}{\overset{q_i}{\rightleftarrows}} W$ in mod A such that (2.32) holds.

Then

$$\mathcal{T} = \begin{pmatrix} q_1*T*m_1 & q_1*T*m_2 \\ q_2*T*m_1 & q_2*T*m_2 \end{pmatrix} \qquad \text{where} \quad q_j*T*m_i \in D(W_j,U_i) \quad (i,j = 1,2)$$

By (i),

$$\begin{aligned} q_1*T*m_1 &= T_1 \Rightarrow q_1*T*m_1p_1 = T_1*p_1 \\ q_1*T*m_2 &= 0 \Rightarrow q_1*T*m_2p_2 = 0 \\ q_2*T*m_1 &= 0 \Rightarrow q_2*T*m_1p_1 = 0 \\ q_2*T*m_2 &= T_2 \Rightarrow q_2*T*m_2p_2 = T_2*p_2 \ . \end{aligned}$$

Thus by (2.32)

$$q_i*T*(m_1p_1+m_2p_2) = T_i*p_i \Rightarrow q_i*T = T_i*p_i \qquad (i = 1,2) \ .$$

Similarly $T*m_i = n_i*T_i$ $(i = 1,2)$.

Thus the maps m_i,n_i,p_i,q_i $(i = 1,2)$ verify (2.31), (2.32), so by above equivalences

$$(T;W,U) \cong (T_1;W_1,U_1) \amalg (T_2;W_2,U_2) \ . \qquad \square$$

We also observe the following:

(2.33) The category $\mathcal{T}$ has kernels. Therefore in this category idempotents split (see (0.10)).

In fact we have:

(2.34) Let $(f,h) : (T;W,U) \to (T';W',U')$ be a morphism in $\mathcal{T}$; then $((T{*}i,W,\ker f)\ ,\ (i,1_W))$ (where $i:\ker f \to U$ is the inclusion map) is a kernel of (f,h) .

<u>Proof</u>: Consider the diagram

$$\begin{tikzcd}
 & (\ ,\mathrm{Ker} f) \arrow[rr, "T_1 = T{*}i"] \arrow[dd, "(\ ,i)"] & & D(W,\ ) \arrow[dd, "D(1_W,\ )"] \\
(\ ,U'') \arrow[ur, dashed, "(\ ,f'')"] \arrow[dr, dashed, "(\ ,f')"'] \arrow[rr, dashed, "T_2"] & & D(W'',\ ) \arrow[ur, dashed, "D(h',\ )"] \arrow[dr, dashed, "D(h',\ )"'] & \\
 & (\ ,U) \arrow[rr, "T"] \arrow[d, "(\ ,f)"'] & & D(W,\ ) \arrow[d, "D(h,\ )"] \\
 & (\ ,U') \arrow[rr, "T'"] & & D(W',\ )
\end{tikzcd}$$

$(i,1_W):(T_1;W,\ker f) \to (T;W,U)$ is a morphism in $\mathcal{T}$ because $T{*}i = 1_W{*}T_1$.

Let $(T_2;W'',U'') \in \mathcal{T}$ and $(f',h'):(T_2;W'',U'') \to (T;W,U)$ be such that $(f,h)(f',h') = (ff',hh') = (0,0)$.

Then clearly there is $f'' : U'' \to \ker f$ such that $f' = if''$.

And, of course, $1_W h' = h'$.

And $T_1{*}f'' = T{*}i{*}f'' = T{*}if'' = T{*}f' = h'{*}T_2$. Thus (f'',h') is a morphism in $\mathcal{T}$.

So $(T{*}i,W,\ker f)$ is a kernel of (f,h) . $\square$

(2.35) Remark: By (2.33) and using (0.9), (0.11) we have:

(T;W,U) is indecomposable in T iff the set of idempotents in $\mathrm{End}_T(T;W,U)$ is $\{(0_U,0_W),\ (1_U,1_W)\}$.

Now let us recall the following theorem (see [CRM], pgs.111 and 119):

(2.36) Theorem: Let $B \neq 0$ be a finite-dimensional k-algebra. Then the following are equivalent:

I. $Id(B) = \{0,1\}$ where $Id(B)$ is the set of idempotents in B .

II. Each element $b \in B$ is either invertible or nilpotent.

III. $B/J(B)$ is a division algebra ($J(B)$ is the Jacobson radical of B).

IV. $J(B)$ is the unique maximal ideal of B . □

Such an algebra B is called a local algebra.

It is clear that:

(2.37) If B is a local algebra, and I is an ideal of B such that $I \neq B$, then B/I is a local algebra. □

Recall also that to prove that an algebra B is local, is equivalent to prove that B/N is local, where N is some nilpotent ideal of B (see for example [Lr], pg.3, using the fact that an idempotent $\neq 0$ is not nilpotent).

The following fact will be useful later:

(2.38) If B_0 is a subalgebra of B such that $B_0 + N = B$, where N is a nilpotent ideal of B, then if B_0 is local, B is local.

(This is so because if B_0 is local, then $\frac{B_0}{B_0 \cap N} \cong \frac{B_0 + N}{N}$, is local, i.e. B/N is local, therefore B is local.)

Now returning to our discussion about $\mathcal{T}$ and mmod A:

(2.39) <u>Lemma</u>: If $(T; W,U)$ is a non-zero object in $\mathcal{T}$, then it is non-zero in $\mathcal{T}/\mathcal{J}$, i.e. the ideal $I = \mathcal{J}((T;W,U),(T;W,U))$ is not equal to $\mathrm{End}_{\mathcal{T}}(T;W,U)$.

<u>Proof</u>: $I = \{(f,h) \in \mathrm{End}\, U \amalg \mathrm{End}\, W : T{*}f = 0 = h{*}T\}$ (see (2.26)).

If $(1_U, 1_W) \in I$, then $T = T{*}1_U = 0$, and this is a contradiction. $\square$

Thus

(2.40) <u>Proposition</u>: $(T;W,U)$ is decomposable in $\mathcal{T}$ iff $H(T;W,U)$ is decomposable in mmod A .

<u>Proof</u>: By (2.23) if $(T;W,U)$ is decomposable in $\mathcal{T}$ then $H(T;,W,U)$ is decomposable in mmod A .

Now suppose that $(T;W,U)$ is indecomposable in $\mathcal{T}$. By (2.35) and (2.36).I, $\mathrm{End}_{\mathcal{T}}(T;W,U)$ is a local algebra.

Lemma (2.39) shows that $I \nsubseteq \mathrm{End}_{\mathcal{T}}(T;W,U)$.

Then by (2.37), $\dfrac{\mathrm{End}_{\mathcal{T}}(T;W,U)}{I}$ is local. But

$$\frac{\mathrm{End}_{\mathcal{T}}(T;W,U)}{I} \cong \mathrm{End}_{\mathrm{mmod}\ A} H(T;W,U) \qquad (2.27)$$

Thus $\mathrm{End}_{\mathrm{mmod}\ A} H(T;W,U)$ is local and this implies that $H(T;W,U)$ is indecomposable in mmod A (0.14). □

§5. Examples of indecomposable finitely presented functors

Let k be a field and A the k-algebra considered in Chapter I, §1, i.e. $A = A_q = \text{k-alg} <z:z^q = 0>$.

In this section we consider some examples of indecomposable elements in mmod A_q .

(1) Let $q = 3$ and let V_1, V_2, V_3 be the indecomposable A_3-modules.

Suppose $T = \begin{pmatrix} \pi_{12} & \pi_{13} \\ 0 & \pi_{33}z \end{pmatrix} \in D(W,U)$ where $W = V_1 \amalg V_3$, $U = V_2 \amalg V_3$. Consider $(T;W,U) \in \mathcal{T}$ (see §3) .

Let $B = \mathrm{End}_{\mathcal{T}}(T;W,U) = \{(f,h) \in \mathrm{End}\ U \amalg \mathrm{End}\ W : T^*f = h^*T\}$.

Let $N = \mathrm{rad}\,(\mathrm{End}\,U \amalg \mathrm{End}\,W)$ and

$$B_0 = \{(f,h) \in \mathrm{End}\,U \amalg \mathrm{End}\,W : f = \begin{pmatrix} au_{22} & 0 \\ 0 & bu_{33} \end{pmatrix},\ h = \begin{pmatrix} cu_{11} & 0 \\ 0 & du_{33} \end{pmatrix}$$

$: a,b,c,d \in k$ and $T*f = h*T\}$.

Then $B_0 + (N \cap B) = B$ and $N \cap B$ is nilpotent.

In order to prove that B is local it is enough to prove that B_0 is local (by 2.38).

By (2.36) we can prove that B_0 is local by showing that the only idempotents of B_0 are $(0_U, 0_W)$ and $(1_U, 1_W)$:

$$T*f = h*T \Rightarrow \begin{pmatrix} \pi_{12} & \pi_{13} \\ 0 & \pi_{33}z \end{pmatrix} * \begin{pmatrix} au_{22} & 0 \\ 0 & bu_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} cu_{11} & 0 \\ 0 & du_{33} \end{pmatrix} * \begin{pmatrix} \pi_{12} & \pi_{13} \\ 0 & \pi_{33}z \end{pmatrix} \Rightarrow \begin{pmatrix} a\pi_{12} & b\pi_{13} \\ 0 & b\pi_{33}z \end{pmatrix} = \begin{pmatrix} c\pi_{12} & c\pi_{13} \\ 0 & d\pi_{33}z \end{pmatrix}$$

$\Rightarrow a = c = b = d$.

Now clearly the only idempotents in B_0 are the trivial ones.

Using this method it is easy to prove that all matrices in (1.47) give indecomposable elements in T , and so correspond to indecomposable elements in mmod A (see 2.40).

<u>Remark</u>: It can be shown now that no two of the functors given by the matrices (1.47) are isomorphic, by considering for each $T \in D(W,U)$ in (1.47), the map

$$\alpha_T : (\ ,U) \to D(W,\ )$$

such that $\alpha_T(U)(1_U) = T$ (by Yoneda's Lemma (0.15)) and then constructing the modules $F_T(C) = \operatorname{Im} \alpha_T(C) \cong (C,U)/\{f \in (C,U) : T^*f = 0\}$ $(= M_T)$ where $C = V_1 \amalg V_2 \amalg V_3$ and $F_T = H(T;W,U)$.

It can be shown that no two of the 21 modules M_T are isomorphic.

Some examples of these modules are given in Chapter III and a complete list of them (given by considering their radical series and socle series) appears in the graph (3.27).

Now, if for some T,T' in (1.47), $F_T \cong F_{T'}$, then $M_T \cong M_{T'}$, a contradiction. It is also clear that no two matrices of (1.47) are equivalent (by 2.20).

(2) Now consider the following example:

Let $q \geq 4$ and $U = W = V_2^n \amalg V_4^n$ for some $n \in \mathbb{N}$.

Let

$$T = T_n = \begin{pmatrix} \pi_{22}P & \pi_{24}zI \\ \pi_{42}zI & \pi_{44}z^2I \end{pmatrix}_{2n\times 2n} \in D(W,U) \qquad \text{where } P \text{ is}$$

the matrix $\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \dots & & & \dots & \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}_{n\times n}$

Consider the element $(T\ ;W,U) \in \mathcal{T}$ and let $B = \mathrm{End}\ (T;W,U) = \{(f,h) \in \mathrm{End}\ U \amalg \mathrm{End}\ W : T^*f = h^*T\}$.

We may write f in the form $\begin{pmatrix} F_{22}u_{22} & F_{24}u_{24} \\ F_{42}u_{42} & F_{44}u_{44} \end{pmatrix}$ and h in the form $\begin{pmatrix} H_{22}u_{22} & H_{24}u_{24} \\ H_{42}u_{42} & H_{44}u_{44} \end{pmatrix}$ where F_{ij} , H_{ij} are $n\times n$ matrices of elements in A .

Let

$$B^* = \{(f,h) \in \mathrm{End}\ U \amalg \mathrm{End}\ W : F_{24} = 0 = F_{42} = H_{24} = H_{42}$$

and $F_{22}^{(i)} = F_{44}^{(i)} = H_{22}^{(i)} = H_{44}^{(i)} = 0$ if $i \geq 1\}$ where by $F_{kj}^{(i)}$ we denote the matrix whose elements are the coefficients of the terms of degree i of the entries of F_{kj} .

Let $B_0 = B^* \cap B$. This is clearly a subalgebra of B .

Let $N = \mathrm{rad}\ (\mathrm{End}\ U \amalg \mathrm{End}\ W) \cap B$.

Then

$$B_0 + N = B$$

as follows:

Clearly $B_0 + N \subset B$.

Conversely let $(f,h) \in B$; then $f = f_0+n$ $h = h_0+m$ where

f_0 is given by $\begin{pmatrix} F_{22}^{(0)} & 0 \\ 0 & F_{44}^{(0)} \end{pmatrix}$, h_0 is given by $\begin{pmatrix} H_{22}^{(0)} & 0 \\ 0 & H_{44}^{(0)} \end{pmatrix}$

and $n \in J(\mathrm{End}\ U)$, $m \in J(\mathrm{End}\ W)$.

So $(f,h) = (f_0,h_0) + (n,m)$ where $(n,m) \in \mathrm{rad}(\mathrm{End}\ U \amalg \mathrm{End}\ W)$.

Now

$$T^*f = h^*T \Rightarrow \begin{pmatrix} \pi_{22}(PF_{22}+zF_{42}) & \pi_{24}(z^2PF_{24}+zF_{44}) \\ \pi_{42}(zF_{22}+z^2F_{42}) & \pi_{44}(z^3F_{24}+z^2F_{44}) \end{pmatrix} =$$

$$= \begin{pmatrix} \pi_{22}(H_{22}P+zH_{24}) & \pi_{24}(zH_{22}+z^2H_{24}) \\ \pi_{42}(z^2H_{42}P+zH_{44}) & \pi_{44}(z^3H_{42}+z^2H_{44}) \end{pmatrix} \Rightarrow \begin{array}{l} PF_{22}^{(0)} = H_{22}^{(0)}P \\ H_{44}^{(0)} = F_{44}^{(0)} \\ H_{22}^{(0)} = F_{44}^{(0)} \\ H_{44}^{(0)} = F_{22}^{(0)} \end{array}$$

$$\Leftrightarrow \begin{pmatrix} \pi_{22}PF_{22}^{(0)} & \pi_{24}zF_{44}^{(0)} \\ \pi_{42}zF_{22}^{(0)} & \pi_{44}z^2F_{44}^{(0)} \end{pmatrix} = \begin{pmatrix} \pi_{22}H_{22}^{(0)}P & \pi_{24}zH_{22}^{(0)} \\ \pi_{42}zH_{44}^{(0)} & \pi_{44}z^2H_{44}^{(0)} \end{pmatrix} \Leftrightarrow$$

$\Leftrightarrow$ $T^*f_0 = h_0^*T$ $\Rightarrow$ $(f_0,h_0) \in B_0$.

Since (f,h) , $(f_0,h_0) \in B$ then $(n,m) \in B$, thus $(n,m) \in N$. Therefore $(f,h) \in B_0 + N$.

By (2.38), we must prove that B_0 is local.

Using the calculations just done it is easy to see that

$$B_0 = \left\{ \left(\begin{pmatrix} F_{22}^{(0)} u_{22} & 0 \\ 0 & F_{44}^{(0)} u_{44} \end{pmatrix} \begin{pmatrix} H_{22}^{(0)} u_{22} & 0 \\ 0 & H_{44}^{(0)} u_{44} \end{pmatrix} \right) : \begin{array}{l} PF_{22}^{(0)} = H_{22}^{(0)} P \\ H_{44}^{(0)} = F_{44}^{(0)} = H_{22}^{(0)} = F_{22}^{(0)} \end{array} \right\}$$

$$= \left\{ \left(\begin{pmatrix} F_{22}^{(0)} u_{22} & 0 \\ 0 & F_{22}^{(0)} u_{44} \end{pmatrix} \begin{pmatrix} F_{22}^{(0)} u_{22} & 0 \\ 0 & F_{22}^{(0)} u_{44} \end{pmatrix} \right) : PF_{22}^{(0)} = F_{22}^{(0)} P \right\}$$

Simple calculations give that:

$$PF_{22}^{(0)} = F_{22}^{(0)} P \Rightarrow F_{22}^{(0)} = \begin{pmatrix} a_1 & 0 & \cdots & & 0 \\ a_2 & a_1 & \cdots & & 0 \\ a_3 & a_2 & \cdots & & 0 \\ \cdots & & & & \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix}$$

(for some $a_1, \ldots a_n \in k$.)

$$= a_1 I + a_2 P + a_3 P^2 + \ldots + a_n P^{n-1} .$$

Since P is nilpotent, it is clear that every element of B_0 is invertible or nilpotent. Thus by (2.36), B_0 is a local algebra. Then by (2.38), B is local.

Thus

$$(2.41)\qquad F_n = H\left(T_n = \begin{pmatrix} \pi_{22}P & \pi_{24}zI \\ \pi_{42}zI & \pi_{44}z^2I \end{pmatrix}_{2n\times 2n} ;\ V_2^n \amalg V_4^n\ ,\ V_2^n \amalg V_4^n \right) \in$$

$\in$ mmod A_q , is indecomposable, $\forall\ n \in \mathbb{N}$.

Let

$$\alpha_n : (\ ,V_2^n \amalg V_4^n) \longrightarrow D(V_2^n \amalg V_4^n\ ,\ )$$

be the map such that

$$\alpha_n(V_2^n \amalg V_4^n)(1_{V_2^n \amalg V_4^n}) = T_n$$

(use Yoneda's Lemma (0.15)).

Thus

$\qquad$ Ker $\alpha_n \leq$ rad$(\ ,V_2^n \amalg V_4^n)$ i.e. α_n is a projective cover of F_n (see [AI] pg.208).

To prove this it is enough to show that

$$\text{Ker } \alpha(V_2^n \amalg V_4^n) \leq \text{rad}(\text{End } V_2^n \amalg V_4^n)$$

(by Fitting's theorem ([CRM], pg.462)).

But

$$\text{Ker } \alpha(V_2^n \amalg V_4^n) = \{f \in \text{End}(V_2^n \amalg V_4^n) : T_n{*}f = 0\}\ .$$

Writing f in the form $\begin{pmatrix} F_{22}u_{22} & F_{24}u_{24} \\ F_{42}u_{42} & F_{44}u_{44} \end{pmatrix}$ we have

$$T_n{*}f = 0 \Rightarrow \begin{pmatrix} \pi_{22}(PF_{22} + zF_{44}) & \pi_{24}(z^2PF_{24} + zF_{44}) \\ \pi_{42}(zF_{22} + z^2F_{42}) & \pi_{44}(z^3F_{24} + z^2F_{44}) \end{pmatrix} = 0$$

$$\Rightarrow \left\{ \begin{array}{l} F_{22}^{(0)} = 0 \\ F_{44}^{(0)} = 0 \end{array} \right. \Rightarrow f \in \text{rad}\,(\text{End}\; V_2^n \amalg V_4^n)$$

Since a projective cover is unique up to isomorphism (see [AI] pg. 209), and $(\;,V) \cong (\;,U)$ in mmod A iff $V \cong U$ in mod A , it is clear that if $n \neq m$, then $F_n \not\cong F_m$.

Thus

(2.42) $\{F_n = H(T_n; V_2^n \amalg V_4^n\;,\; V_2^n \amalg V_4^n) : n \in \in N\}$

given in (2.41) is an infinite family of non-isomorphic indecomposable functors in mmod A_q .

§6. More about the category mmod A

The category of finitely presented functors is well known (see for example [A]), but the interpretation given by Lemma (2.25) which derives from the important results (2.1) of Auslander-Reiten and (2.15) of Green, provides a different way of viewing mmod A , which may bring a better understanding of this category.

We finish this chapter with some facts about mmod A (where A is any finite-dimensional k-algebra), in which we use the characterization of this category given by lemma (2.25).

The facts contained in this section are not necessary for the continuation of this work.

One may ask the question:

Since it is clear that different elements of T may correspond to isomorphic functors in mmod A (see e.g. (2.20)), find a necessary and sufficient condition for this to happen.

The answer to this question is a corollary of the following proposition:

(2.43) <u>Proposition</u>: Let $F = H(T;W,U)$, $F' = H(T';W',U') \in$ mmod A. Let $\phi: F \to F'$ be such that $\phi = H(f,g)$ where $f \in (U,U')$, $g \in (W,W')$. Then:

(i) ϕ is an epimorphism iff there exists $h: U' \to U$ such that $T'{*}fh = T'$.

(ii) ϕ is a monomorphism iff there exists $h: W' \to W$ such that $hg{*}T = T$.

<u>Proof</u>: The diagram

$$\begin{tikzcd}
(\ ,U) \arrow[r, "\alpha"] \arrow[d, "(\ ,f)"'] & F = \mathrm{Im}\,\alpha \arrow[r, hook, "i"] \arrow[d, "\phi"] & D(W, \ ) \arrow[d, "D(g, \ )"] \\
(\ ,U') \arrow[r, "\alpha'"'] & F' = \mathrm{Im}\,\alpha' \arrow[r, hook, "i'"'] & D(W', \ )
\end{tikzcd}$$

is commutative.

(i) Suppose that there exists $h:U' \to U$ such that $T'*fh = T'$.

Let $Z \in F'(X)$. Then $Z = \alpha'(X)(t)$ for some $t \in (X,U')$.

Thus

$$Z = \alpha'(X)(t) = T'*t = T'*fh*t = T'*f*ht =$$
$$= g*(T*ht) = \phi(X)\alpha(X)(ht) \quad .$$

Therefore ϕ is epimorphism.

Conversely, suppose ϕ is an epimorphism, so $T' \in F'(U')$ is such that $T' = \phi(U')\alpha(U')(h)$ for some $h \in (U',U)$. Thus $T' = \alpha'(U')(U',f)(h) = \alpha'(U')(fh) = T'*fh$.

(ii) Suppose that there exists $h:W' \to W : hg*T = T$.

Let $Z \in F(X)$ be such that $\phi(X)(Z) = 0 \in F'(X)$.

Thus $D(g,X)(Z) = 0$ and also $Z = \alpha(X)(\ell)$ for some $\ell \in (X,U)$, i.e. $Z = T*\ell$.

Therefore

$$0 = D(g,X)\alpha(X)(\ell) = \alpha'(X)(X,f)(\ell) = \alpha'(X)(f\ell) = T'*f\ell = g*T*\ell \;;$$

So $0 = hg*T*\ell = T*\ell = Z$.

Thus ϕ is monomorphism.

Conversely suppose ϕ is monomorphism. Consider the diagram:

$$\begin{tikzcd}
D(W,\ ) & & \\
F \arrow[u, "i"] \arrow[r, "\phi"'] & F' \arrow[r, hook, "i'"'] & D(W',\ ) \arrow[ull, dashed, "\theta"']
\end{tikzcd}$$

Since $D(W,\ )$ is injective, there exists $\theta:D(W',\ ) \to D(W,\ )$ such that this diagram commutes.

Thus $D\theta:(W,\ ) \to (W',\ )$ i.e. $D\theta = (h,\ )$ for some $h:W' \to W$. Therefore $\theta = D(h,\ )$.

But $\theta i'\phi = i \Rightarrow D(h,\ )i'\phi\alpha = i\alpha \Rightarrow D(h,U)i'(U)\phi(U)\alpha(U)(1_U) = $ $= i(U)\alpha(U)(1_U) \Rightarrow D(h,U)i'(U)\phi(U)(T) = T \Rightarrow D(h,U)D(g,U)i(U)(T)$ $= T \Rightarrow D(hg,U)(T) = T \Rightarrow hg*T = T$. □

(2.44) Corollary: Let F, F', ϕ be as in (2.43). Then ϕ is isomorphism iff there exist $t:U' \to U$, $h:W' \to W$ such that $hg*T = T, T'*ft = T'$. Also $\phi^{-1} = H(t,h)$.

Proof: The first part is obvious.

Now $hg*T = T \Rightarrow h*T'*f = T \Rightarrow h*T'*ft = T*t \Rightarrow h*T' = T*t$. Thus $(t,h):(T';W',U') \to (T;W,U)$ is a morphism in $\mathcal{T}$, so $H(t,h)$ is a morphism in mmod A.

Clearly $T*tf = T$ and $gh*T' = T'$. Thus, by (2.43), $H(t,h)$ is an isomorphism.

Since $hg*T = T$ and $T*tf = T$, then $(\overline{1_U, 1_W}) = (\overline{tf, hg})$ (see (2.26)); so $(\overline{1_U, 1_W}) = \overline{(t,h)}\,\overline{(f,g)}$ in $\mathcal{T}\mathcal{J}$. Also, since $gh*T' = T'$ and $T'*ft = T'$, $(\overline{1_{U'}, 1_{W'}}) = (\overline{ft, gh}) = \overline{(f,g)}\,\overline{(t,h)}$.

Thus $1_F = H(t,h)H(f,g) = H(t,h)\phi$ and $1_{F'} = \phi\, H(t,h)$, so $H(t,h) = \phi^{-1}$. □

Remark: Since conditions $hg*T = T$ and $T'*ft = T'$ can be written in the form $h*T'*f = T$ and $g*T*t = T'$, respectively, it is clear that (2.20) is a particular case of (2.44).

Now we consider some examples of monomorphisms and epimorphisms:

(2.45) Let $U, U_1, W \in \text{mod } A$ and $f \in (U_1, U)$. Then

$$\psi = H(f, 1_W) : H(T*f, W, U_1) \to H(T; W, U)$$

is a monomorphism.

Moreover the family of subfunctors of $H(T;W,U)$ in mmod A is:

$$\{H(T*f; W, U_1) : U_1 \in \text{mod } A \text{ , } f \in (U_1, U)\}$$

and ψ is the inclusion map.

Proof: The first part is obvious.

Let $F_1 = H(T*f, W, U_1)$, $F = H(T;W,U)$ where U_1, U, W, f satisfy the given conditions.

Then $\psi = H(f, 1_W)$ is such that

$$\psi = D(1_W, \ )\Big|_{F_1} = 1_{D(W, \ )}\Big|_{F_1} \qquad \text{(see proof of 2.15(ii)).}$$

Thus $F_1(X) = \psi(X)F_1(X) \subseteq F(X)$, and $\psi(X)$ is natural in X.

This means that F_1 is a subfunctor of F and ψ is the inclusion map.

Conversely let $F_1 \leq F = H(T;W,U)$ in mmod A. Then $F \leq D(W,\ )$, so $F_1 \leq D(W,\ )$. Then we can construct the diagram:

$$\gamma = (\ ,f) \qquad \begin{tikzcd} (\ ,U_1) \arrow[r, "\alpha_1"] \arrow[d, dashed] & F_1 \arrow[r] \arrow[d, "i"] & D(W,\ ) \arrow[d, "D(1_W,)"] \\ (\ ,U) \arrow[r] & F \arrow[r] & D(W,\ ) \end{tikzcd}$$

where U_1, α_1 are such that $F_1 \cong (\ ,U_1)/\ker \alpha_1$ (we know that such module and map exist). Since $(\ ,U_1)$ is a projective functor and α is a epimorphism, there exists a map γ such that this diagram commutes, and γ has the form $(\ ,f)$ with $f \in (U_1,U)$. Then $\alpha_1(U_1)(1_{U_1}) = i(U_1)\alpha_1(U_1)(1_{U_1}) = \alpha(U_1)(f,U_1)(1_{U_1}) = T^*f$.

Thus $F_1 = H(T^*f,W,U_1)$. $\square$

We have by (2.43)(i) :

(2.46) Let $W,W_1,U \in$ mod A and $g \in (W,W_1)$; then

$$\phi = H(1_U,g) : H(T;W,U) \to H(g^*T:W_1,U)$$

is an epimorphism.

Moreover the family of quotient functors of $H(T;W,U)$ in mmod A is:

$$\{H(g^*T;W_1,U) : W_1 \quad \text{mod } A \ , \quad g \in (W,W_1)\}$$

and ϕ is the natural epimorphism.

Proof: Since ϕ is an epimorphism the elements of mmod A with the form $H(g^*T;W_1,U)$ are quotient functors of $H(T;W,U)$.

Conversely if $G = H(T^*f;W,U_1) \leq H(T;W,U) = F$ then F/G is finitely presented and there is an epimorphism $(\ ,U) \xrightarrow[\alpha']{} F/G$. Then there exists $W_1 \in$ mod A such that $F/G \leq D(W_1, \ )$. Consider the diagram:

$$\begin{tikzcd}
(\ ,U) \arrow[r, "\alpha"] \arrow[d, dash, "(\ ,1_U)"'] & F \arrow[r] \arrow[d, "n"] & D(W, \ ) \arrow[d, dashed, "D(g,) = \delta"] \\
(\ ,U) \arrow[r, "\alpha'"'] & F/G \arrow[r] & D(W_1, \ )
\end{tikzcd}$$

Since $D(W_1, \ )$ is injective there exists $\delta = D(g, \ )$ such that this diagram commutes. So $F/G = H(g^*T;W_1,U)$. $\square$

We can also observe the following:

Let ϕ be given by

(2.47) $\phi = H(f,g):F = H(T;W,U) \rightarrow F' = H(T';W',U')$.

The following diagram commutes:

$$\begin{tikzcd}
(\ ,U) \arrow[r, "\alpha"] \arrow[d, "(\ ,1_U)"'] & F \arrow[r] \arrow[d, "\phi"] & D(W, \ ) \arrow[d, "D(g, \ )"] \\
(\ ,U) \arrow[r, "\alpha^*"] \arrow[d, "(\ ,f)"'] & \mathrm{Im}\ \phi \arrow[r] \arrow[d] & D(W', \ ) \arrow[d, "D(1_{W'}, \ )"] \\
(\ ,U') \arrow[r, "\alpha'"'] & F' \arrow[r] & D(W', \ )
\end{tikzcd}$$

where $\alpha^* = \phi\alpha$. And $\alpha^*(U)(1_U) = \phi(U)\alpha(U)(1_U) = \phi(U)(T) = D(g, \ )(T) =$ $= g^*T = T'^*f$.

Thus

(2.48) $\quad \mathrm{Im}\ \phi = H(g^*T;W',U) = H(T'^*f;W',U)$.

And

(2.49) $\quad H(T;W,U) \xrightarrow{H(1_U,g)} H(g^*T;W',U) \xrightarrow{H(f,1_{W'})} H(T';W',U)$

is the "canonical decomposition" of $\phi = H(f,g)$.

To obtain a description of the injective and projective objects in mmod A we can proceed as follows:

Recall that given W mod A , if $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} W \to 0$ is a projective resolution of W then

(2.50) $\quad (\ ,NP_1) \xrightarrow{(\ ,Np_1)} (\ ,NP_0) \xrightarrow{b} D(W, \ ) \to 0$

is a projective presentation of $D(W,\ )$, where $b = D(p_0,\ )\alpha_{p_0}^{-1}$ (see (2.4)).

Then

(2.51) The injective objects of mmod A are

$$D(W,\ ) = H(\ b(NP_0)\ (1_{NP_0})\ ,\ W, NP_0)$$

where $W \in \text{mod } A$, P_0 is a projective module such that there exists an epimorphism $p_0 : P_0 \to W$ and $b = D(p_0,\ )\ \alpha_{p_0}^{-1}$ (where α_{p_0} is given by (2.4)).

We can give a similar description for the projective objects in mmod A :

Since these are of the form $(\ ,U)$ we must find $U, W \in \text{mod } A$ and $\alpha : (\ ,U) \to D(W,\ )$ such that α is monomorphism.

Let U be any A-module and

(2.51) $\quad 0 \to U \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1$

an injective resolution for U .

Apply $(X,\ )$ to (2.51) :

$$0 \to (X,U) \xrightarrow{(X,i_0)} (X,I_0) \xrightarrow{(X,i_1)} (X,I_1)\ .$$

Apply $M = dD$, which is left exact, to (2.51) and let $B = \operatorname{Coker} Mi_1$.

Then

$$0 \to MU \xrightarrow{Mi_0} MI_0 \xrightarrow{Mi_1} MI_1 \to B \to 0$$

is exact.

Apply $(\ ,X)$ (left exact, contravariant)

$$(MU,X) \longleftarrow (MI_0,X) \leftarrow (MI_1,X) \leftarrow (B,X) \leftarrow 0$$

Now

$$D(MI_0,X) \to D(MI_1,X) \to D(B,X) \to 0$$

is exact.

Recall that $\alpha_P : D(P,\ ) \to (\ ,DdP)$ (2.4) is isomorphism, (see [Gr 2] pg.17) when P is in projective module.

If $P = MI$ where I is injective, then $NP \cong I$. Thus

$$\alpha_{MI} : D(MI,\ ) \to (\ ,I) \text{ is an isomorphism.}$$

Let

$$\beta_I = \alpha_{MI}^{-1} .$$

Then we have the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (X,U) \arrow[r, "{(X,i_0)}"] \arrow[d, "{(X,1_U)}"] & (X,I_0) \arrow[r, "{(X,i_1)}"] \arrow[d, "\beta_{I_0}(X)"] & (X,I_1) \arrow[d, "\beta_{I_1}(X)"] & & \\
0 \arrow[r] & (X,U) \arrow[r, "\beta_{I_0}(X)(X,i_0)"'] & D(MI_0,X) \arrow[r] & D(MI_1,X) \arrow[r] & D(B,X) \arrow[r] & 0
\end{tikzcd}$$

Let

$$c = \beta_{I_0} \circ (\ ,i_0) = \alpha^{-1}_{MI_0}(\ ,i_0) \quad .$$

This is a monomorphism and we have:

(2.52) The projective objects in mmod A are

$$(\ ,U) = H(c(U)(1_U) \ , \ MI_0,U)$$

where $U \in$ mod A , I_0 is an injective module such that there exists a monomorphism $i_0 : U \to I_0$ and $c = \alpha^{-1}_{MI_0}(\ ,i_0)$.

Chapter III : Representation type of R_q and the Auslander-Reiten quiver of R_3 .

§1. Representation type of R_q

In this chapter we consider again the Auslander Algebra $R_q = \mathrm{End}_{A_q}(V_1 \amalg \ldots \amalg V_q)$, where A_q is the k-algebra $\langle z : z^q = 0\rangle$, and $\{V_1, \ldots, V_q\}$ is a full set of indecomposable objects in mod A_q (see Chapter I, §1).

Using some of the facts established in Chapter II we can prove now the following theorem:

(3.1) Theorem: The Auslander algebra R_q of $A_q =$ $= \text{k-alg}\langle z : z^q = 0\rangle$ is of finite representation type if $q \leq 3$ and of infinite representation type if $q \geq 4$.

For this we must consider the following equivalence of categories (see [AI], pg. 191 to 193):

$$(3.2) \qquad e_C : \mathrm{mmod}\, A_q \longrightarrow \mathrm{mod}' R_q$$
$$F \longrightarrow F(C)$$

where $C = V_1 \amalg \ldots \amalg V_q$.

$F(C)$ is considered a right R_q-module with the rule:

If $\xi \in F(C)$, $h \in R_q$ then $\xi h = F(h)(\xi)$.

We want to decide when the number of isomorphism classes of indecomposable modules in mod R_q (or equivalently in mod'R_q, since the number is the same ...) is finite or infinite.

By (3.2), we see that this number is the number of isomorphism classes of indecomposable functors in mmod A_q.

But this number has already been calculated in the case $q = 3$:

In §5, Chapter II we saw that the functors $F = H(T;W,U)$, given by the matrices (1.47) of Chapter I are indecomposable and non-isomorphic, and from §7, Chapter I, (using (2.40), (2.30)) we deduce that they are the only indecomposable functors.

Thus:

If $q = 3$, there are 21 isomorphism classes of indecomposable functors, and therefore there are 21 isomorphism classes of indecomposable modules in mod R_3. Therefore R_3 is of finite representation type.

If $q \geq 4$ there is an infinite number of non-isomorphic indecomposable functors in mmod A_q, since (2.41) is an infinite family of such functors.

Thus if $q \geq 4$, R_q is of infinite representation type.

If $q \leq 2$, R_q is a serial algebra, so it is of finite representation type (see [Ft] prop. 16.11 and 16.14 pg.58, 61). □

In the following sections of this Chapter we construct the Auslander-Reiten quiver of the Auslander Algebra of finite representation type, R_3 .

§2. mod R_3 and mod'R_3

As we saw in §1, Chapter I , $e_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1_{V_i} & & \\ & & & & 0 & \\ & & & & & \ddots \; 0 \end{pmatrix} \begin{pmatrix} & & \vdots & & \\ & & 0 & & \\ \cdots & 0 & u_{ii} & 0 & \cdots \\ & & 0 & & \\ & & \vdots & & \end{pmatrix}$

$i = 1,2,\ldots,q$, are the primitive orthogonal idempotents in $R = R_q$ so the principal indecomposable modules in mod R are Re_i and in mod'R are e_iR , $i = 1,\ldots,q$.

(3.3) <u>Definition</u>: Let $\underline{a} = (a_{ij})_{i,j \in \{1,..,q\}}$ be a q×q matrix of integers a_{ij} , such that $i \wedge j \leq a_{ij} \leq i$.

Then define the A-submodule $S(\underline{a})$ of R as follows:

$$S(\underline{a}) := \bigoplus_{i,j} M_{ij}(a_{ij}) = \bigoplus_{i,j} Az^{a_{ij}-(i \wedge j)}.u_{ij}$$

(using the notation of (1.2)(b) and (1.4)).

As examples we may consider the following:

(3.4) (1) $R = S(\underline{a})$ with $\underline{a} = (i \wedge j)_{i,j}$.

(2) By Fitting's theorem

$$f \in J(R) \iff f_{ij} : V_j \to V_i \text{ is non-isomorphism } \forall\, i,j$$

$$\iff f_{ii} \in \mathrm{rad}(V_i,V_i) \forall i \iff f_{ii} \in Azu_{ii} = M_{ii}(1) .$$

Thus

$$(3.5) \quad J(R) = S(\underline{b}) \text{ with } \begin{cases} b_{ij} = 1 & i = j \\ b_{ij} = i{\sim}j & i \neq j \end{cases} .$$

In particular if $q = 3$: $\underline{b} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$.

(3) $f \in e_iR \iff (f_{ij})_{i,j} = \begin{pmatrix} & 0 & \\ f_{i1} & f_{i2} & \dots f_{iq} \\ & 0 & \end{pmatrix} \iff$

$$\iff \begin{cases} f_{kj} = 0 \in M_{kj}(k) & \text{if } k \neq i \\ f_{ij} \in (V_j,V_i) = M_{ij}\ (i{\sim}j) \end{cases} .$$

Thus if $q = 3$:

$$(3.6) \quad \begin{aligned} e_1R &= S(\underline{c}_1) \text{ with } \underline{c}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \\ e_2R &= S(\underline{c}_2) \qquad \underline{c}_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 3 & 3 \end{pmatrix} \\ e_3R &= S(\underline{c}_3) \qquad \underline{c}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{aligned} .$$

Example (3) can be generalized as follows:

(3.7) <u>Proposition</u>:

(i) Every right R-submodule M of e_iR is such that

$$M = S(\underline{a}) \quad \text{with} \quad \underline{a} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ i-1 & \cdots & i-1 \\ a_{i1} & \cdots & a_{iq} \\ i+1 & & i+1 \\ \vdots & & \vdots \\ q & & q \end{pmatrix} \tag{3.8}$$

(ii) Every left R-submodule N of Re_i is such that

$$N = S(\underline{b}) \quad \text{with} \quad \underline{b} = \begin{pmatrix} 1 & \cdots & 1 & b_{1i} & 1 & \cdots & 1 \\ 2 & & 2 & b_{2i} & 2 & & 2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ q & \cdots & q & b_{qi} & q & \cdots & q \end{pmatrix} \tag{3.9}$$

<u>Proof</u>: (i) One has $e_iRe_j = \{\begin{pmatrix} & 0 & \\ 0\ldots & f_{ij} & 0\ldots \\ & 0 & \end{pmatrix} : f_{ij} \in (V_j,V_i)\}$

$\cong (V_j,V_i) = M_{ij}(i\sim j)$ (1.2c).

Let $M \underset{R}{\leq} e_iR$; then $Me_j \leq M$ and $Me_j \leq e_iRe_j$.

Conversely if $m \in M \cap e_iRe_j$, then $m = e_i\, r\, e_j$, some $r \in R$, so $me_j = m$.

Thus $Me_j = M \cap e_i Re_j$, so it is a submodule of (V_j, V_i) and therefore $Me_j = M_{ij}(a_{ij})$, $i \geq a_{ij} \geq i \sim j$ (by 1.2(c)).

Then $M = M.1 = \bigoplus_{j=1}^{q} Me_j = \bigoplus_{j=1}^{q} M_{ij}(a_{ij}) = S(\underline{a})$ with $\underline{a}$ as in (3.8).

(ii) Similar. □

(3.10) Remark: According to (3.7) every right R-submodule M of $e_i R$ can be given by the i^{th} row of the matrix $\underline{a}$.

Thus we may write $M = (a_{i1}, \ldots, a_{iq})$.

Also in (ii) we can write $N = \begin{pmatrix} b_{1i} \\ \vdots \\ b_{qi} \end{pmatrix}$.

So example (3) says that

$$e_1 R = (0 \quad 0 \quad 0) \qquad e_2 R = (1 \quad 0 \quad 0) \qquad e_3 R = (2 \quad 1 \quad 0) .$$

One can also see that:

$$Re_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad Re_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad Re_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

(3.11) Lemma: If $\underline{a} = (a_{ij})_{i,j}$ $\underline{b} = (b_{ij})_{i,j}$ are $q \times q$ matrices such that $i \sim j \leq a_{ij}, b_{ij} \leq i$, then

$$S(\underline{a}) . S(\underline{b}) = S(\underline{c})$$

where $\underline{c} = (c_{ij})_{i,j}$ is given by

$$c_{ij} = \min_{t \in \{1,..q\}}(a_{it} + b_{tj}, i) .$$

Proof: $S(\underline{a}).S(\underline{b}) = (\bigoplus_{i,k} Az^{a_{ik}-(i\sim k)} .u_{ik})(\bigoplus_{\ell,j} Az^{b_{\ell j}-(\ell\sim j)} u_{\ell j})$

$$= \bigoplus_{i,j} (\bigoplus_{t} Az^{a_{it}-(i\sim t)+b_{tj}-(t\sim j)} .u_{it}.u_{tj}) =$$

$$= \bigoplus_{i,j} (\bigoplus_{t} Az^{a_{it}-(i\sim t)+b_{tj}-(t\sim j)+(i\sim t)+(t\sim j)-(i\sim j)} .u_{ij}) =$$

(by 1.8)

$$= \bigoplus_{i,j} (\bigoplus_{t} Az^{a_{it}+b_{tj}-(i\sim j)} .u_{ij}) = \bigoplus_{i,j} Az^{\min_t(a_{it}+b_{tj},i)-(i\sim j)} .u_{ij}$$

because these modules form a chain and if $a_{it} + b_{tj} \geq i$, they are zero.

So $S(\underline{a}).S(\underline{b}) = S(\underline{c})$ with $c = \min_t(a_{it}+b_{tj},i)$. □

(3.12) Example

Since $e_1J(R) \simeq e_1R.J$ and e_1R is given by (3.6), J by (3.5) then using, this lemma we see that

$$e_1J \text{ is given by } S(\underline{c}) \text{ with } \underline{c} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} ,$$

thus $e_1J = (100)$ using notation (3.10).

Also $Je_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

The following consequence of Lemma (3.11) gives a description of the $S(\underline{a})$ which are left or right ideals of R .

(3.13) <u>Proposition</u>: (a) $S(\underline{a}) \trianglelefteq_{\ell} R$ with $\underline{a} = (a_{ij})$ iff $a_{i+1j} = a_{ij}$ or $1+a_{ij}$.

(b) $S(\underline{b}) \trianglelefteq_{r} R$ with $b = (b_{ij})$ iff $b_{ij+1} = b_{ij}$ or $-1+b_{ij}$.

<u>Proof</u>:

(i) $I = S(\underline{a}) \trianglelefteq_{\ell} R$ iff $RI = I$. This is equivalent to say that

$$S(\sigma).S(\underline{a}) = S(\underline{a}) \quad \text{with} \quad \sigma = (i \sim j)_{ij} \quad \text{by (3.4)} .$$

By (3.11),

$$RI = S(\underline{b}) \quad \text{with} \quad b_{ij} = \min_{\ell \in \{1,..,q\}} (i \sim \ell + a_{\ell j}, i) \quad .$$

In particular:

$$b_{i+1j} = \min(i+a_{1j}, i-1+a_{2j}, \dots, 1+a_{ij}, a_{i+1j}, \dots, i+1)$$

$$b_{ij} = \min(i-1+a_{1j}, i-2+a_{2j}, \dots, 1+a_{i-1j}, a_{ij}, a_{i+1j}, \dots, i) \quad .$$

But $RI = S(\underline{a})$, so $\underline{a} = \underline{b}$.

Therefore

$$b_{i+1j} = a_{i+1j} \text{ and so } a_{i+1j} \leq 1 + a_{ij}$$

$$b_{ij} = a_{ij} \text{ and so } a_{ij} \leq a_{i+1j} \ .$$

Thus $a_{i+1j} = a_{ij}$ or $1 + a_{ij}$.

Conversely suppose $a_{ij} \leq a_{i+1j} \leq 1 + a_{ij}$, $\forall$ i,j .

Then

$$a_{ij} \leq a_{i+1j} \leq a_{i+2j} \leq \ldots \leq a_{qj} \text{ by first inequality}$$

$$a_{ij} \leq 1+a_{i-1j} \leq 2+a_{i-2j} \leq \ldots \leq i-1+a_{1j} \text{ by second inequality.}$$

Thus

$$a_{ij} = \min\{i-1+a_{1j}, i-2+a_{2j}, \ldots, 1+a_{i-1j}, a_{ij}, a_{i+1j}, a_{i+2j}, \ldots, a_{qj}, i\} =$$

$$= \min_{\ell \in \{1,\ldots,q\}} \{i \sim \ell + a_{\ell j}, i\} = b_{ij} \ .$$

Therefore $S(\sigma).S(\underline{a}) = S(\underline{a})$ and so $S(\underline{a}) \trianglelefteq_{\ell} R$.

(b) Similar. □

This proposition gives a method to calculate all R-submodules of Re_i and e_iR .

For example:

In mod'R_3, e_2J is given by (11 0) and e_2J^2 by (2 11) (using notation (3.10)). But clearly there are R-submodules M_i such that $e_2J > M_i > e_2J^2$. Using (3.13) we see that there are two such submodules, namely $M_1 = (210)$ $M_2 = (111)$.

<u>Notation</u>: Denote the simple modules in mod R , by T_i $i = 1,\ldots,q$, and in mod'R , by S_i .

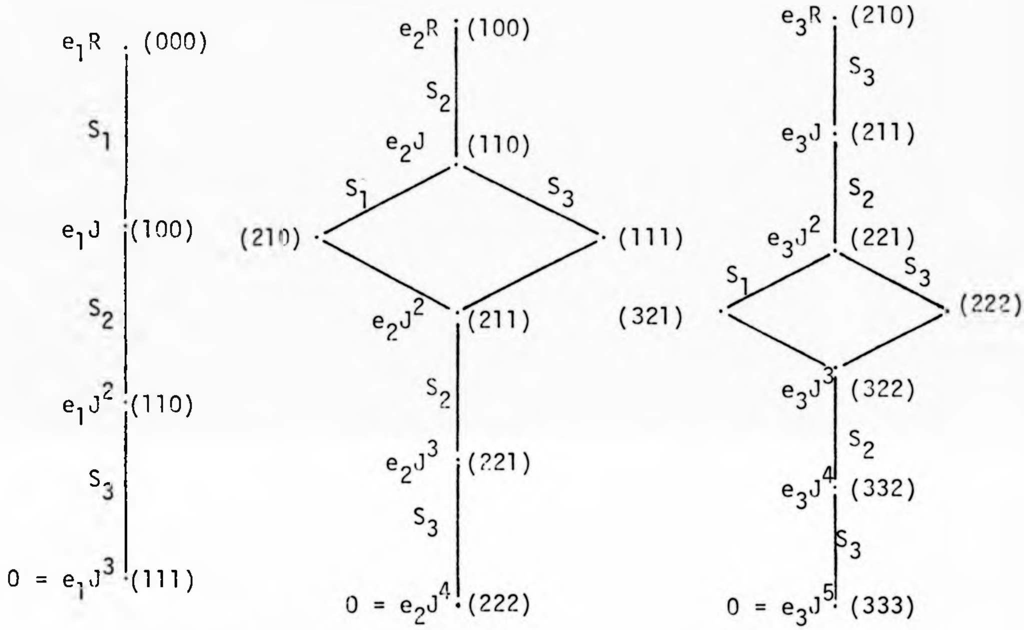
Then $e_1R/e_1J \cong S_1 \cong e_2J/(210)$, $e_2J/(111) \cong S_3$ and we can draw the lattice

$$\begin{tikzcd}
 & (110) \arrow[dl, dash, "S_1"'] \arrow[dr, dash, "S_3"] & \\
(210) \arrow[dr, dash] & & (111) \arrow[dl, dash] \\
 & (211) &
\end{tikzcd}$$

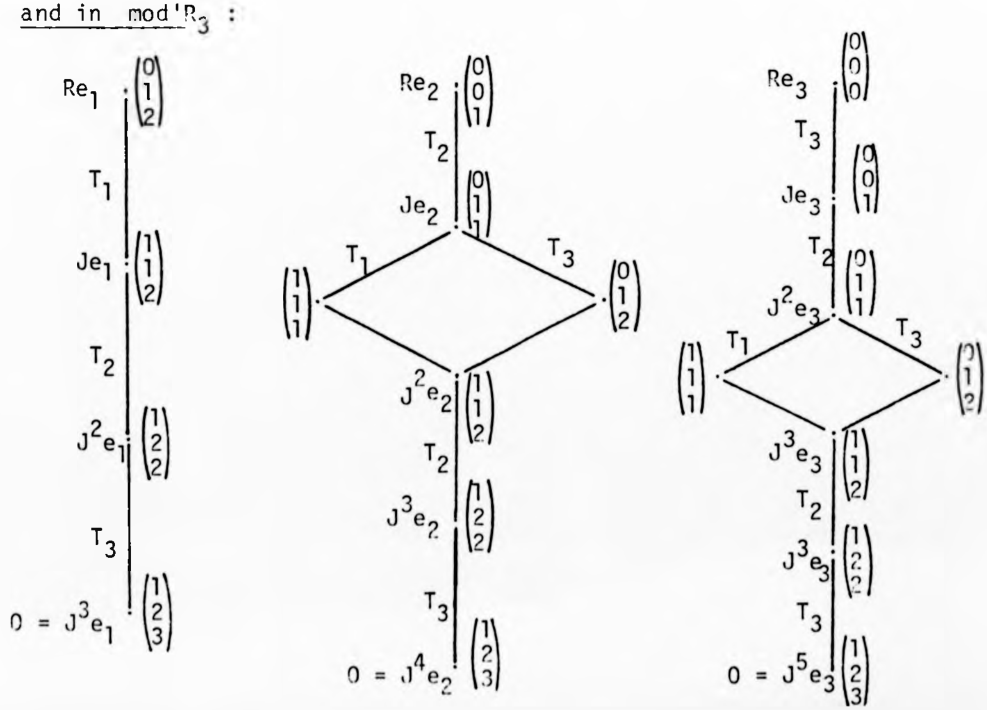
Now we have all the tools to calculate the series of R-submodules of e_iR and Re_i , which,in case $q = 3$, are given by (3.14). These contain, among others, the modules in the radical series and socle series, which are the same, in this particular example.

Since $D(M)/D(M/N) \cong DN$, where N is a submodule of M and $D^2M \cong M$ one can easily calculate the series of submodules for the injective indecomposable modules of R_3 . These are given by (3.15).

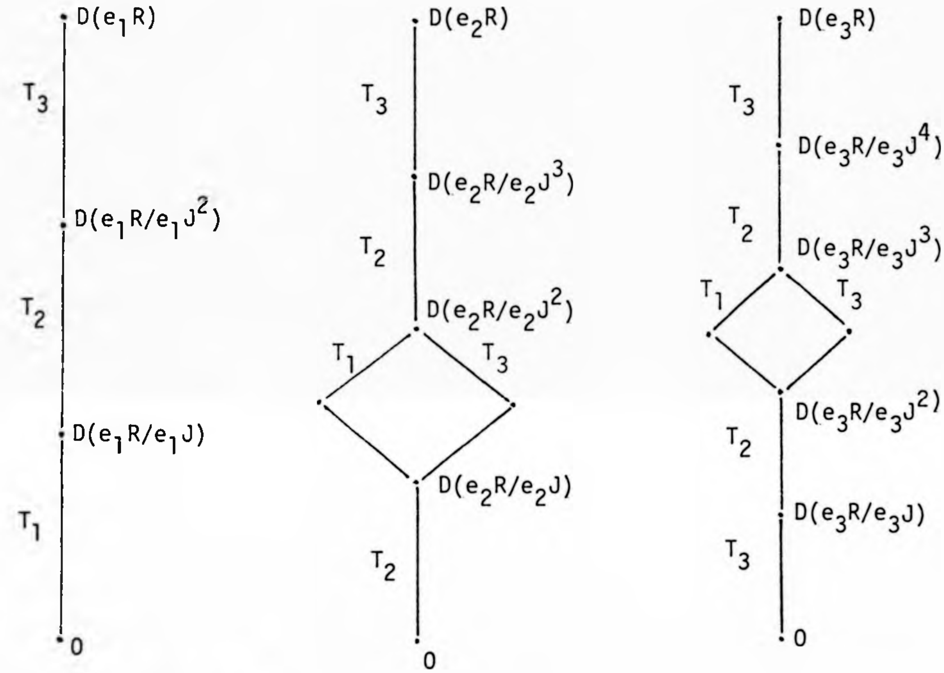
(3.14) Series of submodules of the projective indecomposable modules in mod R_3 :



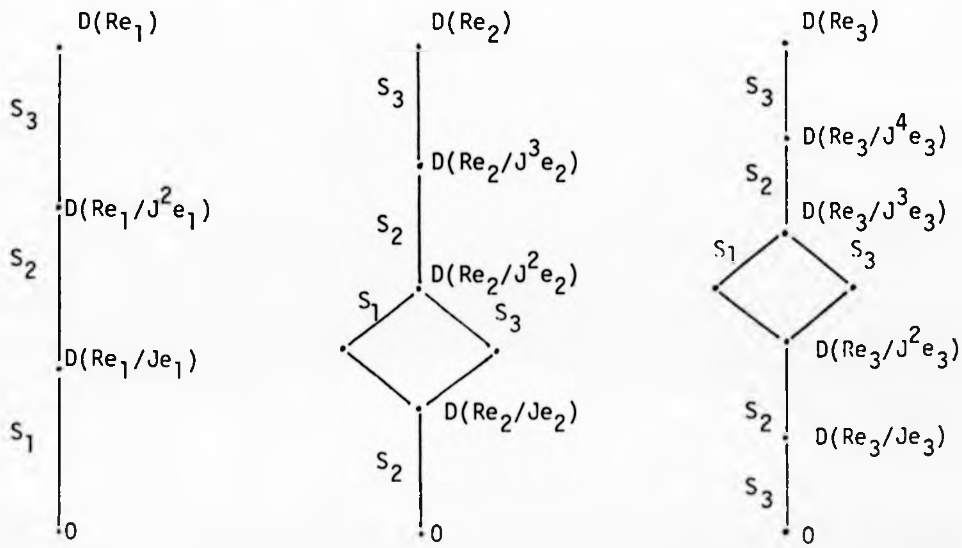
and in mod'R_3 :



(3.15) Series of submodules for the injective indecomposable modules in $\mathrm{mod}'R_3$:



and in $\mathrm{mod}\ R_3$:



§3. The Auslander-Reiten quiver for R_3

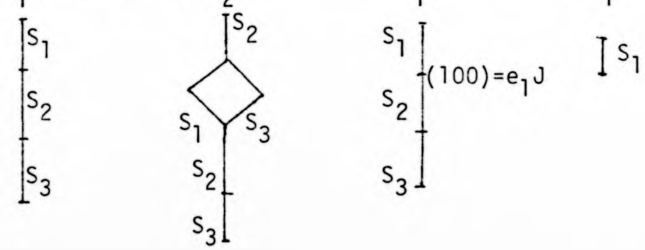
In this section we apply the method described in §5. Chapter 0, to construct the Auslander-Reiten quiver of R .

Since we must start with an indecomposable non-injective R-module, we can take a simple module, for example T_1 .

Its dual is $S_1 \cong e_1R/e_1J$.

A projective resolution of S_1 is:

$$0 \longrightarrow e_1R \xrightarrow{p_2} e_2R \xrightarrow{p_1} e_1R \xrightarrow{p_0} S_1 \longrightarrow 0$$



where the maps are as follows:

p_0 is the natural epimorphism

p_1 is such that $p_1(e_2) = a \in e_1 R e_2$. We can take

$$a = \begin{pmatrix} 0 & u_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ ,since then } p_1(e_2R) =$$

$$= \left\{ ar : r = \begin{pmatrix} f_{11}u_{11} & f_{12}u_{12} & f_{13}u_{13} \\ f_{21}u_{21} & f_{22}u_{22} & f_{23}u_{23} \\ f_{31}u_{31} & f_{32}u_{32} & f_{33}u_{33} \end{pmatrix} \in R \right\} = \left\{ \begin{pmatrix} 0 & f_{22}u_{12} & f_{23}u_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} =$$

$= (100) = e_1J = \ker p_0$.

p_2 is such that $p_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ u_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in e_2Re_1$

(in fact we do not need to know p_2).

Now we apply d which is left exact:

$$\ldots \longleftarrow d(e_2R) \cong Re_2 \xleftarrow{dp_1} d(e_1R) \cong Re_1 \xleftarrow{dp_0} dS_1 \longleftarrow 0$$

Then

$$0 \longleftarrow M = Re_2/\mathrm{Im}\, dp_1 \xleftarrow{n} Re_2 \xleftarrow{dp_1} Re_1 \xleftarrow{dp_0} dS_1 \longleftarrow 0$$

where n is the natural epimorphism, is exact.

dp_1 is such that $dp_1(r) = ra$, $\forall\, r \in Re_1$ and

$$\mathrm{Im}\, dp_1 = \{ra : r \in R\} = \left\{ \begin{pmatrix} 0 & f_{11}u_{12} & 0 \\ 0 & zf_{21}u_{22} & 0 \\ 0 & zf_{31}u_{32} & 0 \end{pmatrix} \quad f_{ij} \in A \right\} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} .$$

Thus $M = Re_2/\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

We will write $M = \begin{array}{l} \bullet \\ \mid T_2 \\ \bullet \\ \mid T_3 \\ \bullet \end{array}$ where this diagram is the minimal lattice of submodules of M, that contains the module on both Loewy series of M. (In this case the radical series and the socle series coincide.)

We often will use the simplified notation $M = \begin{array}{l} \cdot \\ |2 \\ \cdot \\ |3 \\ \cdot \end{array}$.

Now we consider the "push-out" diagram:

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "1_M"] & Re_2 \arrow[l, "n"'] \arrow[d, "\ell"] & Re_1 \arrow[l, "dp_1"'] \arrow[d, "\psi"] & \\
0 & M \arrow[l] & F(\psi) \arrow[l, "f"] & DS_1 \cong Re_1/Je_1 = T_1 \arrow[l, "g"] & 0 \arrow[l]
\end{tikzcd}$$

The only possibilities for ψ are 0 and $\lambda.\mathrm{nat}$ $(\lambda \in k)$ and 0 is not in the required conditions. Thus we can take ψ as the natural epimorphism.

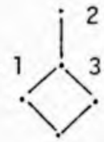
$F(\psi)$ is the pushout over ψ and dp_1 i.e.

$$F(\psi) = \frac{DS_1 \amalg Re_2}{\{(\psi(x), -dp_1(x)) : x \in Re_1\}}$$

ℓ is such that $\ell : y \to [0,y] \in F(\psi)$. Since ψ is epimorphism, ℓ is also epimorphism.

And $\ker \ell = \{y \in Re_2 : \ell(y) = [0,y] = 0\}$.

But $[0,y] = 0 \iff (\psi(x), -dp_1(x)) = (0,y)$ for some $x \in Re_1 \iff \psi(x) = 0$, $-dp_1(x) = y \iff y \in dp_1(\ker \psi) = dp_1(Je_1) =$
$= Je_1\, a = Ja = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

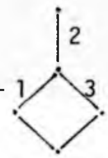
So $\ker \ell = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $F(\psi) \cong Re_2/\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} =$  .

This module is clearly indecomposable.

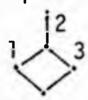
Thus

$$0 \leftarrow Re_1/\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \longleftarrow Re_2/J^2e_2 \longleftarrow T_1 \leftarrow 0$$

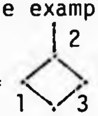
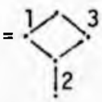
or in another notation:

(3.16) 0 ———  $\longleftarrow$  $\longleftarrow$  $\longleftarrow 0$

is an almost split sequence.

The next step consists in taking the middle term of this sequence, $N =$  and construct the almost split sequence that begins with N .

We shall look into this example with some detail, also, because it gives an almost split sequence whose middle term is decomposable and it involves some techniques that have not been used in the first rather simple example.

If $N =$  , then $DN =$  .

A projective resolution for DN may start as follows:

$$(3.17)\quad \ldots \longrightarrow e_1R \amalg e_3R \xrightarrow{p_0} DN \longrightarrow 0$$

And $DN = \operatorname{Im} p_0 \cong \dfrac{e_1R \amalg e_3R}{\ker p_0}$

The existence of a map p_0, is equivalent to the existence of an R-invariant blinear form:

$$(3.18)\quad \eta : (e_1R \amalg e_3R) \times Re_2 \longrightarrow k$$

such that the left kernel of η, $L(\eta) = \{r \in e_1R \amalg e_3R :$ $\eta(r,g) = 0 \,,\ \forall\, g \in Re_2\}$ is $\ker p_0$ and the right kernel of η, $R(\eta) = \{r \in Re_2 : \eta(g,r) = 0 \,, \forall\, g \in e_1R \amalg e_3R\}$

is $\begin{pmatrix}1\\1\\2\end{pmatrix} = \{r = \begin{pmatrix}0 & r_{12} & u_{12} & 0\\ 0 & r_{22} & u_{22} & 0\\ 0 & r_{32} & u_{32} & 0\end{pmatrix} : r_{ij} \in A$ and $r_{12}^{(0)} = r_{22}^{(0)} = r_{32}^{(0)} = 0\}$

where $r_{ij}^{(t)}$ is the coefficient of the term of degree t of r_{ij}, when this element of A is considered as a polynomial in z.

To give an R-invariant bilinear form (3.18) is equivalent to give a matrix

$$(3.19)\quad W = (\pi_{21}w_{21} \quad \pi_{23}w_{23})$$

where π_{ji} is an A-generator of the module $D(Au_{ij}) = D(e_i R e_j)$.
(since this is $\cong D(V_j,V_i)$ we use the same notation as in (1.14))
and $w_{21}, w_{23} \in A$.

This equivalence is given by the formula:

$$\eta((e_1r,e_3s), r'e_2) = (\pi_{21} w_{21})(e_1rr'e_2) +$$
$$+ (\pi_{23} w_{23})(e_3sr'e_2).$$

We need to know $\ker p_0$, so that we can find the second term of the projective resolution (3.17). This will be done by finding the matrix W (3.19), by imposing that $R(\eta) = \begin{pmatrix}1\\1\\2\end{pmatrix}$. Then using W we can easily determine $L(\eta)$.

Write $w_{21} = w_{21}^{(0)}$

$$w_{23} = w_{23}^{(0)} + w_{23}^{(1)}z \qquad w_{21}^{(0)}, w_{23}^{(0)}, w_{23}^{(1)} \in k.$$

We want to find elements $w_{21}^{(0)}$, $w_{23}^{(0)}$, $w_{23}^{(1)}$ of k such that the following is true:

$$\eta(g,r) = 0 \ \forall g, \text{k-generator of } e_1R \amalg e_3R \iff r \in R(\eta)$$

This can be done as follows:

g (k-generator of $e_1R \amalg e_3R$)	$\eta\left(g\ ,\ \begin{pmatrix} 0 & r_{12}\,u_{12} & 0 \\ 0 & r_{22}\,u_{22} & 0 \\ 0 & r_{32}\,u_{32} & 0 \end{pmatrix}\right) = 0$
$\begin{pmatrix} u_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(\pi_{21}w_{21})(r_{12}u_{12}) = w_{21}^{(0)}\ r_{12}^{(0)} = 0$
$\begin{pmatrix} 0 & u_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$(\pi_{21}w_{21})(r_{22}u_{12}) = w_{21}^{(0)}\ r_{22}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & u_{13} \\ 0 & 0 & 0 \end{pmatrix}$	$(\pi_{21}w_{21})(zr_{32}\ u_{12}) = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ u_{31} & 0 & 0 \end{pmatrix}$	$(\pi_{23}w_{23})(zr_{12}\ u_{32}) = w_{23}^{(0)}\ r_{12}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & u_{32} & 0 \end{pmatrix}$	$(\pi_{23}w_{23})(r_{22}\ u_{32}) = w_{23}^{(0)}\ r_{22}^{(1)} + w_{23}^{(1)}\ r_{22}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & zu_{32} & 0 \end{pmatrix}$	$(\pi_{23}w_{23})(zr_{22}\ u_{32}) = w_{23}^{(0)}\ r_{22}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u_{33} \end{pmatrix}$	$(\pi_{23}w_{23})(r_{32}\ u_{32}) = w_{23}^{(0)}\ r_{32}^{(1)} + w_{23}^{(1)}\ r_{32}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & zu_{33} \end{pmatrix}$	$(\pi_{23}w_{23})(zr_{32}\ u_{32}) : w_{23}^{(0)}\ r_{32}^{(0)} = 0$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z^2u_{33} \end{pmatrix}$	$(\pi_{23}w_{23})(z^2r_{32}\ u_{32}) = 0$

If we take $w_{21}^{(0)} = 1$ $w_{23}^{(0)} = 0$ $w_{23}^{(1)} = 1$ then this system of equations is $\begin{cases} r_{12}^{(0)} = 0 \\ r_{22}^{(0)} = 0 \\ r_{32}^{(0)} = 0 \end{cases}$ and $R(\eta) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Thus $W = (\pi_{21}\ \pi_{23}z)$.

Now $L(\eta) = \{s \in e_1R \amalg e_3R : \eta(s,h) = 0,\ \forall\ h \in$ k- generators of $Re_2\}$.

Writing $s = \begin{pmatrix} s_{11}u_{11} & s_{12}u_{12} & s_{13}u_{13} \\ 0 & 0 & 0 \\ s_{31}u_{31} & s_{32}u_{32} & s_{33}u_{33} \end{pmatrix}$ and using a method similar to the one just used we can see that the conditions $\eta(s,h) = 0$ where h is a k-generator of Re_2 , are:

$$\begin{cases} s_{11}^{(0)} = 0 \\ s_{12}^{(0)} + s_{32}^{(0)} = 0 \iff s_{12}^{(0)} = -s_{32}^{(0)} \\ s_{33}^{(0)} = 0 \end{cases} .$$

Thus $$L(\eta) = \left\{ \begin{pmatrix} 0 & -s_{32}u_{12} & s_{13}u_{13} \\ 0 & 0 & 0 \\ s_{31}u_{31} & s_{32}u_{32} & zs_{33}u_{33} \end{pmatrix} : s_{ij} \in A \right\}.$$

And

$$L(\eta)J = \left\{ \begin{pmatrix} 0 & 0 & g_{13}u_{13} \\ 0 & 0 & 0 \\ g_{31}u_{31} & zg_{32}u_{32} & -zg_{13}u_{33} \end{pmatrix} : g_{ij} \in A \right\}.$$

Thus: $\quad L(\eta)/L(\eta)J \cong S_2 \amalg S_3$.

So we can complete the projective resolution (3.17) as follows:

$$(3.20)\quad 0 \to e_2R \longrightarrow e_2R \amalg e_3R \xrightarrow{p_1} e_1R \amalg e_3R \xrightarrow{p_0} DN \to 0$$

This is so because we can take p_1 as the left multiplication by $b = \begin{pmatrix} 0 & u_{12} & u_{13} \\ 0 & 0 & 0 \\ 0 & -u_{32} & 0 \end{pmatrix}$ and so $\ker p_1 = e_3J \cong e_2R$.

Now we apply d:

$$\ldots \leftarrow Re_2 \amalg Re_3 \xleftarrow{dp_1} Re_1 \amalg Re_3 \xleftarrow{dp_0} d(DN) \leftarrow 0$$

If $M = \dfrac{Re_2 \amalg Re_3}{\mathrm{Im}\, dp_1}$ then:

$$0 \leftarrow M \xleftarrow{nat} Re_2 \amalg Re_3 \xleftarrow{dp_1} Re_1 \amalg Re_3 \xleftarrow{dp_0} dDN \leftarrow 0$$

is exact.

To get a better description of M we calculate its radical series:

$$\mathrm{Im}\, dp_1 = \left\{ \begin{pmatrix} 0 & f_{12}u_{12} & f_{13}u_{13} \\ 0 & zf_{22}u_{22} & zf_{23}u_{23} \\ 0 & f_{32}u_{32} & z^2f_{33}u_{33} \end{pmatrix} \cdot \begin{array}{l} f_{12} = f_{13} \\ f_{ij} \in A \end{array} \right\} < Je_2 \amalg Je_3$$

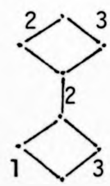
Thus $M/JM \cong \dfrac{Re_2 \amalg Re_3}{Je_2 \amalg Je_3} \cong T_2 \amalg T_3$.

$$J^2e_2 \amalg J^2e_3 + \operatorname{Im} dp_1 = \left\{ \begin{pmatrix} 0 & g_{12}u_{12} & g_{13}u_{13} \\ 0 & zg_{22}u_{22} & zg_{23}u_{23} \\ 0 & g_{32}u_{32} & zg_{33}u_{33} \end{pmatrix} : g_{ij} \in A \right\}$$

Then $$JM/J^2M = \frac{\dfrac{Je_2 \amalg Je_3}{\operatorname{Im} dp_1}}{\dfrac{J^2e_2 \amalg J^2e_3 + \operatorname{Im} dp_1}{\operatorname{Im} dp_1}} \cong \frac{Je_2 \amalg Je_3}{J^2e_2 \amalg J^2e_3 + \operatorname{Im} dp_1} \cong T_2 \ .$$

Using similar calculations, $J^2M/J^3M \cong T_1 \amalg T_3$ and $J^3M = 0$.

Thus the lattice of the radical series of M is



Now we want to find the socle series; this can be done as follows:

With the method used after (3.19) we can show that:

The bilinear form $\eta' : (e_1R \amalg e_3R) \times (Re_2 \amalg Re_3) \to k$ given by

$$W' = \begin{pmatrix} \pi_{21} & 0 \\ -\pi_{31} & z\pi_{33} \end{pmatrix}$$

is such that

$$R(\eta') = \operatorname{Im} dp_1 \ . \quad \text{Then} \quad DM \cong \frac{e_1R \amalg e_3R}{L(\eta')} \ .$$

If we consider the non-singular, R-invariant bilinear form

$$\eta^* : \frac{e_1R \amalg e_3R}{L(\eta')} \times \frac{Re_2 \amalg Re_3}{\operatorname{Im} dp_1} \to k$$

induced by η' ,

then we may calculate the socle of M as follows:

$\operatorname{Soc} M = \{m \in M : Jm = 0\}$. But $Jm = 0 \iff$

$\iff \eta^*(n,Jm) = 0 \, , \forall \, n \in DM \iff \eta^*(nJ,m) = 0 \, , \forall \, n \in DM$

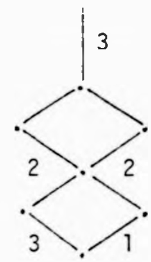
$\iff \eta^*(n',m) = 0 \, , \forall \, n' \in (DM)J$.

The conditions $\eta(n',m) = 0$, where n' is a k-generator of $(DM)J$, and $m = \begin{pmatrix} 0 & m_{12}u_{12} & m_{13}u_{13} \\ 0 & m_{22}u_{22} & m_{33}u_{33} \\ 0 & m_{32}u_{32} & m_{33}u_{33} \end{pmatrix}$, are $m_{23}^{(0)} = 0$ $m_{22}^{(0)} = 0$ $m_{33}^{(0)} = 0$. Let Q be the set of these elements.

Thus $\operatorname{soc} M = \dfrac{Q}{\operatorname{Im} dp_1} \cong T_1 \amalg T_3$.

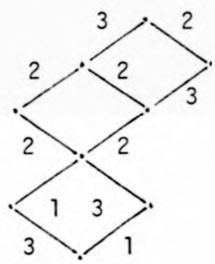
Then $\operatorname{soc}^2 M = \{m \in M : J^2 m = 0\} = \{m \in M : \eta^*(n'',m) = 0 \, , \forall \, n'' \in NJ^2\}$, etc.

Proceeding analogously we conclude that the socle series is given by



Thus the minimal lattice of submodules that contains both Loewy series is:

(3.21) $M =$



Now we must construct the "push-out" diagram:

(3.22)
$$\begin{tikzcd}
0 & M \arrow[l] & Re_2 \amalg Re_3 \arrow[l] \arrow[d] & Re_1 \amalg Re_3 \arrow[l, "\delta = dp_1"'] \arrow[d, "\psi"] & \\
0 & M \arrow[l] \arrow[u, dashed, "1_M"] & F(\psi) = \dfrac{Re_2 \amalg Re_3 \amalg N}{\{(\delta(x), -\psi(x)) : x \; Re_1 \amalg Re_3\}} \arrow[l] & N \cong \dfrac{Re_2}{J^2 e_2} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

The only possible endomorphisms of DN are 0 and automorphisms. Thus $\operatorname{rad} \operatorname{End}(DN) = 0$.

So we can take any non-zero map to be ψ .

Let ψ be given by:

$$\psi(xe_1 + ye_3) = (xe_1 + ye_3)\begin{pmatrix} 0 & u_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + J^2e_2 \ .$$

It is easily seen that $\ker\psi = Je_1 \amalg Re_3$ and so

$$\operatorname{Im}\psi \cong \frac{Re_1 \amalg Re_3}{Je_1 \amalg Re_3} \cong T_1 \ .$$

For simplicity instead of diagram (3.22) we may consider

$$(3.23)\qquad \begin{array}{ccccccc} 0 \leftarrow M & \longleftarrow & \dfrac{Re_2 \amalg Re_3}{\delta(\ker\psi)} & \xleftarrow{\ \bar{\delta}\ } & Re_1 \amalg Re_3 & & \\ \downarrow 1_M & & \downarrow & & \downarrow & & \\ 0 \leftarrow M & \longleftarrow & \bar{F}(\psi) & \longleftarrow & N \cong Re_2/J^2e_2 & \leftarrow 0 \end{array}$$

where $\bar{\delta}$ is induced by δ , and

$$\bar{F}(\psi) = \frac{\dfrac{Re_2 \amalg Re_3}{\delta(\ker\psi)} \amalg N}{\{(\bar{\delta}(x), -\psi(x)) : x \in Re_1 \amalg Re_3\}} \ .$$

Since $\bar{F}(\psi) \cong F(\psi)$, this is the same almost split sequence as in (3.22) (up to isomorphism).

Now we want to study the decomposability of $F(\psi)$.

We have the

(3.24) Lemma: Let $\bar{F}$ be the pushout of $\bar{\delta}$, ψ :

$$\begin{tikzcd} & M' = Y/\delta(\ker\psi) \arrow[d] & X \arrow[l, "\bar{\delta}"'] \arrow[d, dashed, "\psi"] \\ \dfrac{M' \amalg X}{\{(\bar{\delta}(x),-\psi(x)):x\in X\}} & = \bar{F} & N \arrow[l] \end{tikzcd}$$

If there exists a map $w:N \to M'$ such that $w\psi = \bar{\delta}$, then $\bar{F} \cong M' \amalg N/\mathrm{Im}\,\psi$.

Proof: If such an w exists let $\eta:\bar{F} \to N/\mathrm{Im}\,\psi$ be such that:

$$\begin{tikzcd} M' \arrow[d] & X \arrow[l, "\bar{\delta}"'] \arrow[d, "\psi"] \\ \bar{F} \arrow[dr, dashed, shift left, "\eta"'] & N \arrow[l] \arrow[ul, dashed, "w"'] \arrow[d, dashed] \\ & N/\mathrm{Im}\,\psi \arrow[ul, dashed, shift left, "\xi"'] \end{tikzcd}$$

$\eta[\bar{y},n] = n+\mathrm{Im}\,\psi$

where:

$[\bar{y},n] = \{(\bar{y}+\bar{\delta}(x),n-\psi(x)):x \in X\} \in \bar{F}$

Let $\xi:N/\mathrm{Im}\,\psi \to \bar{F}$ be such that $\xi(n+\mathrm{Im}\,\psi) = [-w(n),n]$.

ξ is well-defined since: if $n \in \mathrm{Im}\,\psi$, i.e. $n = \psi(x)$ for some $x \in X$, then $-w(n) = -w\psi(x) = -\bar{\delta}(x)$. Thus $[-w(n),n] =$ $= [-\bar{\delta}(x),\psi(x)] = 0$.

And $\eta\xi(n + \mathrm{Im}\,\psi) = \eta[-w(n),n] = n + \mathrm{Im}\,\psi$, thus $\eta\xi = 1_{N/\mathrm{Im}\,\psi}$.

So $\bar{F} \cong \operatorname{Im} \xi \amalg \ker \eta$ and $\operatorname{Im} \xi \cong N/\operatorname{Im} \psi$ since ξ is a monomorphism also $\ker \eta = \{[\bar{y},n] \in \bar{F} : n = \psi(x),\ x \in X\} \cong M'$ as follows:

$$\begin{aligned} \text{Let} \quad \alpha : M' &\longrightarrow \ker \eta \\ \bar{y} &\longrightarrow [\bar{y},0] \ . \end{aligned}$$

Then $\bar{y} = 0 \Rightarrow y \in \delta(\ker \psi) \Rightarrow y = \delta(x)$, $\psi(x) = 0 \Rightarrow \bar{y} = \bar{\delta}(x)$, $-\psi(x) = 0 \Rightarrow [\bar{y},0] = [\bar{\delta}(x),-\psi(x)] = 0 \Rightarrow (\bar{y},0) = (\bar{\delta}(x),-\psi(x)) \Rightarrow$ $\Rightarrow \bar{y} = \bar{\delta}(x),\ x \in \ker \psi \Rightarrow \bar{y} = 0$; so α is monomorphism.

And $[\bar{y},n] \in \ker \eta \Rightarrow n = \psi(x)$, $x \in X \Rightarrow [\bar{y},n] = [\bar{y}+\bar{\delta}(x),n-\psi(x)] =$ $= [\bar{y}+\bar{\delta}(x),0] = [\bar{y}',0]$ with $\bar{y}' = \bar{y} + \bar{\delta}(x)$. Thus $\bar{y}' \xrightarrow{\alpha} [\bar{y},n]$.

So $\bar{F} \cong N/\operatorname{Im} \psi \amalg M'$. $\square$

If $N = Re_i/L$ then to define a map $w: N \longrightarrow M'$, it is necessary the existence of an element $\bar{m} \in M'$ such that $e_i\bar{m} = \bar{m}$ and $\ell\bar{m} = 0,\ \forall\ \ell \in L$. If such an element $\bar{m}$ exists then we may define w such that $w(e_i + L) = \bar{m}$. If $w\psi = \bar{\delta}$ then this map satisfies the conditions of Lemma (3.24) and $\bar{F}$ is decomposable.

Returning to our example: $N = Re_2/\begin{pmatrix}1\\1\\2\end{pmatrix}$

$e_2\bar{m} = \bar{m} \Leftrightarrow (1-e_2)\bar{m} = 0 \Leftrightarrow (1-e_2)m \in \delta(\ker \psi)$

$\ell\bar{m} = \bar{0},\ \forall\ \ell \in \begin{pmatrix}1\\1\\2\end{pmatrix} \Leftrightarrow \ell m \in \delta(\ker \psi)$, $\forall\ \ell \in \begin{pmatrix}1\\1\\2\end{pmatrix}$ and $m \in Re_2 \amalg Re_3$.

But

$$\delta(\ker\psi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & zg_{22}u_{22} & zg_{23}u_{23} \\ 0 & g_{32}u_{32} & z^2g_{33}u_{33} \end{pmatrix}$$

Writing $m = \begin{pmatrix} 0 & c_{12}u_{12} & c_{13}u_{13} \\ 0 & c_{22}u_{22} & c_{23}u_{23} \\ 0 & c_{32}u_{32} & c_{33}u_{33} \end{pmatrix}$ then:

$$(1-e_2)m = \begin{pmatrix} 0 & c_{12}u_{12} & c_{13}u_{13} \\ 0 & 0 & 0 \\ 0 & c_{32}u_{32} & c_{33}u_{33} \end{pmatrix} \in \delta(\ker\ \psi) \iff \begin{array}{l} c_{12}, c_{13} \in zA \\ c_{33} \in z^2A \end{array} .$$

Then $m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22}u_{22} & d_{23}u_{23} \\ 0 & d_{32}u_{32} & z^2d_{33}u_{33} \end{pmatrix}$. If $\ell \in \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix}$, then one can easily see that $\ell m \in \delta(\ker\psi)$.

Thus

$$\bar{m} = m + \delta(\ker\psi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d_{22}u_{22} & d_{23}u_{23} \\ 0 & 0 & 0 \end{pmatrix} + \delta(\ker\psi) .$$

Defining w by $w(e_2 + \begin{vmatrix} 1 \\ 1 \\ 2 \end{vmatrix}) = \bar{m}$ then

$$w\psi(xe_1 + ye_3) = w\psi\begin{pmatrix} f_{11}u_{11} & 0 & f_{13}u_{13} \\ f_{21}u_{21} & 0 & f_{23}u_{23} \\ f_{31}u_{31} & 0 & f_{33}u_{33} \end{pmatrix} =$$

$$= w\left(\begin{pmatrix} 0 & f_{11}u_{12} & 0 \\ 0 & zf_{21}u_{22} & 0 \\ 0 & zf_{31}u_{32} & 0 \end{pmatrix} + J^2e_2\right) = \begin{pmatrix} 0 & f_{11}u_{12} & 0 \\ 0 & zf_{21}u_{22} & 0 \\ 0 & zf_{31}u_{31} & 0 \end{pmatrix} m + \delta(\ker\psi) =$$

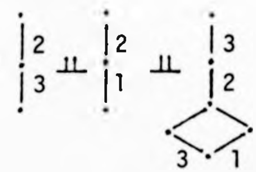
$$= \begin{pmatrix} 0 & f_{11}d_{22}u_{12} & f_{11}d_{23}u_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta(\ker\ \psi) .$$

We want that this equals $\bar{\delta}(xe_1 + ye_3) = \begin{pmatrix} 0 & f_{11}u_{12} & f_{11}u_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta(\ker\psi)$.

If $d_{22} = d_{23} = 1$ then clearly $w\psi = \bar{\delta}$.

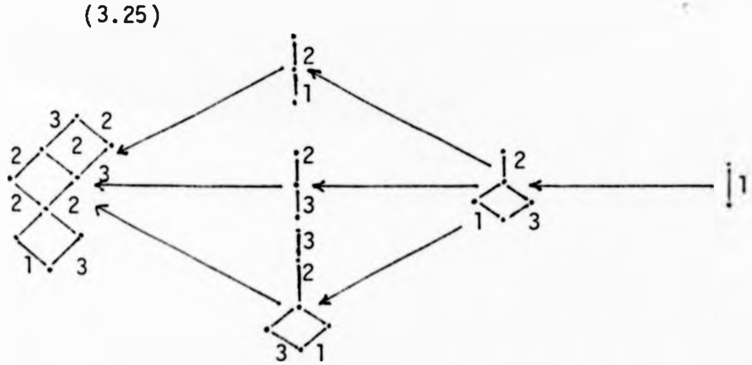
Thus $\bar{\bar{F}}(\psi) \cong N/\mathrm{Im}\ \psi \amalg \dfrac{Re_2 \amalg Re_3}{\delta(\ker\psi)} =$

$\cong N/\mathrm{Im}\ \psi \amalg Re_2/\begin{pmatrix}1\\1\\1\end{pmatrix} \amalg Re_3/\begin{pmatrix}1\\1\\2\end{pmatrix} =$



Thus

(3.25)



is a part of the Auslander-Reiten quiver of R.

Now we construct the almost split sequences that start with the direct summands of $\bar{F}(\psi)$.

After a certain number of calculations similar to those described above we obtain the graph shown in (3.27). Since we already know that there are no more than 21 isomorphism classes of indecomposable modules (§1, Chapter III), this graph is the Auslander-Reiten quiver of R_3 .

Remark: Since R_3 is a connected Algebra (because if $i \neq j$, $\mathrm{Hom}_R(Re_i, Re_j) = e_i\, Re_j \neq 0$) , the fact that its Auslander-Reiten quiver has a connected component (3.27) implies that the Auslander-Reiten quiver of R_3 , is this connected component (see [Ga] pg.43,44).

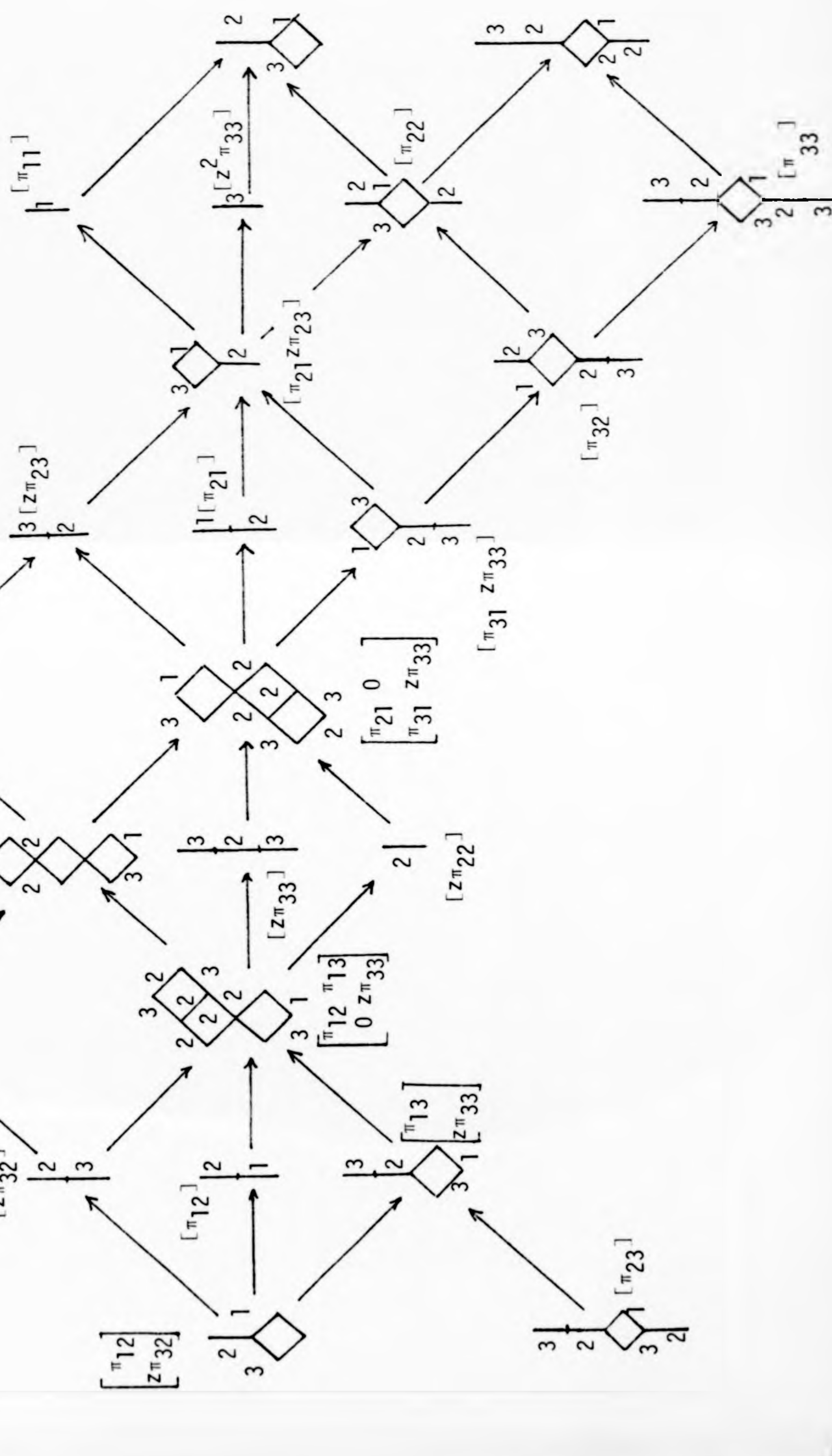
The matrices T that occur next to each indecomposable modules M are the elements of $D(W,U)$ (for some $W,U \in \mathrm{mod}\, A$) , that correspond to M by the rule:

$$(3.26) \quad M \cong \frac{(V_1 \amalg V_2 \amalg V_3, U)}{\{f \in (V_1 \amalg V_2 \amalg V_3, U) : T^*f = 0\}} \quad (\cong T^*(V_1 \amalg V_2 \amalg V_3, U)) \ .$$

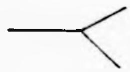
Indeed a module M is such that $M = e_C(F) = F(C)$ (see (3.2)) for some $F \in \mathrm{mmod}\, A$. But $F = \mathrm{Im}\alpha$ where $\alpha : (\ ,U) \to D(W,\ )$ for some $U,W \in \mathrm{mod}\, A$ (see (2.1)) and by Yoneda's Lemma (0.15), α is completely determined by $T = \alpha(U)(1_U) \in D(W,U)$ (see §2, Chapter II).

Thus $F = \mathrm{Im}\,\alpha \Rightarrow M = \mathrm{Im}\,\alpha(C)$ with $C = V_1 \amalg V_2 \amalg V_3$ and

(3.27)



$$\operatorname{Im} \alpha(C) = T^*(C,U) \cong (C,U)/\ker \alpha(C) \ , \quad \text{so we have (3.26).}$$

Removing the projective and injective modules (see (3.14), (3.15)) we have the "stable quiver". Its "tree class" (see [Rt 1], pg.208) is given by the graph  .

We end this chapter with a brief explanation of the symmetry of (3.27) about the two axes formed by the auto-dual modules:

If R is any k-algebra a map $\alpha : R \to R$ such that $\alpha(rs) = \alpha(s)\,\alpha(r)$, $\alpha^2 = 1_R$ is an <u>involution</u>. Then, if $U \in \operatorname{mod} R$, $(U,k) \in \operatorname{mod} R$ with the rule:

$$(r\psi)(u) = \psi(\alpha(r)u) \ , \quad \forall\ \phi \in (U,k) \ , \ r \in R \ , \ u \in U \ .$$

The functor

$$F = \operatorname{Hom}_k(\ ,k) : \operatorname{mod} R \to \operatorname{mod} R$$

is k-linear, contravariant, exact, and $F^2 \cong \mathrm{Id}$, transforms projective modules into injective modules and vice-versa (as the duality functor $\operatorname{Hom}_k(\ ,k) : \operatorname{mod} R \to \operatorname{mod}' R$ (§3. Chapter 0)).

It is also trivial to see that F transforms irreducible maps (0.26) into irreducible maps and if

$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an almost split sequence

then $0 \leftarrow FL \xleftarrow[Ff]{} FM \xleftarrow[Fg]{} FN \leftarrow 0$ is also an almost split sequence.

Now let $R = R_q = \sum_{i,j} Au_{ij}$ and let $\alpha : R \to R$ be such that $\alpha(u_{ij}) = u_{ji} \ \forall\ i,j = 1,\dots,q$.

Then $\alpha^2 = 1_R$ and $\alpha(u_{ij}.u_{j\ell}) = \alpha(z^{(i\sim j)+(j\sim\ell)-(i\sim\ell)}.u_{i\ell})$

(by 1.8) $= z^{(i\sim j)+(j\sim\ell)-(i\sim\ell)}.u_{\ell i} = z^{(\ell\sim j)+(j\sim i)-(\ell\sim i)}.u_{\ell i} =$

$= u_{\ell j}.u_{ji} = \alpha(u_{j\ \ }).\alpha(u_{ij})$.

This can be extended to any element of R , so α is an involution.

Thus reversing all arrows and "turning the modules upside-down" we must get exactly the same quiver. This of course can only happen if the graph has the symmetry mentioned above.

PART B

Chapter IV : Notes on almost split sequences II

§1. Introduction

In this chapter we introduce the notation used in this second part, and outline the results contained in some manuscript notes, by J.A. Green, written under the title "Notes on almost split sequences II", since these have not been published. This prepares the deduction of a "trace formula" (this name derives from the parallel with the trace formula described in [Gr 2] §3) which will be the object of Chapter V.

Let R be a complete discrete rank 1 valuation ring, with maximal ideal $M = R\pi$. It is well known that R is a principal ideal domain, whose ideals are $R\pi^n$ $(n \in \mathbb{N}_0)$.

Let K be the quotient field of R , and A a finite dimensional separable K-algebra, i.e. a K-algebra such that for every extension field E over K , $A^E = E \otimes_K A$ is a semisimple E-algebra (see [CRM] pg. 142). Of course, A itself is semi-simple.

Let Λ be an R-order in A , i.e. Λ is a subring of A which (as an R-module) is such that $\Lambda = Ra_1 \oplus \ldots \oplus Ra_n$ where $\{a_1,\ldots,a_n\}$ is some K-basis of A (see [CRM] pgs. 523, 524).

A typical example, that we shall consider later in Chapter V is $A = KG$, the group algebra, and $\Lambda = RG$, the group ring, where R is the complete ring of p-adic integers (see [D], pg.317), K the quotient field of R, and G a finite group.

As in the first part we use the notation $\mathrm{Mod}\,\Lambda$ for the category of left Λ-modules, $\mathrm{mod}\,\Lambda$ for the category of finitely generated left Λ-modules.

Denote by $\mathrm{mod}^{o}\Lambda$ the category of left Λ-lattices.

Recall that a Λ-module X is a Λ-lattice if X is free and finitely generated as an R-module (see [CRM] pg. 524, having in mind that over a PID a module is projective iff it is free) and that $\mathrm{rank}\, X := n$ if X has a free R-basis of n elements.

Recall also that if X is a Λ-lattice, then $K \otimes_R X$ can be regarded as an A-module and $\dim_K(K \otimes_R X) = \mathrm{rank}\, X$ (because if $\{x_1, \ldots x_n\}$ is an R-basis of X then $\{1_K \otimes x_1, \ldots, 1_K \otimes x_n\}$ is a K-basis of $K \otimes_R X)$.

Denote by $\mathrm{mod}^{t}\Lambda$ the category of the Λ-modules Y, which are finitely generated and torsion as R-modules i.e. $\forall\, y \in Y$, there exists an $n(y) \in \mathbb{N} : \pi^{n(y)}y = 0$. Since Y is a finitely generated R-module this is equivalent to say that there exists $N \in \mathbb{N} : \pi^{N}Y = 0$.

Observe that the quotient X/X_0 of a Λ-lattice X may be a torsion Λ-module. This happens when $\mathrm{rank}\, X = \mathrm{rank}\, X_0$.

Taking the special case $A = K$, $\Lambda = R$ we have the categories mod R , $\mathrm{mod}^{o}R$, and $\mathrm{mod}^{t}R$ of the finitely generated left R-modules, finitely generated free left R-modules, and finitely generated torsion left R-modules respectively.

We have the following:

If X is R-submodule of $Y \in \mathrm{mod}^{o}R$ then $X \in \mathrm{mod}^{o}R$ (because X is finitely generated and torsion-free, so free as an R-module, since R is a P.I.D.).

One can define almost split sequences in $\mathrm{mod}^{o}\Lambda$ as in (0.28), taking all modules and maps in this category.

Then one has the following theorem ([RS] pg.894).

(4.1) <u>Theorem</u>: (Auslander, Roggenkamp, Schmidt)

If $S \in \mathrm{mod}^{o}\Lambda$ is non-projective and indecomposable then there exists an almost split sequence

$$\mathcal{E} : 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

in $\mathrm{mod}^{o}\Lambda$. Moreover $\mathcal{E}$ is unique up to isomorphism. □

Then using the same reasoning as in [Gr 2] pgs. 3,4, we have:

(4.2) <u>Proposition</u>: Let $S \in \mathrm{mod}^{o}\Lambda$ be indecomposable and non-projective. Let

$$E : 0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

be a short exact sequence in $\mathrm{mod}^o\Lambda$.

Then E is almost split iff N is indecomposable and

$$\mathrm{Im}(X,g) = R(X,S) \ , \quad \forall\ X \in \mathrm{mod}^o\Lambda$$

where

$$R(X,S) := \{h \in (X,S) : ht \in \mathrm{rad\ End}\ S \ , \quad \forall\ t \in (S,X)\} \ . \quad \square$$

§2. Dualities and the Nakayama functor

One requires three contravariant R-linear functors:

(4.3) Definitions:

(1) $D^* : \mathrm{Mod}\ R \to \mathrm{Mod}\ R$, such that $D^* = \mathrm{Hom}_R(\ ,I)$ where I is the injective cover of the residue field $R/R\pi$.

(2) $D:\mathrm{Mod}\ \Lambda \to \mathrm{Mod}\ \Lambda^{op}$ is the functor $\mathrm{Hom}_R(\ ,R)$, and given $X \in \mathrm{Mod}\ \Lambda$, DX is regarded as a right Λ-module with rule

$$(f\lambda)(x) = f(\lambda x) \ , \quad \forall\ f \in DX \ , \ \lambda \in \Lambda \ , \ x \in X \ .$$

(3) $d:\mathrm{Mod}\ \Lambda \to \mathrm{Mod}\ \Lambda^{op}$ is the functor $\mathrm{Hom}_\Lambda(\ ,{}_\Lambda\Lambda)$ with: if $X \in \mathrm{Mod}\ \Lambda$, then dX is regarded as a right Λ-module with the rule:

$$(f\lambda)(x) = f(x)\lambda \quad \forall\ f \in dX\ ,\ \lambda \in \Lambda\ ,\ x \in X\ .$$

One needs some facts:

The injective cover I of $R/R\pi$ can be considered as the direct limit

$$I = \lim_{n\to\infty} R/R\pi^n$$

whose elements are classes with the form:

$$[a + R\pi^n] = \{a\pi^s + R\pi^{n+s} : s \in \mathbb{N}_0\} \quad .$$

So we can say that a typical element is $a + R\pi^n$ such that $a + R\pi^n$ is identified with $a\pi^s + R\pi^{n+s}$, $\forall\ s \in \mathbb{N}$.

I is not finitely generated. Clearly $D^*R \cong I$ and so D^* does not map $\mathrm{mod}\ R$ into $\mathrm{mod}\ R$.

But D^* maps $\mathrm{mod}^t R$ into $\mathrm{mod}^t R$, since $D^*X \cong X$, $\forall\ X \in \mathrm{mod}^t R$. This is because $D^*(R/R\pi^n) \cong R/R\pi^n$ $(n \in \mathbb{N})$ and $R/R\pi^n$, $n \in \mathbb{N}$, are the indecomposable modules in $\mathrm{mod}^t R$.

D clearly maps $\mathrm{mod}^o \Lambda$ into $\mathrm{mod}^o \Lambda^{op}$.

It also maps any $M \in \mathrm{mod}^t \Lambda$ into 0 . In fact if $M \in \mathrm{mod}^t \Lambda$, there is an $N \in \mathbb{N} : \pi^N M = 0$. If $\phi \in DM$, then $\phi(u) = r \in R$, $\forall\ u \in M$. But $\pi^N \phi(u) = \phi(\pi^N u) = 0$, thus $\pi^N r = 0$. Since R is an integral domain, $r = 0$. Therefore $\phi(u) = 0$, $\forall\ u \in M$, so $\phi = 0$.

D is left exact but it maps a short exact sequence in $\operatorname{mod}^{o}\Lambda$ into a short exact sequence in $\operatorname{mod}^{o}\Lambda^{op}$ because the elements of this category are free as R-modules. Thus we may say that $D:\operatorname{mod}^{o}\Lambda \to \operatorname{mod}^{o}\Lambda^{op}$ is contravariant, exact and $D^2 \cong id$.

Also D sends projectives (resp. injectives) in $\operatorname{mod}^{o}\Lambda$ to injectives (resp. projectives) in $\operatorname{mod}^{o}\Lambda^{op}$.

d is left-exact and maps $\operatorname{mod}^{o}\Lambda$ into $\operatorname{mod}^{o}\Lambda^{op}$.

(If $X \in \operatorname{mod}^{o}\Lambda$, then $dX = \operatorname{Hom}_{\Lambda}(X,\Lambda) \subseteq \operatorname{Hom}_{R}(X,\Lambda)$ which is free and finitely generated, hence dX is a free and finitely generated R-module).

Since $d(\Lambda e) \cong e\Lambda$ for any idempotent $e \in \Lambda$, d maps projective modules into projective modules.

$N = Dd : \operatorname{Mod}\Lambda \to \operatorname{Mod}\Lambda$ is the Nakayama functor.

It clearly maps $\operatorname{mod}^{o}\Lambda$ into $\operatorname{mod}^{o}\Lambda$.

§3. Some maps

Let $X,Y \in \operatorname{mod}^{o}\Lambda$ and let

$$(4.4)\qquad \beta_Y(X) : dY \underset{\Lambda}{\otimes} X \longrightarrow (Y,X)$$

be such that

$$f \otimes x \longrightarrow \beta_{f,x} : y \to f(y).x \; .$$

(4.5) Notation: Denote by $P(Y,X)$ the space of all maps of (Y,X) which factor through some projective object in $\mathrm{mod}^{o}\Lambda$ (i.e. the space of all projective maps from Y to X in $\mathrm{mod}^{o}\Lambda$) .

Then

(4.6) $\mathrm{Im}\ \beta_Y(X) = P(Y,X)$ (see [RD] pg. V7).

Also,one has:

(4.7) Proposition: (i) If $P \in \mathrm{mod}\ \Lambda$ is projective then

$$\beta_P : dP \underset{\Lambda}{\otimes} - \to (P,\ )$$

is an isomorphism.

(ii) If $P \in \mathrm{mod}\ \Lambda$ is projective then $\beta_Y(P) : dY \underset{\Lambda}{\otimes} P \to (Y,P)$ is an isomorphism.

Pf: see [AR III] pg. 249. □

One has,

(4.8) $$D\beta_Y(X) : D(Y,X) \longrightarrow D(dY \underset{\Lambda}{\otimes} X)$$
$$h \longmapsto h \circ \beta_Y(X)$$

Consider also the "adjoint isomorphism " (see [Ro] pg.37):

$$(4.9)\quad \sigma_Y(X) : D(dY \hat{\otimes} X) \longrightarrow (X, NY)$$
$$\qquad\qquad g \longmapsto (x \to (\underset{\in dY}{f} \to g(f \otimes x))$$

then

$$(4.10)\quad \sigma_Y(X)^{-1} : (X, NY) \longrightarrow D(dY \hat{\otimes} X)$$
$$\qquad\qquad h \longmapsto (\underset{\in dY}{f} \otimes \underset{\in X}{x} \longrightarrow h(x)(f))$$

Now define

$$(4.11)\quad \alpha_Y(X) = \sigma_Y(X) \circ D\beta_Y(X) : D(Y,X) \to (X, NY)$$

(4.12) <u>Remark</u>: $\alpha_Y(X)$ maps $h \in D(Y,X)$ to $\xi \in (X, DdY)$ defined as follows:

$$\xi(x)(f) = h(y \to f(y)x)$$

$\forall\, x \in X$, $f \in dY$.

In fact $\alpha_Y(X)$ is the composition

$$h \to h \circ \beta_Y(X) \to (x \to (f \to h \circ \beta_Y(X)(f \otimes x))$$

but

$$h \circ \beta_Y(X)\,(f \otimes x) = h(\beta_{f,x}) \quad \text{(see (4.4))}$$

with $\beta_{f,x} \in (Y,X)$ such that $\beta_{f,x}(y) = f(y).x$, $\forall\, y \in Y$.

So $\alpha_Y(X)$ is such that

$$h \longrightarrow \xi : X \to NY \text{ such that}$$

$$\xi(x)(f) = h(\beta_{f,x}) = h(y \to f(y).x) .$$

The next proposition is of crucial importance, but, since its proof is not necessary for the purposes of this chapter we omit it.

(4.13) <u>Proposition</u>: Let $U \in \text{mod } R$, $V \in \text{mod}^o R$ and $\beta \in \text{Hom}_R(U,V)$, be such that

$$\beta_k = 1_K \otimes \beta : K \underset{R}{\otimes} U \to K \underset{R}{\otimes} V \text{ is a } K\text{-isomorphism.}$$

Then

(i) ker β , Coker β are torsion modules.

(ii) If $T = \text{Coker } \beta$, then there is a short exact sequence

(4.14) $0 \to DV \xrightarrow{D\beta} DU \xrightarrow{\delta} D^*T \to 0$

in mod R , with the map δ defined as follows:

If $\mu \in DU$, $v \in V$, then:

(4.15) $\delta(\mu)(v + \text{Im } \beta) = [\mu(u) + \pi^N R]$ where $u \in U$, $N \in \mathbb{N}$ are such that $\pi^N v = \beta(u)$. $\square$

Remark: Observe that given $v \in V$, there exists $N \in \mathbb{N}$ such that $\pi^N v \in \mathrm{Im}\beta$, because $\mathrm{Coker}\,\beta = V/\mathrm{Im}\,\beta$ is a torsion module.

Now, returning to the maps we were considering...

Using (4.6) we see that:

$$(4.16)\qquad dY \otimes_{\Lambda} X \xrightarrow[\beta=\beta_Y(X)]{} (Y,X) \xrightarrow[\text{nat}]{} (Y,X)/P(Y,X) \to 0$$

is exact.

The map $\beta = \beta_Y(X)$ satisfies the conditions of proposition (4.13):

In fact $K \otimes_R (dY \otimes_{\Lambda} X) \cong (K \otimes_R dY) \otimes_A (K \otimes_R X)$ (see [Ro]pg.103)

$\cong d(K \otimes_R Y) \otimes_A (K \otimes_R X)$ (because $K \otimes_R dY = K \otimes_R (Y, {}_{\Lambda}\Lambda) \cong$

$\cong \mathrm{Hom}_A(K \otimes_R Y, K \otimes_R {}_{\Lambda}\Lambda) = \mathrm{Hom}_A(K \otimes_R Y, {}_A A) = d(K \otimes_R Y))$

and

$$K \otimes_R (Y,X) \cong \mathrm{Hom}_A(K \otimes_R Y, K \otimes_R X) \qquad \text{(see [CRM] pg. 74).}$$

But

$$d(K \otimes_R Y) \otimes_A (K \otimes_R X) \underset{\beta(K\otimes X)}{\overset{K\otimes Y}{\cong}} \mathrm{Hom}_A(K \otimes_R Y, K \otimes_R X)$$

because $K \otimes_R X$ (and $K \otimes_R Y$) is projective, since A is semisimple.

Thus

$$K \underset{R}{\otimes} (dY \underset{\Lambda}{\otimes} X) \cong K \underset{R}{\otimes} (Y,X) .$$

Then, proposition (4.13) tells us that $(Y,X)/P(Y,X)$ is a torsion module, and there is an exact sequence:

$$(4.17) \quad D(Y,X) \xrightarrow{D\beta_Y(X)} D(dY \underset{\Lambda}{\otimes} X) \xrightarrow{\delta = \delta_Y(X)} D^*((Y,X)/P(Y,X)) \to 0 \quad .$$

Let:

$$(4.18) \quad \gamma_Y(X) = \delta_Y(X)\ \sigma_Y(X)^{-1} \quad .$$

Considering (4.11) and (4.18) we conclude that:

$$(4.19) \quad D(Y,X) \xrightarrow{\alpha_Y(X)} (X,NY) \xrightarrow{\gamma_Y(X)} D^*((Y,X)/P(Y,X)) \to 0$$

is exact.

Therefore:

$$(4.20) \quad \operatorname{Coker} \alpha_Y(X) \cong D^*((Y,X)/P(Y,X)) \quad .$$

§4. The Roggenkamp diagram

In this section we describe a method given in "Notes on almost split sequences II", to construct almost split sequences.

This is a particular case of the following problem:

Given $S \in \mathrm{mod}^{O}\Lambda$, construct a short exact sequence

$$0 \to N \xrightarrow{f} E \xrightarrow{g} S \to 0$$

in $\mathrm{mod}^{O}\Lambda$, in such a way that one has an explicit expression for the subfunctor $\mathrm{Im}(\ ,g)$ of $(\ ,S)$.

Green solves this problem by constructing what he calls the "Roggenkamp diagram" (because this construction is based in some results by K. Roggenkamp).

This is done by using a method similar to the one used in the construction of the Auslander-Reiten-Gabriel diagram (see [Gr 2]; see also (2.5), Chapter II).

Let $M \in \mathrm{mod}^{O}\Lambda$ and let

(4.21) $$P \xrightarrow{p_0} M \to 0$$

be a projective resolution of M. Then

(4.22) $$0 \to \Omega M \to P \xrightarrow[p_0]{} M \to 0$$

is a short exact sequence, with P projective and $\Omega M = \ker p_0$.

Apply d to (4.22). Then

$$0 \to dM \xrightarrow[dp_0]{} dP \to d\Omega M$$

is exact in $\mathrm{mod}^o \Lambda^{op}$.

Let $C = \mathrm{Coker}\ dp_0$.

Then

$$(4.23) \qquad 0 \to dM \xrightarrow{dp_0} dP \xrightarrow{nat} C \to 0$$

is exact, and $C \leq d\,\Omega\,M$, thus $C \in \mathrm{mod}^o \Lambda^{op}$.

Therefore (4.23) is an exact sequence in $\mathrm{mod}^o \Lambda^{op}$.

Apply D and write $DC = \underline{B}M$. Thus

$$(4.24) \qquad 0 \to \underline{B}M \xrightarrow[j = D\ nat]{} NP \xrightarrow[Np_0]{} NM \to 0$$

is a short exact sequence.

This is such that if M is indecomposable non-projective and (4.21) is minimal (i.e. $\mathrm{Ker}\ p_0 \leq \mathrm{rad}\ P$) , then $\underline{B}M$ is indecomposable (see [R] prop. 2, pg.1369).

Applying $D(\ ,X)$ to (4.21) we get the exact sequence:

$$D(P,X) \xrightarrow{D(p_0,X)} D(M,X) \longrightarrow 0 \quad .$$

Applying $(X,\ )$ to (4.24) we obtain the exact sequence:

$$0 \to (X, \underline{B}M) \xrightarrow[(X,j)]{} (X,\mathcal{N}P) \xrightarrow[(X,\mathcal{N}p_0)]{} (X,\mathcal{N}M) \quad .$$

Then we may consider the diagram:

$$\begin{tikzcd}
 & & D(P,X) \arrow[r, "D(p_0{,}X)"] \arrow[d, "\alpha_P(X)"'] & D(M,X) \arrow[r] \arrow[d, "\alpha_M(X)"] & 0 & \\
0 \arrow[r] & (X,\underline{B}M) \arrow[r, "(X{,}j)"'] & (X,\mathcal{N}P) \arrow[r, "(X{,}\mathcal{N}p_0)"'] & (X,\mathcal{N}M) \arrow[r, "\gamma_M(X)"'] & D^*((M,X)/P(M,X)) \arrow[r] & 0
\end{tikzcd} \quad .$$

It is commutative because $\alpha_Y(X)$ is natural in Y; and the sequences are exact. Namely the exactness at $(X, \mathcal{N}M)$ is for the following reason:

Since $\alpha_P(X)$ is an isomorphism and $D(p_0,X)$ is an epimorphism, the commutativity of the diagram gives $\operatorname{Im}(X,\mathcal{N}p_0) = \operatorname{Im} \alpha_M(X)$. By (4.19), $\operatorname{Im} \alpha_M(X) = \ker \gamma_M(X)$.

Now consider any module $S \in \operatorname{mod}^o\Lambda$ and any map $\theta \in (S, \mathcal{N}M)$ and construct the pull-back diagram :

(4.25)

$$\begin{tikzcd}
 & 0 \arrow[r] & \underline{B}M \arrow[r, "j"] & NP \arrow[r, "Np_0"] & NM \arrow[r] & 0 \\
E(\theta) & 0 \arrow[r] & \underline{B}M \arrow[r, "f"'] \arrow[u, "id"'] & E(\theta) \arrow[r, "g"'] \arrow[u, "\ell"'] & S \arrow[r] \arrow[u, "\theta"'] & 0
\end{tikzcd}$$

Let $a_\theta(X) = \gamma_M(X)(X,\theta)$. Then one can construct the "Roggenkamp diagram", which is commutative, and where all rows are exact, as follows:

(4.26)

$$\begin{tikzcd}
 & & D(P,X) \arrow[r, "D(p_0{,}X)"] \arrow[d, "\alpha_P(X)"] & D(M,X) \arrow[r] \arrow[d, "\alpha_M(X)"] & 0 & \\
0 \arrow[r] & (X,\underline{B}M) \arrow[r, "(X{,}j)"] & (X,NP) \arrow[r, "(X{,}Np_0)"] & (X,NM) \arrow[r, "\gamma_M(X)"] & D^*((M,X)/P(M,X)) \arrow[r] & 0 \\
0 \arrow[r] & (X,\underline{B}M) \arrow[r, "(X{,}f)"'] \arrow[u, "id"'] & (X,E(\theta)) \arrow[r, "(X{,}g)"'] \arrow[u, "(X{,}\ell)"'] & (X,S) \arrow[r, "a_\theta(X)"'] \arrow[u, "(X{,}\theta)"'] & D^*((M,X)/P(M,X)) \arrow[u, "id"'] &
\end{tikzcd}$$

In particular

(4.27) $\mathrm{Im}(\ ,g) = \ker a_\theta$.

By Yoneda's lemma (0.15), a_θ is completely determined by the element

$$(4.28) \quad T_\theta = a_\theta(S)\,(1_S) = \gamma_M(S)(S,\theta)\,(1_S) = \gamma_M(S)(\theta)$$

$$\in D^*((M,S)/P(M,S)) \quad .$$

Next one can see how an almost split sequence is a particular case of $E(\theta)$ in (4.25).

One can identify $D^*((M,S)/P(M,S))$ with the R-module consisting of those elements $T \in D^*(M,S)$, which vanish on $P(M,S)$.

Thus each $T \in D^*((M,S)/P(M,S))$ defines a map

$$a_T : (\ ,S) \longrightarrow D^*(M,\ )$$

(by Yoneda's Lemma (0.15)). In particular T_θ (4.28) defines $a_{T_\theta} = a_\theta$.

Using the commutativity of the diagram:

$$\begin{tikzcd}
(S,S) \arrow[r, "a_T(S)"'] \arrow[d, "{(f,S)}"'] & D^*(M,S) \arrow[d, "{D^*(M,f)}"] \\
(X,S) \arrow[r, "a_T(X)"'] & D^*(M,X)
\end{tikzcd}$$

one sees that:

$$[a_T(X)(f)](g) = [D^*(M,f)(T)](g) = T(fg) \ , \quad \forall\ g \in (M,X) \ .$$

Thus

$$\mathrm{Ker}\ a_T(X) = \{f \in (X,S) : T(fg) = 0 , \quad \forall g \in (M,X)\} \quad .$$

Since T vanishes on $P(M,S)$ it is clear that $P(X,S) \leq$ $\mathrm{Ker}\ a_T(X)$ or, in functorial terms, $P(\ ,S) \leq \mathrm{Ker}\ a_T$.

Now take $T = T_\theta$ (4.28) , $M = S$; then:

$$\mathrm{Ker}\ a_\theta(X) = \{f \in (X,S) : T_\theta(fg) = 0 , \forall g \in (S,X)\} =$$

$$= \{f \in (X,S) : fg \in \mathrm{Ker}\ T_\theta, \quad \forall g \in (S,X)\} = \{f \in (X,S) : f(S,X) \leq \mathrm{Ker}\ T_\theta\} \ .$$

But $f.(S,X)$ is a right ideal of $\mathrm{End}(S)$. Thus

(4.29) $\mathrm{Ker}\ a_\theta(X) = \{f \in (X,S) : fg \in$ maximal right ideal of $\mathrm{End}(S)$ contained in $\mathrm{Ker}\ T_\theta$, $\forall g \in (S,X)\}$.

Now suppose S is indecomposable, take $\theta \in (S, NS)$, and construct $E(\theta)$ as in (4.25).

Proposition (4.2) tells us that $E(\theta)$ is almost split sequence if and only if $\underline{B}M$ (= $\underline{B}S$) is indecomposable and $\mathrm{Ker}\ a_\theta = R(\ ,S)$, where:

(4.30) $\forall X \in \mathrm{mod}^{o}\Lambda$, $R(X,S) = \{f \in (X,S) : fg \in \mathrm{rad}\ \mathrm{End}(S), \forall g \in (S,X)\}$.

From (4.29), (4.30) we have:

Ker $a_\theta = R(\ ,S) \Longleftrightarrow$ (maximal right ideal of End S contained in Ker T_θ) = rad End S .

Since rad End S is the unique maximal right ideal of End S (because S is indecomposable), this is equivalent to

$$T_\theta \neq 0 \ , \ T_\theta(\text{rad End } S) = 0 \ .$$

And $\underline{B}S$ is indecomposable if S is indecomposable non-projective and the presentation (4.21) is minimal as we saw before.

Thus:

(4.31) <u>Proposition</u>: Let $S \in \text{mod}^o\Lambda$ be indecomposable non-projective. Take $M = S$, assume that (4.21) is minimal and let $\theta \in (S,NS)$. Then construct $E(\theta)$ as in (4.25).

Then $E(\theta)$ is almost split iff $T_\theta = \gamma_S(S)(\theta) \in D^*((S,S)/P(S,S))$ (see (4.28)) satisfies the conditions

(4.32) $T_\theta \neq 0$, $T_\theta(\text{rad End } S) = 0$. □

<u>Remark</u>: Since S is not projective, $P(S,S) \neq \text{End } S$. Thus $P(S,S) \leq \text{rad End } S < \text{End } S$, so there exist an element $T \in D^*(S,S)$ which satisfy (4.32) and $T(P(S,S)) = 0$, so $T \in D^*((S,S)/P(S,S))$. Since the map $\gamma_S(S)$ (see 4.26) is surjective, there exists $\theta \in (S,NS)$ such that $T_\theta = \gamma_S(S)(\theta)$.

Chapter V : A "trace formula" for T_θ

§1. Introduction

As in §3 of [Gr 2], it is possible to present an explicit formula for T_θ (4.28). But the method used to deduce this formula is very different from that used in that paper.

We need to have in mind a few facts relative to separable algebras. These are a particular case of Frobenius algebras:

Let K be a field.

A Frobenius algebra A is a finite dimensional K-algebra such that ${}_A A \cong D(A_A)$ (see [CR] pg.413).

It can be proved that this is equivalent to the existence of a non-degenerate bilinear form:

$$(5.1) \quad f : A \times A \longrightarrow K$$

such that $f(ab,c) = f(a,bc)$ ([CR] pg.414, 415).

The following facts are taken from [CR] pg. 481, 482:

Given a basis $\{a_1,\ldots,a_n\}$ of A let $\{b_1,\ldots b_n\}$ be a dual basis with respect to f, i.e. such that $f(a_i,b_j) = \delta_{ij}$ $\forall$ $i,j=1,\ldots,n$.

For each $a \in A$ we can consider the element $c(a) = \sum_1^n b_i\, a\, a_i$, which is in the center of A, $C(A)$.

Then we may consider the ideal of $C(A)$:

(5.2) $\Gamma(A) = \{c(a) : a \in A\} \subseteq C(A)$

which is independent of f , and of the chosen basis $\{a_i\}$, $\{b_i\}$.

And one has the following characterization of separable algebras:

(5.3) Proposition (D.G. Higman): The K-algebra A is separable iff A is a Frobenius algebra and $\Gamma(A) = C(A)$. $\square$

Now we want an explicit formula for:

$$T_\theta = a_\theta(S)(1_S) = \gamma_M(S)(\theta) \qquad \text{(see (4.28))}$$

where $\gamma_M = \delta_M \circ \sigma_M^{-1}$ (see (4.18)).

Thus T_θ is the result of the following sequence of maps:

(5.4)
$$(S,S) \xrightarrow{(S,\theta)} (S,DdM) \xrightarrow{\sigma_M(S)^{-1}} D(dM \underset{\Lambda}{\otimes} S) \xrightarrow{\delta_M(S)} D^*((M,S)/\mathrm{Im}\ \beta_M(S))$$

$$1_S \longrightarrow \theta \longrightarrow ((\rho \otimes s) \underset{\mu}{\longrightarrow} \theta(s)(\rho)) \longrightarrow \delta(\mu) = T_\theta$$

$\delta(\mu)$ is such that, for given $h \in (M,S)$,

$$\delta(\mu)(h + \mathrm{Im}\ \beta_M(S)) = f(\mu,h)$$

and f is given by:

$$f : D(dM \underset{\Lambda}{\otimes} S) \times (M,S) \longrightarrow I \quad \text{(injective cover of } R/R\pi)$$

$$(\mu,h) \longrightarrow [\mu(\sum_i \rho_i \otimes s_i) + \pi^N R] =$$

$$= [\sum_i \theta(s_i)(\rho_i) + \pi^N R]$$

where $s_i \in S$, $\rho_i : M \to \Lambda$, $N \in \mathbb{N}$ are such that:

$$(5.5) \qquad \beta_M(S)\,(\sum_i \rho_i \otimes s_i) = \pi^N h$$

(see proposition (4.13)).

Therefore,

given $h:M \to S$ we must find $N \in \mathbb{N}$ such that $\pi^N h \in \operatorname{Im} \beta_M(S)$ and $\rho_i \in dM$, $s_i \in S$ such that (5.5) is verified.

§2. A projective endomorphism of M

The first question we have to answer is:

Given $h \in (M,S)$, find $N \in \mathbb{N}$ such that $\pi^N h \in \operatorname{Im} \beta_M(S)$, i.e. such that $\pi^N h$ is a projective map (see (4.6)).

Observe that, since the set of projective maps $P(M,S)$ forms an ideal in the category $\operatorname{mod}^o\Lambda$ (see (0.12)) it is enough to answer the question:

(5.6) Find $N \in \mathbb{N}$ such that $\pi^N.1_M$ is a projective endomorphism of M .

Recall that we are assuming that R, K, Λ, A, satisfy the conditions given in Chapter IV, §1. In particular $\Lambda = R\,a_1 \oplus \dots \oplus R\,a_n$, where $\{a_1,\dots,a_n\}$ is a K-basis of A , and A is a separable K-algebra, with a non-degenerate associative bilinear form $f: A \times A \to K$ (see 5.1)).

Let $\{b_1,\dots,b_n\}$ be a dual basis of $\{a_1,\dots,a_n\}$ with respect to f .

Since A is separable, and $1 \in C(A)$, the center of A , proposition (5.3) tells us that $1 \in \Gamma(A)$ (5.2), thus

$$1 = \sum_{i=1}^{n} b_i\, a\, a_i$$

for some $a \in A$.

If $\tilde{a}_i = b_i a$, then

$$1 = \sum_{i=1}^{n} \tilde{a}_i . a_i \ .$$

Consider the element:

$$(5.7) \qquad b = \sum_{i=1}^{n} \tilde{a}_i \otimes a_i \in A \otimes_K A \ .$$

Since $A \otimes_K A = K(\Lambda \otimes_R \Lambda)$, there exists $r_0 \in R$ such that

$$(5.8) \qquad r_0 b \in \Lambda \otimes_R \Lambda \ .$$

(Observe that it is enough to find $r_0 : r_0 \tilde{a}_i \in \Lambda$, $i = 1,\dots,n$).

Now consider the map:

$$m : A \underset{K}{\otimes} A \longrightarrow A$$

such that $\quad x \otimes y \longrightarrow x.y$.

Denote also by m its restriction to $\Lambda \underset{R}{\otimes} \Lambda$:

$$(5.9) \qquad m : \Lambda \underset{R}{\otimes} \Lambda \longrightarrow \Lambda \; .$$

Let

$$f : A \longrightarrow A \underset{K}{\otimes} A$$

be such that $\quad 1 \longrightarrow b$.

Then $mf(1) = m(\sum_1^n \tilde{a}_i \otimes a_i) = 1$, i.e. $mf = 1_A$.

Let f_0 be the restriction of $r_0 f$ to Λ :

$$(5.10) \qquad f_0 = r_0 f : \Lambda \longrightarrow \Lambda \underset{R}{\otimes} \Lambda \; .$$

Then $mf_0(\lambda) = m(r_0 f(\lambda)) = r_0\, mf(\lambda) = r_0\, \lambda$, $\forall \lambda \in \Lambda$, i.e.

$$(5.11) \qquad mf_0 = r_0\, 1_\Lambda \; .$$

Now apply the functor $- \underset{\Lambda}{\otimes} M$ to (5.9):

$$(5.12)\qquad p = m \underset{\Lambda}{\otimes} 1_M : \Lambda \underset{R}{\otimes} \Lambda \underset{\Lambda}{\otimes} M \cong \Lambda \underset{R}{\otimes} M \to \Lambda \underset{\Lambda}{\otimes} M \cong M \ .$$

p is clearly an epimorphism and $\Lambda \underset{R}{\otimes} M$ is projective. So this is a projective presentation for M .

Now applying the same functor $- \underset{\Lambda}{\otimes} M$ to (5.10) we get:

$$(5.13)\qquad w = f_0 \underset{\Lambda}{\otimes} 1_M : M \cong \Lambda \underset{\Lambda}{\otimes} M \longrightarrow \Lambda \underset{R}{\otimes} \Lambda \underset{\Lambda}{\otimes} M \cong \Lambda \underset{R}{\otimes} M$$

$$m(= 1 \otimes m) \longrightarrow \sum_1^n r_0 \tilde{a}_i \underset{R}{\otimes} a_i m =$$

$$= r_0 \, b \underset{\Lambda}{\otimes} m$$

and

$$p \circ w = (m \underset{\Lambda}{\otimes} 1_M)(f_0 \underset{\Lambda}{\otimes} 1_M) = mf_0 \underset{\Lambda}{\otimes} 1_M = r_0 \, 1_\Lambda \otimes 1_M = r_0 \, 1_M \ .$$

Thus $r_0 \cdot 1_M = p \circ w$, and so $r_0 1_M$ factors through the projective module $\Lambda \underset{R}{\otimes} M$ i.e. $r_0 1_M$ is a projective endomorphism.

Then:

(5.14) <u>Proposition</u>: Let A be a finite dimensional separable K-algebra, Λ an R-order in A such that $\Lambda = R a_1 \oplus \dots \oplus Ra_n$ for some K-basis $\{a_1, \dots, a_n\}$ of A , $\tilde{a}_1, \dots, \tilde{a}_n$ elements of A such that $1 = \sum_{i=1}^n \tilde{a}_i a_i$, and $r_0 \in R$ such that $r_0 \tilde{a}_i \in \Lambda , \forall \ i = 1, \dots, n$.

Then

$\forall \ M \in \mathrm{mod}^0\Lambda$, $r_0 \cdot 1_M$ is a projective endomorphism of M . □

(5.15) Remark: In the particular case we are considering we can take $r_0 = \pi^N.u$, where u is a unit in R , and $N \in \mathbb{N}_0$.

§3. The map $\beta_M(X)$

The next proposition gives a way of finding an element $\tau \in dM \underset{\Lambda}{\otimes} M$ such that $\beta_M(M)(\tau)$ is a given element of $P(M,M)$.

(5.16) Proposition: Let $M \in \mathrm{mod}^o\Lambda$ be such that

$$(5.17) \quad 0 \to M' \to \bigoplus_{i=1}^{s} \Lambda\, y_i = P \xrightarrow{p_0} M \to 0$$

is a projective presentation of M . Let $g \in P(M,M)$, so g factors through P , i.e. there exists $w{:}M \to P : p_0 w = g$.

Consider the following element $\tau \in dM \underset{\Lambda}{\otimes} M$;

$$(5.18) \qquad \tau = \sum_{i=1}^{s} \mu_i \otimes m_i$$

such that

(1) $m_i = p_0(y_i)$, $i = 1,\ldots,s$.

(2) $\mu_i \in dM$ is defined by

$$(5.19) \quad w(m) = \sum_{i=1}^{s} \mu_i(m).y_i \ , \qquad \forall\, m \in M .$$

Then

$$(5.20) \quad \beta_M(M)(\tau) = g .$$

<u>Proof</u>: Since $P = \bigoplus_{i=1}^{s} \Lambda y_i$ is a direct sum, the expression (5.19) makes sense and gives a definition for the maps $\mu_i \in dM = (M, {}_\Lambda\Lambda)$.

Let $m \in M$. Then $[\beta_M(M)(\tau)](m) = \sum_{i=1}^{s} \mu_i(m).m_i =$

$= \sum_{i=1}^{s} \mu_i(m).p_0(y_i) = p_0(\sum_{i=1}^{s} \mu_i(m).y_i) = p_0\, w(m) = g(m)$.

(See (4.4).) $\square$

In particular we may consider the conditions of proposition (5.14), that $g = r_0 . 1_M$, and that we have a projective presentation as in (5.12):

Suppose $M = \bigoplus_{i=1}^{s} Rm_i$; then $P = \Lambda \otimes_R M = \bigoplus_{1}^{s} \Lambda(1 \otimes_R m_i)$

and $P = \Lambda \otimes_R M \xrightarrow[p_0]{} M \to 0$ is a projective presentation of M with $p_0(1 \otimes m_i) = m_i$, $i = 1,\dots,s$.

Also

$$w(m) = \sum_{j=1}^{s} \mu_j(m)(1 \otimes m_j) .$$

But

$$w(m) = (f_0 \otimes 1_M)(m) = \sum_{i=1}^{n} r_0\, \tilde{a}_i \otimes_R a_i m \qquad \text{(see (5.13))}.$$

Let $a_i m = \sum_{j=1}^{s} r_{ij}(m).m_j$ with $r_{ij}(m) \in R$.

Then $w(m) = \sum_{j=1}^{s} r_0 \left(\sum_{i=1}^{n} \tilde{a}_i\; r_{ij}(m)\right) \underset{R}{\otimes} m_j =$

$$= \sum_{j=1}^{s} r_0\left(\sum_{i=1}^{n} \tilde{a}_i\; r_{ij}(m)\right)\left(1 \underset{R}{\otimes} m_j\right) .$$

Comparing these two expressions of $w(m)$ we get

$$\mu_j(m) = r_0 \sum_{i=1}^{n} \tilde{a}_i\; r_{ij}(m) .$$

Thus:

(5.21) <u>Proposition</u>: Suppose conditions of proposition (5.14) are verified. Let $M = \bigoplus_{j=1}^{s} Rm_j \in \mathrm{mod}^{O}\Lambda$ and suppose that $a_i.m \in M$ is given by the expression

$$a_i.m = \sum_{j=1}^{s} r_{ij}(m).m_j \quad \text{with} \quad r_{ij}(m) \in R .$$

Consider the element $\tau = \sum_{j=1}^{s} \mu_j \otimes m_j \in dM \underset{\Lambda}{\otimes} M$, where μ_j is such that $\mu_j(m) = r_0 \cdot \sum_{i=1}^{n} \tilde{a}_i\; r_{ij}(m)$.

Then

$$\beta_M(M)(\tau) = r_0\, 1_M .$$

<u>Proof</u>: $\beta_M(M)(\tau)(m) = \sum_{j=1}^{s} \mu_j(m).m_j =$

$$= \sum_{j=1}^{s} (r_0 \sum_{i=1}^{n} \tilde{a}_i\, r_{ij}(m))m_j = r_0 \sum_{i=1}^{n} \tilde{a}_i (\sum_{j=1}^{s} r_{ij}(m).m_j) =$$

$$= r_0 \sum_{i=1}^{n} \tilde{a}_i.a_i m = r_0\, 1.m = r_0 m = r_0\, 1_M(m) \quad . \qquad \square$$

Using naturality of $\beta_M(X)$ in X, and given any $S \in \mathrm{mod}^o\Lambda$, $h \in (M,S)$,

$$\begin{tikzcd}
dM \otimes_\Lambda M \arrow[rr, "\beta_M(M)"] \arrow[ddd, "dM\otimes h"'] & & (M,M) \arrow[ddd, "(M{,}h)"] \\
\sum_{j=1}^{s} \mu_j \otimes m_j \arrow[r] \arrow[d] & r_0\, 1_M \arrow[d] & \\
\sum_{j=1}^{s} \mu_j \otimes h(m_j) \arrow[r] & r_0 h & \\
dM \otimes_\Lambda S \arrow[rr, "\beta_M(S)"'] & & (M,S)
\end{tikzcd}$$

we obtain $\beta_M(S)(\sum_{j=1}^{s} \mu_j \otimes h(m_j)) = r_0 h$.

Now we can return to the expression of T_θ (5.4) and take the following <u>conclusion</u>:

(5.22) <u>Theorem</u>: Let R be a complete discrete rank 1 valuation ring, with maximal ideal $R\pi$; let K be the quotient field of R and

A a separable f.d. K-algebra. Let Λ be an R-order such that $\Lambda = Ra_1 \oplus \ldots \oplus Ra_n$ for some K-basis $\{a_1,\ldots,a_n\}$ of A, and $\tilde{a}_1,\ldots,\tilde{a}_n$ elements of A such that $1 = \sum_{i=1}^{n} \tilde{a}_i . a_i$.

Let $N \in \mathbb{N}$ be such that $\pi^N \tilde{a}_i \in \Lambda$, $\forall$ $i = 1,\ldots,n$.

Let $M \in \mathrm{mod}^o\Lambda$ and suppose that $M = \bigoplus_1^s R\, m_i$.

Let $\rho_j \in dM$ be given by

$$\rho_j(m) = \pi^N \sum_{i=1}^{n} \tilde{a}_i\, r_{ij}(m) \qquad j = 1,\ldots,s$$

where the $r_{ij}(m) \in R$ are such that

$$a_i m = \sum_{j=1}^{s} r_{ij}(m).m_j \quad .$$

Let $S \in \mathrm{mod}^o\Lambda$. Then:

Identifying $D^*((M,S)/P(M,S))$ with the R-module consisting of those $T \in D^*(M,S)$ which vanish in $P(M,S)$, we have:

For each $\theta \in (S,\Lambda M)$, $T_\theta = \gamma_M(S)(\theta) \in D^*(M,S)$ is given by:

$$(5.23) \quad T_\theta(h) = [\sum_{j=1}^{s} (\theta(h(m_j)))(\rho_j) + \pi^N R] \ , \quad \forall\ h \in (M,S) \ . \quad \square$$

<u>Remark</u>: (5.23) may be called a "Trace formula" for T_θ in parallel with formula (3.11), pg. 18 of [Gr.2].

§4. Case where A is symmetric

Now we assume that A is symmetric, i.e. the non-degenerate associative, bilinear form f (see (5.1)) satisfies the condition:

$$f(a,b) = f(b,a) \quad \forall\, a,b \in A$$

(see [CR] pg.440).

Suppose also that f induces a non-degenerate R-integral form in Λ.

Let $\lambda : A \to K$ be such that

$$\lambda(a) = f(a,1)\ (= f(1,a))\ , \quad \forall\, a \in A\ .$$

Then

$$\lambda(ab) = f(ab,1) = f(a,b) \quad \forall\, a,b \in A\ .$$

Since f induces a non-degenerate R-integral form in Λ, then $\lambda(\ell) \in R$, $\forall\, \ell \in \Lambda$. So we may consider

$$\lambda : \Lambda \to R\ .$$

As in [Gr. 2] pg.22, for each $X \in \mathrm{mod}^{o}\Lambda$ consider the map

$$\begin{aligned} U(X) : dX &\longrightarrow DX \\ g &\longrightarrow \lambda g\ . \end{aligned}$$

This is an isomorphism in $\mathrm{mod}^o\Lambda^{op}$:

In fact $(\lambda g)(x) = 0 , \forall\, x \in X \Rightarrow f(1,g(x)) = 0 , \forall\, x \in X$

$\Rightarrow f(\ell,g(x)) = f(1,\ell g(x)) = f(1,g(\ell x)) = 0 , \forall\, \begin{smallmatrix}\ell \in \Lambda \\ x \in X\end{smallmatrix}$

$\Rightarrow g(x) = 0 , \forall\, x \in X \Rightarrow g = 0$, so $U(X)$ is injective and

if $h \in DX$, let $g : X \to {}_\Lambda\Lambda$ be such that $x \to h(x) . 1_\Lambda$.

Then

$\lambda(g(x)) = \lambda(h(x).1_\Lambda) = h(x).\lambda(1_\Lambda) = h(x)$, thus $U(X)$ is surjective.

Since, given $t : X \to Y$ $X,Y \in \mathrm{mod}^o\Lambda$ the diagram

$$\begin{tikzcd} dX \arrow[d, "U(X)"'] & dY \arrow[l, "dt"'] \arrow[d, "U(Y)"] \\ DX & DY \arrow[l, "Dt"] \end{tikzcd}$$

commutes, U is a natural isomorphism

between the functors d and D .

Then

$W = DU$ gives a natural isomorphism between $D^2 \cong Id$ and $Dd = N$.

Remark: We see that $\underset{\sim}{B}M$ (c.f. (4.24)) is ΩM , in this particular case.

Thus

$$(5.24) \qquad W(X) : X \longrightarrow NX$$
$$x \longrightarrow (g \to (\lambda g)(x)) \qquad \forall\, g \in dX$$

is an isomorphism.

Then for every $S \in \mathrm{mod}^o\Lambda$, one has the isomorphism (since S is projective as R-module)

$$\begin{array}{ccc} (S,X) & \xrightarrow{(S,W(X))} & (S,NX) \\ \psi & \longrightarrow & W(X)\circ\psi \qquad \forall\ \psi \in (S,X) . \end{array}$$

Now consider formula (5.23):

Let $\theta \in (S,NM)$. Then $\theta = W(M)\circ\phi$ for some $\phi \in (S,M)$.

Then, for all $h \in (M,S)$

$$(\theta(h(m_j)))(\rho_j) = ((W(M) \circ \phi)(h(m_j))(\rho_j) =$$

$$= (W(M)(\phi h(m_j)))(\rho_j) = (\lambda\rho_j)((\phi h)(m_j)) \quad \text{by (5.24)} .$$

Let

$$\sigma_j = \lambda\, \rho_j \in DM .$$

Thus formula (5.23) gives the following, where we write U_ϕ instead of $T_{W(M)\circ\phi}$:

$$U_\phi(h) = [\sum_{j=1}^{s} \sigma_j((\phi h)(m_j)) + \pi^N R] \qquad , \forall\ h \in (M,S) .$$

Observe that σ_j is such that

$$\sigma_j(m) = \pi^N \sum_{i=1}^{n} r_{ij}(m)\lambda(\tilde{a}_i)$$

$j = 1,\dots,s$

where $r_{ij} \in DM$ is such that

$$a_i m = \sum_{j=1}^{s} r_{ij}(m).m_j \quad .$$

And

$\lambda(\tilde{a}_i) = \lambda(b_i a) = f(b_i, a)$ where $\{b_i\}$ is the basis dual of $\{a_i\}$ with respect to f .

Let $a = \alpha_1 a_1 + \ldots + \alpha_n a_n$, $\alpha_i \in K$.

Then

$$f(b_i, a) = f(b_i, \alpha_1 a_1 + \ldots + \alpha_i a_i + \ldots \alpha_n a_n) =$$
$$= \alpha_i \; f(a_i, a_i) = \alpha_i \; .$$

Thus

$\lambda(\tilde{a}_i)$ is the coefficient of a_i , when a is written in terms of the basis $\{a_1, \ldots, a_n\}$.

Therefore we have:

(5.25) Proposition: Let R, K, A, $\Lambda = Ra_1 \oplus \ldots \oplus Ra_n$ verify the conditions in (5.22). Suppose, further, that A is symmetric and $f: A \times A \to K$ is a non-degenerate associative symmetric bilinear form which induces a non-degenerate R-integral form in Λ .

Let $\{b_i\}_{i=1,\ldots,n}$ be the basis dual to $\{a_i\}_{i=1,\ldots,n}$ with respect to f . Let $a \in A$ be such that

$$1 = \sum_{i=1}^{n} b_i \, a \, a_i$$

and suppose that $a = \alpha_1 a_1 + \ldots + \alpha_i a_i + \ldots + \alpha_n a_n \quad (\alpha_i \in K \, , \, i = 1,\ldots n)$.

Let $N \in \mathbb{N}$ be such that $\pi^N b_i . a \in \Lambda \quad \forall \, i = 1,\ldots,n$.

Suppose $M,S \in \mathrm{mod}^O \Lambda$, where $M = \bigoplus_{1}^{s} R \, m_i$.

Let $r_{ij} \in DM$ be such that

$$a_i \, m = \sum_{j=1}^{s} r_{ij}(m) m_j$$

and $\sigma_j \in DM$ be such that

$$\sigma_j(m) = \pi^N \sum_{i=1}^{n} r_{ij}(m)\alpha_i \, , \quad \forall \, m \in M \, , \, j = 1,\ldots,s \, .$$

Then

$$\forall \, \phi \in (S,M) \, , \quad U_\phi = T_{W(M)\circ\phi} \in D^*((M,S)/P(M,S)) \quad \text{is given by}$$

(5.26) $\quad U_\phi(h) = [\sum_{j=1}^{s} \sigma_j((\phi h)(m_j)) + \pi^N R]$

$\forall \, h \in (M,S)$.

§5. Case where Λ is the group ring

Let $G = \{1_G = x_1, x_2, \ldots x_n\}$ be a finite group of order $n > 1$.

Suppose that R is a complete discrete rank 1 valuation ring with maximal ideal $P = (\pi)$, such that the characteristic of its field of fractions K does not divide n. Let $n = u.\pi^N$ where u is a unit in R and $N \in \mathbb{N}$.

If the characteristic of the field of fractions K of R does not divide n, by Maschke's Theorem (see [CRM] pg.42), KG is semisimple, i.e. $\text{rad } KG = 0$.

Then KG is separable (see [CRM] Theorem 7.10, pg. 147).

(5.27) Example: If π is a fixed prime, consider the ring of π-adic integers i.e. the subring of Q consisting on all rational numbers a/b such that $\pi \nmid b$. Let R be the "complete ring of π-adic integers" (see [D] pg. 316, 317). Then K is an extension of Q so has characteristic zero.

It is well-known that KG is a symmetric algebra, with

$$f : KG \times KG \to K$$
$$\left(\sum_{x \in G} a_x x, \sum_{x \in G} b_x x \right) \to \sum_{xy=1} a_x . b_y \quad .$$

Clearly f induces a non-degenerate R-integral form in $\Lambda = RG$.

Thus conditions of (5.25) are verified.

We can take $\{1 = x_1, x_2, \ldots x_n\} = \{a_i\}_{i=1,\ldots,n}$ and then $\{1_G = x_1^{-1}, x_2^{-1}, \ldots x_n^{-1}\} = \{b_i\}_{i=1,\ldots,n}$.

Since $1 = \frac{1}{|G|} \sum_{x \in G} x.x^{-1}$ then $a = \frac{1}{|G|} 1_G$.

Thus

$$\alpha_1 = \frac{1}{|G|}, \ \alpha_2 = 0, \ldots, \alpha_n = 0 \quad .$$

Since $n = u.\pi^N$, $N \in \mathbb{N}$ is such that $\pi^N b_i.a \in RG \quad i = 1, \ldots, n$.

Then if $M = \bigoplus_{i=1}^{s} Rm_i$, let $\sigma_j \in DM$ be such that

$$\sigma_j(m) = \pi^N r_{1j}(m) \frac{1}{|G|}$$

where the r_{1j} are such that

$$1.m = \sum_{j=1}^{s} r_{1j}(m).m_j \ .$$

i.e. $\{r_{1j}\}_{j=1,\ldots,s}$ is a basis of DM dual to $\{m_j\}_{j=1,\ldots,s}$.

Thus

$$r_{1j} = m_j^* \text{ and } \sigma_j = \frac{\pi^N}{|G|} m_j^* \quad .$$

Thus

$$\forall\, h \in (M,S),\ U_\phi(h) = [\sum_{j=1}^{s} \frac{\pi^N}{|G|} m_j^*((\phi h)(m_j)) + \pi^N R] =$$

$$= \frac{\pi^N}{|G|} [\sum_{j=1}^{s} m_j^*((\phi h)(m_j)) + \pi^N R] =$$

$$= \frac{\pi^N}{|G|} [\mathrm{Tr}(\phi h) + \pi^N R] \quad \text{where } \mathrm{Tr}(\phi h) \text{ is the}$$

trace of the endomorphism ϕh of M.

Thus:

(5.28) <u>Theorem</u>: Let G be a finite group, R a complete discrete rank 1 valuation ring with maximal ideal $P = (\pi)$, such that the characteristic of its field of fractions K does not divide $|G|$. Let $N \in \mathbb{N}$ be such that $|G| = \pi^N.u$ (u is a unit in R). Let $M = \bigoplus_1^s Rm_j$, $S \in \mathrm{mod}^o RG$.

Then for each $\phi \in (S,M)$, $U_\phi = T_{W(M)\circ\phi} \in D^*((M,S)/P(M,S))$ is given by

$$(5.29) \qquad U_\phi(h) = \frac{\pi^N}{|G|} [\mathrm{Tr}(\phi h) + \pi^N R] \quad \forall\, h \in (M,S). \qquad \square$$

§6. <u>Examples</u>

We end this chapter by considering some examples of application of theorem (5.28) to the construction of almost split sequences.

(i) Let p be a fixed prime.

Let R be a complete discrete valuation ring with maximal ideal πR, such that $p \in \pi R$.

Let G be a p-group with $|G| = n = \pi^N.u > 1$ where u is a unit in R, and $N \in \mathbb{N}$.

Let $R_G = R$, be the trivial RG-module i.e. R with the action: if $g \in G$, $\lambda \in R$ then $g\lambda = \lambda$.

R is an indecomposable non-projective RG-module (because $n > 1$).

Consider the following projective presentation of R:

$$(5.30) \qquad 0 \longrightarrow \Omega R \xrightarrow{i} \Lambda = RG \xrightarrow{\varepsilon} R \to 0$$

where

$\varepsilon(\sum_{x\in G} r_x x) = \sum_{x\in G} r_x$ is the augmentation map ([CRM], pg.189),

i is the inclusion map and

$$\Omega R = \operatorname{Ker} \varepsilon = \{ \sum_{x\in G} r_x x : \sum r_x = 0\} = \bigoplus_{x\in G-\{1\}} R(x-1) \quad .$$

Since G is a p-group, $\Omega R \subsetneq \operatorname{rad} RG = \pi G + (\bigoplus_{x\in G-\{1\}} R(x-1))$ ([CRM] pg.115), thus (5.30) is minimal.

Now we must find $\phi \in \operatorname{End}_{RG}R$ such that:

(1) $U_\phi(\operatorname{End}_{RG}R) \neq 0$.

(2) U_ϕ (rad $\mathrm{End}_{RG}R$) $\neq 0$

where U_ϕ is given by (5.29), i.e.

$$U_\phi(h) = \frac{\pi^N}{|G|}[\mathrm{Tr}(\phi h) + \pi^N R] = u^{-1}.[\mathrm{Tr}(\phi h) + \pi^N R]$$

$\forall\ h \in \mathrm{End}_{RG}R$.

Since $\mathrm{End}_{RG}R = \{\lambda.1_R : \lambda \in R\}$, condition (1) is equivalent to $U_\phi(1_R) \neq 0$.

But $U_\phi(1_R) = u^{-1}[\mathrm{Tr}\ \phi + \pi^N R] = u^{-1}[\phi(1) + \pi^N R]$ so $U_\phi(1_R) \neq 0$ iff $\phi(1) \notin \pi^N R$.

One has $\pi R \subseteq J(R)$ and $R/\pi R$ is a division ring, so $\pi R = J(R)$; since $\mathrm{End}_{RG}R \cong R$, rad $\mathrm{End}_{RG}R \cong \pi R$ so condition (2) is equivalent to $U_\phi(\pi.1_R) = 0$.

But $U_\phi(\pi.1_R) = u^{-1}[\pi\phi(1) + \pi^N R] = 0$ iff $\phi(1) \in \pi^{N-1}R$.

Thus we may take $\phi = \pi^{N-1}\ 1_R$.

Now consider the pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{x\in G-\{1\}} R(x-1) \arrow[r, "i"] & RG \arrow[r, "\varepsilon"] & R \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{x\in G-\{1\}} R(x-1) \arrow[r, "f"'] \arrow[u, "||"] & F \arrow[r, "g"'] \arrow[u, "\ell"] & R \arrow[r] \arrow[u, "\phi = \pi^{N-1}1_R"'] & 0
\end{tikzcd} \tag{5.31}$$

with $F = \{(r, \sum_{x\in G} r_x x) \in R \oplus RG \mid \pi^{N-1} r = \sum_{x\in G} r_x\}$

$\cong \{ \sum_{x\in G} r_x x \mid \sum r_x \in \pi^{N-1} R\}$ (since R is an integral domain)

$= (\bigoplus_{x\in G-\{1\}} R(x-1)) \oplus R\pi^{N-1}.1$.

Then (5.31) is an almost split sequence.

The middle term, F, is indecomposable. To show this we start by proving the

(5.32) <u>Lemma</u>: Let $a \in KG$ and let $\rho_a : KG \to KG$ denote the "right multiplication by a". Then

$$\mathrm{End}_{RG} F = \{\rho_a|_F : a \in C\}$$

where

$$C = R\pi E_1 \oplus (\sum_{x\in G-\{1\}} Rx) \text{ , with } E_1 = \frac{1}{n} \sum_{x\in G} x \text{ .}$$

<u>Proof</u>: The R-basis $\{\pi^{N-1} 1 , x-1 \mid x \in G-\{1\}\}$ of F is a K-basis for KG (where K is the field of fractions of R), so

$$K \otimes_R F \cong KG \otimes_{RG} F \cong KG \text{ .}$$

Therefore any RG-endomorphism of F can be extended to a KG-endomorphism of KG . In other words:

$$\mathrm{End}_{RG}F = \{f|_F : F \to F \mid f \in \mathrm{End}_{KG}KG \text{ and } f(F) \subseteq F\} .$$

It is well known that $\mathrm{End}_{KG}KG = \{\rho_a : a \in KG\} \cong (KG)^{op}$. (From now on, $\rho_a|_F$ will be written ρ_a , if $Fa \subseteq F$) . Then:

$$\mathrm{End}_{RG}F = \{\rho_a : a \in KG \text{ and } Fa \subseteq F\} .$$

But

$Fa \subseteq F \iff ba \in F$, $\forall\, b \in F \iff ba \in RG$ and $\varepsilon(ba) \in \pi^{N-1}R$ (see definition of F) , $\forall\, b \in F$.

Since $\varepsilon(ba) = \varepsilon(b).\varepsilon(a)$ and $\varepsilon(b) \in \pi^{N-1}R$, then $\varepsilon(ba) \in \pi^{N-1}R$, $\forall\, b \in F$, iff $\varepsilon(a) \in R$.

It is enough to consider the condition $ba \in RG$ whenever b is an element of the basis of F .

Thus if $a = \sum_{x\in G} a_x.x$, $a_x \in K$, we must have:

$$\pi^{N-1}a = \sum_{x\in G} \pi^{N-1}a_x.x \in RG \iff \pi^{N-1}a_x \in R , \ \forall\, x \in G$$

and

$$\forall\, g \in G,\ (g-1)a = \sum_{x\in G} a_x(gx-x) = \sum_{y\in G} a_{g^{-1}.y}(y-g^{-1}.y) =$$

$$= \sum_{y\in G} a_{g^{-1}.y}\,y - \sum_{y\in G} a_y.y = \sum_{y\in G} (a_{g^{-1}.y}-a_y)y \in RG \iff a_{g^{-1}.y}-a_y \in R , \ \forall\, g,y \in G .$$

Then: $\mathrm{End}_{RG}F = \{\rho_a : a \in C\}$, where

$$C = \{a = \sum_{x\in G} a_x . x \in KG \mid \pi^{N-1}a_x \in R \text{ and } a_x - a_y \in R\ ,\ \forall\ x,y \in G\}\ .$$

Let $a = \sum_{x\in G} a_x . x \in C$. Then $a = \sum_{x\in G} a_1 x + \sum_{x\in G-\{1\}} (a_x - a_1)x$.

By definition of C , $a_x - a_1 \in R$ and $\pi^{N-1}a_1 = r_1' \in R$,
thus $a_1 = \frac{\pi}{\pi^N} r_1' = \frac{\pi u}{\pi^N u} r_1' = \pi . \frac{1}{n}(ur_1') = \pi . \frac{1}{n} r_1$ with $r_1 = ur_1'$,
so $\sum_{x\in G} a_1 x = \pi\, r_1 (\frac{1}{n} \sum_{x\in G} x)$.

Let $E_1 = \dfrac{\sum_{x\in G} x}{n}$, $r_x = a_x - a_1$. Then

(5.33) $\quad a = r_1\, \pi E_1 + \sum_{x\in G-\{1\}} r_x . x$.

Conversely, given an element a with an expression of the form (5.33), it is trivial to see that it belongs to C .

Since the expression (5.33) of an element of C is unique, we can write:

(5.34) $\quad C = R\,\pi\, E_1 \oplus (\sum_{x\in G-\{1\}} Rx)$. $\square$

We also have:

(5.35) C is an R-order in KG with $\{\pi\, E_1, x \mid x \in G-\{1\}\}$ as an R-basis.

Observe that $1 \in C$ has the following expression in terms of this R-basis:

$$1 = \frac{|G|}{\pi} (\pi E_1) - \sum_{x \in G-\{1\}} x \quad .$$

Since $\mathrm{End}_{RG} F$ is anti-isomorphic to C, to prove that F is indecomposable it is enough to prove that C is local.

Let k be the residue field $R/\pi R$, and let J be the ideal $(\pi E_1)C + \pi C$ of C.

Then C/J is a k-algebra with basis

$$\{x + J \mid x \in G-\{1\}\}$$

and multiplication given by

$$(x+J)(y+J) = xy + J \quad \text{if} \quad x,y,xy \in G-\{1\}$$

$$(x+J)(x^{-1}+J) = -\sum_{y \in G-\{1\}} (y+J), \quad \forall\, x \in G-\{1\} \quad .$$

Consider the k-algebra kG/T where $T = k(\sum_{x \in G} x)$.

It has the basis $\{x+T : x \in G-\{1\}\}$ and multiplication

$$(x+T)(y+T) = xy + T \quad \text{if} \quad x,y,xy \in G-\{1\}$$

$$(x+T)(x^{-1}+T) = 1+T = -\sum_{y \in G-\{1\}} (y+T) \quad .$$

Thus clearly

$$C/J \cong kG/T$$

and this is a local algebra because kG is local and $T \leq \text{rad } kG$

(since $\sum_{x\in G} x = \sum_{x\in G-\{1\}} (x-1) + \pi^N$, and $N \neq 0$) .

Also $J \leq \text{rad } C$ (since $J^2 \subseteq \pi C \subseteq \text{rad } C$) so C/J local implies C local.

Thus F is indecomposable.

(ii) Suppose R,G verify the conditions of (i) with $G = \langle x\rangle$, a cyclic p-group of order $n = \pi^N u$ with $N \geq 2$.

Then

$$F = \left(\bigoplus_{i=1}^{n-1} R(x^i-1)\right) \oplus R\pi^{N-1}.1 = RG(x-1) + RG\,\pi^{N-1}$$

is indecomposable by (i) and non-projective (because $N \geq 2$) .

Let $\gamma:(RG)^2 \longrightarrow F = RG(x-1) + RG\,\pi^{N-1}$ be such that

$$(1,0) \longrightarrow x-1$$
$$(0,1) \longrightarrow \pi^{N-1} \quad .$$

Then

$$\gamma\left(\sum_{i=0}^{n-1} a_i.x^i, \sum_{j=0}^{n-1} b_j x^j\right) = \sum_{i=0}^{n-1} a_i x^i (x-1) + \sum_{j=0}^{n-1} \pi^{N-1} b_j x^j =$$

$$= \sum_{j=0}^{n-1} \pi^{N-1} b_j + \sum_{i=1}^{n-1} (a_{i-1} - a_i + \pi^{N-1} b_i)(x^i-1)$$

and

$$\ker \gamma = \{ (\sum_{i=0}^{n-1} a_i x^i , \sum_{j=0}^{n-1} b_j x^j) : \sum_{j=0}^{n-1} b_j = 0 \; ; \; a_{i-1} - a_i + \pi^{N-1} b_i = 0 ,$$

$i = 1, \ldots, n-1 \}$.

Now consider formula (5.29)

$$U_\phi(h) = [\mathrm{Tr}(\phi h) + \pi^N R], \quad \forall\, h \in \mathrm{End}_{RG} F \quad .$$

We know that

$$\mathrm{End}_{RG} F = \{ \rho_a : a \in R \pi E_1 \oplus (\sum_{u \in G - \{1\}} R x) \}$$

where $E_1 = \frac{1}{\pi^N} (\sum_{i=0}^{n-1} x^i)$ (see (5.32)) .

With respect to the basis $\{x^i - 1, \pi^{N-1} 1 : i = 1, \ldots, n-1\}$ of F ,

$$\rho_{\pi E_1} \quad \text{has matrix} \quad Y = \begin{pmatrix} & & 1 \\ & & 1 \\ & 0 & \vdots \\ & & 1 \\ & & \pi \end{pmatrix}_{n \times n}$$

and

$$\rho_x \quad \text{has matrix} \quad X = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 & -1 & \pi^{N-1} \\ 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 1 \end{pmatrix}_{n \times n}$$

Thus the elements of $End_{RG}F$ are such that its matrix with respect to the given basis of F is

(5.36) $\Phi = r_1X + r_2X^2 + \ldots + r_{n-1}X^{n-1} + sY \quad (r_i, s \in R)$.

Then

$$Tr\ \Phi = Tr(r_1X) + \ldots + Tr(r_{n-1}X^{n-1}) + Tr(sY) = s\pi \quad .$$

Thus

(5.37) $\pi \mid Tr\ \phi\ , \quad \forall\ \phi \in End_{RG}F$.

Also

$$rad(End_{RG}F) \underset{(ANTI)}{\cong} rad\ C = (\bigoplus_{i=1}^{n-1} R(x^i-1) \oplus R\pi)(R\pi E_1 \oplus (\sum_{i=1}^{n-1} Rx^i))$$

is R-generated by:

$$(x^i-1)\pi E_1 = 0\ ,\ (x^i-1)x^j\ ,\ \pi^2E_1\ ,\ \pi x^j \quad (i,j = 1,\ldots,n-1)\ .$$

Then with respect to the same basis of F we can show that:

$\rho_{(x^i-1)x^j}$ has matrix $A_{ij} =$

$$\begin{array}{c} \\ j \\ \\ \\ \\ i+j \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccccc} 1 & 1 & 1 & \cdots & 1 & 1 & -\pi^{N-1} \\ 1 & 0 & 0 & \cdots & & & \\ 0 & 1 & 0 & \cdots & & & \\ & & & \ddots & & & \\ -1 & -1 & -1 & & -1 & -1 & \pi^{N-1} \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & & & \\ & & & \ddots & & & \end{array}\right)$$

and

$Tr(A_{ij}) = 0 \quad \forall\ i,j = 1,\dots,n-1$

$\rho_{\pi^2 E_1}$ has matrix $Y = \begin{pmatrix} & \pi \\ & \pi \\ 0 & \vdots \\ & \pi \\ & \pi^2 \end{pmatrix}$

and $Tr(Y) = \pi^2$

$\rho_{\pi x^j}$ has matrix $Z_j =$

$$\begin{array}{c} j \\ \\ \\ \\ \\ \end{array}\left(\begin{array}{ccccc} -\pi & -\pi & \cdots & -\pi & \pi^N \\ \pi & 0 & & & \\ 0 & \pi & & & \\ & & \ddots & & \\ & & & \ddots & \pi \end{array}\right)$$

and $Tr(Z_j) = 0\ \forall\ j = 1,\dots,n-1$

Thus

(5.38) $\quad \pi^2 \mid Tr\, h \quad \forall\ h \in rad\ End_{RG}F$.

Now suppose that with respect to the same basis of F , ϕ is given by the matrix

$$(5.39) \qquad \phi = \begin{pmatrix} & \pi^{N-2} \\ 0 & \pi^{N-2} \\ & \vdots \\ & \pi^{N-2} \\ & \pi^{N-1} \end{pmatrix}$$

Then

$$[\mathrm{Tr}(\phi\ 1_F) + \pi^N R] \neq 0 .$$

Let $h \in \mathrm{rad}\ \mathrm{End}_{RG}F$ be given by $H = [h_{ij}]_{n\times n}$.

Then

$$\phi h \text{ has matrix } \begin{pmatrix} \pi^{N-2}h_{n1} & \cdots & \pi^{N-2}h_{nn} \\ \pi^{N-2}h_{n1} & \cdots & \pi^{N-2}h_{nn} \\ \pi^{N-2}h_{n1} & \cdots & \pi^{N-2}h_{nn} \\ \pi^{N-1}h_{n1} & \cdots & \pi^{N-1}h_{nn} \end{pmatrix}_{n\times n}$$

and

$$\mathrm{Tr}(\phi h) = \pi^{N-2}(h_{n1} + h_{n2} + \cdots + h_{nn-1} + \pi h_{nn}) .$$

Now observe that $\begin{pmatrix} & 1 \\ 0 & 1 \\ & \vdots \\ & \pi \end{pmatrix} \cdot H = \begin{pmatrix} h_{n1} & \cdots & h_{nn} \\ h_{n1} & \cdots & h_{nn} \\ \pi h_{n1} & \cdots & \pi h_{nn} \end{pmatrix} \in$

$\in \mathrm{rad}\ \mathrm{End}_{RG}F$, thus $\pi^2 \mid (h_{n1} + h_{n2} + \cdots + \pi h_{nn})$ by (5.38) .

Thus $\quad Tr(\phi h) \in \pi^N R$

i.e. $\quad [Tr(\phi h) + \pi^N R] = 0 \ , \quad \forall\, h \in \text{rad}\ End_{RG} F \quad .$

Now consider the pull-back diagram where ϕ is given by (5.39):

$$\begin{tikzcd}
0 \arrow[r] & \ker\gamma \arrow[r] & (RG)^2 \arrow[r, "\gamma"] & F = \bigoplus_{i=1}^{n-1} R(x^i-1) \oplus R\pi^{N-1} \arrow[r] & 0 \\
0 \arrow[r] & \ker\gamma \arrow[r] \arrow[u, dash] & L \arrow[r] \arrow[u] & F = \bigoplus_{i=1}^{n-1} R(x^i-1) \oplus R\pi^{N-1} \arrow[r] \arrow[u, "\phi"'] & 0
\end{tikzcd} \tag{5.40}$$

Then (5.40) is an almost split sequence.

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