

## S-duality in lattice super Yang-Mills

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We present a progress report on studying S-duality in lattice  $\mathcal{N} = 4$  super Yang-Mills. This is being done through a computation of 1/2-BPS states on the Coulomb branch, especially the 't Hooft–Polyakov monopole and the W boson. Key to these calculations is the use of twisted and C-periodic boundary conditions. In addition we describe a variational method to disentangle operators with definite scaling dimension, particularly the Konishi and supergravity operators.

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## 1. Introduction

We report on certain aspects of our continuing theoretical studies of  $\mathcal{N} = 4$  super Yang-Mills (SYM) using lattice gauge theory techniques. There are many reasons for formulating and studying such theories using this first principles approach, with the goal in mind of repeating the successes of lattice quantum chromodynamics—which are quite substantial. Continuum tools such as non-renormalization theorems, holomorphy, anomaly matching and the computation of BPS protected quantities are quite powerful and have allowed for impressive progress in understanding supersymmetric field theories over several decades. However, there remain many unanswered questions. These include the nonperturbative spectrum in strongly coupled gauge theories,<sup>1</sup> holographic duality for quantities that are not BPS protected, renormalization of nonholomorphic quantities such as the Kähler potential, and many other aspects of these theories that need to be studied at a more detailed, quantitative level.

Dualities are useful because this understanding of seemingly different theories in fact unifies them under an umbrella of equivalent descriptions. In many examples there is a self-duality, where the dual theory has an identical action except that the parameters are transformed; the self-dual point for parameters is often associated with a critical point, as in the Kramers-Wannier duality ( $T \sim 1/T$ , with  $T$  temperature) of the two-dimensional (classical) Ising model. Also, supersymmetric gauge theories have a rich vacuum structure because of the extension of spacetime symmetries; complexities of the vacuum are much better understood by studying the implications of dualities, such as has been done in [1]. Global aspects of gauge theories, such as consistency constraints on line operators, are very well addressed by a concrete lattice formulation such as we describe here. Our current studies focus on a strongly coupled field theory where many exact results are available—an essential aspect since formulating supersymmetric systems on the lattice is a difficult problem.<sup>2</sup>

## 2. Status of lattice $\mathcal{N} = 4$ SYM

Over the last few years we have been studying a lattice formulation of  $\mathcal{N} = 4$  SYM that is based on a particular (Marcus) topological twist [2] of the continuum theory [3], and which is equivalent to formulating the theory through orbifolding a matrix model [4]. The covariant derivatives are chosen in such a way that spectral doubling is avoided, based on old works involving the formulation of Kähler-Dirac fermions on the lattice [5–8]. All of the numerical tests that we have performed on the theory show that there is no sign problem for the fermion measure (which is a pfaffian in this case, rather than a determinant), provided we use antiperiodic boundary conditions for the fermions and the 't Hooft coupling is not too large [9].

We have studied the renormalization of the theory and have shown that no new relevant or marginal operators are generated in the flow to long distances [10]. Instead, the coefficients of the

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<sup>1</sup>On the Coulomb branch of  $\mathcal{N} = 4$  SYM, the theory is gapped and there will be a particle spectrum. In fact the BPS saturated states are quite interesting to us in testing S-duality, and will be described below. One of the chief goals of our research is to verify a continuum formula for the spectrum of particles that is supposed to be exact—Eq. (3.1).

<sup>2</sup>Lattice discretization necessarily breaks supersymmetry at the scale of the lattice spacing because the full supersymmetry algebra closes on the generators of infinitesimal spacetime translation. To recover supersymmetry in the continuum limit, it must emerge as a symmetry at long distances, just like Lorentz invariance.

terms that are already in the action will be modified from their tree-level values. (To avoid one new relevant operator, we invoke our results regarding moduli space not being lifted at any finite order in perturbation theory [10], and the assumption that this will also be the case nonperturbatively [11].) We have found that the  $\beta$  functions for these coefficients all vanish at one loop in lattice perturbation theory, which shows that the lattice theory is equivalent to the continuum theory at one loop. These coefficients all experience a logarithmic flow at higher orders because they correspond to marginal operators near the Gaussian fixed point. All of the flow is due to lattice artifacts; after taking account of the ability to redefine the fields, we find [11] that the fine-tuning necessary to restore the full  $\mathcal{N} = 4$  supersymmetry consists of tuning to a submanifold in the space of couplings with codimension 1. In this sense, the fine-tuning is no worse than for Wilson fermions in lattice QCD. This is to be compared to formulating  $\mathcal{N} = 4$  SYM with Wilson fermions, which would require eight parameters to be fine-tuned [11].

We are continuing our studies of this theory by using Monte Carlo renormalization group techniques to identify the correct tuning as a function of the lattice spacing. Initial studies of this were performed for Wilson loops in [11]. We found that the Wilson loops on a blocked fine lattice and a coarse lattice agreed with each other within errors without any fine-tuning being necessary.<sup>3</sup> We view this as a reflection of the fact that the coefficients are running very slowly due to the approximate supersymmetry of the lattice action. Further studies are underway using operators constructed out of fermions, which should provide tests that are in some sense “orthogonal” to or “independent” of those involving Wilson loops.

We have found in recent work [12] that any fine-tuning that restores a discrete version of the  $SU(4)_R$  global symmetry of  $\mathcal{N} = 4$  will automatically recover the full 16-supercharge supersymmetry.<sup>4</sup> Thus we are able to test the amount of supersymmetry breaking by measuring the difference between  $n \times n$  Wilson loops<sup>5</sup> and the corresponding loops rotated by the discrete  $R$  transformation. The violation of the  $R$  symmetry is  $\mathcal{O}(10)$  per cent [9], and we expect that this will also be the case for the other 15 supercharges  $\mathcal{Q}_a, \mathcal{Q}_{ab}$  ( $a, b = 1, \dots, 5$ ) that are not scalars in the twisted formulation. This indicates that there will ultimately be a need for some fine-tuning in order to recover the desired continuum theory.

### 3. S-duality

One of our current efforts is to study a key feature of  $\mathcal{N} = 4$  SYM, namely S-duality. We will do this by measuring part of the spectrum of  $\frac{1}{2}$ -BPS states, especially the W-boson and 't Hooft–Polyakov magnetic monopole on the Coulomb branch. This spectrum is given by

$$M_{p,q} = \nu g |p + q\tau| = \nu g \sqrt{\left(p + \frac{\theta}{2\pi}q\right)^2 + \left(\frac{4\pi q}{g^2}\right)^2} \quad (3.1)$$

Here  $p$  is the electric charge,  $q$  is the magnetic charge,  $\nu$  is the scalar field expectation value that spontaneously breaks the gauge symmetry on the Coulomb branch of  $\mathcal{N} = 4$  SYM moduli space,

<sup>3</sup>Optimization of a blocking parameter was utilized.

<sup>4</sup>This is a consequence of the exact scalar supercharge and enhanced point group symmetry that we preserve on the  $A_4^*$  lattice. A hypercubic lattice would not have this nice property.

<sup>5</sup>As described below, our link variables include scalars, and therefore transform nontrivially under the  $R$  symmetry.

and  $\tau$  is the complexified coupling:  $\tau = (\theta/2\pi) + i(4\pi/g^2)$ . In this last expression  $g$  is the gauge coupling and  $\theta$  is the parameter that described the partition function in terms of topological sectors  $Z = \sum_{\nu} e^{i\nu\theta} Z_{\nu}$ . The index  $\nu$  is the topological charge of a particular sector. The full duality group for  $SU(N)$  gauge theory (which is simply-laced) is  $SL(2, \mathbf{Z})$ . This acts on the charges of the  $\frac{1}{2}$ -BPS states (which are generically dyons) according to  $(a, b, c, d \in \mathbf{Z})$

$$\begin{pmatrix} p \\ q \end{pmatrix} \rightarrow \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}, \quad ad - bc = 1 \quad (3.2)$$

while for the complexified coupling a projective transformation is made:  $\tau \rightarrow \tau' = (a\tau + b)/(c\tau + d)$ . Verifying all of these features in our numerical simulations, to the extent that it is possible,<sup>6</sup> will provide a nonperturbative check on S-duality using a first principles approach. In particular, the W boson  $M_{1,0}$  and the 't Hooft–Polyakov monopole  $M_{0,1}$  are mapped into each other under

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.3)$$

so we will measure both of these states as a function of the coupling  $g$ . Furthermore, because the spectrum in (3.1) is BPS, it is an exact prediction which we can aim to verify numerically. Doing so will give further confidence in both the lattice techniques and the continuum arguments.

In order to perform this study, we have to push the lattice theory out onto the Coulomb branch where  $U(N) \rightarrow U(1)^N$  spontaneous gauge symmetry breaking occurs. We do this by adding a small negative mass-squared for one of the scalars and then remove it in the thermodynamic limit. The form of that mass is

$$\Delta S = -F \sum_x \text{Tr} P^2(x), \quad P = (\mathcal{U}_m^\dagger \mathcal{U}_m)_{\text{traceless}}, \quad \text{n.s.m} \quad (3.4)$$

where  $\mathcal{U}_m$ ,  $m = 1, \dots, 5$  are the  $GL(N, \mathbf{C})$  valued link fields, due to complexification of the gauge field (this, together with the 4d  $\rightarrow$  5d lift, is how scalars are incorporated in the twisted/orbifold approach). The mass term can be seen by considering the continuum limit of this expression, which is determined by the following expansion of the link fields:

$$\mathcal{U}_m = \frac{1}{a} + \mathcal{A}_m, \quad \mathcal{A}_m = A_m + iB_m \quad (3.5)$$

Here,  $A_m = A_m^i t^i$  corresponds to the gauge field (up to a subtlety regarding the sixth scalar<sup>7</sup>),  $B_m = B_m^i t^i$  are scalars, and we use anti-Hermitian generators  $t^i$  of  $U(N)$ . In order to recover the theory with the continuum symmetries, it is necessary to remove the mass term in the thermodynamic limit,  $F \sim 1/V$ , where  $V$  is the spacetime volume (we use a 4d torus in our lattice formulation).

The mass of the 't Hooft–Polyakov monopole has been computed on the lattice previously in the simpler Georgi–Glashow model [13–15]. One computes the free energy difference between

<sup>6</sup>Unfortunately, the simulations must be restricted to  $\theta = 0$  to avoid a sign problem.

<sup>7</sup>To be precise, the sixth scalar is given by  $\phi_6 = (1/\sqrt{5}) \sum_{m=1}^5 A_m$  for the  $A_4^*$  lattice that we are using in our formulation.

partition functions with twisted boundary conditions<sup>8</sup>

$$U_\mu(x + N\hat{j}) = U_\mu^*(x) = \sigma_j U_\mu(x) \sigma_j, \quad \Phi(x + N\hat{j}) = \Phi^*(x) = -\sigma_j \Phi(x) \sigma_j \quad (3.6)$$

and  $C$ -periodic boundary conditions

$$U_\mu(x + N\hat{j}) = U_\mu^*(x) = \sigma_2 U_\mu(x) \sigma_2, \quad \Phi(x + N\hat{j}) = \Phi^*(x) = -\sigma_2 \Phi(x) \sigma_2 \quad (3.7)$$

where  $j = 1, 2, 3$  and  $N$  is the number of sites in each of the spatial directions. The point is that the former boundary condition only allows odd numbers of monopoles, whereas the latter boundary condition only permits even numbers of monopoles. In the limit of large inverse temperature  $\beta \rightarrow \infty$  (this limit corresponds to extrapolating the temporal extent of the lattice to infinity), the configurations with the fewest possible monopoles dominate, and so the mass of the monopole is obtained from  $M = -\lim_{\beta \rightarrow \infty} (1/\beta) \ln(Z_{\text{tw}}/Z_C)$ . In practice one must obtain this quantity as an integral with respect to some bare lattice parameter. A scalar mass has been used in previous studies, and we will continue this practice in our own calculations (directly related to the parameter  $F$  above). Thus we obtain a finite difference equation that is to be numerically integrated:

$$M(m_{i+1}^2) - M(m_i^2) = -\frac{1}{\beta} \ln \frac{\langle \exp(-(m_{i+1}^2 - m_i^2) \sum_x \text{Tr} \Phi^2) \rangle_{m_i^2, \text{tw}}}{\langle \exp(-(m_{i+1}^2 - m_i^2) \sum_x \text{Tr} \Phi^2) \rangle_{m_i^2, C}} \quad (3.8)$$

The W boson mass is also rather involved. In this case the difficulty relates to Gauss' law on a torus: we cannot put an isolated charge on a timeslice. The way that we will circumvent this is to use the  $C$ -periodic boundary conditions described above; these project out the zero mode of the photon field  $A_0$ , which would otherwise lead to Gauss' law as a constraint equation when it is integrated in the path integral. Additionally, we must form a local gauge transformation invariant interpolating operator. This will be done by inserting the W boson operators onto Polyakov lines that wrap around the lattice:

$$C(t) = \langle \text{Tr}(\mathcal{W}_0^-(x) U_0(x + \hat{0}) U_0(x + 2\hat{0}) \cdots U_0(x + (t-1)\hat{0}) \mathcal{W}_0^+(x + t\hat{0}) U_0(x + (t+1)\hat{0}) \cdots U_0(x + (T-1)\hat{0})) \rangle \quad (3.9)$$

In addition, the operators  $\mathcal{W}_0^-(x)$  must be formed by projections to unitary gauge based on the local value of the Higgs field  $\Phi(x)$ .

#### 4. Variational analysis of scaling dimensions

In addition to studying the dualities, we also are investigating the scaling dimensions of operators. Currently our focus is on the Konishi and supergravity operators constructed from scalar fields; even this is nontrivial because the scalars are wrapped up with the gauge fields in the twisted formulation. There are essentially two methods to access the scalars. One is to use  $\mathcal{U}_a(x) \overline{\mathcal{U}}_a(x) - 1 = 2iB_a(x) + \text{quadratic}$ , based on (3.5). The other is to perform a polar decomposition  $\mathcal{U}_a(x) = H_a(x) U_a(x)$  and then take the logarithm of the Hermitian matrix,  $B_a = \ln H_a(x)$ .

<sup>8</sup>Here we specify to the gauge group  $SU(2)$  for purposes of illustration; it is known how to generalize this to  $SU(N)$ ,  $N > 2$ .

#### 4.1 Konishi operator

Here we use the interpolating operator  $\mathcal{O}_{\text{K.I.}} = \sum_{a=1}^5 \text{Tr} B_a^2$  where for comparison  $B_a$  is constructed by the two separate methods just described. The  $B_a$  fields are related to the untwisted scalars  $\phi_i$ ,  $i = 1, \dots, 6$  according to<sup>9</sup>  $B_a = \sum_{b=1}^5 P_{ab}^{-1} \phi_b$  where  $P_{ab}$  are projection operators that relate the twisted and untwisted theories. Then we find the interpolating operator has an untwisted interpretation of  $\mathcal{O}_{\text{K.I.}} = \sum_{a,b,c=1}^5 P_{ab}^{-1} P_{ac}^{-1} \text{Tr} \phi_b \phi_c = \sum_{a=1}^5 \text{Tr} \phi_a^2$ , where the fact that the operators  $P$  satisfy  $P^T P = 1$  has been used. Then taking into account the basic definitions of the Konishi and supergravity operators as irreducible representations of  $SO(6)_R$ ,

$$\Phi^K = \sum_{i=1}^6 \text{Tr} \phi_i^2, \quad \Phi_{ij}^S = \underline{20'}_{ij} = \text{Tr}(\phi_i \phi_j) - \delta_{ij} \frac{1}{6} \sum_{k=1}^6 \text{Tr}(\phi_k^2) \quad (4.1)$$

straightforward algebra shows that  $\mathcal{O}_{\text{K.I.}} = \frac{5}{6} \Phi^K - \Phi_{66}^S$ . Hence when we measure the dimension of the ‘‘Konishi’’ operator in the current approach, what we are actually getting is a weighted average of the supergravity operator and the Konishi operator. It is not straightforward to get around this in a way that is also lattice gauge invariant. The trick is that we need the scalar part of the ‘‘gauge field,’’  $\phi_6 = \sum_{a=1}^5 P_{5a} A_a$ . The ambiguities, and gauge dependence, have been found empirically to lead to nonsensical results if we attempt to build a ‘‘pure’’ Konishi operator, which necessarily involves  $\phi_6$ . Thus we turn to an alternative approach that is free of these problems.

#### 4.2 Variational analysis

There is a linear combination of operators  $\mathcal{O}_i(x)$  that we create on the lattice that has definite scaling dimension  $\Delta_\alpha$ :

$$\Phi_\alpha(x) = \sum_i d_{\alpha i} \mathcal{O}_i(x) \quad (4.2)$$

We define the correlation matrix  $C_{ij}(r) = \langle \mathcal{O}_i(x) \mathcal{O}_j(y) \rangle$ ,  $r = \|x - y\|$ . If the operators  $\Phi_\alpha(x)$  are primary operators of the CFT, then  $\langle \Phi_\alpha(x) \Phi_\beta(y) \rangle = \delta_{\alpha\beta} \kappa_\beta r^{-\Delta_\beta}$ . Substituting (4.2), we have

$$\sum_{ij} d_{\alpha i} d_{\beta j} C_{ij}(r) = \left( \frac{r}{r_0} \right)^{-\Delta_\beta} \delta_{\alpha\beta} \kappa_\beta r_0^{-\Delta_\beta} = \left( \frac{r}{r_0} \right)^{-\Delta_\beta} \sum_{ij} d_{\alpha i} d_{\beta j} C_{ij}(r_0) \quad (4.3)$$

Differentiating this equation w.r.t.  $d_{\alpha i}$ , we find the generalized eigenvalue problem

$$\sum_j C_{ij}(r) d_{\beta j} = \left( \frac{r}{r_0} \right)^{-\Delta_\beta} \sum_j C_{ij}(r_0) d_{\beta j} \quad (4.4)$$

Note that this only differs from the usual variational analysis in that we have replaced  $e^{-E_n(t-t_0)} \rightarrow (r/r_0)^{-\Delta_\beta}$  because we have a CFT with a spectrum of primary operators,<sup>10</sup> rather than a gapped theory with a spectrum of energy eigenvalues  $E_n$ .

<sup>9</sup>Note that here and in the following, Latin letters from the beginning of the alphabet will have range  $a, b, c = 1, \dots, 5$  while Latin letters  $i, j, k = 1, \dots, 6$ .

<sup>10</sup>In this discussion we are not on the Coulomb branch, but are instead at the superconformal point in moduli space.

We take advantage of the fact that  $C(r)$  and  $C(r_0)$  are symmetric matrices and perform the Cholesky decomposition  $C(r_0) = Q^T Q$ . Then the generalized eigenvalue problem can be rewritten as

$$(Q^T)^{-1}C(r)Q^{-1}d_\beta = \left(\frac{r}{r_0}\right)^{-\Delta_\beta} d_\beta \quad (4.5)$$

The matrix  $(Q^T)^{-1}C(r)Q^{-1}$  is symmetric, hence its eigenvalues are real.

## 5. Outlook

In forthcoming work, we will demonstrate that the lowest lying  $\frac{1}{2}$ -*BPS* states satisfy the tree level relations in the fully quantum nonperturbative theory. This will verify a prediction of S-duality, since the  $\frac{1}{2}$ -*BPS* solitons fill out a multiplet under  $SL(2, \mathbf{Z})$ . The variational approach will be exploited to disentangle the Konishi and supergravity scaling dimensions (the latter is protected at  $\Delta_S = 2$ , which is an important check on the lattice theory). Further down the road, we will fine tune the lattice to recover the full 16-supercharge supersymmetry using Monte Carlo renormalization group methods.

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