Transitory Mortality Jump Modeling with Renewal Process and Its Impact on Pricing of Catastrophic Bonds*

Selin Özen[1] *, Şule Şahin[2]

Department of Actuarial Sciences and Risk Management, Karabük University, Karabük, Turkey

Department of Mathematical Sciences, Institute for Financial and Actuarial Mathematics, University of Liverpool, Liverpool, UK & Department of Actuarial Sciences, Hacettepe University, Ankara, Turkey

Abstract

A number of stochastic mortality models with transitory jump effects have been proposed for the securitisation of catastrophic mortality risks. Most of the studies on catastrophic mortality risk modeling assumed that the mortality jumps occur once a year or used a Poisson process for their jump frequencies. Although the timing and the frequency of catastrophic events are unknown, the history of the events might provide information about their future occurrences. In this paper, we propose a specification of the Lee-Carter model by using the renewal process and we assume that the mean time between jump arrivals is no longer constant. Our aim is to find a more realistic mortality model by incorporating the history of catastrophic events. We illustrate the proposed model with mortality data from the US, the UK, Switzerland, France, and Italy. Our proposed model fits the historical data better than the other jump models for all countries. Furthermore, we price hypothetical mortality bonds and show that the renewal process has a significant impact on the estimated prices.

*Declarations of interest: none.

*Corresponding author

Email addresses: `selindegirmenci@karabuk.edu.tr` (Selin Özen[1]), `Sule.Sahin@liverpool.ac.uk` (Şule Şahin[2])

[1] Adress: Karabük University, Faculty of Business, 78000, Turkey

[2] Address: Mathematical Sciences Building, Liverpool, L69 7ZL, UK

Keywords: renewal process, mortality risks, jump-diffusion process, stochastic mortality, Merton model, catastrophic bonds

1. Introduction

Insurance companies and pension plans are exposed to the risk of uncertainty in future mortality. This risk may arise due to improvements in mortality or shocks such as catastrophic mortality events [26]. The latter is called catastrophic mortality risk, which is the risk that, over short periods of time, mortality rates are much higher than expected [9]. Due to a shorter lifetime of an individual or group than expected, an insurer or a pension plan may have to make sudden pay-outs to many policyholders. Hence, severe adverse financial consequences can potentially arise, such as breaches in regulatory solvency and capital requirements [25]. As a result, the management of catastrophic mortality risk is fundamental for insurance companies and pension plans.

Catastrophic events, such as infectious diseases/pandemics, natural disasters, terrorist attacks, wars, and accidents, may cause sudden increases in mortality curves, which are called mortality jumps. For instance, the Spanish flu virus killed 40 to 50 million people in 1918 and caused a huge jump in mortality rates. More recently, avian flu in 2006 and the Ebola virus in 2014 caused approximately 1 million deaths [3].

According to statistics from the Emergency Events Database (EM-DAT), the frequency, magnitude, and duration of natural disasters have increased since 1975. The World Disasters Report in 2016 stated that rising global temperatures caused global climate change and more natural disasters. These climate changes and natural disasters led to catastrophic events that caused many diseases and deaths in recent years. In the 1970s there were roughly 100 catastrophic events per year. This number has consistently increased more than three times in the last decade. Between 1994 and 2013, the EM-DAT recorded 6,873 natural disasters that claimed 1.35 million lives on average each year. Furthermore, in 2018, there were 348 climate-related and geophysical disaster events recorded in International Disaster Database reports and 68 million people were affected around the world.

The occurrence of catastrophic events could cause a large number of deaths and hence a large number of unexpected death claims. Consequently, the financial impacts of catastrophic events on an insurer's solvency require effective risk management to eliminate and reduce the risk [27]. In the United States, the three largest natural disasters recorded before Hurricane Katrina in 2005 caused a total insured loss of $23 billion and a few reinsurers went insolvent to pay claims [56]. Moreover, a worst pandemic could result in approximately €45 billion of additional claim expenses in Germany according to the estimations of Stracke and Heinen (2006). This amount is equivalent to 100% of the policyholder bonus reserves in the German life insurance market. Some public health experts think that a pandemic is overdue and another will inevitably occur due to the nature of inter-species transmission, intra-species variation, and altered virulence [3].

The frequency of catastrophic events and the degree to which they are accurately priced are serious concerns in managing extreme mortality risks. In recent years, catastrophic bonds have been used by insurers as a risk management tool. The first catastrophic bond was issued by Swiss Re, called Vita I, in 2003 to reduce the impact of catastrophic events. Due to the great success of that bond, many other catastrophic mortality bonds are now being issued (see [6], [5]). Several stochastic models have been developed to capture these jump effects in mortality and to value catastrophic bonds. These models differ in the type of mortality jumps and the severity of jumps. For instance, Cox et al. (2006) combined a geometric Brownian motion and compound Poisson process to model age-adjusted rates. Cox et al. (2006) modelled permanent mortality jumps by considering Poisson jump counts. Chen and Cox (2009) used a normal distribution for jump severity, while Chen et al. (2010) combined two types of jumps in their model. Similarly, Deng et al. (2012) considered the mortality time index as a double-exponential jump process. In contrast to those studies, Liu and Li (2015) investigated the age pattern of jump effects on mortality.

All of these mentioned jump models in the literature assumed that mortality jumps occur once a year, or they used a Poisson process for their jump frequencies. Due to their low probability and high impact nature, the timing and the frequency of future catastrophic events and hence mortality jumps are unpredictable [12]. On the other hand, the history of events can give information about their future occurrences. In the Poisson process, inter-arrival times between events are independent and exponentially distributed. However, the Poisson process has a limitation arising from the memoryless property of the exponential distribution. In this paper, we aim to include the history of catastrophic events. One way to incorporate the history of the events is to use *duration dependence models*. Instead of the constant hazard function, these models have time-varying hazard functions. This property is important for duration analysis since the hazard function is used to capture the duration dependence. The hazard function reflects the waiting times between events. For instance, an increasing hazard function represents longer waiting times between events compared to a decreasing hazard function. In these models, events are dependent in the sense that the arrival of at least one event (in contrast to none) up to time t influences the probability of a further arrival in $t + \Delta t$. There is thus a link between the counting model and timing process. This class is known as renewal processes [28].

Winkelmann (1995) was the first to derive a counting process by using the renewal process with gamma distributed inter-arrival times. Many other models were derived by using different inter-arrival times afterwards. McShane et al. (2008) used the Weibull distribution for inter-arrival times while lognormal distribution was used by Bradlow et al. (2002), Everson and Bradlow (2002), and Miller et al. (2006) [31].

In this paper, we propose a new approach for modeling the arrivals of mortality jumps. Inter-arrival time implies the time between two jumps, and we use the renewal process for modeling. For this purpose, we detect jumps in the mortality time series and perform statistical tests for inter-arrival times of mortality jumps to show that we can use the renewal process as a counting process.

After that, we use the Lee-Carter model with a jump-diffusion process to model mortality and the lognormal renewal process to model jump count probabilities. We test our model with historical data and compare the goodness of fit of the models with jump sizes and jump count processes for the US, the UK, Switzerland, Italy, and France. To the best of the authors' knowledge, using the renewal process for jump counts is new in mortality modeling.

It is reasonable to assume that the renewal process has an impact on the pricing of catastrophic mortality bonds. To verify this impact, we use our mortality model to price a catastrophic mortality bond. In an incomplete market, the pricing problem is not explicit, although it might be met by no-arbitrage methods ([9]; [12]; [35]; [38]), insurance-based methods ([13]; [51]) or economic methods [54].

The no-arbitrage approach was often used in previous research on pricing mortality-linked securities. In the no-arbitrage condition, the market price of risks cannot uniquely be identified. As a result, an arbitrary assumption is necessary for pricing. One might also use canonical valuation to create a risk-neutral probability measure. Canonical valuation can be applied without making any arbitrary decisions [55]. For this reason, we use canonical valuation to create a risk-neutral probability measure and to obtain mortality risk premiums, which was first proposed by Stutzer (1996) and then applied to the market for insurance-linked securities by Chen et al. (2013), Li (2010), and Li and Ng (2010). In this paper, we use the Swiss Re mortality bond as a martingale constraint. This method identifies a risk-neutral probability measure, and thus the price of the hypothetical mortality bonds can be estimated. By using a risk-neutral measure, we price the hypothetical bonds in an incomplete market.

The rest of the paper is organized as follows. Section 2 defines the renewal count process. In Section 3, mortality data are presented. Section 4 provides the specifications of the proposed model and the statistical analysis of mortality jumps. Section 5 demonstrates a numerical example of pricing mortality-linked security. Finally, Section 6 concludes the paper.

2. Renewal Process

Since we want to include the history of the events in our jump frequency model, we need to use a renewal process. A renewal process is a stochastic model for events that occur randomly in time (generally called renewals or arrivals). The times between the successive arrivals are independent and identically distributed and the renewal process might be used as a foundation for building more realistic models.

A counting process $\{N_t\}_{t\geq 0}$ is a renewal process with independent and identically distributed nonnegative inter-arrival times. The times between successive events are called waiting times or inter-arrival times, and for a renewal process we have $N(t) = \max\{n : S_n \leq t\}$ and $N(t) < \infty, t \geq 0$, where $S_0 = 0, S_n = Z_1 + Z_2 + ... + Z_n$ for $n \geq 0$. In a renewal process $Z_1, Z_2, ...$ are independent inter-arrival times with a common distribution F. The process is called a renewal process because it starts over for each arrival period. The distribution function of the count process might be determined by using the inter-arrival times because they are independent and identically distributed. Let F be the distribution function of Z_i and F_n be the distribution function of S_n. Since $S_n = \sum_{i=1}^{n} Z_i$ and Z_i are distributed as F, the distribution function of arrival times might be obtained by:

$$F_n(t) = P(S_n \leq t), \quad t \in [0, \infty). \tag{1}$$

By using the relationship between arrival times and count process, $\{S_n \leq t\} = \{N(t) \geq n\}$, distribution function of count process might be determined in terms of the distribution function of inter-arrival times. For $t \geq 0$ and $n \in \mathbf{N}$ [45]:

- $P(N(t) \geq n) = F_n(t)$,
- $P(N(t) = n) = F_n(t) - F_{n+1}(t)$.

$F_n(t)$ is n-fold convolution of the common inter-arrival time distribution and we can obtain renewal count probabilities by using $R_n(t) = F_n(t) - F_{n+1}(t)$. We need to evaluate convolutions of the form $\int_0^t F(t-s)f(s)ds$. To solve this integral, we need to obtain the following recursive relationship:

$$R_n(t) = \int_0^t = F_{n-1}(t-s)f(s)ds - \int_0^t F_n(t-s)f(s)ds. \tag{2}$$

$$= \int_0^t R_{n-1}(t-s)f(s)ds$$

We note that $F_0(t) = 1$ for all t and $F_1(t) = F(t)$. Then, we have $R_0(t) = F_0(t) - F_1(t) = 1 - F(t)$, which leads us to the survival function. Using Equation (2), we can compute $R_1(t)$:

$$R_1(t) = \int_0^t R_0(t)f(s)ds.$$

and we can obtain $R_n(t)$ probabilities by using the recursive formula [43].

Let us now assume that $m(t)$ is the expected value of the process, $m(t) = E[N(t)]$ and can be expressed in terms of F as:

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \tag{3}$$

which is used to calculate the expected number of arrivals per period.

3. Data Description

We use mortality data for the US, the UK, Switzerland, France, and Italy. The US mortality data is obtained from the National Center for Health Statistics (NCHS) for the period of 1900-2017 for all ages. The data from the UK, Switzerland, and France are obtained from the Human Mortality Database (HMD) for the period of 1922-2016 for all ages. The Italian mortality data is obtained from the HMD for the period of 1922-2014 for all ages. The data are arranged in 10-year age intervals as follows: $< 1, 1-4, 5-14, 15-24, ..., 75-84, 85+$.

4. Transitory Mortality Jump Modeling with Renewal Process

4.1. The Lee-Carter Model

In the Lee-Carter model [34], $m_{x,t}$ denotes the central death rate of age group x in year t. The model is expressed as

$$\ln(m_{x,t}) = a_x + b_x k_t + e_{x,t},$$

where a_x is an average of $\ln m_{x,t}$ over time t and $\exp(a_x)$ represents the general shape of the mortality rates. Mortality time index k_t which captures the variation of log mortality rates over time, is modulated by an age response b_x that represents how slowly or rapidly mortality at each age varies when the mortality index changes [20]. $e_{x,t}$ is the error term, which captures the age-specific effects not reflected in the model. Parameters of the model can be estimated by using a two-stage singular value decomposition (SVD) or the maximum likelihood method. As indicated by Brouhns et al. (2002), estimation results from both methods are almost the same. We use the SVD method with the following constraints:

$$\sum_x b_x = 1 \quad and \quad \sum_t k_t = 0.$$

Then a_x is the average value of $ln(m_{x,t})$ over time and given as:

$$a_x = \frac{1}{T}\sum_{t=1}^{T} \ln(m_{x,t}),$$

where T is the length of the time series of mortality data. The SVD method is applied to the matrix of $ln(m_{x,t}) - a_x$ to obtain the estimates of b_x and k_t. In the second stage, the time-varying terms are re-estimated by iteration, given the values of a_x and b_x. This makes the actual sum of death at time t equal the implied sum of deaths at time t:

$$D_t = \sum_x (P_{x,t} \exp(a_x + b_x k_t)),$$

where D_t is the actual sum of deaths at time t, and $P_{x,t}$ is the population in age group x at time t.

By implementing the SVD two-stage procedure on historical US, UK, Swiss, Italian, and French mortality data for their time periods, we obtain the fitted a_x and b_x values given in Table 1 and the time-varying mortality index k_t as in Figure 1. The decreasing trend of the time-varying mortality index k_t shows the improvement of mortality over time for all countries. Moreover, sudden increases that cause mortality jumps in the 1910s and 1970s may be seen from

Figure 1.

Table 1 Fitted Values of a_x and b_x From the Lee-Carter Model

	US		UK		Switzerland		Italy		France	
Age Group	a_x	b_x	a_x	b_x	a_x	b_x	a_x	b_x	a_x	b_x
< 1	-3.593	0.146	-4.022	0.147	-4.204	0.133	-3.662	0.151	-3.947	0.152
$1-4$	-6.450	0.192	-6.968	0.193	-7.033	0.165	-6.520	0.206	-6.828	0.172
$5-14$	-7.401	0.153	-7.877	0.152	-7.846	0.136	-7.621	0.136	-7.781	0.135
$15-24$	-6.399	0.096	-7.037	0.106	-6.892	0.094	-6.785	0.106	-6.677	0.109
$25-34$	-6.085	0.097	-6.737	0.097	-6.678	0.094	-6.542	0.099	-6.344	0.103
$35-44$	-5.575	0.082	-6.105	0.077	-6.194	0.085	-6.061	0.079	-5.793	0.078
$45-54$	-4.858	0.061	-5.208	0.063	-5.341	0.073	-5.276	0.057	-5.036	0.057
$55-64$	-4.095	0.051	-4.285	0.051	-4.442	0.067	-4.414	0.046	-4.230	0.053
$65-74$	-3.317	0.048	-3.367	0.047	-3.514	0.066	-3.475	0.047	-3.492	0.057
$75-84$	-2.483	0.043	-2.459	0.041	-2.533	0.056	-2.470	0.043	-2.552	0.053
> 85	-1.661	0.029	-1.589	0.026	-1.575	0.031	-1.549	0.029	-1.597	0.032

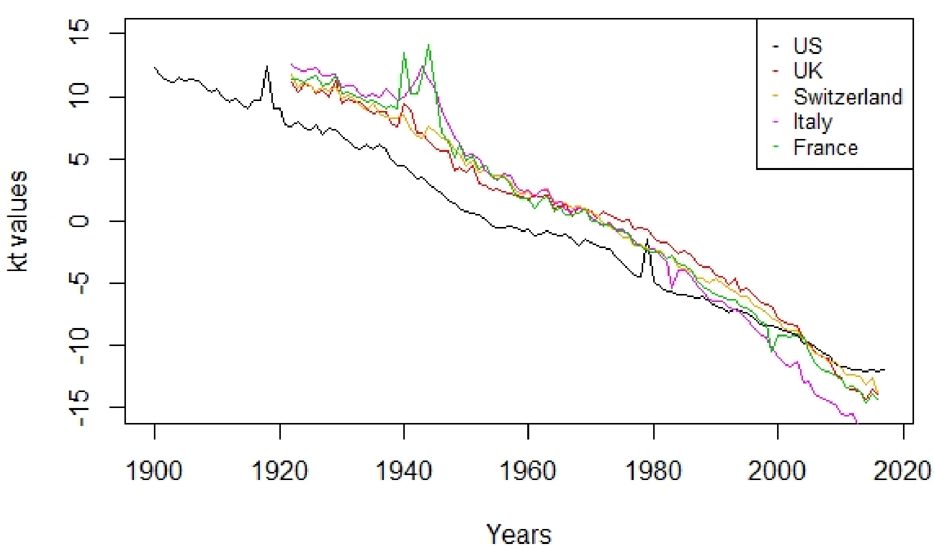

Fig.1. Estimation of k_t for all Countries.

After estimating the parameters of the Lee-Carter model, we need a model to capture the features of the shape, trend, and jumps of the time-varying mortality index k_t. Although the Lee-Carter model is a long-term mortality model, its time-varying mortality index should capture these short-term effects for effective mortality modeling. In the original Lee-Carter model, the k_t parameters are modeled by using a random walk with drift. However, we can model k_t as a stochastic process to deal with the uncertainty over mortality trends. Moreover, k_t includes both positive and negative values and geometric Brownian motion does not fit the process since it does not generate a negative value from the positive starting value. Therefore, we will use standard Brownian motion. Besides, we need to model k_t as a jump-diffusion model due to the existence of transient mortality jumps in Figure 1. We need to choose an appropriate jump-diffusion process to capture the shape, the trend, and the jumps of the time-varying mortality index k_t.

The descriptive statistics of $\Delta k_t = k_{t+1} - k_t$ indicate a leptokurtic distribution for all countries, as shown in Table 2.

Table 2 Skewness of Δk_t for all countries.

	US	UK	Switzerland	Italy	France
Skewness	−0.598	−1.061	−1.197	−1.237	−0.427

The Δk_t distributions are skewed to the left and have higher peaks and heavier tails than a normal distribution, as shown in Figure 2. Therefore, we need to consider a heavy-tailed distribution instead of the normal distribution for the jump severities.

We model k_t as a Merton jump-diffusion model to include the leptokurtic features of Δk_t. The model is specified as follows (Merton, 1976): [42]

$$dk_t = \mu dt + \sigma W_t + d(\sum_{i=1}^{N(t)} (V_i - 1)), \tag{4}$$

where W_t is standard Brownian motion, $N(t)$ is a counting process and V_i is a squence of the independent and identically distributed nonnegative variables that represent the sizes of the jumps.

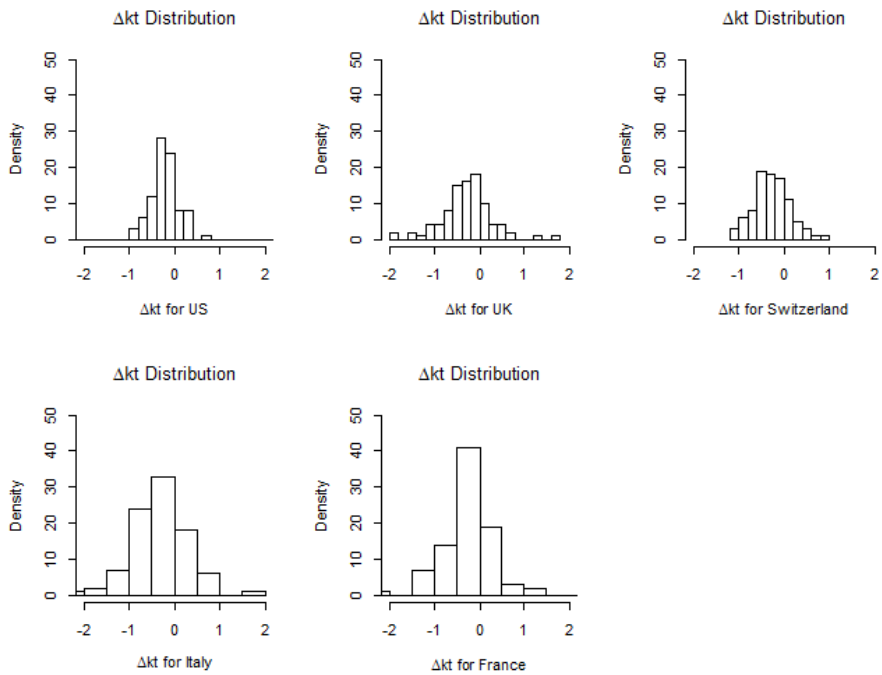

Fig. 2. Distribution of k_t for all countries.

By integrating Equation (4), we obtain (Deng et al., 2012):

$$k_t = k_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \sum_{i=1}^{N(t)} Y_i, \tag{5}$$

where Y_i is defined as $Y = log(V)$.

In the original Merton model, the jump sizes, $Y = log(V)$, are normally distributed. However, the $\Delta k_t s$ have higher peaks and heavier tails than the normal distribution for all countries. Thus, we consider that the jump sizes have exponential distribution. Although our aim is to introduce a renewal process to model the mortality jumps, we analyse different jump sizes and count processes to compare and choose the best model. Therefore, we propose four models: normal jump and Poisson process, exponential jump and Poisson process, normal jump and renewal process, and exponential jump and renewal process. The originality of our paper lies in introducing the renewal process in these models for the mortality jump counts. We also compare the jump sizes by using normal and exponential distributions.

4.2. Jump Detection Process

In order to use the renewal process, inter-arrival times between mortality jumps must be modeled. For this reason, the fitted k_t values should be obtained. After fitting the Lee-Carter model parameters, we can proceed to the outlier analysis of the mortality index, k_t, to detect the jumps in the mortality curve. We do this by using the method of Chen and Liu (1993) to search for outliers present in the mortality index and we show that the inter-arrival times between the outliers can be expressed as a renewal process. Finally, we find the distribution of the inter-arrival times of jumps to make predictions for future arrivals of outliers.

There are four types of outliers. These include an innovational outlier (IO), an additive outlier (AO), a level shift (LS), and a temporary change (TC) [14]. In this paper, we use additive outliers, which have an immediate and one-shot effect on the observed series. We obtain the test statistics by applying the method of Chen and Liu (1993). The first step of this method is to find an appropriate time series model for mortality time index and AR(1) model fits best the datasets of all selected countries we used. The outlier is detected based on the test statistics obtained from the residuals of the fitted model and we employ the critical value as 2.5 as recommended by Liu and Hudak (1994) for a reasonable level of sensitivity. The obtained outliers in the mortality index for five countries and their test statistics are given in Table 3 (see [46] for more details).

Although we adopted the same outlier detection methodology as Li and Chan (2005) and Li and Chan (2007), the detected outliers are different for the US and the UK due to the use of different datasets and different periods. These differences affected the time series models selected for the mortality time indexes. We fitted AR(1) model to both data while Li and Chan (2007) fitted ARIMA(0,1,0) to the US data and Li and Chan (2005) fitted ARIMA(0,1,1) to the England and Whales data. Therefore, the estimated parameters, residuals and the values of the test statistics for the outlier analysis are different.

Table 3 The years of the detected outliers and test statistic values

US												
Year	1901	1918	1919	1920	1921	1937	1938	1941	1974	1975	1978	1980
Test statistic	-2.6	6.9	-8.3	-3.3	-3.5	-3.1	-3.2	-2.5	-3.1	-2.7	7.3	-9.8
UK												
Year	1929	1939	1941	1942	1952	2009						
Test statistic	3.9	2.7	-3.8	-2.9	-2.6	-2.5						
Switzerland												
Year	1941	1949	2016									
Test statistic	-3.4	-2.7	-4.5									
France												
Year	1939	1941	1943	1945	1946	1947	1998	2005				
Test statistic	6.2	-4.7	5.9	-9.9	-6.2	-2.8	-3.7	-3.2				
Italy												
Year	1945	1946	1947	1982								
Test statistic	-2.9	-3.2	-2.6	-3.2								

4.2.1. Statistical Analysis of the Outliers

We need to analyse the inter-arrival times of the outliers that cause jumps in the mortality curves to show that the renewal process can be used to model jump frequencies. Several statistical tests should be performed to confirm that a renewal process is appropriate for the arrivals of the jumps. The first indicator that the process is not a Poisson process is the uniformity test and a formal statistical test on uniformity can be performed by using the Kolmogorov-Smirnov test. All obtained p-values are lower than 0.05, so we reject the uniformity and constant mean for the process [2].

We need to analyse the inter-arrival times to check if they are stationary, independent, and identically distributed. Based on Ljung-Box test statistics [30], the inter-arrival times are independent and stationary for all five countries.

After confirming that the inter-arrival times between outliers are stationary and independent, we need to determine the distribution of inter-arrival times. For this purpose, the properties of the inter-arrival times should be considered.

The distribution of inter-arrival times cannot be fitted for Switzerland and Italy since the inter-arrival time counts between the outliers are less than four. The estimated skewness coefficients of the inter-arrival times are 1.9, 2.1, and 2.6 for the US, the UK, and France respectively. Since they have positive skewness, right-skewed distributions could be considered, such as Weibull, gamma, and lognormal distributions. We fit these three distributions to the inter-arrival times and the results are presented in Table 4.

The Bayesian information criterion (BIC) values indicate that lognormal distribution fits best the inter-arrival times for three countries. Due to Switzerland and Italy having statistical properties similar to those of the other countries, we assume that their inter-arrival times follow lognormal distribution as well [46].

After deciding the distribution for the inter arrival times, we can proceed to construct and compare the mortality models.

Table 4 Fitted Results

The US	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	shape=0.78, scale=6.05	shape=0.73, rate=0.10	mean=1.15, sd=1.22
Log likelihood	-31. 74	-32.09	-30.24
BIC	68.72	69.35	65.71

The UK	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	shape=0.62, scale=9.34	shape=0.51, rate=0.04	mean=1.41, sd=1.56
Log likelihood	-16.99	-17.25	-16.37
BIC	37.20	37.71	35.95

France	Weibull Distribution	Gamma Distribution	Lognormal Distribution
Parameters	shape=0.62, scale=6.16	shape=0.50, rate=0.05	mean=1.05, sd=1.38
Log likelihood	-21.06	-21.73	-19.56
BIC	46.02	47.46	43.12

4.3. A Model with Normal Jump and Poisson Process

In the original Merton model, $N(t)$ is a Poisson process with rate λt and $Y = log(V)$ follows a normal distribution with the following density:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(y-m)^2}{2s^2}}. \quad (6)$$

We need to find the density function of the one-period increments $r_i = \Delta k_i = k_i - k_{i-1}$ to estimate parameters and make forecasts. If we consider the one-period increment for the Merton model, conditional on the event $(N_t = n)$, we can write $X = Y_1+Y_2+...+Y_n$, where $Y_i \sim N(m, s^2)$ and Y_i are independent, and then $X \sim N(nm, ns^2)$. Then we obtain the conditional density for increments, which is the sum of $N(m, s^2)$ and $N((\mu - 0.5\sigma^2), \sigma^2)$. By using the convolution technique, the density is found as $N((\mu - 0.5\sigma^2) + mn, \sigma^2 + ns^2)$ [24]. Then the unconditional density of a one-period increment is:

$$f(r) = \sum_{n=0}^{\infty} Pr(N(t) = n) f(r|n). \tag{7}$$

Let $C = k_0, k_1, ..., k_T$ denote the time-varying mortality factors at equally spaced times $t = 1, 2, ..., T$. Then the log-likelihood of the one-period increment observations is:

$$L(C; \mu, \sigma, m, s, \lambda) = \sum_{i=1}^{T} log(f(r)).$$

We estimate the parameters by maximum likelihood estimation (MLE).

4.4. A Model with Exponential Jump and Poisson Process

Δk_t has a heavier tail than a normal distribution, as shown in Figure 2. Thus, we assume the jump sizes follow an exponential distribution with the following density:

$$f_Y(y) = \eta e^{-\eta y}. \tag{8}$$

The count process is a Poisson process with rate λt. Conditional on the event $(N_t = n)$, we can write $X = Y_1 + Y_2 + ... + Y_n$, where $Y_i \sim \exp(\eta)$ and Y_i are independent, and then $X \sim \Gamma(n, \eta)$. The conditional density of X is:

$$f_X(X|n) = \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X}.$$

Now we can determine two conditional densities for the no-jump case and n-jump case. For the no-jump case the conditional density is:

$$f(r|0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-\mu+0.5\sigma^2)^2}{2\sigma^2}}.$$

For the n-jump case the conditional distribution is the independent sum of $\Gamma(n,\eta)$ and $N((\mu-0.5\sigma^2),\sigma^2)$. By convolution techniques, we obtain:

$$f(r|n) = \int_0^\infty \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-X-\mu+0.5\sigma^2)^2}{2\sigma^2}} dx$$

$$= \frac{\eta^n}{(n-1)!\sqrt{2\pi}\sigma} \int_0^\infty X^{n-1} e^{-\eta X - \frac{1}{2\sigma^2}(r-X-\mu+0.5\sigma^2)^2} dx.$$

The unconditional density of a one-period increment is [47]:

$$f(r) = P(0)f(r|0) + \sum_{n=1}^{\infty} P(n)f(r|n). \tag{9}$$

Thus, the log-likelihood function becomes:

$$L(C;\mu,\sigma,\eta,\lambda) = \sum_{i=1}^{T} log(f(r)).$$

4.5. A Model with Normal Jump and Renewal Process

Now we assume that our counting process, $N(t)$, is a renewal process and we consider the history of the catastrophic events to model k_t. Thus, the model has the information of the frequency of the events and we accomplish this by combining the Merton model and the renewal process. Jump sizes $Y = \log(V)$ follow normal distribution as in the original Merton model with density as follows:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}s} e^{-\frac{(y-m)^2}{2s^2}}.$$

Since the count process is a renewal process and conditional on $(N_t = n)$, we can write $X = Y_1 + Y_2 + ... + Y_n$, where Y_i are independent and $Y_i \sim N(m, s^2)$. Then $X \sim N(nm, ns^2)$. Considering the jumps, we obtain the conditional density for increments as the sum of $N(m, s^2)$ and $N((\mu-0.5\sigma^2),\sigma^2)$. By using the convolution technique, the density is obtained as $N((\mu-0.5\sigma^2)+mn, \sigma^2 + ns^2)$ (Gugole, 2016). Letting $R(n)$ be the renewal process which shows the probability of the nth jump, the unconditional density for $f(r)$ is:

$$f(r) = \sum_{n=0}^{\infty} Pr(N(t) = n)f(r|n).$$

$C = k_0, k_1, ..., k_T$ denote the time-varying mortality factors at equally spaced times $t = 1, 2, ..., T$. Then the log-likelihood of the one-period increment observations is:

$$L(C; \mu, \sigma, m, s, \alpha, \beta) = \sum_{i=1}^{T} \log(f(r)).$$

4.6. A Model with Exponential Jump and Renewal Process

As in Section 4.4, we assume that the jump sizes follow exponential distribution with density as follows:

$$f_Y(y) = \eta e^{-\eta y}.$$

To estimate the parameters and to make forecasts, we need to find the density function of the one-period increments, which might be shown by $r_i = \Delta k_i = k_i - k_{i-1}$. If we consider the one-period increments for the Merton model, conditional on the event $(N_t = n)$, we might write $X = Y_1 + Y_2 + ... + Y_n$, where $Y_i \sim exp(\eta)$ and Y_i are independent, and then $X \sim \Gamma(n, \eta)$. The conditional density of X is:

$$f_X(X|n) = \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X}.$$

Now we might determine two conditional densities for the no-jump case and the n-jump case. For the no-jump case, the conditional density is:

$$f(r|0) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-\mu+0.5\sigma^2)^2}{2\sigma^2}}.$$

For the n-jump case, the conditional distribution is the independent sum of $\Gamma(n, \eta)$ and $N((\mu - 0.5\sigma^2), \sigma^2)$. By using the convolution technique, we obtain:

$$\begin{aligned} f(r|n) &= \int_0^{\infty} \frac{\eta^n}{(n-1)!} X^{n-1} e^{-\eta X} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(r-X-\mu+0.5\sigma^2)^2}{2\sigma^2}} dx \\ &= \frac{\eta^n}{(n-1)!\sqrt{2\pi}\sigma} \int_0^{\infty} X^{n-1} e^{-\eta X - \frac{1}{2\sigma^2}(r-X-\mu+0.5\sigma^2)^2} dx. \end{aligned}$$

Next we derive the unconditional density of the one-period increments, $f(r)$ as:

$$f(r) = R(0) f(r|0) + \sum_{n=1}^{\infty} R(n) f(r|n). \tag{10}$$

Let $C = k_0, k_1, ..., k_T$ denote the time-varying mortality factors at equally spaced times $t = 1, 2, ..., T$. Then the log-likelihood of the one-period increment observations is:

$$L(C; \mu, \sigma, \eta, \alpha, \beta) = \sum_{i=1}^{T} \log(f(r)).$$

Based on the observations $C = k_0, k_1, ..., k_T$, we can estimate the parameters by maximising the following log-likelihood function:

$$\sum_{i=1}^{T} \log\left(R(0) f(r|0) + \sum_{n=1}^{\infty} R(n) f_(r|n)\right) \quad (11)$$

4.7. Estimation Results

Solving the jump process from the diffusion components is a serious challenge in the calibration of the underlying process. The independent increments of the underlying process are captured by the diffusion process with extreme increments captured by jumps. A calibration method is needed to generate the right parameters of low frequency and large severity for jumps. Ait-Sahalia and Hansen (2004) demonstrated that MLE has advantages in solving jumps from diffusion. Meanwhile, jump-diffusion is a linear process with independent increments and an explicit transition density, which fortunately satisfies the requirement of a complete specification of the transition density for using MLE. Therefore, we choose the MLE method to calibrate the parameters.

We estimate the parameters of time-varying mortality factors for the jump-diffusion models introduced in the previous subsections for five countries. Following Cox et al. (2006), we take maximum jump counts of 10 for a year and the estimated parameters are given in Table 5.

As shown in Table 5, the expected rate of change in the mortality factor, μ, is -0.2637 for the renewal process with exponential jumps for the US. This implies that the mortality factor decreases by 0.2637 per year on average. The negative sign of μ is consistent with the fact that the US population mortality improves over time. The volatility of the annual mortality rate of change is 0.1599 for the renewal process with the exponential jump model for the US. The average severity of jumps is equal to 0.6757 ($= 1/\eta$) in a year. Similar comments hold for

Table 5 Estimated parameters for all countries.

The US	NJ with Poisson P.	EJ with Poisson P.	NJ with Renewal P.	EJ with Renewal P.
	μ =-0.2471	μ =-0.2594	μ =-0.2415	μ =-0.2637
	σ = 0.1772	σ =0.1448	σ =0.1713	σ =0.1599
	m =0.0332	η =1.4874	m = 0.0284	η =1.4794
	s =0.5292	λ = 1.0108	s = 0.4414	α =0.0190
	λ =1.006		α = 0.00031	β =0.6051
			β =0.6030	
log(L)	-77.9282	-44.3847	-30.3421	-29.2499
BIC values	179.7098	107.8522	89.31	82.3532
The UK	**NJ with Poisson P.**	**EJ with Poisson P.**	**NJ with Renewal P.**	**EJ with Renewal P.**
	μ =-0.2369	μ =-0.2189	μ =-0.2389	μ =-0.2364
	σ =0.2874	σ =0.1204	σ =0.2195	σ = 0.1788
	m =0.0322	η =1.4794	m = -0.0355	η =1.4963
	s =0.6124	λ = 0.7435	s = 0.3640	α =0.00091
	λ =0.6277		α = 0.0029	β =0.61345
			β =0.6009	
log(L)	-72.8151	-45.6658	-37.44246	-36.7632
BIC values	164.3995	109.5471	102.2082	96.2958
Switzerland	**NJ with Poisson P.**	**EJ with Poisson P.**	**NJ with Renewal P.**	**EJ with Renewal P.**
	μ =-0.2468	μ =-0.2141	μ =-0.2345	μ =-0.2506
	σ =0.3626	σ =0.1641	σ =0.4128	σ = 0.1887
	m =0.0413	η =1.4806	m = 0.0259	η =1.4891
	s =0.1778	λ = 1.0014	s =0.0581	α =0.0118
	λ =1.0472		α = 0.0011	β =0.6127
			β =0.7038	
log(L)	-48.6356	-44.3310	-35.7192	-33.5098
BIC values	120.0406	106.8775	98.7617	89.7889
Italy	**NJ with Poisson P.**	**EJ with Poisson P.**	**NJ with Renewal P.**	**EJ with Renewal P.**
	μ =-0.2293	μ =-0.2243	μ =-0.2498	μ =-0.2232
	σ =0.3694	σ =0.0932	σ =0.3359	σ = 0.1879
	m =-0.0232	η =1.4759	m = -0.0116	η =1.4858
	s =0.4399	λ = 1.0942	s = 0.4346	α =0.0136
	λ =1.0212		α = 0.0019	β =0.6078
			β =0.6301	
log(L)	-77.9737	-56.1175	-51.0489	-47.4151
BIC values	178.6105	130.3653	129.2334	117.4931
France	**NJ with Poisson P.**	**EJ with Poisson P.**	**NJ with Renewal P.**	**EJ with Renewal P.**
	μ =-0.2057	μ =-0.2170	μ =-0.2112	μ =-0.2208
	σ =0.2246	σ =0.0970	σ =0.1863	σ = 0.1384
	m =-0.0736	η =1.4993	m = -0.0353	η =1.4595
	s =0.9114	λ = 1.1631	s = 0.5677	α =0.0194
	λ =1.0019		α = 0.0022	β =0.5524
			β =0.5304	
log(L)	108.7111	-52.2698	-46.6672	-38.6063
BIC values	240.1916	122.7551	120.6577	99.3819

the other countries. Additionally, there are significant differences in the means and the variances of the jump frequency distributions of these two models. One can also see that the distribution of jump severities is important but the process for jump frequencies is more important.

For model selection, we use the Bayesian information criterion (BIC). The results in Table 5 show that the Merton model with the renewal process is the best model for all countries. The reasons for this can be summarised as below.

First, the outliers in the time series data cause fat tails and a high peak in the increment Δk_t distribution, which rules out normal distribution. The Lee-Carter model treats outliers the same as other points in the mortality time-series evolution process. As a result, the outliers increase the volatility of the process and cause an overestimation of the standard deviation σ. Our model applies a renewal process separately from the Brownian motion diffusion process. This avoids the problem of mismatching the fat tail and high peak with the normal distribution and hence provides a better fit. Second, the Poisson process does not include the history of jumps. However, the history of jumps could give information about the future occurrences of jumps. Besides, the non-constant hazard function property of the renewal process enables us to obtain more realistic models for the jump frequency.

We also conduct the likelihood ratio test to compare the renewal and Poisson processes. The null hypothesis and alternative hypothesis of the test are as follows:

$$H_0 = \theta = \theta_0 \quad and \quad H_1 = \theta = \theta_1,$$

where θ_0 represents the Poisson process and θ_1 represents the renewal process. The likelihood ratio test statistic LR is calculated as:

$$LR = 2 \times (\hat{l_1} - \hat{l_0}),$$

where $\hat{l_0}$ is the maximized log-likelihood of the Poisson process and $\hat{l_1}$ is the maximized log-likelihood of the renewal process. The log-likelihood ratio test statistics are presented in Table 6. According to the test results for the models with exponential jumps, all calculated statistics are higher than the chi-square value, 3.84, of the test. Therefore, the null hypothesis is rejected for all cases.

Table 6 LR test statistics.

	US	UK	Switzerland	Italy	France
LR	30.28	17.81	21.64	17.41	27.93

In order to see the difference between the Poisson process and the renewal process in terms of the expected number of mortality jumps, we can calculate the expected values for the jump frequencies by using Equation (3) and the parameters estimated in Table 5. We present the results for the expected jump frequencies in a year for the Poisson process and the renewal process models with exponential jumps in Table 7.

Table 7 Expected Jump Frequencies

Country	EJ with Poisson P.	EJ with Renewal P.
US	1.01	2.71
UK	0.74	2.43
Switzerland	1.00	1.40
Italy	1.09	1.43
France	1.16	1.90

As shown in Table 7, the expected jump frequencies are higher for the model with renewal process which is caused by the use of lognormal distribution and thus time-varying hazard function for the inter arrival times. Although renewal process produces higher values for all countries, the difference is particularly significant for the US and the UK.

5. Swiss Re Mortality Bond

We use the Swiss Re mortality bond to obtain a risk-neutral probability measure for pricing hypothetical bonds. The Swiss Re insurance company issued the first mortality risk contingent securitization in December 2003. When the bond is triggered by a catastrophic evolution of death rates of a certain population, the investors incur the loss in principal and interest. The bond provides the investor higher yield as compensation for the mortality risk taken. The bond was issued through a special purpose vehicle called Vita Capital, which enabled Swiss Re to remove extreme catastrophic risk from its balance sheet.

The bond had a maturity of 3 years, a principal of \$400 million, and a coupon rate of 135 basis points plus LIBOR rate. The mortality index, M_t, was a weighted average of mortality rates over five countries, males and females, and a range of ages. The principal was repayable in full only if the mortality index did not exceed 1.3 times the 2002 base level during any year of the bond's life and was otherwise dependent on the realised values of the mortality index. The precise payment schedules were given by the following f_t function:

$$f_t = \left\{ \begin{array}{ll} \text{LIBOR+spread}, & t = 1, ..., T-1 \\ \text{LIBOR+spread+max}[0, 100\% - \sum_t L_t], & t = T \end{array} \right\}$$

Here, the function L_t specifies the amount of payment that is lost due to mortality experience [19]:

$$L_t = \left\{ \begin{array}{ll} 0, & M_t < 1.3M_0 \\ \frac{M_t - 1.3M_0}{0.2M_0}, & 1.3M_0 \leq M_t \leq 1.5M_0 \\ 1, & 1.5M_0 < M_t \end{array} \right\}.$$

5.1. Risk-Neutral Pricing

The market for catastrophic mortality bonds is incomplete because it is not possible to price securities in this market by constructing a replicating portfolio. A critical step in performing risk-neutral valuation is to identify a risk-neutral probability measure under which prices of mortality bonds might be computed in an incomplete market. The risk-neutral measure could be obtained in several ways. One way to implement this is canonical valuation. This method identifies a risk-neutral measure by minimizing the Kullback-Leibler information criterion, subject to market price constraints.

The first step in implementing this pricing method is to generate a large number of sample paths of future mortality rates from the assumed stochastic mortality model, which is defined under the real-world probability measure. The generated sample paths represent a collection of states of nature, all of which are equally probable. Therefore, if N sample paths are generated, then the probability mass function for the state of nature w under the real-world probability measure $\mathbf{P}$ is given by:

$$Pr(w = w_j) = \pi_j = \frac{1}{N}, \quad j = 1, 2, ..., N.$$

Our aim is to determine the probability distribution of w under a risk-neutral probability measure $\mathbf{Q}$ that is equivalent to $\mathbf{P}$. We use $N = 10000$ in our calculations.

Suppose that the market contains m distinct primary securities, whose values evolve according to the state of nature w. The ith primary security, where $i = 1, 2, ..., m$, has a time-0 price of F_i and, at the risk-free rate, a random discounted payoff of $f_i(w)$. It is necessary that all primary securities be priced correctly under $\mathbf{Q}$, so that the martingale constraints are:

$$E^{\mathbf{Q}}[f_i(w)] = \sum_{j=1}^{N} f_i(w)\pi_j^\star = F_i, \quad i = 1, 2, ..., m, \tag{12}$$

where $\pi_j^\star$, $j = 1, 2, ..., N$, is the probability distribution of w under $\mathbf{Q}$.

In an incomplete market, $m < N$ and hence there are multiple risk-neutral probability measures satisfying Equation (12). Let $\mathcal{Q}$ be the set of all measures that are equivalent to $\mathbf{P}$ and satisfies Equation (12); that is, $\mathcal{Q}$ is the set of all equivalent martingale measures. The next step is to choose a measure in $\mathcal{Q}$. This choice is based on the Kullback-Leibler information criterion ([33]), which is defined by:

$$D(\mathbf{Q}, \mathbf{P}) = E^{\mathbf{P}}\big(\frac{d\mathbf{Q}}{d\mathbf{P}} ln \frac{\mathbf{Q}}{d\mathbf{P}}\big) = \sum_{j=1}^{N} \pi_j^\star ln \frac{\pi_j^\star}{\pi_j}.$$

We choose the equivalent martingale measure $\mathcal{Q}_0$ that minimizes the Kullback-Leibler information criterion as:

$$\mathcal{Q}_0 = arg \min_{\mathbf{Q} \epsilon \mathcal{Q}} \; D(\mathbf{Q}, \mathbf{P}),$$

subject to $\sum_{j=1}^{N} \pi_j^\star = 1$, and the martingale constraints are specified by Equation (12). We refer to $\mathcal{Q}_0$ as the canonical measure.

From a statistical point of view, the justification of the canonical measure is that it incorporates all information contained in the prices of the m primary securities that are traded in the market but no other (irrelevant) information. The canonical measure might also be justified from economic and geometric viewpoints (see [22]; [38] for details).

Given the canonical measure, we might price security that has the same underlying payoff structure. Let us consider the security that has a payoff, discounted to time zero at the risk-free interest rate, of $g(w_j)$ in the jth state of nature. The price of this security implied by $\mathcal{Q}_0$ is $\sum_{j=1}^{N} g(w_j)\pi_j^\star$, where $\pi_j^\star$,$j = 1, 2, ..., N$, is the probability distribution of w under $\mathcal{Q}_0$ [41].

5.2. Derivation of the Canonical Measure

We consider the martingale constraint of $m = 1$, which is based on the price (premium spread) of the Swiss Re mortality bond. The payment structure of the Swiss Re bond was summarized in Section (5.1.). We might derive the risk-neutral measure based on the actively traded mortality-linked securities on the market whose fair price is known and then apply the same measure to the unknown mortality-linked securities. Based on the Swiss Re mortality bond the canonical risk measure might be expressed as:

$$E^{\mathbf{Q}}\left(\sum_{t=2004}^{t=2006} D_t \times f_t \times C\right), \tag{13}$$

where $C = \$400$ million, f_t is defined as previously, and D_t is the risk free discount factor. We assume that the coupon payments are paid annually and the risk-free interest rate is 3% as in Zhou et al. (2013).

Let $V(w_j)$ be the value of $\sum_{t=2004}^{t=2006} D_t \times f_t \times C$ in the jth state of nature (simulated mortality scenario). It can be shown that the distribution of w under the resulting canonical measure is:

$$\hat{\pi_j^\star} = \frac{exp(\hat{\gamma}V(w_j))}{\sum_{j=1}^{N} exp(\hat{\gamma}V(w_j))}, \quad j = 1, 2, ..., N. \tag{14}$$

Table 8 The age weights for all countries.

Age Group	US	UK	Switzerland	Italy	France
< 1	0.013818	0.011436	0.009806	0.009412	0.012336
$1-4$	0.055317	0.045715	0.040615	0.037047	0.049835
$5-14$	0.145565	0.126983	0.115676	0.095351	0.123835
$15-24$	0.138646	0.127065	0.117012	0.106701	0.129935
$25-34$	0.135573	0.136169	0.138486	0.150067	0.134373
$35-44$	0.162613	0.152864	0.167278	0.156862	0.144262
$45-54$	0.134834	0.127953	0.138976	0.131998	0.139694
$55-64$	0.087247	0.113205	0.116313	0.121131	0.102405
$65-74$	0.066037	0.083906	0.081884	0.103889	0.085608
$75-84$	0.044842	0.056444	0.054777	0.067138	0.059768
> 85	0.015508	0.018263	0.019177	0.020405	0.017948

Here, the Lagrangian multiplier $\hat{\gamma}$ is given by [38]:

$$\hat{\gamma} = arg \min_{\gamma} \sum_{j=1}^{N} exp(\gamma(V(w_j) - 400,000,000)).$$

In our calculations, the mortality scenarios are obtained from 10000 simulations of the time-varying factor k_t for 2004-2006 based on the known 2003 mortality time-varying factor. We use the Merton jump-diffusion model given in Equation (4) to simulate the mortality time series with the exponential jump and count processes for all countries. We calculate the mortality rates for different age groups by the formula $m_{x,t} = exp(a_x + k_t b_x)$. The year 2000 standard population and corresponding weights are used to compute the weighted average mortality index M_t for the US. The weights are based on the technique notes of NCHS report GMWK293R. The age weights are calculated based on exposure data for all five countries and are presented in Table 8. We then calculate the distribution of w under the canonical measure by using Equation (14) for the simulated scenarios. We use the same methodology for each mortality model and obtain the risk premiums.

5.3. Pricing Hypothetical Mortality Bonds

The important point about mortality-linked securities is the premium that investors might obtain from the transaction. Now we calculate the premium spreads of the hypothetical mortality bonds by using the risk-neutral measure. The premium spread might be expressed as the premium that compensates investors for taking on the extreme mortality risk.

We assume that our hypothetical bond has a payment structure similar to that of the Swiss Re mortality bond. The three-year bond was issued in 2003 and it is written on a mortality index M_t with base level in the year 2003. The mortality index depends on the US, the UK, Swiss, Italian, and French death rates, respectively. The index is a weighted average across age groups based on the weights for each country. We estimate parameters for 2003 mortality rates and the parameters are used for premium calculations of the proposed model.

In order to attract different types of investors, the bond is structured into two tranches with different lower and upper strikes M and U as shown in Table 9. The payment of each tranche follows the Swiss Re mortality bond payment structure. We calculate the premium spreads for each tranche based on our proposed model and the model with exponential jump and Poisson process.

Table 9 Premium Spreads of Tranche I and II for all countries.

	US		UK		Switzerland		Italy		France	
	I	II	I	II	I	II	I	II	I	II
Tranche Size	\$100	\$100	\$100	\$100	\$100	\$100	\$100	\$100	\$100	\$100
Upper strike U	1.04	1.12	1.04	1.12	1.04	1.12	1.04	1.12	1.04	1.12
Lower strike M	1.02	1.10	1.02	1.10	1.02	1.10	1.02	1.10	1.02	1.10
Premium S. (Poisson P.)	45.41	43.10	65.04	63.74	50.03	47.97	21.52	20.21	73.23	71.24
Premium S. (Renewal P.)	107.53	106.34	106.09	104.80	157.51	155.72	44.14	42.85	93.52	90.93

Tranche I has smaller upper strike and lower strike, which means that the investors in Tranche I are exposed to higher risk of losing some or all of the principals they invested. The estimated premium spreads decrease while the lower and upper strikes increase. The reason for this decrease is that the investors earn less premium spread as the risk of the bond reduces. Therefore, Tranche I produces the higher premium spreads for all countries due to representing higher risk.

The results in Table 9 show that the premium spreads obtained from the renewal process are much higher than the ones obtained from the Poisson process. Independent from the level of the lower and upper strikes of the Tranches, the renewal process indicates higher jump frequencies based on the expectations calculated for lognormally distributed inter arrival times and thus a time-varying hazard function compared to Poisson process which uses exponentially distributed inter arrival times which leads constant hazard function. In other words, considering the historical catastrophic events represented by the outliers detected in the mortality data, the renewal process produces higher jump frequencies and thus higher premium spreads. Although it is clear that the discrepancies between the premium spreads obtained from the Poisson process and renewal process are high, they are particularly significant for the US, Switzerland and Italy. The discrepancies in the premium spreads are important for the investors since obtaining lower premiums than they need might cause financial problems.

6. Conclusions

In this paper, we have investigated the impacts of the history of catastrophic events on mortality modelling. We use the lognormal renewal process with exponential jumps as the counting process for transitory mortality jumps. A specification of the Lee-Carter model has been proposed, which provides a better fit for different countries. We applied the proposed model to the mortality data for the US, the UK, Switzerland, France, and Italy. Our model turned out to be the best model for all countries compared to the models with normal jump with Poisson process and exponential jump with Poisson process.

Statistical analysis of the mortality jumps has shown that the inclusion of the history of events is significant for mortality modeling. These analyses have been done for the US, the UK, Switzerland, Italy and France. Since the mortality time indices have similar statistical properties, we can use the renewal process for jump count probabilities for other countries.

After showing the impact of the renewal process on mortality modeling, we calculated and compared the premium spreads of the hypothetical bonds for five countries. We conclude that the bond with the renewal process has a higher risk premium than the bond with the Poisson process due to producing higher jump frequencies each year.

We have restricted our model to short-term mortality jumps only. One might also use a similar modeling approach to capture changes in long-term jumps which is discussed by Deng et al. (2012). They modelled long-term and short-term jumps together by using a double exponential jump process. Moreover, model risk is an important concern in mortality modeling, and it can be taken into account in future research.

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