Mathematical Notes, vol. 0, no. 0, 1966, p. 1.
Translated from Matematicheskie Zametki, vol. 0, no. 0, 1966, pp. 0–1.

Martin integral representation for nonharmonic functions and discrete co-Pizzetti series

T. Boiko and O. Karpenkov

Received ??, 2019; in final form, ??, 20??

Abstract—In this paper we study the Martin integral representation for nonharmonic functions in discrete settings of infinite homogeneous trees. Recall that the Martin integral representation for trees is analogues to mean value property in Euclidean spaces. In the Euclidean case the mean value property for nonharmonic functions is provided by the Pizzetti (and co-Pizzetti) series. We extend the co-Pizzetti series to the discrete case. This provides us an explicit expression for the discrete mean value property for nonharmonic functions in discrete settings of infinite homogeneous trees.

Key words: *Mean value property, Laplacian, discrete Laplacian, homogeneous trees, Pizzetti series, co-Pizzetti series*

1. INTRODUCTION

{intro}

In this article we extend the Martin integral representation (the mean value property) for harmonic functions to the case of nonharmonic functions on infinite homogeneous trees (see Theorem 1 below).

The study of mean value property for nonharmonic functions in Euclidean spaces was originated by P. Pizzetti [10] in 1909. In his paper P. Pizzetti introduced the first formula for the mean value property. Another elegant proof of the Pizzetti formula is given by J.H. Sampson in [11]. For the relation of the Pizzetti formula to mean and extreme values properties we refer to [9]. The convergence of the Pizzetti series in potential theory was studied by D.H. Armitage and Ü. Kuran in [1]. Pizzetti formula for Stiefel manifolds is described by K. Coulembier, M. Kieburg in [4].

Further we briefly recall the mean value property for harmonic functions, discuss the Pizzetti series for Euclidean spaces, and finally introduce the generalized formula for infinite homogeneous trees.

Mean value property for harmonic functions. Denote by $S^{d-1}(r)$ the $(d-1)$-dimensional sphere in Euclidean space $\mathbb{R}^d$ with radius r and center at the origin. Let $\mathrm{Vol}(S^{d-1}(r))$ be its volume and let $d\mu$ be the standard surface volume measure on each of the spheres.

Recall the classical *mean value property* for a harmonic function f:

$$f(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} f d\mu. \qquad (1.1)$$

{eq0}

See [8] as a general reference to potential theory.

Mean value property for nonharmonic functions in Euclidean space $\mathbb{R}^d$. It is clear that for an arbitrary nonharmonic function Identity (1.1) does not hold. One has two distinct expressions: one on the right hand side of the identity and one on the left hand side respectively. So there are two ways to generalize Identity (1.1):

(Q1): To find an expression for $f(0)$ (the left hand side of Identity (1.1)), adding some correction terms to the integral of the right hand side;

(Q2): To find an expression for the integral of f over a sphere (the right hand side of Identity (1.1)) adding some correction terms to the left hand side of Identity (1.1).

The answers to (Q1) in the Euclidean case are written in terms of powers of the Laplace operator $\triangle$. Set

$$\triangle^0 f = f, \qquad \triangle^1 f = \triangle f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} \quad \text{and} \quad \triangle^{j+1} f = \triangle(\triangle^j f), \quad \text{for } j = 1, 2, \ldots$$

where $(x_1, \ldots, x_d)$ are standard Cartesian coordinates in $\mathbb{R}^d$.

A classical theory of Pizzetti series delivers the answer to (Q2): for a suitable f (see, e.g., in [1])

$$\frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} f d\mu = \sum_{j=0}^{\infty} \frac{\triangle^j f(0) r^{2j}}{2^j j!\, d(d+2)\cdots(d+2j-2)}, \tag{1.2}$$ {eq000}

here the first summand corresponding to $j = 0$ is $f(0)$. The expression from the right hand side of Equation (1.2) was introduced by P. Pizzetti in 1909. Now it is known as the *Pizzetti series* for f. Convergence of the Pizzetti series is not a straightforward subject. First conditions of convergency were introduced in 1992 by D.H. Armitage and Ü. Kuran (for further details, see [1]).

Let us announce the answer to (Q1). We have the following:

$$f(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} \sum_{k=0}^{\infty} \alpha_{k,d} r^{2k} \triangle^k f d\mu. \tag{1.3}$$ {eq00}

The coefficients $\alpha_{k,d}$ are generated as follows

$$\sum_{k=0}^{\infty} \alpha_{k,d} x^{2k} = \frac{(x/2)^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right) I_{\frac{d-2}{2}}(x)},$$

where I_q denotes the modified Bessel function of the first kind.

We call the expression from the right hand side of Identity (1.3) the *co-Pizzetti series* for f. The detailed proof of Expression 1.3 can be found in [2].

Let us mention that in both cases (Q1) and (Q2) any harmonic functions f satisfies $\triangle^k f = 0$ (for $k \geq 1$) and hence we get a classical mean value property.

Mean value property for nonharmonic function for homogeneous trees. In this paper we provide the answer to (Q1) for infinite homogeneous graphs (Theorem 1 and Corollary 1), while (Q2) seems to be non-relevant for graphs.

The study of harmonic function on trees (under a different terms) goes back to the middle of 19-th century when G. Kirchhoff formulated the Kirchhoff's circuit laws in electric networks. The term "harmonic functions" on trees for the first time was used in 1972 by P. Cartier in [3]. The mean value property for infinite trees is referred as *Martin integral representation* for harmonic functions. In this paper we introduce the discrete co-Pizzetti series to generalize the Martin integral representation to the case of nonharmonic functions. (For a general theory of harmonic functions on graphs and, in particular, trees we refer to [5, 6, 12].)

This paper is organized as follows. We start in Section 2 with necessary notions and definitions. Further in Section 3 we formulate the Martin integral representation theorem for nonharmonic functions on homogeneous trees (Theorem 1 and Corollary 1). In Section 4 we study some necessary tools that are further used in the proofs of the main result. We conclude the proof of Theorem 1 in Section 5.

2. NOTIONS AND DEFINITIONS {2n&d}

Consider a homogeneous tree of degree $q+1$ which we denote by T_q (i.e., every vertex of such tree has $q+1$ neighbors). We use the standard *distance* between any pair of vertices $v, w \in T_q$, which is the minimal number of edges for all possible connected edge paths joining v and w. If v and w are connected by an edge we write $v \sim w$.

2.1. *Laplace operator*

In this paper we consider the standard Laplace operator on the space of all real-valued functions on the set of vertices of T_q, which is defined as

$$\triangle f(v) = \frac{\sum\limits_{w \sim v} f(w)}{q+1} - f(v).$$

The compositions $\triangle^k$ are defined inductively in k (as it is done in the Euclidean case). Set $\triangle^0$ the identity operator. For simplicity reasons, below we always write $\triangle^k f(*)$ instead of $(\triangle^k f)(*)$.

Remark. The statements of this article have a straightforward generalization to arbitrary locally finite graphs. For simplicity we restrict ourselves entirely to homogeneous trees.

2.2. *Maximal cones and MP-arcs*

We start with the definition of maximal proper cones.

{d2}

Definition 1. Consider two vertices $v, w \in T_q$ connected by an edge e. The maximal connected component of $T_q \setminus e$ containing v is called the *maximal proper cone with vertex at v (with respect to w)*. We denote it by C^{v-w}.

The *distance* between two vertices $v, w \in T_q$ is the minimal number of edges needed to reach the vertex w starting from the vertex v. For an arbitrary nonnegative integer r and an arbitrary vertex v we denote by $S_r(v)$ the set of all vertices at distance r to v, we call such set the *circle of radius r with center v*. Note that $S_r(v)$ contains exactly $(q+1)q^{r-1}$ points.

{d3}

Definition 2. Let C^{v-w} be a maximal proper cone of T_q and let r be a nonnegative integer. The set

$$C_r^{v-w} = C^{v-w} \cap S_r(v)$$

is called the *maximal proper arc of radius r with center at v with respect to w* (or, the *MP-arc*, for short).

2.3. *Integral series*

For an arbitrary real-valued function on the vertices $f : T_q \to \mathbb{R}$ we write

$$f(C_r^{v-w}) = \frac{1}{q^r} \sum_{u \in C_r^{v-w}} f(u).$$

Let us mention that discrete Laplace operator is defined entirely on the vertices of the graph but not on its boundary (or part of a boundary). Therefore, we should reconsider the definition of integration over the boundary, since we will be addressed to some kind of improper integration of discrete Laplace series. In order to do this we suggest a new definition of the integral as a limit of certain Riemann sums.

{d1}

Definition 3. For a real-valued function f on vertices of the tree T_q, a maximal proper cone C^{v-w}, and a countable collection of sequences $\lambda_0, \lambda_1, \lambda_2, \ldots : \mathbb{Z}_+ \to \mathbb{R}$ such that the following limit exists, we define the *maximal proper cone integral* (or, *MPC-integrals*, for short) as

$$PV \int\limits_{\partial C^{v-w}} \left[\sum_{k=0}^{\infty} \lim_{r\to\infty} \lambda_k(r) \cdot \triangle^k f(t)\right] dt = \lim_{r\to\infty} \left(\sum_{k=0}^{r} \lambda_k(r) \triangle^k f(C_r^{v-w})\right),$$

Respectively we write

$$PV \int\limits_{\partial T_q} \left[\sum_{k=0}^{\infty} \lim_{r\to\infty} \lambda_k(r) \cdot \triangle^k f(t)\right]_v dt = \lim_{r\to\infty} \left(\sum_{k=0}^{r} \lambda_k(r) \sum_{u\in S_r(v)} \frac{\triangle^k f(u)}{q^r}\right).$$

Here we specify by an index v that the series are taken with respect to the vertex v, since in such settings v is defined by the integration domain.

In case when $\lambda_0(r) = 1$ and $\lambda_k(r) = 0$ for all positive integers k and r, the MPC-integral is denoted by $PV \int [f(t)] dt$ for brevity.

Remark. As far as we are aware of Definitions 3–5 and the corresponding notation are new.

Example. In case if $\lambda_0(r) = q^r(r+1)$ and $\lambda_k(r) = 0$ for all positive integers k and r we then have

$$\begin{aligned} PV \int\limits_{\partial C^{v-w}} [\lim_{r\to\infty} (q^r(r+1)) \cdot f(t)] dt &= \lim_{r\to\infty} \Big(q^r(r+1) f(C_{r+1}^{v-w})\Big) \\ &= \lim_{r\to\infty} \Big((r+1) \sum_{u\in C_r^{v-w}} f(u)\Big). \end{aligned}$$

Remark. Notice that the expressions in the right hand sides of the formulae in Definition 3 exist for every harmonic function even if the integral at boundary diverges (see Theorem 2). So the notion of integral series extends the notion of integration of functions at boundary. Here the coefficients $\lambda_k(r)$ play a role of partitions in Riemann sums for the functions $\triangle^k f(t)$.

Remark. In this paper we do not work with the boundary of a tree, but rather with certain limits (in particular the boundary is not used in the formulations and proofs of all statements of the paper). However we would like illustrate a link to the integration over the boundary by the following example. For that reason we give an informal definition of the boundary for T_q, here we do not pretend to be strict due to space restrictions.

First of all let us give an intuitive construction of the boundary of T_q in a complex disc (there are two alternative constructions in hyperbolic plane and with q-adic numbers, which we do not touch here). Consider a tree T_q. Let us embed it to the unit disc $x^2 + y^2 \le 1$.

Step 1: Choose a vertex of T_q and denote it by 0. Let us associate the origin to O. In addition we call the origin as $V_{0,0}$.

Step 2: Associate the vertices at distance k to O with points

$$V_{r,k} = \frac{r}{r+1} \exp(2\pi k / q^{r-1}(q+1)), \quad \text{where } k = 0, \ldots, q^{r-1}(q+1) - 1.$$

Step 3: Connect $V_{r,k}$ to $V_{r+1,kq}, \ldots, V_{r+1,kq+q-2}, V_{r+1,kq+q-1}$ by straight segments respectively. These segments represent the edges of T_q.

Step 4: One can set of the *boundary for* T_q as the unit circle S^1 with the angular parameter φ. This circle is equipped with a standard Lebesgue measure $d\varphi$.

As a result of Steps 1–4 we have an embedding of a graph to the disk. A standard topology on the unit disk together with the above embedding provide us with continuous extensions of functions on T_q to the boundary S^1. Given a function f on the vertices of T_q, it can be lifted to the function $\hat{f}$ on the vertices $V_{r,k}$ and then expanded to the boundary unit circle by continuity if possible. So we end up with a function (in case of convergency) on the boundary unit circle, which we denote by $\hat{f}$.

Example. Set $\lambda_0(r) = 1$ and $\lambda_k(r) = 0$ for all positive integers k and r. Then we have the standard integration over the boundary, i.e.,

$$PV \int\limits_{\partial T_q} \left[\sum_{k=0}^{\infty} \lim_{r\to\infty} (\lambda_k(r)) \cdot \triangle^k f(t) \right]_v dt = PV \int\limits_{\partial T_q} [f(t)]_v \, dt = \int\limits_{S^1} \hat{f}(\varphi) d\varphi.$$

In some sense extended integration is analogous to improper integration on the circle.

3. MAXIMAL PROPER CONE INTEGRAL FORMULA {2main}

In this section we introduce the co-Pizzetti series and formulate the mean value property for certain nonharmonic functions.

3.1. *MPC-integrals for C^{v-w}-summable functions*

We start with the following definition. {C-sum}

Definition 4. We say that a real-valued functions f on the vertices of T_q is C^{v-w}*-summable* if

$$\lim_{r\to\infty} \left(q^r f(C_{2r}^{v-w}) \right) = \lim_{r\to\infty} \Big(\sum_{u \in C_{2r}^{v-w}} \frac{f(u)}{q^r} \Big) = 0.$$

Let us now introduce a discrete analog of co-Pizzetti series. {d6}

Definition 5. Let f be an arbitrary real-valued function on the vertices of the tree T_q and let $v, w \in T_q$ be two vertices connected by an edge. The *partial co-Pizzetti series* for a function f on T_q for the maximal proper cone C^{v-w} is the following expression

$$co\mathcal{P}(f, v-w) = PV \int\limits_{\partial C^{v-w}} \left[\sum_{k=0}^{\infty} \left(\lim_{r\to\infty} \Big((q+1)^k \big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r) \big) \Big) \cdot \triangle^k f(t) \right) \right] dt,$$

where

$$\gamma_{q,k}(r) = c_{q,k,k} r^k + \ldots + c_{q,k,1} r + c_{q,k,0}, \tag{3.1}$$ {eq1}

whose collection of coefficients $c_{q,k,j}$ (for a fixed k) is the solution of the following linear system

$$A_{q,k} \begin{pmatrix} c_{q,k,k} \\ \vdots \\ c_{q,k,1} \\ c_{q,k,0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.2}$$ {systemA}

where $A_{q,k}$ is the following matrix

$$\begin{pmatrix} 0 & \ldots & 0 & 1 \\ 1^k(1+(-1)^k q^1) & \ldots & 1^1(1+(-1)^1 q^1) & 1^0(1+(-1)^0 q^1) \\ 2^k(1+(-1)^k q^2) & \ldots & 2^1(1+(-1)^1 q^2) & 2^0(1+(-1)^0 q^2) \\ \vdots & \ddots & \vdots & \vdots \\ (k-1)^k(1+(-1)^k q^{k-1}) & \ldots & (k-1)^1(1+(-1)^1 q^{k-1}) & (k-1)^0(1+(-1)^0 q^{i-1}) \\ k^k(1+(-1)^k q^k) & \ldots & k^1(1+(-1)^1 q^k) & k^0(1+(-1)^0 q^k) \end{pmatrix}.$$

Remark. Note that the last element in the first line of $A_{q,k}$ is 1 and not 2, as one can expect. This affects coefficients $c_{i,j}$ only in a special case $k=0$. We show that matrices $A_{q,k}$ is invertible in Proposition 7.

The Martin integral representation for nonharmonic functions on homogeneous trees can now be formulated in the following theorem and its corollary.

{TheoremB}

Theorem 1. (on Martin integral representation for nonharmonic functions.) *Consider two vertices $v, w \in T_q$ connected by an edge. If f is C^{v-w}-summable, then*

$$f(v) = co\mathcal{P}(f, v-w).$$

In addition, the convergence of the partial co-Pizzetti series $co\mathcal{P}(f, v-w)$ is equivalent to the condition that f is C^{v-w}-summable.

We prove this theorem later in Section 5.

Let us write a weaker version of Theorem 1 for the integration over the whole boundary.

{main-tree-cor}

Corollary 1. *Consider a vertex $v \in T_q$, and let f be a C^{v-w}-summable function for all vertices w adjacent to v. Then*

$$f(v) = \frac{q}{q+1} PV \int\limits_{\partial T_q} \left[\sum_{k=0}^{\infty} \left(\lim_{r\to\infty} \Big((q+1)^k \big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\big) \Big) \triangle^k f(t) \right) \right]_v dt.$$

The proof of Corollary 1 is given modulo Theorem 1.

Proof. Let us sum up the partial co-Pizzetti series for all maximal proper cones with vertex at v. From one hand there are exactly $q+1$ such cones so the sum equals to $(q+1)f(v)$. From the other hand each vertex of the discrete sphere $S_r(v)$ contributed to maximal proper integrals of q distinct maximal proper cones C_r^{v-w} with vertex at v. Therefore, we get the constant $\frac{q}{q+1}$ in the statement of the corollary. □

Remark. Note that it is possible to write the partial co-Pizzetti series for arbitrary locally-finite trees, although the formulas for the coefficients would be more complicated.

3.2. *MPC-integral formula for harmonic functions*

Let us prove the following more general statement for harmonic functions.

{holom}

Corollary 2. *Consider an arbitrary harmonic C^{v-w}-summable function h on a homogeneous tree T_q. Let v be a vertex of T_q and C^{v-w} be one of its proper maximal cones. Let also f be another holomorphic function on T_q defined as:*

$$f(t) = h(t) - PV \int_{\partial C^{v-w}} [h(u)] du.$$

Then the following holds:

$$h(v) = PV \int\limits_{\partial C^{v-w}} [h(t)] dt + PV \int\limits_{\partial C^{v-w}} \left[\lim_{r\to\infty} \Big(q^r f(t) \Big) \right] dt.$$

Proof. Denote $h(C_r^{v-w})$ by H_r (for $r \geq 0$). Then the sequence H_r satisfies the recurrence relation:

$$q^r H_r = \frac{q \cdot q^{r-1} H_{r-1} + q^{r+1} H_{r+1}}{q+1} \quad (\text{for } r \geq 1),$$

here we sum up the harmonic conditions at all the vertices of C_r^{v-w}. The last is equivalent to

$$qH_{r+1} - (q+1)H_r + H_{r-1} = 0$$

The characteristic polynomial of this constant-recursion condition has roots 1 and q. Therefore,

$$H_r = C_1 + \frac{C_2}{q^r} \quad \text{for some } C_1, C_2 \in \mathbb{R}.$$

Now let us compute all the entries in the expression of Corollary 2:

$$h(v) = H_0 = C_1 + C_2 q^0 = C_1 + C_2;$$
$$PV \int\limits_{\partial C^{v-w}} [h(t)]dt = \lim_{r\to\infty} H_r = \lim_{r\to\infty} (C_1 + C_2 q^{-r}) = C_1;$$
$$PV \int\limits_{\partial C^{v-w}} \left[\lim_{r\to\infty} \left(q^r f(t)\right)\right]dt = PV \int\limits_{\partial C^{v-w}} \left[\lim_{r\to\infty} \left(q^r \left(h(t) - PV \int_{\partial C^{v-w}} [h(u)]du\right)\right)\right]dt$$
$$= \lim_{r\to\infty} q^r \left((C_1 + C_2 q^{-r}) - C_1\right) = C_2.$$

So we have the convergence of integrals and the consistency of the expression in Corollary 2. □

Remark. Suppose that h is a harmonic function that is integrable on ∂C^{v-w} with respect to the standard probability measure $d\mu$ on the boundary. Then

$$\int\limits_{\partial C^{v-w}} h d\mu = PV \int\limits_{\partial C^{v-w}} [h(t)]dt.$$

In case if h is not integrable with respect to probability measure, the MPC-integral nevertheless exists. In some sense MPC-integrability is an improper integrability with respect to integration over probability measure. MPC-integral exists for every harmonic function h and for every cone C^{v-w}.

4. RELATIONS ON SPECIAL LAURENT POLYNOMIAL

{2relations}

In this section we are dealing with two special families of linearly independent Laurent polynomials that span the same subspaces in the space of all Laurent polynomials. It turns out that the transition matrix between such families "almost coincides" with the matrix $A_{q,k}$ mentioned above. Here we study the properties of these families in order to conclude the proof of Theorem 1 in Section 5.

4.1. *Auxiliary statements*

For every nonnegative integer r and integer $q \geq 2$ we set

$$S_{q,r}(x) = \frac{(x-1)^r (x-q)^r}{(q+1)^r x^r}.$$

For a nonnegative $k \leq r$ we also set

$$\hat{S}_{q,k,r} = \sum_{i=-k}^{k} d_{q,-i,k} f(C_{r+i}^{v-w})$$

where the coefficients $d_{q,i,k}$ are generated by

$$S_{q,k}(x) = \sum_{i=-k}^{k} d_{q,i,k} x^i.$$

(Note that we have a reversed order for the powers of generating functions.)

In the following proposition we see that the quantities $\hat{S}_{q,k,r}$ and $\triangle^k(C_r^{v-w})$ are in fact coincide.

{extra1}

Proposition 1. *Let f be a real-valued function on the vertices of T_q and v, w be two vertices of T_q connected by an edge. Then for every positive integers $0 < k < r$ it holds*

$$\triangle^k(C_r^{v-w}) = \hat{S}_{q,k,r}.$$

Proof. Let us prove the existence of expression by induction over k for $k \geq 0$.

Base of induction: The case $k = 0$ is straightforward. Indeed,

$$\triangle^0(C_r^{v-w}) = f(C_r^{v-w}) = \hat{S}_{q,0,r}.$$

The second equality holds, since $S_{q,0}(x) = 1$.

Step of induction: Suppose now the statement holds for k, i.e.,

$$\triangle^k(C_r^{v-w}) = \hat{S}_{q,k,r}.$$

Let us prove it for $k+1$. We have

$$\begin{aligned}\triangle^{k+1}(C_r^{v-w}) = \triangle(\triangle^k(C_r^{v-w})) = \triangle\Big(\sum_{i=-k}^{k} d_{q,-i,k} f(C_{r+i}^{v-w})\Big) = \sum_{i=-k}^{k} \big(\triangle(d_{q,-i,k} f(C_{r+i}^{v-w})\big) \\ = \sum_{i=-k}^{k} \bigg(d_{q,-i,k}\Big(\frac{f(C_{r+i-1}^{v-w}) + q f(C_{r+i+1}^{v-w})}{q+1} - f(C_{r+i}^{v-w})\Big)\bigg) \\ = \sum_{i=-k}^{k} \bigg(d_{q,-i,k}\Big(\frac{f(C_{r+i-1}^{v-w}) - (q+1) f(C_{r+i}^{v-w}) + q f(C_{r+i+1}^{v-w})}{q+1}\Big)\bigg)\end{aligned}$$

The fourth equality follows from the definitions of $\triangle$ and C_*^{v-w}. From the last formula we conclude that the generating functions for $\triangle^{k+1}(C_r^{v-w})$ is obtained from the generating function for $\triangle^k(C_r^{v-w})$ (with reverse order of powers) by multiplying by

$$\frac{(x - (q+1) + x^{-1})}{q+1} = \frac{(x-1)(x-q)}{(q+1)x}.$$

Therefore,

$$\triangle^k(C_r^{v-w}) = \hat{S}_{q,k,r}.$$

This concludes the proof of the induction step. □

Now let us study the Laurent polynomials $S_{q,r}$ in more details. First of all for every integer r and integer $q \geq 2$ we denote

$$D_{q,r}(x) = x^r + \frac{q^r}{x^r}.$$

Note that $D_{q,0}(x) = x^0 + \frac{q^0}{x^0} = 2$.

We have the following recurrent relation for the defined above Laurent polynomials.

{recursion}

Proposition 2. *For every integer r and integer $d \geq 2$ we have*

$$S_{q,1} D_{q,r} = \frac{D_{q,r+1} - (q+1) D_{q,r} + q D_{q,r-1}}{q+1}.$$

Proof. For every integer r and integer $d \geq 2$ (including $r = -1, 0, 1$) it holds

$$\begin{aligned}
S_{q,1}D_{q,r} &= \Big(\frac{(x-1)(x-q)}{(q+1)x}\Big)\Big(x^r + \frac{q^r}{x^r}\Big) \\
&= \frac{x^{r+1}}{q+1} - x^r + \frac{q}{q+1}x^{r-1} + \frac{q^r}{(q+1)x^{r-1}} - \frac{q^r}{x^r} + \frac{q^{r+1}}{(q+1)x^{r+1}} \\
&= \frac{1}{q+1}\Big(x^{r+1} + \frac{q^{r+1}}{x^{r+1}}\Big) - \Big(x^r + \frac{q^r}{x^r}\Big) + \frac{q}{q+1}\Big(x^{r-1} + \frac{q^{r-1}}{x^{r-1}}\Big) \\
&= \frac{D_{q,r+1} - (q+1)D_{q,r} + qD_{q,r-1}}{q+1}. \quad \square
\end{aligned}$$

{uniqueness_SD}

Proposition 3. *For every integer r and integer $q \geq 2$ there exists a unique decomposition*

$$D_{q,r} = \sum_{k=0}^{|r|} a_{q,r,k} S_{q,k}, \tag{4.1}$$ {D-S}

where $a_{q,r,k}$ are real numbers.

Proof. Let us prove the existence of expression by induction over r for $r \geq 0$ (for negative r the proof is analogous).
Base of induction: $D_{q,0} = 2S_{q,0}$, and $D_{q,1} = (q+1)S_{q,1} + (q+1)S_{q,0}$.
Step of induction: Assume that $D_{q,r}$ and $D_{q,r-1}$ are linear combinations of $S_{q,0}, \ldots, S_{q,r}$. Therefore $S_{q,1}D_{q,r}$ is a linear combination of $S_{q,0}, \ldots, S_{q,r+1}$, since $S_{q,1}S_{q,j} = S_{q,j+1}$ for $j = 0, 1, \ldots, r$. Now the induction step follows directly from Proposition 2.

Uniqueness follows from the fact that all the rational functions $S_{q,k}$ (for fixed q) have distinct degrees, and therefore they are linear independent. $\square$

{series-for-a}

Definition 6. Let r be any integer and let an integer q satisfy $q \geq 2$. For $0 \leq k \leq |r|$ we define $a_{q,r,k}$ from Equation (4.1). For $k > |r|$ we set $a_{q,r,k} = 0$. Now we can write $D_{q,r}$ as infinite series:

$$D_{q,r} = \sum_{k=0}^{+\infty} a_{q,r,k} S_{q,k}.$$

Now let us study the coefficients $a_{q,r,k}$. The next statement follows directly from Proposition 2.

{cor_rec}

Corollary 3. *For every positive integer k, integer r, and integer $q \geq 2$ it holds*

$$a_{q,r,k-1} = \frac{a_{q,r+1,k} - (q+1)a_{q,r,k} + qa_{q,r-1,k}}{q+1}.$$

Additionally, for the case $k = 0$ we have

$$0 = a_{q,r+1,0} - (q+1)a_{q,r,0} + qa_{q,r-1,0}.$$

Proof. By the definition we have

$$S_{q,1}S_{q,r} = S_{q,r+1},$$

for all integer r. Propositions 2 and 3 imply

$$\begin{aligned}
\sum_{k=1}^{+\infty} a_{q,r,k-1}S_{q,k} = S_{q,1}D_{q,r} &= \frac{D_{q,r+1} - (q+1)D_{q,r} + qD_{q,r-1}}{q+1} \\
&= \frac{1}{q+1}\Big(\sum_{k=0}^{+\infty} a_{q,r+1,k}S_{q,k} - (q+1)\sum_{k=0}^{+\infty} a_{q,r,k}S_{q,k} + q\sum_{k=0}^{+\infty} a_{q,r-1,k}S_{q,k}\Big).
\end{aligned}$$

Collecting the coefficients at $S_{q,k}$ we get the recurrence relations of the corollary. $\square$

In fact, the coefficients $a_{q,r,k}$ for fixed q and r satisfy a remarkable linear recurrence relation, which is a consequence of the next proposition. Let us first give the following definition.

Definition 7. For a positive integer n we define the linear form $L_{q,n}$ in $2n{+}1$ variables as follows

$$L_{q,n}(y_1,\ldots,y_{2n+1}) = \sum_{i=-n}^{n} d_{q,-i,n}y_{n+i+1},$$

where $d_{q,i,n}$ are defined as the coefficients of $S_{q,n}$, i.e., from the expression

$$S_{q,n}(x) = \frac{(x-1)^n(x-q)^n}{(q+1)^n x^n} = \sum_{i=-n}^{n} d_{q,i,n}x^i.$$

{LinearRecurrences}

Proposition 4. *For every non-negative integer k, integer r, and integer $q \geq 2$ we have*

$$L_{q,k+1}(a_{q,r-k-1,k}, a_{q,r-k+1,k}, \ldots, a_{q,r+k+1,k}) = 0.$$

Proof. We prove the proposition by induction over k.

Base of induction. For the case $k=0$ the statement holds by Corollary 3.

Step of induction. Suppose that the statement holds for $k-1$. Let us prove it for k. We have

$$L_{q,k}(a_{q,r-k,k-1},\ldots,a_{q,r+i,k-1}) = 0.$$

By Corollary 3 and linearity of $L_{q,k+1}$ we have

$$\begin{aligned}
&L_{q,k}(a_{q,r-k,k-1},\ldots,a_{q,r+k,k-1})\\
&\quad= L_{q,k}\Big(\tfrac{a_{q,r-k+1,k}-(q+1)a_{q,r-k,k}+qa_{q,r-k-1,k}}{q+1},\ldots,\\
&\qquad\qquad\qquad\qquad\qquad \tfrac{a_{q,r+k+1,k}-(q+1)a_{q,r-k,k}+qa_{q,r-k-1,k}}{q+1}\Big)\\
&\quad= \frac{1}{q+1}\Big(L_{q,k}(a_{q,r-k+1,k},\ldots,a_{q,r+k+1,k}) - (q+1)L_{q,k}(a_{q,r-k,k},\ldots,a_{q,r+k,k})\\
&\qquad + qL_{q,k}(a_{q,r-k-1,i},\ldots,a_{q,r+k-1,k})\Big)\\
&\quad= L_{q,k+1}(a_{r-k-1,k}, a_{q,r-k,k},\ldots,a_{q,r+k,k},a_{q,r+k+1,k}).
\end{aligned}$$

The last equality holds, since

$$\frac{1}{q+1}\Big(xS_{q,k} - (q+1)S_{q,k} + q\frac{S_{q,k}}{x}\Big) = \frac{(x-1)(x-q)}{(q+1)x}\cdot S_{q,k} = S_{q,k+1}.$$

Therefore, by inductive hypothesis we have

$$\begin{aligned}
L_{q,k+1}(a_{q,r-k-1,k}, a_{q,r-k,k},\ldots,a_{q,r+k,k},a_{q,r+k+1,k}) &= L_{q,k}(a_{q,r-k,k-1},\ldots,a_{q,r+k,k-1})\\
&= 0.
\end{aligned}$$

This concludes the proof of the induction step. ☐

4.2. *Expression for $a_{q,r,k}$*

First of all we observe a general expression for $a_{q,r,k}$. Secondly, we write $a_{q,r,k}$ in terms of solutions to System (3.2).

{p=q}

Proposition 5. *For every nonnegative integer k and integer r we have*

$$a_{q,r,k} = P_{q,k}(r) + q^r P_{q,k}(-r),$$

where $P_{q,k}$ is a polynomial of degree at most k.

Proof. By Proposition 4 the sequence $a_{q,r,k}$ (for every fixed q and k) satisfies a linear recursive relation defined by $L_{q,k+1}$, whose characteristic polynomial has only two roots: 1 and q, both of multiplicity $k+1$.

Therefore, $a_{q,r,k}$ can be written as

$$a_{q,r,k} = P_{q,k}(r) + q^r \hat{P}_{q,k}(r),$$

where $P_{q,k}$ and $\hat{P}_{q,k}$ are polynomials of degree at most k (e.g., see in Section 2.1.1 of [7]).

For every integer r we have

$$D_{q,-r}(x) = x^{-r} + \frac{q^{-r}}{x^{-r}} = \frac{1}{q^r}\Big(\frac{q^r}{x^r} + x^r\Big) = \frac{D_{q,r}(x)}{q^r}.$$

By Proposition 3 the coefficients $a_{q,r,k}$ and $a_{q,-r,k}$ are uniquely defined, therefore,

$$a_{q,r,k} = q^r a_{q,-r,k}.$$

Let us rewrite this equality in terms of polynomials $P_{q,k}$ and $\hat{P}_{q,k}$:

$$P_{q,k}(r) + q^r \hat{P}_{q,k}(r) = q^r(P_{q,k}(-r) + q^{-r}\hat{P}_{q,k}(-r)),$$

and hence

$$P_{q,k}(r) + q^r \hat{P}_{q,k}(r) = \hat{P}_{q,k}(-r) + q^r P_{q,k}(-r).$$

Since this equality is fulfilled for every r we have $\hat{P}_{q,k}(-r) = P_{q,k}(r)$. This concludes the proof. □

Example. Direct calculations show that in case $q = 2$ we have

$$\begin{aligned}
a_{2,r,0} &= 1 + 2^r,\\
a_{2,r,1} &= \tfrac{3^1}{1!}(-r + 2^r r),\\
a_{2,r,2} &= \tfrac{3^2}{2!}\big(r^2 + 3r + 2^r(r^2 - 3r)\big),\\
a_{2,r,3} &= \tfrac{3^3}{3!}\big(-r^3 - 9r^2 - 26r + 2^r(r^3 - 9r^2 + 26r)\big),\\
&\dots
\end{aligned}$$

Observe the following general properties of $a_{q,r,k}$.

{2prop2}

Proposition 6. *For every integer $q \geq 2$ and integers r, k as above we have*

(*a*) $k > |r| \geq 0$: $a_{q,r,k} = 0$;
(*b*) $k = 0, r \in \mathbb{Z}$: $a_{q,r,0} = 1 + q^r$;
(*c*) $k = r > 0$: $a_{q,k,k} = (q+1)^k$.

Proof. *Item* (a) is actually a convention of Definition 6.

Item (b) From Proposition 5 we have that

$$a_{q,r,0} = \alpha + q^r \alpha$$

for some real α. Since $D_{q,0} = 2S_{q,0}$, we have $a_{q,0,0} = 2$. Therefore, $\alpha = 1$. This concludes the proof of Item (b).

Finally we prove *Item* (c) of the proposition by induction over k.

Base of induction. For the case $k = 1$ we have $D_{q,1} = (q+1)S_{q,0} + (q+1)S_{q,1}$, therefore

$$a_{q,1,1} = q + 1.$$

Step of induction. Let $a_{q,k,k}=(q+1)^k$. Then

$$(q+1)^k=a_{q,k,k}=\frac{a_{q,k+1,k+1}-(q+1)a_{q,k,k+1}+qa_{q,k-1,k+1}}{q+1}=\frac{a_{q,k+1,k+1}}{q+1}.$$

The second equality follows from the recursive formula of Corollary 3. Hence

$$a_{q,k+1,k+1}=(q+1)^{k+1}.$$

This concludes the step of induction. □

Before to proceed we prove invertibility of matrix $A_{q,k}$.

{MatrixA}

Proposition 7. *The determinant of the matrix $A_{q,k}$ in System (3.2) is nonzero (for $q=2,3,\ldots$).*

Proof. Suppose that $w=(w_1,\ldots,w_{k+1})$ satisfies $A_{q,k}(w)=0$. Let us prove that $w=(0,\ldots,0)$.

Set

$$\varphi(x)=w_1x^k+\ldots+w_kx+w_{k+1},\quad \text{and}\quad \Phi(x)=\varphi(x)+\varphi(-x)q^x.$$

Since w satisfies equation $A_{q,k}w=0$ we have

$$\Phi(0)=\Phi(1)=\ldots=\Phi(k)=0.$$

Assume that $\Phi(k+1)=a$ for some real number a.

From a general observation

$$\Phi(-r)=\varphi(-r)+\varphi(r)q^{-r}=\frac{\varphi(r)+\varphi(-r)q^r}{q^r}=\frac{\Phi(r)}{q^r}$$

we get

$$\Phi(-1)=\ldots=\Phi(-k)=0\quad \text{and}\quad \Phi(-k-1)=\frac{a}{q^{k+1}}.$$

Now recall a general fact that the sequence $\Phi(r)$ satisfies the constant-recursion condition with characteristic polynomial: $(x-1)^{k+1}(x-q)^{k+1}$ (e.g., see in Section 2.1.1 of [7]). Hence substituting the values $\Phi(-k-1),\ldots,\Phi(k+1)$ to the constant-recursive condition we have

$$a+0+\ldots+0+a=0,$$

and therefore $a=0$.

Therefore, $\Phi(r)$ has $2k+3$ consequent zeroes, which implies that $\Phi(r)=0$ for all integer r (as the characteristic polynomial is of degree $2k+2$).

Finally, note that $\varphi(x)+\varphi(-x)q^x$ has an exponential growth at $+\infty$ if $\varphi(x)$ is not identically zero, which contradict to the fact that $\Phi(n)=0$ for all integer n. Thus $\varphi(x)$ is identically zero, and hence all w_i are zeroes. Therefore, the kernel of $A_{q,k}$ is empty. □

Let us prove a general theorem on numbers $a_{q,r,k}$.

{polynomials}

Theorem 2. *For a non-negative integer r, a non-negative integer k, and $q\geq 2$ it holds*

$$a_{q,r,k}=(q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big),$$

where $\gamma_{q,k}$ are polynomials as above (see Equation (3.1)) whose coefficients are the solutions of System (3.2).

Proof. First, let us study the tautological case $k=0$. From Proposition 6(b) we have:

$$a_{q,r,0} = 1 + 1 \cdot q^r.$$

Therefore, $P_{q,0}(r) = 1 \cdot r^0 = 1$. Since $A_{q,0} = (1)$ we have:

$$A_{q,0}(c_{q,0,0}) = 1,$$

and hence the coefficient $c_{q,0,0} = 1$. Therefore, $P_{q,0} = \gamma_{q,0}$. This concludes the proof for the case $k=0$.

Let now $k \geq 1$. From Proposition 5 we have:

$$a_{q,r,k} = P_{q,k}(r) + q^r P_{q,k}(-r),$$

where $P_{q,k}$ is a polynomial of degree at most k.

Since the degree of $P_{q,k}$ is at most k, the coefficients of the polynomial $P_{q,k}$ are uniquely defined by values at $j = 0, \dots, k$:

$$P_{q,k}(j) + q^j P_{q,k}(-j) = a_{q,j,k} = \begin{cases} 0, & \text{for } j = 0, \dots, k-1, \\ (q+1)^k, & \text{for } j = k, \end{cases}$$

where the expressions hold by Proposition 6(a) and 6(c). These conditions form a linear system on the coefficients of the polynomial $\frac{P_{q,k}}{(q+1)^k}$, which coincides with equations of System (3.2) with the only exception. The first equation (for $j = 0, k \neq 0$) is $2P_{q,k}(0) = 0$, while the corresponding equation of System (3.2) corresponds to the multiple one: $P_{q,k}(0) = 0$.

So both the coefficients of $\frac{P_{q,k}}{(q+1)^k}$ and the coefficients of $\gamma_{q,k}$ are solutions of System (3.2). Since System (3.2) has a unique solution, the polynomials $P_{q,k}$ and $(q+1)^k \gamma_{q,k}$ coincide. Therefore, by Proposition 5 it holds

$$a_{q,r,k} = P_{q,k}(r) + q^r P_{q,k}(-r) = (q+1)^k \big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\big).$$

This concludes the proof of Theorem 2. $\square$

5. PROOF OF THEOREM 1

{2conclude}

Finally we have all necessary tools to prove of Theorem 1. We start with the following lemma.

{expression}

Lemma 1. *Let f be a real-valued function on the vertices of T_q and v, w be two vertices of T_q connected by an edge. Then for every positive integer r it holds*

$$f(v) + q^r f(C_{2r}^{v-w}) = \sum_{k=0}^{r} \Big((q+1)^k \big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\big) \triangle^k f(C_r^{v-w}) \Big),$$

where $\gamma_{q,k}$ are polynomials given by Equation (3.1) whose coefficients are the solutions of System (3.2).

Proof. For $0 < k \leq r$ set

$$\hat{D}_{q,r} = f(C_0^{v-w}) + q^r f(C_{2r}^{v-w}),$$
$$\hat{S}_{q,k,r} = \sum_{i=-k}^{k} d_{q,-i,k} f(C_{r+i}^{v-w}),$$

where the coefficients $d_{q,i,k}$ are generated by

$$S_{q,k} = \frac{\big((x-1)(x-q)\big)^k}{(q+1)^k x^k} = \sum_{i=-k}^{k} d_{q,i,k} x^i.$$

Notice that $\hat{D}_{q,r}$ and $\hat{S}_{q,k,r}$ are obtained from $D_{q,r}$ and $S_{q,k}$ by replacing powers x^i by the values $f(C^{v-w}_{r+i})$ for all $i=-k,\ldots,k$. Therefore, the expression

$$D_{q,r}=\sum_{k=0}^{+\infty} a_{q,r,k}S_{q,k}=\sum_{k=0}^{r} a_{q,r,k}S_{q,k}.$$

of Definition 6 implies the expression for $\hat{D}_{q,r}$ and $\hat{S}_{q,k,r}$:

$$\hat{D}_{q,r}=\sum_{k=0}^{r} a_{q,r,k}\hat{S}_{q,k,r}.$$

Therefore, we have

$$f(v)+q^r f(C^{v-w}_{2r})=\hat{D}_{q,r}=\sum_{k=0}^{r} a_{q,r,k}\hat{S}_{q,k,r}, \tag{5.1}$$ {interm}

First, note that by Proposition 1 we get

$$\hat{S}_{q,k,r}=\triangle^k(C^{v-w}_r).$$

Secondly by Theorem 2 we have

$$a_{q,r,k}=(q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big),$$

where $\gamma_{q,k}$ are polynomials of Equation (3.1) whose coefficients are the solutions of System (3.2).

Substituting the last two expression to Equation (5.1) we obtain

$$f(v)+q^r f(C^{v-w}_{2r})=\sum_{k=0}^{r}\Big((q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big)\triangle^k f(C^{v-w}_r)\Big).$$

This concludes the proof. $\square$

Proof of Theorem 1. From Lemma 1 we have

$$f(v)+q^r f(C^{v-w}_{2n})=\sum_{i=0}^{r}\Big((q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big)\triangle^k f(C^{v-w}_r)\Big).$$

Hence,

$$\begin{aligned}
PV\int\limits_{\partial C^{v-w}} &\left[\sum_{k=0}^{\infty}\left(\lim_{r\to\infty}\Big((q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big)\Big)\cdot\triangle^k f(t)\right)\right]dt\\
&=\lim_{r\to\infty}\sum_{k=0}^{r}\left(\Big((q+1)^k\big(\gamma_{q,k}(r)+q^r\gamma_{q,k}(-r)\big)\Big)\triangle^k f(C^{v-w}_r)\right)\\
&=\lim_{r\to\infty}\big(f(v)+q^r f(C^{v-w}_{2r})\big)=f(v)+\lim_{r\to\infty}\big(q^r f(C^{v-w}_{2r})\big).
\end{aligned}$$

In case if f is C^{v-w}-summable (see Definition 4), we have

$$\lim_{r\to\infty}\big(q^r f(C^{v-w}_{2r})\big)=0$$

and hence the above MPC-integral converges to $f(v)$. This concludes the proof of Theorem 1. $\square$

Example. Let us check Theorem 1 for the function χ_v that is zero everywhere except for the point v and $\chi_v(v) = 1$. We have

$$\triangle^k \chi_v(C_r^{v-w}) = \frac{1}{q^r} \sum_{u \in C_r^{v-w}} \chi_v(u) = \begin{cases} 0, & \text{if } k < r; \\ \frac{1}{(q+1)^r}, & \text{if } k = r. \end{cases}$$

(notice that C_r^{v-w} contains exactly q^r vertices). Therefore, the partial co-Pizzetti series $co\mathcal{P}(\chi_v, v-w)$ are

$$\begin{aligned} co\mathcal{P}(\chi_v, v-w) &= PV \int_{\partial C^{v-w}} \left[\sum_{k=0}^{\infty} \left(\lim_{n\to\infty} \left((q+1)^k \big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\big) \right) \cdot \triangle^k \chi_v(t) \right) \right] dt \\ &= \lim_{r\to\infty} \left(\sum_{k=0}^{r} (q+1)^k \Big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\Big) \triangle^k \chi_v(C_r^{v-w}) \right) \\ &= \lim_{r\to\infty} \Big((q+1)^r \Big(\gamma_{q,r}(r) + q^r \gamma_{q,r}(-r)\Big) \triangle^r \chi_f(C_r^{v-w}) \Big). \end{aligned}$$

By Theorem 2 the expression

$$(q+1)^r \Big(\gamma_{q,r}(r) + q^r \gamma_{q,r}(-r)\Big)$$

is equal to $a_{q,r,r}$ from Definition 6, which by Proposition 6(c) below is equal to $(q+1)^r$. Therefore, we conclude

$$\lim_{r\to\infty} \Big((q+1)^r \Big(\gamma_{q,r}(r) + q^r \gamma_{q,r}(-r)\Big) \triangle^r \chi_v(C_r^{v-w}) \Big) = \lim_{r\to\infty} (q+1)^r \frac{1}{(q+1)^r} = 1 = \chi_v(v).$$

It is clear from this example that it is not always possible to exchange the sum operator and the limit operator. For the function χ_v we have

$$\sum_{k=0}^{\infty} \lim_{r\to\infty} \left((q+1)^k \Big(\gamma_{q,k}(r) + q^r \gamma_{q,k}(-r)\Big) \triangle^k \chi_v(C_r^{v-w}) \right) = \sum_{k=0}^{\infty} 0 = 0 \neq 1 = \chi_v(v).$$

Acknowledgements. The authors are grateful to Prof. Woess for constant attention to this work and to the unknown reviewers for providing useful references regarding Pizzetti series and other useful remarks. This first author is supported by the Austrian Science Fund (FWF): W1230, Doctoral Program "Discrete Mathematics". The second author is partially supported by EPSRC grant EP/N014499/1 (LCMH).

BIBLIOGRAPHY

1. D.H. Armitage, U. Kuran, "The convergence of the Pizzetti series in potential theory", J. Math. Anal. Appl., **171(2)**, 516-531 (1992).
2. T. Boiko, O. Karpekov, "Mean value propery for non-harmonic functions in $\mathbb{R}$", preprint (2019).
3. P. Cartier, "Fonctions harmoniques sur un arbre", in Symposia Mathematica, volume IX, pages 203–270 (Academic Press, London and New-York, 1972).
4. K. Coulembier, M. Kieburg, "Pizzetti formulae for Stiefel manifolds and applications", Lett. Math. Phys., **105(10)**, 1333-1376 (2015).
5. J.L. Doob, *Classical potential theory and its probabilistic counterpart. Classics in Mathematics* (Springer-Verlag, Berlin, 2001).
6. E.B. Dynkin, "The boundary theory of Markov processes (discrete case)", Uspehi Mat. Nauk, **24(2 (146))** 3–42 (1969).
7. D.H. Greene; D.E. Knuth, *Mathematics for the Analysis of Algorithms (2nd ed.)*, (Birkhäuser, 1982).

8. L.L. Helms, *Introduction to potential theory* (Robert E. Krieger Publishing Co., Huntington, N.Y., 1975).
9. J.S. Ovall, “The Laplacian and mean and extreme values”, Amer. Math. Monthly, **123(3)**, 287-291 (2016).
10. P. Pizzetti, “Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera”, Rend. Reale Accad. Lincei, **18(5)**, 182–185 (1909).
11. J.H. Sampson, “On Pizzetti’s formula”, Boll. Un. Mat. Ital. A, **(6) 2, no. 1**, 99-102 (1983).
12. W. Woess, *Denumerable Markov chains. Generating functions, boundary theory, random walks on trees* (EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2009).

T. Boiko
University of Liverpool, Liverpool, UK
E-mail: `t.boiko@liverpool.ac.uk, Liverpool`

O. Karpenkov
University of Liverpool, Liverpool, UK
E-mail: `karpenk@liverpool.ac.uk, Liverpool`