ON DEPTH ZERO L-PACKETS FOR CLASSICAL GROUPS

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ABSTRACT. By computing reducibility points of parabolically induced representations, we construct, to within at most two unramified quadratic characters, the Langlands parameter of an arbitrary depth zero irreducible cuspidal representation π of a classical group (which may be not-quasi-split) over a nonarchimedean local field of odd residual characteristic. From this, we can explicitly describe all the irreducible cuspidal representations in the union of one, two, or four L-packets, containing π. These results generalize the work of DeBacker–Reeder (in the case of classical groups) from regular to arbitrary tame Langlands parameters.

1. INTRODUCTION

The representation theory of p-adic groups has largely been motivated, over the last half-century, by the Langlands conjectures, seeking an understanding of the absolute Galois (or Weil) groups of local and global fields. Many parts of the local conjectures are now theorems, notably for representations of GL_n [20, 21], of SL_n [17, 22] and, more recently, of classical groups [2, 44, 30].

At the same time as having local Langlands correspondences, one would like to be able to use them to translate fine arithmetical data between representations of p-adic groups and representations of the local Weil group. To this end, one seeks to make the correspondence explicit/effective. For GL_n, this has been the subject of a series of papers by Bushnell–Henniart [7, 8, 9, 11]; for other groups, work has concentrated on *regular depth zero irreducible cuspidal representations* [14, 25, 27] and *epipelagic irreducible cuspidal representations* [47, 19, 26, 48, 28, 10], with the most general work by Kaletha [29] on *regular cuspidal representations.*

In this spirit, we look here at depth zero irreducible cuspidal representations of a classical group G – by which we mean a symplectic, (special) orthogonal, or unitary group, which may be non-quasi-split – over a nonarchimedean locally compact local field *of odd residual characteristic* (this is the only restriction on the field). When these representations are also *regular* (more precisely, the corresponding Langlands parameter is *tame regular semisimple in general position*), these have already been considered, for more general groups but with some conditions on the field, by DeBacker–Reeder [14] and Kaletha [25, 27]; however, our approach here is different, and allows us to treat all depth zero irreducible cuspidal representations. Thus our work, and methods, are complementary to those of [29].

Given a Langlands parameter for G, the Langlands correspondence should determine an L-*packet* of irreducible smooth complex representations of G. These representations should share many properties; for example, they should have all the same L-functions,

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at least where these have been defined. Since, by the results of Shahidi [49], poles of L-functions correspond to reducibility points of parabolic induction, we detect representations in the same L-packet by computing these reducibility points, and this does not require, for example, genericity of the representation.

For now, we are not able completely to compute reducibility points, but only up to *twist by a certain unramified character* (see below for more details). However, an even-dimensional irreducible tame representation of the Weil group is symplectic if and only if this unramified twist is orthogonal; thus, using the Langlands correspondence, for example for symplectic groups, we can see which of the twists must occur, and the only ambiguity is in the reducibility points for quadratic characters of GL_1. In any case, we are able to recover the irreducible cuspidal representations in the union of either one, two or four L-packets.

This paper can be regarded as a first step in a programme to treat all discrete series representations of classical groups – see [6] for the case of arbitrary irreducible cuspidal representations of symplectic groups. Depth zero is the base case, since general irreducible cuspidal representations are built from a "wild part" and a depth zero part (see [52]). In the depth zero case, we avoid the complication of wild ramification; on the other hand, the geometric complications arise essentially from the depth zero part so that the results and techniques here already resolve many difficulties for the general case.

Now let us state our results more carefully; although we have interpreted them above via the Langlands correspondence, they are in fact results on the automorphic side. Let $\mathrm{F}/\mathrm{F}_\mathsf{o}$ be an extension of degree at most two of nonarchimedean local fields of odd residual characteristic, and let G be (the group of rational points of) a symplectic, special orthogonal or unitary group over F_o, the connected component of the group of isometries of an $\mathrm{F}/\mathrm{F}_\mathsf{o}$-hermitian space; this group may be non-quasi-split. We also write $\mathcal{W}_\mathrm{F}$ for the Weil group of F and $\widehat{\mathrm{G}}$ for the complex dual group of G, acting naturally on a vector space of dimension $N_{\widehat{\mathrm{G}}}$.

In their classification of discrete series representations of (quasi-split) p-adic classical groups [43, 38, 40], Mœglin–Tadić use the notion of a *Jordan set* attached to an irreducible discrete series representation of G. For an irreducible cuspidal representation π of G, this can be described via the *reducibility set* $\mathrm{Red}(\pi)$ as follows.

We denote by $\mathcal{A}^\sigma(\mathrm{F})$ the set of (equivalence classes of) self-dual irreducible cuspidal representations of some $\mathrm{GL}_n(\mathrm{F})$ (see Section 3). For $\rho \in \mathcal{A}^\sigma(\mathrm{F})$ there is at most one real number $s = s_\pi(\rho) \geqslant 0$ such that the normalized parabolically induced representation $\mathrm{Ind}\,\rho|\det(\cdot)|_\mathrm{F}^s \otimes \pi$ is reducible, where $|\cdot|_\mathrm{F}$ is the normalized absolute value on F; when there is no such real number (which can happen only for even-dimensional special orthogonal groups and ρ a quadratic character of $\mathrm{GL}_1(\mathrm{F})$), we set $s_\pi(\rho) = 0$. Then

$$\mathrm{Red}(\pi) \;=\; \{(\rho, m) : \rho \in \mathcal{A}^\sigma(\mathrm{F}),\ m \in \mathbf{N} \text{ with } 2s_\pi(\rho) = m+1\}\,.$$

Mœglin proves in [39] that, again for π irreducible cuspidal, the Jordan set is

$$\begin{aligned} \mathrm{Jord}(\pi) &= \{(\rho, m) : \rho \in \mathcal{A}^\sigma(\mathrm{F}),\ m \in \mathbf{N} \text{ with } 2s_\pi(\rho) - (m+1) \in 2\mathbf{Z}_{\geqslant 0}\} \\ &= \{(\rho, m) : (\rho, m_\rho) \in \mathrm{Red}(\pi),\ m \in \mathbf{N} \text{ and } m_\rho - m \in 2\mathbf{Z}_{\geqslant 0}\}\,, \end{aligned} \tag{1.1}$$

so that $\mathrm{Red}(\pi)$ is the set of maximal elements of $\mathrm{Jord}(\pi)$.

It is expected (and in at least when G is quasi-split, known – see, for example, [42]) that the Jordan set should precisely predict the Langlands parameter $\varphi : \mathcal{W}_\mathrm{F} \times \mathrm{SL}_2(\mathbf{C}) \to$

$\widehat{\mathrm{G}} \rtimes \mathcal{W}_{\mathrm{F}}$ whose L-packet Π_φ contains π, by

$$\varphi \;=\; \bigoplus_{(\rho,m)\in \mathrm{Jord}(\pi)} \varphi_\rho \otimes \mathrm{st}_m,$$

where φ_ρ is the (irreducible) representation of the Weil group $\mathcal{W}_{\mathrm{F}}$ corresponding to ρ via the Langlands correspondence for general linear groups, and st_m is the m-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. In particular, writing n_ρ for the unique natural number such that ρ is a representation of $\mathrm{GL}_{n_\rho}(\mathrm{F})$, we should have

$$\sum_{(\rho,m)\in \mathrm{Jord}(\pi)} m n_\rho \;=\; N_{\widehat{\mathrm{G}}}. \tag{1.2}$$

For a fixed ρ, the integer

$$\sum_{(\rho,m)\in \mathrm{Jord}(\pi)} m$$

is the *multiplicity* of ρ (or of φ_ρ) in $\varphi|_{\mathcal{W}_{\mathrm{F}}}$. Since, for any integer $m' \geqslant -1$, we have

$$\sum_{\substack{0\leqslant m\leqslant m',\\ m\equiv m' \ (\mathrm{mod}\ 2)}} m = \left\lfloor \left(\tfrac{m'+1}{2}\right)^2 \right\rfloor,$$

where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x, it follows from (1.1) that the equality (1.2) is equivalent to

$$\sum_{\rho\in\mathcal{A}^\sigma(\mathrm{F})} \left\lfloor (s_\pi(\rho))^2 \right\rfloor n_\rho \;=\; N_{\widehat{\mathrm{G}}}. \tag{1.3}$$

Note that almost all terms in this sum are zero since $s_\rho(\pi) < 1$ for all but finitely many $\rho \in \mathcal{A}^\sigma(\mathrm{F})$.

Suppose now that the representation π is of depth zero; equivalently, the Langlands parameter is *tame* (i.e. trivial on restriction to the wild inertia subgroup of $\mathcal{W}_{\mathrm{F}}$). For clarity of exposition, we specialize temporarily to the case of a symplectic group G, in which case $\widehat{\mathrm{G}}$ is a special orthogonal group with $N_{\widehat{\mathrm{G}}}$ odd. On the other hand, by a result of Blondel [3], there are self-dual irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ only for n even or $n = 1$; in the latter case, we get the pair of unramified characters of order dividing two, and the pair of (tamely) ramified quadratic characters. Since $N_{\widehat{\mathrm{G}}}$ is odd, equation (1.2) implies that there is exactly one pair of characters (either the unramified pair or the ramified pair) for which the multiplicities of the two characters in $\varphi|_{\mathcal{W}_{\mathrm{F}}}$ have the same parity; we denote by φ' the Langlands parameter obtained from φ by exchanging the multiplicities of the two characters in this pair. (In particular, we have $\varphi' = \varphi$ when the multiplicities are equal.) In this paper, we do the following;

(i) Given a tame Langlands parameter φ for a symplectic group as above, we give an explicit algorithm to describe the cuspidal representations in the union of L-packets $\Pi_\varphi \cup \Pi_{\varphi'}$.
(ii) Conversely, given a depth-zero cuspidal irreducible representation π of a symplectic group as above, there we give an explicit description of the pair $\{\varphi, \varphi'\}$ of tame Langlands parameters, such that $\pi \in \Pi_\varphi \cup \Pi_{\varphi'}$.

We remark that, in the situation of *regular* depth zero irreducible cuspidal representations, the multiplicities of the characters are all at most one, so that $\varphi' = \varphi$; thus we recover the description of the representations in an L-packet consisting solely of cuspidal representations from [14] in this case.

We return to the case of depth zero representations of a general classical group G and describe the result here, which is a reinterpretation of the theorem above in terms of the set $\mathrm{Red}(\pi)$. More precisely, denote by $[\rho]$ the *inertial equivalence class* of $\rho \in \mathcal{A}^\sigma(\mathrm{F})$, that is, the set of unramified twists of ρ; note that $[\rho] \cap \mathcal{A}^\sigma(\mathrm{F}) = \{\rho, \rho'\}$ consists of exactly two (inequivalent) representations. We have $\rho' = \rho\chi$, for χ an unramified character with $\rho\chi^2 = \rho$; since ρ has depth zero, the character χ has order $2n_\rho$.

Here, we compute the *inertial reducibility multiset*

$$\mathrm{IRed}(\pi) \;=\; \{\{([\rho], m) : (\rho, m) \in \mathrm{Red}(\pi)\}\}.$$

This is often in fact a set: since π has depth zero, the only inertial classes $[\rho]$ which can occur with multiplicity are the quadratic characters of $\mathrm{GL}_1(\mathrm{F})$. Indeed, for $\rho \in \mathcal{A}^\sigma(\mathrm{F})$ of depth zero with $n_\rho > 1$ and ρ' its self-dual unramified twist, the exterior square L-function of exactly one of ρ, ρ' has a pole at $s = 0$, while for the other representation it is the symmetric square L-function which has a pole (the same comments apply to the Asai and twisted Asai L-functions when $\mathrm{F}/\mathrm{F}_\circ$ is quadratic); thus the parity of m, such that $(\rho, m) \in \mathrm{Red}(\pi)$ should be independent of π (i.e. depend only on ρ), and the parity will be the opposite of that for ρ'. Moreover, this means that by computing $\mathrm{IRed}(\pi)$, we in fact know all elements of $\mathrm{Red}(\pi)$ apart from those associated to characters of $\mathrm{GL}_1(\mathrm{F})$ of order at most two, where an ambiguity may remain.

The results we prove here can be described by the following: Let π be a depth zero irreducible cuspidal representation of G.

(i) We have

$$\sum_{\rho \in \mathcal{A}^\sigma(\mathrm{F})} \left\lfloor (s_\pi(\rho))^2 \right\rfloor n_\rho \;\geqslant\; N_{\widehat{\mathrm{G}}}. \tag{1.4}$$

(See also Section 8 for a discussion of the opposite inequality, which is already known in many cases by results of Mœglin.)

(ii) We give an explicit description of the multiset $\mathrm{IRed}(\pi)$ in terms of the local data defining π as a compactly induced representation.

(iii) We give an explicit description of the set of irreducible cuspidal representations π' of G with $\mathrm{IRed}(\pi) = \mathrm{IRed}(\pi')$ in terms of the local data defining π. Moreover, the number of such representations is the expected number in one, two or four L-packets, this number depending again on the local data.

For a discussion of the expected number of irreducible cuspidal representations in an L-packet, see the beginning of Section 9. One also needs to take care with this in the case of even orthogonal groups (see Example 9.6).

Structure of the algorithm. We now give a description of the algorithm alluded to above. Let G be a classical group, acting naturally on a nondegenerate N-dimensional $\mathrm{F}/\mathrm{F}_\circ$-$\varepsilon$-hermitian space V. The standard maximal parahoric subgroups $\mathrm{J}^\circ_{N_1,N_2}$ of G are indexed by non-negative integers N_1, N_2 with $N_1 + N_2 = N$, and have reductive quotient $\mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2}$ a product of two classical groups over finite fields (see Section 2). Then, from results of Morris, any irreducible cuspidal depth zero representation of G has the form $\pi = \text{c-Ind}_{\mathrm{J}_\pi}^{\mathrm{G}} \lambda_\pi$, with J_π the normalizer of a (unique) standard maximal parahoric subgroup $\mathrm{J}^\circ_{N_1,N_2}$ and λ_π an irreducible representation such that $\lambda_\pi|_{\mathrm{J}^\circ_{N_1,N_2}}$ contains an irreducible representation λ°_π inflated from an irreducible cuspidal representation $\tau_\pi \simeq \tau^{(1)}_\pi \otimes \tau^{(2)}_\pi$ of the reductive quotient $\mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2}$ (see Section 3). From Lusztig's Jordan decomposition, for $i = 1, 2$, there is a unique conjugacy class (s_i) in the dual group $\mathcal{G}_{N_i}^{(i),*}$ such that $\tau_\pi^{(i)}$ is in the Lusztig series $\mathscr{E}(\mathcal{G}_{N_i}^{(i)}, s_i)$, and we denote the characteristic polynomial of s_i by

$$\prod_P P(X)^{a_P^{(i)}},$$

where the product runs over irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomials in $k_{\mathrm{F}}[X]$ (see Section 7). Results of Lusztig imply that there are integers $m_P^{(i)} \geqslant 0$ such that:

- if $k_{\mathrm{F}} \neq k_{\mathsf{o}}$ or $P(X) \neq (X \pm 1)$ then $a_P^{(i)} = \frac{1}{2} m_P^{(i)}(m_P^{(i)} + 1)$;
- if $k_{\mathrm{F}} = k_{\mathsf{o}}$ and $P(X) = (X \pm 1)$ then we write $m_+^{(i)} = m_{(X-1)}^{(i)}$ and $m_-^{(i)} = m_{(X+1)}^{(i)}$, and these satisfy additional conditions below in (7.2).

Now, for each irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomial P of degree n_P in $k_{\mathrm{F}}[X]$, there is a unique irreducible cuspidal representation τ_P of $\mathrm{GL}_{n_P}(k_{\mathrm{F}})$ which lies in the Lusztig series $\mathscr{E}(\mathrm{GL}_{n_P}(k_{\mathrm{F}}), s_P)$, where s_P has characteristic polynomial P. Inflating this to an irreducible representation λ_P of $\mathrm{J}_P = \mathrm{GL}_{n_P}(\mathfrak{o}_{\mathrm{F}})$, extending trivially on a uniformizer and inducing, we obtain an irreducible self-dual cuspidal representation ρ_P of $\mathrm{GL}_{n_P}(\mathrm{F})$. Then, when $n_P > 1$, we have

$$([\rho_P], m_P^{(1)} + m_P^{(2)}), ([\rho_P], |m_P^{(1)} - m_P^{(2)}| - 1) \in \mathrm{IRed}(\pi),$$

where we understand that terms $([\rho_P], m)$ are ignored when the multiplicity m is non-positive. For $P(X) = X \pm 1$, the multiplicities depend on the type of group (see Section 8); for example in the case of symplectic groups we have the following elements of $\mathrm{IRed}(\pi)$:

$$([\mathbf{1}], 2(m_+^{(1)} + m_+^{(2)}) + 1), ([\mathbf{1}], 2|m_+^{(1)} - m_+^{(2)}| - 1),$$
$$([\omega_1], 2(m_-^{(1)} + m_-^{(2)}) - 1), ([\omega_1], 2|m_-^{(1)} + m_-^{(2)}| - 1),$$

where $\mathbf{1}$ is the trivial character and ω_1 is a ramified quadratic character of $\mathrm{GL}_1(\mathrm{F})$, and again we ignore terms with non-positive multiplicity.

Now, to describe the other cuspidal representations with the same inertial reducibility set, we set

$$\mathrm{Q}(\pi) \;=\; \left\{\text{irreducible self-dual monic } P \in k_{\mathrm{F}}[X] : m_P^{(1)}(\pi) \neq m_P^{(2)}(\pi)\right\}$$

and put $q(\pi) := \#\mathrm{Q}(\pi)$. The irreducible cuspidals π' with $\mathrm{IRed}(\pi') = \mathrm{IRed}(\pi)$ are parametrized by certain maps $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$, with conditions depending on the type of group. (See Section 9 for the detail of these conditions; there are, for example, no conditions in the symplectic case.) Given a suitable map $\boldsymbol{\varepsilon}$, we extend it to all irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomials by setting $\boldsymbol{\varepsilon}(P) = 1$ for $P \notin \mathrm{Q}(\pi)$. Then we can find, for $i = 1, 2$, a semisimple element $s_{\boldsymbol{\varepsilon}}^{(i)}$ in a suitable classical group with characteristic polynomial

$$\prod_P P(X)^{a_P^{(i)}(\boldsymbol{\varepsilon})},$$

where the integers $a_P^{(i)}(\boldsymbol{\varepsilon})$ are related to integers $m_P^{(i)}(\boldsymbol{\varepsilon})$ as above, with

$$m_P^{(i)}(\boldsymbol{\varepsilon}) \;=\; m_P^{(i \cdot \boldsymbol{\varepsilon}(P))},$$

where the index is understood modulo 3. Correspondingly, we have irreducible cuspidal representations $\tau_{\boldsymbol{\varepsilon}} = \tau_{\boldsymbol{\varepsilon}}^{(1)} \otimes \tau_{\boldsymbol{\varepsilon}}^{(2)}$ of the reductive quotient $\mathcal{G}_{N_1'}^{(1)} \times \mathcal{G}_{N_2'}^{(2)}$ of a standard parahoric; note that, for each $\boldsymbol{\varepsilon}$, there may be several representations $\tau_{\boldsymbol{\varepsilon}}$, depending on the number of cuspidal representations in each Lusztig series, but this number is independent of the choice of $\boldsymbol{\varepsilon}$. Inflating each $\tau_{\boldsymbol{\varepsilon}}$ to the maximal parahoric subgroup $\mathrm{J}_{N_1',N_2'}$, extending to its normalizer and inducing to G, we get an irreducible cuspidal representation, and this gives all the irreducible cuspidal representations π' with $\mathrm{IRed}(\pi') = \mathrm{IRed}(\pi)$.

Structure of the proof. Finally, we describe the ideas behind the proof. The computation of reducibility points required is achieved using Bushnell–Kutzko's theory of covers [12], together with results of Blondel [4], which translate the problem to the computation of parameters in the Hecke algebra of a cover (see Sections 5–6).

Given π, ρ_P as in the description of the algorithm, we can consider $\rho_P \otimes \pi$ as a cuspidal representation of a Levi subgroup $\mathrm{GL}_{n_P}(\mathrm{F}) \times \mathrm{G}$ of a larger classical group. The type $(\mathrm{J}_P \times \mathrm{J}_\pi, \lambda_P \otimes \lambda_\pi)$ admits a cover (J, λ) constructed by Morris (see also [37]) and the corresponding spherical Hecke algebra is an algebra on an infinite dihedral group, generated by two elements satisfying quadratic relations. The recipe of Blondel allows one to translate the knowledge of these quadratic relations into the pair of reducibility points $s_\pi(\rho_P), s_\pi(\rho'_P)$, where ρ'_P is the self-dual unramified twist of ρ_P inequivalent to ρ_P.

To compute these quadratic relations, the results of [37] first reduce to the calculation of quadratic relations in *finite* Hecke algebras for $\tau_P \otimes \tau_\pi^{(i)}$, for $i = 1, 2$, which are described by Howlett–Lehrer [23]. To compute these values, we use results of Lusztig on representations of finite reductive groups (recalled in Section 7), which reduce the computation to *unipotent* cuspidal representations, in which case Lusztig has performed the computation [35]. Note, however, that care must be taken since the finite reductive groups which occur do not, in general, have connected centre. There is particular difficulty for (even-dimensional) special orthogonal groups and the results we obtain here may be of independent interest; in particular, we compute when an irreducible cuspidal representation of an even-dimensional special orthogonal group over a finite field extends to the full orthogonal group (see Proposition 7.10), generalizing results of Lusztig and Waldspurger.

Having computed these parameters, we can then put all the ingredients together to prove the main results in Sections 8 and 9; the latter includes some illustrative examples.

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2. Notation and background

We fix some notation for the rest of the paper (with the exception of Section 7, whose notation is independent). Let F_o be a locally compact nonarchimedean local field of *odd* residual characteristic p, and let $\mathrm{F}/\mathrm{F}_\mathsf{o}$ be an extension of degree at most 2. We write $\sigma : \lambda \mapsto \overline{\lambda}$ for the generator of the Galois group of $\mathrm{F}/\mathrm{F}_\mathsf{o}$. For E any field containing F_o, we write $\mathfrak{o}_\mathrm{E}$ for its ring of integers, $\mathfrak{p}_\mathrm{E}$ for its maximal ideal, and $k_\mathrm{E} = \mathfrak{o}_\mathrm{E}/\mathfrak{p}_\mathrm{E}$ for its residue field, of cardinality q_E; in particular we abbreviate $q = q_\mathrm{F}$. We also abbreviate $\mathfrak{o}_\mathsf{o} = \mathfrak{o}_{\mathrm{F}_\mathsf{o}}$ etc. We also fix a uniformizer ϖ_F of F such that $\overline{\varpi_\mathrm{F}} = -\varpi_\mathrm{F}$ if F/F_o is quadratic ramified, and $\overline{\varpi_\mathrm{F}} = \varpi_\mathrm{F}$ otherwise, and write $\mathrm{N}_{\mathrm{F}/\mathrm{F}_\mathsf{o}} : \mathrm{F}^\times \to \mathrm{F}_\mathsf{o}^\times$ for the norm map, which is given by $\lambda \mapsto \lambda\overline{\lambda}$ if $[\mathrm{F} : \mathrm{F}_\mathsf{o}] = 2$.

We fix a sign $\varepsilon = \pm 1$, and let (V, h) be a nondegenerate $\mathrm{F}/\mathrm{F}_\mathsf{o}$-$\varepsilon$-hermitian space of Witt index N and dimension $2N + N^{\mathsf{an}}$; thus V is an F-vector space, the form h satisfies

$$h(\lambda v, \mu w) = \lambda\overline{\mu}h(v, w) = \varepsilon\lambda\overline{\mu}\overline{h(w, v)}, \qquad \text{for } v, w \in \mathrm{V},\ \lambda, \mu \in \mathrm{F},$$

and we have a Witt decomposition

$$\mathrm{V} = \mathrm{V}^- \oplus \mathrm{V}^{\mathsf{an}} \oplus \mathrm{V}^+,$$

with $\dim_{\mathrm{F}} \mathrm{V}^\pm = N$ and $\dim_{\mathrm{F}} \mathrm{V}^{\mathsf{an}} = N^{\mathsf{an}}$, such that the restriction of h to $\mathrm{V}^\pm$ is totally isotropic, while its restriction h^{an} to V^{an} is anisotropic. We denote by $\mathrm{H} = \mathrm{H}^- \oplus \mathrm{H}^+$ the hyperbolic plane; that is, $\mathrm{H}^\pm$ is a 1-dimensional F-vector space with basis $\mathrm{e}_\pm$ and H is equipped with the form h_{H} given by

$$h_{\mathrm{H}}(\lambda_-\mathrm{e}_- + \lambda_+\mathrm{e}_+, \mu_-\mathrm{e}_- + \mu_+\mathrm{e}_+) = \lambda_-\overline{\mu_+} + \varepsilon\lambda_+\overline{\mu_-}, \qquad \text{for } \lambda_\pm, \mu_\pm \in \mathrm{F}.$$

Thus the restriction of h to $\mathrm{V}^- \oplus \mathrm{V}^+$ is (isometric to) an orthogonal direct sum of N copies of H. We choose a Witt basis for V, that is: $\mathrm{e}_1^+, \ldots, \mathrm{e}_N^+$ a basis for V^+, with dual basis $\mathrm{e}_1^-, \ldots, \mathrm{e}_N^-$ for V^-, and $\mathrm{e}_1^{\mathsf{an}}, \ldots, \mathrm{e}_{N^{\mathsf{an}}}^{\mathsf{an}}$ a basis V^{an} with respect to which h^{an} has diagonal Gram matrix. We order this basis

$$\mathrm{e}_N^-, \ldots, \mathrm{e}_1^-, \mathrm{e}_1^{\mathsf{an}}, \ldots, \mathrm{e}_{N^{\mathsf{an}}}^{\mathsf{an}}, \mathrm{e}_1^+, \ldots, \mathrm{e}_N^+.$$

For $n \geqslant 0$, we denote by $n\mathrm{H}$ the orthogonal direct sum of n copies of H, and put

$$\mathrm{V}_n = \mathrm{V} \oplus n\mathrm{H}$$

with the form $h_n = h \oplus h_{\mathrm{H}} \oplus \cdots \oplus h_{\mathrm{H}}$, so that the decomposition above is orthogonal and we have a Witt decomposition

$$\mathrm{V}_n = \mathrm{V}_n^- \oplus \mathrm{V}^{\mathsf{an}} \oplus \mathrm{V}_n^+, \qquad \text{with } \mathrm{V}_n^\pm = \mathrm{V}^\pm \oplus n\mathrm{H}^\pm.$$

Thus $(\mathrm{V}_n : n \geqslant 0)$ is a Witt tower over $\mathrm{V}_0 = \mathrm{V}$. Writing $\mathrm{e}_{N+i}^\pm$ for the image in V_n of $\mathrm{e}_\pm$ in the i^{th} copy of H, the space V_n has the ordered Witt basis

$$\mathrm{e}_{N+n}^-, \ldots, \mathrm{e}_1^-, \mathrm{e}_1^{\mathsf{an}}, \ldots, \mathrm{e}_{N^{\mathsf{an}}}^{\mathsf{an}}, \mathrm{e}_1^+, \ldots, \mathrm{e}_{N+n}^+.$$

For $n \geqslant 0$, we put $\mathrm{G}_n^+ = \mathrm{U}(\mathrm{V}_n)$, the group of F_o-rational points of the reductive algebraic group over F_o determined by (V_n, h_n), so that

$$\mathrm{G}_n^+ = \{g \in \mathrm{Aut}_{\mathrm{F}}(\mathrm{V}_n) : h_n(gv, gw) = h_n(v, w) \text{ for all } v, w, \in \mathrm{V}\};$$

thus G_n^+ is (the group of points of) a unitary, symplectic or (full) orthogonal group. We also put $\mathrm{G}_n = \mathrm{U}(\mathrm{V}_n)^\circ$, the group of F_o-rational points of the connected component, so that

$$\mathrm{G}_n = \{g \in \mathrm{G}_n^+ : \mathrm{N}_{\mathrm{F}/\mathrm{F}_\mathsf{o}} \det_{\mathrm{F}}(g) = 1\};$$

thus $\mathrm{G}_n = \mathrm{G}_n^+$ unless G_n^+ is an orthogonal group (so $\mathrm{F} = \mathrm{F}_\mathsf{o}$ and $\varepsilon = 1$), in which case G_n is the special orthogonal group, of index 2 in G_n^+. We will abbreviate $\mathrm{G} = \mathrm{G}_0$ and $\mathrm{G}^+ = \mathrm{G}_0^+$.

The stabilizer in G_n of the decomposition

$$\mathrm{V}_n = n\mathrm{H}^- \oplus \mathrm{V} \oplus n\mathrm{H}^+$$

is a Levi subgroup M_n of G_n, which is standard with respect to the chosen Witt basis, and we have an isomorphism $\mathrm{M}_n \simeq \mathrm{GL}_n(\mathrm{F}) \times \mathrm{G}$ given by $g \mapsto (g|_{n\mathrm{H}^-}, g|_{\mathrm{V}})$; moreover, the stabilizer of the subspace $n\mathrm{H}^-$ is a standard parabolic subgroup P_n of G_n, with Levi component M_n. Thus, writing elements of G_n as matrices with respect to the Witt basis, the group P_n is block upper triangular and M_n is block diagonal.

We end this section with a description of the maximal parahoric subgroups of G and of their reductive quotients (see also [45]). For L an $\mathfrak{o}_{\mathrm{F}}$-lattice in V, we denote by $\mathrm{L}^{\#}$ the dual lattice

$$\mathrm{L}^{\#} = \{v \in \mathrm{L} : h(v, \mathrm{L}) \subseteq \mathfrak{p}_{\mathrm{F}}\}.$$

We say that L is *almost self-dual* if

$$\mathrm{L} \supseteq \mathrm{L}^{\#} \supseteq \mathfrak{p}_{\mathrm{F}}\mathrm{L};$$

in that case, the stabilizer $\mathrm{J} = \mathrm{J}_{\mathrm{L}}$ in G of L is a maximal compact subgroup of G, and every maximal compact subgroup arises in this way for a unique self-dual lattice L. We write J^1 for the pro-unipotent radical of J, that is the subgroup consisting of those elements g which induce the identity map on the k_{F}-vector spaces $\overline{\mathrm{V}}_{(1)} := \mathrm{L}/\mathrm{L}^{\#}$ and $\overline{\mathrm{V}}_{(2)} := \mathrm{L}^{\#}/\mathfrak{p}_{\mathrm{F}}\mathrm{L}$.

The form h induces nondegenerate $k_{\mathrm{F}}/k_{\mathsf{o}}$-forms on $\overline{\mathrm{V}}_{(1)}$ and $\overline{\mathrm{V}}_{(2)}$ by

$$\begin{cases} h_{\overline{\mathrm{V}}}(v + \mathrm{L}^{\#}, w + \mathrm{L}^{\#}) := h(v, w) + \mathfrak{p}_{\mathrm{F}}, & \text{for } v, w \in \mathrm{L}, \\ h_{\overline{\mathrm{V}}^{\#}}(v' + \mathfrak{p}_{\mathrm{F}}\mathrm{L}, w' + \mathfrak{p}_{\mathrm{F}}\mathrm{L}) := \varpi_{\mathrm{F}}^{-1} h(v', w') + \mathfrak{p}_{F}, & \text{for } v', w' \in \mathrm{L}^{\#}. \end{cases}$$

The form $h_{\overline{\mathrm{V}}_{(1)}}$ is ε-hermitian, while the form $h_{\overline{\mathrm{V}}_{(2)}}$ is $(-\varepsilon)$-hermitian if $\mathrm{F}/\mathrm{F}_{\mathsf{o}}$ is quadratic ramified, and ε-hermitian otherwise, by our choice of uniformizer. Thus we get an induced map

$$\mathrm{J} \to \mathrm{U}(\overline{\mathrm{V}}_{(1)}) \times \mathrm{U}(\overline{\mathrm{V}}_{(2)}),$$

with kernel J^1, and hence the quotient $\mathcal{G} = \mathcal{G}_{\mathrm{L}} = \mathrm{J}/\mathrm{J}^1$ is naturally a subgroup of the finite reductive group $\mathrm{U}(\overline{\mathrm{V}}_{(1)}) \times \mathrm{U}(\overline{\mathrm{V}}_{(2)})$. In fact, $\mathcal{G}$ identifies with the subgroup

$$\left\{(g_1, g_2) \in \mathrm{U}(\overline{\mathrm{V}}_{(1)}) \times \mathrm{U}(\overline{\mathrm{V}}_{(2)}) : \mathrm{N}_{k_{\mathrm{F}}/k_{\mathsf{o}}}(\det_{k_{\mathrm{F}}}(g_1)\det_{k_{\mathrm{F}}}(g_2)) = 1\right\},$$

which has connected component $\mathcal{G}^{\circ} = \mathrm{U}(\overline{\mathrm{V}}_{(1)})^{\circ} \times \mathrm{U}(\overline{\mathrm{V}}_{(2)})^{\circ}$. We denote by $\mathrm{J}^{\circ} = \mathrm{J}^{\circ}_{\mathrm{L}}$ the inverse image in J of $\mathcal{G}^{\circ}$; this is a parahoric subgroup of G and J is its normalizer in G. It is *not* always a maximal parahoric subgroup of G (it is so if and only if neither factor $\mathrm{U}(\overline{\mathrm{V}}_{(i)})^{\circ}$ is a two-dimensional special orthogonal group) but every maximal parahoric subgroup does arise in this way. If either $\mathrm{F}/\mathrm{F}_{\mathsf{o}}$ is quadratic ramified and the orthogonal space among $\overline{\mathrm{V}}_{(1)}, \overline{\mathrm{V}}_{(2)}$ is non-zero, or $\mathrm{F} = \mathrm{F}_{\mathsf{o}}$, $\varepsilon = 1$ and both $\overline{\mathrm{V}}_{(1)}, \overline{\mathrm{V}}_{(2)}$ are non-zero, then J° has index 2 in J; otherwise we have $\mathrm{J} = \mathrm{J}^{\circ}$.

Restricting first to the case of the hermitian space $(\mathrm{V}^{\mathsf{an}}, h^{\mathsf{an}})$, there is a unique almost self-dual lattice L_{an} in V^{an}, and the corresponding group $\mathrm{G}_{\mathsf{an}} = \mathrm{U}(\mathrm{V}^{\mathsf{an}})^{\circ}$ is compact and normalizes the unique (maximal) parahoric subgroup $\mathrm{J}^{\circ}_{\mathsf{an}} = \mathrm{J}^{\circ}_{\mathrm{L}_{\mathsf{an}}}$, with connected component $\mathcal{G}^{\circ}_{\mathsf{an}}$. We set

$$\overline{\mathrm{V}}^{\mathsf{an}}_{(1)} = \mathrm{L}_{\mathsf{an}}/\mathrm{L}^{\#}_{\mathsf{an}}, \qquad \overline{\mathrm{V}}^{\mathsf{an}}_{(2)} = \mathrm{L}^{\#}_{\mathsf{an}}/\mathfrak{p}_{\mathrm{F}}\mathrm{L}_{\mathsf{an}},$$

and $N^{\mathsf{an}}_i = \dim_{k_{\mathrm{F}}} \overline{\mathrm{V}}^{\mathsf{an}}_{(i)}$, so that $N^{\mathsf{an}} = N^{\mathsf{an}}_1 + N^{\mathsf{an}}_2$. Then we have the following possibilities for $\mathcal{G}^{\circ}_{\mathsf{an}} = \mathrm{U}(\overline{\mathrm{V}}^{\mathsf{an}}_{(1)})^{\circ} \times \mathrm{U}(\overline{\mathrm{V}}^{\mathsf{an}}_{(2)})^{\circ}$, where we write $\mathrm{SO}(M_1, M_2, k_{\mathrm{F}})$ for the special orthogonal group with form of Witt index M_2 and anisotropic part of dimension $M_1 - M_2 \leqslant 2$:

- If $\mathrm{F} = \mathrm{F}_{\mathsf{o}}$ and $\varepsilon = -1$ then $N^{\mathsf{an}} = 0$ so $\mathcal{G}^{\circ}_{\mathsf{an}}$ is trivial.
- If $\mathrm{F} = \mathrm{F}_{\mathsf{o}}$ and $\varepsilon = 1$ then $\mathcal{G}^{\circ}_{\mathsf{an}} \simeq \mathrm{SO}(N^{\mathsf{an}}_1, 0, k_{\mathrm{F}}) \times \mathrm{SO}(N^{\mathsf{an}}_2, 0, k_{\mathrm{F}})$, with $N^{\mathsf{an}}_i \leqslant 2$.
- If $\mathrm{F}/\mathrm{F}_{\mathsf{o}}$ is unramified quadratic, then $\mathcal{G}^{\circ}_{\mathsf{an}} \simeq \mathrm{U}(N^{\mathsf{an}}_1, k_{\mathrm{F}}/k_{\mathrm{F}_{\mathsf{o}}}) \times \mathrm{U}(N^{\mathsf{an}}_2, k_{\mathrm{F}}/k_{\mathrm{F}_{\mathsf{o}}})$, the product of two unitary groups with $N^{\mathsf{an}}_i \leqslant 1$.

- If $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is ramified quadratic then
$$\mathcal{G}^\circ_{\mathsf{an}} \simeq \begin{cases} \mathrm{SO}(N_1^{\mathsf{an}}, 0, k_\mathrm{F}) & \text{if } \varepsilon = +1, \\ \mathrm{SO}(N_2^{\mathsf{an}}, 0, k_\mathrm{F}) & \text{if } \varepsilon = -1, \end{cases}$$
with $N_i^{\mathsf{an}} \leqslant 2$ and only one N_i^{an} non-zero.

Returning to the general case of the space (V, h), the *standard* almost self-dual lattices are those of the following form: for $0 \leqslant N_1, N_2$ with $N_1 + N_2 = N$, set
$$\mathrm{L}_{N_1,N_2} := \mathfrak{o}_\mathrm{F}\mathrm{e}_N^- \oplus \cdots \oplus \mathfrak{o}_\mathrm{F}\mathrm{e}_1^- \oplus \mathrm{L}^{\mathsf{an}} \oplus \mathfrak{o}_\mathrm{F}\mathrm{e}_1^+ \oplus \cdots \oplus \mathfrak{o}_\mathrm{F}\mathrm{e}_{N_1}^+ \oplus \mathfrak{p}_\mathrm{F}\mathrm{e}_{N_1+1}^+ \oplus \cdots \oplus \mathfrak{p}_\mathrm{F}\mathrm{e}_N^+,$$
where L^{an} is the unique almost self-dual lattice in V^{an}. We write J_{N_1,N_2} for the stabilizer of L_{N_1,N_2} and $\mathrm{J}^\circ_{N_1,N_2}$ for the corresponding parahoric subgroup. Every almost self-dual lattice has the form $g\mathrm{L}_{N_1,N_2}$, for some $g \in \mathrm{G}$ and a unique standard lattice L_{N_1,N_2}; thus every maximal compact (respectively, maximal parahoric) subgroup is conjugate to a unique standard one J_{N_1,N_2} (respectively, $\mathrm{J}^\circ_{N_1,N_2}$). The choice of Witt basis and the forms on $\overline{\mathrm{V}}_{(1)}, \overline{\mathrm{V}}_{(2)}$ then give us the following identifications for the connected reductive quotients $\mathcal{G}^\circ_{N_1,N_2} = \mathrm{J}^\circ_{N_1,N_2}/\mathrm{J}^1_{N_1,N_2}$:

- If $\mathrm{F} = \mathrm{F}_\mathsf{o}$ and $\varepsilon = -1$ then
$$\mathcal{G}^\circ_{N_1,N_2} \simeq \mathrm{Sp}(2N_1, k_\mathrm{F}) \times \mathrm{Sp}(2N_2, k_\mathrm{F}).$$
- If $\mathrm{F} = \mathrm{F}_\mathsf{o}$ and $\varepsilon = 1$ then
$$\mathcal{G}^\circ_{N_1,N_2} \simeq \mathrm{SO}(N_1 + N_1^{\mathsf{an}}, N_1, k_\mathrm{F}) \times \mathrm{SO}(N_2 + N_2^{\mathsf{an}}, N_2, k_\mathrm{F}).$$
- If $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is quadratic unramified then
$$\mathcal{G}^\circ_{N_1,N_2} \simeq \mathrm{U}(2N_1 + N_1^{\mathsf{an}}, k_\mathrm{F}/k_\mathsf{o}) \times \mathrm{U}(2N_2 + N_2^{\mathsf{an}}, k_\mathrm{F}/k_\mathsf{o}).$$
- If $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is quadratic ramified then
$$\mathcal{G}^\circ_{N_1,N_2} \simeq \begin{cases} \mathrm{SO}(N_1 + N_1^{\mathsf{an}}, N_1, k_\mathrm{F}) \times \mathrm{Sp}(2N_2, k_\mathrm{F}) & \text{if } \varepsilon = +1, \\ \mathrm{Sp}(2N_1, k_\mathrm{F}) \times \mathrm{SO}(N_2 + N_2^{\mathsf{an}}, N_2, k_\mathrm{F}) & \text{if } \varepsilon = -1. \end{cases}$$

Writing $\overline{\mathrm{H}}$ for hyperbolic space over k_F, we can unify these by writing
$$\mathcal{G}^\circ_{N_1,N_2} \simeq \mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2},$$
with $\mathcal{G}^{(i)}_{N_i} = \mathrm{U}(N_i\overline{\mathrm{H}} \oplus \overline{\mathrm{V}}^{\mathsf{an}}_{(i)})^\circ$, for $i = 1, 2$.

We note that $\mathrm{J}^\circ_{N_1,N_2}$ is a maximal parahoric subgroup except where one of the factors here is $\mathrm{SO}(1, 1, k_\mathrm{F})$ but G is not itself a 2-dimensional special orthogonal group; that is, in the following cases:

- $\mathrm{F} = \mathrm{F}_\mathsf{o}$, $\varepsilon = 1$, $(N, N^{\mathsf{an}}) \neq (1, 0)$, with $(N_i, N_i^{\mathsf{an}}) = (1, 0)$, for $i = 1$ or 2;
- $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is quadratic ramified, $N^{\mathsf{an}} = 0$ and $(\varepsilon, N_1) = (1, 1)$ or $(\varepsilon, N_2) = (-1, 1)$.

3. Depth zero cuspidal representations

In this section, we recall the classification of the depth zero irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ and of the classical group G, beginning with the former.

We write $\mathcal{A}_n(\mathrm{F})$ for the set of equivalence classes of irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$ and put $\mathcal{A}(\mathrm{F}) = \bigcup_{n\geqslant 1} \mathcal{A}_n(\mathrm{F})$. We will abuse notation by writing $\rho \in \mathcal{A}(\mathrm{F})$ to mean ρ is an irreducible cuspidal representation of some $\mathrm{GL}_n(\mathrm{F})$, where $n = n_\rho$ is of course uniquely determined by ρ. For $\rho \in \mathcal{A}_n(\mathrm{F})$, we denote by ρ^σ the representation
$$\rho^\sigma(g) = \rho(\sigma(g^{-1})^\mathrm{T}), \qquad \text{for } g \in \mathrm{GL}_n(\mathrm{F}),$$

where $\sigma(g)$ denotes the matrix obtained by applying the generator σ of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_\circ)$ to each entry, and g^{T} denotes the transpose matrix. We say that ρ is *self-dual* if $\rho^\sigma \simeq \rho$, and write $\mathcal{A}_n^\sigma(\mathrm{F})$ for the set of equivalence classes of self-dual irreducible cuspidal representations of $\mathrm{GL}_n(\mathrm{F})$, and $\mathcal{A}^\sigma(\mathrm{F}) = \bigcup_{n\geqslant 1} \mathcal{A}_n^\sigma(\mathrm{F})$ for the set of equivalence classes of self-dual representations in $\mathcal{A}(\mathrm{F})$.

We do not recall here the general notion of depth, only that a representation $\rho \in \mathcal{A}(\mathrm{F})$ is said to be of *depth zero* if it has fixed vectors under the pro-unipotent radical of the maximal parahoric subgroup $\mathrm{GL}_{n_\rho}(\mathfrak{o}_\mathrm{F})$ of $\mathrm{GL}_{n_\rho}(\mathrm{F})$. We denote by $\mathcal{A}_{[0]}(\mathrm{F})$ the set of equivalence classes of depth zero representations in $\mathcal{A}(\mathrm{F})$, and by $\mathcal{A}_{[0]}^\sigma(\mathrm{F})$ the set of equivalence classes of self-dual depth zero representations in $\mathcal{A}(\mathrm{F})$.

Any depth zero representation $\rho \in \mathcal{A}_n(\mathrm{F})$ can be written

$$\rho \;=\; \text{c-Ind}_{\mathbf{J}_\rho}^{\mathrm{GL}_n(\mathrm{F})} \Lambda_\rho,$$

where $\mathbf{J}_\rho = \mathrm{F}^\times \mathrm{GL}_n(\mathfrak{o}_\mathrm{F})$ is the normalizer of the maximal parahoric subgroup $\mathrm{J}_\rho = \mathrm{GL}_n(\mathfrak{o}_\mathrm{F})$ of $\mathrm{GL}_n(\mathrm{F})$, and Λ_ρ is an irreducible representation of $\mathbf{J}_\rho$ whose restriction $\lambda_\rho = \Lambda_\rho|_{\mathrm{J}_\rho}$ is the inflation of an irreducible cuspidal representation τ_ρ of the reductive quotient $\mathrm{J}_\rho/\mathrm{J}_\rho^1 \simeq \mathrm{GL}_n(k_\mathrm{F})$. Moreover, the (equivalence class of the) representation $\tau = \tau_\rho$ is uniquely determined by ρ. Further, ρ is self-dual if and only if τ is self-dual; that is, denoting again by σ the generator of $\mathrm{Gal}(k_\mathrm{F}/k_\circ)$ and by τ^σ the representation $\tau^\sigma(g) = \tau_\rho(\sigma(g^{-1})^{\mathrm{T}})$, for $g \in \mathrm{GL}_n(k_\mathrm{F})$, we have $\tau^\sigma \simeq \tau$.

The (equivalence classes of) irreducible cuspidal representations τ of $\mathrm{GL}_n(k_\mathrm{F})$ were first classified by Green [18], and are parametrized by regular characters of the multiplicative group of the degree n extension of k_F, or, equivalently (after making choices), by monic irreducible degree n polynomials $P = P_\tau \in k_\mathrm{F}[X]$ with $P(0) \neq 0$. Writing $\sigma(P)$ for the polynomial obtained by applying σ to the coefficients of P, the representation τ^σ then corresponds to the polynomial $P^\sigma(X) := \sigma(P(0))^{-1} X^{\deg(P)} \sigma(P)(1/X)$. Thus τ is self-dual if and only if $P_\tau = P_\tau^\sigma$. If $k_\mathrm{F} = k_\circ$, such polynomials exist if and only if $n = 1$ or n is even (see [1]); if $k_\mathrm{F}/k_\circ$ is quadratic, then such polynomials exist if and only if n is odd (see [31, §5.4]).

Similarly, we write $\mathcal{A}(\mathrm{G})$ for the set of equivalence classes of irreducible cuspidal representations of G, and $\mathcal{A}_{[0]}(\mathrm{G})$ for the subset of equivalence classes of depth zero representations. Then, for $\pi \in \mathcal{A}_{[0]}(\mathrm{G})$, we can write

$$\pi \;=\; \text{c-Ind}_{\mathrm{J}_\pi}^{\mathrm{G}} \lambda_\pi,$$

where $\mathrm{J}_\pi = \mathrm{J}_{N_1,N_2}$ is the (compact open) normalizer of a standard maximal parahoric subgroup J_π° and λ_π is an irreducible representation of J_π whose restriction $\lambda_\pi^\circ = \lambda_\pi|_{\mathrm{J}_\pi^\circ}$ is a sum of conjugates (under J_π) of the inflation of an irreducible cuspidal representation τ_π of the reductive quotient $\mathcal{G}_{N_1,N_2}^\circ$. By [46, 53], the standard maximal parahoric subgroup J_π° is uniquely determined by π, so that N_1, N_2 here are determined by π, and the representation τ_π is determined up to conjugacy by an element of $\mathcal{G}_{N_1,N_2}$. Since the group $\mathcal{G}_{N_1,N_2}^\circ$ decomposes as $\mathcal{G}_{N_1}^{(1)} \times \mathcal{G}_{N_2}^{(2)}$, we can also write $\tau_\pi = \tau_\pi^{(1)} \otimes \tau_\pi^{(2)}$, with $\tau_\pi^{(i)}$ an irreducible cuspidal representation of $\mathcal{G}_{N_i}^{(i)}$.

The irreducible cuspidal representations τ of the groups $\mathcal{G}_{N_1,N_2}^\circ$ were classified by Lusztig [33, 34], in terms of semisimple elements s of the dual group and unipotent irreducible cuspidal representations of the centralizer of s, generalizing the classification of Green. (The only unipotent irreducible cuspidal representation of $\mathrm{GL}_n(k_\mathrm{F})$ is the trivial representation of $\mathrm{GL}_1(k_\mathrm{F})$.) We will recall this later, in Section 7, when we require it.

4. Reducibility of parabolic induction

In this section, we recall some basic results, in particular due to Silberger, on reducibility of parabolic induction. We continue with the same notation, so that $\mathrm{G} = \mathrm{U}(\mathrm{V})^\circ$ is our classical group. We recall that we have the group $\mathrm{G}_n = \mathrm{U}(\mathrm{V}_n)^\circ$, with Levi subgroup $\mathrm{M}_n \simeq \mathrm{GL}_n(\mathrm{F}) \times \mathrm{G}$ (with the isomorphism determined by the chosen Witt basis) and standard parabolic subgroup $\mathrm{P}_n = \mathrm{M}_n\mathrm{N}_n$. Let $\rho \in \mathcal{A}_n(\mathrm{F})$ and $\pi \in \mathcal{A}(\mathrm{G})$, so that we can consider $\rho \otimes \pi$ as a representation of M_n.

We are interested in the (ir)reducibility of the normalized parabolically induced representation

$$I(\rho, \pi, s) = \mathrm{Ind}_{\mathrm{P}_n}^{\mathrm{G}_n} \rho|\det(\cdot)|_\mathrm{F}^s \otimes \pi,$$

for $s \in \mathbf{C}$, where $|\cdot|_\mathrm{F}$ is the normalized absolute value on F (with image $q^{\mathbf{Z}}$) and det is the determinant on $\mathrm{GL}_n(\mathrm{F})$. We note that replacing ρ by an unramified twist just has the effect of translating the parameter s; that is

$$I(\rho|\det(\cdot)|_\mathrm{F}^{s_0}, \pi, s) = I(\rho, \pi, s + s_0).$$

Thus we lose no information if we replace our base-point ρ with any unramified twist. We have the following fundamental result of Silberger: the first part comes from [50, Corollaries 5.4.2.2–3] and the second from [51, Theorem 1.6].

Theorem 4.1. (i) *If $I(\rho, \pi, s)$ is reducible for some $s \in \mathbf{R}$, then there exists $s_0 \in \mathbf{R}$ such that $\rho|\det(\cdot)|_\mathrm{F}^{s_0}$ is self-dual.*

(ii) *If ρ is self-dual and $I(\rho, \pi, s)$ is reducible for some $s \in \mathbf{R}$, then there is a (unique) real number $s_\pi(\rho) \geqslant 0$ such that, for $s \in \mathbf{R}$,*

$$I(\rho, \pi, s) \text{ is reducible if and only if } s = \pm s_\pi(\rho).$$

Remark 4.2. In the situation of Theorem 4.1, it is almost true that, if ρ is self-dual then $I(\rho, \pi, s)$ is reducible for some $s \in \mathbf{R}$. The only exception comes from even special orthogonal groups, where we have an extra subtlety (see [24] for more details): if π is an irreducible cuspidal representation of G which is *not* normalized by the full orthogonal group G^+ and $n = 1$ (so that ρ is a trivial or quadratic character of $\mathrm{F}^\times$) then $I(\rho, \pi, s)$ is irreducible for all $s \in \mathbf{R}$. On the other hand, in this situation, putting $\pi^+ = \mathrm{Ind}_\mathrm{G}^{\mathrm{G}^+} \pi$, which is an irreducible cuspidal representation of G^+, then $I(\rho, \pi^+, s)$ does have reducibility, at $s = 0$.

From Theorem 4.1, for a fixed $\pi \in \mathcal{A}(\mathrm{G})$, we get a map

$$s_\pi : \mathcal{A}^\sigma(\mathrm{F}) \to \mathbf{R}_{\geqslant 0},$$

where we define $s_\pi(\rho) = 0$ if $I(\rho, \pi, s)$ is irreducible for all $s \in \mathbf{R}$. Part of the well-known "Basic Assumption" made in [43] is that the image of this map is in fact in $\frac{1}{2}\mathbf{Z}$ (indeed, this is now known in many cases – at least when G is quasi-split – through the work of Arthur, Mœglin, Waldspurger). We will prove here (independently) that this is indeed the case for depth zero representations π.

Silberger's results in fact give a little more than stated here, since we have stated them only for *real* values of s. Indeed, if $\rho \in \mathcal{A}^\sigma(\mathrm{F})$ then there are (up to equivalence) exactly *two* unramified twists of ρ which are self-dual: ρ and another one ρ'. If, moreover, ρ is a depth zero representation then this second representation is easy to describe: it is $\rho' := \rho|\det(\cdot)|_{\mathrm{F}}^{\pi \mathrm{i}/n \log q}$. Thus Silberger's result in fact gives a qualitative description of all complex s for which $I(\rho, \pi, s)$ is reducible.

In general, we will here only be able to compute the pair of numbers $\{s_\pi(\rho), s_\pi(\rho')\}$, rather than distinguishing them individually. However, this is sufficient to prove the equality (1.4).

5. Covers and Hecke algebras

The theory of types and covers was developed by Bushnell–Kutzko to give a strategy and framework to describe the structure of the category of smooth representations of a connected reductive group. Here we are interested only in a rather special case (in particular, we have only maximal proper Levi subgroups and depth zero representations for classical groups) so we do not give definitions and results in their full generality. In particular, we are specializing to depth zero the results of [4, §3.2].

We continue in the notation of the previous section but restrict to depth zero. Thus we have $\rho \in \mathcal{A}^{\sigma}_{[0]}(\mathrm{F})$, a representation of $\mathrm{GL}_n(\mathrm{F})$, and $\pi \in \mathcal{A}_{[0]}(\mathrm{G})$, giving us a representation $\rho \otimes \pi$ of the Levi subgroup $\mathrm{M}_n \simeq \mathrm{GL}_n(\mathrm{F}) \times \mathrm{G}$ of G_n. We write $\rho = \text{c-Ind}_{\mathbf{J}_\rho}^{\mathrm{GL}_n(\mathrm{F})} \Lambda_\rho$ and $\pi = \text{c-Ind}_{\mathrm{J}_\pi}^{\mathrm{G}} \lambda_\pi$, as in Section 3, and use all the associated notation from there. We write $\mathrm{P}_n = \mathrm{M}_n \mathrm{N}_n$ and denote by $\mathrm{P}_n^- = \mathrm{M}_n \mathrm{N}_n^-$ the opposite parabolic subgroup.

We put $\mathrm{J}_\mathrm{M} = \mathrm{J}_\rho \times \mathrm{J}_\pi$, a compact open subgroup of M_n, and $\lambda_\mathrm{M} = \lambda_\rho \otimes \lambda_\pi$, an irreducible representation of J_M. From [37, Theorem 1.1], there is a *cover* (J, λ) of $(\mathrm{J}_\mathrm{M}, \lambda_\mathrm{M})$, that is

- J is a compact open subgroup of G_n which has an Iwahori decomposition with respect to $(\mathrm{M}_n, \mathrm{P}_n)$ and such that $\mathrm{J} \cap \mathrm{M}_n = \mathrm{J}_\mathrm{M}$;
- λ is an irreducible representation of J whose restriction to J_M is λ_M and whose restriction to $\mathrm{J} \cap \mathrm{N}_n^{\pm}$ is a multiple of the trivial representation;
- the Hecke algebra $\mathscr{H}(\mathrm{G}_n, \lambda)$ contains an invertible element whose support is the (J, J)-double coset of a strongly positive element of the centre of M_n.

Moreover, we have a description of the Hecke algebra $\mathscr{H}(\mathrm{G}_n, \lambda)$ given by [37, Theorem 1.2]:

(i) If there is some $g \in \mathrm{G}_n \backslash \mathrm{M}_n$ which normalizes M_n and such that the conjugate by g of $\rho \otimes \pi$ is equivalent to an unramified twist of $\rho \otimes \pi$, then $\mathscr{H}(\mathrm{G}_n, \lambda)$ is a generic Hecke algebra on an infinite dihedral group; that is, it is generated by T_1, T_2, each supported on a single (J, J)-double coset, with relations

$$(T_i - q^{f_i})(T_i + 1) = 0,$$

for some half-integers $f_i \geqslant 0$. Moreover, there is a recipe to compute the f_i, which we revisit in Section 6 below.

(ii) Otherwise, $\mathscr{H}(\mathrm{G}_n, \lambda)$ is abelian, isomorphic to $\mathbf{C}\left[Z^{\pm 1}\right]$.

In the second case, the induced representation $I(\rho, \pi, s)$ is irreducible for any $s \in \mathbf{C}$ so we restrict our interest to the first case. Since ρ is self-dual, the condition in (i) is always satisfied, unless G is an even-dimensional special orthogonal group, the representation π is *not* normalized by the full orthogonal group G^+, and $n = 1$. (See Remark 4.2.)

We write $\mathfrak{R}^{[\rho,\pi]}(\mathrm{M}_n)$ for the full subcategory of (smooth complex) representations of M_n all of whose irreducible subquotients are unramified twists of $\rho \otimes \pi$, and $\mathfrak{R}^{[\rho,\pi]}(\mathrm{G}_n)$ for the full subcategory of representations of G_n all of whose irreducible subquotients have supercuspidal support an unramified twist of $\rho \otimes \pi$. Then, since (J, λ) is a cover of $(\mathrm{J_M}, \lambda_\mathrm{M})$ which is a type, we have a normalized embedding of Hecke algebras $t : \mathscr{H}(\mathrm{M}_n, \lambda_\mathrm{M}) \hookrightarrow \mathscr{H}(\mathrm{G}_n, \lambda)$ giving us a commutative diagram

$$\begin{array}{ccc} \mathfrak{R}^{[\rho,\pi]}(\mathrm{G}_n) & \xrightarrow{\mathfrak{h}} & \text{Mod-}\mathscr{H}(\mathrm{G}_n, \lambda) \\ \uparrow{\scriptstyle \mathrm{Ind}_{\mathrm{P}_n}^{\mathrm{G}_n}} & & \uparrow{\scriptstyle t_*} \\ \mathfrak{R}^{[\rho,\pi]}(\mathrm{M}_n) & \xrightarrow{\mathfrak{h}_\mathrm{M}} & \text{Mod-}\mathscr{H}(\mathrm{M}_n, \lambda_\mathrm{M}) \end{array}$$

Here, the functor t_* maps a right $\mathscr{H}(\mathrm{M}_n, \lambda_\mathrm{M})$-module X to $\mathrm{Hom}_{\mathscr{H}(\mathrm{M}_n,\lambda_\mathrm{M})}(\mathscr{H}(\mathrm{G}_n, \lambda), X)$, where the $\mathscr{H}(\mathrm{M}_n, \lambda_\mathrm{M})$-module structure on $\mathscr{H}(\mathrm{G}_n, \lambda)$ is given by t. The horizontal arrows are equivalences of categories: the functor $\mathfrak{h}_\mathrm{M}$ is given by $\xi \mapsto \mathrm{Hom}_{\mathrm{J_M}}(\lambda_M, \xi)$, and similarly for $\mathfrak{h}$.

The Hecke algebra $\mathscr{H}(\mathrm{M}_n, \lambda_\mathrm{M})$ is isomorphic to $\mathbf{C}[Z^{\pm 1}]$, where Z is supported on $\zeta \mathrm{J_M}$ and ζ is the element of the centre of M_n which acts on $\mathrm{V}_n = n\mathrm{H}^- \oplus \mathrm{V} \oplus n\mathrm{H}^+$ as 1 on V and as ϖ_F on $n\mathrm{H}^+$. The element $T_2T_1 \in \mathscr{H}(\mathrm{G}_n, \lambda)$ is supported on the double coset $\mathrm{J}\zeta\mathrm{J}$ and, since $t(Z)$ is supported on the same double coset, we may (and do) normalize Z so that $t(Z) = T_2T_1$.

Now, from [4, Proposition 3.12], we get that, if $I(\rho, \pi, s)$ is reducible, then the real part of s belongs to the set

$$\left\{ \pm \frac{(f_1 \pm f_2)}{2n} \right\}. \tag{5.1}$$

We will see these values are always half-integers. Thus, in the notation of Section 4, we have

$$\{s_\pi(\rho), s_\pi(\rho')\} = \left\{ \frac{|f_1 \pm f_2|}{2n} \right\},$$

where $|\cdot|$ is the usual (real) absolute value, and we recall that ρ' is the unique self-dual unramified twist of ρ which is not equivalent to ρ.

6. Reduction to the finite case

We now describe how to relate the parameters f_i of the previous section to a problem in the representation theory of finite reductive groups, and rephrase the equality (1.3) in these terms. The recipe for computing these parameters is described in [37, §6.3], which is considerably simplified by only treating depth zero representations; in particular, the character χ_0 of *loc. cit.* is trivial. Thus the content of this section is all proved in *loc. cit.* and here we just explicate it in our special case.

We recall that $\pi = \text{c-Ind}_{\mathrm{J}_\pi}^{\mathrm{G}} \lambda_\pi$, where J_π is the normalizer of a standard maximal parahoric subgroup $\mathrm{J}^\circ_{N_1,N_2}$, with N_1, N_2 uniquely determined, and $\lambda_\pi|_{\mathrm{J}^\circ_{N_1,N_2}}$ contains an irreducible representation λ°_π inflated from an irreducible cuspidal representation $\tau_\pi \simeq \tau_\pi^{(1)} \otimes \tau_\pi^{(2)}$ of the (connected) reductive quotient $\mathcal{G}^\circ_{N_1,N_2} \simeq \mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2}$. More explicitly, as in Section 2, we write

$$\mathcal{G}^{(i)}_{N_i} = \mathrm{U}(N_i\overline{\mathrm{H}} \oplus \overline{\mathrm{V}}^{\mathsf{an}}_{(i)})^\circ, \qquad \text{for } i = 1, 2.$$

We will also write $\mathcal{G}^{(i)+}_{N_i}$ for the full finite classical group of which $\mathcal{G}^{(i)}_{N_i}$ is the connected component. We will need to distinguish the cases when $\tau_\pi^{(i)}$ is normalized by $\mathcal{G}^{(i)+}_{N_i}$ from those where it is not.

By construction, the group J in G_n has reductive quotient isomorphic to $\mathrm{GL}_n(k_\mathrm{F}) \times \mathcal{G}_{N_1,N_2}$, and we denote by J° the inverse image of its connected component. The parahoric subgroup J° is contained in precisely two maximal compact subgroups of G_n, namely the standard maximal compact $\mathrm{J}_2 := \mathrm{J}_{N_1,N_2+n}$ and a maximal compact group J_1 conjugate to J_{N_1+n,N_2}. More precisely, both $\mathrm{J}_1, \mathrm{J}_2$ would be standard with respect to the ordered Witt basis

$$\mathrm{e}_N^-, \dots, \mathrm{e}_{N_1+1}^-, \mathrm{e}_{N+n}^-, \dots \mathrm{e}_{N+1}^-, \mathrm{e}_{N_1}^-, \dots, \mathrm{e}_1^-, \mathrm{e}_1^{\mathsf{an}}, \dots, \mathrm{e}_{N^{\mathsf{an}}}^{\mathsf{an}},$$
$$\mathrm{e}_1^+, \dots, \mathrm{e}_{N_1}^+, \mathrm{e}_{N+1}^+, \dots, \mathrm{e}_{N+n}^+, \mathrm{e}_{N_1+1}^+, \dots, \mathrm{e}_N^+.$$

These maximal compact subgroups have reductive quotients $\mathcal{G}_1, \mathcal{G}_2$ isomorphic to $\mathcal{G}_{N_1+n,N_2}$ and $\mathcal{G}_{N_1,N_2+n}$ respectively, and the image of J° in each of these is a parabolic subgroup with Levi component isomorphic to $\mathrm{GL}_n(k_\mathrm{F}) \times \mathcal{G}^\circ_{N_1,N_2}$. More explicitly, the $\mathcal{G}_i$ have connected components

$$\mathcal{G}_1^\circ \simeq \mathrm{U}((N_1+n)\overline{\mathrm{H}} \oplus \overline{\mathrm{V}}^{\mathsf{an}}_{(1)})^\circ \times \mathcal{G}^{(2)}_{N_2} \supseteq \left(\mathrm{GL}_n(k_\mathrm{F}) \times \mathcal{G}^{(1)}_{N_1}\right) \times \mathcal{G}^{(2)}_{N_2},$$
$$\mathcal{G}_2^\circ \simeq \mathcal{G}^{(1)}_{N_1} \times \mathrm{U}((N_2+n)\overline{\mathrm{H}} \oplus \overline{\mathrm{V}}^{\mathsf{an}}_{(2)})^\circ \supseteq \mathcal{G}^{(1)}_{N_1} \times \left(\mathrm{GL}_n(k_\mathrm{F}) \times \mathcal{G}^{(2)}_{N_2}\right).$$

Moreover, we have certain Weyl group elements $s_i \in \mathrm{J}_i$, defined in [37, §5.6]. (See also [52, §6.2] where, in most cases, they are denoted s_1^ϖ, s_1 respectively, though there is an added complication when $n = 1$ and G is a special orthogonal group, explained in [37, §5.6].) More explicitly, both s_1, s_2 exchange (up to scalars) the vectors e^+_{N+j} and e^-_{N+j}, for $1 \leqslant j \leqslant n$ and preserve the subspace V of V_n.

Now let (J, λ) be the cover of $(\mathrm{J}_\rho \times \mathrm{J}_\pi, \lambda_\rho \otimes, \lambda_\pi)$ from Section 5. The proof of the existence of this cover, in [37], goes via first constructing a cover $(\mathrm{J}^\circ, \lambda^\circ)$ of $(\mathrm{J}_\rho \times \mathrm{J}^\circ_\pi, \lambda_\rho \otimes \lambda^\circ_\pi)$. The Hecke algebras $\mathscr{H}(\mathrm{G}_n, \lambda^\circ)$ and $\mathscr{H}(\mathrm{G}_n, \lambda)$ are not in general isomorphic, but they are closely related, as we now describe.

The Hecke algebra $\mathscr{H}(\mathrm{G}_n, \lambda)$ is generated by two elements T_1, T_2, with T_i supported on $\mathrm{J}s_i\mathrm{J}$ and satisfying a quadratic relation $(T_i - q^{f_i})(T_i + 1) = 0$. Then:

(i) If $n = 1$, and either $\mathcal{G}^{(i)+}_{N_i}$ is the trivial orthogonal group (i.e. the orthogonal group on a trivial space) or the irreducible cuspidal representation $\tau^{(i)}_\pi$ of $\mathcal{G}^{(i)}_{N_i}$ is *not* normalized by $\mathcal{G}^{(i)+}_{N_i}$, then we always have $f_i = 0$; that is, $T_i^2 = 1$. Note that our assumption that we have reducibility implies that this can be the case for at most one value of i, and this is never the case if G is either symplectic or unramified unitary.

(ii) Otherwise, there is a corresponding element $T_i^\circ \in \mathscr{H}(\mathrm{G}_n, \lambda^\circ)$ with support $\mathrm{J}^\circ s_i \mathrm{J}^\circ$ and satisfying the same quadratic relation as T_i (with the same parameter f_i).

It remains to describe the parameter f_i in the latter case and we assume from now on that we are in that situation.

By inflation, we have a support-preserving algebra injection

$$\mathscr{H}(\mathcal{G}_i^\circ, \tau_\rho \otimes \tau_\pi^\circ) \hookrightarrow \mathscr{H}(\mathrm{G}_n, \lambda^\circ),$$

and T_i° is in the image of this map. We also have isomorphisms

$$\mathscr{H}(\mathcal{G}_i^\circ, \tau_\rho \otimes \tau_\pi^\circ) \simeq \mathscr{H}(\mathrm{U}((N_i+n)\overline{\mathrm{H}} \oplus \overline{\mathrm{V}}^{\mathsf{an}}_{(i)})^\circ, \tau_\rho \otimes \tau_\pi^{(i)}),$$

and the algebra on the right is described (at least in the case that the ambient finite classical group has connected centre) in [36, Theorem 8.6]: they are two-dimensional, generated by an element satisfying the same quadratic relation, with q^{f_i} the quotient of the dimensions of the two irreducible factors of the representation parabolically induced from $\tau_\rho \otimes \tau_\pi^{(i)}$. Moreover, one can compute this by using Lusztig's Jordan decomposition of characters, as we describe in the next section.

7. Computation of parameters

In this section, we undertake the computation of the parameters in the finite Hecke algebras from above. When the finite reductive group arising has connected centre, this can more-or-less be read off from the Jordan decomposition of characters and the case of unipotent irreducible cuspidals, for which there are tables in [35]. In general, one must first embed the group into one with connected centre, and then make the comparison. A special case is already carried out in [32], where they look at the Hecke algebra coming from inducing a self-dual irreducible cuspidal representation of the Siegel parabolic of a classical group.

In order to fit with the usual notations for finite reductive groups, the notation in this section is independent of that in the rest of the paper. We will not recall here the definitions of *geometric* and *rational Lusztig series*, both of which give partitions of the set of irreducible representations of a connected finite reductive group coming from Deligne–Lusztig induction; we refer the reader instead, for example, to [13] or [15].

7.1. **Self-dual polynomials.** We begin with a brief section on irreducible self-dual polynomials over finite fields, since these will be used to parametrize the irreducible cuspidal representations of our finite reductive groups. We fix $\mathbf{F}_q$ a finite field of odd cardinality q and let $\mathbf{F}_{q_\circ}$ be a subfield of index at most 2. We denote by σ the automorphism generating $\mathrm{Gal}(\mathbf{F}_q/\mathbf{F}_{q_\circ})$, and use the same notation for the induced automorphism of the polynomial ring $\mathbf{F}_q[X]$, obtained by applying σ to all coefficients.

For $P \in \mathbf{F}_q[X]$, we put

$$P^\sigma(X) \ := \ \sigma(P(0))^{-1}X^{\deg(P)}\sigma(P)(1/X).$$

We say that a monic polynomial $P \in \mathbf{F}_q[X]$ is $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-*self-dual* if $P = P^\sigma$; thus P is $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-self-dual if and only if:

(i) when $\mathbf{F}_q = \mathbf{F}_{q_\circ}$, for each root ζ of P (in some algebraic closure of $\mathbf{F}_q$), the element ζ^{-1} is also a root of P;
(ii) when $\mathbf{F}_q \neq \mathbf{F}_{q_\circ}$, for each root ζ of P, the element $\zeta^{-q_\circ}$ is also a root of P.

When $\mathbf{F}_q = \mathbf{F}_{q_\circ}$, we will just speak of *self-dual* polynomials; these might more often elsewhere be called *reciprocal*.

If we now restrict to *irreducible* $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-self-dual monic polynomials P, the possibilities are somewhat constrained:

(i) when $\mathbf{F}_q = \mathbf{F}_{q_\circ}$, either $P(X) = X \pm 1$ or else $\deg(P)$ is even;
(ii) when $\mathbf{F}_q \neq \mathbf{F}_{q_\circ}$, we must have that $\deg(P)$ is odd.

7.2. **Connected centre.** We now turn to the problem at hand, excluding for now the case of unitary groups, whose treatment is postponed to Section 7.7. We begin with the case of a group with connected centre, so that the centralizer of any semisimple element of the dual group is connected. Let $\mathcal{G}$ be a connected reductive group of *classical type*, over a finite field $\mathbf{F}_q$ of odd characteristic p, *with connected centre* and with Frobenius map $\mathcal{F}$. By classical type here we mean that $\mathcal{G}$ is one of:

(i) an odd-dimensional special orthogonal group SO_{2N+1};
(ii) a group of symplectic similitudes GSp_{2N};
(iii) a group of orthogonal similitudes GSO_{2N}^+ or GSO_{2N}^-, of Witt index N, $N-1$ respectively. (Note that we mean here that $\mathrm{GSO}_{2N}^\pm$ is the connected component of the full group of orthogonal similitudes $\mathrm{GO}_{2N}^\pm$.) In this case we do *not* allow the group GSO_2^+.

In each case, the Frobenius map $\mathcal{F}$ is the standard one. We denote by $\mathcal{G}^*$ the dual group and write $\mathcal{F}$ again for the (dual) Frobenius on it. The dual group acts naturally on an $\overline{\mathbf{F}}_q$-vector space $\mathcal{V}$, with an $\mathbf{F}_q$-structure and a form, of dimension $2N, 2N+1, 2N$ respectively in the three cases above. In case (iii), we say that $\mathcal{V}$ is of type $+1$ if it has Witt index N, and of type -1 otherwise; we say that the zero space has type $+1$.

Write $\mathscr{E}(\mathcal{G}^{\mathcal{F}})$ for the set of equivalence classes of irreducible (complex) representations of $\mathcal{G}^{\mathcal{F}}$. Then (see for example [33, §7.6]) there is a partition into *geometric Lusztig series*

$$\mathscr{E}(\mathcal{G}^{\mathcal{F}}) \;=\; \bigcup_s \mathscr{E}(\mathcal{G}^{\mathcal{F}}, s),$$

where s runs over the conjugacy classes of semisimple elements of $\mathcal{G}^{*,\mathcal{F}}$. (Note that rational and geometric conjugacy classes coincide as the centre of $\mathcal{G}$ is connected.) The partition is given as follows: for any $\mathcal{F}$-stable maximal torus $\mathcal{T}$ of $\mathcal{G}^*$ containing s, we have the Deligne–Lusztig representation $\mathrm{R}_{\mathcal{T}}^{\mathcal{G}} s$; then an irreducible representation π of $\mathcal{G}^{\mathcal{F}}$ lies in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ if and only if there is such a torus $\mathcal{T}$ with

$$\langle \pi, \mathrm{R}_{\mathcal{T}}^{\mathcal{G}} s\rangle \;\neq\; 0.$$

(Here, $\langle\cdot,\cdot\rangle$ denotes the natural $\mathcal{G}$-invariant inner product on class functions, and we identify (equivalence classes of) representations with their characters.)

Given a semisimple element $s \in \mathcal{G}^{*,\mathcal{F}}$, the centralizer $\mathcal{G}_s^*$ is a connected reductive group of the same rank as $\mathcal{G}$, though in general it is not a Levi subgroup. Then the Jordan decomposition of characters [33, Corollary 7.10] (see also [13, Theorem 15.8]) gives a bijection

$$\psi_s^{\mathcal{G}} : \mathscr{E}(\mathcal{G}^{\mathcal{F}}, s) \to \mathscr{E}(\mathcal{G}_s^{*,\mathcal{F}}, 1)$$

with the following properties (see [33, §7.8]):

- for any irreducible representation π in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ and any $\mathcal{F}$-stable maximal torus $\mathcal{T}$ containing s,

$$\varepsilon_{\mathcal{G}}\langle \pi, \mathrm{R}_{\mathcal{T}}^{\mathcal{G}} s\rangle = \varepsilon_{\mathcal{G}_s}\langle \psi_s^{G}(\pi), \mathrm{R}_{\mathcal{T}}^{\mathcal{G}_s^*} 1\rangle, \tag{7.1}$$

  where $\varepsilon_{\mathcal{G}} = (-1)^{\mathbf{F}_q\text{-rank}(\mathcal{G})}$ (see [15, Definition 8.3]);
- there is a uniform constant c_s such that

$$\dim \pi \;=\; c_s \dim \psi_s^{\mathcal{G}}(\pi),$$

  for all π in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$; explicitly, writing $d_{p'}$ for the maximal divisor of d coprime to p, for any positive integer d, we have $c_s = |\mathcal{G}^{*,\mathcal{F}}|_{p'} |\mathcal{G}_s^{*,\mathcal{F}}|_{p'}^{-1}$;
- if the identity components of the centres of $\mathcal{G}^*$ and $\mathcal{G}_s^*$ have the same $\mathcal{F}_q$-rank then π in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ is cuspidal if and only if $\psi_s^{\mathcal{G}}(\pi)$ is cuspidal; otherwise, no representation in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ is cuspidal.

(We remark that all the results so far extend to the case of disconnected centre once we replace the geometric Lusztig series with the rational series – see below in section 7.3 for the notion of rational Lusztig series, which coincides with the geometric one for groups with connected centre.)

Thus we get a classification of the irreducible cuspidal representations of $\mathcal{G}^{\mathcal{F}}$ from a classification of pairs (s, τ), with s a semisimple element of $\mathcal{G}^{*,\mathcal{F}}$ (up to conjugacy) such that the identity components of the centres of $\mathcal{G}^*$ and $\mathcal{G}_s^*$ have the same $\mathcal{F}_q$-rank, and τ an irreducible cuspidal unipotent representation of $\mathcal{G}_s^{*,\mathcal{F}}$ (up to equivalence). Lusztig classified the irreducible cuspidal unipotent representations of classical groups – in particular, there is at most one irreducible cuspidal unipotent representation for each such group – and we find (see [33, p172]) the following.

For $s \in \mathcal{G}^{*,\mathcal{F}}$ semisimple, we denote by $P_s \in \mathbf{F}_q[X]$ its characteristic polynomial (as an automorphism of the space $\mathcal{V}$ on which $\mathcal{G}^*$ acts naturally), by $\mathcal{V}_+$ the $(+1)$-eigenspace and by $\mathcal{V}_-$ the (-1)-eigenspace. Then there is a bijection between the (equivalence classes of) irreducible cuspidal representations of $\mathcal{G}^{\mathcal{F}}$ and the set of conjugacy classes of semisimple elements $s \in \mathcal{G}^{*,\mathcal{F}}$ such that

$$P_s(X) \;=\; \prod_P P(X)^{a_P},$$

where the product runs over all irreducible self-dual monic polynomials over $\mathbf{F}_q$ and the integers a_P satisfy:

(7.2).
- $\sum_P a_P \deg(P) = \dim \mathcal{V}$;
- for $P(X) \neq (X \pm 1)$, we have $a_P = \frac{1}{2}(m_P^2 + m_P)$, for some integer $m_P \geqslant 0$;
- writing $a_+ := a_{(X-1)}$ and $a_- := a_{(X+1)}$, there are integers $m_+, m_- \geqslant 0$ such that
  (i) if $\mathcal{G} = \mathrm{SO}_{2N+1}$ then $a_+ = 2(m_+^2 + m_+)$ and $a_- = 2(m_-^2 + m_-)$,
  (ii) if $\mathcal{G} = \mathrm{GSp}_{2N}$ then $a_+ = 2(m_+^2 + m_+) + 1$ and $a_- = 2m_-^2$,
  (iii) if $\mathcal{G} = \mathrm{GSO}_{2N}^{\pm}$ then $a_+ = 2m_+^2$ and $a_- = 2m_-^2$,
  where, in case (iii), $\mathcal{V}_\pm$ is an even-dimensional orthogonal space of type $(-1)^{m_\pm}$, and the same in case (ii) for $\mathcal{V}_-$ only.

Let s be a semisimple element of $\mathcal{G}^{*,\mathcal{F}}$ and suppose that we have an $\mathcal{F}$-stable Levi subgroup $\mathcal{L}^*$, which is the Levi component of an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}^*$ of $\mathcal{G}^*$, such that $s \in \mathcal{L}^*$. Correspondingly, we have an $\mathcal{F}$-stable Levi subgroup $\mathcal{L}$, which is the Levi component of an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$. Then $\mathcal{L}_s^*$ is an $\mathcal{F}$-stable Levi subgroup of $\mathcal{G}_s^*$ (though it need not be a *proper* Levi subgroup), and is the Levi component of the $\mathcal{F}$-stable parabolic $\mathcal{P}_s^*$. Then we have a diagram

$$\begin{array}{ccc}
\mathbf{Z}\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s) & \xrightarrow{\;\psi_s^{\mathcal{G}}\;} & \mathbf{Z}\mathscr{E}(\mathcal{G}_s^{*,\mathcal{F}}, 1) \\
\uparrow{\scriptstyle \mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}} & & \uparrow{\scriptstyle \mathrm{Ind}_{\mathcal{L}_s^*,\mathcal{P}_s^*}^{\mathcal{G}_s^*}} \\
\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s) & \xrightarrow{\;\psi_s^{\mathcal{L}}\;} & \mathscr{E}(\mathcal{L}_s^{*,\mathcal{F}}, 1)
\end{array}$$

(The vertical arrows here are parabolic induction – i.e. Harish-Chandra induction – and we have abbreviated from $\mathrm{Ind}_{\mathcal{L}^{\mathcal{F}},\mathcal{P}^{\mathcal{F}}}^{\mathcal{G}^{\mathcal{F}}}$ since the notation is already heavy.) This diagram commutes (this is a result of Shoji, which can be extracted from the appendix to [16]); in the cases that interest us here it can be seen fairly directly:

- If $\mathcal{G}_s^* \subseteq \mathcal{L}^*$ then the vertical arrows preserve irreducibility (this is a special case of [33, (7.9.1)]) and the diagram commutes by (7.1). In fact, this also generalizes to the case where the parabolic $\mathcal{P}$ is not $\mathcal{F}$-stable, replacing $\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}$ by the Deligne–Lusztig map $\varepsilon_{\mathcal{G}}\varepsilon_{\mathcal{L}}\mathrm{R}_{\mathcal{L}\subset\mathcal{P}}^{\mathcal{G}}$; indeed, in that case the diagram commutes *by definition* of the map $\psi_s^{\mathcal{G}}$ (see the proof of [33, Proposition 7.9]).
- Suppose $\mathcal{L}$ is a *maximal* proper Levi subgroup and $\tau \in \mathscr{E}(\mathcal{L}, s)$ is cuspidal. Then $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$ has order 1 or 2. We are interested in the case where $\mathrm{Ind}_{\mathcal{L}}^{\mathcal{G}}\tau$ is reducible; equivalently, $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$ has order 2 and, writing w for a representative of the nontrivial coset, w normalizes τ. In this case the induced representation decomposes as

$$\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\tau \;=\; \pi_1 \oplus \pi_2, \qquad \dim(\pi_1) > \dim(\pi_2),$$

(the inequality is strict by [36, Theorem 8.6]) and $\mathrm{End}_{\mathcal{G}^{\mathcal{F}}}(\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau)$ is a two-dimensional algebra with a quadratic generator T satisfying a relation of the form

$$(T+1)(T-q^{f_\tau}) \;=\; 0, \qquad q^{f_\tau} = \frac{\dim(\pi_1)}{\dim(\pi_2)}.$$

Moreover, the same is true for $\mathrm{Ind}_{\mathcal{L}_s^*,\mathcal{P}_s^*}^{\mathcal{G}_s^*}\,\psi_s^{\mathcal{L}}(\tau)$ and the recipe given in [36, §8] (see *op. cit.* Theorem 8.6 and (8.2.3)) to calculate f_τ depends only on the Weyl group $\mathcal{N}_{\mathcal{G}_s^*}(\mathcal{L}_s^*)^{\mathcal{F}}/\mathcal{L}_s^{*,\mathcal{F}}$ (see, *op. cit.* (8.5.7)) which is *identical* for the two induced representations. (Indeed, this matching is the idea behind the inductive proof of the Jordan decomposition of characters.) The recipe is somewhat complicated but fortunately, on the side of the centralizer $\mathcal{G}_s^*$, we have unipotent representations and the parameter q^{f_τ} can be read off from [35, Table II, page 33]. (In the special case that $s^2=1$ (to which one could reduce) one can also read off the parameter from [33, Proposition 8.3].)

7.3. **Disconnected centre.** We now consider the case which really interests us here. So we suppose that $\mathcal{G}$ is a *classical group* over a finite field $\mathbf{F}_q$ of odd characteristic p, with Frobenius map $\mathcal{F}$. By classical group here, we mean that $\mathcal{G}$ is one of:

(i) an odd-dimensional special orthogonal group SO_{2N+1};
(ii) a symplectic group Sp_{2N};
(iii) an even-dimensional special orthogonal group SO_{2N}^+ or SO_{2N}^-, of Witt index N, $N-1$ respectively, where again we do not allow the group SO_2^+.

In each case, the Frobenius map $\mathcal{F}$ is again the standard one. Case (i) has already been treated above, so we only consider cases (ii),(iii) here.

In each case, we embed $\mathcal{G}$ in a group $\widetilde{\mathcal{G}}$ with connected centre of the type considered in Section 7.2. Then we get a map $\widetilde{\mathcal{G}}^* \to \mathcal{G}^*$ which maps conjugacy classes of semisimple elements in $\widetilde{\mathcal{G}}^{*,\mathcal{F}}$ to $\mathcal{G}^{*,\mathcal{F}}$-conjugacy classes of semisimple elements in $\mathcal{G}^{*,\mathcal{F}}$. (Recall that the map $\widetilde{\mathcal{G}} \to \mathcal{G}$ remains surjective on Frobenius-fixed points.)

The geometric conjugacy class of a semisimple element s in $\mathcal{G}^{*,\mathcal{F}}$ splits into two $\mathcal{G}^{*,\mathcal{F}}$-conjugacy classes if and only if its centralizer $\mathcal{G}_s^*$ is disconnected, which happens if and only both 1 and -1 are eigenvalues of s. Here we also have a partition (into *rational* Lusztig series) of the set of equivalence classes of irreducible representations of $\mathcal{G}^{\mathcal{F}}$,

$$\mathscr{E}(\mathcal{G}^{\mathcal{F}}) \;=\; \bigcup_s \mathscr{E}(\mathcal{G}^{\mathcal{F}}, s),$$

where s runs over the $\mathcal{G}^{*,\mathcal{F}}$-conjugacy classes of semisimple elements of $\mathcal{G}^{*,\mathcal{F}}$. Each geometric Lusztig series is the union of at most two rational Lusztig series, corresponding to the rational conjugacy classes in a geometric conjugacy class. Moreover, if $\tilde{s} \in \widetilde{\mathcal{G}}^{*,\mathcal{F}}$ maps to $s \in \mathcal{G}^{*,\mathcal{F}}$, then the rational series $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ is precisely the set of irreducible components of the restriction to $\mathcal{G}^{\mathcal{F}}$ of the representations in the Lusztig series $\mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s})$ (see [13, Proposition 15.6]).

We begin by considering the irreducible cuspidal representations of $\mathcal{G}^{\mathcal{F}}$. An irreducible representation of $\mathcal{G}^{\mathcal{F}}$ is cuspidal if and only if it is a component of the restriction of an irreducible cuspidal representation of $\widetilde{\mathcal{G}}^{\mathcal{F}}$. In general, an irreducible cuspidal representation of $\widetilde{\mathcal{G}}^{\mathcal{F}}$ will decompose as a sum of at most two pieces on restriction to $\mathcal{G}^{\mathcal{F}}$, inequivalent but of the same dimension when there are two (since they are conjugate by $\widetilde{\mathcal{G}}^{\mathcal{F}}$). Precisely what happens is essentially determined by the following lemma of Lusztig, which treats the special case of quadratic unipotent representations.

Lemma 7.3 ([33, Lemma 8.9])**.** *Let $\tilde{s} \in \widetilde{\mathcal{G}}^{*,\mathcal{F}}$ be such that its image $s \in \mathcal{G}^{*,\mathcal{F}}$ is an involution and such that $\mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s})$ contains an irreducible cuspidal representation π; then $\pi|_{\mathcal{G}^{\mathcal{F}}}$ is irreducible if and only if $s = \pm 1$.*

(Note that $s = -1$ is in fact not possible when $\mathrm{G} = \mathrm{Sp}_{2n}$.)

This is enough to deal with the general case because of the following (see [13, Theorem 8.27]). Suppose $\mathcal{L}$ is an $\mathcal{F}$-stable Levi subgroup of $\mathcal{G}$ and $\mathcal{P}$ is a parabolic subgroup (not necessarily $\mathcal{F}$-stable) with Levi component $\mathcal{L}$, and let $\mathcal{L}^*$, $\mathcal{P}^*$ be the corresponding subgroups of $\mathcal{G}^*$. Suppose $s \in \mathcal{G}^{*,\mathcal{F}}$ is a semisimple element such that $\mathcal{G}_s^{*,\circ}\mathcal{G}_s^{*,\mathcal{F}} \subseteq \mathcal{L}^*$, where $\mathcal{G}^{*,\circ}$ denotes the connected component of $\mathcal{G}^*$. Then Deligne–Lusztig twisted induction $\mathrm{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ gives a bijection

$$\varepsilon_{\mathcal{G}}\varepsilon_{\mathcal{L}}\mathrm{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathscr{E}(\mathcal{L}^{\mathcal{F}}, s) \to \mathscr{E}(\mathcal{G}^{\mathcal{F}}, s).$$

Moreover (cf. [33, §7.9]), there is a constant $c_{\mathcal{L},\mathcal{G}} = \left|\mathcal{G}^{\mathcal{F}}\right|_{p'} \left|\mathcal{L}^{\mathcal{F}}\right|_{p'}^{-1}$ such that, for any $\pi \in \mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$,

$$\dim\left(\varepsilon_{\mathcal{G}}\varepsilon_{\mathcal{L}}\mathrm{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\pi)\right) \;=\; c_{\mathcal{L},\mathcal{G}}\dim(\pi).$$

Finally, this map respects cuspidality when the connected centres of $\mathcal{G}^*$ and $\mathcal{L}^*$ have the same $\mathbf{F}_q$-rank: i.e. in that case, $\pi \in \mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$ is cuspidal if and only if $\varepsilon_{\mathcal{G}}\varepsilon_{\mathcal{L}}\mathrm{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}\pi$ in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ is cuspidal.

Now, suppose we are given a semisimple element $\tilde{s}$ of $\widetilde{\mathcal{G}}^{*,\mathcal{F}}$, mapping to $s \in \mathcal{G}^{*,\mathcal{F}}$, such that $\mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s})$ contains a cuspidal representation. Denote by $\mathcal{L}^*$ the minimal $\mathcal{F}$-stable Levi subgroup containing the centralizer of s; if we set $\mathcal{V}_0 = \mathrm{Ker}(s^2 - 1)$ then in fact $\mathcal{L}^* = \mathcal{G}^*_s$ whenever $s|_{\mathcal{V}_0} = \pm 1$. The connected centres of $\mathcal{L}^*$ and $\mathcal{G}^*$ have the same $\mathbf{F}_q$-rank so we have a bijection between the irreducible cuspidal representations in $\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ and in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$. Moreover, denoting by $\widetilde{\mathcal{L}}, \widetilde{\mathcal{P}}$ the inverse images of $\mathcal{L}, \mathcal{P}$ in $\widetilde{\mathcal{G}}$ respectively, we have (see [13, (15.5)]) a commutative diagram

$$\begin{array}{ccc}
\mathscr{E}(\widetilde{\mathcal{L}}^{\mathcal{F}}, \tilde{s}) & \xrightarrow{\mathrm{R}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{L}}\subset\widetilde{\mathcal{P}}}} & \mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s}) \\
\Big\downarrow{\scriptstyle \mathrm{Res}^{\widetilde{\mathcal{L}}}_{\mathcal{L}}} & & \Big\downarrow{\scriptstyle \mathrm{Res}^{\widetilde{\mathcal{G}}}_{\mathcal{G}}} \\
\mathbf{Z}\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s) & \xrightarrow{\mathrm{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}} & \mathbf{Z}\mathscr{E}(\mathcal{G}^{\mathcal{F}}, s),
\end{array}$$

where we have omitted the superscripts $\mathcal{F}$ in the functors, for ease of notation.

Finally, we put this together. Note that $\widetilde{\mathcal{L}}$ is a product of general linear groups (with twisted Frobenius) and a single classical group $\widetilde{\mathcal{G}}_0$ with connected centre, whose dual group $\widetilde{\mathcal{G}}_0^*$ acts naturally on $\mathcal{V}_0$; then any irreducible representation $\pi_{\widetilde{\mathcal{L}}} \in \mathscr{E}(\widetilde{\mathcal{L}}^{\mathcal{F}}, \tilde{s})$ decomposes as a product of irreducible representations of unitary group and an irreducible $\pi_0 \in \mathscr{E}(\widetilde{\mathcal{G}}_0^{\mathcal{F}}, \tilde{s}|_{\mathcal{V}_0})$. Similarly, $\mathcal{L}$ is a product of general linear groups and a classical group $\mathcal{G}_0$ and then the (ir)reducibility of $\pi_{\widetilde{\mathcal{L}}}|_{\mathcal{L}^{\mathcal{F}}}$ is determined by the (ir)reducibility of $\pi_0|_{\mathcal{G}_0^{\mathcal{F}}}$.

Thus, for $\pi = \mathrm{R}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{L}}\subset\widetilde{\mathcal{P}}}(\pi_{\widetilde{\mathcal{L}}})$ an irreducible cuspidal representation in $\mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s})$, the restriction $\pi|_{\mathcal{G}^{\mathcal{F}}}$ is irreducible if and only if $\pi_{\widetilde{\mathcal{L}}}|_{\mathcal{L}^{\mathcal{F}}}$ is irreducible (from the commutative diagram), which happens if and only if $\pi_0|_{\mathcal{G}_0^{\mathcal{F}}}$ is irreducible, in the notation of the previous paragraph. Moreover, it follows from Lemma 7.3 that this occurs if and only if the restriction $s|_{\mathcal{V}_0} = \pm 1$. Thus, in fact, the restriction remains irreducible if and only if the centralizer $\mathcal{G}^*_s$ is connected. We have proved:

Lemma 7.4. *Let $\pi \in \mathscr{E}(\mathcal{G}^{\mathcal{F}}, s)$ be an irreducible cuspidal representation and let $\tilde{s} \in \widetilde{\mathcal{G}}^{*,\mathcal{F}}$ be a semisimple element mapping to s. The following are equivalent:*

(i) *π extends to an irreducible representation in $\mathscr{E}(\widetilde{\mathcal{G}}^{\mathcal{F}}, \tilde{s})$;*
(ii) *the centralizer $\mathcal{G}^*_s$ is connected;*
(iii) *at most one of ± 1 is an eigenvalue of s.*

Now we turn to the computation of the parameter. Thus, as in Section 7.2, we have s a semisimple element of $\mathcal{G}^{*,\mathcal{F}}$ and we suppose that $\mathcal{L}^*$ is a maximal proper $\mathcal{F}$-stable Levi subgroup, which is the Levi component of an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}^*$ of $\mathcal{G}^*$, such that $\mathcal{G}^*_s \subseteq \mathcal{L}^*$. Correspondingly, we have $\mathcal{F}$-stable Levi and parabolic subgroups $\mathcal{L}, \mathcal{P}$ in $\mathcal{G}$. We lift s to $\tilde{s} \in \widetilde{\mathcal{G}}^*$ and likewise have lifts $\widetilde{\mathcal{L}}^*$ and $\widetilde{\mathcal{P}}^*$, with $\widetilde{\mathcal{G}}^*_{\tilde{s}} \subseteq \widetilde{\mathcal{L}}^*$, and corresponding subgroups $\widetilde{\mathcal{L}}, \widetilde{\mathcal{P}}$ in $\widetilde{\mathcal{G}}$ into which $\mathcal{L}, \mathcal{P}$ respectively embed.

We are given a cuspidal representation τ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$, so that τ appears (with multiplicity one) in the restriction of a cuspidal representation $\tilde{\tau}$ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, \tilde{s})$, with $\tilde{s} \in \widetilde{\mathcal{L}}^{*,\mathcal{F}}$ mapping to s. Let w be any representative for the non-trivial element of $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$. We will eventually be interested in the case where w normalizes τ (so that the parabolically induced representation from τ is reducible) but, in that case, it is not immediately clear whether or not w normalizes $\tilde{\tau}$. We will need to know exactly when this is the case but we can already say something about the parameters in the Hecke algebras. For ease of notation, we again omit the superscripts $\mathcal{F}$ in the following.

Lemma 7.5. (i) *Suppose w normalizes $\tilde{\tau}$. Then w also normalizes τ and we have an isomorphism of Hecke algebras $\mathrm{End}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau) \simeq \mathrm{End}_{\widetilde{\mathcal{G}}}(\mathrm{Ind}_{\widetilde{\mathcal{L}},\widetilde{\mathcal{P}}}^{\widetilde{\mathcal{G}}}\,\tilde{\tau})$ which preserves support: that is, it maps any element supported on the double-coset $\mathcal{P}w\mathcal{P}$ to an element supported on the double-coset $\widetilde{\mathcal{P}}w\widetilde{\mathcal{P}}$.*
(ii) *Suppose w does not normalize $\tilde{\tau}$ but does normalize τ. Then $\mathrm{End}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau)$ has generator T supported on the double-coset $\mathcal{P}w\mathcal{P}$, satisfying $(T+1)(T-1) = 0$.*

Proof. We write

$$\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau \;=\; \bigoplus_{i=1}^{\ell} \pi_i, \qquad \text{where } \ell = \begin{cases} 2, & \text{if } w \text{ normalizes } \tau, \\ 1, & \text{otherwise.} \end{cases}$$

If $\ell = 2$ then we order the terms so that $\dim(\pi_1) \geqslant \dim(\pi_2)$, and $\mathrm{End}_{\mathcal{G}}(\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau)$ is a two-dimensional algebra with a quadratic generator T satisfying a relation of the form

$$(T+1)(T-q^{f_\tau}) \;=\; 0, \qquad q^{f_\tau} = \frac{\dim(\pi_1)}{\dim(\pi_2)}.$$

In either case, from Mackey we also have

$$\mathrm{Res}_{\mathcal{G}}^{\widetilde{\mathcal{G}}}\,\mathrm{Ind}_{\widetilde{\mathcal{L}},\widetilde{\mathcal{P}}}^{\widetilde{\mathcal{G}}}\,\tilde{\tau} \;=\; \bigoplus_{\gamma\in\widetilde{\mathcal{L}}/\mathcal{N}_{\widetilde{\mathcal{L}}}(\tau)} \mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau^{\gamma} \;=\; \bigoplus_{\gamma\in\widetilde{\mathcal{L}}/\mathcal{N}_{\widetilde{\mathcal{L}}}(\tau)} \bigoplus_{i=1}^{\ell} \pi_i^{\gamma}. \tag{7.6}$$

(i) Suppose w normalizes $\tilde{\tau}$, in which case

$$\mathrm{Ind}_{\widetilde{\mathcal{L}},\widetilde{\mathcal{P}}}^{\widetilde{\mathcal{G}}}\,\tilde{\tau} \;=\; \tilde{\pi}_1 \oplus \tilde{\pi}_2, \qquad \dim(\tilde{\pi}_1) > \dim(\tilde{\pi}_2).$$

Since the dimensions are distinct they do not form a single $\widetilde{\mathcal{L}}$-orbit so, comparing with (7.6), we must have $\ell = 2$ and

$$\mathrm{Res}^{\widetilde{\mathcal{G}}}_{\mathcal{G}} \tilde{\pi}_i = \bigoplus_{\gamma \in \widetilde{\mathcal{L}}/\mathcal{N}_{\widetilde{\mathcal{L}}}(\tau)} \pi_i^{\gamma};$$

in particular we get $\dim(\pi_1)/\dim(\pi_2) = \dim(\tilde{\pi}_1)/\dim(\tilde{\pi}_2)$ so that, by [23, Theorem 4.14] (see also *op. cit.* Definition 3.19), the relations of the quadratic generators in $\mathrm{End}_{\mathcal{G}}(\mathrm{Ind}^{\mathcal{G}}_{\mathcal{L},\mathcal{P}} \tau)$ and $\mathrm{End}_{\widetilde{\mathcal{G}}}(\mathrm{Ind}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{L}},\widetilde{\mathcal{P}}} \tilde{\tau})$ are the same.

(ii) Suppose now that w does not normalize $\tilde{\tau}$ but does normalize τ (so that $\ell = 2$). Then $\mathrm{Ind}^{\widetilde{\mathcal{G}}}_{\widetilde{\mathcal{L}},\widetilde{\mathcal{P}}} \tilde{\tau}$ is irreducible so the two pieces of $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{L},\mathcal{P}} \tau$ are conjugate under $\widetilde{\mathcal{G}}^{\mathcal{F}}$ so of the same dimension; in particular we get $f_\tau = 0$, as required. □

It remains now to compute when the representation $\tilde{\tau}$ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, \tilde{s})$ containing τ is normalized by w. Ultimately, the answer depends on the family of groups in question, but we can already make a preliminary reduction. We assume from now on that w does indeed normalize τ.

We write $\mathcal{L}^{\mathcal{F}} \simeq \mathrm{GL}_n^{\mathcal{F}} \times \mathcal{G}_0^{\mathcal{F}}$, where $\mathcal{G}_0$ is a (possibly trivial) classical group in the same family as $\mathcal{G}$, and $\tau = \tau_1 \otimes \tau_0$, so that τ_1 is a self-dual irreducible cuspidal representation of $\mathrm{GL}_n^{\mathcal{F}}$. We write $\widetilde{\mathcal{G}}_0$ for the similitude group into which $\mathcal{G}_0$ embeds; if $\mathcal{G}_0$ is a trivial group (i.e. acts on a trivial space) then $\widetilde{\mathcal{G}}_0$ is the multiplicative group.

We also have an isomorphism $\widetilde{\mathcal{L}}^{\mathcal{F}} \simeq \mathrm{GL}_n^{\mathcal{F}} \times \widetilde{\mathcal{G}}_0^{\mathcal{F}}$, which can be seen as follows. We choose a Witt basis $\mathrm{e}_N^-, \ldots, \mathrm{e}_1^-, \mathrm{e}_1^+, \ldots, \mathrm{e}_N^+$ for the space on which $\mathcal{G}^{\mathcal{F}}$ acts, with respect to which $\mathcal{L}^{\mathcal{F}}$ is standard (so it is the stabilizer of the subspace $\langle \mathrm{e}_N^-, \ldots, \mathrm{e}_{N-n+1}^- \rangle$), where we have changed from our usual notation in the case of a non-split special orthogonal group, with $\mathrm{e}_1^-, \mathrm{e}_1^+$ a basis for the anisotropic part of the space. We write $\mu : \widetilde{\mathcal{G}}_0^{\mathcal{F}} \to \mathbf{F}_q^\times$ for the similitude map, which is the identity map when $\mathcal{G}_0$ is trivial. Then we get an isomorphism $\mathrm{GL}_n^{\mathcal{F}} \times \widetilde{\mathcal{G}}_0^{\mathcal{F}} \to \widetilde{\mathcal{L}}$ from the map

$$(g, h) \ \mapsto \ \mathrm{diag}(g, h, \mu(h) w_n (g^{-1})^{\mathrm{T}} w_n^{-1}), \qquad \text{for } g \in \mathrm{GL}_n^{\mathcal{F}},\ h \in \widetilde{\mathcal{G}}_0^{\mathcal{F}},$$

where w_n is the antidiagonal element of $\mathrm{GL}_n^{\mathcal{F}}$ with all non-zero entries equal to 1 (a representative for the longest element of the Weyl group), g^{T} denotes the transpose matrix, and the matrix on the right hand side is block diagonal. (Note that, when $\mathcal{G}_0$ is trivial, the central term h in the block diagonal matrix is acting on a trivial space, so is not really present.) Thus we can write $\tilde{\tau} = \tau_1 \otimes \tilde{\tau}_0$, where $\tilde{\tau}_0$ is an irreducible cuspidal representation of $\widetilde{\mathcal{G}}_0^{\mathcal{F}}$ whose restriction to $\mathcal{G}_0^{\mathcal{F}}$ contains τ_0.

We denote by ε the sign such that $\mathcal{G}$ preserves an ε-symmetric form. Then, if either $\mathcal{G}$ is a symplectic group or n is even, we can take the representative w given by

$$w(\mathrm{e}_i^{\pm}) \ = \ \begin{cases} \mathrm{e}_i^{\pm} & \text{if } 1 \leqslant i \leqslant N - n, \\ \varepsilon \mathrm{e}_i^{\mp} & \text{if } N - n < i \leqslant N; \end{cases}$$

thus w is block antidiagonal (for the block sizes corresponding to $\mathcal{L}$), and the conjugation action of w on $\widetilde{\mathcal{L}}$, transported back to $\mathrm{GL}_n^{\mathcal{F}} \times \widetilde{\mathcal{G}}_0^{\mathcal{F}}$, is given by

$$(g, h) \ \mapsto \ (\mu(h)(g^{-1})^{\mathrm{T}}, h). \tag{7.7}$$

Since τ_1 is self-dual, if its central character is also trivial then the map (7.7) clearly intertwines $\tau_1 \otimes \tilde{\tau}_0$ with itself. The only case when the central character is non-trivial is when τ_1 is the quadratic character, where we see that (7.7) intertwines $\tau_1 \otimes \tilde{\tau}_0$ with itself if and only if $\tilde{\tau}_0 \simeq \tilde{\tau}_0 \otimes (\tau_1 \circ \mu)$.

This leaves the case of even special orthogonal groups with $n = 1$, where we need a different representative w. Note that, in this case, we do not have that $\mathcal{G}_0$ is the trivial group, since we have excluded the case $\mathcal{G} = \mathrm{SO}_2^+$. We have an identification $\mathcal{G}_0 \simeq \mathrm{SO}^{\pm}_{2(N-1)}$, from the action on $\langle \mathrm{e}_{N-1}^-, \ldots, \mathrm{e}_{N-1}^+ \rangle$, and we pick $c_0 \in \mathrm{O}^{\pm}_{2(N-1)} \backslash \mathrm{SO}^{\pm}_{2(N-1)}$. Then we can take w to be the element given by

$$\begin{cases} w(\mathrm{e}_N^{\pm}) = \mathrm{e}_N^{\mp}, \\ w|_{\langle \mathrm{e}_{N-1}^-, \ldots, \mathrm{e}_{N-1}^+ \rangle} = c_0. \end{cases}$$

The action of w on $\widetilde{\mathcal{L}}$ is given by

$$(g, h) \mapsto (\mu(h) g, c_0 h c_0^{-1})$$

and we see that $\tilde{\tau}$ is normalized by w if and only if $\tilde{\tau}_0 \simeq \tilde{\tau}_0^{c_0} \otimes (\tau_1 \circ \mu)$.

If we set $c_0 = 1$ in the case of symplectic groups, we can unify the discussion above into the following statement.

Lemma 7.8. (i) *If $n > 1$ then $\tilde{\tau}$ is normalized by w.*
(ii) *If $n = 1$ then $\tilde{\tau}$ is normalized by w if and only if $\tilde{\tau}_0 \simeq \tilde{\tau}_0^{c_0} \otimes (\tau_1 \circ \mu)$.*

In the following subsections, we analyze precisely when the conditions in Lemma 7.8(ii) are satisfied, in terms of the eigenvalues of the semisimple element s such that $\tau \in \mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$.

7.4. **Symplectic groups.** We begin with the case of symplectic groups, so that we are in case (ii) of Section 7.3. Thus we have a cuspidal representation τ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$, for some maximal proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}$, and a cuspidal representation $\tilde{\tau}$ in $\mathscr{E}(\widetilde{\mathcal{L}}^{\mathcal{F}}, \tilde{s})$ whose restriction to $\mathcal{L}$ contains τ. We denote by w a representative for the non-trivial element of $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$, which we assume normalizes τ.

We write $\mathcal{L}^{\mathcal{F}} \simeq \mathrm{GL}_n^{\mathcal{F}} \times \mathcal{G}_0^{\mathcal{F}}$, where $\mathcal{G}_0$ is a (possibly trivial) symplectic group, and $\tau = \tau_1 \otimes \tau_0$, so that τ_1 is a self-dual irreducible cuspidal representation of $\mathrm{GL}_n^{\mathcal{F}}$. Then $\tau_1 \in \mathscr{E}(\mathrm{GL}_n^{\mathcal{F}}, s_1)$ and $\tau_0 \in \mathscr{E}(\mathcal{G}_0^{\mathcal{F}}, s_0)$, for some semisimple elements s_0, s_1 of the respective dual groups. Note that, if $\mathcal{G}_0$ is the trivial symplectic group then its dual group is SO_1 so that $s_0 = 1$.

Lemma 7.9. *The representation $\tilde{\tau}$ is normalized by w unless τ_1 is the non-trivial quadratic character of $\mathrm{GL}_1^{\mathcal{F}}$ and -1 is not an eigenvalue of s_0.*

We remark that, since 1 is always an eigenvalue of s_0 in the case of symplectic groups, the condition that -1 not be an eigenvalue of s_0 is equivalent to the condition that τ_0 extend to the similitude group $\widetilde{\mathcal{G}}_0^{\mathcal{F}}$, by Lemma 7.4.

Proof. By Lemma 7.8, we only need to consider the case that τ_1 is the non-trivial quadratic character; in that case, w normalizes $\tilde{\tau}$ if and only if $\tilde{\tau}_0 \simeq \tilde{\tau}_0 \otimes \chi$, where we recall that $\tilde{\tau}_0$ is an irreducible cuspidal representation of the similitude group $\widetilde{\mathcal{G}}_0^{\mathcal{F}} = \mathrm{GSp}_{2(N-1)}^{\mathcal{F}}$ containing τ_0, and $\chi = \tau_1 \circ \mu$ is the non-trivial quadratic character of $\widetilde{\mathcal{G}}_0^{\mathcal{F}}$. Denote by $\mathscr{Z}(\widetilde{\mathcal{G}}_0^{\mathcal{F}})$ the centre of $\widetilde{\mathcal{G}}_0^{\mathcal{F}}$; then $\tilde{\tau}_0 \otimes \chi \simeq \tilde{\tau}_0$ if and only if the restriction of $\tilde{\tau}_0$ to the index two subgroup $\mathscr{Z}(\widetilde{\mathcal{G}}_0^{\mathcal{F}})\mathcal{G}_0^{\mathcal{F}}$ is reducible, i.e. $\tilde{\tau}_0$ restricts reducibly to $\mathcal{G}_0^{\mathcal{F}}$. Thus $\tilde{\tau}^w \simeq \tilde{\tau}$ if and only if τ_0 does *not* extend to $\widetilde{\mathcal{G}}_0^{\mathcal{F}}$, and we are done. ∎

7.5. **Even special orthogonal groups.** Now we turn to the case of even-dimensional orthogonal groups, so that we are in case (iii) of Section 7.3. Here, as well as proving the analogue of Lemma 7.9, we need to consider the cases, seen in Section 6, where the parameter in the affine Hecke algebra is zero, rather than matching the parameter in the finite Hecke algebra for the connected reductive quotient. This happens precisely when either we have a "trivial orthogonal group" or an irreducible cuspidal representation of an even-dimensional special orthogonal group which is not normalized by the full orthogonal group. Thus we must also identify when this happens, in the language of the previous sections. We begin with this question, since the answer is also needed for the proof of the analogue of Lemma 7.9.

Proposition 7.10. *Let $\tau \in \mathscr{E}(\mathrm{SO}_{2N}^{\pm,\mathcal{F}}, s)$ be an irreducible cuspidal representation, and let $\tilde\tau \in \mathscr{E}(\mathrm{GSO}_{2N}^{\pm,\mathcal{F}}, \tilde s)$ be an irreducible cuspidal representations, where $\tilde s$ is a semisimple element mapping to s.*

(i) *$\tilde\tau$ extends to a representation of $\mathrm{GO}_{2N}^{\pm,\mathcal{F}}$ if and only if 1 is an eigenvalue of s.*
(ii) *τ extends to a representation of $\mathrm{O}_{2N}^{\pm,\mathcal{F}}$ if and only if at least one of ± 1 is an eigenvalue of s.*

We remark that in the case that $s^2 = 1$ (that is, τ is a *quadratic unipotent representation*), (i) is proved in [33, Lemma 8.9], while (ii) is proved in [54] (see the proof of *op. cit.* Proposition 4.3). Our proofs in the general case are similar.

Proof. In order to prove this, we need to recall a little about the dual group of $\mathrm{GSO}_{2N}^{\pm}$. This dual group is the *special Clifford group* $\mathrm{C}^0(\mathcal{V})$, which sits in an exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{C}^0(\mathcal{V}) \to \mathrm{SO}(\mathcal{V}) \to 1,$$

which is exact on points by Hilbert's Theorem 90. It is the connected component of the full Clifford group $\mathrm{C}(\mathcal{V})$, which sits in a similar exact sequence, mapping onto the full orthogonal group $\mathrm{O}(\mathcal{V})$. The Clifford group is a subgroup of the group of invertible elements of the Clifford algebra $\mathrm{A} = \mathrm{A}(\mathcal{V})$, which is $\mathbf{Z}/2\mathbf{Z}$-graded $\mathrm{A} = \mathrm{A}^0 \oplus \mathrm{A}^1$; then $\mathrm{C}(\mathcal{V}) = \mathrm{C}^0(\mathcal{V}) \sqcup \mathrm{C}^1(\mathcal{V})$, with $\mathrm{C}^i(\mathcal{V}) = \mathrm{C}(\mathcal{V}) \cap \mathrm{A}^i$, and the special Clifford group is a subgroup of index two in the full Clifford group.

We will need to know when the semisimple element $\tilde s \in \mathrm{C}^0(\mathcal{V})$ of the special Clifford group is centralized by some element of $\mathrm{C}^1(\mathcal{V})$. If 1 is an eigenvalue of s then it has an anisotropic eigenvector v with eigenvalue 1 and one checks that $\mathrm{v} \in \mathrm{C}^1(\mathcal{V})$ is centralized by $\tilde s$. If 1 is not an eigenvalue of s, then the elements of A^1 which commute with $\tilde s$ are linear combinations of elements of the form

$$\mathrm{v}_1 \cdots \mathrm{v}_r,$$

with v_i linearly independent eigenvectors of s with eigenvalue ζ_i, such that $\prod_{i=1}^r \zeta_i = 1$ and r is odd. However, any two such elements anti-commute (since r is odd) and any such element squares to 0, since the v_i are isotropic unless $\zeta_i = -1$ and we cannot have all $\zeta_i = -1$, since their product is 1 and r is odd. Thus any element of $a \in \mathrm{A}^1$ which commutes with s satisfies $a^2 = 0$ so is non-invertible. Thus no element of $\mathrm{C}^1(\mathcal{V})$ commutes with $\tilde s$.

Finally, we pick $c \in \mathrm{O}_{2N}^{\pm,\mathcal{F}} \backslash \mathrm{SO}_{2N}^{\pm,\mathcal{F}}$ and we are ready to begin the proof.

(i) The representation $\tilde\tau$ extends to $\mathrm{GO}_{2N}^{\pm,\mathcal{F}}$ if and only if it is normalized by c. Now $\tilde\tau^c \in \mathscr{E}(\mathrm{GSO}_{2N}^{\pm,\mathcal{F}}, \tilde c^{-1}\tilde s\tilde c)$, for some $\tilde c \in \mathrm{C}^1(\mathcal{V})$. On the other, hand, these Lusztig series contain only one cuspidal representation each, so $\tilde\tau \simeq \tilde\tau^c$ if and only if $\tilde s$ is conjugate in $\mathrm{C}^0(\mathcal{V})$ to $\tilde c^{-1}\tilde s\tilde c$, that is, if and only if the centralizer in $\mathrm{C}(\mathcal{V})$ of $\tilde s$ is *not* contained in the special

Clifford group $\mathrm{C}^0(\mathcal{V})$. However, we have seen above that this happens if and only if 1 is an eigenvalue of s.

(ii) The proof begins in a similar way. The representation τ extends to $\mathrm{O}_{2N}^{\pm,\mathcal{F}}$ if and only if it is normalized by c. In this case, $\tau^c \in \mathscr{E}(\mathrm{SO}_{2N}^{\pm,\mathcal{F}}, c^{-1}sc)$ and, if this Lusztig series contains only one cuspidal representation, then $\tau \simeq \tau^c$ if and only if s is conjugate in $\mathrm{SO}_{2N}^{\pm,\mathcal{F}}$ to $c^{-1}sc$, that is, if and only if the centralizer in $\mathrm{O}_{2N}^{\pm,\mathcal{F}}$ of s is *not* contained in the special orthogonal group. On the other hand, we already know that the Lusztig series contains only one cuspidal representation if and only if at most one of ± 1 is an eigenvalue of s, while the centralizer of s is contained in $\mathrm{SO}_{2N}^{\pm,\mathcal{F}}$ if and only if neither ± 1 is an eigenvalue of s.

This leaves the case when *both* ± 1 are eigenvalues of s. Let $\tilde\tau$ be an irreducible cuspidal representation in $\mathscr{E}(\mathrm{GSO}_{2N}^{\pm,\mathcal{F}}, \tilde s)$ whose restriction to $\mathrm{SO}_{2N}^{\pm,\mathcal{F}}$ contains τ. By (i), this representation is normalized by c.

Now we consider the representation $\mathbf{1}\otimes\tau$ of $\mathcal{L}^{\mathcal{F}} = \mathrm{GL}_1^{\mathcal{F}} \times \mathrm{SO}_{2N}^{\pm,\mathcal{F}}$, a Levi subgroup of $\mathrm{SO}_{2(N+1)}^{\pm,\mathcal{F}}$. As at the end of Section 7.3, we choose a Witt basis $\mathrm{e}_{N+1}^-, \ldots, \mathrm{e}_{N+1}^+$ with respect to which $\mathcal{L}^{\mathcal{F}}$ is standard and denote by w the element of $\mathrm{SO}_{2(N+1)}^{\pm,\mathcal{F}}$ given by

$$\begin{cases} w(\mathrm{e}_{N+1}^{\pm}) = \mathrm{e}_{N+1}^{\mp}, \\ w|_{\langle \mathrm{e}_N^-,\ldots,\mathrm{e}_N^+\rangle} = c. \end{cases}$$

Similarly, we have the representation $\mathbf{1}\otimes\tilde\tau$ of $\widetilde{\mathcal{L}}^{\mathcal{F}} = \mathrm{GL}_1^{\mathcal{F}} \times \mathrm{GSO}_{2N}^{\pm,\mathcal{F}}$, which is normalized by w, by Lemma 7.8(ii). But then Lemma 7.5(i) implies that w also normalizes $\mathbf{1}\otimes\tau$, whence τ is normalized by c. Thus τ extends to $\mathrm{O}_{2N}^{\pm,\mathcal{F}}$, as required. □

Now we return to the notation at the end of section 7.3. Thus, with $\mathcal{G} = \mathrm{SO}_{2N}^{\pm}$ and $\widetilde{\mathcal{G}} = \mathrm{GSO}_{2N}^{\pm}$, we have a cuspidal representation τ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$, for some maximal proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}$, and a cuspidal representation $\tilde\tau$ in $\mathscr{E}(\widetilde{\mathcal{L}}^{\mathcal{F}}, \tilde s)$ whose restriction to $\mathcal{L}$ contains τ. We denote by w a representative for the non-trivial element of $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$, which we assume normalizes τ.

We write $\mathcal{L}^{\mathcal{F}} \simeq \mathrm{GL}_n^{\mathcal{F}} \times \mathcal{G}_0^{\mathcal{F}}$, where $\mathcal{G}_0$ is a (possibly trivial) special orthogonal group, and $\tau = \tau_1 \otimes \tau_0$, so that τ_1 is a self-dual irreducible cuspidal representation of $\mathrm{GL}_n^{\mathcal{F}}$. Then $\tau_1 \in \mathscr{E}(\mathrm{GL}_n^{\mathcal{F}}, s_1)$ and $\tau_0 \in \mathscr{E}(\mathcal{G}_0^{\mathcal{F}}, s_0)$, for some semisimple elements s_0, s_1 of the respective dual groups. Note that, if $\mathcal{G}_0$ is the trivial special orthogonal group then its dual group is the trivial group so that s_0 has no eigenvalues.

Corollary 7.11. *The representation $\tilde\tau$ is normalized by w in all but the following cases:*

(i) *τ_1 is the trivial character of $\mathrm{GL}_1^{\mathcal{F}}$ and 1 is not an eigenvalue of s_0;*

(ii) *τ_1 is the quadratic character of $\mathrm{GL}_1^{\mathcal{F}}$ and -1 is not an eigenvalue of s_0.*

We remark that, among the exceptional cases, we cannot have that $\mathcal{G}_0$ is the trivial group, since otherwise we would have $\mathcal{G} = \mathrm{SO}_2^+$, which we have excluded.

Proof. By Lemma 7.8, we need only consider the case $n = 1$, so that τ_1 is a trivial or quadratic character; then $\tilde\tau$ is normalized by w if and only if $\tilde\tau_0 \simeq \tilde\tau_0^{c_0} \otimes (\tau_1 \circ \mu)$, where we recall that $\tilde\tau_0$ is an irreducible cuspidal representation of the similitude group $\widetilde{\mathcal{G}}_0^{\mathcal{F}} = \mathrm{GSO}_{2(N-1)}^{\pm,\mathcal{F}}$ containing τ_0, and $c_0 \in \mathrm{O}_{2(N-1)}^{\pm,\mathcal{F}} \backslash \mathrm{SO}_{2(N-1)}^{\pm,\mathcal{F}}$.

Suppose first that τ_1 is trivial. Then $\tilde\tau$ is normalized by w if and only if $\tilde\tau_0$ is normalized by c_0, which happens if and only if $\tilde\tau_0$ extends to the full similitude group; by Proposition 7.10, this happens if and only if 1 is an eigenvalue of s_0, and we are done.

Now suppose that τ_1 is the non-trivial quadratic character and put $\chi_1 = \tau_1 \circ \mu$. We also denote by c_1 an element of $\mathrm{GSO}^{\pm,\mathcal{F}}_{2(N-1)}$ which is not in $\mathscr{Z}(\mathrm{GSO}^{\pm,\mathcal{F}}_{2(N-1)})\,\mathrm{SO}^{\pm,\mathcal{F}}_{2(N-1)}$; thus $\tilde\tau_0$ is an extension of τ_0 if and only if τ_0 is normalized by c_1, if and only if $\tilde\tau_0 \not\simeq \tilde\tau_0 \otimes \chi_1$. On the other hand, we already know from Lemma 7.4 that τ_0 extends to an irreducible representation of $\mathrm{GSO}^{\pm,\mathcal{F}}_{2(N-1)}$ if and only if at most one of ± 1 is an eigenvalue of s_0.

If both ± 1 are eigenvalues of s_0, then $\tilde\tau_0 \simeq \tilde\tau_0 \otimes \chi_1$, while $\tilde\tau_0$ extends to $\mathrm{GO}^{\pm,\mathcal{F}}_{2(N-1)}$, by Proposition 7.10(i). Thus $\tilde\tau_0^{c_0} \otimes \chi_1 \simeq \tilde\tau_0 \otimes \chi_1 \simeq \tilde\tau_0$.

If neither ± 1 is an eigenvalue of s_0, then $\tilde\tau_0^{c_0} \otimes \chi_1$ is an extension of $\tau_0^{c_0}$, which is not equivalent to τ_0 by Proposition 7.10(ii). Thus $\tilde\tau_0^{c_0} \otimes \chi_1 \not\simeq \tilde\tau_0$.

Finally, if exactly one of ± 1 is an eigenvalue of s_0, then $\tau_0^{c_0} \simeq \tau_0$, by Proposition 7.10(ii). Then $\tilde\tau_0^{c_0}$ contains τ_0 on restriction, so is equivalent either to $\tilde\tau_0$ or to $\tilde\tau_0 \otimes \chi_1$, since it agrees with $\tilde\tau_0$ on the index two subgroup $\mathscr{Z}(\mathrm{GSO}^{\pm,\mathcal{F}}_{2(N-1)})\,\mathrm{SO}^{\pm,\mathcal{F}}_{2(N-1)}$ of $\mathrm{GSO}^{\pm,\mathcal{F}}_{2(N-1)}$. By Proposition 7.10(i), the former happens if and only if 1 is an eigenvalue of s_0, in which case $\tilde\tau_0^{c_0} \otimes \chi_1 \simeq \tilde\tau_0 \otimes \chi_1 \not\simeq \tilde\tau_0$. Thus the latter happens if and only if -1 is an eigenvalue s_0, in which case $\tilde\tau_0^{c_0} \otimes \chi_1 \simeq \tilde\tau_0$. □

7.6. **Summary.** We summarize the results of all these calculations, including looking up parameters in Lusztig's tables in [35], in the following table. We are given an irreducible cuspidal representation τ in $\mathscr{E}(\mathcal{L}^{\mathcal{F}}, s)$, for some maximal proper Levi subgroup $\mathcal{L}$ of $\mathcal{G}$, which is normalized by $\mathcal{N}_{\mathcal{G}}(\mathcal{L})$. If $w \in \mathcal{N}_{\mathcal{G}}(\mathcal{L}) \backslash \mathcal{L}$, then the Hecke algebra $\mathrm{End}_{\mathcal{G}}(\mathrm{Ind}^{\mathcal{G}}_{\mathcal{L},\mathcal{P}}\,\tau)$ is two-dimensional and is generated by an element T supported on $\mathcal{P}w\mathcal{P}$ and satisfying a quadratic relation

$$(T+1)(T-q^{f_\tau}) = 0.$$

By Lemma 7.5, the parameter q^{f_τ} is either 1 or else coincides with a similar parameter in the case of connected centre, and which of these occurs is determined by Lemma 7.8 and the results of Sections 7.4–7.5. Moreover, the parameter in the case of connected centre can be read from [35, Table II, page 33], as described at the end of Section 7.2

We write $\mathcal{L}^{\mathcal{F}} \simeq \mathrm{GL}_n^{\mathcal{F}} \times \mathcal{G}_0^{\mathcal{F}}$, where $\mathcal{G}_0$ is a (possibly trivial) classical group of the same type as $\mathcal{G}$, and $\tau = \tau_1 \otimes \tau_0$, so that τ_1 is a self-dual irreducible cuspidal representation of $\mathrm{GL}_n^{\mathcal{F}}$. Then $\tau_1 \in \mathscr{E}(\mathrm{GL}_n^{\mathcal{F}}, s_1)$ and $\tau_0 \in \mathscr{E}(\mathcal{G}_0^{\mathcal{F}}, s_0)$, for some semisimple elements s_0, s_1 of the respective dual groups.

We write

$$P_{s_0}(X) = \prod_P P(X)^{a_P}(X-1)^{a_+}(X+1)^{a_-}$$

for the characteristic polynomial of s_0, where the product is over all irreducible self-dual monic polynomials over $\mathbf{F}_q$ of even degree, and the integers $a_P, a_\pm$ are related to integers $m_P, m_\pm$ as in the description in (7.2). We also write Q for the characteristic polynomial of $s_1 \in \mathrm{GL}_n^{*,\mathcal{F}}$; thus either $Q(X) = (X \pm 1)$ or Q is an irreducible self-dual monic polynomial of even degree $n = n_Q$. In the table, the cases (i)–(iii) refer to the different possible classical groups, as in Section 7.3.

degree n	polynomial Q	case	f_τ
1	$X-1$	(i),(ii) (iii)	$2m_+ + 1$ $2m_+$
1	$X+1$	(i) (ii),(iii)	$2m_- + 1$ $2m_-$
n_Q even	Q	(i),(ii),(iii)	$(2m_Q+1)\frac{n_Q}{2}$

7.7. **Unitary groups.** Finally, we consider the case of unitary groups. We could have included this in the cases of Sections 7.2–7.6 above but it would have further complicated the notation. Instead, we indicate here the differences with the previous cases and summarize the final results.

Let $q = q_\circ^2$ be an even power of an odd prime p, take $\mathcal{G} = \mathrm{GL}_n$ over the finite field $\mathbf{F}_{q_\circ}$, and let $\mathcal{F}$ be the twisted Frobenius map, so that $\mathcal{G}^{\mathcal{F}}$ is a unitary group (which we can think of as a subgroup of $\mathrm{GL}_n(\mathbf{F}_q)$). Then $\mathcal{G}^* = \mathrm{GL}_n$ act naturally on an n-dimensional vector space $\mathcal{V}$ with an $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-hermitian form. For $s \in \mathcal{G}^{*,\mathcal{F}}$ semisimple, we denote by $P_s(X) \in \mathbf{F}_q[X]$ its characteristic polynomial as an automorphism of $\mathcal{V}$.

From [33, §9], the equivalence classes of irreducible cuspidal representations of $\mathcal{G}^{\mathcal{F}}$ are in bijection with the set of conjugacy classes of semisimple elements s in $\mathcal{G}^{*,\mathcal{F}}$ whose characteristic polynomial is of the form

$$P_s(X) = \prod_P P(X)^{a_P},$$

where the product runs over all irreducible $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-self-dual monic polynomials in $\mathbf{F}_q[X]$ (see Section 7.1), and $a_P = \frac{1}{2}(m_P^2 + m_P)$, for some integer $m_P \geqslant 0$.

Now suppose $\mathcal{L}$ is a maximal proper $\mathcal{F}$-stable Levi subgroup of $\mathcal{G}$, which is the Levi component of an $\mathcal{F}$-stable parabolic subgroup $\mathcal{P}$. We write $\mathcal{L}^{\mathcal{F}} \simeq \mathrm{GL}_m(\mathbf{F}_q) \times \mathcal{G}_0^{\mathcal{F}}$, with $\mathcal{G}_0^{\mathcal{F}}$ again a unitary group. Let τ be an irreducible cuspidal representation of $\mathcal{L}^{\mathcal{F}}$ with the property that any representative w for the non-trivial element of $\mathcal{N}_{\mathcal{G}}(\mathcal{L})/\mathcal{L}$ normalizes τ. Thus we may decompose $\tau = \tau_1 \otimes \tau_0$, with τ_1 a (conjugate)-self-dual irreducible cuspidal representation of $\mathrm{GL}_m(\mathbf{F}_q)$ and τ_0 an irreducible cuspidal representation of $\mathcal{G}_0^{\mathcal{F}}$.

In this situation, the induced representation $\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau$ decomposes again as $\pi_1 \oplus \pi_2$, with $\dim(\pi_1) > \dim(\pi_2)$, and $\mathrm{End}_{\mathcal{G}^{\mathcal{F}}}(\mathrm{Ind}_{\mathcal{L},\mathcal{P}}^{\mathcal{G}}\,\tau)$ is a two-dimensional algebra with a quadratic generator T satisfying a relation of the form

$$(T+1)(T - q^{f_\tau}) \;=\; 0, \qquad q^{f_\tau} = \frac{\dim(\pi_1)}{\dim(\pi_2)} > 1.$$

As in the connected case above, the parameter may be computed via the Jordan decomposition of characters and Lusztig's tables, as follows. For s_0 a semisimple element of $\mathcal{G}_0^{*,\mathcal{F}}$ such that $\tau_0 \in \mathscr{E}(\mathcal{G}_0^{\mathcal{F}}, s_0)$, we write its characteristic polynomial

$$P_{s_0}(X) = \prod_P P(X)^{a_P},$$

for integers $a_P = \frac{1}{2}(m_P^2 + m_P)$ as above. We also write Q for the irreducible characteristic polynomial of an element $s_1 \in \mathrm{GL}_m^{*,\mathcal{F}}$ such that $\tau_1 \in \mathscr{E}(\mathrm{GL}_m^{\mathcal{F}}, s_1)$; thus Q is an irreducible $\mathbf{F}_q/\mathbf{F}_{q_\circ}$-self-dual monic polynomial, of some odd degree $n = n_Q$. Then we get

$$f_\tau = (2m_Q + 1)\frac{n_Q}{2}.$$

8. Synthesis

In this section, we put together the previous results to verify the inequality (1.4), for π a depth zero irreducible cuspidal representation of the classical group G. Recall that $N_{\widehat{\mathrm{G}}}$ is the dimension of the vector space on which the complex dual group $\widehat{\mathrm{G}}$ acts naturally. In fact we prove that, in (1.4), the sum over depth zero self-dual irreducible cuspidal representations already gives us $N_{\widehat{\mathrm{G}}}$, that is:

$$\sum_{\rho\in\mathcal{A}^{\sigma}_{[0]}(\mathrm{F})} \left\lfloor (s_\pi(\rho))^2 \right\rfloor n_\rho \;=\; N_{\widehat{\mathrm{G}}}, \tag{8.1}$$

Remark 8.2. In many cases, the opposite inequality to (1.4) was already proved by Mœglin in [39] (so that (1.3) follows); alternatively, the techniques used here, together with the results in [37], easily show that, for ρ a positive depth self-dual irreducible cuspidal representation we have $s_\pi(\rho)\in\left\{0,\pm\frac{1}{2}\right\}$, so that these do not contribute to the sum. (See also [6], where this is carried out in a more general situation.) The details are left as an exercise.

Thus we return to the notation of Sections 2–6: we have $\pi = \text{c-Ind}_{\mathrm{J}_\pi}^{\mathrm{G}}\,\lambda_\pi$ an irreducible cuspidal depth zero representation of a classical group G, with J_π the normalizer of a standard maximal parahoric subgroup $\mathrm{J}^{\circ}_{N_1,N_2}$, and $\lambda_\pi|_{\mathrm{J}^{\circ}_{N_1,N_2}}$ contains an irreducible representation λ°_π inflated from an irreducible cuspidal representation $\tau_\pi\simeq\tau_\pi^{(1)}\otimes\tau_\pi^{(2)}$ of the reductive quotient $\mathcal{G}^{\circ}_{N_1,N_2}\simeq\mathcal{G}^{(1)}_{N_1}\times\mathcal{G}^{(2)}_{N_2}$. For $i=1,2$, there is a unique conjugacy class (s_i) in $\mathcal{G}^{(i),*}_{N_i}$ such that $\tau_\pi^{(i)}$ is in the Lusztig series $\mathscr{E}(\mathcal{G}^{(i)}_{N_i},s_i)$, and we denote the characteristic polynomial of s_i by

$$\prod_P P(X)^{a_P^{(i)}},$$

where the product runs over irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomials in $k_{\mathrm{F}}[X]$, and the powers $a_P^{(i)}$ satisfy the conditions of (7.2); in particular, there are integers $m_P^{(i)}\geqslant 0$ such that:

- if $k_{\mathrm{F}}\neq k_{\mathsf{o}}$ or $P(X)\neq(X\pm1)$ then $a_P^{(i)}=\frac{1}{2}m_P^{(i)}(m_P^{(i)}+1)$;
- if $k_{\mathrm{F}}=k_{\mathsf{o}}$ and $P(X)=(X\pm1)$ then we write $m_+^{(i)}=m_{(X-1)}^{(i)}$ and $m_-^{(i)}=m_{(X+1)}^{(i)}$, to match the notation of Section 7, and these satisfy the conditions in (7.2).

Remark 8.3. It may be that $N_i=N_i^{\mathsf{an}}=0$, for $i=1$ or 2; in this case the group $\mathcal{G}^{(i)}_{N_i}$ is trivial, but we must interpret it as the "right" trivial group. That is, if G is symplectic then the group is a trivial symplectic group; if G is special orthogonal it is a trivial special orthogonal group; if G is unramified unitary it is a trivial unitary group; and if G is ramified unitary then it is a trivial symplectic group if $\varepsilon=(-1)^i$, and a trivial special orthogonal group otherwise. In particular, if the group is trivial symplectic then the characteristic polynomial of s_i is $X-1$; in the other cases, the characteristic polynomial of s_i is the constant polynomial 1.

Now, for ρ a self-dual irreducible cuspidal depth zero representation of some $\mathrm{GL}_n(\mathrm{F})$, we have a unique self-dual irreducible cuspidal representation τ_ρ of $\mathrm{GL}_n(k_{\mathrm{F}})$ such that ρ contains the representation λ_ρ of $\mathrm{GL}_n(\mathfrak{o}_{\mathrm{F}})$ obtained from τ_ρ by inflation. Then τ_ρ is in the Lusztig series associated to some conjugacy class in $\mathrm{GL}_n(k_{\mathrm{F}})$ with irreducible self-dual characteristic polynomial $Q=Q_\rho$ of degree n.

We suppose first that $k_{\mathrm{F}}\neq k_{\mathsf{o}}$ or $Q(X)\neq(X\pm1)$; thus either $n>1$ or G is an unramified unitary group, and the parameters q^{f_i} of the Hecke algebra are always computed from the Hecke algebra in the finite group. Then the formulae in Sections 7.6–7.7 give

$$f_i = f_{\tau_\rho\otimes\tau_\pi^{(i)}} = (2m_Q^{(i)}+1)\frac{n}{2}$$

and, from (5.1) we get reducibility points

$$\left\{\pm s_\pi(\rho), \pm s_\pi(\rho')\right\} = \left\{\pm\frac{(m_Q^{(1)}+m_Q^{(2)}+1)}{2}, \pm\frac{(m_Q^{(1)}-m_Q^{(2)})}{2}\right\}.$$

Since one of these is an integer and the other a half-integer, we get

$$\lfloor s_\pi(\rho)^2\rfloor + \lfloor s_\pi(\rho')^2\rfloor = \left(\frac{(m_Q^{(1)}+m_Q^{(2)}+1)}{2}\right)^2 + \left(\frac{(m_Q^{(1)}-m_Q^{(2)})}{2}\right)^2 - \frac{1}{4} = a_Q^{(1)} + a_Q^{(2)}.$$

Thus we are already done in the case of unramified unitary groups: summing, we get

$$\sum_{\rho\in\mathcal{A}^\sigma_{[0]}(\mathrm{F})} \lfloor s_\pi(\rho)^2\rfloor n_\rho = \sum_P (a_P^{(1)} + a_P^{(2)})\deg(P) = (2N_1 + N_1^{\mathsf{an}}) + (2N_2 + N_2^{\mathsf{an}}) = 2N + N^{\mathsf{an}},$$

as required.

For the cases $k_\mathrm{F} = k_\mathsf{o}$ and $Q(X) = (X \pm 1)$, we will split according to the type of group G, since the values for the parameters do not admit such a uniform description.

8.1. **Symplectic groups.** We suppose first that $Q(X) = X - 1$, so that ρ, ρ' are the trivial character and the unramified character of order 2, and write $m_Q^{(i)} = m_+^{(i)}$. Since both $\mathcal{G}_{N_i}^{(i)}$ are symplectic groups, we get

$$f_i = (2m_+^{(i)} + 1),$$

with reducibility points

$$\left\{\pm s_\pi(\rho), \pm s_\pi(\rho')\right\} = \left\{\pm(m_+^{(1)} + m_+^{(2)} + 1), \pm(m_+^{(1)} - m_+^{(2)})\right\}.$$

Thus

$$\lfloor s_\pi(\rho)^2\rfloor + \lfloor s_\pi(\rho')^2\rfloor = (m_+^{(1)} + m_+^{(2)} + 1)^2 + (m_+^{(1)} - m_+^{(2)})^2 = a_+^{(1)} + a_+^{(2)} - 1.$$

Now suppose that $Q(X) = X + 1$, so that ρ, ρ' are (tamely) ramified characters of order 2, and write $m_Q^{(i)} = m_-^{(i)}$. Then we get

$$f_i = 2m_-^{(i)},$$

with reducibility points

$$\left\{\pm s_\pi(\rho), \pm s_\pi(\rho')\right\} = \left\{\pm(m_-^{(1)} \pm m_-^{(2)})\right\}.$$

Thus

$$\lfloor s_\pi(\rho)^2\rfloor + \lfloor s_\pi(\rho')^2\rfloor = (m_-^{(1)} + m_-^{(2)})^2 + (m_-^{(1)} - m_-^{(2)})^2 = a_-^{(1)} + a_-^{(2)}.$$

Finally, summing we get

$$\sum_{\rho\in\mathcal{A}^\sigma_{[0]}(\mathrm{F})} \lfloor s_\pi(\rho)^2\rfloor n_\rho = \sum_P (a_P^{(1)} + a_P^{(2)})\deg(P) - 1 = (2N_1 + 1) + (2N_2 + 1) - 1 = 2N + 1.$$

8.2. **Ramified unitary groups.** In this case, the groups $\mathcal{G}_{N_i}^{(i)}$ are one symplectic and one orthogonal; for ease of exposition, we will assume that $\mathcal{G}_{N_1}^{(1)}$ is symplectic (otherwise exchange 1 and 2).

We begin again with the case $Q(X) = X - 1$ and write $m_+^{(i)}$ in place of $m_Q^{(i)}$. Thus we get

$$f_1 = 2m_+^{(1)} + 1,\ f_2 = \begin{cases} 2m_+^{(2)} + 1, & \text{if } N_2^{\mathsf{an}} = 1, \\ 2m_+^{(2)}, & \text{otherwise.} \end{cases}$$

Thus we get reducibility points

$$\left\{\pm s_\pi(\rho), \pm s_\pi(\rho')\right\} = \begin{cases} \left\{\pm(m_+^{(1)} + m_+^{(2)} + 1), \pm(m_+^{(1)} - m_+^{(2)})\right\}, & \text{if } N_2^{\mathsf{an}} = 1, \\ \left\{\pm\left(m_+^{(1)} + m_+^{(2)} + \frac{1}{2}\right), \pm\left(m_+^{(1)} - m_+^{(2)} + \frac{1}{2}\right)\right\}, & \text{otherwise.} \end{cases}$$

Thus, if $N_2^{\mathsf{an}} = 1$, we get

$$\lfloor s_\pi(\rho)^2 \rfloor + \lfloor s_\pi(\rho')^2 \rfloor = (m_+^{(1)} + m_+^{(2)} + 1)^2 + (m_+^{(1)} - m_+^{(2)})^2 = a_+^{(1)} + a_+^{(2)};$$

and otherwise, since both reducibility points are half-integers, we get

$$\lfloor s_\pi(\rho)^2 \rfloor + \lfloor s_\pi(\rho')^2 \rfloor = \left(m_+^{(1)} + m_+^{(2)} + \tfrac{1}{2}\right)^2 + \left(m_+^{(1)} - m_+^{(2)} + \tfrac{1}{2}\right)^2 - \tfrac{1}{2} = a_+^{(1)} + a_+^{(2)} - 1.$$

The case $Q(X) = X + 1$ is similar, the main difference being that $f_1 = 2m_-^{(1)}$. Then we get reducibility points

$$\left\{\pm s_\pi(\rho), \pm s_\pi(\rho')\right\} = \begin{cases} \left\{\pm(m_-^{(1)} + m_-^{(2)} + \frac{1}{2}), \pm(m_-^{(1)} - m_-^{(2)} - \frac{1}{2})\right\}, & \text{if } N_2^{\mathsf{an}} = 1, \\ \left\{\pm\left(m_-^{(1)} \pm m_-^{(2)}\right)\right\}, & \text{otherwise.} \end{cases}$$

Now in both cases we get

$$\lfloor s_\pi(\rho)^2 \rfloor + \lfloor s_\pi(\rho')^2 \rfloor = a_-^{(1)} + a_-^{(2)}.$$

Noting that we have

$$\sum_P a_P^{(1)} \deg(P) = 2N_1 + 1, \quad \sum_P a_P^{(2)} \deg(P) = \begin{cases} 2N_2, & \text{if } N_2^{\mathsf{an}} = 1, \\ 2N_2 + N_2^{\mathsf{an}}, & \text{otherwise,} \end{cases}$$

we once again see that, summing over all depth zero self-dual irreducible cuspidal representations of all $\mathrm{GL}_n(\mathrm{F})$, equation 8.1 is satisfied.

8.3. **Special orthogonal groups.** The case of special orthogonal groups is exactly analogous and we do not give the details. One can check the equality in (8.1) by working through the cases according to the parities of $N_1^{\mathsf{an}}, N_2^{\mathsf{an}}$. For example, if both are odd, then with $Q(X) = X \pm 1$ we get

$$\lfloor s_\pi(\rho)^2 \rfloor + \lfloor s_\pi(\rho')^2 \rfloor = a_\pm^{(1)} + a_\pm^{(2)} + 1.$$

The additions of the extra 1 here exactly compensate for the fact that the dual groups of $\mathcal{G}_{N_i}^{(i)}$ have dimension $2N_i = 2N_i + N_i^{\mathsf{an}} - 1$.

8.4. **Summary.** In all cases, we have now checked that the equality (8.1) holds. We have also seen that $s_\pi(\rho) \in \frac{1}{2}\mathbf{Z}$ in all cases.

9. L-packets and Examples

In this final section, we examine the implications of the results here for the computation of L-packets and give some examples. Firstly, we recall some facts about the (expected) sizes of discrete series L-packets containing an irreducible cuspidal representation, and the (expected) number of cuspidal representations in them.

Let $\varphi : \mathcal{W}_{\mathrm{F}} \times \mathrm{SL}_2(\mathbf{C}) \to \widehat{\mathrm{G}} \rtimes \mathcal{W}_{\mathrm{F}}$ be a Langlands parameter for G whose L-packet Π_φ contains an irreducible cuspidal representation π of G. Then, as recalled in the introduction, we should have

$$\varphi \;=\; \bigoplus_{(\rho,m)\in \mathrm{Jord}(\pi)} \varphi_\rho \otimes \mathrm{st}_m,$$

where φ_ρ is the (irreducible) representation of the Weil group $\mathcal{W}_{\mathrm{F}}$ corresponding to ρ via the Langlands correspondence for general linear groups, and st_m is the m-dimensional irreducible representation of $\mathrm{SL}_2(\mathbf{C})$. Putting $\ell(\pi) = \#\,\mathrm{Jord}(\pi)$, the number of representations one expects in the packet Π_φ is $2^{\ell(\pi)-1}$, since this is the number of characters of the component group of $\mathrm{Cent}_{\widehat{\mathrm{G}}}(\mathrm{Im}(\varphi))$ trivial on the centre of $\widehat{\mathrm{G}}$. We also set

$$\mathrm{E}(\pi) \;=\; \{\rho \in \mathcal{A}^\sigma(\mathrm{F}) : s_\pi(\rho) \in \mathbf{N}\}$$

and $e(\pi) = \#\mathrm{E}(\pi)$; this is the number of $\rho \in \mathcal{A}^\sigma(\mathrm{F})$ such that $(\rho, m) \in \mathrm{Jord}(\pi)$ for some *odd* integer m. Finally, put

$$e_0(\pi) \;=\; \begin{cases} 1 & \text{if there is } \rho \in \mathcal{A}^\sigma(\mathrm{F}) \text{ such that } s_\pi(\rho) \in 1 + 2\mathbf{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the number of irreducible cuspidal representations one expects in the L-packet Π_φ is $2^{e(\pi)-e_0(\pi)}$, since this is the number of characters of the component group of $\mathrm{Cent}_{\widehat{\mathrm{G}}}(\mathrm{Im}(\varphi))$ which are trivial on the centre of $\widehat{\mathrm{G}}$ and *alternating*, in the sense of, for example, [41, Section 8].

Remark 9.1. The reference to the work of Mœglin in [41, Section 8] is in fact only for unitary groups, though it also holds when G is a quasi-split unitary, orthogonal, symplectic or GSpin group (see [40]) and it seems reasonable to expect it to hold in more generality. Although our results here on the representations with given inertial reducibility set do not require it, for the purposes of discussion, from now on we make the following assumption:

(A) The description of the number of irreducible cuspidal representation in an L-packet above is valid for the group G.

However, extra care must be taken when G is an even-dimensional special orthogonal group – see Example 9.6.

Now we describe how our results allow us to find all irreducible cuspidal representations with the same inertial reducibility set, hence all irreducible cuspidal representations in a union of one, two or four L-packets (assuming (A)). We suppose we are in the general situation of the previous section, with π an irreducible cuspidal depth zero representation of G. Recall that

$$\mathrm{IRed}(\pi) \;=\; \{\{([\rho], m) : \rho \in \mathcal{A}^\sigma(\mathrm{F}),\ m \in \mathbf{N} \text{ with } 2s_\pi(\rho) = m + 1\}\},$$

where $[\rho]$ denotes the inertial equivalence class of ρ.

Our representation π is induced from a representation containing the inflation of an irreducible cuspidal representation $\tau_\pi \simeq \tau_\pi^{(1)} \otimes \tau_\pi^{(2)}$ of the reductive quotient $\mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2}$ of a maximal parahoric subgroup. For $i = 1, 2$ and P an irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomial in $k_{\mathrm{F}}[X]$, we denote by $m_P^{(i)}$ the associated non-negative integer as in (7.2).

The formulae obtained above show that, for each irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomial P in $k_{\mathrm{F}}[X]$, the pair of integers $\{m_P^{(1)}, m_P^{(2)}\}$ can be recovered from the reducibility points $\{s_\pi(\rho), s_\pi(\rho')\}$, for $\rho = \rho_P$ a representation in $\mathcal{A}_{[0]}^{\sigma}(\mathrm{F})$ with associated characteristic polynomial P, and ρ' its self-dual unramified twist. Indeed, one gets the following, where we write $|\cdot|_\infty$ for the usual (archimedean) absolute value on $\mathbf{R}$:

- if $P(X) \neq X \pm 1$ or $k_{\mathrm{F}} \neq k_{\mathsf{o}}$ then
$$\left\{m_P^{(1)}, m_P^{(2)}\right\} = \left\{\left\lfloor \left|s_\pi(\rho) \pm s_\pi(\rho')\right|_\infty \right\rfloor\right\};$$
- if $P(X) = X \pm 1$ and $k_{\mathrm{F}} = k_{\mathsf{o}}$, so that ρ is a character of $\mathrm{GL}_1(\mathrm{F})$ of order at most 2, then
$$\left\{m_P^{(1)}, m_P^{(2)}\right\} = \left\{\left\lfloor \frac{|s_\pi(\rho) \pm s_\pi(\rho')|_\infty}{2} \right\rfloor\right\}.$$

Thus one obtains the same inertial reducibility set as for π only for irreducible cuspidal representations π' with $\{m_P^{(1)}(\pi), m_P^{(2)}(\pi)\} = \{m_P^{(1)}(\pi'), m_P^{(2)}(\pi')\}$, for every irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomial P in $k_{\mathrm{F}}[X]$. Hence, in order to obtain other representations with the same inertial reducibility set, it is enough to exchange (some of) the integers $m_P^{(1)}(\pi), m_P^{(2)}(\pi)$. Note, however, that it is not always possible to do this, for parity reasons. We set

$$\mathrm{Q}(\pi) = \left\{\text{irreducible self-dual monic } P \in k_{\mathrm{F}}[X] : m_P^{(1)}(\pi) \neq m_P^{(2)}(\pi)\right\}$$

and put $q(\pi) := \#\mathrm{Q}(\pi)$.

Remark 9.2. If $P \in k_{\mathrm{F}}[X]$ is an irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomial, with $P(X) \neq X \pm 1$ or $k_{\mathrm{F}} \neq k_{\mathsf{o}}$, and ρ_P, ρ'_P are the corresponding self-dual irreducible cuspidal representations of some $\mathrm{GL}_n(\mathrm{F})$, then one of $s_\pi(\rho_P), s_\pi(\rho'_P)$ is integral and the other non-integral. In particular, we see that (exactly) one of ρ_P, ρ'_P is in $\mathrm{E}(\pi)$ if and only if $P \in \mathrm{Q}(\pi)$.

A similar analysis can be done for $P(X) = X \pm 1$ and $k_{\mathrm{F}} = k_{\mathsf{o}}$, but it depends on the type of group. We summarize the results in the separate cases below.

We will parametrize the irreducible cuspidals π' with $\mathrm{IRed}(\pi') = \mathrm{IRed}(\pi)$ by maps $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$, noting that not all such maps are permissible, and that each map may give rise to more than one representation. We split again according to the type of group.

9.1. **Symplectic groups.** We begin with the case that $\mathrm{G} = \mathrm{Sp}_{2N}(\mathrm{F})$ is a symplectic group, in which case there are no restrictions on the maps $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$. We put

$$\delta(\pi) = \#\left\{i : m_{X+1}^{(i)}(\pi) \neq 0\right\}.$$

Now suppose we have $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$ and, for irreducible self-dual monic $P \notin \mathrm{Q}(\pi)$, we set $\boldsymbol{\varepsilon}(P) = 1$. Then we can find, for $i = 1, 2$, a semisimple element $s_{\boldsymbol{\varepsilon}}^{(i)}$ in a suitable odd special orthogonal group $\mathrm{SO}_{2N'_i+1}(k_{\mathrm{F}})$ with characteristic polynomial

$$\prod_P P(X)^{a_P^{(i)}(\boldsymbol{\varepsilon})},$$

where the product is taken over all irreducible self-dual monic polynomials in $k_{\mathrm{F}}[X]$ and the integers $a_P^{(i)}(\boldsymbol{\varepsilon})$ are related to integers $m_P^{(i)}(\boldsymbol{\varepsilon})$ as in (7.2), with

$$m_P^{(i)}(\boldsymbol{\varepsilon}) \;=\; m_P^{(i\cdot\boldsymbol{\varepsilon}(P))},$$

where the index is understood modulo 3. Correspondingly, we have irreducible cuspidal representations $\tau_{\boldsymbol{\varepsilon}} = \tau_{\boldsymbol{\varepsilon}}^{(1)} \otimes \tau_{\boldsymbol{\varepsilon}}^{(2)}$ of $\mathcal{G}_{N_1',N_2'} = \mathrm{Sp}_{2N_1'}(k_{\mathrm{F}}) \times \mathrm{Sp}_{2N_2'}(k_{\mathrm{F}})$; note that, for each $\boldsymbol{\varepsilon}$, the number of such representations is $2^{\delta(\pi)}$. Inflating each $\tau_{\boldsymbol{\varepsilon}}$ to the maximal parahoric subgroup $\mathrm{J}_{N_1',N_2'}$ and inducing to G, we get an irreducible cuspidal representation. Thus we get $2^{q(\pi)+\delta(\pi)}$ irreducible cuspidal representations of G.

The analysis of Remark 9.2, along with that for the cases $P(X) = X \pm 1$, shows that

$$e(\pi) \;=\; \begin{cases} q(\pi) + \delta(\pi) + 1 & \text{if } \delta(\pi) \leqslant 1; \\ q(\pi) + \delta(\pi) & \text{if } \delta(\pi) = 2. \end{cases}$$

On the other hand, we always have $e_0(\pi) = 1$, since either $s_\pi(\mathbf{1})$ or $s_\pi(\omega_0)$ is an odd integer, where $\mathbf{1}$ is the trivial character of $\mathrm{GL}_1(\mathrm{F})$ and ω_0 is the unramified character of order two. Hence we have constructed the irreducible cuspidal representations in a union of two L-packets if $\delta(\pi) = 2$, or in a single L-packet otherwise.

Thus, in some cases we are able to identify all the representations in a single L-packet, but in others we cannot distinguish between the representations in two L-packets without further work. We give some examples to illustrate these phenomena. In the following, we write $\omega_1, \omega_2 = \omega_0\omega_1$ for the (tamely) ramified characters of $\mathrm{GL}_1(\mathrm{F})$ of order two.

Example 9.3. We begin with an example where we are able to recover all the cuspidal representations in a single L-packet. We take $\mathrm{G} = \mathrm{Sp}_6(\mathrm{F})$ and begin with the parahoric subgroup $\mathrm{J}_{2,1}$, which has reductive quotient $\mathcal{G}_{2,1} \simeq \mathrm{Sp}_4(k_{\mathrm{F}}) \times \mathrm{SL}_2(k_{\mathrm{F}})$.

We take the representation θ_{10} of $\mathrm{Sp}_4(k_{\mathrm{F}})$, that is the unique cuspidal representation in the Lusztig series $\mathscr{E}(\mathrm{Sp}_4(k_{\mathrm{F}}), 1)$ (so that the associated characteristic polynomial is $(X-1)^5$). We also take an irreducible cuspidal representation τ of $\mathrm{SL}_2(k_{\mathrm{F}})$ in a Lusztig series with associated characteristic polynomial $(X-1)(X+1)^2$. Thus τ is a representation of dimension $\frac{q-1}{2}$, of which there are two. We denote by λ_π the representation of $\mathrm{J}_{2,1}$ inflated from $\theta_{10} \otimes \tau$ and put $\pi = \text{c-Ind}_{\mathrm{J}_{2,1}}^{\mathrm{G}} \lambda_\pi$, an irreducible cuspidal representation of G.

Following the recipe in Section 8, we find that $s_\rho(\pi) \in \{0, \pm\frac{1}{2}\}$ unless ρ is a character of $\mathrm{GL}_1(\mathrm{F})$. On the other hand, we get

$$m_{X-1}^{(1)} = 1,\ m_{X-1}^{(2)} = 0, \qquad m_{X+1}^{(1)} = 0,\ m_{X+1}^{(2)} = 1,$$

and hence

$$\{s_\pi(\mathbf{1}), s_\pi(\omega_0)\} = \{2, 1\}, \qquad \{s_\pi(\omega_1), s_\pi(\omega_2)\} = \{1\};$$

thus $\mathrm{IRed}(\pi)$ is the multiset $\{\!\{([\mathbf{1}], 2), ([\mathbf{1}], 1), ([\omega_1], 1), ([\omega_1], 1)\}\!\}$. In this case we know more since the Langlands parameter φ_π corresponding to π has image in $\mathrm{SO}_7(\mathbf{C})$ so, in particular, has determinant 1; thus it must be

$$\varphi_\pi = \mathbf{1} \otimes (\mathrm{st}_3 \oplus \mathrm{st}_1) \oplus \omega_0 \oplus \omega_1 \oplus \omega_2,$$

since exchanging $\mathbf{1}$ and ω_0 would give a representation with determinant ω_0.

In the notation above, we have $\mathrm{E}(\pi) = \{\mathbf{1}, \omega_0, \omega_1, \omega_2\}$ so that $e(\pi) = 4$. Thus the L-packet containing π consists of 16 irreducible representations, 8 of which are cuspidal. On the other hand, we have $\mathrm{Q}(\pi) = \{X+1, X-1\}$ so that $q(\pi) = 2$, and $\delta(\pi) = 1$, so that we can construct exactly the cuspidal representations in the L-packet, as follows:

(i) There is one rational conjugacy class (s) in $\mathrm{SO}_3(k_{\mathrm{F}})$ such that its characteristic polynomial is $(X-1)(X+1)^2$ *and* the corresponding Lusztig series $\mathscr{E}(\mathrm{SL}_2(k_{\mathrm{F}}), s)$ contains two irreducible cuspidal representations τ, τ'.

Now we can inflate the representations $\theta_{10}\otimes\tau$ and $\theta_{10}\otimes\tau'$ of $\mathrm{Sp}_4(k_{\mathrm{F}})\times\mathrm{SL}_2(k_{\mathrm{F}})$ to either $\mathrm{J}_{2,1}$ or $\mathrm{J}_{1,2}$ and then induce to G. This gives us four inequivalent irreducible cuspidal representations of G (one of which is π).

(ii) There is one rational conjugacy class (s_1) in $\mathrm{SO}_7(k_{\mathrm{F}})$ such that its characteristic polynomial is $(X-1)^5(X+1)^2$ *and* the corresponding Lusztig series $\mathscr{E}(\mathrm{Sp}_6(k_{\mathrm{F}}), s_1)$ contains two irreducible cuspidal representation τ_1, τ_1'.

We inflate these representations to either $\mathrm{J}_{3,0}$ or $\mathrm{J}_{0,3}$ and induce to G, giving us another four inequivalent irreducible cuspidal representations of G, also inequivalent to those in (i).

Example 9.4. Now we look at the simplest example where the information we have so far is only sufficient to recover the cuspidal representations in a union of two L-packets. We take $\mathrm{G} = \mathrm{Sp}_4(\mathrm{F})$ and begin with the parahoric subgroup $\mathrm{J}_{1,1}$, which has reductive quotient $\mathcal{G}_{1,1} \simeq \mathrm{SL}_2(k_{\mathrm{F}}) \times \mathrm{SL}_2(k_{\mathrm{F}})$.

We take irreducible cuspidal representations τ_1, τ_2 of $\mathrm{SL}_2(k_{\mathrm{F}})$ each in a Lusztig series with associated characteristic polynomial $(X-1)(X+1)^2$, as in the previous example. We denote by λ_π the representation of $\mathrm{J}_{1,1}$ inflated from $\tau_1\otimes\tau_2$ and put $\pi = \text{c-Ind}_{\mathrm{J}_{1,1}}^{\mathrm{G}}\,\lambda_\pi$, an irreducible cuspidal representation of G. Following the recipe, this time we obtain

$$\mathrm{IRed}(\pi) = \{\!\{([\mathbf{1}], 1), ([\omega_1], 2)\}\!\}$$

and, using the fact that the corresponding Langlands parameter φ_π has determinant 1, we have

$$\varphi_\pi = \mathbf{1} \oplus \omega \otimes (\mathrm{st}_3 \oplus \mathrm{st}_1), \tag{9.5}$$

where ω is either ω_1 or ω_2. However, without further work, we cannot distinguish which ramified quadratic character occurs here. This reflects the fact that $m_{X+1}^{(1)} = m_{X+1}^{(2)} = 1$ so that $\delta(\pi) = 2$.

Thus, at this stage, we can only identify the 4 cuspidal representations occurring in the union of the two L-packets corresponding to $\omega = \omega_1, \omega_2$ in (9.5): they are given by independently choosing the τ_i to be one of the two irreducible cuspidal representations of $\mathrm{SL}_2(k_{\mathrm{F}})$ of dimension $\frac{q-1}{2}$.

Distinguishing these two L-packets (and identifying the two discrete series representation in each of them) requires further analysis: for this particular example, this is carried out in [5].

In general, distinguishing the representations when we have two L-packets as in Example 9.4 will probably require, as a first step, the classification of quadratic-unipotent irreducible cuspidal representations of finite classical groups, and the compatibility of this classification with Deligne–Lusztig induction, which is done by Waldspurger in [54].

9.2. **Unramified unitary groups.** Suppose now that G is an unramified unitary group of dimension $2N + N^{\mathsf{an}}$. On the one hand this case is simpler, and we will see that the set of representations with given inertial reducibility (multi)set is a single L-packet. On the other hand, we cannot arbitrarily exchange the integers $m_P^{(1)}, m_P^{(2)}$ as above, due to parity constraints – swapping would sometimes lead to representations of the isometry group of a non-isometric hermitian space.

Recall that π is induced from the inflation of an irreducible cuspidal representation $\tau_\pi^{(1)} \otimes \tau_\pi^{(2)}$ of $\mathcal{G}^{(1)}_{N_1} \times \mathcal{G}^{(2)}_{N_2}$, where $\mathcal{G}^{(i)}_{N_i} = \mathrm{U}(\overline{\mathrm{V}}_{(i)})$, with $\overline{\mathrm{V}}_{(i)}$ a hermitian space of dimension $2N_i + N_i^{\mathsf{an}}$. Moreover, if $\mathcal{G}^{(1)}_{N'_1} \times \mathcal{G}^{(2)}_{N'_2}$ is the reductive quotient of another maximal parahoric subgroup of G then, for $i = 1, 2$, the corresponding space $\overline{\mathrm{V}}'_{(i)}$ must have dimension of the same parity as that of $\overline{\mathrm{V}}_{(i)}$.

Recall also that we have a semisimple element s_i of the dual group of $\mathcal{G}^{(i)}_{N_i}$, such that $\tau_\pi^{(i)}$ is the (unique) irreducible cuspidal representation in the corresponding Lusztig series, and that s_i has characteristic polynomial

$$\prod_P P(X)^{a_P^{(i)}},$$

where the product runs over all irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic polynomials in $k_{\mathrm{F}}[X]$, and $a_P^{(i)} = \frac{1}{2} m_P^{(i)}(m_P^{(i)} + 1)$. Since the degree of each such polynomial P is odd, we have

$$\deg\left(P(X)^{a_P^{(i)}}\right) \text{ is } \begin{cases} \text{odd} & \text{if } m_P^{(i)} \equiv 1, 2 \pmod 4, \\ \text{even} & \text{if } m_P^{(i)} \equiv 0, 3 \pmod 4. \end{cases}$$

Thus, if one $m_P^{(i)}$ is $1, 2 \pmod 4$ and the other is $0, 3 \pmod 4$, then $m_P^{(1)}$ cannot be exchanged with $m_P^{(2)}$ independently of other changes. This is exactly reflected in the (expected) size of the L-packet as follows.

We put

$$\mathrm{Q}_0(\pi) \;=\; \left\{ P \in \mathrm{Q}(\pi) \,\middle|\, \left\lceil m_P^{(1)}/2 \right\rceil \not\equiv \left\lceil m_P^{(2)}/2 \right\rceil \pmod 2 \right\}.$$

We saw in Remark 9.2 that, in this case, we have $q(\pi) = e(\pi)$. Moreover, the formula for the reducibility points shows that $P \in \mathrm{Q}_0(\pi)$ if and only if one of $s_\pi(\rho_P), s_\pi(\rho'_P)$ is an odd integer; thus $\mathrm{Q}_0(\pi)$ is empty if and only if $e_0(\pi) = 0$.

Now suppose we are given a map $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$ *such that* $\#\{P \in \mathrm{Q}_0(\pi) : \boldsymbol{\varepsilon}(P) = 2\}$ *is even*; for irreducible $k_{\mathrm{F}}/k_{\mathsf{o}}$-self-dual monic $P \notin \mathrm{Q}(\pi)$, we set $\boldsymbol{\varepsilon}(P) = 1$. Then we can find, for $i = 1, 2$, a semisimple element $s_{\boldsymbol{\varepsilon}}^{(i)}$ in a suitable unitary group $\mathrm{U}(2N'_i + N_i^{\mathsf{an}}, k_{\mathrm{F}})$ with characteristic polynomial

$$\prod_P P(X)^{a_P^{(i)}(\boldsymbol{\varepsilon})},$$

where $a_P^{(i)}(\boldsymbol{\varepsilon}) = \frac{1}{2} m_P^{(i)}(\boldsymbol{\varepsilon})(m_P^{(i)}(\boldsymbol{\varepsilon}) + 1)$ and

$$m_P^{(i)}(\boldsymbol{\varepsilon}) \;=\; m_P^{(i \cdot \boldsymbol{\varepsilon}(P))},$$

with the index understood modulo 3. Correspondingly, we have a unique irreducible cuspidal representation $\tau_{\boldsymbol{\varepsilon}} = \tau_{\boldsymbol{\varepsilon}}^{(1)} \otimes \tau_{\boldsymbol{\varepsilon}}^{(2)}$ of $\mathcal{G}_{N'_1, N'_2}$ and, by inflation and compact induction, a unique irreducible cuspidal representation of G.

In this way, we construct $2^{q(\pi) - e_0(\pi)}$ inequivalent irreducible cuspidal representations of G with the same inertial reducibility set as π, which is exactly the number of cuspidal representations in the L-packet of π.

9.3. **Special orthogonal and ramified unitary groups.** A similar analysis can be made in the cases of special orthogonal and unitary groups G. The constraints for the maps $\boldsymbol{\varepsilon}$ are like those in the unramified case, since the anisotropic dimensions of the groups $\mathcal{G}^{(i)}_{N_i}$ are determined by the group G, as is the sum of the dimensions of the spaces on which $\mathcal{G}_{N_i}^{(i)}$ act. Since the details are rather similar to the cases above, we only sketch them.

We are given an irreducible cuspidal representation π of G. For P a self-dual monic polynomial in $k_{\mathrm{F}}[X]$, as in previous cases, we get integers $m_P^{(i)}$, for $i = 1, 2$. We modify slightly the definition of $\mathrm{Q}(\pi)$, replacing it with

$$\begin{cases} \mathrm{Q}(\pi)\backslash\{X-1\} & \text{if G is even-dimensional ramified unitary;} \\ \mathrm{Q}(\pi)\backslash\{X+1\} & \text{if G is odd-dimensional ramified unitary;} \\ \mathrm{Q}(\pi)\backslash\{X-1, X+1\} & \text{if G is odd-dimensional orthogonal.} \end{cases}$$

We also put

$$\mathrm{Q}'(\pi) = \begin{cases} \mathrm{Q}(\pi) & \text{if G is even-dimensional orthogonal;} \\ \mathrm{Q}(\pi)\backslash\{X-1, X+1\} & \text{otherwise.} \end{cases}$$

For P a self-dual monic polynomial in $k_{\mathrm{F}}[X]$, we set

$$f_P = \begin{cases} 1 & \text{if } \deg(P) = 1, \\ 2 & \text{otherwise,} \end{cases}$$

and then define

$$\mathrm{Q}_0(\pi) \;=\; \left\{ P \in \mathrm{Q}'(\pi) \,\middle|\, \left\lceil m_P^{(1)}/f_P \right\rceil \not\equiv \left\lceil m_P^{(2)}/f_P \right\rceil \pmod 2 \right\}.$$

Then we again constrain our map $\boldsymbol{\varepsilon} : \mathrm{Q}(\pi) \to \{1, 2\}$ such that $\#\{P \in \mathrm{Q}_0(\pi) : \boldsymbol{\varepsilon}(P) = 2\}$ is even. For each such $\boldsymbol{\varepsilon}$ we can construct a finite set of irreducible cuspidal representations of G. The total number of cuspidal representations obtained in this way is one, two or four times the expected number of cuspidal representations in the packet, *or half this number*; the latter can occur only in the case of even orthogonal groups.

We illustrate this, in particular the last case, with examples, using the same notation for the quadratic characters of $\mathrm{GL}_1(\mathrm{F})$ as in Examples 9.3 and 9.4.

Example 9.6. Let $\mathrm{G} = \mathrm{SO}(\mathrm{V})$ be the (split) special orthogonal group of an 8-dimensional orthogonal space V with Witt index 4. Denote by $\mathrm{J}_{4,0}, \mathrm{J}_{0,4}$ the maximal compact subgroups whose reductive quotients are $\mathrm{SO}_8^+(k_{\mathrm{F}})$. Denote by τ the unipotent irreducible cuspidal representation of $\mathrm{SO}_8^+(k_{\mathrm{F}})$, which we may inflate to either $\mathrm{J}_{4,0}$ or $\mathrm{J}_{0,4}$, and thus obtain irreducible cuspidal representations $\pi = \text{c-Ind}_{\mathrm{J}_{4,0}}^{\mathrm{G}} \tau$ and $\pi' = \text{c-Ind}_{\mathrm{J}_{0,4}}^{\mathrm{G}} \tau$. We have

$$\mathrm{IRed}(\pi) = \mathrm{IRed}(\pi') = \{\{([\mathbf{1}], 2), ([\mathbf{1}], 2)\}\}$$

and the corresponding Galois parameter has the form

$$\varphi_\pi = \varphi_{\pi'} = \mathbf{1} \otimes (\mathrm{st}_3 \oplus \mathrm{st}_1) \oplus \omega_0 \otimes (\mathrm{st}_3 \oplus \mathrm{st}_1).$$

According to the discussion at the beginning of the section, the packet should contain four cuspidal representations, but π, π' are the only two cuspidal representations with this inertial reducibility set. This disparity comes from the difference between the group G and the full orthogonal group $\mathrm{G}^+ = \mathrm{O}(\mathrm{V})$. By Proposition 7.10(ii), the representation τ extends to a representation of $\mathrm{O}_8^+(k_{\mathrm{F}})$, in two ways, and inducing the inflation of these two representations from $\mathrm{J}_{4,0}^+$ and $\mathrm{J}_{0,4}^+$ to G^+, we obtain four inequivalent irreducible cuspidal representations, two restricting to π and the other two to π'.

This example illustrates that, for even orthogonal groups, the expected number of cuspidal representations in a packet should be interpreted for the *full* orthogonal group, rather than for the special orthogonal group.

Example 9.7. Let $\mathrm{G} = \mathrm{SO(V)}$ be the special orthogonal group of a 5-dimensional orthogonal space V with Witt index 2, and denote by $\mathrm{J}_{2,0}$ the maximal compact subgroup whose reductive quotient $\mathcal{G}_{2,0}$ has connected component $\mathcal{G}_{2,0}^{\circ} \simeq \mathrm{SO}_4^+(k_\mathrm{F}) \times \mathrm{SO}_1(k_\mathrm{F})$.

In the dual of the finite group $\mathrm{SO}_4^+(k_\mathrm{F})$ there is an element s with characteristic polynomial $(X-1)^2(X+1)^2$ such that the Lusztig series $\mathscr{E}(\mathrm{SO}_4^+(k_\mathrm{F}), s)$ contains a cuspidal representation τ (in fact, two such representations). The inflation of τ has two extensions to $\mathcal{G}_{2,0}$ and we denote by λ_π the inflation to $\mathrm{J}_{2,0}$ of one such extension. Then $\pi = \text{c-Ind}_{\mathrm{J}_{2,0}}^{\mathrm{G}} \lambda_\pi$ is an irreducible cuspidal representation of G, for which

$$\mathrm{IRed}(\pi) = \{\!\{([\mathbf{1}], 2), ([\mathbf{1}], 1), ([\omega_1], 2), ([\omega_1], 1)\}\!\},$$

and the corresponding Galois parameter φ_π has the form

$$\varphi_\pi = \omega \otimes \mathrm{st}_2 \oplus \omega'\omega_1 \otimes \mathrm{st}_2,$$

for some unramified characters ω, ω' of order at most 2. For each choice of ω, ω', the corresponding packet should contain a unique cuspidal representation, while we have constructed four such representations. Thus we have the irreducible cuspidal representations in a union of four L-packets.

Example 9.8. Let $\mathrm{G} = \mathrm{SO(V)}$ be the special orthogonal group of a 20-dimensional orthogonal space V with Witt index 8 and anisotropic part of dimension 4. Denote by $\mathrm{J}_{4,4}$ the maximal compact subgroup whose reductive quotient $\mathcal{G}_{4,4}$ has connected component $\mathcal{G}_{4,4}^{\circ} \simeq \mathrm{SO}_{10}^-(k_\mathrm{F}) \times \mathrm{SO}_{10}^-(k_\mathrm{F})$.

In the dual of the finite group $\mathrm{SO}_{10}^-(k_\mathrm{F})$ there are elements s_1, s_2 with characteristic polynomials $(X-1)^8(X+1)^2$ and $(X-1)^2(X+1)^8$ respectively, and such that the corresponding Lusztig series $\mathscr{E}(\mathrm{SO}_{10}^-(k_\mathrm{F}), s_i)$ contains a cuspidal representation τ_i (in fact, two such representations). The representation $\tau_1 \otimes \tau_2$ has two extensions to $\mathcal{G}_{4,4}$ and we denote by λ_π the inflation to $\mathrm{J}_{4,4}$ of one such extension. Then $\pi = \text{c-Ind}_{\mathrm{J}_{4,4}}^{\mathrm{G}} \lambda_\pi$ is an irreducible cuspidal representation of G, for which

$$\mathrm{IRed}(\pi) = \{\!\{([\mathbf{1}], 3), ([\mathbf{1}], 1), ([\omega_1], 3), ([\omega_1], 1)\}\!\}.$$

The corresponding Galois parameter φ_π has the form

$$\varphi_\pi = \omega \otimes (\mathrm{st}_5 \oplus \mathrm{st}_3 \oplus \mathrm{st}_1) \oplus \omega\omega_0 \oplus \omega'\omega_1 \otimes (\mathrm{st}_5 \oplus \mathrm{st}_3 \oplus \mathrm{st}_1) \oplus \omega'\omega_2,$$

for some unramified characters ω, ω' of order at most 2. For each choice of ω, ω', the corresponding packet should contain 8 irreducible cuspidal representations.

From the two choices for each of τ_1, τ_2 above, and the two choices of extension to $\mathcal{G}_{4,4}$, we get 8 representations. However, we also get 8 more by exchanging the roles of τ_1, τ_2, and these also have the same inertial reducibility (multi)set. However, each of these irreducible cuspidal representations has two extensions to the full orthogonal group G^+. Thus we in fact have the irreducible cuspidal representations in the union of four L-packets for the full orthogonal group G^+.

In this case, there are also 16 other representations of the split special orthogonal group $\mathrm{H} = \mathrm{SO(V')}$, where V′ is a 20-dimensional orthogonal space with Witt index 10, with the same inertial reducibility set, obtained as follows. We denote by $\mathrm{J}_{8,2}$ a maximal compact subgroup of H whose reductive quotient has connected component $\mathrm{SO}_{16}^+(k_\mathrm{F}) \times \mathrm{SO}_4^+(k_\mathrm{F})$. In the duals of the isotropic finite groups $\mathrm{SO}_{16}^+(k_\mathrm{F})$ and $\mathrm{SO}_4^+(k_\mathrm{F})$ there are elements s_1, s_2 respectively, with characteristic polynomials $(X-1)^8(X+1)^8$ and $(X-1)^2(X+1)^2$ respectively, such that the corresponding Lusztig series contain cuspidal representations τ_1, τ_2 respectively (two such representations in each series). The representation $\tau_1 \otimes \tau_2$ has two extensions to the reductive quotient of $\mathrm{J}_{8,2}$ and, inflating and then inducing to G, we obtain an irreducible cuspidal representation. Since we can also inflate to $\mathrm{J}_{2,8}$, we obtain 16 inequivalent representations in this way. Again, each of these representations extends in two ways to the full orthogonal group H^+.

Example 9.9. Let $\mathrm{F}/\mathrm{F}_\mathsf{o}$ be a ramified quadratic extension and let $\mathrm{G} = \mathrm{U}(\mathrm{V})$ be the unitary group of a 14-dimensional hermitian space V with Witt index 6 and anisotropic part of dimension 2. Denote by $\mathrm{J}_{0,6}$ the maximal compact subgroup whose reductive quotient $\mathcal{G}_{0,6}$ has connected component $\mathcal{G}_{0,6}^\circ \simeq \mathrm{SO}_2^-(k_\mathrm{F}) \times \mathrm{Sp}_{12}(k_\mathrm{F})$.

We fix an irreducible self-dual monic polynomial $P \in k_\mathrm{F}[X]$ of degree two. Then there are semisimple elements s_1, s_2 in the dual groups of $\mathrm{SO}_2^-(k_\mathrm{F})$, $\mathrm{Sp}_{12}(k_\mathrm{F})$ respectively, with characteristic polynomials $P(X), P(X)^6$ respectively, such that the corresponding Lusztig series contain unique irreducible cuspidal representations τ_1, τ_2 respectively. Then there is a unique irreducible representation λ_π of $\mathrm{J}_{0,6}$ inflated from a representation of $\mathcal{G}_{0,6}$ containing $\tau_1 \otimes \tau_2$ (since τ_1 does not extend to $\mathrm{O}_2^-(k_\mathrm{F})$), and $\pi = \text{c-Ind}_{\mathrm{J}_{0,6}}^{\mathrm{G}} \lambda_\pi$ is irreducible and cuspidal. We have

$$\mathrm{IRed}(\pi) = \{\{([\rho_P], 5/2), ([\rho_P], 1)\}\},$$

where ρ_P, ρ'_P are the self-dual irreducible cuspidal representations of $\mathrm{GL}_2(\mathrm{F})$ corresponding to P, and the corresponding Galois parameter has the form

$$\varphi_\pi = \varphi_P \otimes (\mathrm{st}_4 \oplus \mathrm{st}_2) \oplus \varphi'_P,$$

where φ_P, φ'_P are the Galois parameters corresponding to ρ_P, ρ'_P respectively. The corresponding packet should contain a unique irreducible cuspidal representation, which is π.

As in Example 9.8, we also find an irreducible cuspidal representation of the 14-dimensional quasi-split ramified unitary group with the same inertial irreducibility (multi)set, by exchanging the characteristic polynomials of s_1, s_2.

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