A Quantum Group Approach to some Exotic States in Quantum Optics

Thesis

31-10-4 UNRESTRICTED

A Quantum Group Approach to some Exotic States in Quantum Optics

Roger McDermott, B.Sc., M.Sc., A.R.C.S.

A thesis submitted for the degree of Doctor of Philosophy
in the Faculty of Mathematics and Computing
of The Open University.

31st December 1994

Date of submission: 16 December 1994
Date of award: 14 February 1995

Abstract

This subject of this thesis is the physical application of deformations of Lie algebras and their use in generalising some exotic quantum optical states.

We begin by examining the theory of quantum groups and the q–boson algebras used in their representation theory. Following a review of the properties of conventional coherent states, we describe the extension of the theory to various deformed Heisenberg–Weyl algebras, as well as the q–deformations of $su(2)$ and $su(1,1)$. Using the Deformed Oscillator Algebra of Bonatsos and Daskaloyannis, we construct generalised deformed coherent states and investigate some of their quantum optical properties. We then demonstrate a resolution of unity for such states and suggest a way of investigating the geometric effects of the deformation.

The formalism devised by Rembielinski et al is used to consider coherent states of the q–boson algebra over the quantum complex plane. We propose a new unitary operator which is a q–analogue of the displacement operator of conventional coherent state theory. This is used to construct q–displaced vacuum states which are eigenstates of the annihilation operator. Some quantum mechanical properties of these states are investigated and it is shown that they formally satisfy a Heisenberg–type minimum uncertainty relation.

After briefly reviewing the theory of conventional squeezed states, we examine the various q–generalisations. We propose a q–analogue of the squeezed vacuum state and use this in conjunction with the unitary q–displacement operator to construct a general q–squeezed state, parameterised by noncommuting variables. It is shown that, like their conventional counterparts, such states satisfy the Robertson–Schrodinger Uncertainty Relation.

We conclude with a brief discussion about the appearance of noncommuting variables in the states that have been considered.

For my wife, Rachel,
and my parents, Roger and Sheila.

A.M.D.G.

Acknowlegements

It gives me great pleasure to be able to thank the many people whose contributions have enabled me to complete this thesis.

Firstly, I would like to thank my supervisor, Professor Allan Solomon, for introducing me to the subject of quantum groups and for his constant encouragement and support during the period of my research. I am extremely grateful for the kindness and generosity that he has shown, as well as the way in which he has been able to bring out positive aspects of my research. I wish to also express my gratitude for his critical reading of this text and the many improvements that he has suggested.

I would also like to thank Professors Joe Birman, Mario Rasetti, Jacob Katriel and Costas Daskaloyannis for their help and advice on the subjects of quantum groups, condensed matter theory, quantum optics, and physics in general. I am grateful for the help of Professor Ruggero Santilli on Lie admissible algebras. Thanks are also due to the students and lecturers of the 1994 *Quantum Groups and their Applications in Physics* course of the International School of Physics "Enrico Fermi" at Varenna, Italy, for numerous discussions and clarifications on many aspects of mathematical physics.

I would like to express my appreciation to the Open University for the postgraduate studentship which enabled me to undertake this work. I am grateful to the academic staff of the Faculty of Mathematics and Computing and to the members of the Deanery who have helped make my studentship so enjoyable. I would also like to thank the Higher Degrees Office for helping me to survive the administrative difficulties of being a research student.

I am especially grateful to my friends, David Wooding and Cristyn Williams, whose help and advice on enduring the trials and tribulations of postgraduate life have been invaluable. I would also thank David Wooding for proof–

reading the text.

I would especially like to thank my parents and all the members of my family for their constant support throughout the past three years.

Finally, no thanks can be enough for my wife, Rachel, whose love has always been the primary inspiration for my work.

Contents

Chapter 1

Introduction

The investigation of deformations of physical theories has a long and profitable tradition in mathematical physics: indeed, it has been claimed that the two paradigmatic changes in physics over the last century are precisely concerned with the deformation of the older classical theories [Kib:92]. If one analyses the language used by physicists when they talk about relating the theories of quantum mechanics or relativity theory to classical physics, the concept of a special limit of some important parameter often occurs. Newtonian mechanics, invariant under Galilean relativistic transformations can, in some sense, be thought of as the $\kappa \rightarrow 0$ undeformed limit of the theory of special relativity, the deformation parameter being $\kappa = c^{-1}$. At the microscopic level, classical mechanics is often said to be the $\hbar \rightarrow 0$ limit of quantum mechanics which indicates that the magnitude of the quantum of action, $\hbar$, is thought of as a deformation parameter. The idea of deformation provides an elegant way in which older models can be generalised and so be subsumed into newer theories.

The subject of this thesis is the physical application of deformations of some well–known Lie algebras and their use in generalising some exotic quantum mechanical states found in quantum optics. The deformations discussed can trace their conceptual lineage back to Heisenberg near the foundation of quantum mechanics but the main source of inspiration of this research has been the study of quantum groups and the algebras associated with their representations, especially the q–boson algebras. To set the scene for this thesis, we therefore give a brief historical survey of some of the ideas which have contributed to its development.

1.1 Deformations of the Quantum Oscillator Algebra

As has been previously mentioned, deformations of physical systems, especially quantum mechanical systems, have been studied for quite some time. One of the first such studies was the quantum mechanical oscillator whose algebraic properties are described by the Heisenberg–Weyl Lie algebra. The simple harmonic oscillator was one of the first systems to be quantised using algebraic quantum mechanical techniques and it still plays a fundamental role in our understanding of many aspects of the theory. It provides the key example for the canonical quantisation procedure and, under the guise of the second quantised creation and annihilation operators, forms the basis of the subjects of quantum field theory and quantum statistical mechanics. It is interesting then, that what are now called q–deformations of the oscillator algebra were known to Heisenberg (as mentioned in [RSW:65]). These deformations were rediscovered at various times by different authors and are now important elements in the theory of representations of quantum groups.

During the 1930's, the relation between spin and statistics, namely that integer–spin particles obey Bose statistics and odd–half–integer spin particles obey Fermi statistics (first stated by Pauli [Pau:36]) was proved from the basic requirement of local commutivity of observables. This meant that the theory of particles whose creation and annihilation operators obeyed deformed commutation relations was linked to the search for particles with exotic properties; in particular those obeying some sort of *intermediate statistics*. The first attempt to go beyond Bose and Fermi statistics appears to have been that proposed by Gentile [Gen:40]. This scheme, which interpolated between fermions and bosons, proved unsatisfactory due to the fact that the maximum number of particles allowed in one state was not invariant under changes of basis. A more successful generalisation was the *parastatistics* of H.S. Green [Gree:53]. He used trilinear relations between the creation and annihilation operators resulting in a theory which yielded an infinite class of statistics, each parameterised by an integer called the order. Later analysis of parastatistics by Doplicher et al [DHR:69] using algebraic field theoretic techniques confirmed that they give consistent field theories.

In the late 1960's, another deformation of the boson commutation relations was investigated by Santilli [San:67]. Essentially, he proposed to replace the commutator bracket found in the conventional theory by a weighted sum of commutator and anticommutator terms. This leads to a theory in which there is a change in the multiplication between elements of the Lie algebra with a coresponding change in the unit and results in a generalisation of the theory of Lie algebras and the formulation of concepts such as Lie Admissibility and Lie Isotopy (see [San:94] and references therein). These provided the theoretical framework for the extensive investigation of what we shall call *Arik–Coon*–type q–deformed bosons carried out by Jannussis et al throughout the 1980's.

The Dual Resonance Model in the string theory of the early 1970s was the motivation behind the q–boson algebra introduced by Arik and Coon [ArC:76]. Using techniques drawn from classical q–analysis, they were able to construct a set of q–coherent states. Deformations of the commutation relations were also studied by Kuryshkin [Kur:80, Kur:88]. As has been mentioned above, Santilli's theory of Lie–admissible algebras provided the background for much of the work done by Jannussis et al in the field of dissipative quantum systems [JPS:80, JBS:81, JBSPPS:82, SJBP:83, JBPKPI:83, Jan:84, Jan:85] during the 1980's. This group produced many of the familiar technical results of what might be called *q–boson* theory including the bosonisation schemes, coherent states and even a type of q–

analogue unitary displacement operator.

One type of deformed oscillator algebra, developed in the late 1980's which also gives a consistent field theory is the so-called *quon* algebra developed by Greenberg [Gre:90]. This has commutation relation

$$a_k a_l^\dagger - q a_l^\dagger a_k = \delta_k^l \tag{1.1}$$

and so can be thought of as a multimode generalisation of the q–oscillator of Arik and Coon. One of the strange features of this model is that no commutation relations can be specified between two creation operators or two annihilation operators, except for the $q = 1$ case. Greenberg found that many of the properties of the quon algebra were qualitatively illustrated in the simplest $q = 0$ case. This algebra had been known for some time from the work of Cuntz [Cun:77] in the context of operator algebras. Investigation of the implications of a field theory based on quon–statistics are continuing with generalisations of (1.1) appearing in the literature [Fiv:90, Gre:91, Zag:92, GT:93].

While the examples given above illustrate the argument that the search for deformations of quantum mechanical algebras has a long history, it is the recent interest in quantum groups that has been the biggest source of inspiration.

1.1.1 Quantum Groups and the q–Boson Algebra

Quantum groups were first discovered through the study of integrable systems [Fad:84] especially those arising from the *Quantum Inverse Scattering Method* developed by the (then) Leningrad School of Faddeev et al in the late 1970's and early 1980's (see [Fad:87, FT:87] and references therein). The Q.I.S.M. is a technique for obtaining exact solutions for integrable quantum field theories in 1+1 dimensions and classical models of statistical mechanics in 2 dimensions. Such solutions or *exactly solvable models* have a long history going back to the work of Bethe [Bet:31] who devised the ansatz for diagonalising the Hamiltonian of the one–dimensional spin chain that nowadays bears his name. Later developments include such achievements as Onsager's solution of the Ising model in classical statistical mechanics [Ons:44], the work of Yang [Yan:67] on factorisable S–matrices, Baxter [Bax:82] on the lattice models and modern field theoretical work by Zamalodchikov and Zamalodchikov [ZZ:78, ZZ:79]. This latter work led eventually to the discovery that the ultraviolet limit of certain exactly solvable models are conformal field theories and the realisation that the braid group and exchange algebras played an important role in 2–dimensional quantum field theories [FRS:89, Ger:90]. Given the importance of such structures to string theory, it is not surprising that integrable systems in general and the Quantum Inverse Scattering Method in particular, became a important area of research.

Although it was known quite early on that the structure of the objects underlying the theory was some deformation of the conventional Lie algebra found in the classical case, their true mathematical description was given by the work of Drinfel'd [Dri:83, Dri:85]. From studies of the formal quantisation of Poisson–Lie groups, he was able to show that the underlying structure was that of a noncommutative, noncocommutative Hopf algebra [Dri:86]. A similar picture emerged from the work of Jimbo [Jim:85] around the same time. It was found that the noncocommutivity of the Hopf algebra was effectively controlled by an algebraic element called the universal $\mathcal{R}$–element (also somewhat confusingly called the universal $\mathcal{R}$–matrix). For the Hopf algebras of interest, this object was found to be quasitriangular, which essentially meant that it could be used to provide a realisation of the generators of the Braid group [Art:47] and obeyed the Yang–Baxter Equation [Yan:67, Bax:72, Jim:89]. This equation had previously arisen in many diverse areas of mathematics and physics such as statistical mechanics [Bax:82], knot theory [Tur:88, Jon:89, Kau:91] and field theory [Wit:90, AlC:92]. Moreover, a matrix representation of $\mathcal{R}$ called the (numerical) R–matrix had appeared in the Q.I.S.M. as a solution to the same equation. Some time after this, Woronowicz [Wor:87a] found realisations of these objects in the context of the noncommutative geometry of Connes [Con:86] using matrix pseudogroups with coefficients in various C^*–algebras. From the physicist's point of view, however, probably the most intuitive picture of quantum groups (at least of the matrix group type) is that provided by Manin [Man:88]. He considered quantum matrix groups to be invariance "groups" for noncommutative structures called quantum planes. This led to the construction, initially by Woronowicz [Wor:87b] and later by by Wess and Zumino [WZ:90, Zum:92], of noncommutative differential calculi on quantum groups, which once more provided explicit examples of the kind of noncommutative geometric structures proposed by Connes. Since that time, there has been an explosion of interest in the use of quantum groups to provide a noncommutative model of spacetime structures. One can mention, for example, Majid's programme of using quantum group techniques to work in spaces which have non–trivial braiding between tensor products (see [Maj:93] and references therein), the quantum group Cartan calculus of Schupp [Sch:93], or the work of Kulish et al [DKR:94] or Castellanni and Aschieri [AsC:93] in developing a q–analogue of Minkowski space.

One result of the interest generated in quantum groups was the systematic investigation of their representation theory. For generic values of the deformation parameter, the unitary irreducible representations of the q–analogues of the compact simple Lie algebras are in one–to–one correspondence with their undeformed counterparts [Ros:87, Ros:89, Lus:88]. This means that much of the machinery for looking at the representation theory of Lie algebras extends quite naturally to the quantum group case. Since boson realisations of Lie algebras through the Jordan–Schwinger map or its noncompact variant have found so many applications in physics, it was only natural to look to see if there was some kind of deformed oscillator structure that could be used in the q–case. The result of this search was the q–boson, first described by Macfarlane [Mac:89] and Biedenharn [Bie:89], which was used to realise the generators of $su_q(2)$. These techniques have since been applied to many different

q–algebras [Hay:90, Fu:91, SmK:92, BCN:93, Que:93]. In addition, both the quantum optical [CEK:90, SK:90, KS:91a, CJ:92b, WK:93] and statistical mechanical [Mar:91, GeS:91, NU:92, BB:92, HL:93, SDI:93, CGM:93] properties of the q–bosons themselves have been considered.

The thesis presented here uses the algebraic properties of such q–analogue bosons to investigate their quantum optical properties. In the next section, we give a short overview of the work that it contains.

1.2 Overview of the Thesis

The thesis starts with an introduction to quantum groups. This not only illustrates some of the concepts lying behind the work but also serves to clarify notation. The concept of a quasitriangular Hopf algebra is built up in stages and the examples are given from the different approaches of Drinfel'd, Faddeev et al and Manin.

The third chapter is concerned with the probably the most important algebra found in quantum mechanics, the Heisenberg–Weyl Lie algebra. The deformations of this algebra will provide the central theme running throughout this work. We discuss the different aspects of the algebra which appear in physical theories. In the undeformed case, one is essentially dealing with one algebra (with more or less added structure) playing various roles. This is not the case, however, upon deformation. It is seen that the form of the deformation dictates the structure of the new algebra and this, in turn, constrains the role that it plays in the generalised theory. The q–analogues of the Boson algebra are discussed in this context.

Chapter 4 considers the application of the algebraic and group–theoretical techniques of quantum mechanics to the problem of coherent states. The concept of a coherent state is one of the most important ideas to have come out of the algebraic approach to quantum mechanics. Their use is widespread in almost all aspects of mathematical physics. In this chapter, an introduction is given to the theory of coherent states starting from their role as states of the electromagnetic field which minimise the uncertainty product. The properties of the Glauber coherent states are reviewed and Perelomov's abstraction of the group theoretical content of this analysis is discussed. The coherent states for the Lie groups $SU(2)$ and $SU(1,1)$ are briefly described.

Chapter 5 gives a description of the extensions of the theory of coherent states to the q–deformations of the boson algebra and the quantum deformations of the semisimple Lie algebras $su(2)$ and $su(1,1)$. Given the different geometrical setting of conventional Lie algebras and quantum groups, it is to be expected that the elegant group–theoretical formulation of the theory of coherent states decribed in the previous chapter should break down. The result is a theory based on the somewhat *ad hoc* method of q–exponentiating the raising–operator associated with the relevant algebra. The different ways of deforming the boson algebra manifest themselves in different sets of coherent states. Some applications of the states associated with $su_q(2)$ and $su_q(1,1)$ are also given.

The fact that there are several types of q–deformed boson coherent states each produced by different algebras suggests that there is a more fundamental structure underlying this system which could be of use in the construction of better or more general physical models. Several authors have attempted to analyse the basic properties of the q–deformed boson algebras as the first stage in developing such physical theories. There have been many proposals for general deformation schemes which subsume the different q–oscillators, as well as conventional (para)bosons and (para)fermions, as special cases. Probably the most well–known is the deformation scheme of Daskaloyannis et al which uses a general *structure function* to define the oscillator algebra. In chapter 6, we discuss various deformation schemes before considering the Daskaloyannis–type deformations. We then investigate the quantum optical properties of the coherent states associated with the general algebra and show that, given certain conditions on the structure function, the boson field described by this type of state exhibits unconventional quantum noise properties such as simultaneous squeezing in both field quadratures. The relationship between the deformed oscillators and the conventional boson operators is considered and an overcompleteness relation for the deformed coherent states is proposed. Some suggestions for further work are also made. The author's work on this subjects has been published in the following articles [MS:93a, MS:93b, SM:93, MS:94a, MS:94c]

The seventh chapter begins a new part of the thesis devoted to the use of noncommuting variables in the construction of quantum optical operators and states. The work of Rembielinski et al on the differential calculus of the complex quantum plane is discussed in the context of the use of annihilation operator eigenstates parameterised by noncommuting variables. A modification of the theory is proposed, leading to the construction of a unitary operator which can be thought of as the q–analogue of the Heisenberg–Weyl displacement operator. This is used to construct a new set of q–displaced vacuum states which are also eigenstates of the annihilation operator. The quantum optical properties of these states are formally investigated. The results of this analysis have been published in [MS:94b]. This is followed by a brief discussion of an alternative proposal for the q–analogue of the displacement operator.

Following the investigation of coherent states and their deformed analogues, we then consider another set of quantum optical states — squeezed states. Such states, characterised by having a minimum uncertainty product which is asymmetric in the field quadratures, have no classical analogue but have been produced in the laboratory. The noise reduction in one field component relative to the vacuum state (at a corresponding noise amplification in the other component) means that they have found use in experiments requiring extreme sensitivity, such as the detection of gravity waves. After a description of conventional squeezed states in chapter 8, their q–analogues are discussed in chapter 9. As with the definition of q–deformed coherent states, two different approaches can be identified. The first parameterises the states by c–numbers. The second builds on the work done in chapter 7 using q–numbers. A squeezed state, constructed using noncommutative variables, is described and is found to obey the Robertson–Schrodinger minimum uncertainty relation. We conclude this thesis with a brief discussion concerning the appearance of noncommuting variables in the states that were proposed.

Note on the Bibliography

The list of references cited in this thesis may seem somewhat excessive running as it does to over twenty pages. However, it should be noted that the study of quantum groups is a new and very fast–moving field. In fact, the articles mentioned in the bibliography represent only a fraction of the published body of work on the subject. For a rather more complete exposition of the mathematical aspects which lie at the heart of the subject such as Poisson–Lie groups, quasitriangular Hopf algebras and noncommuting geometry, together with an accompanying list of references, see Chari and Pressley [CP:94].

Chapter 2

Quantum Groups and Associated Deformations of Algebras

The major impetus for the recent development of deformed oscillator techniques has been the advances made in the study of quantum groups. We will therefore give a description of these mathematical objects. This chapter gives a brief but standard introduction to Quasitriangular Hopf Algebras following the lines of, say, Majid [Maj:90] or Tjin [Tji:92].

2.1 Classical Hopf Algebras

A Quantum Group is a Quasitriangular Hopf algebra. Since the definition of this object is quite involved, it is constructed in several stages.

An *associative (unital) algebra* $(\mathcal{A}, m, \eta; k)$ over a field k (often assumed to be $\mathbb{C}$) is a linear space $\mathcal{A}$ with a bilinear map m such that for $a, b \in \mathcal{A}$

$$m : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \tag{2.1}$$

$$(a, b) \longmapsto ab \tag{2.2}$$

where m satisfies the associativity condition

$$m \circ (m \otimes id) = m \circ (id \otimes m) \tag{2.3}$$

$$(ab)c = a(bc) \tag{2.4}$$

and id is the identity map.

That $\mathcal{A}$ is a unital algebra is expressed by the fact that there is a special non–zero element $1_{\mathcal{A}} \in \mathcal{A}$ such that

$$1_{\mathcal{A}}\, a = a = a\, 1_{\mathcal{A}} \tag{2.5}$$

which, in turn, can be rephrased as the existence of a map η from k to $\mathcal{A}$ such that $\eta(1) = 1_{\mathcal{A}}$ and

$$m \circ (\eta \otimes id) = id = m \circ (id \otimes \eta) \tag{2.6}$$

where we make the canonical identification

$$k \otimes \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \otimes k \tag{2.7}$$

Given an algebra $(\mathcal{A},\, m,\, \eta; k)$, a linear space V and a map, ρ, from $\mathcal{A}$ to the space of linear operators on V, the pair (V, ρ) is called a representation of $\mathcal{A}$ in V if ρ is linear and

$$\rho(xy) = \rho(x)\rho(y) \tag{2.8}$$

for all $x, y \in \mathcal{A}$.

It is often necessary to compose two representations (e.g in the physical problems such as the addition of angular momentum [BL:81]), and this means that we must consider tensor product representations of the underlying algebra. However, there is no canonical way of doing this unless additional algebraic structure is imposed.

If $\mathcal{A}$ is an algebra with identity element $1_{\mathcal{A}}$, then the tensor product algebra $\mathcal{A} \otimes \mathcal{A}$ naturally takes on the structure of an algebra with identity element $1_{\mathcal{A}} \otimes 1_{\mathcal{A}}$ if we define the product of $a \otimes b$ and $c \otimes d$ as $ac \otimes bd$.

A *coassociative coalgebra with counit* [Swe:69, Abe:80] $(\mathcal{A}, \Delta, \varepsilon; k)$ is a linear space $\mathcal{A}$ with linear mappings $\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ (comultiplication) and

$\varepsilon : \mathcal{A} \longrightarrow k$ (counit) such that the following conditions hold

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \tag{2.9}$$

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta \tag{2.10}$$

These are dual to the axioms that defined the structure of a unital algebra. Consequently equation (2.9) is known as the *coassociativity* condition and equation (2.10) expresses the existence of the counit.

If V is a linear space we can define the linear *twist* operator τ on $V \otimes V$

$$\tau : V \otimes V \longrightarrow V \otimes V \tag{2.11}$$

$$\tau(v_1 \otimes v_2) \longmapsto v_2 \otimes v_1 \tag{2.12}$$

The multiplication m in the algebra $(\mathcal{A}, m, \eta; k)$ is called *commutative* if

$$m \circ \tau = m \tag{2.13}$$

Dually, the comultiplication Δ in the coalgebra $(\mathcal{A}, \Delta, \varepsilon; k)$ is called *cocommutative* if

$$\tau \circ \Delta = \Delta \tag{2.14}$$

It is customary to use *Sweedler's notation* [Swe:69] for the coproduct. Since for $x \in \mathcal{A}$, $\Delta(x) \in \mathcal{A} \otimes \mathcal{A}$, we denote

$$\Delta(x) = \sum_i x_{(1)i} \otimes x_{(2)i} \tag{2.15}$$

$$= x_{(1)} \otimes x_{(2)} \tag{2.16}$$

where the summation is understood.

A *Bialgebra* $(\mathcal{A}, m, \eta, \Delta, \varepsilon; k)$ is a linear space $\mathcal{A}$ which has the structure of both an algebra and a coalgebra in a compatible way. The compatibility condition is that the mappings Δ and ε are algebra homomorphisms.

Given the map Δ, it is now possible to compose representations. If (V_1, ρ_1) and (V_2, ρ_2) are two representations of $\mathcal{A}$, then the tensor product representation ρ_{12} in $V_1 \otimes V_2$ is given by

$$\rho_{12} = (\rho_1 \otimes \rho_2)\Delta \tag{2.17}$$

Thus the coproduct Δ can be thought of as the minimal extra structure that an algebra must have in order to form tensor product representations. In addition, the counit implies the existence of a trivial one–dimensional representation of the algebra given by ε.

A *Hopf algebra* $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S; k)$ [Hop:41, MM:65, Swe:69, Abe:80] is a bialgebra together with an antilinear homomorphism, S (the antipode), such that

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta \tag{2.18}$$

In practice, the field is often taken to be $\mathbb{C}$ and this is omitted from the data of the definition.

Many algebraic objects used in mathematical physics can be shown to have a unified structure as examples of Hopf algebras. Two notable cases are the algebra of functions on a compact topological group and the universal enveloping algebra of a Lie algebra.

Example 1: Let $\mathcal{G}$ be a compact topological group and let $C(\mathcal{G})$ be the space of continuous functions on $\mathcal{G}$. Then $C(\mathcal{G})$ is a commutative Hopf algebra with the Hopf–structure maps (m, Δ, η, ε and S) given by

- m: $(f \cdot h)(g) = f(g)h(g)$
- Δ: $\Delta(f)(g_1, g_2) = f(g_1 g_2)$
- η: $\eta(x) = x1$ where $1(g) = 1$ for all $g \in G$
- ε: $\varepsilon(f) = f(e)$ where e is the unit element of G
- S: $S(f)(g) = f(g^{-1})$

where g, g_1 , $g_2 \in G$, $x \in k$ and f, $h \in C(\mathcal{G})$.

Example 2: Let $\mathcal{L}$ be a Lie algebra and $\mathcal{U}(\mathcal{L})$ its universal enveloping algebra, then $\mathcal{U}(\mathcal{L})$ is a cocommutative Hopf algebra with structure maps given by

- m: ordinary multiplication in $\mathcal{U}(L)$
- Δ: $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\Delta(1) = 1 \otimes 1$
- η: $\eta(\alpha) = \alpha 1_{\mathcal{U}(\mathcal{L})}$
- ε: $\varepsilon(1_{\mathcal{U}(\mathcal{L})}) = 1$ and $\varepsilon(x) = 0$.
- S: $S(x) = -x$

where $x \in \mathcal{L} \subset \mathcal{U}(\mathcal{L})$ and $1_{\mathcal{U}(\mathcal{L})}$ is the unit element of this algebra.

If $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$ is a Hopf algebra and $\mathcal{H}^*$ is its dual space then the structure maps on $\mathcal{H}$ induce a Hopf structure on $\mathcal{H}^*$ by duality. If the brackets $\langle \cdot, \cdot \rangle$ denotes the evaluation map of $\mathcal{H}^*$ on $\mathcal{H}$ (or indeed of $\mathcal{H}$ on $\mathcal{H}^*$) then $(\mathcal{H}^*, m^*, \eta^*, \Delta^*, \varepsilon^*, S^*)$ is a Hopf algebra with structure maps defined by

- $\langle m^*(f \otimes g), x \rangle = \langle f \otimes g, \Delta(x) \rangle$
- $\langle \Delta^*(f), x \otimes y \rangle = \langle f, xy \rangle$
- $\langle \eta^*(\alpha), x \rangle = \alpha \cdot \varepsilon(x)$
- $\varepsilon^*(f) = \langle f, 1_{\mathcal{H}} \rangle$
- $\langle S^*(f), x \rangle = \langle f, S(x) \rangle$

where $f, g \in \mathcal{H}^*$, $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

Since we consider the dual of the multiplication map to give the comultiplication map, this implies that we identify the spaces $(\mathcal{H} \otimes \mathcal{H})^*$ (which is the codomain of m^*) and $\mathcal{H}^* \otimes \mathcal{H}^*$ (which is the codomain of Δ). If $\mathcal{H}$ was finite dimensional, this would not be a problem. Unfortunately, for infinite–dimensional algebras, $(\mathcal{H} \otimes \mathcal{H})^*$ is bigger than $\mathcal{H}^* \otimes \mathcal{H}^*$ and so m^* does not necessarily map $\mathcal{H}^*$ to $\mathcal{H}^* \otimes \mathcal{H}^*$. To rectify this difficulty, it is necessary to enlarge $\mathcal{H}^* \otimes \mathcal{H}^*$ to include elements such as $\sum_k^\infty x_k \otimes y_k$, i.e. work with the completion of the space.

It is a well–known result of Lie theory (e.g. [Hel:78]) that if $\mathcal{G}$ is a simply–connected Lie group, $C_c^\infty(\mathcal{G})$ denotes the algebra of C^∞–functions (with compact support) on $\mathcal{G}$, g is the Lie algebra of $\mathcal{G}$ and $U(\mathrm{g})$ its associated universal enveloping algebra, then $U(\mathrm{g})$ may be identified with the subalgebra (with unit) of $(C_c^\infty(\mathcal{G}))^*$ generated by tangent vectors at the identity of $\mathcal{G}$. Consequently it can be seen that the examples of Hopf algebras given above are dual to each other not only as linear spaces but also as Hopf algebras.

2.2 Quasitriangular Hopf Algebras

In the examples given above, the commutative multiplication in one Hopf algebra induces cocommutivity in the coproduct of its dual. The question therefore arises whether there are any examples which are both noncommutative and noncocommutative. One positive answer to this question is provided by a special type of Hopf algebra which forms the basis of the study of Quantum Groups — the *Quasitriangular Hopf algebra* [Dri:86].

A *Quasitriangular Hopf algebra* $(\mathcal{H}, \mathcal{R}; m, \eta, \Delta, \varepsilon, S)$ (abbreviated to $(\mathcal{H}, \mathcal{R})$) is a Hopf algebra for which there exists a universal invertible element, $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, which intertwines between the coproduct Δ and the *opposite coproduct* $\Delta' = \tau \circ \Delta$, i.e. for all $x \in \mathcal{H}$

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \tag{2.19}$$

In addition, $\mathcal{R}$ should satisfy

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.20}$$

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{2.21}$$

where $\mathcal{R}_{ij} \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ acts as $\mathcal{R}$ in the ith and jth tensor product space and as the identity in the remaining one (e.g. $\mathcal{R}_{12} = \mathcal{R} \otimes I_{\mathcal{H}}$).

One consequence of this definition is that for all $x \otimes y \in \mathcal{H} \times \mathcal{H}$

$$\mathcal{R}_{12}(\Delta \otimes id)(x \otimes y) = (\Delta' \otimes id)(x \otimes y)\mathcal{R}_{12} \tag{2.22}$$

whereupon taking $x \otimes y = \mathcal{R}$, we obtain the *Quantum Yang–Baxter equation.*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.23}$$

As has been remarked in the Introduction, this is a fundamental equation in many areas of mathematical physics. Essentially one can say that the universal $\mathcal{R}$-element, through the quasitriangular structure of the Hopf algebra, keeps the noncocommutivity under control.

If ρ^i_j is a matrix representation of $\mathcal{H}$ in some module V, then

$$(\rho^i_k \otimes \rho^j_l)\mathcal{R} = R^{ij}_{kl} \tag{2.24}$$

is a $n^2 \times n^2$ matrix of End$(V \otimes V)$ called the (numerical) R–matrix. The Yang–Baxter equation can then be formulated as a matrix equation in $V \otimes V \otimes V$.

If we define the element $\hat{\mathcal{R}}$ by

$$\hat{\mathcal{R}} = \tau \circ \mathcal{R} \tag{2.25}$$

where τ is the twist map, then the Yang–Baxter equation (2.23) can be rewritten as

$$\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_{23}\hat{\mathcal{R}}_{12} = \hat{\mathcal{R}}_{23}\hat{\mathcal{R}}_{12}\hat{\mathcal{R}}_{23} \tag{2.26}$$

which shows that $\hat{\mathcal{R}}$ provides a representation of the generators of the *Braid Group* [Art:47].

The examples of Hopf algebra considered previously are trivially quasitriangular, the universal $\mathcal{R}$-element being simply the identity in the tensor square $\mathcal{H} \otimes \mathcal{H}$. Just as in the classical case where the most common examples fall into two mutually dual types — the group-like type and the universal enveloping algebra type, so it is in the Quantum Group or *quantised* case. The quantised universal enveloping algebra (QUEA) of a Lie algebra is a noncommutative, noncocommutative Hopf algebra, as is the quantised algebra of functions on a (Lie) group. There are several approaches to the study of quantum groups depending on the type of algebra under investigation. We will first consider the approach of Drinfel'd and Jimbo to QUEA's using the example of $\mathcal{U}_q(sl(2))$ or $sl_q(2)$, and then look at that of Faddeev and the St. Petersburg School. Finally we consider the approach of Manin. The C^*-algebra approach of Woronowicz will not be considered here.

2.2.1 The Drinfel'd–Jimbo Approach: $sl_q(2)$

Drinfel'd's approach to the study of quantum groups was motivated by the structure of Lie–Poisson bialgebras [Dri:83] and their formal quantisation. Consequently, the algebras that he investigated were of the universal enveloping algebra type.

The quantum group $sl_q(2)$ is a single parameter q-deformation of the universal enveloping algebra of the Lie algebra $sl(2)$, generated by the elements $\{H, E_+, E_-\}$ with relations

$$[H, E_\pm] = \pm E_\pm \tag{2.27}$$

$$[E_+, E_-] = \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{2.28}$$

where $q \in \mathbb{C}$ is a free parameter and the right hand side of equation (2.28) is understood in terms of a power series expansion. In the limit $q \rightarrow 1$, the deformed commutation relation (2.28) becomes the usual commutation relation for the Lie algebra $sl(2)$.

The additional Hopf–structure of $sl_q(2)$ is given by

$$\Delta(H) = H \otimes I + I \otimes H \tag{2.29}$$

$$\Delta(E_\pm) = E_\pm \otimes q^{H/2} + q^{-H/2} \otimes E_\pm \tag{2.30}$$

and

$$\varepsilon(I) = 1, \quad \varepsilon(H) = 0, \quad \varepsilon(E_\pm) = 0 \tag{2.31}$$

$$S(H) = -H, \quad S(E_\pm) = -q^{\mp\frac{1}{2}} E_\pm \tag{2.32}$$

The coproduct clearly shows the noncocommutivity of the algebra. In this case, (with q not a root of unity), the universal $\mathcal{R}$–element can be given explicitly as

$$\mathcal{R} = q^{H\otimes H/2} \sum_{n\geq 0}^{\infty} \frac{(1-q^{-2})^n}{[\![n]\!]_q!} (q^{H/2} E_+ \otimes q^{-H/2} E_-)^n q^{n(n-1)/2} \tag{2.33}$$

where $[\![n]\!]_q!$ is a function related to the basic factorial of classical q–analysis (see section 3.2.3). This $\mathcal{R}$–element does not actually lie in $sl_q(2) \otimes sl_q(2)$ but in some completion of it.

As in the classical case, the imposition of a compatible $*$–structure on the algebra allows the definition of the quantum groups $su_q(2)$ and $su_q(1,1)$. Just as higher rank Lie algebras also obey the *Serre* relations, so their q–analogues have to obey q–Serre relations.

The representation theory [Ros:87, Ros:89, Lus:88] of these algebras is very similar to that of the underlying Lie algebra for generic q (i.e. q not a root of unity). In fact, for $su_q(2)$ the unitary irreducible representations are in one–to–one correspondence with those of $su(2)$ and have the same dimension. If we take $E_+ = J_+ = (J_-)^\dagger = (E_-)^\dagger$ and $J_0^\dagger = J_0 = H$, then for every $j = 0, \frac{1}{2}, 1, \ldots$ there is an irreducible representation of $su_q(2)$ in a $(2j+1)$–dimensional Hilbert space spanned by vectors $|-j\rangle, |-j+1\rangle, \ldots, |j\rangle$. The action of the generators is

$$J_0|j,m\rangle = m|j,m\rangle \tag{2.34}$$

$$J_+|j,m\rangle = \sqrt{[\![j-m]\!]_q\,[\![j+m+1]\!]_q}|j,m+1\rangle \tag{2.35}$$

$$J_-|j,m\rangle = \sqrt{[\![j-m+1]\!]_q\,[\![j+m]\!]_q}|j,m-1\rangle \tag{2.36}$$

where the function $[\![n]\!]_q$ is given by

$$[\![n]\!]_q = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{2.37}$$

2.2.2 The F.R.T. Approach: $GL_q(2)$

The observation (2.24) that a matrix representation of the Hopf algebra gives rise to a numerical matrix version of the universal $\mathcal{R}$–element was used extensively by the St. Petersburg school of Faddeev, Reshetikhin, Takhtajan, Kulish, Sklyanin et al in their approach to quantum groups. Since the coordinate functions on a matrix group have an obvious matrix representation, equation (2.24) can be rewritten as

$$\langle \mathcal{R}, T^i_k \otimes T^j_l \rangle = R^{ij}_{kl} \tag{2.38}$$

where T^i_j is the matrix of coordinate functions whose coproduct is given by matrix multiplication

$$\Delta T = T\dot{\otimes}T \qquad \text{i.e. } \Delta(T^i_j) = T^i_k \otimes T^k_j \tag{2.39}$$

From this it can be shown that the matrix elements of T satisfy the so–called RTT–relations

$$R^{ij}_{kl} T^k_m T^l_n = T^j_s T^i_r R^{rs}_{mn} \tag{2.40}$$

or in more concise notation

$$R_{12} T_1 T_2 = T_2 T_1 R_{12} \tag{2.41}$$

where $T_1 = T \otimes I$, $T_2 = I \otimes T$ and I is the unit matrix. This is the *F.R.T.* approach (see [RTF:90] and references therein) and has been successfully applied to all the classical groups as well as their corresponding Lie algebras.

If we consider the algebra $Fun(GL(2))$ generated by the ring of coordinate functions on the group $GL(2)$, this can be quantised to give the quantum group $Fun_q(GL(2)) \equiv GL_q(2)$, which is generated by elements T^i_j, where

$$T^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.42}$$

The R–matrix for this quantum group is

$$R^{ij}_{kl} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q-q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{2.43}$$

where the rows and columns of the matrix have a multiple index running (11), (12), (21), (22). This then yields the following commutation relations for the elements of T^i_j

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc \tag{2.44}$$

$$bc = cb, \quad ad - da = (q - q^{-1})bc \tag{2.45}$$

with a bialgebra structure given by

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dot{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.46}$$

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.47}$$

$$\tag{2.48}$$

where $\dot{\otimes}$ is a combination of tensor product and matrix multiplication, i.e. $\Delta(a) = a \otimes a + b \otimes c$, etc, and $\varepsilon(a) = \varepsilon(d) = 1$, $\varepsilon(b) = \varepsilon(c) = 0$.

If we define the quadratic product $\mathcal{D} = \det_q T = ad - qbc$, then it can be shown that it is both central and group–like. If the algebra is then extended by formally adjoining to it the element $\mathcal{D}^{-1}$, the antipode matrix, $S(T)$, may be defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathcal{D}^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \in GL_{q^{-1}}(2) \tag{2.49}$$

This provides a full Hopf algebra structure. Moreover since $\mathcal{D}$ is central and group–like, it may be set to unity with the result that the quantum group becomes the q–analogue of $SL(2)$, i.e. $SL_q(2)$. Given an appropriate $*$–structure on the elements, it is possible to define other subgroups of $GL_q(2)$ such as $SU_q(2)$. Among the remarkable properties of these matrices, (more easily explained by Manin's approach to the subject) is the fact that if T and T' are mutually commuting copies of a particular matrix quantum group, then their product T'' defined by

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{2.50}$$

satisfy the same relations.

2.2.3 The Manin Approach

One approach to matrix quantum groups (or matrix pseudogroups in the terminology of Woronowicz [Wor:87a]) pioneered by Manin [Man:88] was based on the fact that they can be considered as comodule algebras for the so–called *Manin* or *Quantum Plane.* If we consider again the example of $GL_q(2)$, then this is the "automorphism" algebra of a complex algebra $A_q^{2|0}$, generated by two elements x and y (which commute with all elements of $GL_q(2)$) such that

$$xy \doteq q\, yx \tag{2.51}$$

The fact that $GL_q(2)$ is a comodule algebra (or alternatively that there is a *coaction of* $GL_q(2)$ *on the quantum plane* $A_q^{2|0}$) means that if we define x' and y' by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.52}$$

then $x'y' = q\, y'x'$.

Alternatively, there exists a left coaction, i.e. an algebra homomorphism $\Delta_L : A_q^{2|0} \longrightarrow GL_q(2) \otimes A_q^{2|0}$, such that

$$\Delta_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.53}$$

This, however, produces only half the relations for the matrix elements of T. The rest are obtained by requiring $GL_q(2)$ be a right coaction for another algebra $A_q^{0|2}$ (the q–deformed Grassmann plane), generated by two elements η and ξ obeying

$$\eta\xi + q\,\xi\eta \;=\; 0 \tag{2.54}$$

$$\eta^2 = \xi^2 \;=\; 0 \tag{2.55}$$

This right coaction is then given by

$$\Delta_R \begin{pmatrix} \eta, \xi \end{pmatrix} = \begin{pmatrix} \eta, \xi \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.56}$$

In fact the algebra $A_q^{2|0}$ is invariant under the coaction of a larger quantum group than $GL_q(2)$, namely the multiparameter quantum group $GL_{q,s}(2)$ [Sud:90, Schi:91]. This is the the quantum matrix group with fundamental representation T where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.57}$$

whose elements have commutation relations

$$\begin{array}{lll} ab = sq^{-1}ba & ac = q\,ca & bc = s^{-1}q^2cb \\ bd = q\,db & cd = sq^{-1}dc & ad = da + q^{-1}(s-1)bc \end{array} \tag{2.58}$$

Clearly, the single–parameter quantum group is the same as the multiparameter quantum group with the restriction that $s = q^2$.

We note in passing that the quantum plane is also invariant under a group action, namely that of the group of scaling transformations on the elements of $A_q^{2|0}$. This group is the diagonal subgroup of $GL(2)$. The coordinate ring of this subgroup is isomorphic to the coordinate ring of the abelian diagonal sub–quantum–group of $GL_q(2)$ (or even $GL_{q,s}(2)$).

Chapter 3

The Heisenberg–Weyl Algebra and its Quantum Group Counterparts

3.1 The Classical Heisenberg–Weyl Algebra

Of all the Lie algebras used in quantum mechanics, probably the most important is the Heisenberg–Weyl algebra. This basic structure, together with its various representations, forms a family of algebras which underpin the mathematical structure of both the first and second quantised versions of the theory. Classically (i.e. in a non deformed setting) there are several ways in which it may be introduced.

We may abstractly define the *Heisenberg-Weyl Lie algebra* $\mathcal{HW}(3)$ as the universal enveloping algebra of the Lie algebra whose generators $\{H, E_{\pm}\}$ have the commutation relations

$$[E_-, E_+] = H, \qquad [H, E_{\pm}] = 0 \tag{3.1}$$

This Lie algebra may, for example, be considered a contraction of the Lie algebra $u(2)$ (see, e.g., p460 in [Gil:74]).

In the construction above, the generator H is central. We may therefore consider representations of the algebra in which H is given by a multiple of the unit operator (or alternatively consider the quotient algebra $\mathcal{HW}(3)/B$ where B is the two-sided ideal generated by $H - I$, and I is the unit element of the algebra). This algebra will be denoted $\mathcal{H}(3)$.

One reason that the above algebra is so important is that it has a realisation in terms of the coordinate and derivative operators of single variable calculus

$$\left[\frac{d}{dx}, x\right] = I \tag{3.2}$$

This clearly has immense implications for its importance in both classical and quantum mechanics where position and momentum have coordinate/differential realisations. Indeed, if a $*$–structure is imposed upon the $\mathcal{H}(3)$ algebra such that the elements $E_\pm$ are realised by hermitian (self–adjoint) operators and the central element is realised by iI, then the commutation relation for the Lie algebra expresses the fundamental commutation relation linking the position and momentum operators in first–quantised quantum mechanics.

The algebra $\mathcal{H}(3)$, together with an appropriate Hilbert (Fock) space and $*$–structure, is also realized as the algebra of boson creation and annihilation operators of second–quantised quantum mechanics. This is the algebra generated by three elements $\{b, b^\dagger = (b)^*, I\}$ with commutation relations

$$[b, b^\dagger] = I \tag{3.3}$$

$$[b, I] = [b^\dagger, I] = 0 \tag{3.4}$$

If we consider the universal enveloping algebra of this Lie algebra, we can also form the element $N = b^\dagger b$ which clearly has relations

$$[N, b] = -b, \quad [N, b^\dagger] = b^\dagger, \quad [N, I] = 0 \tag{3.5}$$

The algebra generated by the elements $\{b, b^\dagger, I\}$ together with the element $\{N = b^\dagger b\}$ will be called the *Boson algebra* $\mathcal{H}(4)$.

The Fock space in which the operators act is built up from a lowest weight *vacuum* vector $|0\rangle$ by the action of the creation operator $b^\dagger$. This vacuum vector is annihilated by the annihilation operator. The multiparticle states are eigenvectors of the operator N, and are labelled by its eigenvalues. For this reason, N is called the *Number* operator. The orthonormal number eigenstates are

$$|n\rangle = \frac{(b^{\dagger})^n}{\sqrt{n!}}|0\rangle \tag{3.6}$$

and the action of the generators on the number states is given by

$$\begin{aligned} N|n\rangle &= n \quad |n\rangle \\ b|n\rangle &= \sqrt{n} \quad |n-1\rangle \\ b^{\dagger}|n\rangle &= \sqrt{n+1} \quad |n+1\rangle \end{aligned} \tag{3.7}$$

This algebra is covariant under the action of the group $Sp(2,\mathbb{R}) \cong SU(1,1)$,

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} b \\ b^{\dagger} \end{pmatrix} = \begin{pmatrix} b' \\ b^{\dagger\prime} \end{pmatrix} \tag{3.8}$$

where b' and $b^{\dagger\prime}$ obey the same relations as b and $b^{\dagger}$.

The boson algebra is of great use in providing an elegant approach to Lie group symmetry by means of the *Jordan–Schwinger* map [Jor:35, Schw:65]. If we consider the Lie algebra g of a compact Lie group $\mathcal{G}$, we may realise the generators of g by $n \times n$ matrices $(g_{\alpha})_{ij}$ of the fundamental irreducible representation. Given n independent (i.e. commuting) sets of boson creation and annihilation operators

$$[b_i, b_j^{\dagger}] = \delta_{ij} \qquad i,j = 1,\ldots n \tag{3.9}$$

we may define a new realisation by means of the Lie algebra homomorphism $\mathcal{J} : (g_{\alpha}) \to X_{\alpha}$, where X_{α} are operators defined by

$$X_{\alpha} = \sum_{ij}^{n} b_i^{\dagger}(g_{\alpha})_{ij} b_j \tag{3.10}$$

It is therefore possible to construct irreducible representations of Lie algebras associated with compact Lie groups in the usual Fock space of quantum field theory.

One further occurrence of the boson algebra is in the construction of modules for an irreducible representation of $SU(N)$, i.e. elements of an $SU(n)$–spinor. If we consider the simplest example of $SU(2)$, we find that if $\{b_1, b_1^\dagger\}$ and $\{b_2, b_2^\dagger\}$ are a pair of independent (i.e. commuting) sets of creation and annihilation operators, then the transformation

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \tag{3.11}$$

together with the relations obtained by hermitian conjugation, preserve the commutation relations, i.e. $\{b_1', b_1^{\dagger\prime}\}$ and $\{b_2', b_2^{\dagger\prime}\}$ are another set of independent boson operators. The annihilation operators (resp. creation operators) therefore form a basis of a fundamental (resp. conjugate) representation of the special unitary group.

3.2 The q–deformed variants of the Heisenberg Weyl Algebra

In the Lie algebraic setting of conventional quantum mechanics, the Heisenberg–Weyl/Boson algebra appears as one unified structure playing many roles. Unfortunately, this is not the case when it comes to the q–deformations of the structure.

3.2.1 The Heisenberg–Weyl Quantised Universal Enveloping Algebra, $\mathcal{H}_q$.

If one considers the deformation of the abstract Heisenberg–Weyl algebra, one can obtain a QUEA structure by contracting the two–dimensional unitary quantum group $su_q(2)$ [CGST:90, CGST:91]. For example, if we make the scaling transformation

$$\begin{pmatrix} E_- \\ E_+ \\ H \\ \omega \end{pmatrix} = \begin{pmatrix} \varepsilon^{1/2} & 0 & 0 & 0 \\ 0 & \varepsilon^{1/2} & 0 & 0 \\ 0 & 0 & 2\varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon^{-1} \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_0 \\ \log q \end{pmatrix} \tag{3.12}$$

where $\{J_0, J_+, J_-\}$ generate $su_q(2)$, then in the limit, $\varepsilon \to 0$, we obtain a Hopf algebra with relations

$$[E_-, E_+] = \frac{\sinh(\omega H/2)}{\omega/2} \tag{3.13}$$

$$[E_+, H] = [E_-, H] = 0 \tag{3.14}$$

and Hopf structure

$$\Delta(H) = 1 \otimes H + H \otimes 1 \tag{3.15}$$

$$\Delta(E_+) = e^{-(\omega/4)H} \otimes E_+ + E_+ \otimes e^{(\omega/4)H} \tag{3.16}$$

$$\Delta(E_-) = e^{-(\omega/4)H} \otimes E_- + E_- \otimes e^{(\omega/4)H} \tag{3.17}$$

In addition,

$$\varepsilon(X) = 0, \qquad S(X) = -e^{(\omega/2)H} X e^{-(\omega/2)H} = -X \tag{3.18}$$

where $X \in \{H, E_\pm\}$. It is possible to see that the contraction procedure is consistent with the Hopf structure of the system. It is also possible to introduce another primitive generator N with the relations

$$[N, E_+] = E_+; \qquad [N, E_-] = -E_- \tag{3.19}$$

$$\Delta(N) = N \otimes 1 + 1 \otimes N \tag{3.20}$$

$$\varepsilon(N) = 0, \qquad S(N) = -N \tag{3.21}$$

This has the advantage that the Hopf algebra formed is now fully quasi-triangular, i.e. a quantum group. This particular Hopf algebra was called $H(1)_q$ by Celeghini et al [CGST:90, CGST:91, GS:93, BM:93], but we will denote it as $\mathcal{H}_q$. It may be thought of as a different contraction from a central extension of $su_q(2)$, this time the J_0 generator mapping onto the operator N and the central extension mapping onto H. It is important to note that, at the Hopf-algebraic level, the generator H is primitive and so it is impossible to represent H as a multiple of the unit operator since the unit of a Hopf algebra is always group-like.

3.2.2 The Arik–Coon q–Oscillator Algebra, $\mathcal{A}_q$.

Well before the advent of quantum groups, Arik and Coon [ArC:76] produced a minimal q–deformation of the ordinary boson algebra for use in extending the theory of coherent states. They considered a one–parameter deformation $\mathcal{F}_q$, of the usual Fock space $\mathcal{F}$, (with the parameter $q \in (0,1)$), spanned by orthonormal vectors $|n\rangle$ generated from the vacuum state $|0\rangle$ by the action of a deformed creation operator $a^\dagger$. The creation operator and its hermitian conjugate, a, obeyed not the conventional commutation relation but rather

$$aa^\dagger - qa^\dagger a = I \tag{3.22}$$

together with

$$[N,a] = -a \ , \qquad [N,a^\dagger] = a^\dagger \tag{3.23}$$

where N is the hermitian number operator

$$N|n\rangle = n|n\rangle \tag{3.24}$$

This algebra will be denoted $\mathcal{A}_q$.

The spectral properties of such operators have been considered by a number of authors (e.g. [Fiv:91]). Defining

$$[n] = \frac{1-q^n}{1-q} \tag{3.25}$$

$$[n]! = \prod_{k=1}^{n}[k] \tag{3.26}$$

$$[0]! \equiv 1 \tag{3.27}$$

the base vectors can be calculated explicitly as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle \tag{3.28}$$

where the normalised ground state is annihilated by a. For $q \in (0,1)$, the operators a and $a^\dagger$ are bounded with $||a|| = ||a^\dagger|| = (1-q)^{-\frac{1}{2}}$. As $q \to 1$, a, $a^\dagger$ become unbounded.

The Casimir of the algebra is given by

$$\hat{C} = q^{-N}([N] - a^{\dagger}a) \tag{3.29}$$

In the representation space $\mathcal{F}_q$ however, we have the relation that

$$a^{\dagger}a = [N] = \frac{q^N - 1}{q - 1} \tag{3.30}$$

so the casimir eigenvalue is zero.

In the conventional case, the operator N has an expansion in terms of the creation and annihilation operators (namely $N = b^{\dagger}b$). This is also true in the deformed case but the expansion is more involved. In fact there are two possibilties: firstly given equation (3.30), it is clear that

$$N = \frac{1}{s}\ln(1 + (q-1)a^{\dagger}a) \ , \qquad s = \ln q \tag{3.31}$$

A second, more interesting expansion was found by Chakrabarti and Jagannathan [CJ:92a]

$$N = \sum_{n=1}^{\infty} \frac{(1-q)^n}{(1-q^n)}(a^{\dagger})^{n-1}a^{n-1} \tag{3.32}$$

The functions defined in (3.25), (3.26) are the *basic numbers* of classical q–analysis and as such have been used extensively in the study of basic hypergeometric series (see, e.g. [Ext:83]). Moreover, as in the conventional case, there exists a differential realisation of the algebra. In this, the analogue of the annihilation operator is not the ordinary derivative operator but a finite difference *q–derivative* [Jac:08, Jac:51] ${}_qD_x$ defined by

$${}_qD_x f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{3.33}$$

While there is no known Hopf structure for the algebra $\mathcal{A}_q$, it can be obtained from the quantum group $su_q(2)$ by means of a contraction at fixed q [Kul:91]

$$\begin{aligned} a &= \lim_{f\to 0} -f\omega^{1/2}J_+ && (3.34)\\ a^{\dagger} &= \lim_{f\to 0} -f\omega^{1/2}J_- && (3.35)\\ q^N &= \lim_{f\to 0} -fq^{-\frac{1}{2}J_0} && (3.36)\end{aligned}$$

with $\omega = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$.

If this contraction is done in the first space of the coproduct of $su_q(2)$, then most of the Hopf maps do not survive. The comultiplication however still gives a sensible result and can be interpreted as a coaction of $su_q(2)$ on $\mathcal{A}_q$.

$$\begin{aligned}
\Delta_R : \mathcal{A}_q &\rightarrow \mathcal{A}_q \otimes su_q(2) & (3.37)\\
\Delta_R(N) &= N \otimes I - I \otimes J_0 & (3.38)\\
\Delta_R(a) &= a \otimes q^{J_0/2} - \sqrt{\omega} q^{N/2} \otimes J_+ & (3.39)\\
\Delta_R(a^\dagger) &= a^\dagger \otimes q^{J_0/2} - \sqrt{\omega} q^{N/2} \otimes J_- & (3.40)
\end{aligned}$$

This coaction has no $q = 1$ analogue.

Another coaction, this time one which has a well-defined $q = 1$ limit in the form of the Bogoliubov transformation (3.8), is the $SU_q(1,1)$ coaction

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \qquad (3.41)$$

where α, β, β^* and $\alpha^* \in SU_q(1,1)$ and $*$ denotes the involutive automorphism of the $SU_q(1,1)$ algebra.

3.2.3 The Macfarlane–Biedenharn q–Boson Algebra, $\mathcal{B}_q$.

The deformed oscillator algebra which is the counterpart of the conventional boson algebra in the q–analogue of the Jordan–Schwinger construction was discovered independently by a number of different authors [Mac:89, Bie:89, Hay:90]. It is generated by three elements $\{a_q, a_q^\dagger \equiv (a_q)^*, N\}$ with the commutation relations

$$a_q a_q^\dagger - q^{\frac{1}{2}} a_q^\dagger a_q = q^{-\frac{N}{2}} \qquad (3.42)$$

$$[N, a_q] = -a_q \ , \quad [N, a_q^\dagger] = a_q^\dagger \qquad (3.43)$$

Again, the $*$ denotes an involution of the algebra which reduces to hermitian conjugation on representations. This algebra will be denoted $\mathcal{B}_q$.

The Fock space of this oscillator is constructed in the same manner as that of the ordinary boson algebra, the base vectors being the orthonormal eigenstates of the number operator N. The operator a_q annihilates the lowest weight vacuum ket $|0\rangle$ and the multiparticle states are constructed by multiple applications of the q–creation operator $a_q^\dagger$. If, for all $x \in \mathbb{C}$ and $m \in \mathbb{N}$, we define the functions $[\![x]\!]_q$ and $[\![m]\!]_q!$ by

$$[\![x]\!]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{3.44}$$

$$= \frac{\sinh(\gamma x)}{\sinh(\gamma)} \tag{3.45}$$

$$[\![m]\!]_q! = \prod_{k=1}^{m} [\![k]\!]_q \tag{3.46}$$

with $\gamma = \ln \sqrt{q}$ and $[\![0]\!]_q! \equiv 1$, then the n–particle states are

$$|n\rangle = \frac{(a_q^\dagger)^n}{\sqrt{[\![n]\!]_q!}}|0\rangle \tag{3.47}$$

The action of the operators on this q–deformed Fock space is then

$$\begin{aligned} N|n\rangle &= n\ |n\rangle \\ a_q|n\rangle &= \sqrt{[\![n]\!]_q}\ |n-1\rangle \\ a_q^\dagger|n\rangle &= \sqrt{[\![n+1]\!]_q}\ |n+1\rangle \end{aligned} \tag{3.48}$$

One difference between this q–deformed algebra and the undeformed case is that the product $a_q^\dagger a_q$ no longer equals the number operator N. Instead we have

$$a_q^\dagger a_q = [\![N]\!]_q = \frac{\sinh(\gamma N)}{\sinh(\gamma)} \tag{3.49}$$

where the last term is understood as a power series expansion. Clearly in the limit $q \to 1$, $[\![x]\!]_q \to x$ and so we recover the ordinary boson algebra. One notable symmetry property of the q–boson states is their invariance under the transformation $q \to q^{-1}$.

The Casimir operator, $\hat{C}$, for the algebra is given by

$$\hat{C} = q^{-N}([\![N]\!]_q - a_q^\dagger a_q) \tag{3.50}$$

and is identically zero in the Fock–space representation. It has recently been pointed out [OS:94] that the definition of the q–oscillator is dependent on the represetation space. Thus, for example, although the equations (3.42) and (3.48) imply each other in the representation outlined above, there are other representations (labelled by a non–zero Casimir eigenvalue) in which this is not the case.

The algebra of this q–boson in the Fock–space representation does not have a Hopf structure [OS:94] but has been shown [CPT:91] to close (with other generators) under the action of the graded–commutator to give the super–Hopf algebra (*quantum supergroup*) $osp_q(1|2)$.

The procedure [Bie:89, Hay:90, Pol:90] for constructing the Jordan–Schwinger representations of the q–deformed Lie algebras is similar to the undeformed case except that the diagonal matrix generators are obtained by replacing $a_q^\dagger a_q$ by the number operator N. For the case of the simplest QUEA, $su_q(2)$, the fundamental matrix representation coincides with the $q = 1$ case. The representations ρ of the generators $\{J_0, J_\pm\}$ are

$$\rho(J_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad (3.51)$$

and so the Jordan–Schwinger map $\mathcal{J}$ in terms of two sets of independent q–bosons, $\{a_{(1)q}, a^\dagger_{(1)q}, N_{(1)}\}$ and $\{a_{(2)q}, a^\dagger_{(2)q}, N_{(2)}\}$ gives the realisation as

$$\mathcal{J}(J_0) = \tfrac{1}{2}(N_{(1)} - N_{(2)}), \qquad \mathcal{J}(J_+) = a^\dagger_{(1)q} a_{(2)q}, \qquad \mathcal{J}(J_-) = a^\dagger_{(2)q} a_{(1)q} \qquad (3.52)$$

Then the states which form the $su_q(2)$–module are

$$|j, m\rangle = |n_1\rangle\,|n_2\rangle = \frac{(a^\dagger_{(1)q})^{j+m}(a^\dagger_{(2)q})^{j-m}}{([\![j+m]\!]_q!\;[\![j-m]\!]_q!)^{1/2}}|0\rangle|0\rangle \qquad (3.53)$$

where $j + m = n_1$ and $j - m = n_2$. The generators of the $su_q(2)$–algebra defined in (3.52) act on these states to give the required action. It is important to note that the Jordan–Schwinger map only yields the correct result for states that terminate with the vacuum state. This is unlike the conventional case where the generators close abstractly.

For $q \in \mathbb{R}$, there is a simple invertible deformation map between the algebra $\mathcal{B}_q$ and the algebra $\mathcal{A}_q$ defined in (3.22) namely

$$a = q^{N/4} a_q, \qquad a^\dagger = a_q^\dagger q^{N/4}, \qquad N_{\mathcal{A}_q} = N_{\mathcal{B}_q} = N \tag{3.54}$$

The invertibilty of the map means that it is possible to use the oscillator algebra $\mathcal{A}_q$ to realise the generators of a quantum group in the same way as for the algebra $\mathcal{B}_q$.

There is also a deformation map from the algebra $\mathcal{B}_q$ (and hence from $\mathcal{A}_q$) to the ordinary boson algebra $\mathcal{H}(4)$ [Pol:90, CGZ:91].

$$a_q = b\sqrt{\frac{[N]_q}{N}}, \qquad a_q^\dagger = \sqrt{\frac{[N]_q}{N}}\, b^\dagger, \qquad N_{\mathcal{A}_q} = N_{\mathcal{H}(4)} = N \tag{3.55}$$

These two maps are special cases of a more general transformation discussed later.

Just as the algebra $\mathcal{A}_q$ has a realisation in terms of the coordinate operator x and a q-derivative ${}_qD_x$, so $\mathcal{B}_q$ has a similar realisation in terms of the x–multiplication operator, the *symmetric* q–derivative operator ${}_qS_x$ and the q–dilation operator $\hat{T}_q$. The representation is defined by

$$a_q^\dagger \to x, \qquad a_q \to {}_qS_x, \qquad q^N \to \hat{T}_q = q^{\frac{1}{2}x\frac{d}{dx}} \tag{3.56}$$

where the action of $\hat{T}_q$ and ${}_qS_x$ on an arbitrary function of x is

$$\hat{T}_q f(x) = f(q^{\frac{1}{2}}x) \tag{3.57}$$

$${}_qS_x f(x) = \left((q^{\frac{1}{2}} - q^{-\frac{1}{2}})x\right)^{-1} (\hat{T}_q - \hat{T}_{1/q}) f(x) \tag{3.58}$$

$$= \frac{f(q^{\frac{1}{2}}x) - f(q^{-\frac{1}{2}}x)}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})x} \tag{3.59}$$

3.2.4 The $\mathrm{SU}_q(\mathrm{N})$ Spinor Algebra

While the q–bosons described above are very useful in constructing Jordan–Schwinger representations, they are not covariant under $SU_q(N)$ transformations. Essentially, one requires the existence of a quantum group coaction on a module composed of annihilation operators. Since the quantum matrix group preserves a quantum plane structure, it is clear that the elements of an $SU_q(N)$–module must have non–trivial commutation relations among themselves. Consequently, simply taking n independent (i.e. commuting) sets of q–bosons will not be successful. The problem is resolved by building in covariance from the outset [PW:89, WZ:90, Kem:93].

It was observed by Zumino [WZ:90] that the statement that a set of non-commuting coordinates $\{x_i\}$ belong to the N–dimensional quantum plane can be re–expressed as

$$x_i x_j = q^{-1} R_{ij,kl} x_l x_k \tag{3.60}$$

where, for convenience, all indices are written as subscripts. If $\{A_i\}$ is a set of deformed annihilation operators, these will be covariant under the coaction of $SU_q(N)$ if

$$A_i A_j = q^{-1} R_{ij,kl} A_l A_k \tag{3.61}$$

If we require the relation between the annihilation operators and their counterpart creation operators to be of the same form but with a minimal central extension, then this leads to the relation

$$A_i A_j^\dagger = q R_{ki,jl} A_k^\dagger A_l + \delta_{ij} \tag{3.62}$$

Invariance of this relation under hermitian conjugation means that q must be real and R must satisfy

$$R_{ij,kl} = R^*_{lk,ji} \tag{3.63}$$

which in turn implies that

$$A_i^\dagger A_j^\dagger = q^{-1} R_{ji,kl} A_k^\dagger A_l^\dagger \tag{3.64}$$

Clearly the algebra should have the same size as the commutative one. This condition manifests itself in relations obtained by using the above commutation relations to reorder triples such as $A_iA_jA_k$ and $A_iA_jA_k^\dagger$ and then equating like–order terms. If this is done, the cubic terms equate if R satisfies the Yang–Baxter equation and the linear terms equate if R satisfies the Hecke condition which can be written as

$$(PR + q^{-1}I)(PR - qI) = 0 \tag{3.65}$$

where $P = P_{ij,kl} = \delta_{il}\delta_{jk}$ is a permutation which implements the twist operation in a pair of vector spaces. The Hecke condition is fulfilled by the R–matrices of the quantum unitary matrix groups and so the equations (3.61), (3.62), (3.64) define an $SU_q(N)$–covariant q–boson algebra.

Explicitly, the commutation relations come out as

$$\begin{aligned}
A_iA_j &= qA_jA_i && \text{for} \quad i<j && (3.66)\\
A_i^\dagger A_j^\dagger &= qA_j^\dagger A_i^\dagger && \text{for} \quad i>j && (3.67)\\
A_iA_j^\dagger &= qA_j^\dagger A_i && \text{for} \quad i=j && (3.68)\\
A_iA_i^\dagger - q^2A_i^\dagger A_i &= 1 + (q^2-1)\sum_{j<i} A_j^\dagger A_i && && (3.69)
\end{aligned}$$

3.2.5 Multiparameter Deformation of the Boson Algebra

As well as deformations of the boson algebra depending on one parameter, there are also multiparameter generalisations. One such is the (p,q)*–deformed boson algebra* [CJ:92b, ADTEM:92] generated by elements $\{a, a^\dagger, N\}$ with relations

$$aa^\dagger - q^{\frac{1}{2}}a^\dagger a = p^{-\frac{N}{2}} \tag{3.70}$$

$$[N,a] = -a, \qquad [N,a^\dagger] = a^\dagger \tag{3.71}$$

The corresponding deformed Fock–space is spanned by vectors of the form

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[\![n]\!]_{p,q}!}}|0\rangle \tag{3.72}$$

where

$$\llbracket x \rrbracket_{p,q} = \frac{q^{x/2} - p^{-x/2}}{q^{1/2} - p^{-1/2}} \tag{3.73}$$

$$\llbracket m \rrbracket_{p,q}! = \prod_{k=1}^{m} \llbracket k \rrbracket_{p,q} \tag{3.74}$$

which implies that in the usual Fock–space,

$$aa^{\dagger} - p^{-\frac{1}{2}}a^{\dagger}a = q^{\frac{N}{2}} \tag{3.75}$$

This deformation clearly subsumes both the algebras $\mathcal{A}_q$ and $\mathcal{B}_q$ as special cases.

It is possible to find the q–deformed calculus characterised by the deformed derivative

$$_{q,p}D_x f(x) = \frac{f(qx) - f(p^{-1}x)}{(q - p^{-1})x} \tag{3.76}$$

The (q,p)–boson algebra can be used to form the Jordan–Schwinger representations of the two-parameter quantum deformation of $su(2)$ as well as its noncompact counterpart $su_{q,p}(1,1)$ [Schi:91, CJ:92b]. The R–matrix for these algebras is given by

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & qp^{-1} & 0 & 0 \\ 0 & (q - p^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{3.77}$$

which leads to the deformed commutation relations

$$[J_0, J_\pm] = \pm J_\pm \tag{3.78}$$

$$J_+J_- - q^{-1}pJ_-J_+ = \llbracket 2J_0 \rrbracket_{q,p} \tag{3.79}$$

for $su_{q,p}(2)$, and

$$[K_0, K_\pm] = \pm K_\pm \tag{3.80}$$

$$K_-K_+ - qp^{-1}K_+K_- = \llbracket 2K_0 \rrbracket_{q,p} \tag{3.81}$$

for $su_q(1,1)$.

The comultiplication map for the two algebras is the obvious one given the two–parameter nature of the deformation, namely

$$\Delta(X_0) = X_0 \otimes I + I \otimes X_0 \;, \qquad \Delta(X_\pm) = q^{X_0} \otimes X_\pm + X_\pm \otimes p^{-X_0} \quad (3.82)$$

where X is J or K depending on the algebra.

The deformed boson realisations of these quantum algebras then follow in the same way as for the single–parameter deformations, e.g., the one–mode (q,p)–boson $su_{q^2,p^2}(1,1)$ realisation is

$$K_0 = \frac{1}{2}(N + 1/2) \;, \qquad K_+ = (K_-)^\dagger = [\![2]\!]_{q,p}^{-1}\, a^{\dagger 2} \quad (3.83)$$

There are also multimode systems of two parameter deformed oscillators which are covariant under the actions of the two–parameter quantum group $GL_{p,q}(n)$ and its quantum supergroup counterpart [Vok:91, JSVCKSS:92]. A discussion of these systems would proceed along the lines of the single–parameter case but will not be undertaken here since it is not pertinent to the present investigation.

Chapter 4

Coherent States

In 1926, Schrödinger [Schr:26] introduced a system of wavefunctions to describe the dynamical evolution of classical, i.e. non-spreading wave-packets for quantum harmonic oscillators. However, the fact that these wavefunctions were non-orthogonal mitigated against their use in further work. In 1932, in his work on the quantum mechanical measurement problem, von Neumann [Neu:32] considered a subset of the Schrödinger wavefunctions parameterised by points on a regular lattice in a complex phase space. Both these examples show that the concept of the coherent state was implicit in the work of some of the founders of Quantum Mechanics. It was, however, the application of states modelling *coherent* light by Glauber [Gla:63a] and Sudarshan [Suda:63] which provided the impetus for the many advances in the use of coherent state techniques which have emerged since the early 1960's.

4.1 Overview

In one of his early papers [Gla:63b], Glauber gave three definitions for the coherent states associated with the quantum harmonic oscillator.

1. Coherent States $|\alpha\rangle$ are eigenstates of the photon annihilation operator, b:

$$b|\alpha\rangle = \alpha|\alpha\rangle \tag{4.1}$$

where $\alpha \in \mathbb{C}$.

2. Coherent States are the states obtained by applying a unitary displacement operator $D(\alpha)$ to the oscillator ground state $|0\rangle$:

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{4.2}$$

where $D(\alpha)$ is defined by

$$D(\alpha) = \exp(\alpha b^{\dagger} - \alpha^{*} b) \tag{4.3}$$

3. Coherent States are the quantum states that minimise the Heisenberg Uncertainty Relation [Hei:27]

$$(\Delta X)^2(\Delta P)^2 \geq \left(\frac{1}{2}\right)^2 \tag{4.4}$$

where the field coordinate and momentum operators are defined as

$$X = \frac{1}{\sqrt{2}}(b + b^{\dagger}) \tag{4.5}$$

$$P = \frac{1}{i\sqrt{2}}(b - b^{\dagger}) \tag{4.6}$$

and $(\Delta A)^2 = \langle\alpha|A^2 - \langle A\rangle_{\alpha}^2|\alpha\rangle$; $\langle A\rangle_{\alpha} = \langle\alpha|A|\alpha\rangle$.

About the same time, Klauder [Kla:63], in his work on continuous representations of states in a Hilbert Space, described the minimum requirements for a set of states to be coherent. According to Klauder [KlS:85], coherent states $|\alpha\rangle$ are a set of state vectors in a Hilbert Space $\mathcal{H}$ (finite or countably infinite dimensional), parameterised by an element of a (possibly multidimensional) label space $\mathcal{L}$ which is endowed with an appropriate topology.

Coherent states have two properties:

1. The vector $|\alpha\rangle$ is a strongly continuous function of the label α.
2. There exists a positive measure $d\mu(\alpha)$ on $\mathcal{L}$ such that the coherent states admit a resolution of unity when integrated over α.

$$\int |\alpha\rangle\langle\alpha|\, d\mu(\alpha) = I \tag{4.7}$$

It is notable that the continuity property rules out both discrete orthogonal vectors $\{|n\rangle : n = 0, 1, 2, \ldots$ with $\langle n|m\rangle = \delta_{n,m}\}$ and normalized orthogonal continuum vectors $\{|x\rangle : -\infty < x < \infty$, with $\langle x'|x\rangle = \delta(x - x')\}$. In the former case, the vectors do not form a continuous set, whereas in the latter the vectors are not continuous in the labels. In fact, the eigenvectors of any self-adjoint operator never constitute a set of coherent states [KlS:85].

The existence of a resolution of unity gives rise to representations of vectors and operators. For example, if $|\phi\rangle$, $|\psi\rangle$ are arbitrary vectors in $\mathcal{H}$

$$\langle\psi|\phi\rangle = \int \langle\psi|\alpha\rangle\langle\alpha|\phi\rangle \, d\mu(\alpha) \tag{4.8}$$

and if B is some operator,

$$\langle\alpha|B|\phi\rangle = \int \langle\alpha|B\alpha'\rangle\langle\alpha'|\phi\rangle d\mu(\alpha') \tag{4.9}$$

The functional representations of the states induced by the coherent states

$$\phi(\alpha) = \langle\alpha|\phi\rangle \tag{4.10}$$

gives rise to vector representatives that are continuous functions. Moreover, since, for all $\phi \in \mathcal{H}$

$$\langle\phi|\phi\rangle = \int |\langle\alpha|\phi\rangle|^2 d\mu(\alpha) < \infty \tag{4.11}$$

the functions of the continuous representation are all square integrable. If we further impose the condition that the Hilbert Space should be equipped with a *Reproducing Kernel* $\mathcal{K}(\alpha, \alpha')$

$$\mathcal{K}(\alpha, \alpha') = \langle\alpha|\alpha'\rangle = \langle\alpha'|\alpha\rangle^* \tag{4.12}$$

such that

$$\langle\alpha|\beta\rangle = \int \langle\alpha|\alpha'\rangle\langle\alpha'|\beta\rangle d\mu(\alpha') \tag{4.13}$$

or

$$\mathcal{K}(\alpha, \beta) = \int \mathcal{K}(\alpha, \alpha')\mathcal{K}(\alpha', \beta) d\mu(\alpha') \tag{4.14}$$

then it is possible to show that only a subset of all square-integrable functions is needed to represent vectors in $\mathcal{H}$.

4.2 Quantum Noise and Uncertainty Principles

One of the most striking differences between classical physics and quantum mechanics is that, in the latter theory, simultaneous measurements of variables described by noncommuting operators cannot be made with arbitrary precision. This statement was formalised in a number of ways, the most notable being the *Uncertainty Principle* first proposed by Heisenberg [Hei:27]. Since much of the following work will involve the calculation of various uncertainty products, we briefly recall the calculation of the general formula, following the derivation found in [Gar:91].

We consider two operators X and Y and define

$$\delta X = X - \langle X\rangle\,, \qquad \delta Y = Y - \langle Y\rangle \tag{4.15}$$

where $\langle\cdot\rangle$ denotes the quantum mechanical average over the range of states in question. We can write both the commutator of X and Y, as well as the variances $(\Delta X)^2$ and $(\Delta Y)^2$, in terms of the new operators:

$$[X,Y] = [\delta X, \delta Y] \tag{4.16}$$

$$(\Delta X)^2 = \langle X^2 - \langle X\rangle^2\rangle = \langle(\delta X)^2\rangle \tag{4.17}$$

$$(\Delta Y)^2 = \langle Y^2 - \langle Y\rangle^2\rangle = \langle(\delta Y)^2\rangle \tag{4.18}$$

It is known from spectral theory that, for any operator Z with hermitian conjugate $Z^\dagger$,

$$\langle ZZ^\dagger\rangle \geq 0 \tag{4.19}$$

Consequently, if we form the operator $Z = \delta X + |\lambda|e^{i\theta}\delta Y$ and substitute into (4.19), we obtain

$$\langle(\delta X)^2\rangle + |\lambda|\{\cos\theta\langle[\delta X,\delta Y]_+\rangle - i\sin\theta\langle[\delta X,\delta Y]\rangle\} + |\lambda|^2\langle(\delta Y)^2\rangle \geq 0 \tag{4.20}$$

where $[\ ,\ ]_+$ denotes the anticommutator. Equation (4.20) is a quadratic in $|\lambda|$ and so the inequality for the discriminant gives

$$\{\cos\theta\langle[\delta X,\delta Y]_+\rangle - i\sin\theta\langle[\delta X,\delta Y]\rangle\}^2 \leq 4\langle(\delta X)^2\rangle\langle(\delta Y)^2\rangle \tag{4.21}$$

Since this must be true for all θ, we may maximise the left hand side to yield

$$\langle(\delta X)^2\rangle\langle(\delta Y)^2\rangle \geq \frac{1}{4}|\langle[\delta X,\delta Y]\rangle|^2 + \frac{1}{4}\langle[\delta X,\delta Y]_+\rangle^2 \tag{4.22}$$

The term involving the anticommutator is the quantum mechanical analogue of the square of the covariance ΔXY where

$$\Delta XY \equiv \frac{1}{2}\langle[X,Y]_+\rangle - \langle X\rangle\langle Y\rangle = \frac{1}{2}\langle[\delta X,\delta Y]_+\rangle \tag{4.23}$$

and so (4.22) can be rewritten as

$$\sqrt{(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2} \geq \frac{1}{2}|\langle[X,Y]\rangle| \tag{4.24}$$

which is the *Robertson–Schrodinger Uncertainty Relation* [Schr:30, Rob:30]. Since the square of the covariance is positive or zero, a less accurate lower bound on the uncertainty product is obtained by neglecting its contribution on the left hand side of (4.24). (Alternatively, if the two operators are uncorrelated, the covariance is identically zero). We therefore obtain the more familiar *Heisenberg Uncertainty Relation* (H.U.R.),

$$\Delta X\Delta Y \geq \frac{1}{2}|\langle[X,Y]\rangle| \tag{4.25}$$

4.3 Coherent states of the electromagnetic field

The physical motivation for the consideration of photon coherent states was the application of Quantum Field Theory to Quantum Optics. Specifically, Glauber wished to factorise the correlation functions of the electromagnetic field to all orders.

Ehrenfest's theorem [Ehr:27] for a particle in a quadratic potential (such as a harmonic oscillator) states that the motion of the centre of the wavepacket obeys the classical evolution equation. In terms of the quantum mechanical operators $\hat{X}(t)$ and $\hat{P}(t)$ and the corresponding classical variables $X_c(t)$ and $P_c(t)$, this means that [MeS:90]

$$\begin{aligned} \langle\psi|\hat{X}(t)|\psi\rangle &= X_c(t) && (4.26)\\ \langle\psi|\hat{P}(t)|\psi\rangle &= P_c(t) && (4.27) \end{aligned}$$

Classically, the energy of a field oscillator of unit angular frequency is given by the Hamiltonian function H_c,

$$H_c = \frac{P_c^2 + X_c^2}{2} \tag{4.28}$$

$$= \frac{1}{2}\left\{\langle\psi|\hat{P}(t)|\psi\rangle^2 + \langle\psi|\hat{X}(t)|\psi\rangle^2\right\} \tag{4.29}$$

In terms of the boson creation and annihilation operators b and $b^\dagger$, this can be rewritten as

$$H_c = \langle\psi|b^\dagger|\psi\rangle\langle\psi|b|\psi\rangle \tag{4.30}$$

The corresponding quantum mechanical oscillator has energy

$$\langle H\rangle = \langle\psi|H|\psi\rangle = \langle\psi|b^\dagger b|\psi\rangle \tag{4.31}$$

and so the requirement that the quantum mechanical energy be equal to the classical energy leads to the *factorisation condition*

$$\langle\psi|b^\dagger b|\psi\rangle = \langle\psi|b^\dagger|\psi\rangle\langle\psi|b|\psi\rangle \tag{4.32}$$

Glauber named the states that fulfilled such a condition to all orders, *Coherent States.*

Equation (4.32) can be rewritten as

$$|\langle\psi|b^\dagger|\psi\rangle|^2 = \langle\psi|b^\dagger b|\psi\rangle \tag{4.33}$$

If the identity is now resolved into the projector onto $|\psi\rangle$ and the projector onto the orthogonal complement of $|\psi\rangle$ (which we will write as $\sum_k |\psi_k^\perp\rangle\langle\psi_k^\perp|$), i.e.

$$I = |\psi\rangle\langle\psi| + \sum_k |\psi_k^\perp\rangle\langle\psi_k^\perp| \tag{4.34}$$

then we can insert this resolution into $\langle\psi|b^\dagger b|\psi\rangle$ to get

$$\langle\psi|b^\dagger b|\psi\rangle = \langle\psi|b^\dagger|\psi\rangle\langle\psi|b|\psi\rangle + \sum_k\langle\psi|b^\dagger|\psi_k^\perp\rangle\langle\psi_k^\perp|b|\psi\rangle \tag{4.35}$$

$$= |\langle\psi|b^\dagger|\psi\rangle|^2 + \sum_k |\langle\psi_k^\perp|b|\psi\rangle|^2 \tag{4.36}$$

Therefore, using (4.33), we see that

$$\sum_k |\langle \psi_k^\perp | b | \psi \rangle|^2 = 0 \tag{4.37}$$

Since each term in the sum is positive definite,

$$\langle \psi_k^\perp | b | \psi \rangle = 0 \tag{4.38}$$

for all k, i.e. $b|\psi\rangle$ must be orthogonal to any $|\psi_k^\perp\rangle$ and so must be proportional to $|\psi\rangle$. Consequently coherent states are eigenstates of the annihilation operator.

Such coherent states have found immense application in the area of quantum optics. In such circumstances, consideration of the algebraic content of the equations show that the dynamical properties of the field (which is essentially a sum of terms linear in the photon creation, annihilation and number operators) emerge from the Hamiltonian and its Hilbert space. Such operators realise the (Heisenberg–Weyl) Boson algebra, $\mathcal{H}(4)$. A detailed investigation of this algebra was made in chapter 2. We recall that this Lie algebra is spanned by three elements $\{b, b^\dagger, I\}$ with fundamental commutation relations

$$[b, b^\dagger] = I \quad [b, I] = [b^\dagger, I] = 0 \tag{4.39}$$

with an extra element $N = b^\dagger b$, which also forms a system closed under commutation

$$[N, b] = -b \ ; \quad [N, b^\dagger] = b^\dagger \ ; \quad [N, I] = 0 \tag{4.40}$$

The Hilbert Space (in this case, Fock Space) is spanned by the eigenstates of the Number operator $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots, |n\rangle, \ldots\}$ where

$$N|n\rangle = n|n\rangle \tag{4.41}$$

and it is easy to show that the n–th normalised number state is given by

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{4.42}$$

The vacuum state, $|0\rangle$, is annihilated by the lowering operator, b, and forms the ground state of the system.

Given this Hilbert Space, the photon coherent states are generated by unitary transformations of the ground state, $|0\rangle$.

$$
\begin{aligned}
|\alpha\rangle &= D(\alpha)|0\rangle && (4.43)\\
&= \exp(\alpha b^{\dagger} - \alpha^{*} b)|0\rangle && (4.44)\\
&= \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha b^{\dagger})\exp(-\alpha^{*} b)|0\rangle && (4.45)\\
&= \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha b^{\dagger})|0\rangle && (4.46)\\
&= \exp(-\tfrac{1}{2}|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|0\rangle && (4.47)
\end{aligned}
$$

where we have used the Baker–Campbell–Hausdorff formula to get a decomposition of the coherent state as a superposition of number states.

If we set the adjoint vector $\langle\alpha| \equiv |\alpha\rangle^{\dagger}$, the non–orthogonality of the coherent states is evident

$$\langle\alpha|\alpha'\rangle = \exp(-\tfrac{1}{2}|\alpha|^2 + \alpha\alpha' - \tfrac{1}{2}|\alpha'|^2) \qquad (4.48)$$

which is a nowhere–vanishing continuous function of the variable α and α'.

The relation

$$\exp(-\alpha b^{\dagger})\, b\, \exp(\alpha b^{\dagger}) = b + \alpha \qquad (4.49)$$

is a direct consequence of the basic commutation relation, so clearly

$$b \exp(\alpha\, b^{\dagger})|0\rangle = \alpha \exp(\alpha\, b^{\dagger})|0\rangle \qquad (4.50)$$

This shows that the coherent states are indeed eigenstates of the annihilation operator with eigenvalue α. Moreover, by explicit calculation, it is seen that the field variances $(\Delta\, Q)^2$ and $(\Delta\, P)^2$ are both equal to $\frac{1}{2}$ which minimises the H.U.R. as stated by Glauber.

The photon coherent states, i.e. coherent states associated with the boson algebra, were successfully applied to a wide variety of problems. Apart from Glauber's work on photon correlations and optical coherence, Sudarshan [Suda:63] established the validity of the semiclassical approach to optical problems. The solution of the harmonic oscillator in a time–dependent potential was solved by Carruthers and Nieto [CN:65]. Cummings and Johnston [CuJ:66] used coherent states to study superfluidity problems and there has been widespread use of such techniques elsewhere in condensed matter physics, as well as plasma physics. There have also been applications of the boson coherent states to atomic and nuclear physics. A bibliography of relevant papers appears in [KlS:85]

4.3.1 Group–theoretical description of Photon coherent states

Such success prompted a number of mathematical physicists [Per:72, Ras:73, Ras:75, Gil:72] to try to extend the notion of coherent state to other physical systems not adequately modelled by the harmonic oscillator. In order to do this, it was necessary to abstract the algebraic and group theoretic content of Glauber's construction and apply this to more complex systems, particularly to other Lie groups [Per:86, ZFG:90]. It is clear that the algebraic or group theoretical content of the Glauber coherent state emerges from consideration of the Hamiltonian of the system. The photon creation and annihilation operators generate an irreducible representation of the Heisenberg–Weyl algebra. The corresponding Lie group, $\mathcal{G}\{\mathcal{H}(4)\}$, can therefore be constructed. The Hilbert space for this group is spanned by eigenvectors of the Number operator and has an extremal (in this case, lowest) weight vector — the vacuum state.

The extremal state is stable only under the action of the *Isotropy* or *Stability* subgroup of the full Heisenberg–Weyl Group. This is the subgroup spanned by group elements corresponding to the operators $\{N, I\}$ and so having the general form

$$h = e^{i(\nu N + \theta I)} \tag{4.51}$$

where $\nu, \theta \in \mathbb{C}$, and so

$$h|0\rangle = e^{i\theta}|0\rangle \tag{4.52}$$

The isotropy subgroup for the Heisenberg–Weyl group is isomorphic to $U(1) \otimes U(1)$. It is possible to form the coset space with respect to the stability subgroup. This provides a unique decomposition for any element of the group. A typical representative of the coset space $\mathcal{G}\{\mathcal{H}(4)\}/U(1) \otimes U(1)$ is given by

$$D(\alpha) = \exp(\alpha b^\dagger - \alpha^* b) \tag{4.53}$$

where $\alpha \in \mathbb{C}$. The coherent states for this system are defined by the action of the coset elements on the extremal state

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{4.54}$$

The operator $D(\alpha)$ provides a one–to–one correspondence between the states $|\alpha\rangle$ and the points in the complex α–plane, with the coset representative generating finite displacements. This mapping is continuous where the metric for the coherent state is taken to be the usual Hilbert space inner product while that of the complex plane is just the ordinary Euclidean metric. This means that the corresponding invariant measure, $d\mu(\alpha)$, is (up to normalisation) just given by

$$d\mu(\alpha) = d^2\alpha = d\alpha\, d\alpha^* \tag{4.55}$$

The resolution of unity can therefore be calculated to be

$$\int |\alpha\rangle\langle\alpha| \frac{d^2\alpha}{\pi} = I \tag{4.56}$$

Since the Hilbert space was assumed to have been spanned by a denumerable set of number states and these coherent states are labelled by a continuous parameter, they are over–complete.

It is evident that this group–theoretical construction of the Heisenberg–Weyl states clearly exhibits Klauder's criteria for coherent states.

4.4 Generalised Coherent States for other semisimple Lie groups

The definition given by Glauber of the HWCS suggest three possible means of extending the concept to other groups. Firstly one could use eigenstates of the lowering operator to define states for arbitrary dynamical systems. This was done by Barut and Giradello [BaG:72] for the non-compact group SU(1,1), the Lie algebra of which is an important spectrum generating algebra. However such a construction is not possible for compact Lie groups such as SU(2) which have finite dimensional Hilbert spaces.

Another extension could be to use *Minimum Uncertainty States* (MUS). For two observables, A and B, the states $|\psi\rangle$ which minimise the quantum mechanical uncertainty relation

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle[A,B]\rangle| \tag{4.57}$$

(with averages taken in $|\psi\rangle$) are given as solutions of the eigenvalue equation

$$(A+i\lambda B)|\psi\rangle = \{\langle A\rangle + i\lambda\langle B\rangle\}\,|\psi\rangle \tag{4.58}$$

Such states were constructed by Aragone et al [AGST:74] who named them *Intelligent* states. They were extensively studied in a series of papers by Nieto and others [NS:78, Nie:84a] for various arbitrary dynamical systems. For a classically integrable system, it is possible to define canonically conjugate functions, $X_c(x_c,p_c)$ and $P_c(x_c,p_c)$, of the standard position and momentum variables x_c and p_c that vary sinusoidally with time. Often this means that the classical Hamiltonian is quadratic in these functions. If this system is quantised canonically, the functionals X_c and P_c become operators X and P with commutation relation

$$[X,P] = iG \tag{4.59}$$

where G is some operator. This gives the dispersion relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}|\langle[X,P]\rangle|^2 = \frac{1}{4}\langle G\rangle^2 \tag{4.60}$$

The states which minimise this uncertainty relation are given by the solutions to the eigenvalue equation

$$\left(X + \frac{i\langle G\rangle}{2(\Delta P)^2}P\right)|\psi\rangle_{MUS} = \left(\langle X\rangle + \frac{i\langle G\rangle}{2(\Delta P)^2}\langle P\rangle\right)|\psi\rangle_{MUS} \qquad (4.61)$$

By requiring that the ground-state of the Schrödinger equation be a solution, Nieto et al solved the eigenvalue equation and calculated the *arbitrary-potential MUS*. There are, however, some problems with this extension of the concept of coherent states. Such states follow the classical motion but show dispersion with time evolution, i.e. do not remain minimum uncertainty states. Moreover the set of states may not be complete or may lack a resolution of unity.

The most widely used generalisation of Heisenberg–Weyl coherent states to other dynamical systems is that proposed by Perelomov and others [Per:86, Ras:75, ZFG:90]. Perelomov considered the case where the dynamical symmetry group was a finite-dimensional Lie group, G. In such a case, one considers its unitary irreducible representations, T(g), acting in a Hilbert space $\mathcal{H}$. We then take a fixed (cyclic) vector $|\psi_0\rangle$ and consider the set $\{|\psi_g\rangle\}$, where for $g \in G$

$$|\psi_g\rangle = T(g)|\psi_0\rangle \qquad (4.62)$$

If H is the Isotropy subgroup, i.e. the maximal subgroup which, for $h \in H$,

$$T(h)|\psi_0\rangle = \exp(i\alpha)|\psi_0\rangle \qquad (4.63)$$

then the set of states $\{\lambda|\psi_g\rangle, \lambda \in \mathbb{C}\}$ is determined by a point $x(g)$ in the homogeneous coset space G/H corresponding to the element g and so we can use this a parameterisation, i.e.

$$|\psi_g\rangle = \exp(i\alpha)|x(g)\rangle \qquad (4.64)$$

The group G may be considered as a fibre bundle with base $x = G/H$ and fibre H. A choice of $g(x)$ then corresponds to a cross-section of the fibre bundle.

Perelomov's construction is not the only one which can be made. The inputs for the above algorithm are essentially the Lie group structure of the dynamical symmetry, the unitary irreducible representation and the reference state. There are other choices which may be made however. Gilmore et al [ZFG:90] considered an arbitrary dynamical symmetry group instead of a Lie group and the representation in the Hilbert space was square–integrable. His fiducial state was not arbitrary, as with Perelomov, but was required to be an extremal state annihilated by a maximal subset of the algebra associated with the symmetry group. This choice of reference state is important as it can greatly simplify calculations. Usually the most useful choice is to make the state the unperturbed physical ground state. This is then an *extremal* state (at least for a discrete spectrum). The coset space and hence the coherent states can then be constructed from a knowledge of the positive root elements of the group which associate to shift operators in the weight space.

4.5 Coherent States for SU(2)

4.5.1 The Irreducible Representations

The Lie group $SU(2)$ has unitary irreducible representations labelled by a non–negative integer or half–integer j. The dimension of the representation is $2j+1$. The Lie algebra of $SU(2)$ is generated by three elements $\{J_0, J_\pm\}$ with commutation relations

$$[J_0, J_\pm] = \pm J_\pm \tag{4.65}$$

$$[J_+, J_-] = 2J_0 \tag{4.66}$$

We can form a finite dimensional Hilbert space basis of eigenstates $|j,m\rangle$ of the Cartan element, J_0, of the Lie algebra. The label m lies in the range $-j \leq m \leq j$ and increases in integer steps. The element J_0 acts on the basis as

$$J_0|j,m\rangle = m|j,m\rangle \tag{4.67}$$

The representation label, j, can be related to the eigenvalue of the Casimir, J^2 which has action on the canonical basis

$$J^2|j,m\rangle = j(j+1)|j,m\rangle \tag{4.68}$$

The other generators act as ladder operators, raising and lowering the label m.

$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m\rangle \tag{4.69}$$

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m\rangle \tag{4.70}$$

and so

$$J_-|j,-j\rangle = 0 \tag{4.71}$$

and

$$|j,m\rangle = \sqrt{\frac{(j-m)!}{(j+m)!\,(2j)!}}\,(J_+)^{j+m}|j,-j\rangle \tag{4.72}$$

The holomorphic realisation of the representation is generated by first order differential operators

$$J_0 = z\tfrac{d}{dz} \;, \qquad J_+ = -z^2\tfrac{d}{dz} + 2jz \;, \qquad J_- = z\tfrac{d}{dz} - j \tag{4.73}$$

where the operators J_+ and J_- are conjugate to each other with respect to the scalar product

$$\langle f_1|f_2\rangle = (2j+1)\int \overline{f_1(z)}\,f_2(z)\,d\mu(z) \tag{4.74}$$

and the measure is given by

$$d\mu(z) = \frac{d^2 z}{(1+|z|^2)^{2j+2}} \tag{4.75}$$

The basis vectors are the monomials f_m

$$f_m = \sqrt{\frac{(2j)!}{(j-m)!\,(j+m)!}}\;z^{j+m} \tag{4.76}$$

4.5.2 The Coherent States

The unitary group–theoretical displacement operator for the $SU(2)$ group is given by

$$
\begin{aligned}
D(\xi) &= \exp(\xi J_+ - \bar{\xi} J_-) && (4.77)\\
&= \exp(\zeta J_+)\exp(\eta J_0)\exp(-\bar{\zeta} J_-) && (4.78)
\end{aligned}
$$

where we have used the Baker–Campbell–Hausdorff formula in the last line to decompose the single exponential into *normal* form. If we write ξ as $(\theta/2)e^{-i\phi}$, for $(0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi)$, then the parameters ζ, η and ζ' are related to the original ones, ξ and ξ^*, by

$$
\begin{aligned}
\zeta &= \tan(\theta/2)\exp(-i\phi) && (4.79)\\
\eta &= \ln(1 + |\zeta|^2) && (4.80)
\end{aligned}
$$

If we choose the lowest weight state $|j, -j\rangle$ to be the cyclic vector, the group–theoretical coherent states take the form

$$
\begin{aligned}
|\zeta\rangle &= \exp(\xi J_+ - \bar{\xi} J_-)|j, -j\rangle && (4.81)\\
&= \exp(\zeta J_+)\exp(\eta J_0)\exp(-\bar{\zeta} J_-)|j, -j\rangle && (4.82)\\
&= \exp(\zeta J_+)(1 + |\zeta|^2)^{J_0}|j, -j\rangle && (4.83)\\
&= (1 + |\zeta|^2)^{-j}\exp(\zeta J_+)|j, -j\rangle && (4.84)
\end{aligned}
$$

Then using (4.72), the coherent states can be written as

$$
|\zeta\rangle = \sum_{-j}^{j} \sqrt{\frac{(2j)!}{(j+m)!\,(j-m)!}}\,(1 + |\zeta|^2)^{-j}\zeta^{j+m}|j, m\rangle \qquad (4.85)
$$

The Lie group $SU(2)$ is locally isomorphic to the group $SO(3)$ of rotations in three dimensional Euclidean space. Consequently, these coherent states have been used extensively in the study of spin systems, especially atomic spin systems where they correspond to semiclassical states on the spherical spin phase manifold. The techniques used to construct $SU(2)$–coherent states can be generalised to other compact Lie groups. Since the dynamical groups of many–fermion systems are compact, it is not surprising that the coherent states of such systems play an important role in such physical phenomena as superconductivity, charge–density waves and spin–density waves [BS:82, SB:84, SB:87].

4.6 SU(1,1)–Coherent States

4.6.1 The Irreducible Representations

The non–compact Lie group $SU(1,1)$, which is locally isomorphic to both $SO(2,1)$ and $Sp(2,\mathbb{R})$, consists of all matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{4.86}$$

where $|\alpha|^2 - |\beta|^2 = 1$. Because the group is non–compact, its unitary irreducible representations are infinite dimensional. Moreover, unlike the group $SU(2)$, it has several series of unirreps: the principal, discrete and supplementsary series. We shall consider only the discrete series.

The Lie algebra of $SU(1,1)$ is generated by three elements K_0, $K_\pm$ with commutation relations

$$[K_0, K_\pm] = \pm K_\pm \ , \quad [K_+, K_-] = 2K_0 \tag{4.87}$$

with Casimir

$$\hat{C} = K_0^2 - \frac{K_+K_- + K_-K_+}{2} \tag{4.88}$$

The irreducible representations are labelled by the number k, determined from the eigenvalues $k(k-1)$ of the Casimir. For the discrete series, k takes values $1, 3/2, 2, \ldots$ and the basis vectors take the form $|k, k+m\rangle$ where m is an integer and $k+m$ is the eigenvalue of K_0.

$$K_0|k, k+m\rangle = (k+m)|k, k+m\rangle \tag{4.89}$$

and $m \in \mathbb{N}$. The action of the ladder operators on the basis vectors is given by

$$K_+|k, k+m\rangle = \sqrt{(2k+m)(m+1)}|k, k+m+1\rangle \tag{4.90}$$

$$K_-|k, k+m\rangle = \sqrt{m(2k+m-1)}|k, k+m-1\rangle \tag{4.91}$$

The Lie algebra generators have a realisation in terms of first-order differential operators

$$K_0 = z\frac{d}{dz} + k \ , \qquad K_+ = z^2\frac{d}{dz} + 2kz \ , \qquad K_- = \frac{d}{dz} \tag{4.92}$$

which act on the space of functions analytic in the unit circle, e.g. the polynomial basis

$$p_m = \sqrt{\frac{(2k+m+1)!}{(2k+1)!\,(m)!}}\, z^m \tag{4.93}$$

One important realisation of the algebra is in terms of single-mode boson operators, b and $b^\dagger$. The bilinear operators

$$K_0 = \tfrac{1}{4}(bb^\dagger + b^\dagger b) \ , \qquad K_+ = \tfrac{1}{2}(b^\dagger)^2 \ , \qquad K_- = \tfrac{1}{2}b^2 \tag{4.94}$$

satisfy the defining commutation relations and give a Casimir operator with eigenvalues $-\frac{3}{16}$. This corresponds to k-values of $\frac{1}{4}$ or $\frac{3}{4}$. The ordinary number basis for the bose oscillator provides a basis for $k = \frac{1}{4}$ realisation with n is even, and a basis for the $k = \frac{3}{4}$ realisation if n is odd. There is also a two-mode realisation which is analagous to the Jordan–Schwinger representation. This has generators

$$K_0 = \tfrac{1}{2}(b_1b_1^\dagger + b_2b_2^\dagger + 1) \ , \qquad K_+ = b_1^\dagger b_2^\dagger \ , \qquad K_- = \tfrac{1}{2}b_1b_2 \tag{4.95}$$

with Casimir

$$\hat{C} = \frac{1}{4}(b_1^\dagger b_1 - b_2^\dagger b_2)^2 - \frac{1}{4} \tag{4.96}$$

A basis for this representation is

$$|n_1, n_2\rangle = \frac{(b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}}{\sqrt{(n_1)!(n_2)!}}|0,0\rangle \tag{4.97}$$

4.6.2 The Coherent States

The unitary displacement operator for the group is

$$\begin{aligned} D(\xi) &= \exp(\xi K_+ - \bar{\xi}K_-) &\quad (4.98)\\ &= \exp(\zeta K_+)\exp(\eta K_0)\exp(-\bar{\zeta}K_-) &\quad (4.99)\end{aligned}$$

where $\zeta = \tanh|\xi|e^{i\phi}$ and $\eta = -\ln(1-|\zeta|^2)$. If we act on the lowest weight state, $|k,0\rangle$, with the displacement operator, we obtain

$$|\zeta\rangle = (1-|\zeta|^2)^k \exp(\zeta K_+)|0\rangle \tag{4.100}$$

which gives a decomposition of the coherent state over the polynomial basis

$$|\zeta\rangle = (1-|\zeta|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(m+2k)}{m!\Gamma(2k)}}\,\zeta^m |k,m\rangle \tag{4.101}$$

These coherent states obey the eigenvalue equation

$$(K_- - 2\zeta K_0 + \zeta^2 K_+)|\zeta\rangle = 0 \tag{4.102}$$

The fact that $SU(1,1)$ is the dynamical group for the harmonic oscillator means that $SU(1,1)$–coherent states techniques have found great application in situations where one is dealing with the physics of bosonic systems. These include many apects of condensed matter theory such as the study of superfluidity [Sol:71]. They have also been used to study *squeezed states* of light in quantum optics (see, e.g., [WE:85]). This latter application will be considered in detail in chapter 9. The techniques applied to the non–compact $SU(1,1)$ group can also be extended, in many cases, to other non–compact groups such as $SO(3,1)$ and $SO(n,1)$ [Per:86].

Chapter 5

Coherent states of q–analogues of semi-simple Lie groups

Given the importance of coherent state techniques to mathematical physics, it is not surprising that the analogues of coherent states for quantum groups were one of the first things to be examined. Unfortunately there is a problem with the definition of such objects. As illustrated in chapter 4, the most useful definition for extending coherent states to semisimple Lie groups was that of Perelomov (or some conceptually equivalent scheme). This was a *group–theoretical* construction based on the embedding of the group manifold in some completion of the universal enveloping algebra. The duality between the space of functions on the group and the enveloping algebra meant that the representation of the group elements was given by exponentiation of the Lie algebra generators. In the case of quantum groups, however, *there is no underlying group manifold.* Consequently there is no representation of the matrix quantum group by an analogue of exponentiation on the generators of the corresponding quantum universal enveloping algebra. This is a serious problem and effectively rules out a straightforward generalisation of the Perelomov approach.

The failure of the group–theoretical method means that less elegant procedures must be applied and the difficulties of alternative definitions, that were detailed for the undeformed case, occur. The most common approach is based on the observation that the undeformed coherent states can be formed from the ground state by the exponential action of the raising operator. If normalised, use of the Baker–Campbell–Hausdorff theorem shows that this is the same state as that produced by Perelomov's method. In the quantum group context, an obvious analogue is that state, produced by the action of the q–exponentiated raising operator. In the case of q–deformations of the Heisenberg algebra, the resulting states are also eigenstates of the deformed annihilation operator.

In the following section, we give a review of the basic results of q–coherent state theory following the approach of Jurčo [Jur:91], before going on in later chapters to see how the formalism may be expanded.

5.1 The Coherent States associated with the q–analogues of the Heisenberg–Weyl group

As we have seen, there are several different ways of deforming the Lie algebra of the Heisenberg–Weyl group, each of which leads to algebras with different properties. If we restrict ourselves to deformations of the one–mode algebra, the following deformations are the most common

- The Arik–Coon q–oscillator algebra, $\mathcal{A}_q$.
- The Macfarlane–Biedenharn q–boson algebra, $\mathcal{B}_q$.
- The quantised universal enveloping algebra of the Heisenberg quantum group, $\mathcal{H}_q$

We will consider these deformations in turn.

5.1.1 The Arik–Coon q–coherent state.

As detailed earlier, the Hilbert space H_q of the Arik–Coon oscillator is spanned by eigenstates $|n\rangle$ of the number operator N. These are generated from the vacuum $|0\rangle$ by the action of the creation operator $a^\dagger$, which obeys the commutation relation

$$aa^\dagger - qa^\dagger a = 1 \qquad q \in (0,1) \tag{5.1}$$

The above equation leads to the following

$$a(a^\dagger)^n - q^n(a^\dagger)^n a = [n]_q (a^\dagger)^{n-1} \tag{5.2}$$

where $[m]_q = \frac{q^m - 1}{q-1}$ is the basic number of classical q–analysis. By acting with (5.2) on the vacuum Arik and Coon [ArC:76] obtained

$$a(a^\dagger)^n|0\rangle = [n]_q (a^\dagger)^{n-1}|0\rangle \tag{5.3}$$

which implies that

$$aE_q(\alpha a^\dagger)|0\rangle = \alpha E_q(\alpha a^\dagger)|0\rangle \tag{5.4}$$

where $E_q(x)$ is the Jackson q–exponential function [Ext:83] (see appendix) and $\alpha \in \mathbb{C}$.

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{5.5}$$

For $q \in (0,1)$, this converges provided $|x| < (1-q)^{-1/2}$.

It is clear that if we define a coherent state of the algebra $\mathcal{A}_q$ by

$$|\alpha\rangle = A(\alpha;q)E_q(\alpha a^\dagger)|0\rangle \tag{5.6}$$

where $A(\alpha;q)$ is a normalisation constant given by

$$A(\alpha;q)^{-2} = E_q(|\alpha|^2) \tag{5.7}$$

then $|\alpha\rangle$ will be a (normalised) eigenstate of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{5.8}$$

The overlap integral between two such states can be calculated

$$\langle\beta|\alpha\rangle = E_q(|\beta|^2)^{-1/2} E_q(\bar{\beta}\alpha) E_q(|\alpha|^2)^{-1/2} \tag{5.9}$$

which reduces to the conventional result for $q = 1$.

A q–analogue of the Bargmann representation [Bar:61] of the Fock space can be provided. For every vector $|\phi\rangle = \sum \phi_n |n\rangle$, there corresponds an analytic function $\phi(z)$ in the region $|z|^2 < [\infty]_q \equiv (1-q)^{-1}$ given by

$$\phi(z) = \langle \bar{z}|\phi\rangle = \sum_{n=0}^{\infty} \phi_n z^n \tag{5.10}$$

Then the creation and annihilation operators have the following realisation

$$\langle \bar{z}|a^\dagger|\phi\rangle = z\,\phi(z); \qquad \langle \bar{z}|a|\phi\rangle = {}_qD_z\phi(z) \tag{5.11}$$

where ${}_qD_z$ is the q–analogue of the derivative operator, i.e.

$${}_qD_z\phi(z) = \frac{\phi(qz) - \phi(z)}{(q-1)z} \tag{5.12}$$

The analogue of the indefinite integral, i.e. the inverse of the q–derivative, was discovered by Jackson [Jac:51, Ext:83]. It can be derived operationally by formally inverting the operator ${}_qD_z$

$$\begin{aligned} ({}_qD_z)^{-1} &= (1-\hat{T}_q)^{-1}(1-q)z & (5.13)\\ &= (1-q)\sum_{k=0}^{\infty} \hat{T}_q{}^k z & (5.14)\end{aligned}$$

where $\hat{T}_q$ is the q–dilation operator. The *Jackson q–integral* can therefore be defined as

$$\int_0^b \phi(z)\, d_q z \equiv (1-q)\, b \sum_{k=0}^{\infty} q^k \phi(q^k b) \tag{5.15}$$

which goes to the Riemann integral of $\phi(z)$ in the $q = 1$ limit.

Using this, it is possible to define a scalar product. First the measure $d\mu(z)$ is given by

$$d\mu(z) = \frac{d_q^2(z)}{E_q(q|z|^2)} \tag{5.16}$$

where for $z = |z|e^{i\theta}$, $d_q^2 z$ indicates a conventional integration over the θ–variable and q–integration over $|z|^2$. The scalar product is then given by

$$\langle\psi|\phi\rangle \doteq \frac{1}{\pi}\int \overline{\psi(z)}\phi(z)d\mu(z) \tag{5.17}$$

The multiplication operator $\hat{z}$ and q–derivative operator ${}_qD_z$ are hermitian conjugate under this scalar product, i.e.

$$\int d\mu(z)\,\overline{\psi(z)}\,{}_qD_z\phi(z) = \int d\mu(z)\,\overline{\{\hat{z}\,\psi(z)\}}\phi(z) \tag{5.18}$$

There is also a q–analogue of the usual overcompleteness relation for conventional bosonic coherent states. A result of classical q–analysis is the analogue of Euler's integral representation of the Gamma function. In terms of the Jackson q–exponential function and the integral, this can be stated as

$$\int_0^{[\infty]} z^n\, d\mu(z) = [n]! \tag{5.19}$$

If $|\alpha\rangle$ is the coherent state defined in (5.6), then the overcompleteness relation can be stated as

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, d\mu(\alpha) = I \tag{5.20}$$

5.1.2 The Macfarlane–Biedenharn q–boson coherent state

If we now consider the algebra $\mathcal{B}_q$, the Macfarlane–Biedenharn q–deformed boson algebra, we find that we may form coherent states in a manner formally similar to that for the algebra $\mathcal{A}_q$ [Bie:89]. Given the commutation relation

$$a_q a_q^\dagger - q^{\frac{1}{2}} a_q^\dagger a_q = q^{-N/2} \tag{5.21}$$

the eigenstate of the annihilation operator $|\beta\rangle$ is given by

$$|\beta\rangle = \exp_q(|\beta|^2)^{-1}\exp_q(\beta a_q^\dagger)|0\rangle \tag{5.22}$$

where $\exp_q(x)$ is the *symmetric* q–exponential function

$$\exp_q(x) = \sum_{n=0}^{\infty}\frac{x^n}{[\![n]\!]_q!} \tag{5.23}$$

$$= \sum_{n=0}^{\infty} q^{n(n-1)/4}\frac{x^n}{[n]_q!} \tag{5.24}$$

The essential step in demonstrating an overcompleteness relation for these q–analogue coherent states is the proof of an analogue of Euler's integral formula for $\Gamma(x)$. In the case of the q–bosons of the algebra $\mathcal{A}_q$, a result from classical q–analysis gives the required formula. The properties of the function $\exp_q(x)$, however, are non–trivially different from the Jackson q–exponential $E_q(x)$ [NG:94]. Moreover, the definition of the q–integral and the measure used in the integration are not the same.

The integral which is the inverse of the symmetric q–derivative can be found by the same methods as the Jackson q–integral and is defined on the interval $[0, a]$ by

$$\int_0^a f(x) d_q x = a\,(q^{-1/2} - q^{1/2}) \sum_{k=0}^{\infty} q^{(2n+1)/2} f(q^{(2n+1)/2} a) \tag{5.25}$$

and on the interval $[0, \infty]$ by

$$\int_0^\infty f(x) d_q x = (q^{-1/2} - q^{1/2}) \sum_{k=-\infty}^{\infty} q^{(2n+1)/2} f(q^{(2n+1)/2}) \tag{5.26}$$

With such formulae, it is possible to give an integral representation of the symmetric q–Gamma function, $\Gamma_q(x)$

$$\int_0^\zeta \exp_q(-x)\, x^n d_q x = [\![n]\!]! \tag{5.27}$$

where $-\zeta$ is the largest zero of $\exp_q(x)$. It is known that $\zeta > 0$ and as $q \to 1$, $\exp_q(x) \to \exp(x)$ and $\zeta \to \infty$.

A resolution of the identity for the coherent states can therefore be given as

$$I = \int |\beta\rangle \langle\beta|\, d\mu\,(\beta) \tag{5.28}$$

where, for $\beta = |\beta| e^{i\theta}$,

$$d\mu(\beta) = \frac{1}{2\pi} \exp_q(|\beta|^2) \exp_q(-|\beta|^2) d_q |\beta|^2 \, d\theta \tag{5.29}$$

the integral over the $|\beta|$ variable being a q–integration and the integral over the θ variable being a conventional one.

Just as in the conventional case, a (q–analogue) Bargmann representation of the Fock space is possible. Vectors $|\psi\rangle$ in the Fock space are mapped into entire functions $\psi(z)$ by

$$\begin{aligned} \psi(z) &= \langle z|\psi\rangle & (5.30) \\ &= \sum_{n=0}^{\infty} \frac{\langle n|\psi\rangle}{\sqrt{[\![n]\!]!}} z^n & (5.31) \end{aligned}$$

where $|z\rangle$ is the unnormalised coherent state

$$|z\rangle = \sum_{0}^{\infty} \frac{z^n}{\sqrt{[\![n]\!]!}} |n\rangle \tag{5.32}$$

and the entire function $\psi(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfies

$$\sum_{n=0}^{\infty} [\![n]\!]! |c_n|^2 < \infty \tag{5.33}$$

If a function $\psi(z)$ satisfies the convergence criterion (5.33), then it corresponds to a vector

$$|\psi\rangle = \sum_{n=0}^{\infty} \sqrt{[\![n]\!]!} c_n |n\rangle \tag{5.34}$$

The scalar product of the function represented by this vector with another entire function $\phi(z) = \sum_{n=0}^{\infty} d_n z^n$ is given by

$$(\phi, \psi) = \langle\phi|\psi\rangle = \sum_{n=0}^{\infty} [\![n]\!]!\, d_n^* c_n \tag{5.35}$$

In terms of the usual integral representation, this scalar product is

$$\langle\phi|\psi\rangle = \int \overline{\phi(z)} \psi(z) \exp_q(-|z|^2)\, d_q^2 z \tag{5.36}$$

5.1.3 Coherent states of the Quantum Heisenberg Universal Enveloping Algebra

We recall that $\mathcal{H}_q$, the (one–mode) quantum group version of the Heisenberg–Weyl algebra [CGST:90, CGST:91, GS:93, BM:93], has deformed commutation relations

$$\begin{aligned} [E_-, E_+] &= \frac{\sinh(\omega H/2)}{\omega/2} & (5.37) \\ [N, E_\pm] &= \pm E_\pm & (5.38) \\ [E_\pm, H] &= [N, H] = 0 & (5.39) \end{aligned}$$

By a simple rescaling of the generators $E_{\pm}$ we may rewrite (5.37) as

$$[E_-, E_+] = \frac{\sinh(\omega H/2)}{\sinh(\omega/2)} = [\![H]\!]_q \tag{5.40}$$

where $q = \exp(\omega)$.

$\mathcal{H}_q$ has a Hilbert space representation given by

$$\begin{aligned} E_-|n\rangle &= [\![h]\!]^{\frac{1}{2}}\sqrt{n}|n-1\rangle & (5.41)\\ E_+|n\rangle &= [\![h]\!]^{\frac{1}{2}}\sqrt{n+1}|n+1\rangle & (5.42)\\ H|n\rangle &= h|n\rangle & (5.43)\\ N|n\rangle &= (k'+n)|n\rangle & (5.44) \end{aligned}$$

where $|n\rangle$, $n = 0, 1, 2, \ldots$ indicates an orthonormal basis,

$$E_-|0\rangle = 0 \tag{5.45}$$

and the analogue of number states are therefore

$$|n\rangle = \frac{(E_+)^n}{\sqrt{[\![h]\!]^n\, n!}}|0\rangle \tag{5.46}$$

The irreducible representations are labelled by the eigenvalues of the two Casimirs, H^2 and $C = [\![H]\!]\, N - E_+E_-$, (i.e h^2 and $[\![h]\!]\, k'$ respectively). We consider here only the $(k' = 0)$–representation in which $\langle n|N|n\rangle = 0$.

We note that the commutator (5.40) is actually a quantum group version of the conventional Heisenberg–Weyl commutator in which the Planck constant is explicitly stated, i.e. it corresponds in the undeformed case to

$$[E_-, E_+] = \hbar I \tag{5.47}$$

However, H is central and simply acts on the number states to give the eigenvalue h. Therefore, if we consider the situation only at the universal enveloping algebra level (as opposed to the Hopf algebraic level), (5.40) can be written

$$[E_-, E_+] = hI \tag{5.48}$$

which implies that the algebra $\mathcal{H}_q$ and $\mathcal{H}(4)$ are isomorphic and therefore that their coherent states have the same properties. Since this work is only concerned with one–mode coherent states, we will not proceed further with this algebra. However, we note that it has been shown that the quantum group $\mathcal{H}_q$ and its braided version [Maj:93, BM:93] are isomorphic and have been used to investigate the time evolution of a multiparticle Fock–space, covariant under the action of the braided group.

5.2 Coherent States of the $su_q(2)$ and $su_q(1,1)$ Algebras

5.2.1 The $su_q(2)$ coherent state

We recall that the quantised universal enveloping algebra, $su_q(2)$, is generated by the elements $\{J_0, J_\pm\}$ subject to the relations

$$[J_0, \pm J_\pm] = \pm J_\pm \tag{5.49}$$

$$[J_+, J_-] = [\![2J_0]\!] \equiv \frac{q^{J_0} - q^{-J_0}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{5.50}$$

where for convenience, we now drop the subscript on the box–functions. These generators act on the $(2j+1)$–dimensional Hilbert space as

$$J_0|j,m\rangle = m|j,m\rangle \tag{5.51}$$

$$J_+|j,m\rangle = \sqrt{[\![j-m]\!]\,[\![j+m+1]\!]}|j,m+1\rangle \tag{5.52}$$

$$J_-|j,m\rangle = \sqrt{[\![j-m+1]\!]\,[\![j+m]\!]}|j,m-1\rangle \tag{5.53}$$

with

$$J_-|j,-j\rangle = 0 \tag{5.54}$$

One candidate for the unnormalised $su_q(2)$–coherent state is therefore

$$|z\rangle = \exp_q(zJ_+)|j,-j\rangle \tag{5.55}$$

$$= \sum_{m=-j}^{m=j} \left(\frac{[\![2j]\!]!}{[\![j-m]\!]!\,[\![j+m]\!]!} \right)^{\frac{1}{2}} z^{j+m}|j,m\rangle \tag{5.56}$$

The resolution of unity is given by

$$I = [\![2j+1]\!] \int |z\rangle\langle z|\, d\mu(z) \tag{5.57}$$

where

$$d\mu(z) = \frac{d_q^2}{B\,[\![1, |z^2|; (2j+2)]\!]} \tag{5.58}$$

and the function $B\,[\![a, b; n]\!]$ for a, $b \in \mathbb{C}$, $n \in \mathbb{N}$, is given by

$$B\,[\![a, b; n]\!] = \sum_{k=0}^{n} \left[\!\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\!\right] a^{n-k} b^k \tag{5.59}$$

$$\equiv \sum_{k=0}^{n} \frac{[\![n]\!]!}{[\![k]\!]!\,[\![n-k]\!]!} a^{n-k} b^k \tag{5.60}$$

Note that as $q \to 1$, $B\,[\![a, b; n]\!] \to (a+b)^n$.

The generators have a differential realisation in terms of the symmetric q–derivative ${}_qS_z$ and q–dilation operator $\hat{T}$

$$J_- = {}_qD_z\,, \qquad J_+ = -q^j z^2\, {}_qD_z + [\![2j]\!]\, z\hat{T}_{q^{-1/2}}\,, \qquad J_0 = z\frac{d}{dz} - j \tag{5.61}$$

and the basis vectors in the representation space are polynomials of the form

$$f_m = \sqrt{\frac{[\![2j]\!]!}{[\![j-m]\!]!\,[\![j+m]\!]!}}\, z^{j+m} \tag{5.62}$$

5.2.2 The $su_q(1,1)$ coherent state

If we consider the q–analogue of $su(1,1)$, we see that a similar procedure can be used to generate its coherent states. The quantum group $su_q(1,1)$ has generators $\{K_0, K_\pm\}$ with commutation relations

$$[K_0, \pm K_\pm] = \pm K_\pm \tag{5.63}$$

$$[K_+, K_-] = [\![-2K_0]\!] \equiv -\frac{q^{K_0} - q^{-K_0}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{5.64}$$

We consider only the discrete series representation and we use the standard basis $|k, k+m\rangle$ on which the action of the generators takes the form

$$K_0|k, k+m\rangle = (k+m)|k, k+m\rangle \tag{5.65}$$

$$K_+|k, k+m\rangle = \sqrt{[\![2k+m]\!]\,[\![m+1]\!]}\,|k, k+m+1\rangle \tag{5.66}$$

$$K_-|k, k+m\rangle = \sqrt{[\![2k+m-1]\!]\,[\![m]\!]}\,|k, k+m-1\rangle \tag{5.67}$$

Defining the unnormalised coherent state $|\xi\rangle$ by

$$|\xi\rangle = \exp_q(\xi K_+)|k,0\rangle \tag{5.68}$$

we obtain a resolution of the state in terms of the standard basis

$$|\xi\rangle = \sum_{m=0}^{\infty} \sqrt{\frac{[\![2k+m+1]\!]!}{[\![m]\!]!\,[\![2k+1]\!]!}}\,\xi^m |k,k+m\rangle \ . \tag{5.69}$$

where $\xi \in D^k = \{|\xi|^2 < q^{k-1}\}$. The normalisation factor, $A(\xi)$, for the state is given by

$$A(\xi) = \left\{ \sum_{m=0}^{\infty} \frac{[\![2k+m+1]\!]!}{[\![m]\!]!\,[\![2k+1]\!]!} |\xi|^{2m} \right\}^{-\frac{1}{2}} \tag{5.70}$$

The resolution of unity can be expressed as

$$[\![2k-1]\!] \int_{D^k} |\xi\rangle\langle\xi| d\mu(\xi) = I \tag{5.71}$$

where the measure is given by

$$d\mu(\xi) = \frac{d_q^2\xi}{B\,[\![1,-|\xi|^2;k]\!]} \tag{5.72}$$

The states obey the eigenvalue equation

$$(K_- + (q^k + q^{-k})\xi\,[\![K_0]\!] + \xi^2 K_+)|\xi\rangle = 0 \tag{5.73}$$

5.2.3 Applications of $su_q(2)$ and $su_q(1,1)$ coherent states

The widespread use of conventional $SU(2)$ and $SU(1,1)$ coherent states has meant that their q–analogues have been used increasingly to study deformed physical models. It would be difficult to give an exhaustive list of all the applications of q–coherent states, but the following indicates the breadth of their use in different areas of q–physics.

Firstly, the influence of deformation on the phase transition from the vibrational to the rotational regime has been studied in the deformed versions of the Lipkin model [AB:93, AMMD:94], the $su(2) \otimes su(2)$ Moszkowski model [MAP:92, BBM:93] and the Pairing model [AvM:93]. In addition to this, $su_q(2)$–coherent states were used in the variational method to investigate the q–deformed Thouless model of the strong–coupling limit of BCS–superconductivity [BAMP:94].

Another intriguing aspect of the study of q–deformed coherent states is their use in examining the geometric effects of deformation on the phase spaces of systems with deformed Heisenberg–Weyl, $su_q(2)$ or $su_q(1,1)$ dynamical symmetries. Ellinas [Ell:93a] used the coherent states of the algebras mentioned above to construct the propagator for the path–integral method and so calculate the symplectic 2–forms and hence the Kähler potentials and metrics of the spaces. He was able to show that one result of q–deformation was the introduction of an effective non–constant curvature.

As a further example of work involving quantum group coherent states, we mention their use in developing q–analogues of the squeezed states of quantum optics. The interaction of such states with two–level atoms in a deformed version of the Jaynes–Cummings model has been investigated by several authors [CEK:90, Buz:91, CJ:92b]. The construction of q–deformed squeezed states will be a major part of this thesis and so such considerations will be deferred until chapter 9.

5.3 Appendix: Properties of the Jackson q–Exponential

One of the most important classes of functions to emerge from the subject of q–deformations is the q–deformed exponential function. Given the variety of ways in which the q–derivative can be deformed, it is not surprising that there are various different deformations corresponding to their eigenfunctions. The best reference for the properties of the Jackson q–exponentials which are associated with the classical Jackson q–derivative (and hence with the Arik–Coon q–boson and the algebra $\mathcal{A}_q$) is Exton [Ext:83], whereas the properties of the q–exponential associated with the symmetric q–derivative (and so with the algebra $\mathcal{B}_q$) have been investigated by a number of authors including Gray and Nelson [GN:90], Nelson and Gartley [NG:94] and Morozov and Vinet [MV:94] . For our purposes, we will mainly use the Jackson q–exponentials and so this section will give a brief summary of the properties needed in later chapters.

The q–exponential $E_q(x)$ is defined to be the eigenfunction of the Jackson q–derivative operator ${}_qD_x$, i.e. it is the unique function satisfying

$$E_q(\lambda x) = \lambda \ {}_qD_x \, E_q(\lambda x) \tag{5.74}$$

subject to the condition that

$$E_q(0) = 1 \tag{5.75}$$

Explicitly, it is given by

$$E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{5.76}$$

where $[n]_q \equiv \frac{q^n-1}{q-1}$ is the basic integer of classical q–analysis. If $|q| < 1$, then the ratio test shows that the series converges absolutely and uniformly with respect to x if $|x| < [\infty] \equiv (1-q)^{-1}$ and diverges if $|x| > (1-q)^{-1}$. If $q > 1$, then convergence is ensured for all finite values of x.

Since $[n]_q \to n$ as $q \to 1$, the q–exponential function clearly goes over to the conventional exponential in this limit. However, the fact that $E_q(x)$ is its own q–derivative, i.e.

$$\frac{E_q(qx) - E_q(x)}{(q-1)x} = E_q(x) \tag{5.77}$$

means that

$$\frac{E_q\left(\frac{qx}{1-q}\right)}{E_q\left(\frac{x}{1-q}\right)} = (1-x) \tag{5.78}$$

Repeated application of this formula gives

$$\frac{E_q\left(\frac{q^n x}{1-q}\right)}{E_q\left(\frac{x}{1-q}\right)} = (1-x)(1-qx)\dots(1-q^{n-1}x) \tag{5.79}$$

from which a further $q \to 1$ limit can be obtained

$$\lim_{q \to 1} \frac{E_q\left(\frac{q^n x}{1-q}\right)}{E_q\left(\frac{x}{1-q}\right)} = (1-x)^n \tag{5.80}$$

As well as the series representation of $E_q(x)$, there is an infinite product representation for $|q| > 1$

$$E_q(x) = \prod_{k=0}^{\infty} \left\{1 + x\,(1 - q^{-1}) q^{-k}\right\} \tag{5.81}$$

If the base of the deformed exponential is changed from q to q^{-1}, a second q–exponential is obtained, denoted by $E_{q^{-1}}(x)$. This has a series representation

$$E_{q^{-1}}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q^{-1}}!} \tag{5.82}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} q^{n(n-1)/2} \tag{5.83}$$

This function is no longer an eigenfunction of the q–derivative operator ${}_qD_x$ but instead satisfies

$$ {}_qD_x E_{q^{-1}}(\lambda x) = \lambda E_{q^{-1}}(q\lambda x) \tag{5.84}$$

For $|q| < 1$, there is an infinite product representation corresponding to (5.81)

$$E_{q^{-1}}(x) = \prod_{k=0}^{\infty} \left\{1 + x\,(1 - q) q^{k}\right\} \tag{5.85}$$

The conventional exponential function has the useful property that its reciprocal can be expressed in terms of the same function, i.e. $\exp(x)\exp(-x) = 1$. For the q–exponential, this is not true. Nevertheless, it is fairly easy to show that

$$E_q(x) E_{q^{-1}}(-x) = 1 \tag{5.86}$$

where x is a complex number. This formula is a special case of a more general expression. If x and y are complex numbers, then

$$E_q(x) E_{q^{-1}}(y) \equiv E_q(x; y) \tag{5.87}$$

$$= \sum_{n=0}^{\infty} \frac{(x; y)^{(n)}}{[n]_q!} \tag{5.88}$$

where the function $(x;y)^{(n)}$ is defined by

$$(x;y)^{(n)} = (x+y)(x+qy)(x+q^2y)\dots(x+q^{n-1}y) \tag{5.89}$$

$$= \sum_{k=0}^{n} \frac{[n]_q!}{[n-k]_q![k]_q!} x^{n-k}y^k q^{k(k-1)/2} \tag{5.90}$$

The most useful property of the conventional exponential function is the additivity of exponents upon multiplication. This property does not have a close analogue for products where the arguments of the q–exponential are c–numbers. However, if we consider the product of two q–exponentials with noncommuting arguments (e.g. operators or noncommuting variables), the following important properties can be verified.

- The *q–Addition Property* [Schu:53, Cig:79]

$$E_q(Y)E_q(X) = E_q(X+Y) \qquad \text{if } XY = qYX \tag{5.91}$$

- The *Faddeev–Volkov Identity* [FV:93]

$$E_q(X)E_q(Y) = E_q(X+Y+[X,Y]) \quad \text{if } XY = qYX \tag{5.92}$$

$$= E_q(Y)E_q([X,Y])E_q(X) \tag{5.93}$$

One consequence of these relations is the following reordering formula

$$E_q(Y)E_q(X) = E_q(X)E_q\left(\frac{1}{1+(1-q^{-2})X}Y\right) \qquad \text{if } XY = qYX \tag{5.94}$$

In the conventional $q = 1$ case, if two operators do not commute, the product of their exponentials is given by the exponential of a series of terms involving nested commutators of the operators in question. The expansion of this is computed using the Baker–Campbell–Hausdorff theorem. Unfortunately no underlying BCH–theorem for the q–exponential is known although expansions to low orders have been computed. Indeed, it has been shown that even if the q–commutator vanishes at one particular order, say n, this does not guarantee that the q–commutators vanish at orders greater than n [KS:91b, KS:94b].

Chapter 6

Generalisations of the q–boson algebra

6.1 The Deformed Boson Algebra

Given the similarities between the deformed boson algebras described in chapter 5, it is natural to try to abstract the important features and so generalise the structure to include more complicated examples. This has been done independently by several authors [OKK:91, Das:91, BeD:91, Jan:93, MS:94a, MMP:94]. The deformed boson algebras considered in chapter 3 are all generated by a deformed creation operator a^+, deformed annihilation operator a and a number operator N which satisfies

$$[N, a^-] = -a^-, \quad [N, a^+] = a^+ \tag{6.1}$$

These operators act in a Fock space of number states which are formed by repeated application of the creation operator, the lowest weight state being destroyed by the annihilation operator.

The most general reordering relation between a^- and a^+ which is linear in the products a^-a^+ and a^+a^- is

$$a^-a^+ - f(N)\, a^+a^- = g(N) \tag{6.2}$$

where $f(N)$ and $g(N)$ are functions of N. Since (6.1) implies that the products a^-a^+ and a^+a^- commute with N, they can be written as

$$a^+a^- = \Phi(N), \qquad a^-a^+ = \Phi(N+1) \tag{6.3}$$

Equation (6.2) can be rewritten in a number of equivalent forms depending on where the N–dependence is placed. For example, by making a simple substitution of the form

$$a^- \to a^{-\prime} = a^- \phi(N), \quad a^+ \to \phi(N)\, a^+ = a^{+\prime} \quad N \to N' = N \tag{6.4}$$

the reordering equation (6.2) can be made equivalent to

$$a^{-\prime}a^{+\prime} - F(N)\, a^{+\prime}a^{-\prime} = I \tag{6.5}$$

while changing the function $\phi(N)$, one is also able to bring it into the form

$$a^{-\prime\prime}a^{+\prime\prime} - \; a^{+\prime\prime}a^{-\prime\prime} = G(N) \tag{6.6}$$

Equations (6.2), (6.5) and (6.6) account for most generalisations of the q–boson algebra.

The creation and annihilation operators are usually assumed to be hermitian conjugate to each other (which imposes reality conditions on the functions of the hermitian number operator found in the reordering equations) but there have been schemes where this is not so, in which case they are linked by

$$a^+ = c(N)\, a^\dagger \tag{6.7}$$

The generalisation which has the most extensive formulation (see [BD:93a, BD:93b, BDL:93, BDK:93] and references therein) is that of Daskaloyannis and Bonatsos and we will use their notation in what follows, with some slight modifications.

6.2 The Daskaloyannis–Bonatsos Deformation Scheme

The D–B deformed oscillator algebra $\mathcal{A}_D$ is generated by the operators $\{a, a^\dagger, N\}$ and a positive analytic *structure function* $\Phi(x)$ satisfying the relations

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger \tag{6.8}$$

and

$$a^\dagger a = \Phi(N), \qquad aa^\dagger = \Phi(N+1) \tag{6.9}$$

If $\Phi(0) = 0$ and $\Phi(m) > 0$ for all $m \in \mathbb{N}$, it is possible to use Φ to define deformed analogues, $[m]$, of the positive integers by

$$[m] = \Phi(m) \tag{6.10}$$

The commutation relation for a and $a^\dagger$ are written in terms of the structure function

$$[a, a^\dagger] = [N+1] - [N] \tag{6.11}$$

In addition to the structure function Φ, it is useful to define the *deformation function*, f, given by

$$f(x) = \frac{[x+1]-1}{[x]} \tag{6.12}$$

The function f measures the degree of deformation of the oscillator system, e.g. when $f \equiv 1$, then $\Phi(x) \equiv x$, and the oscillator reduces to the conventional bosonic case.

We can build up a Fock space basis of eigenstates of the number operator in the usual way by repeated application of the creation operator on the lowest weight vacuum state, $|0\rangle$,

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle \tag{6.13}$$

where we define the generalised factorial to be

$$[n]! := \prod_{k=1}^{n} [k] \tag{6.14}$$

$$[0]! \equiv 1 \tag{6.15}$$

Since $[n+1] = 1 + f(n)[n]$, we see that

$$[n] = 1 + f(n-1) + f(n-1)f(n-2) + \cdots + f(n-1)! \tag{6.16}$$

$$= \sum_{k=0}^{n-1} \frac{f(n-1)!}{f(k)!} \tag{6.17}$$

In terms of the deformation function, the reordering relation (6.11) reads

$$aa^\dagger - f(N)a^\dagger a = I \tag{6.18}$$

The action of the generators of the algebra $\mathcal{A}_D$ on the Fock space basis is

$$\begin{array}{rcrl} N|n\rangle & = & n & |n\rangle \\ a|n\rangle & = & \sqrt{[n]} & |n-1\rangle \\ a^\dagger|n\rangle & = & \sqrt{[n+1]} & |n+1\rangle \, . \end{array} \tag{6.19}$$

where $a|0\rangle = 0$ and $\langle m|n\rangle = \delta_{mn}$.

The deformation scheme of Daskaloyannis and Bonatsos manages to accommodate, in an elegant framework, the previous attempts to unify the deformed boson algebras such as those of Beckers and Debergh [BeD:91], Odaka, Kishi and Kamefuchi [OKK:91], Brodimas et al [BJ:92]. It also allows the consideration of other deformations of the commutation relations such as the trilinear relations of the parafermions and parabosons. For example, the unification scheme of Beckers and Debergh has as its main reordering relation

$$aa^\dagger + g(q)\, a^\dagger a = |[N+1]| + g(q)|[N]| \tag{6.20}$$

which is equivalent to the formalism given above with the structure function

$$\Phi(x) = |[x]| \tag{6.21}$$

while the Odaka–Kishi–Kamefuchi scheme uses the relations

$$[a, N] = a \quad , \qquad [N, a^\dagger] = a^\dagger \tag{6.22}$$

$$[a^\dagger, a]_\alpha = a^\dagger a + \alpha a a^\dagger = G(N) \tag{6.23}$$

where

$$N|n\rangle = (n + n_0)|n\rangle \tag{6.24}$$

It can be shown [BD:93b] that

$$\Phi(x) = \sum_{m=0}^{n} (-1)^m \alpha^{-(m+1)} G_{(n-m)} \tag{6.25}$$

where $G_n = G(n+n_0)$.

If we have two sets of boson operators $\{a_1, a_1^\dagger, N\}$ and $\{a_2, a_2^\dagger, N\}$, described by structure functions Φ_1 and Φ_2 (and so with spectra $[n]_1$ and $[n]_2$), one can define a transformation between them

$$a_2 = a_1\sqrt{\frac{[N]_2}{[N]_1}}\,, \qquad a_2^\dagger = \sqrt{\frac{[N]_2}{[N]_1}}a_1^\dagger \tag{6.26}$$

In particular, if the second set is that of the conventional bose oscillator for which $[n] = n$, we see that (6.26) gives an expression for the deformed oscillator in terms of ordinary Heisenberg–Weyl bosons, $\{b, b^\dagger\}$

$$a = b\sqrt{\frac{[N]}{N}}\,, \qquad a^\dagger = \sqrt{\frac{[N]}{N}}\,b^\dagger \tag{6.27}$$

The above equation has the advantage that the a and $a^\dagger$ are hermitian conjugates but there are also other non–unitary realisations of the conventional boson algebra. For example, T' and T^+ defined by

$$T' = a\,, \qquad T^+ = \frac{N}{[N]}a^\dagger \tag{6.28}$$

satisfy $[T', T^+] = I$. Since a has a realisation as the q–derivative, in which representation the Number operator N is given by the *1st order Euler operator* $(x\frac{d}{dx})$, equation (6.28) gives a realisation of the operator which is canonically conjugate to the q–derivative. This is the operator $\hat{\mathcal{X}}_x$ given by

$$\hat{\mathcal{X}}_x = \frac{(x\frac{d}{dx})}{[x\frac{d}{dx}]}\,x \tag{6.29}$$

where $\frac{(x\frac{d}{dx})}{[x\frac{d}{dx}]}$ is to be understood operationally as the operator with action

$$\frac{(x\frac{d}{dx})}{[x\frac{d}{dx}]}\,x^n = \frac{n}{[n]}\,x^n \tag{6.30}$$

Then

$$[\,{}_qD_x, \hat{\mathcal{X}}_x] = I \tag{6.31}$$

This mapping (6.27) of deformed bose–operators into conventional bose–operators means that we use such boson techniques as the Jordan–Schwinger mapping and its non–compact variants to realise representations of the classical Lie algebras in terms of arbitrarily deformed bosons, e.g. a representation of the generators of $su(1,1)$ is given by

$$K_0 = \frac{1}{4}\left(N + \frac{1}{2}\right) \tag{6.32}$$

$$K_+ = \frac{1}{2}\left\{\sqrt{\frac{N}{[N]}}\,a^\dagger\right\}^2 \tag{6.33}$$

$$K_- = (K_+)^\dagger \tag{6.34}$$

The mapping (6.26) also generalises the transformation (3.54) between the algebras $\mathcal{A}_q$ and $\mathcal{B}_q$. For the first algebra,

$$\Phi_{\mathcal{A}_q}(n) = [n] = \frac{q^n - 1}{q - 1} \tag{6.35}$$

whereas for the second

$$\Phi_{\mathcal{B}_q}(n) = [\![n]\!] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \tag{6.36}$$

Hence the ratio

$$\sqrt{\frac{[N]}{[\![N]\!]}} = q^{(N-1)/4} \tag{6.37}$$

which is essentially the same function that appears in (3.54).

6.3 The Deformed Derivative Operator

If we consider a positive analytic structure function $\Phi(x) = [x]$, we can define an operator D_x such that

$$D_x = \frac{1}{x}\left[x\frac{d}{dx}\right] \tag{6.38}$$

This acts on monomials, x^n, as a generalised derivative operator,

$$D_x\, x^n = [n]\, x^{n-1} \tag{6.39}$$

Clearly, in the limit $f(x) \to 1$, $\Phi(x) \to x$, we have $D_x \to \frac{d}{dx}$.

This procedure considers the form of the derivative as arising from that of the structure function. We can also go in the reverse direction and consider the properties of the derivative as determining the structure function. Given a deformed derivative D_z, we can ask that it obey a deformed Leibniz rule such as

$$D_z\{F(z)G(z)\} = \{D_zF(z)\}\,\hat{Q}G(z) + \hat{P}F(z)\,\{D_zG(z)\} \tag{6.40}$$

where $\hat{Q}$ and $\hat{P}$ are operators with the property

$$\hat{Q}(xy) = \hat{Q}(x)\,\hat{Q}(y) \tag{6.41}$$

The form such operators take will also determine the properties of the derivative which in turn will determine the properties of the structure function. Given the commutivity of the functions, a second equation is implied:

$$D_z\{F(z)G(z)\} = \hat{Q}F(z)\{D_zG(z)\} + \{D_zF(z)\}\hat{P}G(z) \tag{6.42}$$

If we assume that $D_z z = 1$ then we can write the operational form of the generalised derivative as

$$D_z = \frac{1}{(\hat{Q} - \hat{P})\,z}\left(\hat{Q} - \hat{P}\right) \tag{6.43}$$

For example, if one of the operators, $\hat{P}$ say, is the identity operator and the other takes the form of a displacement operator

$$\hat{Q} = \exp\left(h\frac{d}{dz}\right) \tag{6.44}$$

so that

$$\hat{Q}(z) = z + h \tag{6.45}$$

then the deformed derivative is the finite difference operator, Δ

$$\Delta f(z) = \frac{f(z+h) - f(z)}{h} \tag{6.46}$$

while if $\hat{Q}$ takes the form of the dilation operator $q^{z\frac{d}{dz}}$, we obtain the Jackson q–derivative. Alternatively, if instead of $\hat{P}$ being the identity in the previous two examples, it is given by $\hat{Q}^{-1}$, we then obtain the symmetric finite difference operator and symmetric q–derivative respectively. It is interesting that these generalisations of the derivative, based as they are on the action of two fundamental arithmetic operations of addition and multiplication, have led to the most widespread extensions of deformed calculus, namely finite difference methods in the first case and the q–calculus in the second. More general affine transformations have been tried [Keh:93] in the context of a more generally deformed Leibniz rule and are linked to the Stochastic calculus [DiM:93].

If we are to use the deformed derivative to give a differential realisation of the annihilation operator, the fundamental commutation relation for the deformed boson should not be changed by hermitian conjugation. This implies that the operator $\hat{R} = \hat{Q} - \hat{P}$ should be some operator–valued function of the Euler operator, $z\frac{d}{dz}$. In this case, if $\hat{R} = \sum_0^\infty r_n \left(z\frac{d}{dz}\right)^n$,

$$\frac{1}{z}\left[z\frac{d}{dz}\right] = \frac{1}{\hat{R}\,z}\hat{R} = \frac{1}{(\sum_{n=0}^\infty r_n)}\sum_{n=0}^\infty r_n \left(z\frac{d}{dz}\right)^n \tag{6.47}$$

Clearly

$$\left(z\frac{d}{dz}\right)^n z^k = k^n z^k \tag{6.48}$$

so

$$D_z z^k = \frac{1}{z}\left[z\frac{d}{dz}\right] z^k \tag{6.49}$$

$$= \frac{1}{z}\frac{\sum_{n=0}^\infty r_n \left(z\frac{d}{dz}\right)^n}{\sum_{n=0}^\infty r_n} z^k \tag{6.50}$$

$$= \frac{\sum_{n=0}^\infty r_n k^n}{\sum_{n=0}^\infty r_n} z^{k-1} \tag{6.51}$$

which means that the deformed number function is given by

$$[k] = \frac{\sum_{n=0}^\infty r_n k^n}{\sum_{n=0}^\infty r_n} \tag{6.52}$$

6.4 Generalised Deformed Coherent States

The eigenfunction of the generalised derivative operator is given by

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{6.53}$$

Applying the ratio test to the series, we see that it converges provided

$$|x| < [n] \qquad \text{as } n \to \infty \tag{6.54}$$

Using (6.17), we see that, in terms of the deformation function, if $f(n) \geq 1$ as $n \to \infty$, then $E(x)$ converges for all real values of x. If $f(n) < 1$ as $n \to \infty$, then convergence is ensured only for a range of x dependent on the functional nature of f. The function $E(x)$ is a generalisation of the exponential function for deformed structure/number functions Φ. We will therefore denote it as the D-exponential

As in the q-deformed case, we can show that the action of the D-exponentiated deformed creation operator on the vacuum state is to produce an eigenstate of the deformed annihilation operator.

$$\begin{aligned} a\,(a^\dagger)^{k+1}|0\rangle &= [N+1](a^\dagger)^{k-1}|0\rangle && (6.55)\\ &= (a^\dagger)^{k-1}[N+k]|0\rangle && (6.56)\\ &= [k](a^\dagger)^{k-1}|0\rangle && (6.57) \end{aligned}$$

so

$$a\,E(\lambda a^\dagger)|0\rangle = \lambda E(\lambda a^\dagger)|0\rangle \tag{6.58}$$

We can therefore define generalised coherent states, $|\lambda\rangle$, as normalised eigenstates of the annihilation operator

$$|\lambda\rangle = \{E(|\lambda|)\}^{-\frac{1}{2}} E(\lambda a^\dagger)|0\rangle \tag{6.59}$$

One can calculate the overlap between two such states

$$\begin{aligned} \langle\nu|\lambda\rangle &= \langle 0|E(|\nu|^2)^{\frac{1}{2}} E(\bar{\nu}a) E(|\lambda|^2)^{\frac{1}{2}} E(\lambda a^\dagger)|0\rangle && (6.60)\\ &= \frac{E(\bar{\nu}\lambda)}{\sqrt{E(|\nu|^2)E(|\lambda|^2)}} && (6.61) \end{aligned}$$

which clearly goes over to the conventional formula in the undeformed limit.

6.5 Noise Reduction Properties of the Deformed Coherent States

The deformation of the mathematical structure of the boson algebra should manifest itself in terms of a change in the physical properties of the quanta that such a deformed bose field produces. The most obvious boson field to examine is that of the photon. It would therefore be useful to examine the quantum optical properties of these generalised coherent states, in particular, their quantum noise properties.

Given the deformed creation and annihilation operators, there is an obvious difficulty in choosing the correct definition of the field components. In the undeformed limit, the X and P variables are the real and imaginary parts of the canonical boson operators and, most importantly, they obey the same commutation relations. Transformation from boson operators to field operators is therefore just a linear change of basis. Deformation of the algebra changes this simple arrangement and a choice has to be made concerning which of the relevant properties to keep. In what follows, we have decided to retain the hermiticity condition on X and P. While this means that the set of field operators is closed under hermitian conjugation, it also has the unfortunate effect that they no longer obey the same commutation relations as the q–boson creation and annihilation operators.

We form the field components X and P

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger) \quad \text{and} \quad P = \frac{1}{i\sqrt{2}}(a - a^\dagger) \tag{6.62}$$

and define the variances $(\Delta X)^2$ and $(\Delta P)^2$ in the usual way

$$(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2 \quad \text{and} \quad (\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2 \tag{6.63}$$

If we consider the Heisenberg Uncertainty Product, $\Delta X \Delta P$, we find that in the vacuum state

$$(\Delta X)_0(\Delta P)_0 = \frac{1}{2} \tag{6.64}$$

Moreover, just as in the conventional case, the vacuum uncertainty product is a lower bound for all *number* states if Φ is an increasing function.

However, unlike the conventional case, it is not a global lower bound.

Consider the quadrature values in eigenstates, $|\lambda\rangle$, of the generalised annihilation operator. Then

$$\langle X\rangle_\lambda = \langle\lambda|\frac{1}{\sqrt{2}}(a^\dagger + a)|\lambda\rangle = \frac{1}{\sqrt{2}}(\lambda + \bar{\lambda}) \tag{6.65}$$

and

$$\begin{aligned} \langle X^2\rangle_\lambda &= \langle\lambda|\,\frac{1}{2}((a^\dagger)^2 + a^2 + a^\dagger a + aa^\dagger)\,|\lambda\rangle & (6.66)\\ &= \frac{1}{2}\{(\bar{\lambda}+\lambda)^2 + 1 - \varepsilon_{f,\lambda}|\lambda|^2\} & (6.67)\end{aligned}$$

where

$$\varepsilon_{f,\lambda} = 1 - \langle f(N+1)\rangle_\lambda. \tag{6.68}$$

The function $\langle f(N+1)\rangle_\lambda$ can be evaluated by writing $|\lambda\rangle$ in a number state basis

$$\begin{aligned} \langle f(N+1)\rangle_\lambda &= \{E(|\lambda|^2)\}^{-1}\sum_{m,n}^{\infty}\langle m|\frac{\bar{\lambda}^m}{\sqrt{[m]!}}f(N+1)\frac{\lambda^n}{\sqrt{[n]!}}|n\rangle & (6.69)\\ &= \frac{\sum_n^\infty \frac{|\lambda|^{2n}}{[n]!}f(n+1)}{E(|\lambda|^2)} & (6.70)\end{aligned}$$

If for all $n \in \mathbb{N}$, $0 < f(n) < 1$, then $\varepsilon_{f,\lambda}|\lambda|^2 \in (0,1)$ for λ within the radius of convergence of the generalized exponential. Hence

$$(\Delta X)^2_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\}. \tag{6.71}$$

Evaluating the variance for the other component, we find that $(\Delta P)^2_\lambda = (\Delta X)^2_\lambda$. The covariance of X and P is zero, so

$$(\Delta X)_\lambda(\Delta P)_\lambda = \frac{1}{2}\{1 - \varepsilon_{f,\lambda}|\lambda|^2\} < \frac{1}{2}. \tag{6.72}$$

However,

$$\frac{1}{2}|\langle[X,P]\rangle_\lambda| = \frac{1}{2}|i\langle\lambda|aa^\dagger - a^\dagger a|\lambda\rangle| \quad (6.73)$$

$$= \frac{1}{2}|\langle\lambda|1 + f(N)a^\dagger a - a^\dagger a|\lambda\rangle| \quad (6.74)$$

$$= \frac{1}{2}|1 - |\lambda|^2 + \bar{\lambda}\langle\lambda|f(N+1)|\lambda\rangle\lambda| \quad (6.75)$$

$$= \frac{1}{2}|1 - \varepsilon_{f,\lambda}|\lambda|^2| \quad (6.76)$$

so

$$(\Delta X)_\lambda(\Delta P)_\lambda = \frac{1}{2}|\langle[X,P]\rangle_\lambda| \quad (6.77)$$

Thus we see that these generalised q–coherent states satisfy a restricted form of the Heisenberg Minimum Uncertainty Property of conventional coherent states. Additionally we see that there is a general noise reduction in both quadratures compared to their vacuum value. In conventional coherent states there is no noise reduction relative to the vacuum value. In conventional squeezed states, there is noise reduction in only one component. We note that the results given above have recently been reformulated by Santilli [San:94] in terms of his theory of Lie–admissible structures. In terms of this formalism, the deformed boson relation (6.2) is a special case of a non–canonical boson system with non–unitary time–evolution. Similar work on the $\star$–product formulation [BFFLS:78] of deformed oscillators has also been done by Ellinas [Ell:93b].

6.5.1 Examples

This formalism may be applied to all types of deformed oscillator systems modelled by the algebra $\mathcal{A}_D$ to find the uncertainty product. As an example, we will compute the quantum noise product for the deformed bosons belonging to the algebras $\mathcal{A}_q$ and $\mathcal{B}_q$.

1. The $\mathcal{A}_q$–type oscillator

The Jackson q–exponential $E_q(x)$ converges absolutely for any x if $q > 1$ so the coherent state is normalisable. Under these circumstances, (and writing

$\varepsilon_q \equiv \varepsilon_{f,\lambda}$) the function $1 - \varepsilon_q|\lambda|^2$ is greater than one and so the uncertainty is greater than in the conventional case. For $q \in (0,1)$, the q-exponential $E_q(|\lambda|^2)$ conveges provided

$$\varepsilon_q|\lambda|^2 = (1-q)|\lambda|^2 < 1 \tag{6.78}$$

Given this condition on λ, the function $1 - \varepsilon_q|\lambda|^2$ clearly lies in the range $(0,1)$ and so we have

$$(\Delta X)^2_\lambda = (\Delta P)^2_\lambda = (\Delta X)_\lambda(\Delta P)_\lambda = \frac{1}{2}\{1 - \varepsilon_q|\lambda|^2\} < \frac{1}{2} \tag{6.79}$$

Hence for this type of q-boson, we do obtain noise reduction in both field components with respect to the vacuum value.

It is notable that one multimode extension of the $\mathcal{A}_q$ algebra, i.e. the $SU_q(N)$-covariant q-boson system [Kem:93] has recently considered in a similar context. Using the R-matrix formalism, Kempf [Kem:94a, Kem:94b] has shown that if $q > 1$, the Bargmann–Fock representation allows the development of a consistent field theory that is not only covariant under quantum group symmetry but allows the regularisation of some of the ultraviolet infinities that plague conventional theories. There is, however, a price that has to be paid for such an improvement. Because of the non-trivial braiding between the elements of the $SU_q(N)$-spinor, not only do the conjugate field operators not commute, neither do the components of the same field operator in different directions.

2. The $\mathcal{B}_q$–type oscillator

We consider now the Macfarlane–Biedenharn q-oscillator with algebra $\mathcal{B}_q$. The deformed exponential $\exp_q(x)$ is a monotonically increasing function for $x > 0$ and converges for all real values of q and x.

The deformation function for this oscillator is

$$f(n) = \frac{[\![n+1]\!] - 1}{[\![n]\!]} = \frac{q^{(n+2)/2} + 1}{\sqrt{q}(q^{n/2}+1)} \tag{6.80}$$

and $f(0) = \frac{1}{2}[\![2]\!] \geq 1$ for all positive q. If we consider the deformation functional at real values of q, we see that the ratio of successive terms is always greater than unity. Therefore $f(n+1) > f(n) \geq 1$. Consequently, we have that for any normalised state $|\psi\rangle$,

$$\langle\psi|f(N+1) - I|\psi\rangle > 0 \tag{6.81}$$

In particular, for the coherent state $|\lambda\rangle$,

$$\langle\lambda|f(N+1) - I|\lambda\rangle > 0 \tag{6.82}$$

and so $\varepsilon_{f,\lambda} < 0$, i.e. the uncertainty is greater than the conventional case. This is in agreement with [CGN:92].

6.6 An Overcompleteness Relation for the Coherent States

In the classical study of undeformed coherent states outlined in chapter 3, the resolution of unity played a central role and the formation of an overcompleteness relation was a major advance in the study of q–deformed coherent states. Another interesting feature of these states is that they allow a perturbative expansion in terms of the coherent states of the corresponding undeformed Lie group. In chapter 5, we briefly described some work by Ellinas [Ell:93a] on the geometric meaning of q–deformation which showed that for the cases of the deformed boson algebra $\mathcal{B}_q$, as well as the quantised universal enveloping algebras $su_q(2)$ and $su_q(1,1)$ (where the resolutions of unity are well–known), q–deformation induces curvature in the phase–space. The three key ingredients of this procedure were the resolution of unity of the coherent states, the equivalence between the action of the relevant D–exponentiated raising operator on the lowest weight state and the (conventional) exponential action of some more complicated operator on the same lowest weight state, and the perturbative expansion of the structure function about its undeformed value. Given the generality of the DB–deformation scheme, it might be argued that it would be difficult to construct an analogue of the simple q–overcompleteness relation for the algebra $\mathcal{A}_D$ since it would depend on the existence of an q–integration procedure. However, an alternative method does exist and has been detailed by Fan [Fan:94]. He was able to construct a (non-standard) resolution of unity for the $\mathcal{A}_q$–type algebra using non–hermitian conjugate states and the technique of integration within an ordered product (IWOP) [FK:88]. This overcompleteness relation does not actually depend on the specific form of the q–deformation and so can be extended to a non-standard resolution for the deformed oscillator algebra $\mathcal{A}_D$.

6.6.1 The Resolution of Unity.

The transformation between deformed bosons with the same generic structure but different structure functions means that it is possible to write the coherent states in a number of different forms. As an example, we consider the operators

$$T^+ = a^\dagger \frac{(N+1)}{[N+1]} \quad \text{and} \quad T^- = \frac{(N+1)}{[N+1]}\, a \tag{6.83}$$

which were briefly described in connection with the deformed algebra realisations of the Heisenberg–Weyl Lie algebra (6.28). We see that they have quite complicated commutation relations with each other

$$[T^-, T^+] = \frac{(N+1)^2}{[N+1]} - \frac{N^2}{[N]} \tag{6.84}$$

but much simpler commutation relations with the original boson operators.

$$[a, T^+] = I = [T^-, a^\dagger] \tag{6.85}$$

The action of the $T^\pm$ on the vacuum vectors is straightforward

$$T^-|0\rangle = a|0\rangle = 0 \tag{6.86}$$

$$\langle 0|T^+ = \langle 0|a^\dagger = 0 \tag{6.87}$$

Moreover we have the result that

$$T^+ a = a^\dagger T^- = N \tag{6.88}$$

If we calculate the k–th power of T^+,

$$
\begin{aligned}
(T^+)^k &= \left\{a^\dagger \frac{N+1}{[N+1]}\right\}^k & (6.89)\\
&= (a^\dagger)^k \frac{(N+k)!\,[N]!}{[N+k]!\,N!} & (6.90)
\end{aligned}
$$

we obtain

$$
(T^+)^k|0\rangle = \frac{k!}{[k]!}(a^\dagger)^k|0\rangle \qquad (6.91)
$$

We may therefore rewrite the completeness relation for the Fock space as

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} |n\rangle\langle n| & (6.92)\\
&= \sum_{n=0}^{\infty} \frac{1}{[n]!}(a^\dagger)^n|0\rangle\langle 0|a^n & (6.93)\\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(T^+)^n|0\rangle\langle 0|a^n & (6.94)
\end{aligned}
$$

Following Fan et al, we may use the fact that a and T^+ obey the conventional Heisenberg–Weyl Lie algebra relations to define a *Normal ordering* in which the T^+ operators are placed to the left of the annihilation operators. If P denotes the vacuum projector $|0\rangle\langle 0|$ and the normal form of P is denoted by $:P:$, we can write (6.94) as

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{1}{n!}(T^+)^n :P: a^n & (6.95)\\
&= \sum_{n=0}^{\infty} :\frac{1}{n!}(T^+a)^n P: & (6.96)\\
&= :\exp(T^+a)P: & (6.97)
\end{aligned}
$$

which implies that

$$
|0\rangle\langle 0| = :P: = :\exp(-T^+a): \qquad (6.98)
$$

Another consequence of (6.91) is that the D–exponentiated action of $a^\dagger$ on the vacuum is the same as the conventionally exponentiated action of T^+ on the vacuum, i.e.

$$
E(\lambda a^\dagger)|0\rangle = \exp(\lambda T^+)|0\rangle = |\lambda\rangle \qquad (6.99)
$$

where $|\lambda)$ indicates the unnormalised coherent state. If we also define a dual (but not hermitian conjugate) set of states $(\lambda\|$ by

$$(\lambda\| = \langle 0|\exp(\bar{\lambda}a) \tag{6.100}$$

then because T^{+} and a realise the Heisenberg–Weyl algebra, we have the overcompleteness relation

$$\int |\lambda)(\lambda\| d\mu(\lambda) = I \tag{6.101}$$

where the integral is the standard integral and

$$d\mu(\lambda) = \frac{d^2\lambda}{\pi}\exp(-|\lambda|^2) \tag{6.102}$$

There is also a second resolution in terms of the hermition conjugate states of those appearing in (6.101).

Given these resolutions of unity, it may be possible to use Ellinas' method [Ell:93b] to investigate the effects of general deformation. Work of this type is in progress. It has also recently been brought to our attention [DY:92] that if one postulates a measure $d\mu(\bar{z}, z)$ with the property that

$$\int_D d\mu(\bar{z}, z)\, \bar{z}^n z^m \equiv \delta_{n,m}[n]! \tag{6.103}$$

where the symbol $\int_D$ indicates the antiderivative operation, then the overcompleteness relation between coherent states and their hermitian conjugate follows. This may also provide a way of using the deformed coherent states in the path–integral method.

6.7 The Physical Status of the Deformed Bosons

One point which should be considered when dealing with these generalised deformed bosons is their physical status. There are several approaches which can be taken.

Firstly, one can consider the primary objects under discussion to be the usual bosons of conventional quantum field theory. The deformed bosons, formed by the nonlinear mapping (6.27), can be considered instrumentally, simply as calculational devices which, under certain circumstances, will allow a better or more concise formulation of the problem. They show that the mathematical structure of different problems can be unified and so techniques used to solve one can be used to investigate another. An example of this approach is the study of two-dimensional quantum superintegrable systems by Bonatsos et al [BDK:93]. This uses the structure function formalism to give a unified description of a number of different oscillator systems.

Secondly, the deformed bosons can be considered as phenomenological devices, i.e. in certain situations, due to processes not directly amenable to investigation, conventional bosons behave like deformed bosons. One example of this is the use of quantum algebras to describe transitions in atomic nuclei where the use of the q-parameter allows a better fit to be made from the phenomenological theory to the experimental data curve. The physical effects which might cause the deformed bosons to describe the data better than the conventional ones may include things like thermodynamic averaging over some set of states or some set of unknown variables. The former has been proposed by Park et al, using an Inverse Schwinger Method in connection with q-deformed bosons as a model of *dressed photons* [Par:94, ChP:94]. It is interesting in this context that Katriel and Solomon [KS:94a] have recently shown that, for low intensities at least, the photon number count of the conventional laser is better described by the coherent state of an $\mathcal{A}_q$-type q-boson which can give non-Poissonian statistics, rather than the coherent state of an ordinary Heisenberg-Weyl boson which necessarily gives Poissonian statistics. Another example is the use of deformed bosons in lattice approximations of quantum field theory. [DiM:92, DMT:93]

A third possibility is that deformed bosons such as the q-bosons really do describe the physical field under investigation. Thus, the electromagnetic field should not really be considered as a bose–field but a deformed bose–field. For example, if we consider the multimode q–boson [Kem:93], it is this rather than the conventional object that is covariant under quantum group transformations. If quantum group space–time symmetries really are extant in nature, it would be surprising if the field theoretic modifications required were not reflected in a change in the physical structure of the field excitations. q–Bosons would therefore be the quanta associated with physical fields. The new physical phenomena which would result from using quantum group theoretical constructions means that it should be possible to use experiments to put a range on the deformation parameters and so on the type of symmetries allowed by the theory. The prediction of deviations from conventional physical theory have been made for fields symmetric under the κ–Poincaré group [Bac:93a, Bac:93b] or the q–Minkowski Space [Mey:94]. So far however, no experiments have been carried out to investigate these matters.

Chapter 7

The q–Analogue of the Unitary Displacement Operator

7.1 The Complex Quantum Plane

The coherent states studied so far preserve the classical commutative manifold structure of their label space. For example, the q–boson coherent states are labelled by elements of the complex plane and so the states in the Bargmann–Fock realisation are represented by analytic functions of commuting variables. There are however other generalisations of coherent states, notably the fermionic coherent states associated with Grassmann variables [OK:78], for which this is not the case. Grassmann–valued functions do not commute because of the anticommutivity of the variables themselves. Such anticommutivity induces nilpotence in the variables which makes the functions particularly easy to deal with since, at worst, they are linear. However in searching for a quantum generalisation of these states for the q-oscillator this will not be the case since q is not generally a power of unity. Indeed, if we restrict ourselves to the algebra $A_q^{2|0}$, the quantum plane variables or q–numbers, which in some sense, can be thought of as extensions of the Grassmann variables, form functions which have power series expansions of arbitrary length. This makes the manipulation of such functions much more difficult. A large part of the formalism needed for solving this problem was devised by Rembielinski and his co–

workers [BDR:92, BR:92, KR:93]. In order to construct a Bargmann–Fock representation of the coherent states, a geometric approach was adopted, i.e. the emphasis was on setting up a differential calculus on the quantum plane. In what follows, we give a brief exposition of the work together with a description of the coherent states that were constructed. We then show that this formalism may be extended to allow the construction of a q–number valued operator which, in the undeformed limit, is the analogue of the unitary displacement operator. This gives rise to a new set of coherent states. The use of quantum plane variables is not restricted to the work mentioned above. We note that the paper [GF:91] also uses noncommutative variables to develop a q–analogue of the algebra of *Weyl-Symmetrised Polynomials.* In various limits, this may be considered as a deformation of such structures as the Moyal Bracket algebra, the Poisson bracket algebra or a Vertex Operator algebra. The use of $\mathbb{C}_q$–valued functions also considerably simplifies the normal ordering problem for q–bosons. In conventional quantum optics (e.g. chapter 3 of [Lou:73]), the calculation of the action of functions of creation and annihilation operators is facilitated by considering a map from the Boson algebra, $\mathcal{H}$, to the algebra of complex numbers, $\mathbb{C}$. It has been shown by Solomon [Sol:93] that a similar map from $\mathcal{H}_q$ to $\mathbb{C}_q$ gives rise to a definition of normal order for $\mathcal{A}_q$–type q–bosons resulting in q–analogues of the ordered product formulas of [Lou:73].

7.2 The Two–Dimensional Quantum Plane and its Differential Calculi.

We recall from chapter 2 that the two–dimensional Manin or Quantum Plane is the complex algebra $A_q^{2|0}$ generated by two elements x and y with the relation

$$xy = q\,yx \tag{7.1}$$

Noncommutative elements such as x and y will be termed *q–numbers.* This algebra is a comodule for the action of the quantum matrix group $GL_q(2)$ which is the same as saying that the coaction Δ_L from $A_q^{2|0}$ to $GL_q(2) \otimes A_q^{2|0}$

given by equation (2.52) preserves the defining relations. There is also an associated Grassmann plane $A_q^{0|2}$ generated by nilpotent elements η and ξ such that

$$\eta\xi + q\,\xi\eta = 0 \tag{7.2}$$

As previously mentioned, the Manin plane is invariant under a larger symmetry quantum group than $GL_q(2)$, namely the multiparameter quantum general linear group $GL_{q,s}(2)$. However, this extra parameter does not manifest itself in the purely algebraic structure of $A_q^{2|0}$.

The algebra $A_q^{2|0}$ has a bialgebra structure given by the relations [CFFS:89]

$$\Delta(x) = x \otimes x \quad , \qquad \Delta(y) = y \otimes 1 + x \otimes y \tag{7.3}$$

$$\varepsilon(x) = 1 \quad , \qquad \varepsilon(y) = 0 \tag{7.4}$$

The problem of developing a differential calculus on the Manin Plane has has been addressed by a number of authors using techniques derived from the study of differential calculi on quantum groups. Such calculi are explicit realisations of noncommutative geometry. In what follows, we will use the procedures developed by Rembielinski et al.

The basic object of study in a differential calculus is the operator d. This is a linear, nilpotent differential operator obeying the graded Leibniz rule. It allows us to define right partial derivatives $\partial_i \equiv \partial_{x^i}$ by

$$\mathrm{d}f(x^1, x^2) = \mathrm{d}x^1\,\partial_1 f(x^1, x^2) + \mathrm{d}x^2\,\partial_2 f(x^1, x^2) \tag{7.5}$$

where we have set $x = x^1$, $y = x^2$. Then we have

$$\partial_i\, x^j = \delta_i^j \tag{7.6}$$

Higher order partial derivatives are defined in the obvious way by iteration

$$(\partial_i)^m f(x^1, x^2) = \partial_i\,(\partial_i)^{m-1} f(x^1, x^2) \tag{7.7}$$

where $f(x^1, x^2)$ is analytic in x^1 and x^2.

If we write $(x_1, x_2)^T$ as the vector $\underline{\mathbf{x}}$, the commutation relations for the generators can be written as

$$\underline{\mathbf{x}} \otimes \underline{\mathbf{x}} = B\,\underline{\mathbf{x}} \otimes \underline{\mathbf{x}} \tag{7.8}$$

Here B is an element of $End(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Similarly, the commutation relations between the differentials $\mathrm{d}x^i$ and the elements of $A_q^{2|0}$ can be written as

$$\underline{\mathbf{x}} \otimes \underline{\mathbf{dx}} = C\,\underline{\mathbf{dx}} \otimes \underline{\mathbf{x}} \tag{7.9}$$

where $\underline{\mathbf{dx}} = (\mathrm{d}x^1, \mathrm{d}x^2)^T$ and $C \in End(\mathbb{C} \otimes \mathbb{C})$.

The differential algebra that is generated by these relations is an associative graded algebra $\Gamma = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2$

$$\Lambda^0 \equiv A_q^{2|0} \xrightarrow{\mathrm{d}} \Lambda^1 \xrightarrow{\mathrm{d}} \Lambda^2 \tag{7.10}$$

where Λ^1, Λ^2 are modules over Λ^0 and $x^1, x^2 \in \Lambda^0$, $\mathrm{d}x_1, \mathrm{d}x_2 \in \Lambda^1$, and $\mathrm{d}x_1\mathrm{d}x_2 \in \Lambda^2$ (we assume $\mathrm{d}x_1 \wedge \mathrm{d}x_1$ and $\mathrm{d}x_1 \wedge \mathrm{d}x_1$ are zero).

The consistency of the calculus together with associativity requirements imply the *Wess-Zumino conditions* [WZ:90] on the matrices B and C.

$$\begin{aligned} B_{12}B_{23}B_{12} &= B_{23}B_{12}B_{23} && (7.11)\\ (B_{12} - I)(C_{12} + I) &= 0 && (7.12)\\ B_{12}C_{23}C_{12} &= C_{23}C_{12}B_{23} && (7.13)\\ C_{12}C_{23}C_{12} &= C_{23}C_{12}C_{23} && (7.14) \end{aligned}$$

where the usual quantum group notation applies, i.e. B_{ij} acts as B in the i–th and j–th space of $End(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ and as the identity in the remaining space.

For $A_q^{2|0}$, the most general form of the matrix B is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - s^{-1} & qs^{-1} & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.15}$$

while the matrix C has one of two generic forms

$$C_{\mathrm{I}} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \tag{7.16}$$

where p and r are free parameters, or

$$C_{\mathrm{II}} = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s^{-1} & q & 0 \\ 0 & sq^{-1} & 0 & 0 \\ 0 & 0 & 0 & s \end{pmatrix} \tag{7.17}$$

There is also a third form of the matrix but this leads to a calculus that is isomorphic to that of C_{II}.

It was found by Brzezinski et al [BDR:92] that these matrices gave rise to two families of differential calculi. These have commutation relations

Family I

$$\begin{array}{rlrl} x\,y &= q\,y\,x & & \\ x\,\mathrm{d}x &= p\,\mathrm{d}x\,x & x\,\mathrm{d}y &= q\,\mathrm{d}y\,x \\ y\,\mathrm{d}x &= q^{-1}\mathrm{d}x\,y & y\,\mathrm{d}y &= r\,\mathrm{d}y\,y \\ \mathrm{d}x \wedge \mathrm{d}y + q\,\mathrm{d}y \wedge \mathrm{d}x &= 0 & & \\ \partial_x\,x &= 1 + p\,x\,\partial_x & \partial_y\,y &= 1 + p\,y\,\partial_y \\ \partial_y\,x &= q\,x\,\partial_y & \partial_x\,y &= q^{-1}y\,\partial_x \\ \partial_x\,\partial_y &= q\,\partial_y\partial_x & & \end{array} \tag{7.18}$$

where

$$\partial_x f(x,y) = \frac{1}{x}\,\frac{f(px,y) - f(x,y)}{p-1} \tag{7.19}$$

$$\partial_y f(x,y) = \frac{1}{y}\,\frac{f(x,ry) - f(x,y)}{r-1} \tag{7.20}$$

Family II

$$\begin{array}{rcl}
x\,y = q\,y\,x & & \\
x\,\mathrm{d}x = s\,\mathrm{d}x\,x & & x\,\mathrm{d}y = (s-1)\,\mathrm{d}y\,x + q\,\mathrm{d}y\,x \\
y\,\mathrm{d}x = sq^{-1}\mathrm{d}x\,y & & y\,\mathrm{d}y = s\,\mathrm{d}y\,y \\
\mathrm{d}x \wedge \mathrm{d}y + qs^{-1}\,\mathrm{d}y \wedge \mathrm{d}x = 0 & & \\
\partial_x\,x = 1 + s\,x\,\partial_x + (s-1)\,y\,\partial_y & & \partial_y\,y = 1 + s\,y\,\partial_y \\
\partial_y\,x = q\,x\,\partial_y & & \partial_x\,y = sq^{-1}y\,\partial_x \\
\partial_x\,\partial_y = qs^{-1}\partial_y\partial_x & &
\end{array} \tag{7.21}$$

where

$$\partial_x f(x,y) = \frac{1}{x}\,\frac{f(sx,sy) - f(x,sy)}{s-1} \tag{7.22}$$

$$\partial_y f(x,y) = \frac{1}{y}\,\frac{f(x,sy) - f(x,y)}{s-1} \tag{7.23}$$

The second of these families is essentially the same as that found by Wess and Zumino. It is invariant under the coaction of $GL_{q,s}(2)$. The first family is not invariant under the full $GL_{q,s}(2)$ coaction. If $p = r = 1$, the differential structure coincides with the $s = 1$ subset of Family II, and so is invariant with respect to $GL_{q,1}(2)$. If $p \neq r$, then the differential structure is only invariant under the diagonal sub–quantum–group of $GL_q(2)$. This means that the calculus is only *scale invariant* [BR:92].

7.2.1 The Complex Differential Structure of the Quantum Plane

There exists several possibilities for introducing a complex structure onto $A_q^{2|0}$. One method would be to make all parameters real and put $z = x + i\,y$, $\bar{z} = x - i\,y$. However, in this case, the quantum analogues of the Cauchy–Riemann equations have solutions that are not (quantum) holomorphic, i.e. cannot be represented as a formal power series in z only.

Another possibility, for $q \in \mathbb{R}$, is to set the antilinear $*$–involution to be

$$x^* = y \tag{7.24}$$

In this case, the graded differential algebra Γ becomes a complex $*$-algebra, $\Gamma_{\mathbb{C}_q}$, defined by the calculus of Family I, with $p \in \mathbb{R}$ and $rp = 1$. If we now denote the algebra $A_q^{2|0}$ by $\mathbb{C}_q$ with generators $x = z$ and $y = z^*$, then the structure of $\Gamma_{\mathbb{C}_q}$ is given by

$$\begin{array}{lll} z\,z^* = q\,z^* z & & \\ z\,\mathrm{d}z = p\,\mathrm{d}z\,z & z\,\mathrm{d}z^* = q\,\mathrm{d}z^* z & \mathrm{d}z \wedge \mathrm{d}z^* = -q\,\mathrm{d}z^* \wedge \mathrm{d}z \\ \partial_z z = 1 + p\,z\,\partial_z & \partial_z z^* = q^{-1} z^* \partial_z & \partial_z \partial_{z^*} = q\,\partial_{z^*} \partial_z \\ \partial_z \mathrm{d}z = p^{-1} \mathrm{d}z\,\partial_z & \partial_z \mathrm{d}z^* = q^{-1} \mathrm{d}z^*\,\partial_z & \\ (z)^* = z^* & (\mathrm{d}z)^* = \mathrm{d}z^* & (\partial_z)^* = -p^{-1} \partial_{z^*} \end{array} \tag{7.25}$$

The imposition of such a differential $*$–structure has the effect of destroying the coalgebra structure of the quantum plane. Nevertheless, the set of q–holomorphic functions form an algebra and they can be written as a formal power series in one variable. Moreover, it is possible to define integration procedures for such functions and prove q–analogues of Cauchy's and Stokes' theorems [BR:92].

7.3 The q–Boson Algebra and the Complex Quantum Plane

In this section we review the $\mathbb{C}_q$*–deformed Hilbert space formalism* of Kowalski and Rembielinski [KR:93]. In their original presentation, these authors considered a two–parameter deformation of the Hilbert space of the conventional bosonic oscillator. This was done by associating the algebra $\mathbb{C}_q$ with a deformed boson algebra with relation $aa^\dagger - p^{-1}a^\dagger a = I$, i.e. the algebra $\mathcal{A}_{p^{-1}}$. In addition, they considered a one parameter limit in which $p = q$ as well as specific examples such as $p = q = \pm 1$. For the purposes of chapter 7, we only need the one–parameter case but it will be necessary to reformulate their results so that the algebra $\mathbb{C}_q$ is associated with the q–boson algebra $\mathcal{A}_q$. This is obtained from the $p = q$ limit of the original formalism by the transformation $z \leftrightarrow z^*$ in the appropriate definitions of the $\mathbb{C}_q$–valued functions.

We recall that the oscillator with algebra $\mathcal{A}_q$ acts on a Hilbert space H which has a basis $\{|n\rangle\}_{n=0}^{\infty}$ of eigenstates of the Number operator N where

$$N = \sum_{k=1}^{\infty} \frac{(1-q)^k}{(1-q^k)} (a^\dagger)^k a^k \tag{7.26}$$

This means that when we study the generators of the algebra in a Fock space representation, we need only consider the creation and annihilation operators since, in principle, the formula above will allow us to calculate any action on the Number operator.

If we consider the action of elements of the algebra $\mathcal{A}_q$ on the Hilbert space module H, we see that it can be thought of in terms of a map $\mathcal{A}_q \otimes H \longrightarrow \mathbb{C} \otimes H$. The canonical isomorphism $\mathbb{C} \otimes H \equiv H$, then allows us to interpret the result as an element of the original Hilbert space. We note that the trivial braiding relations between elements of H and the algebra $\mathbb{C}$ allow us to consider the Hilbert space as either a left or right $\mathbb{C}$–module. In this circumstance, a Hilbert space eigenvalue equation can be thought of as expressing a coaction from H to $\mathbb{C} \otimes H$ or $H \otimes \mathbb{C}$.

We can generalise the concept of H as a left or right $\mathbb{C}$–module to the Hilbert space H_q which is a left and right $\mathbb{C}_q$-module. For this construction, we slightly enlarge the algebra previously denoted $\mathbb{C}_q$ adding an *identity element*, I as a trivial central extension. (This is needed to express the unitary operation of chapter 4 as a coaction.) Thus $\mathbb{C}_q$ is now the quotient of the complex algebra freely generated by z and z^* modulo the two sided ideal B_q, determined by the relations

$$zz^* = q\,z^*z, \quad z\,I = I\,z = z, \quad z^*I = I\,z^* = z^* \quad I^2 = I \tag{7.27}$$

(We note that this algebra, which is now technically the Quantum Inhomogeneous Complex Plane, still has a quantum group coaction. The relevant quantum group is a deformation of the group of Euclidean motions associated with the complex plane, namely $ISO_q(2)$ [Rem:92]. However, this does not alter the basic analysis given above.)

H_q is therefore the free $\mathbb{C}_q$–module generated by $|n\rangle$

$$|f\rangle = \sum_n f_n(z^*, z)|n\rangle \tag{7.28}$$

where

$$f_n(z^*, z) = \sum_{k,l} c_{nkl}\,(z^*)^k z^l \tag{7.29}$$

and $c_{nkl} \in \mathbb{C}$ We can similarly define a right $\mathbb{C}_q$–module structure. We demand a trivial braiding between elements of $\mathbb{C}_q$ and H_q and that the left and right module structures be consistent i.e.

$$|f\rangle = \sum_n f_n(z^*, z)|n\rangle = \sum_n |n\rangle f_n(z^*, z) \tag{7.30}$$

The "inner product" on H_q can be stated as

$$\langle f|g\rangle = \sum_n \{f_n(z^*, z)\}^* g_n(z^*, z) \tag{7.31}.$$

if we identify H_q with the tensor product $\mathbb{C}_q \otimes H$, $H \otimes \mathbb{C}_q$ or even $\mathbb{C}_q \otimes H \otimes \mathbb{C}_q$. As a result, we have

$$\begin{aligned} \langle f|g\rangle &= (\langle g|f\rangle)^* && (7.32)\\ \langle \psi|h|f\rangle &= h\langle \psi|f\rangle && (7.33) \end{aligned}$$

where $|f\rangle$, $|g\rangle \in H_q$, $|\psi\rangle \in H$ and $h = h(z^*, z)$.

7.3.1 Left Eigenvalue States of the Annihilation Operator

Given the annihilation operator a, one possiblity for the definition of a coherent state is the (normalised) state satisfying

$$a|z, z^*) = z|z, z^*) \tag{7.34}$$

where we use a round–bracket to distinguish the state from others introduced later. Projecting onto the basis of number states, we obtain

$$\langle n|z, z^*) = \frac{z^n}{\sqrt{[n]_q!}} A(z^*, z) \tag{7.35}$$

where the normalisation $A(z^*z)$ can be expressed as

$$A(z^*, z) = \langle 0|z, z^*)^{-\frac{1}{2}} \tag{7.36}$$

Explicit calculation shows that the function $A(z, z^*)$ is given by

$$A(z^*, z) = E_{q^{-1}}(-z^*z)^{\frac{1}{2}} \tag{7.37}$$

and so

$$|z, z^*) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} E_{q^{-1}}(-z^*z)^{\frac{1}{2}}|n\rangle \tag{7.38}$$

$$= E_q(za^\dagger)E_{q^{-1}}(-z^*z)^{\frac{1}{2}}|0\rangle \tag{7.39}$$

Kowalski and Rembielinski were able to construct a Bargmann-Fock representation of the operators in terms of q–deformed differential operators on the space of analytic functions in z^*. From this it is possible to define an inner product, reproducing kernel and q–generalisation of the Gaussian integral [KR:93]. These constructions, although of considerable importance to further extensions of the formalism, will not be detailed here as we wish to concentrate on another aspect, namely the connection between the states defined in (7.39) and states produced by a a q–analogue of the unitary displacement operator for the boson algebra.

7.4 The Unitary q–Displacement Operator

The formalism described above provides a useful basis for the discussion of the q–deformed oscillator algebra over a noncommutative base algebra. However it is quite difficult to calculate the expectation value of interesting quantities in the coherent state $|z, z^*)$ because the evaluation procedure may actually change the state which is being used in the calculation. Clearly, this is due to the fact that the left eigenvalue eigenstates of the annihilation operator and the right eigenvalue eigenstates are not the same. This, in turn, is due to the noncommutivity of the coherent state with the eigenvalues. It is possible to calculate the right eigenstates of the annihilation operator by the means discussed in section (7.3) but a more interesting derivation has been found [MS:94b] by considering a deformed analogue of the unitary Displacement operator.

A näive approach to the problem of constructing a q–displacement operator (and the associated coherent states) would consider the q–analogue of the exponentiated action of an operator such as $(za^\dagger - z^*a)$. However, because the reciprocal of $E_q(x)$ is not $E_q(-x)$ but $E_{q^{-1}}(-x)$, the operator, $E_q(za^\dagger - z^*a)$, is not unitary. Moreover, in the $q = 1$ case, this action of the operator on the vacuum state would be calculated using the Baker–Campbell–Hausdorff theorem which provides a resolution of the exponentiated operator in terms of the exponentials of the generators of the algebra. It would then be a simple matter to construct coherent states by calculating the action on the vacuum. In the case of the q–exponentials associated with the deformed boson algebras, no analogue of the BCH–theorem has been found [KS:91b, KS:94b] and so the action of an operator such as $E_q(za^\dagger - z^*a)$ cannot be readily calculated. This implies that a more sophisticated approach to the problem is needed. The analysis developed in the following sections leads to the construction of an operator which is unitary and gives the correct $q = 1$ limit. This is used to form the appropriate coherent states which are shown to be right–eigenvalued eigenstates of the annihilation operator. The properties of these states are then discussed.

7.4.1 q–Exponential Disentangling Relations

There is a well–known lemma used in the proof of the Baker–Campbell–Hausdorff theorem which states that for any two operators A and B,

$$\begin{aligned}
\exp(A)\,B\exp(-A) &= \exp(\mathrm{ad}_A)B && (7.40)\\
&= \left\{\sum_{k=0}^{\infty}\frac{1}{k!}(\mathrm{ad}_A)^k\right\}B && (7.41)\\
&= B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots && (7.42)
\end{aligned}$$

$$\ldots + \frac{1}{n!}\underbrace{[A[A\ldots[A,B]\ldots]]}_{\text{n times}} + \ldots \quad (7.43)$$

where $\mathrm{ad}_A B = [A, B]$.

This has an analogue for q-exponentials, namely that for any two operators A and B [AKM:87],

$$E_q(A)\,B\,E_{q^{-1}}(-A) = \sum_{k=0}^{\infty}\frac{1}{k!}[A,B]^{(k)} \quad (7.44)$$

$$= B + [A,B] + \frac{1}{[2]_q!}[A,[A,B]]_q + \ldots \quad (7.45)$$

$$+ \frac{1}{[n]_q!}[A[A\ldots[A[A,B]]_q\ldots]_{q^{n-2}}]_{q^{n-1}} + \ldots$$

where $[A,B]_q = AB - qBA$, $[A,B]_0 = B$ and

$$[A,B]^{(n+1)} = A[A,B]^{(n)} - q^n[A,B]^{(n)}A \quad (7.46)$$

If we take the q–boson algebra $\mathcal{A}_q$ given by (3.22) in chapter 3

$$aa^\dagger - q\,a^\dagger a = I \quad (3.22)$$

together with the algebra $\mathbb{C}_p$ with relations

$$zz^* = p\,z^*z \quad (7.47)$$

where the elements of $\mathcal{A}_q$ and $\mathbb{C}_p$ commute with each other, then we may calculate the expression

$$E_r(za^\dagger)z^*aE_{r^{-1}}(-za^\dagger) = z^*a + [za^\dagger, z^*a] + \frac{1}{[2]_r}[za^\dagger,[za^\dagger,z^*a]]_r + \ldots \quad (7.48)$$

We see that

$$[za^\dagger, z^*a] = zz^*a^\dagger a - z^*z\,aa^\dagger \quad (7.49)$$

$$= -z^*z\{aa^\dagger - p\,a^\dagger a\} \quad (7.50)$$

$$= -z^*z\{I + (q-p)a^\dagger a\} \quad (7.51)$$

So if $q = p$,

$$[za^{\dagger}, z^{*}a] = -z^{*}z \tag{7.52}$$

where we have omitted the explicit expression of the unit operator since this is central and has the trivial action of multiplication by unity.

Similarly

$$\begin{aligned} [za^{\dagger}, [za^{\dagger}, z^{*}a]]_r &= [za^{\dagger}, -z^{*}z]_r & (7.53) \\ &= -\{zz^{*} - r\, z^{*}z\}za^{\dagger} & (7.54) \\ &= 0 \qquad \text{if } r = p = q & (7.55) \end{aligned}$$

The series (7.48) therefore terminates with the second term to give

$$E_q(za^{\dagger})\, z^{*}a\, E_{q^{-1}}(-za^{\dagger}) = z^{*}a - z^{*}z \tag{7.56}$$

This in turn implies that

$$E_q(za^{\dagger})E_q(z^{*}a)E_{q^{-1}}(-za^{\dagger}) = E_q(z^{*}a - z^{*}z) \tag{7.57}$$

where we have used the fact that

$$E_{q^{-1}}(-x) = E_q(x)^{-1} \tag{7.58}$$

From this we can obtain

$$E_q(za^{\dagger})E_q(z^{*}a) = E_q(z^{*}a - z^{*}z)E_q(za^{\dagger}) \tag{7.59}$$

By taking the hermitian conjugate of equation (7.55), we see that $z^{*}a$ q–commutes with $z^{*}z$, so that we may use the additive property of the q–exponential to rewrite the above equation as a *reordering equation*

$$E_q(za^{\dagger})E_q(z^{*}a) = E_q(z^{*}a)E_q(-z^{*}z)E_q(za^{\dagger}) \tag{7.60}$$

which is a q–analogue of the *Weyl relation* for the Heisenberg group.

Taking the inverse of (7.60) and letting $z \to -z$, and $z^{*} \to -z^{*}$, we obtain a second reordering equation

$$E_{q^{-1}}(z^{*}a)E_{q^{-1}}(za^{\dagger}) = E_{q^{-1}}(za^{\dagger})E_{q^{-1}}(z^{*}z)E_{q^{-1}}(z^{*}a) \tag{7.61}$$

from which one can show a third relation

$$E_q(za^\dagger)E_{q^{-1}}(z^*a) = E_{q^{-1}}(-z^*z)E_{q^{-1}}(z^*a)E_q(za^\dagger) \qquad (7.62)$$

Using equation (7.60) and the properties of the q–exponential, it is possible to prove formulae such as

$$E_{q^{-1}}(-za^\dagger)E_q(z^*a)E_q(-z^*z)E_q(za^\dagger)E_{q^{-1}}(-z^*a) = 1 \qquad (7.63)$$

This motivates the definition of an operator $U(z, z^*)$

$$U(z, z^*) = E_q(-z^*z)^{\frac{1}{2}}E_q(za^\dagger)E_{q^{-1}}(-z^*a) \qquad (7.64)$$

The hermitian conjugate of this is

$$U(z, z^*)^\dagger = E_{q^{-1}}(-za^\dagger)E_q(z^*a)E_q(-z^*z)^{\frac{1}{2}} \qquad (7.65)$$

If one now calculates the product of these two operators (see appendix), one finds that

$$U(z, z^*)U(z, z^*)^\dagger = U(z, z^*)^\dagger U(z, z^*) = 1 \qquad (7.66)$$

which means that

$$U(z, z^*) = E_q(-z^*z)^{\frac{1}{2}}E_q(za^\dagger)E_{q^{-1}}(-z^*a) \qquad (7.67)$$

is *unitary* , i.e.

$$U(z, z^*)^\dagger = U(z, z^*)^{-1} \qquad (7.68)$$

7.4.2 Properties of the q–Displacement Operator

In the limit $q \to 1$, it can be seen that $U(z, z^*) \to D(z, z^*) \equiv D(z)$, where $D(z)$ is the *Weyl Displacement Operator* of conventional quantum mechanics, (c.f. chapter 3),

$$D(z, z^*) = \exp(\tfrac{1}{2}|z|^2)\exp(zb^\dagger)\exp(-z^*b) = \exp(zb^\dagger - z^*b) \qquad (7.69)$$

with $[b, b^\dagger] = 1$.

We will now show how the properties of the undeformed displacement operator generalise to this new q–analogue object. We make use of the following result:

$$a(a^\dagger)^n - q^n(a^\dagger)^n a = [n]_q(a^\dagger)^{n-1} \qquad (7.70)$$

which, since $[n]_{q^{-1}} = q^{1-n}[n]_q$, leads to the formula (see appendix)

$$E_{q^{-1}}(-za^\dagger)a - aE_{q^{-1}}(-zq^{-1}a^\dagger) = zq^{-1}E_{q^{-1}}(-za^\dagger) \qquad (7.71)$$

Now consider the operator product $U(z,z^*)^\dagger a$.

$$\begin{aligned}
&U(z,z^*)^\dagger a && (7.72)\\
&= E_{q^{-1}}(-za^\dagger)E_q(z^*a)E_q(-z^*z)^{\frac{1}{2}}a && (7.73)\\
&= \left\{aE_{q^{-1}}(-zq^{-1}a^\dagger) + zq^{-1}E_{q^{-1}}(-za^\dagger)\right\}E_q(z^*a)E_q(-z^*z)^{\frac{1}{2}} && (7.74)\\
&= \left\{a\,E_{q^{-1}}(-zq^{-1}a^\dagger)E_q(za^\dagger) + zq^{-1}\right\}U(z,z^*)^\dagger && (7.75)\\
&= \left\{a\,E_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} + zq^{-1}\right\}U(z,z^*)^\dagger && (7.76)
\end{aligned}$$

Consequently, the product $U(z,z^*)^\dagger a\,U(z,z^*)$ is given by

$$\begin{aligned}
&U(z,z^*)^\dagger a\,U(z,z^*) && (7.77)\\
&= \left\{aE_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} + zq^{-1}\right\}U(z,z^*)^\dagger U(z,z^*) && (7.78)\\
&= aE_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} + zq^{-1} && (7.79)
\end{aligned}$$

From (5.78), it can be seen that

$$E_q(za^\dagger)E_q(zq^{-1}a^\dagger)^{-1} = 1 + (q-1)q^{-1}za^\dagger \qquad (7.80)$$

so

$$U(z,z^*)^\dagger a\,U(z,z^*) = a\{1 + (q-1)q^{-1}za^\dagger\} + zq^{-1} \qquad (7.81)$$

Assuming we are in Fock space, we have the result that

$$aa^\dagger = [N+1]_q = \frac{q^{N+1}-1}{q-1} \qquad (7.82)$$

which follows from the definition of the algebra $\mathcal{A}_q$ in equation (3.22), and so

$$U(z,z^*)^\dagger a\,U(z,z^*) = a + zq^N \qquad (7.83)$$

Since $q^N = [a, a^\dagger]$, we can also write this as

$$U(z, z^*)^\dagger a\, U(z, z^*) = a + z\, [a, a^\dagger] \tag{7.84}$$

We therefore have an analogue of the conventional ($q = 1$) Heisenberg–Weyl shift automorphism,

$$\begin{aligned} D(z, z^*)^\dagger\, b\, D(z, z^*) &= b + z\, [b, b^\dagger] & (7.85)\\ &= b + z1 & (7.86) \end{aligned}$$

with $D(z, z^*)$ defined as in (7.69).

The exact meaning of this formula requires some consideration. In the undeformed ($q = 1$) coherent state theory, (7.86) is a statement of the fact that the unitary displacement operator $D(z)$ implements an automorphism of the boson algebra. A similar interpretation in the case of the q–deformed unitary operator $U(z, z^*)$ is not possible because the element $a + zq^N$ no longer lies in the algebra. Instead, we must think of $U(z, z^*)$ as implementing an *algebra coaction* of $\mathbb{C}_q$ on $\mathcal{A}_q$

$$\begin{array}{ccccccc} \mathcal{A}_q & \longrightarrow & \mathbb{C}_q & \otimes & \mathcal{A}_q & & \\ a & \mapsto & a' & = & I \otimes a & + & z \otimes q^N \\ a^\dagger & \mapsto & a^{\dagger\prime} & = & I \otimes a^\dagger & + & z^* \otimes q^N \end{array} \tag{7.87}$$

which, as the elements of $\mathbb{C}_q$ have trivial braiding with those of $\mathcal{A}_q$, is equivalent to a coaction $\mathcal{A}_q \longrightarrow \mathcal{A}_q \otimes \mathbb{C}_q$.

The fact that the inverse of $\exp(x)$ is $\exp(-x)$ means that, in the undeformed theory, the inverse (and hermitian conjugate) of the operator $D(z, z^*)$ is given by $D(-z, -z^*)$. The inverse of $E_q(x)$ is not $E_q(-x)$ but $E_{q^{-1}}(-x)$ so that

$$U(z, z^*)\, U(-z, -z^*) \neq 1 \tag{7.88}$$

Consequently we can construct a second unitary operator $V(z, z^*)$

$$V(z, z^*) = U(-z, -z^*) \tag{7.89}$$

which (co)acts on the generators of $\mathcal{A}_q$ to give

$$V(z,z^*)\,a\,V(z,z^*)^\dagger = a - zq^N \qquad (7.90)$$

and the hermitian conjugate relation.

7.5 q–Analogue Displaced Vacuum States

As has already been pointed out, the Displacement operator is used in quantum mechanics to generate coherent states from the vacuum state. We can do exactly the same for the q–analogue operator constructed above.

We define a set of displaced vacuum states by

$$|z\rangle = U(z,z^*)|0\rangle \qquad (7.91)$$

Then due to the unitarity of U and (7.83), we have

$$a\,U(z,z^*) = U(z,z^*)\{a + zq^N\} \qquad (7.92)$$

Therefore

$$\begin{aligned} a\,U(z,z^*)|0\rangle &= U(z,z^*)\{a + zq^N\}|0\rangle && (7.93)\\ &= U(z,z^*)z|0\rangle && (7.94)\\ &= U(z,z^*)|0\rangle z && (7.95) \end{aligned}$$

So $|z,z^*\rangle$ is an eigenstate of the annihilation operator a, with *right eigenvalue* z,

$$a|z,z^*\rangle = |z,z^*\rangle z \qquad (7.96)$$

In the $q \to 1$ limit, where z, z^* commute and have realisation as elements of $\mathbb{C}$ and a and $a^\dagger \to b$ and $b^\dagger$, the usual boson operators, equation (7.96) just states the familiar result that the displaced vacuum states are eigenstates of the annihilation operator

$$b|z\rangle = z|z\rangle \qquad (7.97)$$

The displaced vacuum states can be expressed in terms of the usual q–boson number states

$$
\begin{aligned}
|z, z^*\rangle &= U(z, z^*)|0\rangle & (7.98)\\
&= E_q(-z^* z)^{\frac{1}{2}} E_q(z a^\dagger) E_{q^{-1}}(-z^* a)|0\rangle & (7.99)\\
&= E_q(-z^* z)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n (a^\dagger)^n}{[n]_q!}|0\rangle & (7.100)\\
&= E_q(-z^* z)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}}|n\rangle & (7.101)
\end{aligned}
$$

So

$$\langle n|z, z^*\rangle = E_q(-z^* z)^{\frac{1}{2}} \frac{z^n}{\sqrt{[n]_q!}} \tag{7.102}$$

7.5.1 The Relationship between Left and Right Eigenvalued Eigenstates

In the previous section, we defined, after Kowalski and Rembielinski, another set of normalised states $|z, z^*)$ by

$$|z, z^*) = E_q(z a^\dagger) E_q(-z^* z)|0\rangle \tag{7.103}$$

which have the property that $|z, z^*)$ are eigenstates of the annihilation operator with left eigenvalue z,

$$a|z, z^*) = z|z, z^*) \tag{7.104}$$

These eigenstates of the annihilation operator are related to the displaced vacuum states $|z, z^*\rangle$ by the transformation

$$|z, z^*) = E_q(z^* z)^{\frac{1}{2}} |z, z^*\rangle E_q(-z^* z)^{\frac{1}{2}} \tag{7.105}$$

This illustrates that if $q \neq 1$, the two conventional definitions of coherent states as either left eigenstates of the annihilation operator or displaced vacuum states are not equivalent for the type of q–deformed system constructed here.

7.5.2 Quantum Noise Properties

To consider the quantum noise dispersion value in the displaced vacuum states, we use the (deformed) field components X and P defined by

$$X = \frac{1}{\sqrt{2}}(a + a^{\dagger}) \quad \text{and} \quad P = \frac{1}{i\sqrt{2}}(a - a^{\dagger}). \tag{7.106}$$

If we formally evaluate the product of the variances of these operators in the vacuum state we obtain

$$(\Delta X)_0^2 (\Delta P)_0^2 = \frac{1}{4} \tag{7.107}$$

In the displaced vacuum state, the same result occurs

$$(\Delta X)_z^2 (\Delta P)_z^2 = \frac{1}{4} \tag{7.108}$$

Moreover, the covariance of X and P is identically zero in both cases. Therefore, just as in the conventional case, both the undisplaced and displaced vacuum states have the same value for the quantum noise dispersion. However, if one calculates the theoretical lower bound for the dispersion using the Heisenberg Uncertainty Relation, one finds that

$$\begin{aligned}
\frac{1}{4}\langle z, z^*|[X, P]|z, z^*\rangle &= \frac{1}{4}\langle z, z^*|[a, a^{\dagger}]|z, z^*\rangle && (7.109)\\
&= \frac{1}{4}\langle z, z^*|1 - (1-q)a^{\dagger}a|z, z^*\rangle && (7.110)\\
&= \frac{1}{4}\{1 - [z^*, z]\} && (7.111)
\end{aligned}$$

Thus if z and z^* commute, the uncertainty relation gives the same result as in the classical case. If not, the conventional result is multiplied by an extra algebraic term $1 - [z^*, z]$.

7.5.3 Other States Associated with the Unitary q–Displacement Operator

Using similar techniques to the conventional case, one can also form other states from the q–displacement operator. For example, if this acts not on the vacuum ket but on some number state, *q-displaced number states* are produced.

$$|z,n\rangle = U(z,z^*)|n\rangle \tag{7.112}$$

If we use the fact that

$$\sum_{m=0}^{\infty} \frac{(-z^*)^m a^m}{[m]_q!}(a^\dagger)^n|0\rangle = \sum_{m=0}^{n} \frac{[n]_q!}{[n-m]_q![m]_q!}(a^\dagger)^{n-m}(-z^*)^m|0\rangle \tag{7.113}$$

we see that

$$|z,n\rangle = E_q(-z^*z)^{\frac{1}{2}} E_q(za^\dagger)\frac{B[a^\dagger,-z^*;n]}{\sqrt{[n]_q!}}|0\rangle \tag{7.114}$$

where the function $B[a,b;n]$ is defined by

$$B[a,b;n] = \sum_{m=0}^{n} \left\{ \frac{[n]_q!}{[n-m]_q![m]_q!} a^{n-m} b^m \right\} \tag{7.115}$$

In the $q \to 1$ limit we recover the standard formula

$$|z,n\rangle = \exp(-\tfrac{1}{2}z^*z + za^\dagger)\frac{(a^\dagger - z^*)^n}{\sqrt{[n]_q!}}|0\rangle \tag{7.116}$$

7.6 Appendix

7.6.1 Other q–analogue Unitary Displacement Operators

It is worth noting that the q–displacement operator presented above is not the only generalisation of the conventional formula that has appeared in the literature. Indeed it has been claimed (in [Zhe:93]) that an operator parameterised by c–numbers was discovered by Jannussis [Jan:83] as far back as 1983. There is also a remarkable generalisation of Jannussis' result due to Zhedanov in the context of his (u,v)–algebra formalism [Zhe:93]. We give a brief description of the pertinent aspects of Zhedanov's work, indicating the difficulties with his approach. For convenience, his notation has been changed to conform with that already in use in this work.

The (u,v)–algebra

Zhedanov defined a deformation of a two–parameter family of algebras interpolating between $su(2)$, $su(1,1)$ and the bosonic oscillator algebra $\mathcal{H}(4)$. It is generated by three elements A_0, A_+ and A_- with defining relations

$$[A_0, A_\pm] = A_\pm \ , \quad [A_-, A_+] = uq^{-A_0} + vq^{A_0} \tag{7.117}$$

where $q = \exp(-2\omega) \in \mathbb{R}$, is a deformation parameter ($\omega > 0$), and u and v are arbitrary real parameters. A_0 is assumed to be hermitian whereas $A_\pm$ are hermitian conjugates. Various special cases of the algebra are

$$\begin{array}{lll} su_q(2) & : & u = -v = -\frac{1}{2}\operatorname{cosech}\omega \ ; \\ su_q(1,1) & : & u = -v = \frac{1}{2}\operatorname{cosech}\omega \ ; \\ \mathcal{A}_q & : & u = 0,\ v = 1 \ ; \\ \mathcal{B}_q & : & u = v = \frac{1}{2} \end{array} \tag{7.118}$$

The (u,v)–algebras have an interesting "co–additive" structure. If A_i and B_i, ($i = 0,\ \pm$), are oscillators of type (u,v) and $(-v,w)$ respectively, then the elements C_i given by

$$C_0 = A_0 + B_0 \ , \quad C_\pm = A_\pm \exp(\omega B_0) + B_\pm \exp(-\omega A_0) \tag{7.119}$$

generate an algebra of type (u,w). Put symbolically, there exist maps, $\bullet$, for which

$$(u,v)_A \bullet (-v,w)_B = (u,w)_C \tag{7.120}$$

If the reparametrisation

$$u = \frac{g + h\omega^{-1}}{2} \ , \quad u = \frac{g - h\omega^{-1}}{2} \tag{7.121}$$

is made and the $\omega \to 0$ limit taken, one obtains

$$[A_-, A_+] = g + 2hA_0 \tag{7.122}$$

which clearly shows the relation with $su(2)$, $su(1,1)$ and $\mathcal{H}(4)$.

In the usual way one may define representations of the positive discrete series of the deformed algebras by considering the action of the generators on some canonical basis $|n\rangle$

$$A_0|n\rangle = (a+n)|n\rangle \tag{7.123}$$

$$A_-|n\rangle = r_n|n-1\rangle \tag{7.124}$$

$$A_+|n\rangle = r_{n+1}|n+1\rangle \tag{7.125}$$

where $r_n \in \mathbb{C}$, the constant a may be used to characterise the representation and $|0\rangle$ is annihilated by A_-.

The Master Relation and Coherent States

The (u,v)–algebra formalism is especially useful for constructing unitary operators because of the so–called *Zhedanov Master Relation* which is an analogue of the Generalised Weyl formula for the undeformed Lie algebras. The main relation is

$$E_q(sA_-)E_q(-ustq^{-A_0})E_q(tA_+) = E_q(tA_+)E_q(vstq^{A_0})E_q(sA_-) \tag{7.126}$$

where A_0, $A_\pm$ generate a (u,v)–algebra and s, t are arbitrary complex numbers. Apparently, the relevant result for the algebra $\mathcal{A}_q$ was known to Jannussis.

Using (7.126) it is possible to prove that

$$U_Z(\alpha;q) = E_{q^{-1}}(-v|\alpha|^2q^{-A_0})^{\frac{1}{2}}E_{q^{-1}}(\alpha A_+)E_q(-\bar{\alpha}A_-)E_q(-u|\alpha|^2q^{-A_0}) \tag{7.127}$$

is unitary. If one considers the various special cases of the algebra (7.118) and then takes the $q=1$ limit, it can be shown, using the properties of the Jackson q-exponential (5.80), that $U_Z(\alpha;q=1)$ is the displacement operator for the group corresponding to the undeformed Lie algebra. For example, the algebra $\mathcal{A}_q$ (with generators $A_0 = N$, $A_- = a$, $A_+ = a^\dagger$), gives the operator

$$U_Z(\alpha;q) = E_{q^{-1}}(-|\alpha|^2q^N)^{\frac{1}{2}}E_{q^{-1}}(-\alpha a^\dagger)E_q(\bar{\alpha}a) \tag{7.128}$$

To make a comparison with the approach in section 7.4 using noncommuting variables, it is more convenient to consider the unitary operator $V_Z(\alpha;q)$ given by $V_Z(\alpha;q) = U_Z(-\alpha;q)^\dagger$. This operator my be used to form coherent states from the vacuum vector

$$\begin{aligned} |\alpha\rangle &= V_Z(\alpha;q)|0\rangle && (7.129)\\ &= E_q(\alpha a^\dagger)E_{q^{-1}}(-\alpha a)E_{q^{-1}}(-|\alpha|^2 q^N)^{\frac{1}{2}}|0\rangle && (7.130)\\ &= E_{q^{-1}}(-|\alpha|^2)^{\frac{1}{2}}E_q(\alpha a^\dagger)|0\rangle && (7.131) \end{aligned}$$

Thus we see that $|\alpha\rangle$ coincides with the definition of the coherent state as the normalised vector formed by the action of the q–exponentiated creation operator on the vacuum and hence it is an eigenstate of the annihilation operator with eigenvalue α. (This contradicts the claim that the states are different, found in [Zhe:93] and due to an error in the calculations.)

Unfortunately the analogue of the Heisenberg–Weyl displacement formula, (7.86), using the operator $U_Z(\alpha;q)$ (or indeed $V_Z(\alpha;q)$) is far from trivial and in the case of the unitary operator for the general (u,v)–algebra (7.127) the calculation is even more complicated with the formula depending on the Casimir operator value, i.e. it is representation dependent. The use of such an operator to calculate uncertainties is therefore very difficult and can proceed only by numerical methods.

7.6.2 Proof of some results

In this section we prove some of the results that were stated in the text.

Unitarity of the q–analogue displacement operator

We show that the operator products $U\,U^\dagger$ and $U^\dagger U$ are equal to one, i.e. that $U(z,z^*)^\dagger$ is a left– and right–inverse for $U(z,z^*)$.

$$\begin{aligned} &U(z,z^*)\,U(z,z^*)^\dagger\\ &\quad= E_q(-z^*z)^{\frac{1}{2}}E_q(za^\dagger)\underline{E_{q^{-1}}(-z^*a)E_{q^{-1}}(-za^\dagger)}E_q(z^*a)E_q(-z^*z)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&= E_q(-z^*z)^{\frac{1}{2}} \underbrace{E_q(za^\dagger)E_{q^{-1}}(-za^\dagger)}_{=1} E_{q^{-1}}(z^*z) \underbrace{E_{q^{-1}}(-z^*a)E_q(z^*a)}_{=1} E_q(-z^*z)^{\frac{1}{2}} \\
&= E_q(-z^*z)^{\frac{1}{2}} E_{q^{-1}}(z^*z) E_q(-z^*z)^{\frac{1}{2}} \\
&= 1
\end{aligned}
$$

In the above calculation, the underlined product is reordered using equation (7.61).

We can also calculate the opposite product

$$
\begin{aligned}
& U(z,z^*)^\dagger U(z,z^*) \\
&\quad = E_{q^{-1}}(-za^\dagger)\underline{E_q(z^*a)E_q(-z^*z)^{\frac{1}{2}} E_q(-z^*z)^{\frac{1}{2}} E_q(za^\dagger)}E_{q^{-1}}(-z^*a) \\
&\quad = \underbrace{E_{q^{-1}}(-za^\dagger)E_q(za^\dagger)}_{=1} \underbrace{E_q(z^*a)E_{q^{-1}}(-z^*a)}_{=1} \\
&\quad = 1
\end{aligned}
$$

where we use (7.60) to reorder the underlined product.

Hence $U(z,z^*)$ is a unitary operator.

Proof of equation (7.71)

Simple induction on

$$aa^\dagger - q\, a^\dagger a = 1 \tag{7.132}$$

yields the result that

$$a(a^\dagger)^n - q^n(a^\dagger)^n a = [n]_q (a^\dagger)^{n-1} \tag{7.133}$$

so

$$(a^\dagger)^n a = q^{-n}\left\{a(a^\dagger)^n - [n]_q (a^\dagger)^{n-1}\right\} \tag{7.134}$$

Therefore

$$
\begin{aligned}
& E_{q^{-1}}(-za^\dagger)\,a \\
&\quad = \sum_{n=0}^{\infty} \frac{(-z)^n}{[n]_{\frac{1}{q}}!} \left\{ (a^\dagger)^n a \right\} \\
&\quad = \sum_{n=0}^{\infty} \frac{(-z)^n}{[n]_{\frac{1}{q}}!} \left\{ q^{-n} a (a^\dagger)^n - q^{-n} [n]_q (a^\dagger)^{n-1} \right\} \\
&\quad = a \sum_n \frac{(-zq^{-1}a^\dagger)^n}{[n]_{\frac{1}{q}}!} - (-z) \sum_{n-1} \frac{q^{-n}[n]_q}{[n]_{q^{-1}}} \frac{(-z)^{n-1}(a^\dagger)^{n-1}}{[n-1]_{\frac{1}{q}}!}
\end{aligned}
$$

Thus, using the fact that

$$
q^{1-n}[n]_q = [n]_{q^{-1}}
$$

we obtain

$$
E_{q^{-1}}(-za^\dagger)\,a = a\,E_{q^{-1}}(-zq^{-1}a^\dagger) + zq^{-1} E_{q^{-1}}(-za^\dagger) \tag{7.135}
$$

as required.

Chapter 8

Squeezed and Correlated Coherent States

The Uncertainty Relations place limits on the accuracy of the simultaneous measurement of the field components. However, the lower bound refers to the product of the dispersions rather than the individual dispersion values themselves. This observation is the basis of the study of so–called ***Squeezed States*** [LK:87] and *Correlated Coherent States* [DM:94] in which the dispersion of one field component is reduced below its value in the vacuum state, with a corresponding increase in the other dispersion. These states are non–classical in the sense that their effects are purely quantum mechanical with no classical analogue. In this chapter we review the theory of conventional squeezed and correlated states as preparation for the next chapter in which similar constructions for q–deformed squeezed states are discussed.

8.1 The Uncertainty Relations

We briefly recall the uncertainty relations discussed in chapter 3. For two operators X and Y, the Robertson–Schrodinger Uncertainty Relation [Schr:30, Rob:30] is

$$\sqrt{(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2} \geq \frac{1}{2}|\langle[X,Y]\rangle| \tag{8.1}$$

where the field operator variances are given by

$$(\Delta X)^2 = \langle X^2 - \langle X\rangle^2\rangle \tag{8.2}$$

$$(\Delta Y)^2 = \langle Y^2 - \langle Y \rangle^2 \rangle \tag{8.3}$$

and the field operator covariance is

$$\Delta XY \equiv \frac{1}{2}\langle XY + YX \rangle - \langle X \rangle \langle Y \rangle \tag{8.4}$$

If the square of the covariance is neglected (since it is positive or zero) or the variables are uncorrelated, we obtain the Heisenberg Uncertainty Relation

$$\Delta X \Delta Y \geq \frac{1}{2}|\langle [X, Y] \rangle| \tag{8.5}$$

If we consider either the vacuum state or a coherent state of the electromagnetic field, it is known that the uncertainties in the field components X and P are equal to $\frac{1}{2}$. Following Loudon and Knight [LK:87], we rescale the field components, X and P, by a factor of $\sqrt{2}$ to give the quadrature operators X' and P', where

$$X' = \frac{1}{\sqrt{2}}X = \frac{1}{2}(b + b^\dagger) \tag{8.6}$$

$$P' = \frac{1}{\sqrt{2}}P = \frac{1}{i2}(b - b^\dagger) \tag{8.7}$$

which have the commutation relation,

$$[X', P'] = i/2 \tag{8.8}$$

Since the amplitude of the coherent state is given by the eigenvalue of the annihilation operator, the quadrature operators can be considered to be the real and imaginary parts of the complex–amplitude operator.

If we consider the uncertainties of these quadrature operators in the coherent state $|\alpha\rangle$, we find

$$\langle X' \rangle_\alpha = \mathrm{Re}(\alpha) \tag{8.9}$$

$$\langle P' \rangle_\alpha = \mathrm{Im}(\alpha) \tag{8.10}$$

Therefore, since

$$\langle X' + iP' \rangle_\alpha = \alpha \tag{8.11}$$

we can describe this graphically by a circular uncertainty contour centred on the tip of the complex amplitude vector whose length is given by the eigenvalue label α. The diameter of the circle then corresponds to the size of the quadrature operator uncertainty. This is illustrated in Figure 1:

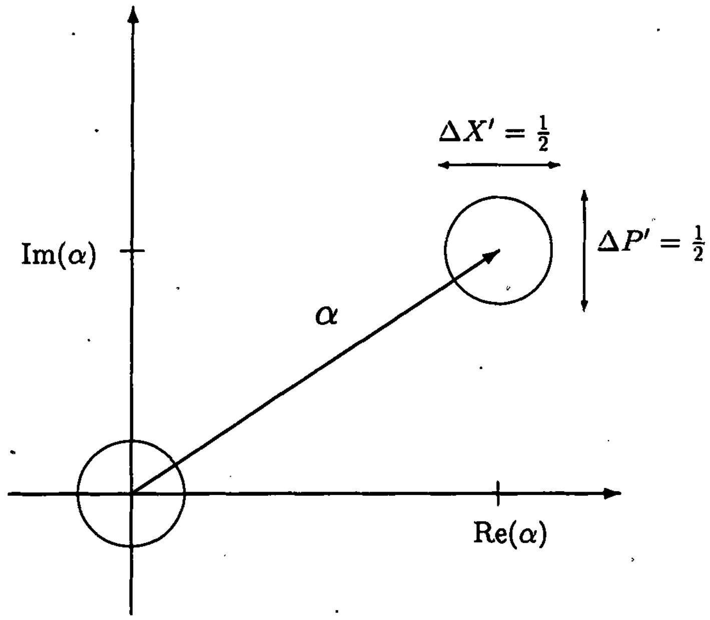

Figure 1: The Uncertainty Circle for the coherent state

The Heisenberg–Weyl displacement operator, described in chapter 4, can be written as

$$D(\alpha) = \exp(\alpha b^\dagger - \bar{\alpha} b) = \exp(2i\{\mathrm{Im}(\alpha)X' - \mathrm{Re}(\alpha)P'\}) \tag{8.12}$$

which implies that $D(\alpha)$ acts to displace the quadrature operators

$$X' \to D(\alpha)^\dagger X' D(\alpha) = X' + \mathrm{Re}\ \alpha \tag{8.13}$$

$$P' \to D(\alpha)^\dagger P' D(\alpha) = P' + \mathrm{Im}\ \alpha \tag{8.14}$$

i.e. $D(\alpha)$ translates the vacuum error contour from being centred at the origin to being centred at $(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha))$, without changing its shape, the diameter remaining equal to $\Delta X' = \Delta P' = \frac{1}{2}$.

8.2 The Squeezing of Photon States

The Photon coherent states described in chapter 4 are minimum uncertainty states for which both field uncertainties are equal to $\frac{1}{\sqrt{2}}$. A natural question is whether there exist states which have an uncertainty product that satisfies the lower bound of the uncertainty relation but have unequal quadrature variances. This has been the subject of a great deal of recent research. On the theoretical side, a number of new techniques have been developed to investigate such novel objects as squeezed states (see [LK:87] and references therein), correlated coherent states [DKM:80, DM:94] and generalised intelligent states [Tri:93, Tri:94], while experimentalists have had great success both in actually producing the optical states and also observing the predicted non–classical effects [SHYMV:85, SLPDW:86, WKHW:86].

Generalisations of the minimum uncertainty states, for which the quadrature values were unequal, arose in the work of several authors in the 1970's although the earliest reference is probably Takahasi [Tak:65] and Robinson [Robi:65]. Just as Glauber had derived the properties of coherent states by considering the photon field correlation function, so this approach lead Stoler [Sto:71] to other non–classical states of light which were obtained from the ordinary coherent states through the action of a further unitary transformation. This formalism was later taken up by Hollenhorst [Hol:79] who coined the term *squeezed state*. The work of Yuen [Yue:76a, Yue:76b] on quantum communication theory was the first to systematically consider production of these states and used the Bogoliubov automorphism of the boson algebra to describe states which had a minimum uncertainty product but with unequal quadrature variances. An equivalent formalism was used by Caves [Cav:81] in his analysis of interferometry–based gravitational wave detectors. Squeezed states were first realised experimentally in the work of Slusher et al using four–wave mixing in Sodium atoms, where the optical noise was reduced below the vacuum fluctuation by 7 to 10 per cent [SHYMV:85]. Some experiments have reported squeezing by as much as 63 per cent below that of the vacuum [WKHW:86].

In the approach of Yuen, squeezed states are the coherent states of quasiexcitations formed from automorphisms of the boson algebra. It has long been known from condensed matter physics that the algebra of creation and annihilation operators has a linear automorphism called the Bogoliubov canonical transformation [Bog:47]. This is the mapping

$$\begin{array}{ccc} b & \rightarrow & b' = \lambda b + \mu b^\dagger \\ b^\dagger & \rightarrow & b'^\dagger = \bar{\lambda} b^\dagger + \bar{\mu} b \end{array} \tag{8.15}$$

where the complex numbers λ and μ satisfy

$$|\lambda|^2 - |\mu|^2 = 1 \tag{8.16}$$

This means that the matrix

$$\begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \bar{\lambda} \end{pmatrix} \tag{8.17}$$

is an element of the Lie group $Sp(2, \mathbb{R}) \cong SU(1,1)$. By Von Neumann's theorem [Neu:31], this automorphism can be realised as a unitary transformation in the boson Hilbert space. The automorphism is accomplished by factoring out the isotropy subgroup from $SU(1,1)$, taking a representative of the coset and using it to implement its adjoint action on the generators of the boson algebra. From chapter 3, the required coset element is $\exp(\xi K_+ - \bar{\xi} K_-)$. The one–mode oscillator considered in (8.15) has an $su(1,1)$ realisation given by (4.94). This means that the unitary operator is

$$U_S(\xi) = \exp\left(\frac{1}{2}\{\xi b^{\dagger 2} - \bar{\xi} b^2\}\right) \tag{8.18}$$

and the Bogoliubov transformation is implemented by

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} \rightarrow U_S(\xi)^\dagger \begin{pmatrix} b \\ b^\dagger \end{pmatrix} U_S(\xi) \tag{8.19}$$

In some senses, therefore, the action of the squeezing operator is to produce states similar to the $SU(1,1)$–coherent states described in chapter 3.

If $|0\rangle_s$ is the vacuum in the new Fock space, we can form the coherent states of the transformed boson operators in the standard way by applying the displacement operator, i.e.,

$$|\alpha\rangle_s = \exp(\alpha b'^\dagger - \bar{\alpha} b')|0\rangle_s \tag{8.20}$$

The operator (8.18) clearly involves the creation and annihilation of pairs of photons and the displacement operator (8.20) simply creates a coherent state in the transformed Fock space. Yuen therefore called such states *two–photon coherent states.*

The equivalent formulation of Caves is given in terms of the original boson operators. The first step is the production of the *squeezed vacuum state* through the action of a unitary *squeezing operator* $S(\xi)$ on the groundstate $|0\rangle$. This squeezed vacuum is then displaced by the action of the conventional displacement operator.

8.2.1 Phase Space Description of Squeezing

If we consider states where the values of the quadrature variances are unequal but still give the minimum value on multiplication, this corresponds to deforming the uncertainty circle into an *uncertainty ellipse* which has major and minor axes parallel to the X' and P' axes. This transformation is given by

$$(\Delta X')^2 = \frac{1}{4}\exp(-2s) \tag{8.21}$$

$$(\Delta P')^2 = \frac{1}{4}\exp(2s) \tag{8.22}$$

and is equivalent to

$$X' \to X'_s = X' \exp(-s)\ , \qquad P' \to P'_s = P' \exp(s) \tag{8.23}$$

Forming the creation and annihilation operators, for which the new operators X'_s and P'_s are the real and imaginary parts, we obtain the *Squeeze Transformation*

$$\begin{aligned} b_s &= b\cosh s - b^\dagger \sinh s \\ b_s^\dagger &= b^\dagger \cosh s - b\sinh s \end{aligned} \tag{8.24}$$

A more general area-preserving transformation is obtained by allowing the uncertainty circle to undergo a phase space rotation through an arbitrary angle, say $\frac{\theta}{2}$, prior to squeezing by (8.24). The rotated quadrature variables X'' and P'' are related to the old ones by

$$X'' + iP'' = (X' + iP')\exp(i\theta/2) \tag{8.25}$$

Equation (8.24) then gives

$$\begin{aligned} b_s &= b\cosh s - b^\dagger \exp(i\theta)\sinh s \\ b_s^\dagger &= b^\dagger \cosh s - b\exp(-i\theta)\sinh s \end{aligned} \tag{8.26}$$

which we will call the *General Squeezing Transformation* and is implemented by the unitary operator

$$S(\xi) = \exp\left(-\frac{1}{2}\{\xi b^{\dagger 2} - \bar{\xi} b^2\}\right) \tag{8.27}$$

where

$$\xi = s\exp(i\theta)\ , \quad 0 \le s < \infty\ , \quad 0 \le \theta \le 2\pi \tag{8.28}$$

The action of this operator on the quadrature variables is given by

$$S(\xi)^\dagger (X'' + iP'')S(\xi) = X''\exp(-r) + iP''\exp(r) \tag{8.29}$$

so we see that in the rotated frame, the squeezing transformation does squeeze the quantum noise in one phase space direction with a consequent amplification in the other.

In terms of the displacement and squeezing operators, the squeezed state in the Stoler/Caves formalism can be written as

$$\begin{aligned} |\xi, \alpha\rangle &= D(\alpha)S(\xi)|0\rangle && (8.30) \\ &= \exp\left(\alpha b^\dagger - \bar{\alpha} b\right) \exp\left(-\frac{1}{2}\{\xi b^{\dagger 2} - \bar{\xi} b^2\}\right)|0\rangle && (8.31) \end{aligned}$$

The equivalent formalism of Yuen gives

$$|\beta;\mu,\nu\rangle = U_L\, D(\beta)|0\rangle \tag{8.32}$$

where U_L is the squeeze operator equivalent to $S(\xi)$ and the different parameters are related by

$$\begin{aligned} \xi &= r\exp(i\theta) \\ \beta &= \mu\alpha + \nu\bar{\alpha} \\ \mu &= \cosh r \\ \nu &= \exp(i\theta)\sinh r \end{aligned} \tag{8.33}$$

The name *squeezed state* was originally used to describe those states for which the noise fluctuations were less than those of the coherent state (and hence the vacuum state). It is known that in the coherent state, the quadratures are uncorrelated so the appropriate uncertainty relation is that of Heisenberg. In terms of the Bogoliubov transformation (8.15), this means that the phases of the parameters must coincide, and in terms of the generalised Squeezing transformation, (8.26), the θ variable must be zero. For $\theta \neq 0$, the covariance is non–zero and so the Robertson–Schrodinger relation is applicable. Consequently, for some values of θ close to $\pi/2$, both variances can be much greater than the corresponding fluctuations in the Glauber coherent state. For this reason, Dodonov and Man'ko [DM:94] have suggested that the states with non–zero covariance between the quadrature operators be called *Correlated Coherent States* to emphasise this statistical dependence. While this is a useful distinction to be made, it has not as yet become the general terminology of the quantum optics community. Consequently, in what follows, we use the term squeezed state to refer to both squeezed and correlated coherent states unless we wish to draw attention to their differences.

8.3 The Squeezed Vacuum State

8.3.1 The standard one–mode realisation

If we reorder the unitary Squeezing operator (8.27) using the Baker–Campbell–Hausdorff theorem, we find that

$$\begin{aligned} S(\xi) &= \exp\left(-\frac{1}{2}(\xi b^{\dagger 2} - \bar{\xi} b^2)\right) & (8.34) \\ &= \exp\left(-\frac{1}{2}\xi b^{\dagger 2}\right) A(N) \exp\left(-\frac{1}{2}\bar{\xi} b^2\right) & (8.35) \end{aligned}$$

where $A(N)$ is some function of the Number operator. It clear that the action on the vacuum of any exponentiated integer–power of the annihilation operator is trivial. The action of the middle term on the ground state is also easily calculated, and is equal to the normalisation function $A(\xi,\bar{\xi})$. Consequently, the states

$$S(\xi)|0\rangle \tag{8.36}$$

and

$$A(\xi,\bar{\xi})\exp\left(-\frac{\xi}{2}b^{\dagger 2}\right)|0\rangle \tag{8.37}$$

are the same even though the operators that act on the vacuum are not. This gives us a way of explicitly constructing the squeezed vacuum state that does not rely on knowledge of the exact form of the Squeezing operator.

Let us consider the state $|\xi\rangle$ given by

$$|\xi\rangle = A(\xi,\bar{\xi})\exp\left(-\frac{\xi}{2}b^{\dagger 2}\right)|0\rangle \tag{8.38}$$

where $A(\xi,\bar{\xi})$ is a normalisation function and ξ is not a phase. From the above considerations, it is clear that this state is the squeezed vacuum state. Direct calculation of the normalisation function shows that the explicit form of the squeezed vacuum is

$$|\xi\rangle = \left\{\sum_{n=0}^{\infty}\left(\frac{|\xi|}{2}\right)^{2n}\frac{(2n)!}{(n!)^2}\right\}^{-\frac{1}{2}}\exp\left(-\frac{\xi}{2}b^{\dagger 2}\right)|0\rangle \tag{8.39}$$

The action of the annihilation operator on $|\xi\rangle$ is

$$b|\xi\rangle = -\xi b^{\dagger}|\xi\rangle \tag{8.40}$$

This allows the calculation of the field variances in this state which gives

$$(\Delta X)^2_\xi = \frac{1}{2}\frac{(1-\xi)(1-\bar{\xi})}{1-|\xi|^2} \tag{8.41}$$

$$(\Delta P)^2_\xi = \frac{1}{2}\frac{(1+\xi)(1+\bar{\xi})}{1+|\xi|^2} \tag{8.42}$$

In addition, the covariance can be calculated to be

$$(\Delta XP) = \frac{1}{i2}\left\{\frac{(\bar{\xi}-\xi)}{1-|\xi|^2}\right\} \tag{8.43}$$

and so is non–zero only if ξ is real.

Calculation of the expectation value of the field component commutator shows that the Robertson–Schrodinger Uncertainty Relation is verified:

$$(\Delta X)_\xi(\Delta P)_\xi - (\Delta XP)^2_\xi = \frac{1}{4}\langle -i[X,P]\rangle_\xi \tag{8.44}$$

If one now acts on the squeezed vacuum state with the displacement operator, general squeezed or correlated states are obtained.

8.3.2 The Holstein–Primakoff realisations

As well as the bosonic realisation of the su(1,1) algebra given above, there is also another common realisation, namely the *Holstein-Primakoff realisation* [HP:40]. This is given by

$$K_+ = \sqrt{(2\sigma - 1 + N)}\, a^{\dagger} \tag{8.45}$$

$$K_- = a\sqrt{(2\sigma - 1 + N)} \tag{8.46}$$

$$K_0 = N + \sigma \tag{8.47}$$

These form an infinite–dimensional unitary irreducible representation with casimir value $\sigma(\sigma-1)$.

There is also a similar realisation of $su(2)$

$$J_+ = \sqrt{(2\sigma + 1 - N)}\, a^\dagger \tag{8.48}$$

$$J_- = a\sqrt{(2\sigma + 1 - N)} \tag{8.49}$$

$$J_0 = N - \sigma \tag{8.50}$$

The states spanning the basis for the $(2\sigma + 1)$–dimensional representation of $su(2)$ are the ordinary boson number states $|n\rangle$ where $0 \leq n \leq 2\sigma$ with the J_0–eigenvalue ranging from $-\sigma$ to σ. The group coherent states constructed from this representation also exhibit squeezing.

It has been shown by Katriel et al [KSDR:86] that the multiboson generalisation of this realisation can be used to generate squeezed states. The realisation makes use of the *Brandt–Greenberg multiphoton operators* [BrG:69, Ras:72, Kat:79], $B_{(k)}$, $B^\dagger_{(k)}$ defined by

$$B_{(k)} = b^k \left(\mathcal{I}\left(\frac{N}{k}\right)\frac{(N-k)!}{N!}\right)^{\frac{1}{2}}, \quad B^\dagger_{(k)} = \left(\mathcal{I}\left(\frac{N}{k}\right)\frac{(N-k)!}{N!}\right)^{\frac{1}{2}}(b^\dagger)^k \tag{8.51}$$

where k is a positive integer and the function $\mathcal{I}(u)$ is defined as the greatest integer less than or equal to u. It can be shown that

$$[B_{(k)}, B^\dagger_{(k)}] = I \tag{8.52}$$

i.e. the multiphoton operators realise the Heisenberg–Weyl algebra. Consequently, the group–theoretic coherent state constructions using boson realisations of the Lie algebras can be applied. For the multiboson $su(1,1)$–realisation, it was shown [KSDR:86] that in various limits, the state reproduced the behaviour of both the generalised boson coherent state of D'Ariano et al [DRV:85], and also the standard $su(1,1)$ realisation (4.94) using generators that are bilinear in the creation and annihilation operators.

It was also shown by Katriel et al [KRS:87] that the semigroup of nonlinear transformations from the boson operator b, to the multiboson operator $B_{(k)}$, can be extended to an Abelian group if k is allowed to take positive rational values. This allows the definition of *fractional bosons.* These can then be used to give Holstein–Primakoff realisations of $su(1,1)$ and $su(2)$, the coherent states of which again show squeezing.

8.4 Higher Order Squeezing

The Displacement Operator

$$D(\alpha) = \exp(\alpha b^{\dagger} - \alpha^{*} b) \tag{8.53}$$

and the Squeezing Operator

$$S(\xi) = \exp\left(\frac{\xi}{2} b^{\dagger 2} - \frac{\bar{\xi}}{2} b^{2}\right) \tag{8.54}$$

suggest that it would be worthwhile considering higher–order operators of the form

$$U_k(\chi) = \exp\left(\chi_k \, (b^{\dagger})^k - \overline{\chi_k} b^k\right) \tag{8.55}$$

Unfortunately, it has been shown by Fisher et al [FNS:84] that this näive generalisation does not work because the vacuum expectation value of the operator $U_k(\chi)$ diverges for $k > 2$. This may seem strange since the operator is apparently unitary. However, while the operator $\mathcal{O}_k = \chi_k (b^{\dagger})^k - \bar{\chi}_k b^k$ is antihermitian (i.e. $i\mathcal{O}_k$ is hermitian), it is not self–adjoint with respect to the Gaussian measure, and consequently cannot be exponentiated to give a unitary operator.

There are, however, other definitions which has been proposed to extend the concept of squeezing to higher order moments of the field quadratures. Hong and Mandel [HM:85] considered the nth–order moments of the real part of the quadrature, where squeezing was said to take place if the relevant value was less than that in the corresponding coherent state [GR:87]. Alternatively, Hillary [Hil:87] showed that squeezing of the square of the field amplitude was also a non–classical effect which was not equivalent to that described in [HM:85]. Multiphoton state nth–order squeezing was shown was to occur in [KSDR:86] while amplitude squeezing was predicted for fractional photons by D'Ariano [D'Ar:90].

Chapter 9

q–Deformed Squeezed and Correlated States

9.1 An Overview of q–Squeezed States

As is often the case with q–deformations, consideration of the different approaches to conventional squeezed states leads to different algebraic objects in the deformed theory. One must therefore pay close attention to the definition that is used.

9.1.1 Undeformed squeezed states generated by q–deformed bosons

It was pointed out by Celeghini et al [CDDRV:93] that, in some sense, the q–boson algebra $\mathcal{A}_q$ is already known to be associated with the generation of *conventional* squeezed states. If we consider the following differential realisation of a conventional set of boson operators $(b, b^{\dagger}, N)$ in the oscillator Hilbert space

$$b = \frac{1}{\sqrt{2}}\left(z + \frac{d}{dz}\right) , \quad b^{\dagger} = \frac{1}{\sqrt{2}}\left(z - \frac{d}{dz}\right) \tag{9.1}$$

this implies that

$$2z\frac{d}{dz}f(z) = (b^2 - b^{\dagger 2})f(z) - f(z) \tag{9.2}$$

One generator of conventional squeezed states is the element

$$S(\zeta) = \exp\left(\frac{\zeta}{2}(b^2 - b^{\dagger 2})\right) \tag{9.3}$$

and so

$$\exp\left(\frac{\zeta}{2}(b^2 - b^{\dagger 2})\right)\psi(z) = \exp\left(2z\frac{d}{dz}\right)\psi(z) \tag{9.4}$$

However, there is a realisation of the deformed algebra $\mathcal{A}_q$ in which $a = z$, $a^\dagger = {}_qD_z$ and $N = z\frac{d}{dz}$. Therefore

$$[a, a^\dagger]\psi(z) = q^N\psi(z) = \exp\left(\zeta z\frac{d}{dz}\right)\psi(z) = \frac{1}{\sqrt{q}}\exp\left(\frac{\zeta}{2}(b^2 - b^{\dagger 2})\right)\psi(z) \tag{9.5}$$

where $q = \exp(\zeta)$. Thus $\zeta = \log q$ plays the role of the squeezing parameter and, up to a factor of $\sqrt{q}$, the commutator $[a, a^\dagger]$ is the squeezing generator with respect to the operators b , $b^\dagger$

$$\sqrt{q}[a, a^\dagger] = S(\zeta) = \exp\left(\frac{\zeta}{2}(b^2 - b^{\dagger 2})\right) \tag{9.6}$$

While such an approach may lead to interesting applications in such areas as the study of dissipative systems and thermal field theory [IV:93], it will not be pursued here since we wish to use the formalism to describe new states with a genuine q–deformation.

9.1.2 The q–analogue Bogoliubov Transformation

The conventional squeezing transforms studied in the preceeding chapter are essentially based on the Bogoliubov automorphism [Bog:47] of the boson algebra. The formation of the squeezed state from the coherent state (or vacuum state) can be understood in terms of diagonalisation of the matrix associated with this transformation. It is fairly easy to show that, for the q–boson algebras $\mathcal{A}_q$ or $\mathcal{B}_q$, there is no complex–linear canonical automorphism. There is, however, a quantum group analogue of the automorphism, namely the $SU_q(1,1)$ coaction described in chapter 3. Unfortunately, the use of this mapping is problematic since the elements of the matrix are no longer complex numbers but non–commuting variables. This makes diagonalisation impossible by the usual means.

Another possibility is the use of some kind of non-linear automorphism of the "Bogoliubov type" which mixes q-boson creation and annihilation operators. This problem was considered by Fillipov et al [FGI:91] for the algebra $\mathcal{A}_q$ and by Van der Jeugt [Van:92] for $\mathcal{B}_q$. In the former case, an automorphism of the type

$$
\begin{aligned}
a' &= a\,u(N) + v(N)\,a^\dagger && (9.7)\\
a^{\dagger\prime} &= u(N)^* a^\dagger + a\,v(N)^* && (9.8)
\end{aligned}
$$

was proposed. This has general solution

$$
\begin{aligned}
u(N) &= |u(N)|\,e^{i\,\alpha(N,q)} && (9.9)\\
v(N) &= q^N W^{1/2} |u(N)|\,e^{i\,\beta(N,q)} && (9.10)
\end{aligned}
$$

with

$$|u(N)| = \left\{ \frac{(1-q^N W)}{(1-q^{2N-1}W)(1-q^{2N+1}W)} \right\}^{\frac{1}{2}} \tag{9.11}$$

and $\alpha(N,q)$, $\beta(N,q)$ are arbitrary operator-valued phase factors and $W \equiv W(q)$ is some function of q only. In the $q \to 1$ limit, taken so that the functions $u(N)$, $v(N)$ are independent of N, we see that

$$
\begin{aligned}
u &= \left(\frac{1}{1-W(1)}\right)^{\frac{1}{2}} e^{i\,\alpha(1)} \in \mathbb{C} && (9.12)\\
v &= \left(\frac{W(1)}{1-W(1)}\right)^{\frac{1}{2}} e^{i\,\beta(1)} \in \mathbb{C} && (9.13)
\end{aligned}
$$

which can be related to the coefficients of the Bogoliubov transformation.

While this procedure does indeed provide an automorphism of the algebra, the complicated nature of the function $|u(N)|$ means that it is extremely difficult to calculate the properties of those transformed states that would be the q-analogue of the Yuen two-photon coherent states. Consequently, this definition of squeezed states will not be pursued.

The Van der Jeugt automorphism of the algebra $\mathcal{B}_q$

$$c = \varepsilon t\gamma(t)a + \frac{\varepsilon'}{t-t^{-1}}a^\dagger\frac{1}{\gamma(t)} \tag{9.14}$$

$$\bar{c} = \varepsilon'\gamma(t)a - \frac{\varepsilon}{t-t^{-1}}a^\dagger\frac{t^{-1}}{\gamma(t)} \tag{9.15}$$

where $\varepsilon^2 = (\varepsilon')^2 = 1$ and $\gamma(t)$ is some arbitrary function in $t = q^N$, is non-unitary and has no analogue in the conventional $(q \to 1)$ case.

9.1.3 su$_q$(1,1) and su$_q$(2)–Squeezed States using q–Boson Realisations

Given the existence [KD:90] of a quadratic $\mathcal{B}_q$–type q–boson realisation of the algebra $su_q(1,1)$, we can form squeezed states by q–exponentiating the relevant raising operator. The properties of such states were analysed by Solomon and Katriel in [SK:90] and found to exhibit squeezing for all real values of q. In addition to this, the dynamics of such states have been examined by several authors [CRV:91, Buz:91] and squeezing has been found.

We can also look for q–analogues of squeezed states which use the q–deformed Holstein–Primakoff realisation. Such states have been extensively investigated for the algebra $\mathcal{B}_q$ by Chaichian et al [CEK:90], Katriel and Solomon [KS:91a] and others [BN:94]. The Holstein–Primakoff realisations for the algebra $su_q(1,1)$, using the bosons of the $\mathcal{B}_q$ algebra, is

$$K_+ = \sqrt{[\![2\sigma - 1 + N]\!]}\, a^\dagger_{(q)} \tag{9.16}$$

$$K_- = a_{(q)}\sqrt{[\![2\sigma - 1 + N]\!]} \tag{9.17}$$

$$K_0 = N + \sigma \tag{9.18}$$

while the realisation for $su_q(2)$ is

$$J_+ = \sqrt{[\![2\sigma + 1 + N]\!]}\, a^\dagger_{(q)} \tag{9.19}$$

$$J_- = a_{(q)}\sqrt{[\![2\sigma + 1 + N]\!]} \tag{9.20}$$

$$J_0 = N - \sigma \tag{9.21}$$

Multi–photon realisations of these relations also occur. Although analysis of the optical behaviour has to proceed by numerical rather than algebraic means, the results show that squeezing does occur for this type of deformed system. There are also q–analogue fractional boson Holstein–Primakoff realisations which, again, exhibit squeezing [SK:93].

9.2 Squeezed States using q–Numbers

The squeezed states described above are characterised by the fact that the label space is classical, i.e. it is parameterised by c–numbers. Given the success in constructing a unitary q–displacement operator for the algebra $\mathcal{A}_q$ by use of q–numbers, it is interesting to discover whether a similar procedure can be used to give a squeezing operator.

If we consider the simplest conventional, one–mode unitary squeezing operator, (8.27), we see that the operator which is exponentiated is essentially bilinear in creation and annihilation operators. Following the theory of chapter 7, this suggests that in the deformed case we should consider the commutation relations between a^2 and $a^{\dagger 2}$.

As pointed out in chapter 3, it is possible to form the two parameter quantum group deformation of the $su(1,1)$ algebra which has a realisation in terms of the (q,p)–bosons of Chakrabarti and Jagannathan [CJ:91]. This has an obvious specialisation ($q \longrightarrow q^2, p \longrightarrow 1$, say) in terms of $\mathcal{A}_q$–type q–bosons and the resulting $su_{q^4,1}(1,1)$ quantum algebra[1] is characterised by the relations

$$\begin{aligned} [K_0, K_\pm] &= \pm K_\pm \\ K_-K_+ - q^2 K_+K_- &= [\![2K_0]\!]_{q^4,1} = [2K_0]_{q^2} \end{aligned} \tag{9.22}$$

where the generators are given by

$$K_0 = \frac{1}{2}(N + 1/2)\ , \qquad K_+ = (K_-)^\dagger = [2]_{q^2}^{-1} a^{\dagger 2} \tag{9.23}$$

[1]The notation here is somewhat unfortunate. In order to keep the mapping from $\mathcal{B}_q$ to $\mathcal{A}_q$ as simple as possible, the two "boxes" $[\cdot]$, and $[\![\cdot]\!]$, are defined using different bases so that $[\![\cdot]\!]_{q^2,1} \equiv [\cdot]_q$.

The q^2–commutation relation between the K_+ and K_- can therefore be written as

$$\frac{a^2}{[2]_{q^2}}\frac{a^{\dagger 2}}{[2]_{q^2}} - q^2\frac{a^{\dagger 2}}{[2]_{q^2}}\frac{a^2}{[2]_{q^2}} = [N+1/2]_{q^2} \tag{9.24}$$

Equation (9.22) is formally similar to the defining relation of the $\mathcal{A}_q$ algebra

$$aa^\dagger - qa^\dagger a = I \ \ (= [I]_q) \tag{3.22}$$

There is, however, one important difference. The Identity operator I, being central, commutes with every other operator and so behaves like a c–number. This has a number of consequences which are crucial to the success of the procedure which led to the construction of the unitary operator in chapter 7.

Firstly, we recall that equation (3.22) was iterated to obtain

$$a(a^\dagger)^n - q^n(a^\dagger)^n a = [n]_q(a^\dagger)^{n-1} \tag{7.70}$$

Iteration of (9.24) leads to the relation

$$\frac{a^2}{[2]_{q^2}}\left\{\frac{a^{\dagger 2}}{[2]_{q^2}}\right\}^n - q^{2n}\left\{\frac{a^{\dagger 2}}{[2]_{q^2}}\right\}^n\frac{a^2}{[2]_{q^2}} = [n]_{q^2}\left\{\frac{a^{\dagger 2}}{[2]_q}\right\}^{n-1}[N+\tfrac{1}{2}+(n-1)]_{q^2} \tag{9.25}$$

which implies

$$K_-K_+^n - q^{2n}K_+^nK_- = [n]_{q^2}K_+^{n-1}[2K_0+(n-1)]_{q^2} \tag{9.26}$$

The operator $[2K_0+(n-1)]_{q^2}$ is neither central nor independent of (what would have been the summation index) n. Consequently, we are not able to form the q–exponential in the same way as was done in chapter 7. This implies that the expansion of the product

$$E_{q^2}(K_+)\,K_-E_{q^{-2}}(-K_+) \tag{9.27}$$

does not terminate after a finite number of terms. Unfortunately, the development of the reordering relations (7.60, 7.61, 7.62) which are at the heart of the technique is dependent upon the termination of the series of nested q–commutators. It therefore seems likely that the construction of a unitary squeezing operator by these means is not possible. It remains an open question, however, whether the quadratic Hopf–algebraic deformations of $sl(2)$ such as those given by Woronowicz [Wor:87a], Witten [Wit:90] or Fairlie [Fai:90], or even more general quantum deformations [FZ:91, FN:94] can be used to make any progress in this area.

If we set aside the search for a squeezing operator, we can nevertheless still look for a construction of the squeezed state in the same way that coherent states can be formed without the explicit knowledge of the displacement operator. One very useful result that gives a clue to the form of the such a state is due to Solomon [Sol:92]. He considered the defining relation of the squeezed state to be given by

$$(a + \rho a^{\dagger})|\sigma; \rho\rangle = \tau|\sigma; \rho\rangle \tag{9.28}$$

where τ is some quantity related to ρ and σ. This is a direct analogue of the conventional case. From (8.37) the conventional one–mode squeezed state may be written

$$|\lambda; \nu\rangle = A(\nu; \lambda) \exp\left(-\frac{1}{2}\nu b^{\dagger 2}\right) \exp(\lambda b^{\dagger})|0\rangle \tag{9.29}$$

where λ, $\nu \in \mathbb{C}$. This gives the eigenvalue equation

$$(a + \nu a^{\dagger})|\lambda; \nu\rangle = \lambda|\lambda; \nu\rangle \tag{9.30}$$

For the q–case, Solomon showed that the unnormalised state

$$|\lambda; \nu) = E_{q^2}\left(-\frac{1}{[2]_q}\nu a^{\dagger 2}\right) E_q(\lambda a^{\dagger})|0\rangle \tag{9.31}$$

satisfied (9.28) provided λ and ν are q–numbers rather than c–numbers. The quantities now have to obey

$$\lambda\nu = q^2\nu\lambda \tag{9.32}$$

The form of (9.31) may require some explanation. For example, it might have been thought that the correct expression should have involved only q–exponentials rather than the q^2–exponential that does appear. However, by use of the standard differential realisation of the algebra $\mathcal{A}_q$ ($a \rightarrow {}_qD_x$, $a^\dagger \rightarrow x$) and the q–deformed Leibniz and Chain rules for the q–derivative [Sol:92, FZ:91], it can be seen that (9.31) does indeed give a sensible definition.

As it stands, the state $|\lambda;\nu)$ given by (9.31) is somewhat problematic. By setting the q–number ν to zero, it can be clearly seen that the state is a squeezed form of the q–coherent states of Kowalski and Rembielinski, not those found in [MS:94b]. As noted there, this means that evaluation of expectation values is extremely difficult. Instead of using the state defined by (9.31) in that form, we will proceed by an indirect method. We will first calculate the squeezed vacuum state and then use the unitary operator defined in chapter 7 to generate a set of states by displacing the squeezed vacuum. These will then be shown to have the required eigenvalue properties.

9.3 The q–Squeezed Vacuum State

The form of the squeezed state proposed by Solomon suggests that we consider as a candidate for the *q–squeezed vacuum*, the vector $|v\rangle$ given by

$$|v\rangle = E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right) A(\xi,\xi^*)|0\rangle \tag{9.33}$$

where ξ, ξ^* are formally conjugate variables. By explicit calculation the following result can be seen to hold (see appendix)

$$a|v\rangle = -\xi a^\dagger|v\rangle \tag{9.34}$$

If

$$[\xi,\xi^*] = 0 \tag{9.35}$$

the left and right eigenvalue eigenstates are identical and we can extend (9.34) to

$$\begin{aligned} a|v\rangle &= -\xi a^\dagger|v\rangle &= -a^\dagger|v\rangle\,\xi \\ \langle v|a^\dagger &= -\xi^*\langle v|a &= -\langle v|\xi^* \end{aligned} \tag{9.36}$$

We can also rewrite $|v\rangle$ as

$$|v\rangle = A(\xi,\xi^*)\, E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle \tag{9.37}$$

where normalisation of the squeezed–vacuum

$$\langle v|v\rangle = 1 \tag{9.38}$$

implies that the normalisation constant $A(\xi,\xi^*)$ is given by

$$A(\xi,\xi^*) = A(\xi\xi^*) = \left\{\sum_{n=0}^{\infty} \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n]_q!}{([n]_{q^2}!)^2}\right\}^{-\frac{1}{2}} \tag{9.39}$$

9.3.1 Noise Properties of the q–Squeezed Vacuum State

In what follows we use the following two lemmas proved in the appendix:

Lemma 1:

$$\langle 0|a^m a^{\dagger n}|0\rangle = [n]_q!\delta_{n,m} \tag{9.40}$$

Lemma 2:

$$\langle 0|a^m\, f(N)\, a^{\dagger n}|0\rangle = [n]_q!\delta_{n,m} f(n) \tag{9.41}$$

Then

$$\begin{aligned}\langle v|a|v\rangle &= A(\xi\xi^*)\langle 0|\left\{\sum_{m,n} \frac{(-\frac{1}{[2]_q}\xi^*)^m}{[m]_{q^2}!} a^{2m} a \frac{(-\frac{1}{[2]_q}\xi)^n}{[n]_{q^2}!}(a^\dagger)^{2n}\right\}|0\rangle A(\xi\xi^*) \\ &= 0 \end{aligned} \tag{9.42}$$

since $\langle v|a^{2m}a(a^\dagger)^{2n}|v\rangle = 0$ for $m,n \in \mathbb{N}$. Similarly

$$\langle v|a^\dagger|v\rangle = 0 \tag{9.43}$$

Since the phase space picture of the squeezing process is problematic, there is no advantage to be gained in using the quadrature operators rather than the field components. We therefore use the conventional definition of the field components. For the X–component, we obtain

$$\begin{aligned}\langle X\rangle_v &= \langle v|X|v\rangle &(9.44)\\ &= \langle v|2^{-1/2}(a+a^\dagger)|v\rangle &(9.45)\\ &= 0 &(9.46)\end{aligned}$$

Using the property (9.36), the quadratic operator averages can be calculated.

$$\begin{aligned}\langle a^2\rangle_v &= \langle v|a^2|v\rangle &(9.47)\\ &= -\langle v|aa^\dagger|v\rangle\,\xi &(9.48)\\ &= -\xi\langle aa^\dagger\rangle_v &(9.49)\end{aligned}$$

and

$$\langle a^{\dagger 2}\rangle_v = -\xi^*\langle aa^\dagger\rangle_v \tag{9.50}$$

Then

$$\begin{aligned}X^2 &= \left\{\frac{1}{\sqrt{2}}(a+a^\dagger)\right\}^2 &(9.51)\\ &= \frac{1}{2}\left\{a^2+a^{\dagger 2}+aa^\dagger+a^\dagger a\right\} &(9.52)\\ &= \frac{1}{2}\left\{a^2+a^{\dagger 2}+(q^{-1}+1)aa^\dagger-q^{-1}\right\} &(9.53)\\ & &(9.54)\end{aligned}$$

We will use another lemma, (proved in the appendix)

Lemma 3:

$$\langle aa^\dagger\rangle_v = \frac{1}{1-q\xi\xi^*} = \frac{1}{1-q\xi^*\xi} \tag{9.55}$$

Therefore we have that

$$\begin{aligned}\langle X^2\rangle_v &= \frac{1}{2}\{(1+q^{-1})-\xi-\xi^*\}\left(\frac{1}{1-q\xi\xi^*}\right)-\frac{q^{-1}}{2}\\ &= \frac{1}{2}\left\{\frac{1-\xi-\xi^*+\xi\xi^*}{1-q\xi\xi^*}\right\}\\ &= \frac{1}{2}\frac{(1-\xi)(1-\xi^*)}{1-q\xi\xi^*} &(9.56)\end{aligned}$$

Then the variance of the X-component of the field in the squeezed vacuum is given by

$$(\Delta X)_v^2 = \langle X^2\rangle_v - \langle X\rangle_v^2 = \frac{1}{2}\frac{(1-\xi)(1-\xi^*)}{1-q\xi\xi^*} \tag{9.57}$$

If we now calculate the variance in the P-component, we find that

$$\langle P\rangle_v = \langle v|\frac{1}{i\sqrt{2}}(a-a^\dagger)|v\rangle = 0 \tag{9.58}$$

and

$$\begin{aligned}
\langle P^2\rangle_v &= -\frac{1}{2}\left\{\langle a^2 + a^{\dagger 2} - aa^\dagger - a^\dagger a\rangle_v\right\} && (9.59)\\
&= -\frac{1}{2}\left\{\langle a^2\rangle_v + \langle a^{\dagger 2}\rangle_v - (1+q^{-1})\langle aa^\dagger\rangle_v + q^{-1}\right\} && (9.60)\\
&= -\frac{1}{2}\left\{q^{-1} - \{(1+q^{-1}) + \xi + \xi^*\}\langle aa^\dagger\rangle_v\right\} && (9.61)\\
&= \frac{1}{2}\left\{\frac{1+\xi+\xi^*+\xi\xi^*}{1-q\xi\xi^*}\right\} && (9.62)\\
&= \frac{1}{2}\frac{(1+\xi)(1+\xi^*)}{1-q\xi\xi^*} && (9.63)
\end{aligned}$$

So the variance of the P-component of the field in the squeezed vacuum is given by

$$(\Delta P)_v^2 = \frac{1}{2}\frac{(1+\xi)(1+\xi^*)}{1-q\xi\xi^*} \tag{9.64}$$

We also calculate the covariance of X and P.

$$(\Delta XP) = \langle\ \frac{1}{2}(XP+PX)\ \rangle - \langle X\rangle\langle P\rangle \tag{9.65}$$

which implies that, in the squeezed vacuum $|v\rangle$,

$$(\Delta XP)_v = \langle\frac{1}{2}(XP+PX)\ \rangle_v \tag{9.66}$$

Now

$$\frac{1}{2}(XP+PX) = -i\,(a^2 - a^{\dagger 2}) \tag{9.67}$$

So

$$\begin{aligned}
(\Delta XP)_v &= \frac{1}{2i}\langle v|(a^2 - a^{\dagger 2})|v\rangle && (9.68)\\
&= \frac{(\xi^* - \xi)}{2i}\langle aa^\dagger\rangle && (9.69)\\
&= \frac{1}{2i}\left(\frac{\xi^*-\xi}{1-q\xi\xi^*}\right) && (9.70)
\end{aligned}$$

Then

$$
\begin{aligned}
& (\Delta X)_v^2\,(\Delta P)_v^2 - (\Delta XP)_v^2 && (9.71)\\
&= \frac{1}{4}\left\{\frac{(1-\xi)(1-\xi^*)(1+\xi)(1+\xi^*)}{(1-q\xi\xi^*)^2}\right\} - \frac{1}{4}\left(\frac{\xi^*-\xi}{1-q\xi\xi^*}\right)^2 && (9.72)\\
&= \frac{1}{4}\frac{(1-\xi\xi^*)^2}{(1-q\xi\xi^*)^2} && (9.73)\\
&= \left\{\frac{1}{2}\left(\frac{1-\xi\xi^*}{1-q\xi\xi^*}\right)\right\}^2 && (9.74)
\end{aligned}
$$

Now

$$[X,P] = i[a,a^\dagger] = iq^N \qquad (9.75)$$

and since

$$aa^\dagger = [N+1]_q = \frac{q^{N+1}-1}{q-1} \qquad (9.76)$$

we have

$$
\begin{aligned}
\langle q^N\rangle_v &= q^{-1}\{(q-1)\langle aa^\dagger\rangle_v + 1\} && (9.77)\\
&= \frac{1-\xi\xi^*}{1-q\xi\xi^*} && (9.78)
\end{aligned}
$$

Therefore the field observables in the squeezed–vacuum state, $|v\rangle$, satisfy

$$(\Delta X)_v^2\,(\Delta P)_v^2 - (\Delta XP)_v^2 = \frac{1}{4}\langle\, -i[X,P]\,\rangle_v \qquad (9.79)$$

which is the lower bound for the Robertson–Schrodinger Uncertainty Principle.

9.4 The Displaced q–Squeezed Vacuum State.

In the conventional theory, the squeezed states of the electromagnetic field are produced by applying the unitary displacement operator to the squeezed vacuum state. For example, if $S(\xi)$ is the operator which, when applied to the vacuum, produces the squeezed vacuum, and $D(z)$ is the displacement operator, then the most general squeezed state would be

$$|\xi, z\rangle = D(z)\,S(\xi)\,|0\rangle \qquad (9.80)$$

Ideally, when looking for a q–deformed version of this, we would like the analogues of $D(z)$ and $S(\xi)$ to be unitary operators. A unitary q–analogue of the displacement operator was given in chapter 7. Unfortunately, the unitary analogue of the *squeezing operator* $S(\xi)$ has not yet been found. Nevertheless, we can produce a q–analogue of the general squeezed state, even if the general form of the *q–squeezing operator* is not known. This is done by applying the q–displacement operator to the q–squeezed vacuum state detailed in the previous section.

We recall that the q–analogue of the displacement operator is the operator $U(z, z^*)$ where

$$U(z, z^*) = E_q(-z^* z)^{\frac{1}{2}} E_q(z a^\dagger) E_{q^{-1}}(-z^* a) \tag{9.81}$$

where

$$U(z, z^*)^\dagger U(z, z^*) = U(z, z^*) U(z, z^*)^\dagger = I \tag{9.82}$$

and

$$U^\dagger a U = a + z q^N \tag{9.83}$$

$$U^\dagger a^\dagger U = a^\dagger + z^* q^N \tag{9.84}$$

So

$$\begin{aligned} U^\dagger X U &= U^\dagger \frac{1}{\sqrt{2}}(a + a^\dagger) U && (9.85)\\ &= \frac{1}{\sqrt{2}}\{a + a^\dagger + (z + z^*) q^N\} && (9.86)\\ &= X + \frac{1}{\sqrt{2}}(z + z^*) q^N && (9.87)\\ & && (9.88) \end{aligned}$$

and

$$\begin{aligned} U^\dagger P U &= U^\dagger \frac{1}{i\sqrt{2}}(a - a^\dagger) U && (9.89)\\ &= \frac{1}{i\sqrt{2}}\{a - a^\dagger + (z - z^*) q^N\} && (9.90)\\ &= P + \frac{1}{i\sqrt{2}}(z - z^*) q^N && (9.91) \end{aligned}$$

We consider the state $|s\rangle$ given by

$$
\begin{aligned}
|s\rangle &= U(z,z^*)|v\rangle && (9.92)\\
&= E_q(-z^*z)^{\frac{1}{2}}E_q(za^\dagger)E_{q^{-1}}(-z^*a)E_{q^2}(-\tfrac{1}{[2]_q}\xi a^{\dagger 2})A(\xi,\xi^*)|0\rangle && (9.93)
\end{aligned}
$$

Then clearly, since $|v\rangle$ is normalised and U is unitary, we have

$$
\langle s|s\rangle = \langle v|\,U^\dagger U\,|v\rangle = \langle v|v\rangle = 1 \tag{9.94}
$$

i.e. $|s\rangle$ is normalised.

If we calculate the expectation values of the creation and annihilation operators in this state, we obtain

$$
\begin{aligned}
\langle a\rangle_s = \langle s|a|s\rangle &= \langle v|U^\dagger a\,U|v\rangle && (9.95)\\
&= \langle v|a + z\,q^N|v\rangle && (9.96)\\
&= \langle v|a\,|v\rangle + \langle v|z\,q^N|v\rangle && (9.97)\\
&= \langle v|z\,q^N|v\rangle && (9.98)
\end{aligned}
$$

where we have used the fact, (9.42), that the expectation value of the annihilation operator in the squeezed vacuum is zero. Similarly we find that

$$
\langle a^\dagger\rangle_s = \langle v|z^* q^N|v\rangle \tag{9.99}
$$

Then the expectation values of the field quadratures X and P are

$$
\begin{aligned}
\langle X\rangle_s &= \langle v|\,U^\dagger\left(\frac{1}{\sqrt{2}}(a+a^\dagger)\right)U\,|v\rangle && (9.100)\\
&= \frac{1}{\sqrt{2}}\langle v|(a+a^\dagger)+(z+z^*)\,q^N|v\rangle && (9.101)\\
&= \frac{1}{\sqrt{2}}\langle v|(z+z^*)\,q^N|v\rangle && (9.102)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle P\rangle_s &= \langle v|\,U^\dagger\left(\frac{1}{i\sqrt{2}}(a-a^\dagger)\right)U\,|v\rangle && (9.103)\\
&= \frac{1}{i\sqrt{2}}\langle v|(a-a^\dagger)+(z-z^*)\,q^N|v\rangle && (9.104)\\
&= \frac{1}{i\sqrt{2}}\langle v|(z-z^*)\,q^N|v\rangle && (9.105)
\end{aligned}
$$

To evaluate these expectations, it is necessary to calculate the action of objects such as z^*q^N on the squeezed vacuum. This is done in the appendix. The main result we use is that if

$$\xi z = q^2 z \xi \tag{9.106}$$

and

$$\xi z^* = q^2 z^* \xi \tag{9.107}$$

then

$$\langle z^* q^N \rangle_v = \langle v | z^* q^N | v \rangle = z^* \tag{9.108}$$

and

$$\langle z\, q^N \rangle_v = \langle v | z\, q^N | v \rangle = z \tag{9.109}$$

so

$$\langle X \rangle_s = \frac{1}{\sqrt{2}}(z + z^*) \tag{9.110}$$

and

$$\langle P \rangle_s = \frac{1}{i\sqrt{2}}(z - z^*) \tag{9.111}$$

Hence $\langle X \rangle_s^2 = \frac{1}{2}(z + z^*)^2$ and $\langle P \rangle_s^2 = -\frac{1}{2}(z - z^*)^2$.

To calculate expectation values of the operators X^2 and P^2, we use the result that, for any function f of z and z^*,

$$\langle v | f(z, z^*) a\, q^N | v \rangle = 0 \tag{9.112}$$

This can be proved using lemma 2 of the previous section. Then

$$\begin{aligned}
\langle X^2 \rangle_s &= \langle s | X^2 | s \rangle && (9.113)\\
&= \langle v | U^\dagger X^2 U | v \rangle && (9.114)\\
&= \langle v | \left\{ X + \frac{1}{\sqrt{2}}(z + z^*) q^N \right\}^2 | v \rangle && (9.115)\\
&= \langle v | X^2 | v \rangle + \frac{1}{\sqrt{2}} \langle v | (z + z^*) \{ X q^N + q^N X \} | v \rangle && (9.116)\\
&\qquad + \frac{1}{2} \langle v | (z + z^*)^2 q^{2N} | v \rangle && (9.117)\\
&= \langle X^2 \rangle_v + \frac{1}{2} \langle v | (z + z^*)^2 q^{2N} | v \rangle && (9.118)
\end{aligned}$$

where equation (9.112) and its conjugate has been used in the last line.

The result

$$z\,q^N|v\rangle = |v\rangle\,z \tag{9.119}$$

has corollaries

$$\begin{aligned}
\langle v|\,z^2q^{2N}|v\rangle &= z^2 && (9.120)\\
\langle v|\,z^{*2}q^{2N}|v\rangle &= z^{*2} && (9.121)\\
\langle v|\,z\,z^*q^{2N}|v\rangle &= z\,z^* && (9.122)\\
\langle v|\,z^*zq^{2N}|v\rangle &= z^*z && (9.123)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2}\langle v|(z+z^*)^2q^{2N}|v\rangle &= \frac{1}{2}\{z^2+z^{*2}+zz^*+z^*z\} && (9.124)\\
&= \frac{1}{2}(z+z^*)^2 && (9.125)
\end{aligned}$$

and so, from (9.118), we see that

$$\langle X^2\rangle_s = \langle X^2\rangle_v + \frac{1}{2}(z+z^*)^2 \tag{9.126}$$

If we now calculate the variance of the field component X, we find that

$$(\Delta X^2)_s = \langle X^2\rangle_s - \langle X\rangle_s^2 \tag{9.127}$$

Using equations (9.110) and (9.126) we obtain

$$\begin{aligned}
(\Delta X^2)_s &= \langle X^2\rangle_v + \frac{1}{2}(z+z^*)^2 - \frac{1}{2}(z+z^*)^2 && (9.128)\\
&= \langle X^2\rangle_v && (9.129)\\
&= (\Delta X^2)_v && (9.130)
\end{aligned}$$

that is, using (9.56),

$$(\Delta X^2)_s = \langle X^2\rangle_v = \frac{1}{2}\,\frac{(1-\xi)(1-\xi^*)}{1-q\xi\xi^*} \tag{9.131}$$

We can calculate the variance in the P–component in a similar manner. We find that

$$\langle P\rangle_s = \frac{1}{i\sqrt{2}}(z-z^*)^2 \tag{9.132}$$

$$\langle P^2\rangle_s = \langle P^2\rangle_v - \frac{1}{2}(z-z^*)^2 \tag{9.133}$$

$$(\Delta P^2)_s = \langle P^2\rangle_v - \frac{1}{2}(z-z^*)^2 + \frac{1}{2}(z-z^*)^2 \tag{9.134}$$

$$= \langle P^2\rangle_v \tag{9.135}$$

so

$$(\Delta P^2)_s = \frac{1}{2}\frac{(1+\xi)(1+\xi^*)}{1-q\xi\xi^*} \tag{9.136}$$

$$= (\Delta P^2)_v \tag{9.137}$$

We can also calculate the covariance (ΔXP)

$$(\Delta XP)_s \equiv \langle \frac{1}{2}(XP+PX)\,\rangle_s - \left\{\frac{1}{2}\left(\langle X\rangle_s\langle P\rangle_s + \langle P\rangle_s\langle X\rangle_s\right)\right\} \tag{9.138}$$

where the noncommutivity of the expectation values, $\langle X\rangle_s$ and $\langle P\rangle_s$, necessitates the symmetrisation procedure in the last term.

From (9.67), we have that

$$\langle\, \frac{1}{2}(XP+PX)\,\rangle_s = \langle -i\,(a^2 - a^{\dagger 2})\,\rangle_s \tag{9.139}$$

$$= -i\left\{\langle a^2\rangle_s - \langle a^{\dagger 2}\rangle_s\right\} \tag{9.140}$$

$$= -i\left\{\left(\langle a^2\rangle_v + z^2\right) - \left(\langle a^{\dagger 2}\rangle_v + z^{*2}\right)\right\} \tag{9.141}$$

$$= -i\,\langle a^2 - a^{\dagger 2}\rangle_v - i\,(z^2 - z^{*2}) \tag{9.142}$$

Also, from (9.110) and (9.111),

$$\frac{1}{2}\left(\langle X\rangle_s\langle P\rangle_s + \langle P\rangle_s\langle X\rangle_s\right) = -i\,(z^2 - z^{*2}) \tag{9.143}$$

so

$$(\Delta XP)_s = -i\langle a^2 - a^{\dagger 2}\rangle_v - i\,(z^2 - z^{*2}) + i\,(z^2 - z^{*2}) \tag{9.144}$$

$$= -i\langle a^2 - a^{\dagger 2}\rangle_v \tag{9.145}$$

$$= (\Delta XP)_v \tag{9.146}$$

Thus, from equations (9.130), (9.137) and (9.146)

$$
\begin{aligned}
&(\Delta X)_s^2\,(\Delta P)_s^2 - (\Delta XP)_s^2 && (9.147)\\
&= (\Delta X)_v^2\,(\Delta P)_v^2 - (\Delta XP)_v^2 && (9.148)\\
&= \left\{\frac{1}{2}\left(\frac{1-\xi\xi^*}{1-q\xi\xi^*}\right)\right\}^2 && (9.149)
\end{aligned}
$$

The *Robertson–Schrodinger* uncertainty term for both the squeezed–vacuum state and the general squeezed state is therefore

$$
\begin{aligned}
\sqrt{(\Delta X)_s^2\,(\Delta P)_s^2 - (\Delta XP)_s^2} &= \sqrt{(\Delta X)_v^2\,(\Delta P)_v^2 - (\Delta XP)_v^2} && (9.150)\\
&= \frac{1}{2}\left(\frac{1-\xi\xi^*}{1-q\xi\xi^*}\right) && (9.151)
\end{aligned}
$$

The definition given above allows the same distinction, as in the conventional theory, to be drawn between *squeezed* states and *correlated coherent* states [DM:94]. We may consistently take the variable ξ to be stable under the $*$–involution (corresponding to a real value in the $q = 1$ case) which makes the covariance between the components vanish and the states obey the Heisenberg Relation. Alternatively, as we have seen above, the $*$–structure on the ξ variable may be such that $\xi \neq \xi^*$ in which case the states may be termed *q–Correlated Coherent States.*

9.4.1 Eigenvalue Properties of the q–Squeezed States

The properties of the q–displacement operator mean that

$$a\,U = U\,a + U\,z\,q^N \tag{9.152}$$

and

$$a^\dagger U = U\,a^\dagger + U\,z^* q^N \tag{9.153}$$

Taken with the eigenvalue properties (9.36) of the squeezed vacuum state, this allows us to calculate the eigenvalue properties of the general q–squeezed state.

$$
\begin{aligned}
a|s\rangle &= a\,U|v\rangle && (9.154)\\
&= U\,a\,|v\rangle + U.z\,q^N|v\rangle && (9.155)
\end{aligned}
$$

$$= U(-a^\dagger|v\rangle\,\xi) + U\,z\,q^N|v\rangle \tag{9.156}$$

$$= -\left\{a^\dagger U + U\,z^* q^N\right\}|v\rangle\,\xi + U\,z\,q^N|v\rangle \tag{9.157}$$

$$= -a^\dagger|s\rangle\,\xi - U\,z^* q^N|v\rangle\,\xi + U\,z\,q^N|v\rangle \tag{9.158}$$

Using (9.119) and its conjugate to move z^*q^N and $z\,q^N$ through $|v\rangle$ and tidying up the equation, we obtain

$$a|s\rangle + a^\dagger|s\rangle\,\xi = |s\rangle\,(z - z^*\xi) \tag{9.159}$$

which in the $q \longrightarrow 1$ limit becomes the familiar eigenvalue equation for the squeezed state

$$(a + \xi a^\dagger)|s\rangle = (z - z^*\xi)|s\rangle \tag{9.160}$$

The form of (9.159) explains why it was not possible to form the squeezed state using the $SU_q(1,1)$–coaction on the q–boson algebra. Multiplying the equation on the left by $\langle s|$, we see that

$$\langle s|a|s\rangle + \langle s|a^\dagger|s\rangle\,\xi = \langle s|s\rangle\,(z - z^*\xi) \tag{9.161}$$

This suggests that any transformation of Bogoliubov–type will be between squeezed expectation values rather than the q–boson operators themselves.

9.4.2 Other Definitions of the q–Squeezed State

If we simply consider the q–squeezed vacuum state, $|v\rangle$, we see that since ξ and ξ^* commute with each other, there is a realisation of the state in which they are conjugate complex numbers. Obviously the procedure detailed above to produce the full q–squeezed state would not be applicable to this realisation since $\xi, \xi^* \in \mathbb{C}$ would commute with the noncommuting variables z and z^*. However, the state $|v\rangle$ with $\xi, \xi^* \in \mathbb{C}$ (which we will henceforth denote as $|r\rangle$) is still of some interest. For example, equation (9.79) indicates that the Robertson–Schrodinger lower bound, $\mathcal{B}_{RS}$, is attained *numerically* rather than just algebraically. Moreover, we now have the numerical lower bound

$$\sqrt{(\Delta X)^2_r\,(\Delta P)^2_r - (\Delta XP)^2_r} = \frac{\rho}{2} \tag{9.162}$$

where ρ is the ratio

$$\rho = \left(\frac{1-|\xi|^2}{1-q\,|\xi|^2}\right) \tag{9.163}$$

If we calculate the bound in the (unsqueezed) vacuum $|0\rangle$,

$$\sqrt{(\Delta X)_0^2\,(\Delta P)_0^2 - (\Delta XP)_0^2} = \frac{1}{2} \tag{9.164}$$

Thus we see that for $q \in (0,1)$, the squeezed state has a lower value for $\mathcal{B}_{RS}$ than the vacuum value. Convergence requirements on ξ for the q^2–exponential means that ρ cannot be negative.

The state $|r\rangle$ was formed without an explicit knowledge of the unitary q–analogue of the squeezing operator, simply by q^2–exponentially acting with the square of the creation operator on the vacuum state and then normalising. In essence, this is the method used by most authors to construct quantum group coherent states. One could extend this procedure to give an analogue of the full squeezed state by exponentially acting with the creation operator on the q–squeezed vacuum $|r\rangle$. Such a state would have the advantage that the parameters are ordinary c–numbers and not non–commuting variables.

If we define a new state

$$|s_1\rangle = |\lambda,\xi\rangle_1 = N(\lambda,\xi)E_q(\lambda a^\dagger)|r\rangle \tag{9.165}$$

where $N(\lambda,\xi)$ is the normalisation, then using (7.70), we see that

$$\begin{aligned}
a|s_1\rangle &= N(\lambda,\xi)\,a\,E_q(\lambda a^\dagger)|r\rangle & (9.166)\\
&= N(\lambda,\xi)\left\{E_q(q\lambda a^\dagger)\,a|r\rangle + \lambda E_q(\lambda a^\dagger)|r\rangle\right\} & (9.167)\\
&= E_q(q\lambda a^\dagger)\{E_q(\lambda a^\dagger)\}^{-1}a|s_1\rangle + \lambda|s_1\rangle & (9.168)
\end{aligned}$$

so that we may use (5.78) to obtain

$$a|s_1\rangle = \{1+(q-1)a^\dagger\}(-\xi a^\dagger)|s_1\rangle + \lambda|s_1\rangle \tag{9.169}$$

i.e.

$$\left\{a + \xi a^\dagger + \xi(q-1)a^{\dagger 2}\right\}|s_1\rangle = \lambda|s_1\rangle \tag{9.170}$$

The deformation induces an $a^{\dagger 2}$ term which goes to zero in the $q \longrightarrow 1$ limit. Unfortunately this means that the algebraic method to find the variances, which was detailed above, no longer works and the uncertainty product has to be calculated numerically. Work on this problem is in progress.

9.5 Appendix: Proof of Results

Proof of Equation (9.36)

We wish to prove that

$$a|v\rangle = -\xi a^{\dagger}|v\rangle = -a^{\dagger}|v\rangle\,\xi \tag{9.171}$$

Firstly, we note that

$$\begin{aligned} a(\xi a^{\dagger 2})^n|0\rangle &= \{\xi^n q^{2n}(a^{\dagger})^{2n}a + \xi^n[2n]_q(a^{\dagger})^{2n-1}\}|0\rangle & (9.172)\\ &= \xi a^{\dagger}[2n]_q\xi^{n-1}(a^{\dagger})^{2(n-1)}|0\rangle & (9.173)\end{aligned}$$

Therefore

$$\begin{aligned} a\frac{(\xi a^{\dagger 2})^n}{[n]_{q^2}!}|0\rangle &= \xi a^{\dagger}\frac{[2n]_q}{[n]_{q^2}}\frac{(\xi a^{\dagger 2})^{n-1}}{[n-1]_{q^2}!}|0\rangle & (9.174)\\ &= [2]_q\xi a^{\dagger}\frac{(\xi a^{\dagger 2})^{n-1}}{[n-1]_{q^2}!}|0\rangle & (9.175)\end{aligned}$$

Consequently

$$a\,E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle = -\xi a^{\dagger}E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle \tag{9.176}$$

and the first part of equation (9.36) follows. Since ξ commutes with all the other elements that make up $|v\rangle$, the second part also follows. The result for $a^{\dagger}$ is obtained simply by taking the hermitian conjugate.

Proof of Lemmas 1 and 2

Clearly, lemma 1 is a special case of lemma 2 with a trivial function f(N). We will therefore only prove Lemma 2.

$$\langle 0|a^m\ f(N)\ a^{\dagger n}|0\rangle = [n]_q!\delta_{n,m}\ f(n) \tag{9.177}$$

We make use of the completeness property of the Fock space basis

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle \tag{9.178}$$

to see that

$$\begin{aligned}
\langle 0|a^m\ f(N)\ a^{\dagger n}|0\rangle &= \langle m|\sqrt{[m]_q!}\ f(N)\ \sqrt{[n]_q!}|n\rangle && (9.179)\\
&= \langle m|\sqrt{[m]_q![n]_q!}\ f(n)\ |n\rangle && (9.180)\\
&= \langle m|n\rangle\ \sqrt{[m]_q![n]_q!}\ f(n) && (9.181)\\
&= \delta_{m,n}\ [n]_q!\ f(n) && (9.182)
\end{aligned}$$

Proof of Lemma 3

We wish to prove the lemma that

$$\langle aa^\dagger\rangle_v = \frac{1}{1-q\xi\xi^*} \tag{9.183}$$

We first note the following preliminary results.

I.)

$$[2n]_q = [n]_{q^2}[2]_q \tag{9.184}$$

so that

$$\frac{[2n-1]_q[2n]_q[2n]_q}{[n]_{q^2}[n]_{q^2}[2]_q[2]_q} = [2n-1]_q \tag{9.185}$$

II.)

$$[2n+1]_q = q[2n]_q + 1 \tag{9.186}$$

III.)

$$\langle v|v\rangle = 1 \tag{9.187}$$

(I) and (II) are clear from the definition of the basic number. (III) can be proved by direct calculation.

$$\langle 0|E_{q^2}\left(-\frac{1}{[2]_q}\xi^* a^2\right) E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle \tag{9.188}$$

$$= \langle 0|\left\{\sum_{m,n}\frac{\left(-\frac{1}{[2]_q}\xi^*\right)^m}{[m]_{q^2}!}a^{2m}\frac{\left(-\frac{1}{[2]_q}\xi\right)^n}{[n]_{q^2}!}(a^\dagger)^{2n}\right\}|0\rangle \tag{9.189}$$

$$= \left\{\sum_{m,n}\frac{\left(-\frac{1}{[2]_q}\xi^*\right)^m}{[m]_{q^2}!}\langle 0|a^{2m}\,(a^\dagger)^{2n}|0\rangle\frac{\left(-\frac{1}{[2]_q}\xi\right)^n}{[n]_{q^2}!}\right\} \tag{9.190}$$

$$= \sum_{n=0}^{\infty}\frac{(\xi\xi^*)^n}{[2]_q^{2n}}\frac{[2n]_q!}{([n]_{q^2}!)^2} \tag{9.191}$$

$$= A(\xi\xi^*)^{-2} \tag{9.192}$$

where we have made use of Lemma 1 in line 3 of the above equation. Hence it is clear that $|v\rangle = A(\xi\xi^*)E_{q^2}\left(-\frac{1}{[2]_q}\xi^* a^2\right)|0\rangle$ is normalised.

IV.) We next prove the relation

$$S(\xi\xi^*) = A(\xi\xi^*)^2\sum_n\left\{\frac{(\xi\xi^*)^n}{[2]_q^{2n}}\frac{[2n]_q![2n]}{([n]_{q^2}!)^2}\right\} = \frac{\xi\xi^*}{1-q\xi\xi^*} \tag{9.193}$$

This is again done by explicit calculation

$$A(\xi\xi^*)^2\sum_n\left\{\frac{(\xi\xi^*)^n}{[2]_q^{2n}}\frac{[2n]_q![2n]}{([n]_{q^2}!)^2}\right\} \tag{9.194}$$

$$= A(\xi\xi^*)^2\sum_{n-1}\left\{(\xi\xi^*)\frac{(\xi\xi^*)^{n-1}}{[2]_q^{2(n-1)}}\frac{[2(n-1)]_q!}{([n-1]_q^2!)^2}\frac{[2n-1]_q[2n]_q[2n]_q}{[n]_{q^2}[n]_{q^2}[2]_q[2]_q}\right\}$$

$$= A(\xi\xi^*)^2\,(\xi\xi^*)\sum_{n-1}\left\{\frac{(\xi\xi^*)^{n-1}}{[2]_q^{2(n-1)}}\frac{[2(n-1)]_q!}{([n-1]_{q^2}!)^2}[2n-1]_q\right\} \tag{9.195}$$

$$= (\xi\xi^*)\,A(\xi\xi^*)^2\sum_k\left\{\frac{(\xi\xi^*)^k}{[2]_q^{2k}}\frac{[2k]_q!}{([k]_{q^2}!)^2}[2k+1]_q\right\} \tag{9.196}$$

$$= (\xi\xi^*)\, A(\xi\xi^*)^2 \sum_n \left\{ \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n]_q!}{([n]_{q^2}!)^2} \{q[2n]_q + 1\} \right\} \tag{9.197}$$

$$= q\,(\xi\xi^*)\, \underbrace{A(\xi\xi^*)^2 \sum_n \left\{ \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n]_q!}{([n]_{q^2}!)^2} [2n]_q \right\}}_{= S(\xi\xi^*)} \quad +$$

$$+ \quad (\xi\xi^*)\, \underbrace{A(\xi\xi^*)^2 \sum_n \left\{ \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n]_q!}{([n]_{q^2}!)^2} \right\}}_{= 1} \tag{9.198}$$

Thus we obtain the relation

$$S(\xi\xi^*) = q\,(\xi\xi^*)\, S(\xi\xi^*) + (\xi\xi^*) \tag{9.199}$$

which clearly proves (9.193)

$$A(\xi\xi^*)^2 \sum_n \left\{ \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n]_q![2n]}{([n]_{q^2}!)^2} \right\} = \frac{\xi\xi^*}{1 - q\xi\xi^*} \tag{9.200}$$

We are now in a position to prove the lemma.

$$\begin{aligned}
\langle aa^\dagger \rangle_v &= A(\xi\xi^*)^2 \langle 0| \left\{ \sum_{m,n} \frac{(-\frac{1}{[2]_q}\xi^*)^m}{[m]_{q^2}!} \frac{(-\frac{1}{[2]_q}\xi)^n}{[n]_{q^2}!} a^{2m+1} (a^\dagger)^{2n+1} \right\} |0\rangle \\
&= A(\xi\xi^*)^2 \sum_{n=0}^{\infty} \frac{(\xi\xi^*)^n}{[2]_q^{2n}} \frac{[2n+1]_q!}{([n]_{q^2}!)^2} & (9.201) \\
&= \frac{S(\xi\xi^*)}{\xi\xi^*} & (9.202)
\end{aligned}$$

where in the last line, we have use equations (9.193), (9.196) and (9.199). The lemma

$$\langle aa^\dagger \rangle_v = \frac{1}{1 - q\xi\xi^*} \tag{9.203}$$

is therefore proved.

Proof of Equations (9.108) and (9.109)

We consider the action of z^*q^N on the squeezed vacuum $|v\rangle$.

$$z^*q^N|v\rangle = z^*q^N E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle\ A(\xi,\xi^*) \tag{9.204}$$

$$= \sum_m \left\{ \frac{\left(-\frac{1}{[2]_q}\right)^m}{[m]_{q^2}!} z^*\xi^m q^N (a^\dagger)^{2m} \right\} |0\rangle\ A(\xi,\xi^*) \tag{9.205}$$

$$= \sum_m \left\{ \frac{\left(-\frac{1}{[2]_q}\right)^m}{[m]_{q^2}!} z^*\xi^m (a^\dagger)^{2m} q^{(N+2m)} \right\} |0\rangle\ A(\xi,\xi^*) \tag{9.206}$$

$$= \sum_m \left\{ \frac{\left(-\frac{1}{[2]_q}\right)^m}{[m]_{q^2}!} z^*\xi^m q^{2m} (a^\dagger)^{2m} \right\} |0\rangle\ A(\xi,\xi^*) \tag{9.207}$$

If

$$\xi z^* = q^2 z^* \xi \tag{9.208}$$

then

$$\xi^m z^* = q^{2m} z^* \xi^m \tag{9.209}$$

so given this constraint upon the noncommutative variable ξ and z^*, (with an associated constraint upon z and ξ^* by conjugation), we have

$$\begin{aligned} z^*q^N|v\rangle &= \sum_m \left\{ \frac{\left(-\frac{1}{[2]_q}\right)^m}{[m]_{q^2}!} \xi^m z^* (a^\dagger)^{2m} \right\} |0\rangle\ A(\xi,\xi^*) \\ &= E_{q^2}\left(-\frac{1}{[2]_q}\xi a^{\dagger 2}\right)|0\rangle\ z^* A(\xi,\xi^*) \\ &= |v\rangle\ A(\xi,\xi^*)^{-1} z^* A(\xi,\xi^*) \end{aligned}$$

Hence

$$\langle v|z^*q^N|v\rangle = \langle v|v\rangle\ A(\xi,\xi^*)^{-1} z^* A(\xi,\xi^*) \tag{9.210}$$

$$= A(\xi,\xi^*)^{-1} z^* A(\xi,\xi^*) \tag{9.211}$$

Now $A(\xi, \xi^*) = A(\xi\xi^*)$ so that we can commute the z^* through the normalisation function provided

$$\xi^* z^* = q^{-2} z^* \xi^* \tag{9.212}$$

i.e.

$$\xi\, z = q^2 z\, \xi \tag{9.213}$$

This explains the choice of commutation relations in (9.106) and (9.107).

Given the above conditions, we obtain the result (9.108) that

$$\langle v | z^* q^N | v \rangle = z^* \tag{9.214}$$

Equation (9.109) is obtained by taking the hermitian conjugate.

Chapter 10

The Use of Noncommuting Variables in Quantum Mechanical States

The use of noncommuting variables in the formalism outlined in chapters 7 and 9 clearly raises some important conceptual problems. The coherent states of the conventional boson algebra are defined in terms of commuting objects, i.e. complex numbers. These numbers have relatively straightforward interpretations in terms of distances in phase space and the associated uncertainties of the physical field. This situation changes dramatically with the introduction of the q–commuting variables of the Manin plane. The phase space description is no longer available and new intuititive pictures are needed to describe the situation. Indeed the conventional probabilistic quantum mechanical interpretation breaks down because square amplitudes no longer represent probabilities.

In some ways this problem is similar to that of the interpretation of anticommuting Grassmann variables that appear in many areas of modern physics, most notably in supersymmetric field theories. At the present time there has been little discussion [DH:87, Nie:92], and almost no consensus, on the physical meaning of such anticommuting terms. Nevertheless, if the present study of *super*–physics is to succeed at all, it must make contact with the measurable physical quantities (i.e. real numbers) that experimentalists obtain in the laboratory. While there is some slight evidence for supersymmetry, it has come from the study of areas such as nuclear [Iac:81] and atomic systems [KNT:88] and is indirect and inconclusive. However, this has not stopped the application of supergroup and superalgebra techniques to a whole host of problems from condensed matter physics (see [DRS:94] and references therein) to quantum optics [Nie:93]. What has motivated the large number of researchers is the fact that there exists an extension of the conventional formalism which has been found to be mathematically consistent. The fact that physicists feel able to proceed without a direct understanding of all the terms in their equations simply shows that the physical interpretation of a theory does not always precede its mathematical development. This in turn is reminiscent of the early days of quantum mechanics when, for example, discussion was aroused by the physical interpretation of imaginary numbers in the wavefunction.

Unfortunately, the problem with q–commuting terms is more complicated than that of Grassmann variables. The nilpotency of the latter objects simplifies the structure of their functions. It is also possible to define some kind of topology on the Grassmann plane, turning it into a Grassmann manifold. As yet, no such construction has appeared for the Manin plane. However, as stated in chapter 7, there have been attempts to use q–numbers (e.g. [GF:91]) in place of the usual field of complex numbers and even attempts to construct a topology by considering the elements of the quantum plane to be noncommuting coordinate functions on the space $\mathbb{R}^2$ [ShM:94]. This suggests that the problem needs to be addressed in the wider context of noncommutative geometry. Here, there is at least some understanding of the problems that spaces of noncommutative variables would solve, such as giving a sound basis to field theoretic renormalisation schemes, alleviating the need for a space–time manifold structure, etc. Recently, Schirrmacher [Schi:94] constructed another deformation of the exponential function which is related to orthogonal quantum symmetries and used it to construct plane–wave solutions with noncommuting variables. It is hoped that further investigation of the noncommutative differential geometry of quantum groups will allow a clarification of the issues involved and eventually lead to a resolution of the problem. Nevertheless, the existence of a formal noncommutative q–deformed extension of the coherent and squeezed states of bosonic operators suggests that it may be profitable to examine further systems which admit simple q–deformation.

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