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Optimal Favoritism in All-Pay Auctions and Lottery Contests*

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Abstract

We analyze the revenue-enhancing potential of favoring specific contestants in complete-information all-pay auctions and lottery contests with several heterogeneous contestants. Two instruments of favoritism are considered: head starts that are added to the bids of specific contestants and multiplicative biases that give idiosyncratic weights to the bids. In the all-pay auction, head starts are more effective than biases while optimally combining both instruments even yields first-best revenue. In the lottery contest, head starts are less effective than biases and combining both instruments cannot further increase revenue. As all-pay auctions revenue-dominate lottery contests under optimal biases, we thus obtain an unambiguous revenue-ranking of all six combinations of contest formats and instruments.

Key Words: All-pay auction, lottery contest, favoritism, head start, revenue dominance.

JEL classification: C72; D44; D72

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1 Introduction

Contests are frequently and increasingly used to allocate scarce resources among competing agents when other allocation mechanism like markets, matching, or bargaining protocols are not feasible, impractical, or not desired. A characteristic feature of contests is that participating agents exert effort or pay non-refundable bids to win an indivisible prize such that all agents incur their respective costs of effort exertion irrespectively of winning the prize or not. Examples range from promotion tournaments within firms to lobbying, from public procurement to rent-seeking, from high school admission to crowdsourcing, and from the allocation of research grants to innovation contests; see Konrad (2009),[29], and Vojnović (2016), [41], for excellent textbooks on contest theory and their applications, as well as Frank and Cook (2010), [16], and English (2005), [11], for popular approaches regarding the related phenomenon of winner-take-all-markets and the ubiquity of contests in arts and culture.

As the organizer of a contest typically has substantial discretionary power in designing the contest rules, there is a tendency for explicit or implicit favoritism with respect to specific agents. Consider, for instance, the preferential treatment of internal or external candidates in hiring decisions or of domestic or small business firms in public procurement, handicap systems in sports, affirmative action in high school admission, or simply discrimination in the sense that the contest organizer favors specific contestants which is manifested by tailoring the conditions of the contest to the advantage of the preferred contestants. In all these cases agents are treated asymmetrically, which might have profound implications for the underlying incentive structure of the contest. Hence, contest organizers should be aware of the consequences of different types and designs of preferential treatment.

In this paper, we consider two different types of preferential treatment, bias and head starts, that alter the respective incentives induced in the contest in a fundamentally different way: Agents that are favored by head starts benefit from an additive bonus on their chosen bid or effort level in the sense that their rivals must first pass the head start to be able to compete on equal footing. In contrast, agents that are favored by a multiplicative bias enjoy a higher weight on their effort in the process of determining the winner of the prize. Both types of favoritism are frequently observed in real world contests: The University of Michigan, for example, added a head start of 20 (out of 150) points to the score of minority applicants for their undergraduate program, while other elite universities seem to (at least implicitly) apply similar policies, see Espenshade et al. (2004), [14]. In public procurement, small businesses or domestic firms are often favored through a multiplicative bias in the form of a bid preference or subsidy, comp. Krasnokutskaya and Seim (2011), [30], and Marion (2007), [32]. There are also instances where both instruments, head start and biases, are applied at the same time. Kirkegaard (2012), [26], reports on a Canadian research promotion program where researchers with excellent past performance receive a head start while the research proposals of junior scientists get a higher weight in the evaluation process.

Naturally, not only the type but also the extent of favoritism affects incentives, having strategic implications for favored agents, their respective rivals, and therefore also on the efforts expended by all contestants. A contest organizer who has the option to fit bias and/or head starts to the underlying heterogeneity of the contestants can therefore influence the aggregate amount of effort, that is, the revenue generated in the contest. Hence, finding the optimal design of those instruments of favoritism becomes the crucial task for a contest organizer who is interested in contest revenue.[1]

Focusing on bias and head starts, we analyze and compare the potential of these two instruments to generate additional revenue in contest games with several heterogeneous contestants. From the perspective of a revenue-interested contest organizer, our analysis provides insights with respect to questions that typically arise in competitive environments with heterogeneous contestants: Should the playing field be leveled by favoring weak contestants to increase competitive pressure? Should preferential treatment induce more contestants to actively participate in the contest or is it better to exclude weak contestants and concentrate only on strong contestants? What type of favoritism, bias or head start, is more effective in generating additional revenue? Our analysis will not only allow us to answer these questions but also to identify the limits and dependencies with respect to the respective instruments and the nature of the competitive process.

To model the competitive process, we concentrate on two frequently used frameworks with complete information: lottery contests and all-pay auctions. Both models are sufficiently tractable and have been extensively used in various contest applications.[2] The fundamental difference between the two framework lies in their decisiveness, that is, the amount of noise in the process that determines the winner. An all-pay auction is highly decisive as the player with the highest effort is deterministically chosen to be the winner. In contrast, the outcome of a lottery contest is probabilistic and therefore less decisive because the probability of a player to win the contest is proportional to her relative effort contribution. From an applied perspective it is ultimately

[1]Arguably, in some of the applications mentioned above, asymmetric treatment of agents is (at least officially) not implemented to increase contest revenue but rather for some normative reasons. However, even if normatively derived deviations from symmetric treatment are applied, the forgone revenue should be an important evaluation criterion for these policies. Our paper provides this benchmark of comparison by deriving the maximal revenue that can be obtained through optimally designing asymmetric treatment of agents.

[2]Lottery contests were introduced by Tullock (1980), [40], and an early analysis of all-pay auctions with complete information can be found in Hillman and Riley (1989), [22]. See the two text books mentioned above and Corchón (2007), [7] for surveys.

an empirical question whether the respective competitive situation is more appropriately captured by a more noisy lottery contest or a highly decisive all-pay auction. For this reason we provide an analysis of both frameworks to be as comprehensive as possible and to allow an applied researcher to refer to the appropriate framework based on the specific application at hand. Moreover, including both frameworks into our analysis allows us to investigate how the revenue-enhancing potential of the two instruments of preferential treatment depends on the decisiveness of the underlying contest.

Whereas revenue-maximizing biases have been studied extensively in the literature (which we discuss below), much less is known about the optimal use of head starts. In this paper, we provide a general analysis of revenue-maximizing head starts, both in isolation and in combination with bias. Combined with the existing findings on optimal biases by Franke et al. (2013 and 2014), [18] and [19], our results enable us to show, among other things, that head starts are unambiguously more effective than biases in increasing revenue in all-pay auctions, whereas the opposite is true in lottery contests. Hence, the usefulness of the instruments depends on the underlying framework, which may explain why both instruments are frequently observed in practice.

For the lottery contest framework, we find that depending on the heterogeneity of contestants the revenue-maximizing head starts (in absence of bias) are either zero or such that only the strongest player actively invests effort (competing only against the head starts of the other players). Moreover, we show that an optimal bias without head starts revenue-dominates any combination of bias and head starts. Intuitively, one reason why head starts are less effective in inducing additional revenue is that a favored player uses his head start to substitute for own effort, whereas for a player who is favored by a bias the incentive to reduce his effort is less pronounced. Hence, a contest organizer who finds himself in a situation resembling the lottery contest framework should rather use biased contest rules to favor specific players, which has interesting normative implications: Under the optimal bias, for instance, entry into the contest is higher because more contestants decide to actively participate in the contest which tends to increase revenue. Moreover, there is some leveling of the playing field which increases competitive pressure among active contestants and therefore also generates additional revenue. Hence, in the lottery contest framework normative concerns like high participation rates and leveling of the playing field are (at least to some degree) aligned with the objective of revenue maximization.

The analysis of the all-pay auction framework with head starts and heterogeneous players is complicated by the fact that there does not exist a complete characterization of the set of equilibria. However, based on a restricted class of head starts such that only two players are active, we are able to construct a lower bound on the revenue under optimal head starts, which is greater than the revenue under the optimal bias. We also demonstrate that, although being sufficient for ranking the two instruments, our lower bound is not tight: Revenue maximization with head starts may require more than two active players. In situations resembling an all-pay auction, a revenue-interested organizer should thus prefer head starts over biases, because head starts are highly effective in the cut-throat competition without noise, which is in contrast to the lottery contest framework. If bias and head starts can be applied simultaneously in the all-pay auction framework, then the optimal bias-head-start combination generates even higher revenue: The optimal combination mimics a take-it-or-leave-it offer to the player with the highest valuation in the sense that this player faces a rival with a head start equal to her valuation. In other words, the strongest player is forced to bid exactly her valuation to overcome the high head start which implies that the entire surplus is extracted from the players and transformed one-to-one into additional revenue. Thus, in the all-pay auction framework head starts are a highly effective instrument for revenue extraction and their effect is even stronger if combined with appropriately designed biases.[3]

As shown by Franke et al. (2014), [19], the revenue generated by an optimal bias is always higher in the all-pay auction than in the lottery contest. Our within-framework rankings of the two instruments imply that this revenue-dominance result extends also to optimal head starts and optimal combinations of the two instruments. Hence, we obtain an unambiguous revenue-ranking among all six combinations of frameworks and instruments. This ranking implies, for instance, that a revenue-maximizing contest organizer who has at least one of the two instruments of favoritism at her disposal would always prefer to be in a situation resembling an all-pay auction than in one than can be described as a lottery contest. Which of the two instruments the organizer prefers, however, is framework-specific: Our results suggest that in highly decisive environments like the all-pay auction, head starts (if possible, in combination with bias) are better suited to generate additional revenue whereas in a noisier situation like the lottery contest, multiplicative biases are superior.[4]

Our theoretical results have also a number of practical implications for the interpretation of real world contests. Observing the occurrence of favoritism, for instance, does not automatically imply that the contest organizer has discriminatory motives or normative concerns for a balanced

[3] A caveat of the optimal bias-head start combination is that the degree of participation is extremely low because in equilibrium only the strongest player actively participates in the contest (while this situation seems to be somewhat contrived, there are similar situations in specific sports competitions, such as pole vault, where in the last round only one contestant remains in the competition).

[4] The incentive effects of additive head starts versus multiplicative biases have also been analyzed experimentally, comp. Balafoutas and Sutter (2012), [3], and Calsamiglia et al. (2013), [6].

playing field per se. Instead, our results demonstrate that plain revenue maximization might be the underlying rationale for favoritism. At the same time our analysis also shows that in some situations favoritism might not be applied although instruments of favoritism are in principle feasible. We point out further empirical implications of our study in the concluding remarks.

The literature on lottery contests has devoted, so far, only little attention to favoritism via head starts. An exception is Konrad (2002), [28], who studies the case of two heterogeneous players but is not concerned with revenue maximization. We generalize the setup to more than two heterogeneous players with arbitrary head starts and we are first to determine optimal head starts (in the absence of bias). Moreover, we provide a complete equilibrium characterization that allows for both bias and head starts, adapting an approach by Wasser (2013), [42] for a modified lottery contest without favoritism. For the special case of two players, the result that an optimal combination of head starts and bias involves zero head starts can be inferred from Nti (2004), [35]. We extend this finding to an arbitrary number of heterogeneous players.[5]

The analysis of head starts in bilateral all-pay auctions has gained recent interest under both complete information (Konrad (2002), [28]; Li and Yu (2012), [31]) and private information (Kirkegaard (2012), [26]; Segev and Sela (2014), [38]; Seel and Wasser (2014), [37]). In particular, for two heterogeneous players, Li and Yu (2012), [31], determine the optimal head starts and show that the induced revenue is higher than under the optimal bias. By considering an arbitrary number of heterogeneous players, we considerably generalize the setting. We show how to make use of the result of Li and Yu (2012), [31], to construct head starts with two active players such that the induced revenue is higher than under the optimal bias, independent of the number of players. At the same time, however, we also demonstrate that head starts with two active players are not optimal in general. This novel observation indicates that beyond the two-player case, determining optimal head starts is non-trivial and calls for new methodological advances to enable a general equilibrium characterization for all-pay auctions with head-starts that accommodates more than two active players.

In contrast to head starts, revenue-maximizing biases in lottery contests and all-pay auctions under complete information have been extensively studied, e.g., by Nti (2004), [35], Fu (2006), [21], Franke (2012), [17], and Epstein et al. (2011 and 2013), [12] and [13]; see also Mealem and Nitzan (2016), [34], for a recent survey on discrimination in contest games. As mentioned, our comparison of the two instruments makes use of Franke et al. (2013 and 2014), [18] and [19], who determined the revenue-maximizing bias in the lottery contest and the all-pay auction, respectively, for more than two heterogeneous players. Under private information, Kirkegaard

[5]For an arbitrary number of homogenous players and without favoritism, Dasgupta and Nti (1998), [9], had shown the optimal homogeneous head start to be zero.

(2013), [27] analyzes a biased all-pay auction with three players, whereas Kirkegaard (2012), [26] determines the optimal combination of bias and head starts for the case of two ex ante asymmetric players.[6]

The paper is organized as follows. In Section 2 we introduce the model setup and the two contest frameworks. In Section 3 we characterize the equilibrium in the lottery contest framework with bias and head starts. Based on this characterization we derive the optimal head starts, summarize the results on the optimal bias and analyze the simultaneous application of bias and head starts. We compare our result for the lottery contest and provide a complete revenue ranking of all instrument combinations in this framework. In Section 4 we analyze the all-pay auction framework by deriving a lower bound on equilibrium revenue based on all-pay auctions with optimal head starts where only two player are active. We then demonstrate that this lower bound is higher than expected revenue under the optimally biased all-pay auction and can sometimes be improved upon by designing head starts such that three players are active. Finally, we identify the optimal combination of bias and head starts and show that it is equivalent to a take-it-or-leave-it offer to the strongest player which therefore yields revenue at the upper bound. In Section 5 we collect our results to compare equilibrium revenue between the two frameworks and provide a complete revenue ranking among all instrument combinations and both frameworks. Section 6 concludes.

2 The Model

There are $n \geq 2$ players of set $\mathcal{N} = \{1, \ldots, n\}$ that compete for an indivisible prize. Players are heterogeneous with respect to their valuations of the prize and can be ordered decreasingly with respect to their valuations: $v_1 \geq v_2 \geq \ldots \geq v_n > 0$. The probability $Pr_i(x_i, x_{-i})$ of player $i \in \mathcal{N}$ to win the prize depends positively on her bid $x_i \in [0, \infty)$ and negatively on the bids $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in [0, \infty)^{n-1}$ of her rivals, where all bids are non-refundable. The expected payoff for player $i \in \mathcal{N}$ is

$$\pi_i(x_i, x_{-i}) := Pr_i(x_i, x_{-i})v_i - x_i.$$

The probability function $Pr_i(x_i, x_{-i})$ depends on the specific design of the contest framework. We focus in our analysis on the two most frequently used contest frameworks; that is,

[6]There is also a recent literature on the optimality of favoritism among ex-ante symmetric players when the organizer cannot observe each player's strength, see Drugov and Ryvkin (2017), [10], Pérez-Castrillo and Wettstein (2016), [36], Matros and Possajennikov (2016), [33], and Kawamura and Moreno de Barreda (2014), [25].

a deterministic all-pay auction and a probabilistic lottery contest. More precisely, we consider asymmetric versions of those two contest frameworks where the bid x_i of each player $i \in \mathcal{N}$ is converted by an idiosyncratic affine transformation consisting of a multiplicative bias parameter $\alpha_i \in [0, \infty)$ and a non-negative head start $\delta_i \in [0, \infty)$ that is added to player i's bid without any costs. Once the decision on the framework is made and the respective instrument combination of bias $\alpha = (\alpha_1, \ldots, \alpha_n)$ and head starts $\delta = (\delta_1, \ldots, \delta_n)$ is designed, the probability for player $i \in \mathcal{N}$ to win the prize can be expressed as follows:

1. For the all-pay auction with bias and head starts (setup BHA),

$$Pr_i^{BHA}(x_i, x_{-i}) := \begin{cases} 1 & \text{if } \alpha_i x_i + \delta_i > \alpha_j x_j + \delta_j \text{ for all } j \neq i, \\ \tau_i(\mathcal{M}) & \text{if } i \in \mathcal{M} = \{m \in \mathcal{N} : \alpha_m x_m + \delta_m \geq \alpha_j x_j + \delta_j \ \forall j \in \mathcal{N}\} \\ & \text{and } |\mathcal{M}| \geq 2, \\ 0 & \text{if } \alpha_i x_i + \delta_i < \alpha_j x_j + \delta_j \text{ for some } j \neq i, \end{cases}$$

   where the tie-breaking rule τ satisfies $\tau_i(\mathcal{K}) \in [0, 1]$ and $\sum_{i \in \mathcal{K}} \tau_i(\mathcal{K}) = 1$ for all $i \in \mathcal{K}$ and $\mathcal{K} \subseteq \mathcal{N}$. If not stated otherwise, our results below hold for all tie-breaking rules τ. The fair tie-breaking rule τ^f, which is prevalent in the literature, is defined as $\tau_i^f(\mathcal{K}) := \frac{1}{|\mathcal{K}|}$ for all $i \in \mathcal{K}$ and $\mathcal{K} \subseteq \mathcal{N}$.

2. For the lottery contest with bias and head starts (setup BHL),

$$Pr_i^{BHL}(x_i, x_{-i}) := \begin{cases} \frac{\alpha_i x_i + \delta_i}{\sum_{j=1}^n (\alpha_j x_j + \delta_j)} & \text{if } \sum_{j=1}^n (\alpha_j x_j + \delta_j) \neq 0, \\ \frac{1}{n} & \text{if } \sum_{j=1}^n (\alpha_j x_j + \delta_j) = 0. \end{cases}$$

These probability functions are also called contest success functions (CSFs). It should be noted that they include two special cases that are of interest in their own right for our analysis: Applying a symmetric bias to all players, e.g., $\alpha = (1, \ldots, 1)$, leads to an unbiased all-pay auction/lottery contest with head starts (setup HA/HL); applying zero head starts to all players, $\delta = (0, \ldots, 0)$, leads to a biased all-pay auction/lottery contest without head starts (setup BA/BL). Tables 1 and 2 clarify our taxonomy for the two frameworks.

We evaluate the revenue potential of the two instruments by comparing the maximal contest revenue (the sum of expected equilibrium bids by all players) that can be induced in each setup by specifying optimal head starts and/or biases. We denote this maximal revenue by $X_F^* := \sum_{i \in \mathcal{N}} E[x_{F,i}^*]$, where $E[x_{F,i}^*]$ denotes the expected equilibrium bid of player $i \in \mathcal{N}$ under an optimal

	No Bias $\alpha = (1, \dots, 1)$	Bias $\alpha \in [0, \infty)^n$
No Head Start $\delta = (0, \dots, 0)$	Standard Lottery Contest	Biased Lottery Contest (BL) Subsection 3.3
Head Start $\delta \in [0, \infty)^n$	Lottery Contest with Head Start (HL) Subsection 3.2	Biased Lottery Contest with Head Start (BHL) Subsection 3.4

Table 1: Lottery Contest Framework

	No Bias $\alpha = (1, \dots, 1)$	Bias $\alpha \in [0, \infty)^n$
No Head Start $\delta = (0, \dots, 0)$	Standard All-Pay Auction	Biased All-Pay Auction (BA) Subsection 4.3
Head Start $\delta \in [0, \infty)^n$	All-Pay Auction with Head Start (HA) Subsection 4.2	Biased All-Pay Auction with Head Start (BHA) Subsection 4.4

Table 2: All-Pay Auction Framework

bias and/or head start in setup $F \in \{BHA, BHL, HA, HL, BA, BL\}$. Analyzing all 2×3 setups separately allows us to isolate the revenue-enhancing effects of the two instruments in separation and to compare them with the optimal affine transformation where both instruments are used simultaneously.

As bias and head starts are framework-specific, our analysis proceeds by firstly characterizing optimal head starts, optimal biases, and the optimal affine transformation for each setup. These instruments are then compared with respect to the induced revenue within each setup and, in a last step, between the two frameworks.[7]

[7] Our analysis can be alternatively framed as the design problem of a contest organizer, whose objective function depends positively on contest revenue, and whose strategy space consists of the type of contest framework with type-dependent instrument (i.e., bias and/or head starts).

3 The Lottery Contest

Our analysis of the lottery contest framework starts with an explicit characterization of the equilibrium for a given head-start-bias combination (δ, α). We then determine optimal head starts without bias (setup HL) and summarize the existing results concerning optimal biases without head starts (setup BL). Finally, we show that the optimal combination of head starts and bias (setup BHL) coincides with the optimal biases with zero head starts. Hence, using head starts in addition to bias is not conducive to generate additional revenue in a lottery contest. In other words, a revenue-maximizing contest organizer would choose to specify optimal biases and refrain from using head starts at all.

3.1 Equilibrium

Consider a lottery contest with head starts $\delta \in [0, \infty)^n$ and bias $\alpha \in [0, \infty)^n$, i.e., the most general setup BHL. Note that for players j with $\alpha_j = 0$ investing positive effort is strictly dominated by zero effort. Hence, define $\tilde{\mathcal{N}} := \{i \in \mathcal{N} : \alpha_i > 0\}$ to be the set of potentially active players.[8]

Let $y_i := \alpha_i x_i + \delta_i$ denote the score of player $i \in \mathcal{N}$ and $Y := \sum_{i \in \mathcal{N}} y_i$ the aggregate score. We will consider the equivalent game where each player $i \in \tilde{\mathcal{N}}$ chooses his score $y_i \in [\delta_i, \infty)$ rather than his effort x_i and obtains payoff

$$\tilde{\pi}_i(y_1, \dots, y_n) = \frac{y_i}{Y} v_i - \frac{y_i - \delta_i}{\alpha_i} \qquad \text{(if } Y > 0\text{)}.$$

As $\tilde{\pi}_i$ is strictly concave in y_i, player i's best response is characterized by the first-order condition

$$\frac{Y - y_i}{Y^2} v_i - \frac{1}{\alpha_i} \leq 0, \quad \text{with equality if } y_i > \delta_i. \tag{1}$$

An effort profile $x_1, \dots, x_n$ in the original contest is a pure-strategy Nash equilibrium if and only if the corresponding score profile $y_1, \dots, y_n$ satisfies (1) for all $i \in \tilde{\mathcal{N}}$ and $x_j = 0$ for all $j \notin \tilde{\mathcal{N}}$.

In the following, we generalize the equilibrium characterization of Wasser (2013), [42], who considered lottery contests with symmetric head starts and without bias and made use of the share function approach by Cornes and Hartley (2005), [8].

[8]Note that a non-active player $j \notin \tilde{\mathcal{N}}$ still wins the contest with positive probability if $\delta_j > 0$.

For all $\mathcal{K} \subseteq \tilde{\mathcal{N}}$, define the function

$$Y(\mathcal{K}) := \begin{cases} \dfrac{|\mathcal{K}| - 1 + \sqrt{(|\mathcal{K}| - 1)^2 + 4\left(\sum_{i\in\mathcal{K}} \frac{1}{\alpha_i v_i}\right)\left(\sum_{j\notin\mathcal{K}} \delta_j\right)}}{2\sum_{i\in\mathcal{K}} \frac{1}{\alpha_i v_i}} & \text{if } \mathcal{K} \neq \emptyset, \\ \sum_{i\in\mathcal{N}} \delta_i & \text{if } \mathcal{K} = \emptyset. \end{cases} \tag{2}$$

This function from the power set of $\tilde{\mathcal{N}}$ to $\mathbb{R}_+$ is central to our equilibrium characterization: As shown in the following proposition, maximizing $Y(\mathcal{K})$ with respect to $\mathcal{K}$ yields the equilibrium aggregate score, with the maximizer being the set of active players that submit non-zero effort in equilibrium.

Proposition 3.1 *There is a unique pure-strategy Nash equilibrium in the BHL setup. Let*

$$\mathcal{K}^* = \arg\max_{\mathcal{K}\subseteq\tilde{\mathcal{N}}} Y(\mathcal{K}).$$

In equilibrium, the aggregate score is $Y(\mathcal{K}^*)$, each player $i \in \mathcal{K}^*$ exerts effort

$$x_i = \frac{1}{\alpha_i}\left(Y(\mathcal{K}^*) - \frac{Y(\mathcal{K}^*)^2}{\alpha_i v_i} - \delta_i\right) > 0, \tag{3}$$

and each player $j \in \mathcal{N} \setminus \mathcal{K}^*$ exerts zero effort.

Proof. Note that condition (1) implies $y_i = \max\left\{Y - \frac{Y^2}{\alpha_i v_i}, \delta_i\right\}$. Following Cornes and Hartley (2005), [8], we define i's share function as

$$\phi_i(Y) := \frac{y_i}{Y} = \max\{f_i(Y), g_i(Y)\},$$

where $f_i(Y) := 1 - \frac{Y}{\alpha_i v_i}$ and $g_i(Y) := \frac{\delta_i}{Y}$. Let $\Phi(Y) := \sum_{i=1}^n \phi_i(Y)$ be the aggregate share function.

A score profile $y_1, \ldots, y_n$ corresponds to a pure-strategy Nash equilibrium if and only if there is an aggregate score Y^* such that $y_i = \phi_i(Y^*)Y^*$ for each i and $\Phi(Y^*) = 1$. Note that for players i who are active in equilibrium ($x_i > 0$ and $y_i > \delta_i$) we have $\phi_i(Y^*) = f_i(Y^*)$, whereas for inactive players ($x_i = 0$ and $y_i = \delta_i$) we have $\phi_i(Y^*) = g_i(Y^*)$.

$\Phi(Y)$ is continuous and strictly decreasing. Moreover, $\Phi\left(\sum_j \delta_j\right) \geq \sum_i g_i\left(\sum_j \delta_j\right) = 1$ and $\Phi(Y) < 1$ for Y large enough. Hence, there is a unique Y^* that solves $\Phi(Y^*) = 1$, implying existence of a unique pure-strategy Nash equilibrium.

Now, consider an equilibrium where the players in $\mathcal{K}^* \subseteq \tilde{\mathcal{N}}$ are active, whereas the remaining players are inactive. Hence, $\Phi(Y^*) = 1$ is equivalent to

$$\Phi(Y^*) = \sum_{i \in \mathcal{K}^*} f_i(Y^*) + \sum_{j \notin \mathcal{K}^*} g_j(Y^*) = |\mathcal{K}^*| - Y^* \sum_{i \in \mathcal{K}^*} \frac{1}{\alpha_i v_i} + \frac{1}{Y^*} \sum_{j \notin \mathcal{K}^*} \delta_j = 1.$$

Solving this equation for Y^* we obtain $Y^* = Y(\mathcal{K}^*)$, where $Y(\cdot)$ was defined in (2).

We will now show that the function $Y(\mathcal{K})$ is maximized at the set of players $\mathcal{K}^*$ that are active in equilibrium. Consider a set of players $\mathcal{M} \subseteq \tilde{\mathcal{N}}$ such that $\mathcal{M} \neq \mathcal{K}^*$. By the definition of $Y(\cdot)$, we have

$$\sum_{i \in \mathcal{M}} f_i(Y(\mathcal{M})) + \sum_{j \notin \mathcal{M}} g_j(Y(\mathcal{M})) = 1.$$

However, since $\mathcal{M}$ does not correspond to the set of players that are active in equilibrium, we must have at least one $i \in \mathcal{M}$ where $f_i(Y(\mathcal{M})) < g_i(Y(\mathcal{M}))$ or one $j \notin \mathcal{M}$ where $g_j(Y(\mathcal{M})) < f_j(Y(\mathcal{M}))$. Hence, $\Phi(Y(\mathcal{M})) > 1$. Since Φ is strictly decreasing, we must have $Y(\mathcal{M}) < Y(\mathcal{K}^*)$, because $\mathcal{K}^*$ is the unique subset of players that satisfies $\Phi(Y(\mathcal{K}^*)) = 1$.

Finally, the equilibrium efforts of active players $i \in \mathcal{K}^*$ given in (3) are obtained from the equilibrium scores $y_i = \phi_i(Y(\mathcal{K}^*))Y(\mathcal{K}^*) = Y(\mathcal{K}^*) - \frac{Y(\mathcal{K}^*)^2}{\alpha_i v_i}$. □

Proposition 3.1 establishes existence and uniqueness of a pure-strategy Nash equilibrium for any combination of head starts and bias and provides a characterization of equilibrium efforts based on the aggregate score. This characterization will prove useful for studying the optimal use of head starts and bias by the contest organizer. Before determining the revenue-maximizing head-start-bias combination, we will proceed by first considering each of the two instruments separately.

3.2 Optimal head starts

We will determine optimal head starts within setup HL by setting $\alpha = (1, \ldots, 1)$ and taking advantage of the equilibrium characterization provided by Proposition 3.1. To clarify the dependence on head starts δ, we will slightly change the notation from the preceding subsection: We will write $Y(\delta, \mathcal{K})$ for the function defined in (2), $\mathcal{K}^*(\delta)$ for the set of players that are active in equilibrium, and $x_i(\delta)$ for the equilibrium effort of player i.

The revenue induced under head starts δ without bias simplifies to the equilibrium aggregate score minus the sum of the head starts.[9] Making use of Proposition 3.1, we have

$$X_{HL}(\delta) := \sum_{i\in\mathcal{N}} x_i(\delta) = Y(\delta, \mathcal{K}^*(\delta)) - \sum_{i\in\mathcal{N}} \delta_i.$$

An important benchmark is the revenue in the standard lottery contests without head starts. We will represent this case by the zero head start vector $\delta_0 := (0, \dots, 0)$. The induced revenue simplifies to the following well-known expression (see, e.g., Hillman and Riley, 1989, [22]):

$$X_{HL}(\delta_0) = Y(\delta_0, \mathcal{K}^*(\delta_0)) = \max_{\mathcal{K}\subseteq\mathcal{N}} Y(\delta_0, \mathcal{K}) = \max_{1<k\leq n} \frac{k-1}{\sum_{i=1}^{k} \frac{1}{v_i}}. \tag{4}$$

As a first step towards finding optimal head starts, the following lemma provides an upper bound on revenue $X_{HL}(\delta)$, which depends on the number of players that are active in equilibrium.

Lemma 3.2 *Let $\delta \neq \delta_0$. If $|\mathcal{K}^*(\delta)| = 1$, then $X_{HL}(\delta) \leq \frac{1}{4} v_1$. If $|\mathcal{K}^*(\delta)| \geq 2$, then $X_{HL}(\delta) < X_{HL}(\delta_0)$.*

Proof. First, suppose δ is such that only one player is active, i.e., $\mathcal{K}^*(\delta) = \{i\}$ for some i. Then

$$X_{HL}(\delta) = Y(\delta, \{i\}) - \sum_{j\in\mathcal{N}} \delta_j = \sqrt{v_i \sum_{j\neq i} \delta_j} - \sum_{j\neq i} \delta_j - \delta_i \leq \sqrt{\sum_{j\neq i} \delta_j} \left(\sqrt{v_i} - \sqrt{\sum_{j\neq i} \delta_j} \right) \leq \frac{v_i}{4} \leq \frac{v_1}{4}.$$

The first inequality is implied by $\delta_i \geq 0$, the second inequality follows from the expression on the LHS being maximized at $\sqrt{\sum_{j\neq i} \delta_j} = \frac{1}{2}\sqrt{v_i}$, and the third inequality uses $v_1 \geq v_i$ for all i.

Now, suppose δ induces $|\mathcal{K}^*(\delta)| \geq 2$. Observe that for any $\mathcal{K} \subseteq \mathcal{N}$ with $|\mathcal{K}| \geq 2$, we have

$$\frac{\partial Y(\delta, \mathcal{K})}{\partial \delta_i} = \begin{cases} 0 & \text{if } i \in \mathcal{K}, \\ \dfrac{1}{\sqrt{(|\mathcal{K}|-1)^2 + 4\left(\sum_{l\in\mathcal{K}} \frac{1}{v_l}\right)\left(\sum_{j\notin\mathcal{K}} \delta_j\right)}} \leq \dfrac{1}{|\mathcal{K}|-1} \leq 1 & \text{if } i \notin \mathcal{K}, \end{cases}$$

where $\frac{\partial Y(\delta,\mathcal{K})}{\partial \delta_i} < 1$ for $i \notin \mathcal{K}$ with $\delta_i > 0$. Hence, $\frac{\partial}{\partial \delta_j}(Y(\delta, \mathcal{K}) - \sum_{i\in\mathcal{N}} \delta_i) \leq 0$ for all $j \in \mathcal{N}$, with strict inequality for at least one j. This leads to the following chain of (in-)equalities, which proves the second result:

$$X_{HL}(\delta) = Y(\delta, \mathcal{K}^*(\delta)) - \sum_{i\in\mathcal{N}} \delta_i \;<\; Y(\delta_0, \mathcal{K}^*(\delta)) \;\leq\; Y(\delta_0, \mathcal{K}^*(\delta_0)) = X_{HL}(\delta_0).$$

□

[9] For the special case where $\delta_1 = \delta_2 = \cdots = \delta_n$, Amegashie (2006), [1], determined the induced revenue if $\mathcal{K}^*(\delta) = \mathcal{N}$ or $\mathcal{K}^*(\delta) = \emptyset$.

Lemma 3.2 differentiates between two cases. If at least two players are active under head starts δ, then revenue can be increased by removing all head starts. For active players, head starts are perfect substitutes for own effort because (1) implies that the best-response score of an active player will not be affected by a reduction of his head start. For inactive players, the proof of Lemma 3.2 reveals that reducing their head starts is also revenue-enhancing. Hence, zero head starts δ_0 induce higher revenue than any other vector of head starts that result in at least two active players.

If, however, exactly one player is active under head starts δ, then setting positive head starts for inactive players can be beneficial. Intuitively, increasing the head starts of inactive players may lead the active player to best-respond by increasing his effort. Lemma 3.2 shows that revenue in this case is bounded above by $\frac{1}{4}v_1$.

The existence of this upper bound has an immediate implication. If revenue without head starts is higher than $\frac{1}{4}v_1$, then Lemma 3.2 implies that optimal head starts are zero. In the opposite case, the following result shows that there exist head starts under which revenue meets the upper bound, i.e., player 1 submits effort $\frac{1}{4}v_1$ and all other players remain inactive.

Proposition 3.3 *If $\frac{1}{4}v_1 \geq X_{HL}(\delta_0)$, then all head starts δ^* where $\delta_1^* = 0$ and $\sum_{j=2}^{n}\delta_j^* = \frac{1}{4}v_1$ are optimal and induce revenue $X_{HL}(\delta^*) = x_1(\delta^*) = \frac{1}{4}v_1$. Otherwise, zero head starts δ_0 are the unique optimal head starts.*

Proof. Suppose $\frac{1}{4}v_1 \geq X_{HL}(\delta_0)$. We will show that any head starts δ^* as defined in the proposition result in equilibrium efforts $x_1(\delta^*) = \frac{1}{4}v_1$ and $x_j(\delta^*) = 0$ for all $j > 1$. The aggregate score in such an equilibrium is $Y = x_1(\delta^*) + \sum_{j=2}^{n}\delta_j^* = \frac{1}{2}v_1$. Recall that player i's best response in terms of individual score is given by (1), which implies

$$y_i = \max\left\{Y\left(1 - \frac{Y}{v_i}\right), \delta_i^*\right\} = \max\left\{\frac{1}{2}v_1\left(1 - \frac{v_1}{2v_i}\right), \delta_i^*\right\}.$$

Hence, the best-response effort of player 1 is indeed $x_1 = y_1 = \frac{1}{4}v_1$. Moreover, recall (4) and note that $\frac{1}{4}v_1 \geq X_{HL}(\delta_0) = \max_{k>1}\frac{k-1}{\sum_{i=1}^{k}\frac{1}{v_i}} \geq \frac{1}{\frac{1}{v_1}+\frac{1}{v_2}}$ implies $3v_2 \leq v_1$. Therefore, $\frac{v_1}{2v_j} \geq \frac{3}{2}$ for all $j > 1$, implying $y_j = \delta_j^*$. Consequently, the best-response effort of each player $j > 1$ is indeed $x_j = 0$.

Now, suppose $\frac{1}{4}v_1 < X_{HL}(\delta_0)$. In this case, Lemma 3.2 implies that $X_{HL}(\delta_0) > X_{HL}(\delta)$ for all $\delta \neq \delta_0$, rendering δ_0 uniquely optimal. $\square$

Proposition 3.3 implies that non-zero head starts are optimal if player 1's valuation is sufficiently higher than the other players' valuations. A simple sufficient condition for $\frac{1}{4}v_1 \geq X_{HL}(\delta_0)$ is $v_1 \geq 4v_2$ (whereas $v_1 \geq 3v_2$ is necessary, as shown in the proof of Proposition 3.3). If non-zero head starts are optimal, then they are set such that the strongest contestant, player 1, competes against the sum of the head starts of all the other players, which are inactive. Hence, from the perspective of player 1 it is as if he were facing just one equally strong opponent. If, however, the difference in valuations between player 1 and the other players is less pronounced, then head starts are not a suitable instrument for increasing revenue.

Using (4), Proposition 3.3 implies the following for revenue under optimal head starts.

Corollary 3.4 *Optimal head starts in the HL setup yield revenue*

$$X^*_{HL} = \max\left\{\frac{v_1}{4}, \max_{1<k\leq n} \frac{k-1}{\sum_{i=1}^{k} \frac{1}{v_i}}\right\}.$$

3.3 Optimal biases

We now turn to setup BL, where the organizer can only set a bias α but no head starts, (i.e., the zero head start vector $\delta = \delta_0 = (0, \ldots, 0)$ is pre-specified). The revenue-maximizing bias for this case has been determined in Franke et al. (2013), [18], which is summarized in the following proposition.

Proposition 3.5 (Franke et al. (2013), [18]) *An optimal bias in the BL setup yields revenue*

$$X^*_{BL} = \frac{1}{4}\left(\sum_{i=1}^{k^*} v_i - \frac{(k^*-2)^2}{\sum_{i=1}^{k^*} \frac{1}{v_i}}\right), \qquad \textit{where} \quad k^* = \max\left\{k \in \mathcal{N} \;\middle|\; \frac{k-2}{v_k} < \sum_{i=1}^{k} \frac{1}{v_i}\right\}.$$

The set of optimal biases consists of all α^* where, for some $c > 0$,

$$\alpha^*_i = 2c\left(v_i + \frac{k^*-2}{\sum_{j=1}^{k^*} \frac{1}{v_j}}\right)^{-1} \textit{ for } i \leq k^* \quad \textit{and} \quad \alpha^*_i < \frac{c}{v_i} \quad \textit{for } i > k^*.$$

Under an optimal bias, each player $i \leq k^*$ is active and each player $i > k^*$ is inactive.

In an optimally biased lottery contest, the k^* contestants with the highest valuations are active in equilibrium. Compared with the unbiased lottery contest, an optimal bias α^* typically encourages additional entry (k^* is greater or equal to the number of active players in an unbiased lottery contest). For example, provided that $n \geq 3$, at least three players will always be active in the optimally biased lottery contest while this lower bound is two in the unbiased contest. In this respect, the way in which optimal biases work is in stark contrast to our characterization of optimal head starts from the preceding subsection: If non-zero head starts are optimal, then they are designed to discourage entry, reducing the number of active players to one.

3.4 Optimal combinations of head starts and bias

We now consider lottery contests where both instruments, head starts and bias, can be used simultaneously. Our main result for this case is that an organizer will never find it profitable to implement non-zero head starts in addition to an optimally chosen bias. In other words, any contest with head starts and bias can be replaced by a contest without head starts and adjusted bias that yields strictly higher revenue.

Consider a contest with strictly positive head starts for some players and suppose we remove them. Each player i that is active under a positive head start will increase his effort due to the fact that head starts are perfect substitutes for own effort exertion. In addition, we prove that by manipulating α_j, each inactive player j can be induced to actively invest effort that makes up for the removed head start.

Proposition 3.6 *For any combination of head starts and bias* (δ, α) *where* $\delta \neq \delta_0$*, there exists a bias* $\hat{\alpha}$ *such that the contest with* $(\delta_0, \hat{\alpha})$ *yields strictly higher revenue than the contest with* (δ, α)*.*

Proof. Let $y_1, \dots, y_n$ denote the equilibrium scores induced under (δ, α) where $\delta \neq \delta_0$ and recall $Y = \sum_{i \in \mathcal{N}} y_i$. Hence, $y_1, \dots, y_n$ are the unique scores that satisfy (1) for each $i \in \tilde{\mathcal{N}}$. Moreover, let $\mathcal{K}^*(\delta, \alpha)$ denote the set of players that are active in this equilibrium (i.e., all i where $y_i > \delta_i$). Define a new bias $\hat{\alpha}$ such that $\hat{\alpha}_i = \alpha_i$ for all $i \in \mathcal{K}^*(\delta, \alpha)$ and $\hat{\alpha}_j = \frac{Y^2}{(Y-\delta_j)v_j} > 0$ for all $j \notin \mathcal{K}^*(\delta, \alpha)$.

We will now show that each individual equilibrium score $\hat{y}_i$ under $(\delta_0, \hat{\alpha})$ is identical to the equilibrium score y_i under (δ, α). Adapting (1), $\hat{y}_1, \dots, \hat{y}_n$ are the equilibrium scores under $(\delta_0, \hat{\alpha})$ if they satisfy, for each $i \in \mathcal{N}$,

$$\frac{\hat{Y} - \hat{y}_i}{\hat{Y}^2} v_i - \frac{1}{\hat{\alpha}_i} \leq 0, \quad \text{with equality if } \hat{y}_i > 0. \tag{5}$$

Hence, we have to show that $\hat{y}_i = y_i$ for each i (and thus $\hat{Y} = Y$) solves (5).

For the originally active players $i \in \mathcal{K}^*(\delta, \alpha)$, we have $\hat{\alpha} = \alpha$ and hence the fact that $y_1, \dots, y_n$ solve (1) for i immediately implies that the same scores also solve (5) for i. For the originally inactive players $j \notin \mathcal{K}^*(\delta, \alpha)$, condition (5) becomes

$$\frac{\hat{Y} - \hat{y}_j}{\hat{Y}^2} v_j - \frac{Y - \delta_j}{Y^2} v_j \leq 0, \quad \text{with equality if } \hat{y}_j > 0.$$

Noting that $y_j = \delta_j$, the above is indeed satisfied for $\hat{Y} = Y$ and $\hat{y}_j = y_j$.

Let x_i and $\hat{x}_i$ denote i's equilibrium effort under (δ, α) and $(\delta_0, \hat{\alpha})$, respectively. Since $y_i = \hat{y}_i$, $\hat{x}_i = x_i + \frac{\delta_i}{\hat{\alpha}_i}$ for all $i \in \mathcal{N}$. Consequently, $\sum_{i \in \mathcal{N}} \hat{x}_i > \sum_{i \in \mathcal{N}} x_i$ because $\delta_i > 0$ for at least one i. □

The crucial implication of Proposition 3.6 is that an optimal bias α^* as provided by Proposition 3.5 combined with zero head starts is also optimal in setup BHL, resulting in maximal revenue $X^*_{BHL} = X^*_{BL}$. Summarizing our results, we obtain the following revenue ranking for the different instruments in the lottery contest framework.

Proposition 3.7 *There is an unambiguous revenue-ranking of lottery contests with optimal head starts and/or biases: If $v_1 = v_2$, then*

$$X^*_{BHL} = X^*_{BL} \geq X^*_{HL}.$$

If $v_1 > v_2$, then

$$X^*_{BHL} = X^*_{BL} > X^*_{HL}.$$

Proof. The equalities and the weak inequality are immediate from Proposition 3.6. It remains to prove the strict inequality $X^*_{BL} > X^*_{HL} = \max\{\frac{1}{4}v_1, X_{HL}(\delta_0)\}$ if $v_1 > v_2$. If $X^*_{HL} = \frac{1}{4}v_1$, then according to Proposition 3.3 at least one head start is strictly positive and the strict inequality follows from Proposition 3.6. If $X^*_{HL} = X_{HL}(\delta_0)$, then the strict inequality follows from the fact that X^*_{BL} is strictly higher than the revenue in absence of any bias if $v_1 > v_2$. To see this, note that if $v_1 > v_2$, Proposition 3.5 implies $\alpha^*_1 < \alpha^*_2$ for each optimal bias α^*. Applying no bias ($\alpha = (1, \ldots, 1)$) is hence not optimal and yields strictly lower revenue. □

From the perspective of a revenue-maximizing contest organizer biasing the contest rule is the preferred instrument of favoritism. Under the optimal bias more contestants are induced to participate and the playing field is more balanced. Both effects imply higher effort exertion and therefore higher revenue. By contrast, head starts are less suitable to induce additional revenue because contestants who are favored by positive head starts use them to reduce their effort level while maintaining their score. In equilibrium the score of other contestants is therefore not affected and revenue is generically lower under positive head starts.

Moreover, for any head start there exists a corresponding bias that yields even higher revenue by inducing non-active contestants to participate. As this substitution works for any degree of heterogeneity in valuations, the respective revenue ranking is unambiguous and confirms the dominance of optimal biases over head starts in the lottery contest framework.

Focusing on combinations of bias and head starts, we have restricted attention to affine bid transformations for implementing favoritism. In the working paper version of this article (Franke et al., 2016, [20]), we show that a lottery contest with optimal bias yields weakly higher revenue than any lottery contest with concave bid transformations. Hence, our results are robust with respect to the large class of concave bid transformations; that is, for the lottery contest framework, biases are also superior to many other forms of favoritism beyond head starts.

4 The All-Pay Auction

Turning to the all-pay auction, we proceed as follows. After stating some preliminary results, we determine and discuss bounds on the revenue generated by optimal head starts in setup HA. Then, we summarize the existing results on optimal biases in setup BA and identify optimal head-start-bias combinations in setup BHA. Our results allow us to compare all instruments with respect to induced revenue which leads to an unambiguous ranking of the three instrument combinations with respect to their potential for revenue extraction in the all-pay auction. In isolation, head starts turn out to be more conducive to generating revenue than biases while combining the two instruments even allows for extracting first-best revenue at the upper bound.

4.1 Preliminaries

We start with two preliminary observations. First, we show that the highest valuation v_1 represents a general upper bound on the revenue that can be obtained in an all-pay auction with head starts and bias. Second, we easily obtain from the analysis of Baye et al. (1993), [4], that this upper bound is reached in a standard all-pay auction if $v_1 = v_2$, implying that using no head starts and no bias maximizes revenue in this case.

Lemma 4.1 *For all head starts δ and biases α, the induced revenue is at most v_1.*

If $v_1 = v_2$, then $X^*_{BHA} = X^*_{HA} = X^*_{BA} = v_1$.

Proof. Note that by investing zero effort, each player can always ensure himself a payoff of at least zero. Hence, in every Nash equilibrium a player's expected effort cannot exceed his valuation times the probability that he wins. The sum of players' valuations weighted with their winning probabilities therefore is an upper bound on revenue, which is less than or equal to v_1.

Now, suppose $v_1 = v_2$ and consider the standard all-pay auction without head starts and bias, i.e., $\alpha = (1, \ldots, 1)$ and $\delta = (0, \ldots, 0)$. Then Theorem 1 in Baye et al. (1993), [4], implies that every Nash equilibrium yields revenue v_1. Consequently, the first-best revenue can be obtained in all three setups by applying no head starts and no bias. □

Having established that in all three setups the optimal use of the instruments yields first-best revenue if $v_1 = v_2$, we will focus on $v_1 > v_2$ in the following.

4.2 Optimal head starts

Let us turn to studying optimal head starts in setup HA where any multiplicative bias is absent such that we can set $\alpha = (1, \ldots, 1)$, for example. For the case of two players, revenue-maximizing head starts in the all-pay auction have been determined by Li and Yu (2012), [31]. They show that it is optimal to assign to the weaker player 2 a head start of $v_1 - v_2$ over the stronger player 1. In equilibrium, player 1 then randomizes his effort uniformly on $[v_1 - v_2, v_1]$ and player 2 chooses zero effort with probability $\frac{v_1 - v_2}{v_1}$ while randomizing uniformly on $[0, v_2]$ otherwise, resulting in an expected payoff of zero for both players. We record their results in the following lemma.

Lemma 4.2 (Li and Yu (2012), [31]) *If $n = 2$, then head starts $\delta_1^* = 0$ and $\delta_2^* = v_1 - v_2$ are optimal in setup HA and induce revenue*

$$X_{HA}^* = v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1}.$$

Compared with the standard all-pay auction without head starts, revenue under optimal head starts for $n = 2$ is higher because the support of player 1's equilibrium effort distribution is shifted upwards by $\delta_2^* = v_1 - v_2$ while player 2's equilibrium strategy remains unchanged. Player 1's increase in effort thus exactly offsets player 2's head start, implying that the same total surplus is realized as in the standard all-pay auction. Under optimal head starts, however, this total surplus is entirely collected by the contest organizer. Put differently, the optimal head starts transform player 1's entire payoff of $v_1 - v_2$ from the standard all-pay auction into additional revenue.

Now, suppose $n > 2$. Note that in the all-pay auction, the organizer can use head starts to exclude any number of players from the contest: Whenever $\delta_j - \delta_i > v_i$ for some j, player i will stay inactive in any equilibrium. One possibility is hence to exclude all but two players from the contest and to set the relative head starts of those two players as in Lemma 4.2. If the identity of those two players is also chosen optimally, then this results in the optimal head starts within the class of head starts with exactly two players being active in equilibrium. The induced revenue represents a lower bound for the revenue under optimal head starts in setup HA. We state this lower bound in the following proposition, where we also show that there exist no head starts that induce first-best revenue v_1 if $v_1 > v_2$.

Proposition 4.3 *Revenue under optimal head starts in setup HA satisfies*

$$X^*_{HA} \geq v_1 - \frac{\min\{v_2(v_1-v_2), v_n(v_1-v_n)\}}{2v_1} = \begin{cases} v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1} & \text{if } v_1 < v_2 + v_n, \\ v_1 - \frac{v_n}{2} + \frac{v_n^2}{2v_1} & \text{if } v_1 > v_2 + v_n. \end{cases}$$

Moreover, $X^*_{HA} < v_1$ *if* $v_1 > v_2$.

Proof. Consider two players $\{i, j\} \subseteq \mathcal{N}$ such that $i < j$ and let the head starts be $\delta_i = e$, $\delta_j = v_i - v_j + e$, and $\delta_k = 0$ for all $k \notin \{i, j\}$, where $e > v_1$. Under these head starts, all players $k \notin \{i, j\}$ remain inactive whereas players i and j compete like in a two-player contest with optimal head starts ($\delta_j - \delta_i = v_i - v_j$ as in Lemma 4.2), resulting in revenue $X(i, j) := v_i - \frac{v_j}{2} + \frac{v_j^2}{2v_i}$.

The lower bound stated in the proposition corresponds to $\max_{\{i,j\}} X(i, j)$. To see this, note that $\frac{\partial X(i,j)}{\partial v_i} = 1 - \frac{1}{2}\left(\frac{v_j}{v_i}\right)^2 > 0$, which implies that it is optimal to set $i = 1$. Moreover, as $\frac{\partial X(1,j)}{\partial v_j} = \frac{v_j}{v_1} - \frac{1}{2}$, revenue $X(1, j)$ is decreasing in v_j for low values of v_j and increasing for high values. Hence, either $j = 2$ or $j = n$ is optimal, i.e.,

$$\max_{\{i,j\}} X(i, j) = \max\{X(1, 2), X(1, n)\} = v_1 - \frac{\min\{v_2(v_1-v_2), v_n(v_1-v_n)\}}{2v_1}.$$

The statement in the proposition holds because $v_2(v_1 - v_2) < v_n(v_1 - v_n)$ if and only if $v_1 < v_2 + v_n$.

It remains to prove that there are no head starts that induce first-best revenue v_1 if $v_1 > v_2$. First note that if $v_1 > v_2$, revenue v_1 can only be obtained in an equilibrium where player 1 invests effort v_1 and wins with probability one. Such an equilibrium exists only if each player $j > 1$ has no incentive to outbid player 1, i.e., only if $v_1 + \delta_1 \geq v_j + \delta_j$. A necessary condition is hence that head starts satisfy $v_1 - v_n \geq \delta_j - \delta_1$ for all $j > 1$. But then player 1 would deviate from bidding v_1 because any bid $x_1 \in (v_1 - v_n, v_1)$ would also result in winning with probability one. □

The optimal head starts within the class of δ with exactly two active players used for the lower bound in Proposition 4.3 let the strongest player compete against either the second strongest or the weakest player, depending on whether $v_1 < v_2 + v_n$ or $v_1 > v_2 + v_n$. Let j denote the player who competes against player 1. Under the optimal head starts of Lemma 4.2, player 1's expected effort $v_1 - \frac{v_j}{2}$ is decreasing in the weaker player's valuation v_j because a lower v_j allows for a higher head start $\delta_j - \delta_1 = v_1 - v_j$ that encourages player 1 to bid more aggressively. By contrast, player j's expected effort $\frac{v_j^2}{2v_1}$ is increasing in v_j. If $v_1 < v_2 + v_n$, i.e., if player 1's valuation is relatively low, then the second effect dominates the first. Hence, it is optimal to make v_j as large as possible by selecting the second strongest player as player 1's active opponent. If player 1's valuation is relatively high, however, the first effect outweighs the second such that it is optimal to let him compete against the weakest player.

The lower bound on revenue in Proposition 4.3 will prove sufficient for obtaining a general revenue ranking of the different instrument combinations. However, one may wonder whether this lower bound is tight or whether an organizer could do strictly better by implementing head starts under which three or more players are active in equilibrium. The following result shows that the latter is indeed possible.

Proposition 4.4 *Suppose $n = 3$, $v_1 > v_2 > v_3$, and $v_1 < v_2 + v_3$. Then, for all $\gamma \in (0, \frac{v_2(v_1-v_2)(v_2-v_3)}{v_1 v_3}]$, the all-pay auction with head starts $\delta_1 = 0$, $\delta_2 = v_1 - v_2$, $\delta_3 = \delta_2 + \gamma$ has an equilibrium with three active players that induces revenue*

$$X_{HA} > v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1}.$$

Proof. See the Appendix. □

Proposition 4.4 implies that whenever $n \geq 3$ and $v_1 < v_2 + v_n$, there are head starts that render three players active and generate strictly more revenue than the optimal head starts within the class of δ with exactly two active players. Hence, not even in a generic sense – as one might have expected – is it true that two active players in equilibrium can yield the optimal revenue. This insight further qualifies the two-player case analyzed by Li and Yu (2012) as a rather exceptional case. The fact that maximizing revenue within setup HA may require more than two players actively competing is in stark contrast to optimal biases within setup BA (discussed below) or optimal exclusion in the standard all-pay auction (Baye et al., 1993, [4]), where two active players are always sufficient.

Note that a complete characterization of the set of equilibria of all-pay auctions with head starts is still an open question. With Proposition 4.4 suggesting the need for considering equilibria with three or more active players, a general characterization of optimal head starts is hence challenging and beyond the scope of this paper.[10]

4.3 Optimal bias

The biased all-pay auction without head starts (setup BA) has been extensively analyzed in Franke et al. (2014), [19], where the optimal bias and the corresponding maximal revenue have been identified following a similar approach like in the previous subsection; that is, deriving a lower bound on equilibrium revenue by using the restricted class with two active players where a unique equilibrium in closed form exists. As in the previous subsection the optimal bias in setup BA neutralizes the difference in valuations among the two active agents. However, in contrast to setup HA the two strongest players will always be active under the optimal bias. Moreover, the lower bound thereby obtained is also an upper bound, implying the following result for maximal revenue in setup BA.

Proposition 4.5 (Franke et al. (2014), [19]) *An optimal bias in the BA setup yields equilibrium revenue* $X^*_{BA} = \frac{v_1+v_2}{2}$.

Using the results from Proposition 4.3 and 4.5 we are now in a position to evaluate the revenue extraction potential of the two instruments by comparing the respectively induced revenue. The following proposition implies that head starts are more effective than biases in an all-pay auction setup, which can be attributed to the cut-throat competition in the all-pay auction: If head starts are present in the all-pay auction then every active player must bid higher than the maximal head start among all players because all other bids are strictly dominated. Hence, head starts are powerful instruments for revenue extraction in an all-pay auction context.

Proposition 4.6 *If* $v_1 > v_2$*, optimal head starts in setup HA strictly revenue dominate optimal biases in setup BA:* $X^*_{HA} > X^*_{BA}$.

Proof. From Proposition 4.3 we know that $X^*_{HA} \geq \frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2}$. Hence it is sufficient to show that $\frac{v_2^2}{2v_1} + v_1 - \frac{v_2}{2} > \frac{v_1+v_2}{2} = X^*_{BA}$. This inequality can be reduced to $(v_1 - v_2)^2 > 0$, which always holds. □

[10] While Siegel (2014), [39], provides important results on all-pay auctions with head starts, they cannot be applied here as the head starts in both Lemma 4.2 and Proposition 4.4 violate his genericity assumption. Moreover, in our single-prize framework the algorithm of Siegel (2014), [39], is restricted to situations where exactly two players are active in equilibrium.

4.4 Optimal combinations of head starts and bias

We now consider revenue-maximizing affine transformations of bids, where non-negative head starts are combined with multiplicative biases (setup BHA). While we showed in the previous subsection that head starts are more effective than biases when used in isolation, the question remains whether combining the two instruments allows for even higher revenue extraction. In contrast to the lottery contest the answer is affirmative in the all-pay auction: Optimal affine transformations of bids yield first-best revenue v_1.

Proposition 4.7 *With an appropriately chosen tie-breaking rule τ, optimal combinations of head starts and bias in the BHA setup yield revenue $X^*_{BHA} = v_1$.*

Proof. Suppose the tie-breaking rule is such that $\tau_1(\mathcal{M}) = 1$ whenever $1 \in \mathcal{M}$. Moreover, let $\alpha_1 = 1$, $\delta_1 = 0$, $\alpha_2 = 0$, $\delta_2 = v_1$, and $\alpha_j = \delta_j = 0$ for $j > 2$. Then it is a Nash equilibrium that player 1 invests effort v_1 and wins with certainty while all other players invest zero effort. □

By combining head starts and bias with an appropriate tie-breaking rule, the contest organizer can extract first-best revenue v_1 by mimicking an optimal take-it-or-leave-it offer to the strongest player: The organizer approaches the strongest player and offers her the prize if she matches a rival's head start equal to her valuation, while keeping all other players inactive by setting their bias parameters equal to zero.[11]

Note that Proposition 4.7 requires that the organizer is able to manipulate the tie-breaking rule in favor of player 1. However, if the organizer cannot design τ and is restricted to use the fair tie-breaking rule τ^f or any other specific tie-breaking rule, he can still implement the first-best revenue in the limit. As the following result shows, revenue v_1 can be approached arbitrarily closely with head starts and biases that induce player 1 to choose effort randomly but close to v_1 and a second player to be active with a small probability, resulting in ties happening with probability zero.

Proposition 4.8 *Suppose $v_1 > v_2$. For any tie-breaking rule τ and any $\varepsilon \in (0, \frac{v_1 - v_2}{2})$, there is a combination of head starts and bias that yields revenue $v_1 - \varepsilon$.*

Proof. For all tie-breaking rules τ, revenue $v_1 - \varepsilon$ can be implemented as follows. Set $\delta_1 = 0$, $\alpha_1 = 1$, $\delta_2 = v_1(1 - \frac{2\varepsilon}{v_1 - v_2})$, and $\alpha_2 = \frac{2\varepsilon v_1}{(v_1 - v_2)v_2}$, and $\alpha_j = \delta_j = 0$ for $j > 2$. In the unique Nash equilibrium, player 1 randomizes uniformly on $[\delta_2, v_1]$ and player 2 bids zero with probability

[11] For $n = 2$, the same observation was made by Kirkegaard (2012), [26].

$\frac{\delta_2}{v_1}$ and with the remaining probability randomizes uniformly on $[0, v_2]$. All other players remain inactive. This yields revenue $\frac{\delta_2+v_1}{2} + (1 - \frac{\delta_2}{v_1})\frac{v_2}{2} = v_1 - \varepsilon$. □

In contrast to the lottery contest, optimal bias-head-start combinations in the all-pay auction are not used to increase competitive pressure by encouraging additional entry of players. Instead, they allow the contest organizer to manipulate the CSF of the all-pay auction in such a way that competition is basically reduced to the strongest player who has to compete against an opponent backed by a head start which is equal (or arbitrarily close) to her valuation. As a result, the payoff of the strongest player is zero and revenue equals (or approaches arbitrarily closely) the first best.

Finally, combining Proposition 4.7 with Propositions 4.6 and 4.3 as well as Lemma 4.1 yields a complete revenue ranking of all combinations of instruments in the all-pay auction framework.

Corollary 4.9 *There is an unambiguous revenue-ranking of all-pay auctions with optimal head starts and/or biases: If $v_1 = v_2$, then*

$$X^*_{BHA} = X^*_{HA} = X^*_{BA}. \tag{6}$$

If $v_1 > v_2$, then

$$X^*_{BHA} > X^*_{HA} > X^*_{BA}. \tag{7}$$

Proof. The equalities in (6) are from Lemma 4.1. The first inequality in (7) follows from Proposition 4.7 and the last statement in Proposition 4.3. The second inequality in (7) is stated in Proposition 4.6. □

5 Revenue Ranking

The separate analysis of the all-pay auction and the lottery contest framework facilitated a within-framework comparison of the different instruments with respect to induced revenue in Proposition 3.7 and Corollary 4.9. It remains to complete the revenue comparison between the two frameworks under the respective instruments. The following result shows that the all-pay auction with optimal bias and/or head start revenue-dominates any lottery contest irrespectively of the used instruments. More importantly, we thus obtain an unambiguous revenue-ranking of all six combinations of instruments and frameworks, which holds for any degree of heterogeneity among contestants.

Proposition 5.1 *There is an unambiguous revenue-ranking of all-pay auctions and lottery contests with optimal head starts and/or biases: If $v_1 = v_2$, then*

$$X^*_{BHA} = X^*_{HA} = X^*_{BA} > X^*_{BL} = X^*_{BHL} \geq X^*_{HL}.$$

If $v_1 > v_2$, then

$$X^*_{BHA} > X^*_{HA} > X^*_{BA} > X^*_{BL} = X^*_{BHL} > X^*_{HL}.$$

Proof. The first two and the last two relations in each line are implied by Proposition 3.7 and Corollary 4.9. The remaining inequality $X^*_{BA} > X^*_{BL}$ is implied by the revenue-dominance theorem in Franke et al. (2014), [19]. □

Proposition 5.1 has two important implications: Firstly, it shows that the all-pay auction with optimal bias-head-start combination revenue-dominates any symmetric or asymmetrically biased all-pay auction or lottery contest and any all-pay auction or lottery contest with or without head starts. Secondly, the revenue-dominance of the (optimally biased) all-pay auction over the lottery contest, established in Franke et al. (2014), [19], also holds more generally for alternative instruments like optimal head starts or optimal affine bid transformations.[12] Hence, a contest organizer interested in maximizing revenue would always prefer the all-pay auction framework as long as at least one of the mentioned instruments can be applied.

6 Conclusion

Favoritism based on affine bid transformations involving bias and head starts can be a powerful instrument for generating additional revenue in contest games. However, the revenue potential of these instruments is highly dependent on the underlying contest framework. While in the less decisive lottery contest optimal biases induce substantial additional revenue because they level the playing field and therefore encourage the entry of formerly non-active weak players, head starts are less effective in this framework as favored agents substitute them for own effort. In the more decisive all-pay auction, by contrast, generically only two agents are active which severely limits the revenue-potential of using biases to induce additional entry of players. Head starts on the other hand are highly effective in the cut-throat competition of the all-pay auction because strong players have to exceed the highest head start of their rivals to maintain a positive

[12]The fact that these revenue rankings do not depend on the heterogeneity of the contestants is in contrast to the unbiased frameworks without head starts where no such unambiguous ranking exists, comp. Fang (2002), [15].

winning probability. These results suggest that the optimal instrument of favoritism (head start or bias) and also the endogenously determined number of active participants indirectly depend on the decisiveness of the contest rule. In principle, the resulting hypothesis could be tested empirically based on data from real world contests where the number of active contests can be easily observed, while the decisiveness of the contest can be either estimated econometrically or proxied based on appropriate measures, see Hwang (2012), [23], and Azmat and Möller (2009), [2], for two empirical approaches along these lines. However, empirical contest analysis faces its own challenges, see Jia et al. (2013), [24], and goes beyond the scope of this paper.

Our theoretical analysis does not only shed light on the optimal design of those two instruments in isolation, but also addresses the optimal design of both instruments, bias and head starts, if they can be applied simultaneously at the same time. For the all-pay auction framework we can show, for instance, that the optimal combination of bias and head starts resembles a take-it-or-leave-it offer to the strongest player. Thus, granting the contest organizer the right to design affine transformations of the bids, results in revenue extraction at the upper bound; however, at the same time competition is basically reduced to one player (the strongest) competing against a head start close to her valuation. Hence, in contrast to the lottery contest game, revenue is maximal in the all-pay auction under the optimal affine transformation, but the resulting setup does not look very competitive in the sense of high participation rates. In this sense, there is a trade-off between revenue maximization on one side and participation in the contest game on the other side.

A pending issue with respect to future research is the complete equilibrium characterization of the all-pay auction with head starts because the algorithm of Siegel (2014), [39], is restricted to those equilibria, where only two players are active. Our equilibrium characterization of an all-pay auction with head starts involving three active players in the appendix constitutes a first step in this direction. A complete characterization of the set of equilibria will also be useful for other applications beyond contest design.

A Appendix: Proof of Proposition 4.4

We will first determine a Nash equilibrium under the given head starts, which is in mixed strategies. Let $y_i = x_i + \delta_i$ denoted the score of player i if he invests effort x_i. Moreover, let G_i denote a cumulative distribution function over scores y_i for player i. As it is more convenient for our analysis, we will represent the equilibrium in terms of score distributions rather than effort distributions. The following lemma identifies equilibrium score distributions under an assumption on γ that is less restrictive than the one stated in the proposition.[13]

Lemma A.1 *Suppose $n = 3$, $v_1 > v_2 > v_3$, and $v_1 < v_2 + v_3$. Then, for all $\gamma \in (0, v_2 - v_3)$, the all-pay auction with head starts $\delta_1 = 0$, $\delta_2 = v_1 - v_2$, and $\delta_3 = \delta_2 + \gamma$ has a Nash equilibrium with score distributions*

$$G_1(y) = \begin{cases} 0 & \textit{if } y < \delta_3, \\ \frac{v_1}{v_3 \hat{y}}(y - \delta_3) & \textit{if } y \in [\delta_3, \hat{y}), \\ \frac{v_1}{v_3 \hat{y}}(\hat{y} - \delta_3) = \frac{1}{v_2}(\hat{y} - \delta_2) & \textit{if } y = \hat{y}, \\ \frac{1}{v_2}(y - \delta_2) & \textit{if } y \in (\hat{y}, v_1], \\ 1 & \textit{if } y > v_1, \end{cases} \qquad G_2(y) = \begin{cases} 0 & \textit{if } y < \delta_2, \\ \frac{\hat{y}}{v_1} & \textit{if } y \in [\delta_2, \hat{y}], \\ \frac{1}{v_1} y & \textit{if } y \in (\hat{y}, v_1], \\ 1 & \textit{if } y > v_1, \end{cases}$$

$$\textit{and } G_3(y) = \begin{cases} 0 & \textit{if } y < \delta_3, \\ \frac{1}{\hat{y}} y & \textit{if } y \in [\delta_3, \hat{y}], \\ 1 & \textit{if } y > \hat{y}, \end{cases}$$

where

$$\hat{y} := \frac{\phi + \delta_2}{2} - \frac{1}{2}\sqrt{(\phi - \delta_2)^2 - 4\phi\gamma}, \quad \hat{y} \in (\delta_3, v_1), \quad \textit{and } \phi := \frac{v_1 v_2}{v_3}.$$

Proof. We will show that no player i has an incentive to deviate from mixing according to G_i, which yields zero expected payoff for each player.

Player 1: Given G_2, G_3, player 1's payoff from choosing score y is

$$G_2(y) G_3(y) v_1 - y = \begin{cases} -y \leq 0 & \text{if } y < \delta_3, \\ \frac{\hat{y}}{v_1} \frac{1}{\hat{y}} y v_1 - y = 0 & \text{if } y \in [\delta_3, \hat{y}], \\ \frac{y}{v_1} v_1 - y = 0 & \text{if } y \in (\hat{y}, v_1], \\ v_1 - y < 0 & \text{if } y > v_1. \end{cases}$$

Hence, randomizing according to G_1 is a best response.

Player 2: Let $u_2(y) := G_1(y) G_3(y) v_2 - y + \delta_2$ denote player 2's payoff from score y. Scores $y \in (\delta_2, \delta_3)$ and $y > v_1$ are clearly dominated as they yield $u_2(y) < 0$ while $u_2(\delta_2) = 0$. Note that $u_2(y)$ is continuous for $y \in [\delta_3, v_1]$. For $y \in [\hat{y}, v_1]$, $u_2(y) = 0$ and for $y \in [\delta_3, \hat{y})$,

$$u_2(y) = \frac{\phi}{\hat{y}^2}(y - \delta_3) y - y + \delta_2.$$

[13] If $v_1 < v_2 + v_3$, then $\frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3} < v_2 - v_3$.

As this is a strictly convex function, $u_2(\delta_3) = -\gamma < 0$ and $u_2(\hat{y}) = 0$ implies $u_2(y) < 0$ for all $y \in [\delta_3, \hat{y})$. Consequently, G_2 is a best response to G_1, G_3.

Player 3: Let $u_3(y) := G_1(y)G_2(y)v_3 - y + \delta_3$ denote player 3's payoff from score $y \geq \delta_3$ and note that this is a continuous function. For $y \in [\delta_3, \hat{y}]$, $u_3(y) = \frac{v_1}{v_3 \hat{y}}(y - \delta_3)\frac{\hat{y}}{v_1}v_3 - y + \delta_3 = 0$ and for $y \in (\hat{y}, v_1]$,

$$u_3(y) = G_1(y)G_2(y)v_3 - y + \delta_3 = \frac{1}{\phi}(y - \delta_2)y - y + \delta_3.$$

Because u_3 is strictly convex on $[\hat{y}, v_1]$, $u_3(\hat{y}) = 0$ and $u_3(v_1) = v_3 - v_2 + \gamma < 0$ implies $u_3(y) < 0$ for all $y \in (\hat{y}, v_1]$. Hence, G_3 is a best response to G_1, G_2. □

On the upper part of the equilibrium score support, only players 1 and 2 are active and mix as in the equilibrium of the two-player case. On the lower part, only players 1 and 3 are active. Player 1's equilibrium score distribution is continuous and atomless, whereas players 2 and 3 both have an atom at their respective head starts (i.e., they choose zero effort with strictly positive probability).

Lemma A.2 *For any $\gamma \in \left(0, \frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3}\right)$ the equilibrium of Lemma A.1 revenue-dominates any all-pay auction with head starts under which exactly two players are active.*

Proof. As player 1's effort is uniformly distributed on $[\delta_3, \hat{y}]$ with probability $G_1(\hat{y})$ and on $[\hat{y}, v_1]$ with probability $1 - G_1(\hat{y})$, 1's expected effort amounts to

$$e_1 := G_1(\hat{y})\frac{\delta_3 + \hat{y}}{2} + (1 - G_1(\hat{y}))\frac{\hat{y} + v_1}{2}.$$

Similarly, player 2's effort is uniformly distributed on $[\hat{y} - \delta_2, v_1 - \delta_2]$ with probability $1 - G_2(\hat{y}) = \frac{v_1 - \hat{y}}{v_1}$ and zero otherwise, resulting in expected effort

$$e_2 := \frac{v_1 - \hat{y}}{v_1} \cdot \frac{\hat{y} - \delta_2 + v_1 - \delta_2}{2} = \frac{(v_1 - \hat{y})(\hat{y} + v_2 - \delta_2)}{2v_1}.$$

To prove the proposition, we will show that $e_1 + e_2 \geq v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1}$ if $\gamma \leq \frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3}$. This implies that revenue is strictly greater than $v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1}$ because in addition to players 1 and 2 also player 3 contributes to revenue.

In the two-player benchmark that yields revenue $v_1 - \frac{v_2}{2} + \frac{v_2^2}{2v_1}$, player 1's expected effort is $\frac{v_1 + \delta_2}{2}$ and player 2's effort is $\frac{v_2^2}{2v_1}$. Comparing these efforts to those in the three-player case, we find, after some rearranging,

$$e_1 - \frac{v_1 + \delta_2}{2} = \frac{(\hat{y} - \delta_2)\gamma}{2v_2} > 0,$$
$$e_2 - \frac{v_2^2}{2v_1} = -\frac{(\hat{y} - \delta_2)^2}{2v_1} < 0.$$

Hence, the presence of player 3 encourages player 1 to exert more effort while player 2 invests less. We now show that the net effect is positive, i.e.,

$$e_1 + e_2 - \frac{v_1 + \delta_2}{2} - \frac{v_2^2}{2v_1} = \frac{(\hat{y} - \delta_2)\gamma}{2v_2} - \frac{(\hat{y} - \delta_2)^2}{2v_1} \geq 0.$$

This inequality can be reduced to $v_1\gamma - v_2(\hat{y} - \delta_2) \geq 0$, which, using the definition of $\hat{y}$, is equivalent to

$$v_2\sqrt{(\phi - \delta_2)^2 - 4\phi\gamma} \geq v_2(\phi - \delta_2) - 2v_1\gamma.$$

It can be shown that the RHS is positive for $\gamma \leq \frac{v_2(v_1(v_2 - v_3) + v_2 v_3)}{2v_1 v_3}$. After squaring both sides, we obtain

$$-v_2^2\phi\gamma \geq v_1^2\gamma^2 - v_2(\phi - \delta_2)v_1\gamma.$$

Substituting for δ_2 and ϕ, we find the above to be equivalent to $\gamma \leq \frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3}$. Note that $\frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3} < \frac{v_2(v_1(v_2 - v_3) + v_2 v_3)}{2v_1 v_3}$ because $v_1 < v_2 + v_3$; hence, $\gamma \leq \frac{v_2(v_1 - v_2)(v_2 - v_3)}{v_1 v_3 1}$ is sufficient. □

The assumption $v_1 < v_2 + v_3$ implies that the highest revenue over all equilibria with only two active players is given when only players 1 and 2 are active. What changes, if we introduce a third payer and give him head start δ_3 with γ as characterized in Lemma A.2? First of all, since $\delta_3 > \delta_2$ the lower bound of the support of players 1's bidding distribution is pushed upwards (from δ_2 to δ_3). This increases the expected bid of player 1. Player 2 now puts an atom on δ_2 as before, but the lower bound of the support of his bidding distribution is also pushed upwards (from δ_2 to $\hat{y}$). This, in fact, may lower his average expected bid as he counters the need to bid higher, if he enters, by a larger probability to stay inactive (i.e., increase the atom at δ_2). Unequivocally, player 3's expected bid is positive. The further contributions of players 1 and 3 to total revenue dominate the possible reduction in Player 2's contribution, if γ is chosen appropriately.

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