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Continuous symmetries and conservation laws in chiral media

Frances Crimin^a, Neel Mackinnon^a, Jörg B. Götter^{a,b}, and Stephen M. Barnett^a

^aDepartment of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, UK

^bCollege of Engineering and Applied Science, Nanjing University, Nanjing 210093, China

ABSTRACT

Locally conserved quantities of the electromagnetic field in lossless chiral media are derived from Noether's theorem, including helicity, chirality, momentum, and angular momentum, as well as the separate spin and orbital components of this last quantity. We discuss sources and sinks of each in the presence of current densities within the material, and in some cases, as also generated by inhomogeneity of the medium. A previously obtained result connecting sources of helicity and energy within chiral materials is explored, revealing that association between the two quantities is not restricted to chiral media alone. Rather, it is analogous to the connection between the momentum, and the spin and orbital components of the total angular momentum. The analysis reveals a new quantity, appearing as the "orbital" counterpart of the helicity density in classical electromagnetism.

Keywords: Chiral media, conserved quantities, Noether's theorem, optical angular momentum, gauge transformations

1. INTRODUCTION

The propagation of electromagnetic fields in chiral media has long been a topic of interest in classical electromagnetism.¹⁻⁴ In the simplest case, a chiral medium is a macroscopic, continuous material made up of indistinguishable chiral objects that are distributed homogeneously throughout, but with random orientation. A chiral object is one which cannot be superimposed upon its mirror image by rotation or translation: such objects exhibit handedness. When a linearly polarised field passes through a chiral medium, the left- and right-circularly polarised components of the field travel at different group velocities,^{4,5} so that, upon leaving the medium and recombining, the polarisation of the transmitted field is rotated with respect to the incident one.⁵⁻⁸ The angle, θ , through which this polarisation is rotated, differs in sign depending upon whether the constituent objects in the medium are left- or right-handed. In this way, chiral matter demonstrates handedness.

The simplest example of a chiral object is a helix, a geometric figure which is also traced by the electric and magnetic field vector of circularly polarised light. There has been a recent resurgence of activity in the study of the chiral content of light,⁹⁻¹³ and the optical helicity^{11,14-20} has been advocated as an appropriate measure of this.^{11,13,21} It has been shown²² that the helicity density, in particular, takes on a peculiar form in chiral media, acquiring an additional term which is proportional to the energy density, with a sign dependent upon the handedness of the material: sources of energy therefore become equivalent to sources of helicity in such media. This was previously observed by Lakhtakia et al.,²³ who found that the far-field radiation emitted by a point electric dipole in a chiral sphere is identical to that of an electric-magnetic dipole pair radiating into vacuum, which itself is a simple model of a source of helicity.^{21,24} The relationship between the helicity and the energy thus warrants further investigation, and so this curious result prompts us to consider whether such a relationship holds between other conserved quantities in chiral media, particularly those naturally associated with the angular momentum content of the field, in the hope that such findings may shed some light upon it.

In this article, we derive the form of some locally conserved quantities in chiral media from Noether's theorem, including the helicity, chirality, momentum and (spin and orbital) angular momentum densities. The forms of sources of these quantities in the presence of a local current are found, and the results discussed. Throughout, we consider only homogeneous, isotropic, and lossless media, unless otherwise stated.

Further author information: (Send correspondence to F.C.)

F.C.: E-mail: frances.crimin@glasgow.ac.uk

2. THE ENERGY DENSITY AND LAGRANGIAN DENSITY IN CHIRAL MEDIA

In this section, we use the known form of the energy density in chiral media^{25,26} to postulate the form of the Lagrangian density, and then go on to test this quantity by various means. The energy density in a chiral medium can be written in the *exact* form^{25,26}

$$w_\beta = \frac{1}{2} \left(\frac{1}{\epsilon} \mathbf{D} \cdot \mathbf{D} + \frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} \right). \quad (1)$$

We use the Drude-Born-Fedorov (DBF) constitutive relations,²⁷ where the electric and magnetic inductions \mathbf{D} and \mathbf{B} have the form

$$\begin{aligned} \mathbf{D} &= \epsilon(\mathbf{E} + \beta \nabla \times \mathbf{E}), \\ \mathbf{B} &= \mu(\mathbf{H} + \beta \nabla \times \mathbf{H}). \end{aligned} \quad (2)$$

Here, the pseudoscalar β describes the chiroptical response of the medium, with ϵ and μ acting as the permittivity and permeability. Using (2), it is straightforward to check the local conservation of the energy density (1). In the presence of a current density \mathbf{j} , we obtain

$$\partial_t w_\beta + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{1}{\epsilon} \mathbf{j} \cdot \mathbf{D}. \quad (3)$$

Expansion of the energy density (1) using (2) highlights its peculiar form in a chiral medium: for each of the electric and magnetic contributions, it contains two terms which do not depend on the handedness of the material, and one, $\mathbf{E} \cdot (\nabla \times \mathbf{E})$ and $\mathbf{H} \cdot (\nabla \times \mathbf{H})$, which does, as it is proportional to β . This structure, the dot product of a field with its curl, is echoed by the form of both the optical chirality density,²⁸ which in vacuum is given by

$$\chi = \frac{\epsilon_0}{2} \left[\mathbf{E} \cdot (\nabla \times \mathbf{E}) + c^2 \mathbf{B} \cdot (\nabla \times \mathbf{B}) \right], \quad (4)$$

and the helicity density in vacuum^{11,18}

$$h = \frac{1}{2} \left[\sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{A}^\perp \cdot (\nabla \times \mathbf{A}) + \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{C}^\perp \cdot (\nabla \times \mathbf{C}) \right], \quad (5)$$

where the transverse part of \mathbf{C} defines the transverse part of the displacement field, $\mathbf{D}^\perp = -\nabla \times \mathbf{C}$.^{29,30} This already hints at the connection between measures of the chiral content of a field and the energy density,²² which only becomes apparent if the medium is chiral.

From (1), we introduce the following form of the Lagrangian density of the free field in a chiral medium:

$$\mathcal{L} = \frac{1}{2} \left(\frac{1}{\epsilon} \mathbf{D} \cdot \mathbf{D} - \frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} \right). \quad (6)$$

Again using the DBF constitutive relations, this is written in terms of the vector and scalar coordinates of the electromagnetic potential, \mathbf{A} and A_0 , in an arbitrary gauge:

$$\mathcal{L} = \frac{1}{2} \left(\epsilon \left(\dot{\mathbf{A}} + \nabla A_0 + \beta \nabla \times \dot{\mathbf{A}} \right)^2 - \frac{1}{\mu} (\nabla \times \mathbf{A})^2 \right). \quad (7)$$

The momentum conjugate to \mathbf{A} is

$$\mathbf{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = -(1 + \beta \nabla \times) \mathbf{D}, \quad (8)$$

and the form of (6) may be verified by the resulting Euler-Lagrange equations, since

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} \right) = -(1 + \beta \nabla \times) \dot{\mathbf{D}} \quad (9)$$

and

$$\frac{\partial \mathcal{L}}{\partial \mathbf{A}} = -(1 + \beta \nabla \times) \nabla \times \mathbf{H}, \quad (10)$$

such that the Maxwell equation $\dot{\mathbf{D}} = \nabla \times \mathbf{H}$ is recovered. It is worthwhile to note that the Lagrangian density (6) can also be written in terms of the electric scalar and vector potentials, C_0 and \mathbf{C} . Following the above treatment for the generalised coordinate \mathbf{A} , the Euler-Lagrange equations in the present case reveal Faraday's law, $\dot{\mathbf{B}} = -\nabla \times \mathbf{E}$. As a further test of (6), we find that the energy density (1) can be recovered by finding $\mathcal{H} \equiv w_\beta = \Pi \dot{\mathbf{A}} - \mathcal{L}$. Analogous results follow using the coordinate \mathbf{C} and associated momentum.

3. CONSERVED QUANTITIES

We wish to test whether various electromagnetic quantities, conserved in vacuum, are also conserved in chiral media, and to find the explicit form of these quantities. To this end, we apply the infinitesimal transformation known to generate each conserved quantity in vacuum to the Lagrangian density (6). Should it remain invariant, we then use Noether's theorem³¹ to find the form of these quantities in chiral media.

3.1 The helicity density in a chiral medium

It is now well-known that the duality transformation is generated by the conservation of helicity in vacuum,^{16,17} and in lossless media in which dual-symmetry is preserved.^{19,22} The form of the helicity density in chiral media has been found previously,^{21,22} by considering its continuity equation directly. Here, we verify these results, and indeed the form of the Lagrangian density (6), by deriving it instead from Noether's theorem. We proceed by applying the infinitesimal form of the duality transformation³²

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \theta \sqrt{\frac{\mu}{\epsilon}} \mathbf{C}, \\ \mathbf{C} &\rightarrow \mathbf{C} - \theta \sqrt{\frac{\epsilon}{\mu}} \mathbf{A}, \end{aligned} \quad (11)$$

to (6). Verifying that the Euler-Lagrange equations still hold upon this transformation, it is straightforward to find:

$$\left(\frac{\partial \mathcal{L}}{\partial \mathbf{A}} \right) \delta \mathbf{A} + \left(\frac{\partial \mathcal{L}}{\partial \mathbf{C}} \right) \delta \mathbf{C} = \theta \left[-\sqrt{\frac{\mu}{\epsilon}} \mathbf{C} \cdot (1 + \beta \nabla \times) \mathbf{D} + \sqrt{\frac{\epsilon}{\mu}} \mathbf{A} \cdot (1 + \beta \nabla \times) \mathbf{B} \right], \quad (12)$$

such that the conserved quantity is given in the square brackets. After integration by parts, setting

$$\int \nabla_i \left[\sqrt{\frac{\epsilon}{\mu}} (\mathbf{A} \times \mathbf{B})_i - \sqrt{\frac{\mu}{\epsilon}} (\mathbf{C} \times \mathbf{D})_i \right] d^3 \mathbf{r} \equiv 0, \quad (13)$$

and dividing the whole quantity by a factor of two, the helicity density in a chiral medium is identified:

$$\begin{aligned} h_\beta &= \frac{1}{2} \left[\sqrt{\frac{\epsilon}{\mu}} \mathbf{B} \cdot (1 + \beta \nabla \times) \mathbf{A}^\perp - \sqrt{\frac{\mu}{\epsilon}} \mathbf{D} \cdot (1 + \beta \nabla \times) \mathbf{C}^\perp \right] \\ &= \frac{1}{2} \left(\sqrt{\frac{\epsilon}{\mu}} \mathbf{A}^\perp \cdot \mathbf{B} - \sqrt{\frac{\mu}{\epsilon}} \mathbf{C}^\perp \cdot \mathbf{D} \right) + \beta \sqrt{\epsilon \mu} \frac{1}{2} \left(\frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} + \frac{1}{\epsilon} \mathbf{D} \cdot \mathbf{D} \right). \end{aligned} \quad (14)$$

We have further specified that only the transverse parts of the vector potentials are included in (14): this enables the helicity density to be written in a manifestly gauge-independent form,¹⁶ and indeed, these are the only parts of the vector potentials picked out by an integration over all space in calculating the total helicity. Eq. (14) is in agreement with the form of the helicity density in a chiral medium found in earlier work.²² It is worthwhile to examine this equation in the presence of a current density \mathbf{j} within the medium. Using the helicity flux density

$$\mathbf{v} = \frac{1}{2} \left(\sqrt{\frac{\epsilon}{\mu}} \mathbf{E} \times \mathbf{A}^\perp + \sqrt{\frac{\mu}{\epsilon}} \mathbf{H} \times \mathbf{C}^\perp \right), \quad (15)$$

and defining $\mathbf{j}^\perp = \nabla \times \mathbf{g}$,¹⁸ the continuity equation for the helicity density produces

$$\partial_t h_\beta + \nabla \cdot \mathbf{v} = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} [\mathbf{g} \cdot (\nabla \times \mathbf{C}) + \mathbf{C} \cdot (\nabla \times \mathbf{g})] - \sqrt{\frac{\mu}{\epsilon}} \beta \mathbf{j}^\perp \cdot \mathbf{D}. \quad (16)$$

This is exact to all $\mathcal{O}(\beta)$, i.e. there are no higher order source terms as previously expected.²¹ This result is comparable to the analogous continuity equation for the energy density, as shown in (3).

3.2 Chirality density in a chiral medium

Following the above treatment, we proceed to find the form of other well-known quantities of the electromagnetic field in a chiral medium, beginning with the chirality density. We consider the following infinitesimal transformation of the magnetic vector potentials \mathbf{A} and \mathbf{C} , known to be generated by the conservation of chirality in vacuum:³³

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \eta \nabla \times \dot{\mathbf{A}}, \\ \mathbf{C} &\rightarrow \mathbf{C} + \eta \nabla \times \dot{\mathbf{C}}. \end{aligned} \quad (17)$$

It is straightforward to check by substitution into the macroscopic Maxwell equations that these constitute a symmetry transformation of the system only when the values of μ , ϵ , and β are constant. We would therefore expect chirality, and not helicity,¹⁹ to be generated at the interface between vacuum and a dual-symmetric chiral medium where the ratio ϵ/μ is constant, but the separate permittivity and permeability may vary. In analogy with the treatment for the helicity density above, we identify, up to surface terms, the following conserved quantity as the chirality density:

$$\chi_\beta = \frac{1}{2} \left[\frac{1}{\epsilon} \mathbf{D} \cdot (\nabla \times \mathbf{D}) + \frac{1}{\mu} \mathbf{B} \cdot (\nabla \times \mathbf{B}) \right]. \quad (18)$$

We proceed to find an exact expression for the source terms on the right-hand side of the chirality continuity equation (up to surface terms) in the presence of a current density \mathbf{j} :

$$\partial_t \chi_\beta + \frac{1}{2} \nabla \cdot [\mathbf{E} \times (\nabla \times \mathbf{H}) - \mathbf{H} \times (\nabla \times \mathbf{E})] = -\frac{1}{2} [\mathbf{j} \cdot (\nabla \times \mathbf{E}) + \mathbf{E} \cdot (\nabla \times \mathbf{j})] - \beta (\nabla \times \mathbf{j}) \cdot (\nabla \times \mathbf{E}). \quad (19)$$

Setting $\beta = 0$ reduces this result to that obtained for achiral media,^{18,34} and it is clear that it parallels the result for the helicity density (16).

3.3 The canonical and kinetic momentum density in a chiral medium

The conservation of electromagnetic momentum is associated with symmetry under translation of the vector potentials:³⁵

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &\rightarrow \mathbf{A}(\mathbf{r} + \boldsymbol{\eta}, t), \\ \mathbf{C}(\mathbf{r}, t) &\rightarrow \mathbf{C}(\mathbf{r} + \boldsymbol{\eta}, t). \end{aligned} \quad (20)$$

We Taylor expand to first order, to write

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &\rightarrow \mathbf{A}(\mathbf{r}, t) + (\boldsymbol{\eta} \cdot \nabla) \mathbf{A}(\mathbf{r}, t), \\ \mathbf{C}(\mathbf{r}, t) &\rightarrow \mathbf{C}(\mathbf{r}, t) + (\boldsymbol{\eta} \cdot \nabla) \mathbf{C}(\mathbf{r}, t), \end{aligned} \quad (21)$$

which is equivalent to

$$\begin{aligned} A_i &\rightarrow A_i + \eta_j \nabla_j A_i, \\ C_i &\rightarrow C_i + \eta_j \nabla_j C_i, \end{aligned} \quad (22)$$

where repeated indices are summed over. Adding a gradient $\nabla_i \chi$ to the translation, and choosing $\chi = -\eta_j A_j$, the variation of the vector potential \mathbf{A} becomes gauge invariant:³⁶

$$\begin{aligned}\delta A_i &= \eta_j \nabla_j A_i - \eta_j \nabla_i A_j, \\ &= \eta_j (\nabla_j A_i - \nabla_i A_j) \\ &\equiv \eta_j \mathcal{F}_{ji}.\end{aligned}\tag{23}$$

We identify \mathcal{F}_{ij} as the electromagnetic field tensor. The corresponding variation of the vector potential \mathbf{C} is

$$\delta C_i = \eta_j \nabla_j C_i - \eta_j \nabla_i C_j.\tag{24}$$

These variations are now equivalent to the transformation

$$\begin{aligned}\mathbf{A} &\rightarrow \mathbf{A} - \boldsymbol{\eta} \times (\boldsymbol{\nabla} \times \mathbf{A}), \\ \mathbf{C} &\rightarrow \mathbf{C} - \boldsymbol{\eta} \times (\boldsymbol{\nabla} \times \mathbf{C}),\end{aligned}\tag{25}$$

which are straightforward to identify as a symmetry transformation of the Lagrangian density (6), with the resultant conserved quantity

$$\mathbf{p}_\beta = \mathbf{D} \times \mathbf{B} + \frac{1}{2} \beta [(\boldsymbol{\nabla} \times \mathbf{D}) \times \mathbf{B} + \mathbf{D} \times (\boldsymbol{\nabla} \times \mathbf{B})],\tag{26}$$

the momentum density in a chiral medium. In the presence of charges and currents, we find

$$\partial_t \mathbf{p}_\beta + \boldsymbol{\nabla} \cdot \bar{\mathbf{T}} = -\frac{1}{2} \left[\rho \left(\mathbf{E} + \frac{1}{\epsilon} \mathbf{D} \right) + 2\mathbf{j} \times \mathbf{B} + \beta [(\boldsymbol{\nabla} \times \mathbf{j}) \times \mathbf{B} + \mathbf{j} \times (\boldsymbol{\nabla} \times \mathbf{B})] \right],\tag{27}$$

where the T_{ij}^{th} component of the tensor $\bar{\mathbf{T}}$ is given by

$$T_{ij} = \frac{1}{2} \delta_{ij} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) - \frac{1}{2} (D_i E_j + E_i D_j + B_i H_j + H_i B_j).\tag{28}$$

We may test the form of the momentum density by inserting the the fields of left- and right-circularly polarised plane waves into (26) and the energy density (1), finding that it correctly produces a momentum of $\hbar k$ per photon. Similarly, in calculating the flux density components T_{ij} for the plane waves, the momentum density is found to travel at the group velocity of the medium.²² The form of \mathbf{p}_β seems to suggest a kinetic momentum density in a chiral medium of the form

$$\mathbf{p}_{k,\beta} = \frac{1}{c^2} \left(\mathbf{E} \times \mathbf{H} + \frac{1}{2} \beta [(\boldsymbol{\nabla} \times \mathbf{E}) \times \mathbf{H} + \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{H})] \right)\tag{29}$$

This can be motivated by considering the ratio of \mathbf{p}_β to $\mathbf{p}_{k,\beta}$ for right- and left-handed circularly polarised plane waves travelling in the \hat{z} direction, which is

$$\frac{\langle \mathbf{p}_\beta^\pm \cdot \hat{z} \rangle}{\langle \mathbf{p}_{k,\beta}^\pm \cdot \hat{z} \rangle} = (n^\pm)^2,\tag{30}$$

where

$$n^\pm = \frac{ck}{\omega} = \frac{\sqrt{\epsilon\mu}(1 \pm \beta k)}{\sqrt{\epsilon_0\mu_0}},\tag{31}$$

with the superscripts + and – denoting the results for the two polarisations. This result, (30), is what we would expect for the ratio of canonical to kinetic momentum in a dielectric.³⁷

3.4 Angular momentum in a chiral medium

Using the transformation generated by the conservation of total angular momentum,³⁸ and following the method above, we find that the angular momentum in a chiral medium, up to surface terms, is given by

$$\mathbf{J}_\beta = \int \mathbf{r} \times \left[\mathbf{D} \times \mathbf{B} + \beta \frac{1}{2} [(\nabla \times \mathbf{D}) \times \mathbf{B} + \mathbf{D} \times (\nabla \times \mathbf{B})] \right] d^3\mathbf{r}. \quad (32)$$

As expected, this is equal to the cross product of the position vector with the linear momentum (26). Following the method of the separation of the angular momentum density into spin and orbital components,³⁸ we re-write (32)

$$\begin{aligned} \mathbf{J}_\beta = \frac{1}{2} \int \left[\mathbf{D} \times \mathbf{A}^\perp + D_j(\mathbf{r} \times \nabla) A_j^\perp + \beta [(\nabla \times \mathbf{D}) \times \mathbf{A}^\perp + (\nabla \times \mathbf{D})_j(\mathbf{r} \times \nabla) A_j^\perp] \right. \\ \left. + \mathbf{B} \times \mathbf{C}^\perp + B_j(\mathbf{r} \times \nabla) C_j^\perp + \beta [(\nabla \times \mathbf{B}) \times \mathbf{C}^\perp + (\nabla \times \mathbf{B})_j(\mathbf{r} \times \nabla) C_j^\perp] \right] d^3\mathbf{r}, \end{aligned} \quad (33)$$

where the surface terms have been dropped. This suggests a spin density of the form

$$\mathbf{s}_\beta = \frac{1}{2} (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp + \beta [(\nabla \times \mathbf{D}) \times \mathbf{A}^\perp + (\nabla \times \mathbf{B}) \times \mathbf{C}^\perp]), \quad (34)$$

with the orbital angular momentum density correspondingly given by

$$\mathbf{l}_\beta = \frac{1}{2} D_j(\mathbf{r} \times \nabla) A_j^\perp + B_j(\mathbf{r} \times \nabla) C_j^\perp + \beta [(\nabla \times \mathbf{D})_j(\mathbf{r} \times \nabla) A_j^\perp + (\nabla \times \mathbf{B})_j(\mathbf{r} \times \nabla) C_j^\perp]. \quad (35)$$

The form of the spin and orbital components, (34) and (35), are in agreement with the results obtained using Noether's theorem and the infinitesimal symmetry transformations generated by the conservation of the respective components.³⁸

Let us examine the β -dependent terms in the spin density (34). Integrating by parts, we find

$$\int \beta [(\nabla \times \mathbf{D}) \times \mathbf{A}^\perp + (\nabla \times \mathbf{B}) \times \mathbf{C}^\perp] d^3\mathbf{r} = 2\beta \int \mathbf{D} \times \mathbf{B} d^3\mathbf{r}, \quad (36)$$

so that the total spin density becomes

$$\begin{aligned} \mathbf{s}_\beta &= \frac{1}{2} (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp) + \beta \mathbf{D} \times \mathbf{B} \\ &\equiv \mathbf{s} + \beta \mathbf{p}, \end{aligned} \quad (37)$$

where \mathbf{s} and \mathbf{p} have the same form as the spin and linear momentum densities in *achiral* media, respectively. This is reminiscent of the presence of the energy density in the chiral contribution to the helicity density (14), although there, it is the *full* energy density which is included, whereas here we do not include the β dependent term in \mathbf{p}_β (26). The continuity equation of the spin density in the presence of sources reveals:

$$\partial_t \mathbf{s}_\beta + \nabla \cdot \bar{\Sigma} = \frac{1}{2} [(\nabla \times \mathbf{A}) \times \mathbf{g} + \mathbf{A}^\perp \times (\nabla \times \mathbf{g}) - 2\beta \mathbf{j}^\perp \times \mathbf{B}], \quad (38)$$

where the ij -th component of the spin flux density $\bar{\Sigma}$ is defined as³⁹

$$\Sigma_{ij} = \frac{1}{2} [\delta_{ij} (\mathbf{A}^\perp \cdot \mathbf{H} - \mathbf{C}^\perp \cdot \mathbf{E}^\perp) - A_i^\perp H_j - H_i A_j^\perp + C_i^\perp E_j^\perp + E_i^\perp C_j^\perp]. \quad (39)$$

Again in analogy with the helicity and energy relationship, by comparison to (27), we see the transverse components of sources of linear momentum contributing a net flux of spin density in a chiral medium.

For completeness, the continuity equation for the orbital component of the angular momentum in the presence of currents and charges is given by

$$\partial_t \mathbf{l}_\beta + \nabla \cdot \bar{\mathbf{A}} = \frac{1}{2} [(\nabla \times \mathbf{A})(\mathbf{r} \times \nabla) \mathbf{g} - (\nabla \times \mathbf{g})(\mathbf{r} \times \nabla) \mathbf{A}^\perp + \beta [\mathbf{B}(\mathbf{r} \times \nabla) \mathbf{j}^\perp - \mathbf{j}^\perp(\mathbf{r} \times \nabla) \mathbf{B}], \quad (40)$$

where³⁹

$$\Lambda_{ij} = \frac{1}{2} [\varepsilon_{ikl} \varepsilon_{jmn} r_k [H_n(\nabla_l A_m^\perp) - E_n^\perp(\nabla_l C_m^\perp)] + H_j A_i^\perp - E_j^\perp C_i^\perp]. \quad (41)$$

We note that in the absence of currents and charges, and in a non-dual-symmetric, achiral medium, the electric and magnetic contributions in the total, spin, and orbital angular momenta are separately conserved.³⁹ Inside a chiral medium, however, both electric and magnetic parts must be used together in order for conservation: dual symmetry is required for their conservation.

The forms of the energy, helicity, linear momentum, and spin densities, in addition to sources of each in the presence of a local current, are summarised in Table 1. The table highlights the connection between the conserved quantities in chiral media, which will be further discussed in the following section.

Table 1. Conserved quantities in chiral media, and sources of each in the presence of a current density \mathbf{j} within the medium. The red highlight indicates the presence of the energy density, and its source term, in the form of the helicity density and its associated sources, with the blue indicating the analogous result for the linear momentum and spin density.

Conserved quantity	Density	Sources
Energy	$\frac{1}{2} (\frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} + \frac{1}{\epsilon} \mathbf{D} \cdot \mathbf{D})$	$-\frac{1}{2} \mathbf{j} \cdot \mathbf{D}$
Helicity	$\frac{1}{2} (\sqrt{\frac{\epsilon}{\mu}} \mathbf{A}^\perp \cdot \mathbf{B} - \sqrt{\frac{\mu}{\epsilon}} \mathbf{C}^\perp \cdot \mathbf{D}) + \beta \frac{1}{c} (\frac{1}{\mu} \mathbf{B} \cdot \mathbf{B} + \frac{1}{\epsilon} \mathbf{D} \cdot \mathbf{D})$	$\frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} (\mathbf{j}^\perp \cdot \mathbf{C} - \mathbf{g} \cdot \mathbf{D} - 2\beta \mathbf{j}^\perp \cdot \mathbf{D})$
Linear momentum	$\mathbf{D} \times \mathbf{B} + \frac{1}{2} \beta ((\nabla \times \mathbf{D}) \times \mathbf{B} + \mathbf{D} \times (\nabla \times \mathbf{B}))$	$-\frac{1}{2} [\rho (\mathbf{E} + \frac{1}{\epsilon} \mathbf{D}) + 2\mathbf{j}^\perp \times \mathbf{B} + \beta ((\nabla \times \mathbf{j}^\perp) \times \mathbf{B} + \mathbf{j}^\perp \times (\nabla \times \mathbf{B}))]$
Spin	$\frac{1}{2} (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp) + \beta \mathbf{D} \times \mathbf{B}$	$\frac{1}{2} (((\nabla \times \mathbf{A}) \times \mathbf{g} + \mathbf{A}^\perp \times ((\nabla \times \mathbf{g}) - 2\beta \mathbf{j}^\perp \times \mathbf{B}))$

4. THE OPTICAL HELICITY AND ENERGY

4.1 The “orbital counterpart” of the helicity density

The spin angular momentum can be derived from the total angular momentum, itself the cross product of the position vector with the linear momentum, and so we propose that the helicity, too, can be constructed from some product of the energy density with the position vector. It turns out there are at least two different methods for doing this, but we shall present the one which most closely resembles the derivation of the spin and orbital angular momentum from the total linear momentum. For simplicity in this section, we will consider fields in vacuum, rather than in chiral media, unless otherwise stated.

From the form of the energy density (1), we construct the quantity

$$\frac{1}{2} \int \left(\frac{1}{\epsilon_0} \mathbf{D} \cdot (\mathbf{r} \times \mathbf{D}) + \frac{1}{\mu_0} \mathbf{B} \cdot (\mathbf{r} \times \mathbf{B}) \right) d^3 \mathbf{r} \equiv 0, \quad (42)$$

where the integrand itself is identically zero, as $\mathbf{B} \cdot (\mathbf{r} \times \mathbf{B}) = \mathbf{r} \cdot (\mathbf{B} \times \mathbf{B}) = 0$, and similarly for the electric term. Focusing on the magnetic half only, we write $\mathbf{B} = \nabla \times \mathbf{A}^\perp$ to produce

$$\int \mathbf{B} \cdot (\mathbf{r} \times \mathbf{B}) d^3 \mathbf{r} = \int (B_i r_j \nabla_i A_j^\perp - B_i r_j \nabla_j A_i^\perp) d^3 \mathbf{r}. \quad (43)$$

The first term on the right-hand side of (43) is integrated by parts, so that the total integral becomes

$$0 = - \int (\mathbf{A}^\perp \cdot \mathbf{B} + B_i (\mathbf{r} \cdot \nabla) A_i^\perp) d^3 \mathbf{r} + \int \nabla_i (B_i (\mathbf{r} \cdot \mathbf{A}^\perp)) d^3 \mathbf{r}. \quad (44)$$

Applying the same treatment to the electric term $\mathbf{D} \cdot (\mathbf{r} \times \mathbf{D})$, we rewrite (42)

$$0 = \frac{1}{2} \int \left(\frac{1}{\mu_0} \mathbf{A}^\perp \cdot \mathbf{B} - \frac{1}{\epsilon_0} \mathbf{C}^\perp \cdot \mathbf{D} \right) d^3\mathbf{r} + \frac{1}{2} \int \left(\frac{1}{\mu_0} B_i(\mathbf{r} \cdot \nabla) A_i^\perp - \frac{1}{\epsilon_0} D_i(\mathbf{r} \cdot \nabla) C_i^\perp \right) d^3\mathbf{r} - \frac{1}{2} \int \left(\frac{1}{\mu_0} \nabla_i (B_i(\mathbf{r} \cdot \mathbf{A}^\perp)) - \frac{1}{\epsilon_0} \nabla_i (D_i(\mathbf{r} \cdot \mathbf{C}^\perp)) \right) d^3\mathbf{r}. \quad (45)$$

The first integrand in (45) is straightforward to identify: multiplying it by the speed of light, we obtain exactly the helicity density in vacuum (5). Accordingly, we multiply the second integrand in (45) by $1/\sqrt{\epsilon_0\mu_0}$ to define

$$h_o = \frac{1}{2} \left(\sqrt{\frac{\epsilon_0}{\mu_0}} B_i(\mathbf{r} \cdot \nabla) A_i^\perp - \sqrt{\frac{\mu_0}{\epsilon_0}} D_i(\mathbf{r} \cdot \nabla) C_i^\perp \right), \quad (46)$$

a quantity with units of a helicity density which seems to play the role of the ‘‘orbital counterpart’’ of this quantity. The final integral of (45) contains the surface terms left over from the integration by parts.

4.2 Conservation of h_o

As is well known, the spin density also plays the role of the helicity flux density:¹¹ the exact relationship is

$$\mathbf{s} = \frac{\mathbf{v}}{c}, \quad (47)$$

where \mathbf{v} is the helicity flux density, and c is the speed of light in vacuum. It would be reasonable to expect, therefore, that the *orbital* angular momentum density also plays the role of the flux of h_o . After some calculation, indeed we find

$$\partial_t h_o + c \nabla \cdot \mathbf{l} = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} [(\nabla \times \mathbf{g})_i(\mathbf{r} \cdot \nabla) C_i^\perp + (\nabla \times \mathbf{C})_i(\mathbf{r} \cdot \nabla) g_i], \quad (48)$$

where the right-hand side reduces to zero in the absence of sources, and \mathbf{l} is the orbital angular momentum density in vacuum, given by setting $\beta = 0$ in (35). The factor of c follows from the relationship between the helicity flux density and the spin angular momentum density (47). The source terms on the right-hand side of (48) clearly have their counterpart in the source terms of helicity (for $\beta = 0$), shown in (16).

The orbital angular momentum density is therefore equivalent to the flux density of h_o , with h_o itself being a conserved quantity in the absence of currents. In contrast to both the energy density and the helicity density which require both the electric and magnetic halves for conservation,²¹ the electric and magnetic halves of h_o are *separately* conserved. Furthermore, it is found that the form of h_o in chiral media remains unchanged from the vacuum definition, apart from the values of the permittivity and permeability, unlike the energy and helicity densities.

4.3 Symmetries and conservation laws of h_o

As h_o is a locally conserved quantity, it must generate a symmetry transformation of the Lagrangian density (6). Let us rewrite the magnetic half of h_o , which follows from (44):

$$B_i(\mathbf{r} \cdot \nabla) A_i^\perp = \mathbf{B} \cdot \nabla(\mathbf{r} \cdot \mathbf{A}^\perp) - \mathbf{B} \cdot \mathbf{A}^\perp = -\mathbf{B} \cdot (\mathbf{A}^\perp - \nabla(\mathbf{r} \cdot \mathbf{A}^\perp)). \quad (49)$$

The quantity in brackets on the right-hand side of (49) has the form of a gauge transformation of the magnetic vector potential. Similarly, for the electric half of h_o , we identify

$$D_i(\mathbf{r} \cdot \nabla) C_i^\perp = -\mathbf{D} \cdot (\mathbf{C}^\perp - \nabla(\mathbf{r} \cdot \mathbf{C}^\perp)). \quad (50)$$

The quantity h_o , whose flux density is given by the orbital angular momentum density, is therefore equivalent to a gauge transformation of the (negative) helicity density. From (43), we identify $B_i r_j \nabla_j A_i = B_i r_j \nabla_i A_j$, and

similarly $D_i r_j \nabla_j C_i = D_i r_j \nabla_i C_j$. The symmetry transformations generated by the conservation of $B_i r_j \nabla_i A_j$ and $D_i r_j \nabla_i C_j$ (and therefore the magnetic and electric components of h_o) are found to be

$$A_i \rightarrow A_i + \eta \sqrt{\frac{\mu}{\epsilon}} (C_i - \nabla_i(\mathbf{r} \cdot \mathbf{C})), \quad C_i \rightarrow C_i - \eta \sqrt{\frac{\epsilon}{\mu}} (A_i - \nabla_i(\mathbf{r} \cdot \mathbf{A})). \quad (51)$$

We recognise these as the symmetry transformation generated by the conservation of helicity, where the potentials \mathbf{A} and \mathbf{C} on the right-hand side are replaced by the gauge transformed quantities.

4.4 Calculating h_o

The surface terms resulting from the integration by parts in forming h and h_o from the energy density are given in the second line of (45), and can be rewritten

$$\int \left(\frac{1}{\mu_0} \mathbf{B} \cdot \nabla(\mathbf{r} \cdot \mathbf{A}^\perp) - \frac{1}{\epsilon_0} \mathbf{D} \cdot \nabla(\mathbf{r} \cdot \mathbf{C}^\perp) \right) d^3 \mathbf{r}. \quad (52)$$

As the gradient functions are purely longitudinal, and \mathbf{B} and \mathbf{D} purely transverse, the integral over all space of each of the magnetic and electric contributions will be zero.⁴⁰ From (45), it therefore follows that

$$\int h_o d^3 \mathbf{r} = - \int h d^3 \mathbf{r} : \quad (53)$$

the total helicity, and its orbital counterpart, are equal and opposite. Thus the integrated value our new quantity, h_o , seems to reveal no new information about the field. We note, however, that it is the *flux* of a quantity which characterises its flow across a surface per unit time,⁴¹ and corresponds to the measured observable. In this case, the flux of h_o , as shown in 4.2, has the form of the orbital angular momentum density of the field, whereas the flux of helicity is the spin density.¹¹ The spin and orbital angular momentum components of light correspond to distinct quantities,³⁸ characterising different features of the field: the spin is determined by the polarisation, whereas the orbital angular momentum measures the phase gradient of the waves. It is therefore worthwhile to examine whether these quantities are similarly related by the transformation of the vector potentials

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} - \nabla(\mathbf{r} \cdot \mathbf{A}), \\ \mathbf{C} &\rightarrow \mathbf{C} - \nabla(\mathbf{r} \cdot \mathbf{C}). \end{aligned} \quad (54)$$

For completeness, the corresponding transformation of the scalar potentials A_0 and C_0 is

$$\begin{aligned} A_0 &\rightarrow A_0 + \partial_t(\mathbf{r} \cdot \mathbf{A}), \\ C_0 &\rightarrow C_0 + \partial_t(\mathbf{r} \cdot \mathbf{C}), \end{aligned} \quad (55)$$

although these do not feature in the present transformation of the spin density, as only transverse components of the \mathbf{A} and \mathbf{C} fields are involved. Let us begin with the spin density in vacuum:

$$\mathbf{s} = \frac{1}{2} (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp). \quad (56)$$

Applying (54), we find:

$$\begin{aligned} \mathbf{s} &\rightarrow (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp - (\mathbf{D} \times \nabla(\mathbf{r} \cdot \mathbf{A}^\perp) + \mathbf{B} \times \nabla(\mathbf{r} \cdot \mathbf{C}^\perp))) \\ &= (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{B} \times \mathbf{C}^\perp - (\mathbf{D} \times \mathbf{A}^\perp + \mathbf{r}(\mathbf{D} \cdot \mathbf{B}) + D_j(\mathbf{r} \times \nabla)A_j^\perp + \mathbf{B} \times \mathbf{C}^\perp - \mathbf{r}(\mathbf{B} \cdot \mathbf{D}) + B_j(\mathbf{r} \times \nabla)C_j^\perp)) \\ &= -\frac{1}{2} (D_j(\mathbf{r} \times \nabla)A_j^\perp + B_j(\mathbf{r} \times \nabla)C_j^\perp) \\ &= -\mathbf{l}, \end{aligned} \quad (57)$$

where the last lines indeed reveal the negative orbital angular momentum density of the field. It therefore follows that the *measurable* components of the helicity density and its orbital counterpart, the spin and orbital angular

momentum densities, are related by the gauge transformation (54). The fact that these fluxes are related by a gauge transformation is in accordance with the fact that h_o integrated over all space measures exactly the helicity of the field: we would not expect a gauge transformed quantity to reveal anything other than the original quantity itself.

It is worthwhile here to again consider chiral media. Although the *full* orbital and spin angular momenta can be derived from the linear momentum density in chiral media (see 3.4), this is not the case for h and h_o found from the energy density, as seen in section 4.1. This is connected to the fact that only the *achiral* part of the linear momentum density appears in the chiral spin density, whereas the *full* energy density appears in the chiral form of the helicity density, as seen in Table 1. This has consequences, too, for the relationship between h and h_o in chiral media: they are no longer related by the gauge transformation (54). In turn, the spin and orbital angular momentum densities are also not related by (54) in a chiral medium.

The quantity h_o arguably has a natural positioning amongst our various angular momentum measures. Indeed, in the same way that the helicity density, spin density, and spin flux density form a “helicity array”,¹¹ it would be natural to form such an array with h_o , the orbital angular momentum density, and its associated flux density. The meaning of h_o , however, remains ambiguous. As is clear from the above analysis, the quantity itself may be of no new significance in measuring the angular momentum content of the field. And yet, what it reveals about the connection between spin and orbital angular momentum components of light surely warrants further study. In addition, the relationship between these components in chiral media may shed further light upon the interaction of light with such matter.

4.5 The gauge transformation

Here we briefly examine the form of the gauge transformation (54). It is reminiscent of the transformation to the Power-Zienau-Wooley (PZW), or multipolar, gauge, familiar from electrodynamics.⁴²⁻⁴⁴ An arbitrary gauge transformation of the magnetic scalar and vector potentials is given by

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\rightarrow \mathbf{A}(\mathbf{r}') = \mathbf{A}(\mathbf{r}) - \nabla\chi(\mathbf{r}), \\ A_0(\mathbf{r}) &\rightarrow A_0(\mathbf{r}') = A_0(\mathbf{r}) + \dot{\chi}(\mathbf{r}),\end{aligned}\tag{58}$$

and the PZW gauge is obtained with the choice $\chi_M(\mathbf{r})$,⁴⁵

$$\chi_M(\mathbf{r}) = - \int d\mathbf{r}' \int_0^1 d\lambda A_j(\mathbf{r}') r_j \delta(\mathbf{r}' - \lambda\mathbf{r}).\tag{59}$$

For comparison, the gauge transformation in (54) is generated by the choice $\chi_h(\mathbf{r}) = A_j(\mathbf{r})r_j$. In the case of the PZW transformation, the new vector potential generated by (59) is⁴⁵

$$\mathbf{A}_M(\mathbf{r}) = \frac{1}{e} \int \boldsymbol{\theta}(\mathbf{r}', \mathbf{r}) \times \mathbf{B}(\mathbf{r}') d\mathbf{r}',\tag{60}$$

where

$$\boldsymbol{\theta}(\mathbf{r}', \mathbf{r}) = -e \int_0^1 d\lambda \lambda \mathbf{r} \delta(\mathbf{r}' - \lambda\mathbf{r}).\tag{61}$$

The vector potential defined by the gauge transformation (54) is

$$\mathbf{A}_h(\mathbf{r}) = -r_j \nabla_i A_j(\mathbf{r}).\tag{62}$$

It should be noted that the PZW gauge transformation is conventionally only applied to the *magnetic* potentials; its application and meaning in transforming the electric potentials \mathbf{C} and C_0 has not, to the authors' knowledge, yet been discussed in the literature. The PZW gauge is commonly used for a system of charges and currents coupling with the fields, as it conveniently expresses this interaction as a function of the fields, rather than of the vector potentials.⁴⁰ Furthermore, the polarisation and magnetisation densities, written as functions of microscopic observables, describe fully the system of charges, such that application to bulk media becomes

straightforward.⁴⁰ The similarity of the generator of the PZW gauge (59) to χ_h , the generator which converts the spin to the orbital angular momentum density, is worthy of further investigation, and will be discussed in later work. As the PZW transformation is only effective in the case of interaction of a system of charges with the field, it will be pertinent to study the relationship between \mathbf{s} and \mathbf{l} in this case. Extension to linear media and then chiral matter will then follow as a matter of course.

5. CONCLUSION

In this article, we have presented the forms of a number of conserved quantities in chiral media, in an effort to study how measures of the chiral content of light change upon interaction with such matter. It has been shown that the helicity, chirality, momentum, and (spin and orbital) angular momentum are conserved in lossless chiral media, where the values of ϵ , μ and β are constant. The previously noted connection between the helicity and energy in a chiral medium has been further examined, and shown to be analogous to the connection between the spin and linear momentum inside the medium. This led to the recognition of the quantity h_o , which plays the role of the orbital counterpart of the helicity density, and whose flux is given by the orbital angular momentum density. It is shown to be related to the helicity by a straightforward gauge transformation, from which it follows that the same connection exists between the spin and orbital components of the angular momentum densities. Far from just being an interesting exercise in vector calculus, these findings may yet reveal some new insight into measuring the angular momentum content of light.

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