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Bifurcation of finitely deformed thick-walled electroelastic spherical shells subject to a radial electric field

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Abstract

This paper is concerned with the bifurcation analysis of a pressurized electroelastic spherical shell with compliant electrodes on its inner and outer boundaries. The theory of small incremental electroelastic deformations superimposed on a radially finitely deformed electroelastic thick-walled spherical shell is used to determine those underlying configurations for which the superimposed deformations do not maintain the perfect spherical shape of the shell. Specifically, axisymmetric bifurcations are analyzed, and results are obtained for three different electroelastic energy functions, namely electroelastic counterparts of the neo-Hookean, Gent and Ogden elastic energy functions. For the neo-Hookean energy function it was reported previously that for the purely mechanical case axisymmetric bifurcations are possible under external pressure only, no bifurcation solutions being possible for internally pressurized spherical shells. In the case of an electroelastic neo-Hookean model bifurcation under internal pressure becomes possible when the potential difference between the electrodes exceeds a certain value, which depends on the ratio of inner to outer undeformed radii. Results obtained for the three classes of model are significantly different and are illustrated for a range of fixed values of the potential difference. Although of less practical significance, results are also shown for fixed charges, and these are both different between the models and different from the case of fixed potential difference.

1 Introduction

In the purely elastic context the problem of bifurcation of a thick-walled spherical elastic shell under internal pressure was examined in [1] and further results were obtained more recently in [2] for different elastic models, including details for the case of external pressure. In this paper we provide a bifurcation analysis for which the spherical shape is not maintained for an electroelastic thick-walled spherical shell with compliant electrodes on its spherical boundaries under internal and/or external pressure and with a potential difference applied across the electrodes. We start by considering a spherically symmetric underlying configuration of a spherical shell, a basic problem that was considered previously in [3]; see also [4]. We use some of the results and notation from these latter works and then we develop a bifurcation analysis along similar lines to that adopted for a thick-walled electroelastic cylinder in [5] and [6]. We note that for an electro-active thin-walled polymeric spherical shell the snap-through instability associated with an increase in the radius of the sphere, with the spherical shape maintained, was examined in [7].

In Sections 2 and 3 the required general equations of electroelasticity and their incremental counterparts are summarized. Next, the equations governing the basic spherically symmetric deformation of a thick-walled spherical shell subjected to an internal or external pressure and a radial electric field are provided in Section 4. This is followed by the lengthy Section 5 in which the equations governing axisymmetric incremental deformations superimposed on the spherically symmetric underlying deformation and the associated boundary conditions are derived without restriction on the form of the electroelastic constitutive model.

Then, in Section 6, the general equations are specialized for a particular class of constitutive models with a view to numerical computations and illustration of the resulting solutions of the governing equations and boundary conditions. Within the considered class three specific models are examined – electroelastic extensions of the purely elastic neo-Hookean, Gent [8] and Ogden 3-term [9] models. For each of these models numerical results are produced which detail the dependence of the inner radius of the shell (as measured by the azimuthal stretch on the inner spherical boundary) on the aspect ratio (i.e. the ratio of inner to outer undeformed radius) of the shell for which bifurcation becomes possible. Curves for different fixed values of the potential difference between flexible electrodes on the inner and outer boundaries are shown along with the zero pressure curves, which show which parts of the bifurcation curves require inner or outer pressure. Also shown, out of general interest, are corresponding curves for which fixed values of the charge are specified instead of the potential difference, although it is recognized that applying a fixed charge on the inner boundary of a spherical shell is not an easy practical proposition.

The paper finishes with a short concluding discussion in Section 7.

2 Basic formulation of nonlinear electroelasticity

We focus on a continuous material body capable of finite deformations and consisting of a dielectric elastic material. We suppose that it occupies a stress-free reference configuration $\mathcal{B}_\mathrm{r}$, with boundary $\partial\mathcal{B}_\mathrm{r}$ in the absence of mechanical loads and electric fields. When an electric field and mechanical loads are applied a deformation is induced and the body then occupies a new configuration, denoted $\mathcal{B}$, with boundary $\partial\mathcal{B}$. Material points in $\mathcal{B}_\mathrm{r}$ are identified by their position vectors $\mathbf{X}$, while their images in the deformed configuration $\mathcal{B}$ are denoted by their position vectors $\mathbf{x}$. The deformation from $\mathcal{B}_\mathrm{r}$ to $\mathcal{B}$ is described by the vector field $\boldsymbol{\chi}$, which relates the position of a particle in the reference configuration to the position of the same particle in the current configuration: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$.

The deformation gradient tensor is given by $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\chi}$, where Grad is the gradient operator with respect to $\mathbf{X}$. Along with the deformation gradient we shall use the right and left Cauchy–Green deformation tensors, respectively defined by

$$\mathbf{C} = \mathbf{F}^\mathrm{T}\mathbf{F}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T}. \tag{1}$$

Attention will be restricted to incompressible materials, so that, for each $\mathbf{X}$, the constraint

$$\det\mathbf{F} = 1 \tag{2}$$

must be satisfied.

2.1 Electrostatics, equilibrium and boundary conditions

When restricted to the purely static context, which is the case here, the relevant reduced forms of Maxwell's equations for a dielectric material are

$$\mathrm{curl}\,\mathbf{E} = \mathbf{0}, \quad \mathrm{div}\,\mathbf{D} = 0, \tag{3}$$

where $\mathbf{E}$ denotes the electric field vector, $\mathbf{D}$ is the electric displacement vector in the configuration $\mathcal{B}$, and the operators curl and div are defined with respect to $\mathbf{x}$. Outside the material body an electric field is in general generated, and the electric and displacement fields there are denoted $\mathbf{E}^\star$ and $\mathbf{D}^\star$, respectively. They are simply related by $\mathbf{D}^\star = \varepsilon_0\mathbf{E}^\star$, where ε_0 is the vacuum permittivity, and they also satisfy equations (3).

The related standard boundary conditions are

$$(\mathbf{E}^\star - \mathbf{E}) \times \mathbf{n} = \mathbf{0}, \quad (\mathbf{D}^\star - \mathbf{D}) \cdot \mathbf{n} = \sigma_\mathrm{f} \quad \text{on} \quad \partial\mathcal{B}, \tag{4}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$ and σ_f is the free surface charge per unit area of $\partial\mathcal{B}$.

Let $\boldsymbol{\tau}$ denote the total Cauchy stress tensor, which, by incorporating *electric* body forces, is symmetric when there are no intrinsic mechanical couples, as is the assumption here. Then, in the absence of *mechanical* body forces the mechanical equilibrium equation can be written simply as

$$\mathrm{div}\,\boldsymbol{\tau} = \mathbf{0}, \tag{5}$$

with $\boldsymbol{\tau}$ depending on the deformation and electric (or electric displacement) field through a constitutive law, which will be discussed in Section 2.3.

We write the traction boundary condition associated with (5) as

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{t}_{\mathrm{a}} + \mathbf{t}^{\star}_{\mathrm{m}} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{t}}, \tag{6}$$

where $\partial\mathcal{B}_{\mathrm{t}}$ is the part of the boundary where the mechanical traction, denoted $\mathbf{t}_{\mathrm{a}}$, is prescribed, while there is an additional mechanical traction, denoted $\mathbf{t}^{\star}_{\mathrm{m}} = \boldsymbol{\tau}^{\star}_{\mathrm{m}}\mathbf{n}$, generated by the external field, $\boldsymbol{\tau}^{\star}_{\mathrm{m}}$ being the so-called Maxwell stress tensor defined by

$$\boldsymbol{\tau}^{\star}_{\mathrm{m}} = \varepsilon_0\mathbf{E}^{\star} \otimes \mathbf{E}^{\star} - \frac{1}{2}\varepsilon_0(\mathbf{E}^{\star} \cdot \mathbf{E}^{\star})\mathbf{I}, \tag{7}$$

and $\mathbf{I}$ is the identity tensor.

2.2 Lagrangian formulation

The Lagrangian forms of the electric field vectors are defined by

$$\mathbf{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \quad \mathbf{D}_{\mathrm{L}} = \mathbf{F}^{-1}\mathbf{D}, \tag{8}$$

the latter being specific for an incompressible material. They satisfy the counterparts of equations (3) in the reference configuration, i.e.

$$\mathrm{Curl}\,\mathbf{E}_{\mathrm{L}} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{D}_{\mathrm{L}} = 0, \tag{9}$$

where the operators Curl and Div are defined with respect to $\mathbf{X}$.

The Lagrangian counterpart of the equilibrium equation (5) is expressed in terms of the total nominal stress tensor $\mathbf{T}$, which, for an incompressible material, is defined by

$$\mathbf{T} = \mathbf{F}^{-1}\boldsymbol{\tau}, \tag{10}$$

analogously to the definition of the nominal stress tensor of nonlinear pure elasticity (see, for example, [10]). This yields the Lagrangian form of (5), namely

$$\mathrm{Div}\,\mathbf{T} = \mathbf{0}. \tag{11}$$

To obtain the associated traction boundary condition analogous to that in (6) we use the connection

$$\boldsymbol{\tau}\mathbf{n}\mathrm{d}s = \mathbf{T}^{\mathrm{T}}\mathbf{N}\mathrm{d}S, \tag{12}$$

which is based on Nanson's formula $\mathbf{n}\mathrm{d}s = \mathbf{F}^{-\mathrm{T}}\mathbf{N}\mathrm{d}S$ (for an incompressible material), where $\mathrm{d}s$ and $\mathrm{d}S$ are infinitesimal area elements on $\partial\mathcal{B}$ and $\partial\mathcal{B}_{\mathrm{r}}$, respectively, and $\mathbf{n}$ and $\mathbf{N}$ are the corresponding outward unit normals to these areas. Equation (6) is then transformed to

$$\mathbf{T}^{\mathrm{T}}\mathbf{N} = \mathbf{t}_{\mathrm{A}} + \mathbf{t}^{\star}_{\mathrm{M}} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{rt}}, \tag{13}$$

where $\partial\mathcal{B}_{\text{rt}}$ is the pre-image of $\partial\mathcal{B}_{\text{t}}$, $\mathbf{t}_{\text{A}}$ and $\mathbf{t}^{\star}_{\text{M}} = \mathbf{T}^{\star\,\text{T}}_{\text{M}}\mathbf{N}$ are the mechanical traction and the Maxwell traction, respectively, per unit area of $\partial\mathcal{B}_{\text{r}}$, with the definition $\mathbf{T}^{\star}_{\text{M}} = \mathbf{F}^{-1}\boldsymbol{\tau}^{\star}_{\text{m}}$.

On use of Nanson's formula and the definitions in (8) the boundary conditions (4) translate to Lagrangian form as

$$(\mathbf{F}^{\text{T}}\mathbf{E}^{*} - \mathbf{E}_{\text{L}}) \times \mathbf{N} = \mathbf{0}, \quad (\mathbf{F}^{-1}\mathbf{D}^{*} - \mathbf{D}_{\text{L}}) \cdot \mathbf{N} = \sigma_{\text{F}} \quad \text{on} \quad \partial\mathcal{B}_{\text{r}}, \tag{14}$$

where σ_{F} is the free surface charge per unit area of $\partial\mathcal{B}_{\text{r}}$.

2.3 Material properties described by constitutive equations

To describe the constitutive properties of the considered material we introduce the total (electromechanical) energy density function, denoted Ω^{*} and defined in [11], depending on the independent variables $\mathbf{F}$ and $\mathbf{D}_{\text{L}}$. From this the total stress tensor $\mathbf{T}$ and the Lagrangian electric field $\mathbf{E}_{\text{L}}$ are obtained as

$$\mathbf{T} = \frac{\partial\Omega^{*}}{\partial\mathbf{F}} - p\mathbf{F}^{-1}, \quad \mathbf{E}_{\text{L}} = \frac{\partial\Omega^{*}}{\partial\mathbf{D}_{\text{L}}}, \tag{15}$$

where p is a Lagrange multiplier necessitated by the constraint (2).

We now restrict attention to isotropic electroelastic materials, so that Ω^{*} depends on $\mathbf{F}$ and $\mathbf{D}_{\text{L}}$ through invariants of the right Cauchy–Green deformation tensor $\mathbf{C}$, which, for an incompressible material, are typically taken to be

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \frac{1}{2}[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)], \tag{16}$$

$$I_4 = \mathbf{D}_{\text{L}} \cdot \mathbf{D}_{\text{L}}, \quad I_5 = \mathbf{D}_{\text{L}} \cdot (\mathbf{C}\mathbf{D}_{\text{L}}), \quad I_6 = \mathbf{D}_{\text{L}} \cdot (\mathbf{C}^2\mathbf{D}_{\text{L}}). \tag{17}$$

Now, by regarding Ω^{*} as a function of the invariants and expanding the formulas in (15) in terms of the derivatives of Ω^{*} with respect to the invariants, we obtain, on use of (10) and $(8)_1$, the total stress $\boldsymbol{\tau}$ and electric field $\mathbf{E}$ as [11]

$$\boldsymbol{\tau} = 2\Omega^{*}_{1}\mathbf{b} + 2\Omega^{*}_{2}(I_1\mathbf{b} - \mathbf{b}^2) - p\mathbf{I} + 2\Omega^{*}_{5}\mathbf{D} \otimes \mathbf{D} + 2\Omega^{*}_{6}(\mathbf{D} \otimes \mathbf{b}\mathbf{D} + \mathbf{b}\mathbf{D} \otimes \mathbf{D}), \tag{18}$$

$$\mathbf{E} = 2(\Omega^{*}_{4}\mathbf{b}^{-1} + \Omega^{*}_{5}\mathbf{I} + \Omega^{*}_{6}\mathbf{b})\mathbf{D}, \tag{19}$$

where Ω^{*}_{i} is shorthand for $\partial\Omega^{*}/\partial I_i$, $i = 1, 2, 4, 5, 6$, and we recall that $\mathbf{b}$ is the left Cauchy–Green tensor defined in $(1)_2$.

3 Incremental analysis

We now move on to derive the incremental equations that govern the incremental deformations and electric displacements that are superimposed on a deformed configuration in which there is a prevailing electric field. For a more detailed account of this general theory we refer to the formulation presented originally in [13] and summarized in [4].

3.1 Incremental Maxwell and equilibrium equations

An increment in a variable is a small change in that variable, so that resulting equations are linearized in the independent increments. Increments are here identified by a superimposed dot on the basic counterpart of the considered variable. For example, $\dot{\mathbf{x}}$ is the increment in the displacement, $\dot{\mathbf{F}} = \mathrm{Grad}\,\dot{\mathbf{x}}$ is the corresponding increment in the deformation gradient and $\dot{\mathbf{D}}_{\mathrm{L}}$ is the increment in the independent variable $\mathbf{D}_{\mathrm{L}}$. The increments $\dot{\mathbf{E}}_{\mathrm{L}}$, $\dot{\mathbf{D}}_{\mathrm{L}}$, $\dot{\mathbf{T}}$ satisfy the incremental governing equations

$$\mathrm{Curl}\,\dot{\mathbf{E}}_{\mathrm{L}} = \mathbf{0}, \quad \mathrm{Div}\,\dot{\mathbf{D}}_{\mathrm{L}} = 0, \quad \mathrm{Div}\,\dot{\mathbf{T}} = \mathbf{0} \tag{20}$$

without the need for linearization. The increments $\dot{\mathbf{D}}^{\star}$ and $\dot{\mathbf{E}}^{\star}$ in the exterior field vectors are connected by $\dot{\mathbf{D}}^{\star} = \varepsilon_0 \dot{\mathbf{E}}^{\star}$ and satisfy the equations

$$\mathrm{curl}\,\dot{\mathbf{E}}^{\star} = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}^{\star} = 0. \tag{21}$$

For ease of subsequent analysis we now introduce the push-forward from $\mathcal{B}_{\mathrm{r}}$ to $\mathcal{B}$ (or updated) versions of the increments in $\dot{\mathbf{E}}_{\mathrm{L}}$, $\dot{\mathbf{D}}_{\mathrm{L}}$ and $\dot{\mathbf{T}}$, which, for an incompressible material, are defined by

$$\dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{F}^{-\mathrm{T}}\dot{\mathbf{E}}_{\mathrm{L}}, \quad \dot{\mathbf{D}}_{\mathrm{L}0} = \mathbf{F}\dot{\mathbf{D}}_{\mathrm{L}}, \quad \dot{\mathbf{T}}_0 = \mathbf{F}\dot{\mathbf{T}}, \tag{22}$$

the updated quantities being distinguished by a zero subscript. These are the incremental counterparts of (8) and (10), which convert Lagrangian to Eulerian variables.

The governing equations (20) are then updated to

$$\mathrm{curl}\,\dot{\mathbf{E}}_{\mathrm{L}0} = \mathbf{0}, \quad \mathrm{div}\,\dot{\mathbf{D}}_{\mathrm{L}0} = 0, \quad \mathrm{div}\,\dot{\mathbf{T}}_0 = \mathbf{0}. \tag{23}$$

These are coupled with the incremental incompressibility constraint, which has the form

$$\mathrm{tr}\,\mathbf{L} \equiv \mathrm{div}\,\mathbf{u} = 0, \tag{24}$$

where $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{u}$, with $\mathbf{u}(\mathbf{x}) = \dot{\mathbf{x}}(\boldsymbol{\chi}^{-1}(\mathbf{x}))$ being the Eulerian version of the displacement $\dot{\mathbf{x}}$.

With the usual summation convention for repeated indices j and k from 1 to 3 we now write $(23)_3$ in the component form

$$\dot{T}_{0ji,j} + \dot{T}_{0ji}\mathbf{e}_k \cdot \mathbf{e}_{j,k} + \dot{T}_{0kj}\mathbf{e}_i \cdot \mathbf{e}_{j,k} = 0 \quad i = 1, 2, 3, \tag{25}$$

with respect to a curvilinear coordinate system with orthonormal basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, where a subscript j following a comma represents the derivative associated with the jth curvilinear coordinate. In Section 5 these component equations will be made explicit for the required specialization to spherical polar coordinates.

3.1.1 Incremental boundary conditions

The incremental form of the boundary condition (13) is

$$\dot{\mathbf{T}}^{\mathrm{T}}\mathbf{N} = \dot{\mathbf{t}}_{\mathrm{A}} + \dot{\boldsymbol{\tau}}_{\mathrm{m}}^{\star}\mathbf{F}^{-\mathrm{T}}\mathbf{N} - \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\mathbf{N} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{r}}, \tag{26}$$

where $\dot{\boldsymbol{\tau}}_{\mathrm{m}}^{\star}$ is the incremental Maxwell stress, which, from (7), is expressed as

$$\dot{\boldsymbol{\tau}}_{\mathrm{m}}^{\star} = \varepsilon_0[\dot{\mathbf{E}}^{\star} \otimes \mathbf{E}^{\star} + \mathbf{E}^{\star} \otimes \dot{\mathbf{E}}^{\star} - (\mathbf{E}^{\star} \cdot \dot{\mathbf{E}}^{\star})\mathbf{I}]. \tag{27}$$

The corresponding incremental forms of the two boundary conditions in (14) are

$$(\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{E}^{\star} + \mathbf{F}^{\mathrm{T}}\dot{\mathbf{E}}^{\star} - \dot{\mathbf{E}}_{\mathrm{L}}) \times \mathbf{N} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{r}}, \tag{28}$$

and

$$(\mathbf{F}^{-1}\dot{\mathbf{D}}^{\star} - \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{D}^{\star} - \dot{\mathbf{D}}_{\mathrm{L}}) \cdot \mathbf{N} = \dot{\sigma}_{\mathrm{F}} \quad \text{on} \quad \partial\mathcal{B}_{\mathrm{r}}. \tag{29}$$

On updating, these incremental boundary conditions become

$$\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} + \dot{\boldsymbol{\tau}}_{\mathrm{m}}^{\star}\mathbf{n} - \boldsymbol{\tau}_{\mathrm{m}}^{\star}\mathbf{L}^{\mathrm{T}}\mathbf{n} \quad \text{on} \quad \partial\mathcal{B}, \tag{30}$$

and

$$(\dot{\mathbf{E}}^{\star} + \mathbf{L}^{\mathrm{T}}\mathbf{E}^{\star} - \dot{\mathbf{E}}_{\mathrm{L}0}) \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\mathcal{B}, \tag{31}$$

$$(\dot{\mathbf{D}}^{\star} - \mathbf{L}\mathbf{D}^{\star} - \dot{\mathbf{D}}_{\mathrm{L}0}) \cdot \mathbf{n} = \dot{\sigma}_{\mathrm{F}0} \quad \text{on} \quad \partial\mathcal{B}. \tag{32}$$

3.2 Incremental constitutive equations

On taking the increments of the constitutive equations in (15) we obtain, on linearizing in the increments $\dot{\mathbf{F}}$ and $\dot{\mathbf{D}}_{\mathrm{L}}$ of the independent variables,

$$\dot{\mathbf{T}} = \boldsymbol{\mathcal{A}}^{*}\dot{\mathbf{F}} + \mathbb{A}^{*}\dot{\mathbf{D}}_{\mathrm{L}} + p\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} - \dot{p}\mathbf{F}^{-1}, \quad \dot{\mathbf{E}}_{\mathrm{L}} = \mathbb{A}^{*\mathrm{T}}\dot{\mathbf{F}} + \mathsf{A}^{*}\dot{\mathbf{D}}_{\mathrm{L}}, \tag{33}$$

where $\boldsymbol{\mathcal{A}}^{*}$, $\mathbb{A}^{*}$, A^{*}, respectively fourth-, third- and second-oder tensors, denote electroelastic moduli associated with the total energy Ω^{*} whose component forms are written

$$\mathcal{A}^{*}_{\alpha i\beta j} = \frac{\partial^2\Omega^{*}}{\partial F_{i\alpha}\partial F_{j\beta}}, \quad \mathbb{A}^{*}_{\alpha i|\beta} = \frac{\partial^2\Omega^{*}}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \quad \mathsf{A}^{*}_{\alpha\beta} = \frac{\partial^2\Omega^{*}}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}. \tag{34}$$

The vertical bar in the component form of $\mathbb{A}^{*}$ separates the first two and the third indices, which are associated, respectively, with a second-order tensor, and a vector.

The component forms of equations (33) are

$$\dot{T}_{\alpha i} = \mathcal{A}^{*}_{\alpha i\beta j}\dot{F}_{j\beta} + \mathbb{A}^{*}_{\alpha i|\beta}\dot{D}_{\mathrm{L}\beta} + pF^{-1}_{\alpha k}\dot{F}_{k\beta}F^{-1}_{\beta i} - \dot{p}F^{-1}_{\alpha i}, \quad \dot{E}_{\mathrm{L}\alpha} = \mathbb{A}^{*}_{\beta i|\alpha}\dot{F}_{i\beta} + \mathsf{A}^{*}_{\alpha\beta}\dot{D}_{\mathrm{L}\beta}, \tag{35}$$

where $F^{-1}_{\alpha i}$ is defined as $(\mathbf{F}^{-1})_{\alpha i}$. Note that the tensor $\mathbb{A}^{*}$ maps a vector into a second-order tensor, while its transpose maps a second-order tensor into a vector. In components we then have $\mathbb{A}^{*}_{\alpha i|\beta} = (\mathbb{A}^{*\mathrm{T}})_{\beta|\alpha i}$.

For an isotropic electroelastic material with Ω^* expressed in terms of the invariants I_m, $m \in \{1, 2, 4, 5, 6\}$, the components in (34) expand out as

$$\mathcal{A}^*_{\alpha i\beta j} = \sum_{m\in\mathcal{I}}\sum_{n\in\mathcal{I}} \Omega^*_{mn} \frac{\partial I_m}{\partial F_{i\alpha}} \frac{\partial I_n}{\partial F_{j\beta}} + \sum_{n\in\mathcal{I}} \Omega^*_n \frac{\partial^2 I_n}{\partial F_{i\alpha}\partial F_{j\beta}}, \tag{36}$$

$$\mathbb{A}^*_{\alpha i|\beta} = \sum_{m=4}^{6}\sum_{n\in\mathcal{I}} \Omega^*_{mn} \frac{\partial I_m}{\partial D_{\mathrm{L}\beta}} \frac{\partial I_n}{\partial F_{i\alpha}} + \sum_{n=5}^{6} \Omega^*_n \frac{\partial^2 I_n}{\partial F_{i\alpha}\partial D_{\mathrm{L}\beta}}, \tag{37}$$

$$\mathsf{A}^*_{\alpha\beta} = \sum_{m=4}^{6}\sum_{n=4}^{6} \Omega^*_{mn} \frac{\partial I_m}{\partial D_{\mathrm{L}\alpha}} \frac{\partial I_n}{\partial D_{\mathrm{L}\beta}} + \sum_{n=4}^{6} \Omega^*_n \frac{\partial^2 I_n}{\partial D_{\mathrm{L}\alpha}\partial D_{\mathrm{L}\beta}}, \tag{38}$$

where $\mathcal{I}$ is the index set $\{1, 2, 5, 6\}$, $\Omega^*_n = \partial\Omega^*/\partial I_n$ and $\Omega^*_{mn} = \partial^2\Omega^*/\partial I_m \partial I_n$, with $m, n \in \{1, 2, 4, 5, 6\}$. Expressions for the derivatives of the invariants with respect to $\mathbf{F}$ and $\mathbf{D}_\mathrm{L}$ required herein are given in [4], to which we refer for details.

On updating the incremental constitutive equations (33) become

$$\dot{\mathbf{T}}_0 = \boldsymbol{\mathcal{A}}^*_0\mathbf{L} + \mathbb{A}^*_0\dot{\mathbf{D}}_{\mathrm{L}0} + p\mathbf{L} - \dot{p}\mathbf{I}, \quad \dot{\mathbf{E}}_{\mathrm{L}0} = \mathbb{A}^{*\mathrm{T}}_0\mathbf{L} + \boldsymbol{\mathsf{A}}^*_0\dot{\mathbf{D}}_{\mathrm{L}0}, \tag{39}$$

where $\boldsymbol{\mathcal{A}}^*_0$, $\mathbb{A}^*_0$, $\boldsymbol{\mathsf{A}}^*_0$ are the updated versions of $\boldsymbol{\mathcal{A}}^*$, $\mathbb{A}^*$, $\boldsymbol{\mathsf{A}}^*$. In component form the updated electroelastic moduli tensors are related to the tensors (34) by

$$\mathcal{A}^*_{0jilk} = F_{j\alpha}F_{l\beta}\mathcal{A}^*_{\alpha i\beta k}, \quad \mathbb{A}^*_{0ji|k} = F_{j\alpha}F^{-1}_{\beta k}\mathbb{A}^*_{\alpha i|\beta}, \quad \mathsf{A}^*_{0ij} = F^{-1}_{\alpha i}F^{-1}_{\beta j}\mathsf{A}^*_{\alpha\beta}, \tag{40}$$

which have the symmetries

$$\mathcal{A}^*_{0jilk} = \mathcal{A}^*_{0lkji}, \quad \mathbb{A}^*_{0ij|k} = \mathbb{A}^*_{0ji|k}, \quad \mathsf{A}^*_{0ij} = \mathsf{A}^*_{0ji}. \tag{41}$$

We note, in addition, for later reference, the connection

$$\mathcal{A}^*_{0jisk} - \mathcal{A}^*_{0ijsk} = (\tau_{js} + p\delta_{js})\delta_{ik} - (\tau_{is} + p\delta_{is})\delta_{jk} \tag{42}$$

given in [13].

At this point the energy function Ω^* in the above formulas is completely general in the case of isotropy, but will be specialized later in Section 6.

4 The basic spherically symmetric configuration

4.1 Geometry and radial deformation

The reference geometry of a spherical shell can be conveniently described by spherical polar coordinates R, Θ, Φ, with

$$0 < A \leq R \leq B, \quad 0 \leq \Theta \leq \pi, \quad 0 \leq \Phi \leq 2\pi, \tag{43}$$

where A and B are the internal and external radii. Assuming that the spherical symmetry is maintained during the deformation, the deformed configuration is described in terms of spherical polar coordinates r, θ, ϕ as

$$a \le r \le b, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi, \tag{44}$$

where a and b are the internal and external radii in the deformed configuration.

Since the material is incompressible the deformation is defined by

$$r = (R^3 + a^3 - A^3)^{1/3}, \quad \theta = \Theta, \quad \phi = \Phi. \tag{45}$$

The resulting deformation gradient with respect to the spherical polar coordinate axes is diagonal, and the associated principal stretches λ_θ and λ_ϕ corresponding to the θ and ϕ directions are equal and henceforth denoted λ, which is given by

$$\lambda = r/R. \tag{46}$$

By incompressibility the principal stretch corresponding to the radial direction is therefore $\lambda_r = \lambda^{-2}$. Using $(45)_1$ we write

$$\lambda = \frac{r}{R} = \left(1 + \frac{a^3 - A^3}{R^3}\right)^{1/3}, \tag{47}$$

and by defining the circumferential stretches at the inner boundary as $\lambda_a = a/A$ we obtain the connection

$$\lambda^3 - 1 = \frac{A^3}{R^3}(\lambda_a^3 - 1). \tag{48}$$

Evaluating the previous relation at $R = B$ and defining $\lambda_b = b/B$ we obtain the connection between the stretches at the inner and outer boundaries:

$$(\lambda_a^3 - 1) = \left(\frac{B}{A}\right)^3 (\lambda_b^3 - 1). \tag{49}$$

Since $B/A > 1$ we conclude that under inflation $\lambda_a > \lambda > \lambda_b > 1$ and under compression $1 > \lambda_b > \lambda > \lambda_a$.

In terms of λ the invariants I_1 and I_2 are simply

$$I_1 = 2\lambda^2 + \lambda^{-4}, \quad I_2 = \lambda^4 + 2\lambda^{-2}. \tag{50}$$

4.2 Electric field and boundary conditions

For the considered spherical geometry the radial electric displacement D_r is the only non-zero component and depends only on r so that equation $(3)_2$ reduces to

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 D_r) = 0. \tag{51}$$

Henceforth we write $D = D(r)$ instead of D_r. Thus,

$$r^2 D = a^2 D(a) = b^2 D(b) = \text{constant}. \tag{52}$$

We now consider that flexible electrodes are affixed to the inner, $R = A$, and outer, $R = B$, spherical boundaries of the shell. When a potential difference is applied across the electrodes equal and opposite charges appear on them and by Gauss's theorem there is then no field in the (free) space outside the shell, so that $\mathbf{D}^\star = \mathbf{0}$ and $\mathbf{E}^\star = \mathbf{0}$. Then, from the boundary condition $(4)_2$,

$$D(a) = \sigma_{\mathrm{f}a}, \quad D(b) = -\sigma_{\mathrm{f}b}, \tag{53}$$

where $\sigma_{\mathrm{f}a}$ and $\sigma_{\mathrm{f}b}$ are the free surface charge densities on the deformed boundaries $r = a$ and $r = b$. If we denote by Q_a and Q_b the total charges on these boundaries then $Q_a + Q_b = 0$ and

$$\sigma_{\mathrm{f}a} = \frac{Q_a}{4\pi a^2}, \quad \sigma_{\mathrm{f}b} = \frac{Q_b}{4\pi b^2}, \quad r^2 D = \frac{Q_a}{4\pi} = -\frac{Q_b}{4\pi}. \tag{54}$$

Since the deformation gradient is diagonal and the only component of $\mathbf{D}$ is the radial component it follows from equation (19) that the only non-zero component of the electric field $\mathbf{E}$ is the radial component E_r, which is henceforth denoted E, and from (19) we have

$$E = 2(\Omega_4^* \lambda^4 + \Omega_5^* + \Omega_6^* \lambda^{-4}) D. \tag{55}$$

With the connection $\mathbf{D}_{\mathrm{L}} = \mathbf{F}^{-1}\mathbf{D}$ from $(8)_2$ reducing to $D_{\mathrm{L}} = \lambda^2 D$ the invariants defined in (17) specialize to

$$I_4 = \lambda^4 D^2 = D_{\mathrm{L}}^2, \quad I_5 = D^2 = \lambda^{-4} I_4, \quad I_6 = \lambda^{-4} D^2 = \lambda^{-8} I_4. \tag{56}$$

We also note from $(8)_1$ that $E = \lambda^2 E_{\mathrm{L}}$.

Because of the spherical symmetry (no dependence on either θ or ϕ) the equation $\mathrm{curl}\mathbf{E} = \mathbf{0}$ is satisfied automatically. This also has the solution $\mathbf{E} = -\mathrm{grad}\mathcal{V}$, where $\mathcal{V}$ is the electrostatic potential. In the present case this reduces to $E = -\mathrm{d}\mathcal{V}/\mathrm{d}r$ with $\mathcal{V}$ a function of r only. It follows, on reference to $(54)_3$, that the connection between the potential difference across $r = a$ and $r = b$, with magnitude V, and Q_a is embodied in the formula

$$V = 2\int_a^b (\Omega_4^* \lambda^4 + \Omega_5^* + \Omega_6^* \lambda^{-4}) D \, \mathrm{d}r. \tag{57}$$

4.3 Stress components and equilibrium

From (18) we obtain the non-zero components of the total Cauchy stress, namely

$$\tau_{rr} = 2\Omega_1^* \lambda^{-4} + 4\Omega_2^* \lambda^{-2} - p + 2\Omega_5^* D^2 + 4\Omega_6^* \lambda^{-4} D^2,$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = 2\Omega_1^* \lambda^2 + 2\Omega_2^* (\lambda^4 + \lambda^{-2}) - p. \tag{58}$$

Recognizing that the invariants (50) and (56) are functions of two independent variables λ and I_4, we introduce the reduced energy function $\omega^*(\lambda, I_4)$ defined by

$$\omega^*(\lambda, I_4) = \Omega^*(I_1, I_2, I_4, I_5, I_6), \tag{59}$$

where, on the right-hand side, the invariants are specialized according to (50) and (56).

This allows us to write

$$\tau_{\theta\theta} - \tau_{rr} = \frac{\lambda\omega^*_\lambda}{2}, \quad E = 2\lambda^4 \frac{\partial \omega^*}{\partial I_4} D, \tag{60}$$

where ω^*_λ denotes the derivative $\partial\omega^*/\partial\lambda$.

Because of the spherical symmetry the equilibrium equation $\operatorname{div} \boldsymbol{\tau} = \mathbf{0}$ reduces to

$$r\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} = 2(\tau_{\theta\theta} - \tau_{rr}) = \lambda\omega^*_\lambda, \tag{61}$$

in which we have used (60). As there is no field outside the shell then, according to (7), the Maxwell stress is zero, and we take the mechanical load to consist of an internal pressure P_{in} applied to the surface at $r = a$ and external pressure P_{out} applied to the surface at $r = b$. Thus, the traction boundary conditions have the form

$$\tau_{rr} = -P_{\text{in}} \quad \text{on} \quad r = a, \quad \tau_{rr} = -P_{\text{out}} \quad \text{on} \quad r = b. \tag{62}$$

Integration of (61) and application of the boundary conditions (62) yields

$$P = P_{\text{in}} - P_{\text{out}} = \int_a^b \lambda\omega^*_\lambda \frac{\mathrm{d}r}{r}, \tag{63}$$

wherein P is defined as the difference between the internal and external pressures. Noting that b depends on a since $b = (B^3 + a^3 - A^3)^{1/3}$, this gives P as a function of a and Q_a, since $I_4 = Q_a^2/16\pi^2 R^4$.

The formula (57) now simplifies to

$$V = 2\int_a^b \lambda^4 \frac{\partial\omega^*}{\partial I_4} D \, \mathrm{d}r, \tag{64}$$

which again provides a connection between V and Q_a.

5 Bifurcation analysis

In the present setting we use spherical polar coordinates θ, ϕ, r with the corresponding unit basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Note the order of spherical polar coordinates, used here for consistency with [1] in the purely elastic context. The derivatives in (25) denoted by subscripts with commas $(\cdot)_{,k}$ can now be specified as $(1/r)\partial(\cdot)/\partial\theta$, $(1/r\sin\theta)\partial(\cdot)/\partial\phi$, $\partial(\cdot)/\partial r$ for $k = 1, 2, 3$, respectively. For spherical polar coordinates the only non-zero scalar products $\mathbf{e}_i \cdot \mathbf{e}_{j,k}$ in (25) are given by

$$-\mathbf{e}_3 \cdot \mathbf{e}_{1,1} = -\mathbf{e}_3 \cdot \mathbf{e}_{2,2} = \mathbf{e}_1 \cdot \mathbf{e}_{3,1} = \mathbf{e}_2 \cdot \mathbf{e}_{3,2} = r^{-1}, \quad \mathbf{e}_1 \cdot \mathbf{e}_{2,2} = -\mathbf{e}_2 \cdot \mathbf{e}_{1,2} = -r^{-1}\cot\theta. \tag{65}$$

5.1 Axisymmetric bifurcations

The increment $\mathbf{u} = \dot{\mathbf{x}}$ in the position vector $\mathbf{x}$ at a point in the spherically symmetric configuration is now written

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3, \tag{66}$$

where, for axisymmetric bifurcations, on which we focus here, u_1 and u_3 are independent of ϕ and $u_2 = 0$. Therefore, the components of $\mathbf{L}$ on the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ can be calculated as

$$[L_{ij}] = \begin{bmatrix} (u_3 + u_{1,\theta})/r & 0 & u_{1,r} \\ 0 & (u_3 + u_1 \cot\theta)/r & 0 \\ (u_{3,\theta} - u_1)/r & 0 & u_{3,r} \end{bmatrix}, \tag{67}$$

where subscripts θ, r following a comma correspond to partial derivatives.

For an incompressible material, from (24), we then have

$$L_{11} + L_{22} + L_{33} \equiv 2u_3 + u_{1,\theta} + u_1 \cot\theta + r u_{3,r} = 0. \tag{68}$$

The incompressibility condition (68) is satisfied if we define u_1 and u_3 in terms of function $\varphi(\theta, r)$ such that

$$u_1 = -\frac{1}{r \sin\theta}\frac{\partial\varphi}{\partial r}, \quad u_3 = \frac{1}{r^2 \sin\theta}\frac{\partial\varphi}{\partial\theta}. \tag{69}$$

The governing equation $(23)_2$ has the same structure as $\operatorname{div}\mathbf{u} = 0$ and hence, similarly to (69), we introduce the function $\psi(\theta, r)$ such that

$$\dot{D}_{\mathrm{L}01} = -\frac{1}{r \sin\theta}\frac{\partial\psi}{\partial r}, \quad \dot{D}_{\mathrm{L}03} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}. \tag{70}$$

For $i = 1$ and $i = 3$ the incremental equilibrium equation (25) gives

$$\dot{T}_{011,1} + \dot{T}_{021,2} + \dot{T}_{031,3} + 2r^{-1}\dot{T}_{031} + r^{-1}\dot{T}_{013} + r^{-1}\cot\theta(\dot{T}_{011} - \dot{T}_{022}) = 0, \tag{71}$$

$$\dot{T}_{013,1} + \dot{T}_{023,2} + \dot{T}_{033,3} + 2r^{-1}\dot{T}_{033} + r^{-1}\cot\theta\,\dot{T}_{013} - r^{-1}(\dot{T}_{011} + \dot{T}_{022}) = 0, \tag{72}$$

while for $i = 2$ the equation is satisfied identically because of the axial symmetry.

For the present case the remaining governing equation $(23)_1$ reduces to

$$\dot{E}_{\mathrm{L}01} + r\frac{\partial \dot{E}_{\mathrm{L}01}}{\partial r} - \frac{\partial \dot{E}_{\mathrm{L}03}}{\partial\theta} = 0. \tag{73}$$

For the considered underlying deformation the deformation gradient has diagonal components with respect to the chosen axes, and with the electric displacement field purely radial the required values of the electroelastic moduli tensors $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$, $\mathbf{A}_0^*$ can be obtained from the general expressions given in [4] or [6]. The underlying spherical symmetry allows us to set $\mathcal{A}^*_{01111} = \mathcal{A}^*_{02222}$, $\mathcal{A}^*_{02233} = \mathcal{A}^*_{01133}$ and $\mathbb{A}^*_{011|3} = \mathbb{A}^*_{022|3}$ while the symmetry

$\mathbb{A}^*_{031|1} = \mathbb{A}^*_{013|1}$ follows from (41). Using these symmetries, together with (68), the components of the constitutive equation $(39)_1$ are given as

$$\dot{T}_{011} = (\mathcal{A}^*_{01111} - \mathcal{A}^*_{01122} + p)L_{11} + (\mathcal{A}^*_{01133} - \mathcal{A}^*_{01122})L_{33} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \quad (74)$$

$$\dot{T}_{022} = (\mathcal{A}^*_{01122} - \mathcal{A}^*_{01111} - p)L_{11} + (\mathcal{A}^*_{01133} - \mathcal{A}^*_{01111} - p)L_{33} - \dot{p} + \mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \quad (75)$$

$$\dot{T}_{033} = (\mathcal{A}^*_{03333} - \mathcal{A}^*_{01133} + p)L_{33} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03}, \quad (76)$$

$$\dot{T}_{013} = \mathcal{A}^*_{01313}L_{31} + (\mathcal{A}^*_{01331} + p)L_{13} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \quad (77)$$

$$\dot{T}_{031} = \mathcal{A}^*_{03131}L_{13} + (\mathcal{A}^*_{01331} + p)L_{31} + \mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \quad (78)$$

where the pairwise symmetry $\mathcal{A}^*_{03113} = \mathcal{A}^*_{01331}$ has been used, and from $(39)_2$ we have

$$\dot{E}_{\mathrm{L}01} = \mathbb{A}^*_{013|1}(L_{31} + L_{13}) + \mathsf{A}^*_{011}\dot{D}_{\mathrm{L}01}, \quad (79)$$

$$\dot{E}_{\mathrm{L}03} = (\mathbb{A}^*_{033|3} - \mathbb{A}^*_{011|3})L_{33} + \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03}. \quad (80)$$

On substituting the expressions (74)–(78) into (71) and (72) and using (68) we obtain

$$\begin{aligned} r\dot{p}_{,\theta} = {} & [r(\mathcal{A}^{*\prime}_{01331} + p') + \mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01111}](u_{3,\theta} - u_1) \\ & + (\mathcal{A}^*_{01331} + \mathcal{A}^*_{01133} - \mathcal{A}^*_{01111})ru_{3,r\theta} + (r\mathcal{A}^{*\prime}_{03131} + 2\mathcal{A}^*_{03131})ru_{1,r} + \mathcal{A}^*_{03131}r^2u_{1,rr} \\ & + r\mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03,\theta} + r\mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01} + r^2\mathbb{A}^{*\prime}_{013|1}\dot{D}_{\mathrm{L}01} + r^2\mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01,r} + 2r\mathbb{A}^*_{013|1}\dot{D}_{\mathrm{L}01}, \end{aligned} \quad (81)$$

and

$$\begin{aligned} r^2\dot{p}_r = {} & [r(\mathcal{A}^{*\prime}_{03333} - \mathcal{A}^{*\prime}_{01133} + p') - 3\mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} \\ & - 4\mathcal{A}^*_{01133} + 2\mathcal{A}^*_{03333} + \mathcal{A}^*_{01111}]ru_{3,r} + (\mathcal{A}^*_{03333} - \mathcal{A}^*_{01331} - \mathcal{A}^*_{01133})r^2u_{3,rr} \\ & + \mathcal{A}^*_{01313}(u_{3,\theta\theta} + u_{3,\theta}\cot\theta + 2u_3) + \mathbb{A}^*_{013|1}(r\dot{D}_{\mathrm{L}01,\theta} + r\cot\theta\dot{D}_{\mathrm{L}01}) \\ & + r^2\mathbb{A}^{*\prime}_{033|3}\dot{D}_{\mathrm{L}03} + \mathbb{A}^*_{033|3}(r^2\dot{D}_{\mathrm{L}03,r} + 2r\dot{D}_{\mathrm{L}03}) - 2r\mathbb{A}^*_{011|3}\dot{D}_{\mathrm{L}03}, \end{aligned} \quad (82)$$

where a prime denotes differentiation with respect to r.

Also, equation (73) gives

$$\begin{aligned} & (\mathbb{A}^*_{013|1} + \mathbb{A}^*_{011|3} - \mathbb{A}^*_{033|3})u_{3,r\theta} + \mathbb{A}^*_{013|1}ru_{1,rr} + \mathbb{A}^{*\prime}_{013|1}(u_{3,\theta} - u_1 + ru_{1,r}) \\ & + \mathsf{A}^*_{011}(\dot{D}_{\mathrm{L}01} + r\dot{D}_{\mathrm{L}01,r}) + r\mathsf{A}^{*\prime}_{011}\dot{D}_{\mathrm{L}01} - \mathsf{A}^*_{033}\dot{D}_{\mathrm{L}03,\theta} = 0. \end{aligned} \quad (83)$$

It is now convenient to introduce the simplified notations

$$\begin{aligned} & a = \mathcal{A}^*_{01313}, \quad 2b = \mathcal{A}^*_{01111} + \mathcal{A}^*_{03333} - 2\mathcal{A}^*_{01133} - 2\mathcal{A}^*_{01331}, \quad c = \mathcal{A}^*_{03131}, \\ & d = \mathbb{A}^*_{013|1} = \mathbb{A}^*_{031|1}, \quad e = \mathbb{A}^*_{033|3} - \mathbb{A}^*_{013|1} - \mathbb{A}^*_{011|3}, \\ & f = \mathsf{A}^*_{011}, \quad g = \mathsf{A}^*_{033}, \quad h = \mathcal{A}^*_{01331} + \mathcal{A}^*_{01313} + \mathcal{A}^*_{01122} - \mathcal{A}^*_{01111}, \end{aligned} \quad (84)$$

and to note from (42) that $\mathcal{A}^*_{01331} + p = \mathcal{A}^*_{03131} - \tau_{33}$.

Then, by differentiating $\dot{p}_\theta$ from (81) with respect to r and $\dot{p}_r$ from (82) with respect to θ, then eliminating the terms in $\dot{p}$ and using (69) and (70), we obtain the governing equation

$$
\begin{aligned}
&cr^4\varphi_{,rrrr} + 2br^2\varphi_{,rr\theta\theta} + a\varphi_{,\theta\theta\theta\theta} + 2rc'r^3\varphi_{,rrr} - 2a\cot\theta\varphi_{,\theta\theta\theta} - 2b\cot\theta r^2\varphi_{,rr\theta}\\
&+2(rb'-2b)r\varphi_{,r\theta\theta} + [h + 4b - rh' - 4rb' + r^2\tau_{33}'' - r^2c'' + (4+3\cot^2\theta)a]\varphi_{,\theta\theta}\\
&+(4b-2rb')\cot\theta r\varphi_{,r\theta} + (r^2c'' - 2rc' + r\tau_{33}' - h)r^2\varphi_{,rr}\\
&+[4rb' + rh' - 4b - h + r^2c'' - r^2\tau_{33}'' - a(3\cot^2\theta + 5)]\cot\theta\varphi_{,\theta}\\
&-(2r^2c'' - 2rc' - r^2\tau_{33}'' + r\tau_{33}' + rh' - 2h)r\varphi_{,r}\\
&+er^2\psi_{,r\theta\theta} - e\cot\theta r^2\psi_{,r\theta} + dr^4\psi_{,rrr} + 2(rd'+d)r^3\psi_{,rr} + r(e'+d')r\psi_{,\theta\theta}\\
&+(r^2d'' + 2rd' - 2d)r^2\psi_{,r} - r^2(e'+d')\cot\theta\psi_{,\theta} = 0 \qquad (85)
\end{aligned}
$$

in terms of functions φ and ψ. Note the connections

$$
r\tau_{33}' = 2(a-c), \quad r^2\tau_{33}'' = 2(ra' - rc' - a + c), \qquad (86)
$$

which can be obtained from (42) and (61).

Similarly, from (83) we obtain the second governing equation

$$
\begin{aligned}
&dr^3\varphi_{,rrr} + er\varphi_{,r\theta\theta} + (rd'-2d)r^2\varphi_{,rr} - e\cot\theta r\varphi_{,r\theta} - (rd'+2e)\varphi_{,\theta\theta} - 2(rd'-d)r\varphi_{,r}\\
&+(rd'+2e)\cot\theta\varphi_{,\theta} + fr^3\psi_{,rr} + gr\psi_{,\theta\theta} + rf'r^2\psi_{,r} - g\cot\theta r\psi_{,\theta} = 0 \qquad (87)
\end{aligned}
$$

relating φ and ψ.

5.1.1 Boundary conditions

We now specialize the boundary condition (30) for the present case in which the electric field is generated by a potential difference across the electrodes and there is no field outside the material. We have

$$
\dot{\mathbf{T}}_0^{\mathrm{T}}\mathbf{n} = \dot{\mathbf{t}}_{\mathrm{A}0} = \begin{cases} P_{\mathrm{in}}\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}_{\mathrm{in}}\mathbf{n} & \text{on } r=a \\ P_{\mathrm{out}}\mathbf{L}^{\mathrm{T}}\mathbf{n} - \dot{P}_{\mathrm{out}}\mathbf{n} & \text{on } r=b, \end{cases} \qquad (88)
$$

where $\dot{P}_{\mathrm{in}}$ and $\dot{P}_{\mathrm{out}}$ are prescribed constants.

Using (78), the connection $\mathcal{A}_{03131} - \mathcal{A}_{01331} = \tau_{33} + p$ obtained from (42), and the values of the stress at the boundaries given in (62), we obtain

$$
c(ru_{1,r} + u_{3,\theta} - u_1) + dr\dot{D}_{\mathrm{L}01} = 0 \quad \text{on} \quad r = a, b. \qquad (89)
$$

Using (76), (42), (68) and (62) we obtain

$$
(\mathcal{A}^*_{03333} - \mathcal{A}^*_{01133} + \mathcal{A}_{03131} - \mathcal{A}_{01331})u_{3,r} - \dot{p} + \mathbb{A}^*_{033|3}\dot{D}_{\mathrm{L}03} = \begin{cases} -\dot{P}_{\mathrm{in}} & \text{on } r=a \\ -\dot{P}_{\mathrm{out}} & \text{on } r=b. \end{cases} \qquad (90)
$$

The remaining component of (88) is satisfied automatically.

In terms of φ and ψ the boundary condition (89) becomes

$$
c(r^2\varphi_{,rr} - \varphi_{,\theta\theta} + \cot\theta\varphi_{,\theta} - 2r\varphi_{,r}) + dr^2\psi_{,r} = 0 \quad \text{on} \quad r = a, b. \qquad (91)
$$

In (90) we differentiate with respect to θ and use (81) to eliminate $\dot{p}_{,\theta}$, and then in terms of φ and ψ we obtain the boundary condition

$$\begin{aligned} &cr^3\varphi_{,rrr} + (2b+c)r\varphi_{,r\theta\theta} - (rc' - r\tau_{33}' + 4b + h + 2c)\varphi_{,\theta\theta} - (2b+c)\cot\theta\, r\varphi_{,r\theta} + rc'r^2\varphi_{,rr} \\ &+ (rc' - r\tau_{33}' + 4b + h + 2c)\cot\theta\varphi_{,\theta} - (2rc' - r\tau_{33}' + h)r\varphi_{,r} \\ &+ dr^3\psi_{,rr} + (e+d)r\psi_{,\theta\theta} + (rd' + 2d)r^2\psi_{,r} - (e+d)\cot\theta r\psi_{,\theta} = 0 \quad \text{on} \quad r = a, b. \end{aligned} \tag{92}$$

The electric boundary condition (31) reduces to

$$\dot{E}_{\mathrm{L}01} = 0 \quad \text{on} \quad r = a, b, \tag{93}$$

which can be rewritten as

$$d(r^2\varphi_{,rr} - \phi_{,\theta\theta} + \cot\theta\varphi_{,\theta} - 2r\varphi_{,r}) + fr^2\psi_{,r} = 0 \quad \text{on} \quad r = a, b. \tag{94}$$

The combination of (91) and (94) allows these two boundary conditions to be simplified to

$$r^2\varphi_{,rr} - \varphi_{,\theta\theta} + \cot\theta\varphi_{,\theta} - 2r\varphi_{,r} = 0, \quad \psi_{,r} = 0 \quad \text{on} \quad r = a, b \tag{95}$$

provided $cf - d^2 \neq 0$, which is certainly the case for the particular models used later and it is therefore reasonable to impose this condition.

The boundary condition (32), which reduces to

$$\dot{D}_{\mathrm{L}03} = \begin{cases} -\dot{\sigma}_{\mathrm{F}0b} & \text{on } r = b, \\ \dot{\sigma}_{\mathrm{F}0a} & \text{on } r = a, \end{cases} \tag{96}$$

where $\dot{\sigma}_{\mathrm{F}0a}$ and $\dot{\sigma}_{\mathrm{F}0b}$ are the increments of the free surface charges $\sigma_{\mathrm{F}A}$ and $\sigma_{\mathrm{F}B}$, respectively, measured per unit deformed area. This condition merely determines the values of $\dot{D}_{\mathrm{L}03}$ on the two boundaries when the incremental potential difference is specified, so the only incremental electric boundary condition needed in this case is (93).

5.1.2 Form of solution

To arrange for the equations to be consistent with those in [1] in the purely elastic situation we write

$$\varphi = -\frac{1}{m}r^2 f_n(r)\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta), \quad \psi = -\frac{1}{m}g_n(r)\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta), \tag{97}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of degree n and $m = n(n+1)$.

Using the standard identity

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}P_n(\cos\theta) + \cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_n(\cos\theta) + n(n+1)P_n(\cos\theta) = 0 \tag{98}$$

and (97) the governing equations (85) and (87) reduce to

$$\begin{aligned}
&cr^4 f_n'''' + 2(4c + rc')r^3 f_n''' + [10rc' + r^2c'' + 12c + r\tau_{33}' - (2mb + h)]r^2 f_n'' \\
&+ [6rc' + 3r\tau_{33}' + 2r^2c'' + r^2\tau_{33}'' - 2(2mb + h) - r(2mb' + h')]rf_n' \\
&+ (m - 2)(r^2c'' - r^2\tau_{33}'' + rh' - h + ma)f_n + dr^2 g_n''' + 2(d + rd')rg_n'' \\
&+ (-me + r^2d'' + 2rd' - 2d)g_n' - m(e' + d')g_n = 0,
\end{aligned} \tag{99}$$

$$dr^2 f_n''' + (rd' + 4d)rf_n'' + (2rd' - me)f_n' + (m - 2)d'f_n + fg_n'' + f'g_n' - mgg_n/r^2 = 0, \tag{100}$$

respectively.

The boundary conditions (95) and (92) become

$$r^2 f_n'' + 2rf_n' + (m - 2)f_n = 0, \quad g_n' = 0 \quad \text{on} \quad r = a, b, \tag{101}$$

and

$$\begin{aligned}
&cr^3 f_n''' + (rc' + 6c)r^2 f_n'' + [2rc' + r\tau_{33}' - h - m(2b + c) + 6c]rf_n' \\
&+(m - 2)(rc' - r\tau_{33}' + h)f_n + drg_n'' + (2d + rd')g_n' - m(e + d)g_n/r = 0 \quad \text{on} \quad r = a, b.
\end{aligned} \tag{102}$$

Equation (100) provides an expression for g_n'', which is substituted into the boundary condition (102), and by differentiating (100) with respect to r we obtain an expression for g_n''' which is used in (99) along with the expression for g_n''. Then, following the procedure used in [6], the equations are arranged as a first-order system in the form

$$\mathbf{y}' = \boldsymbol{\mathcal{M}}\mathbf{y}, \tag{103}$$

where $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$, with $y_1 = f_n, y_2 = y_1', y_3 = y_2', y_4 = y_3', y_5 = g_n, y_6 = y_5'$, a prime indicates differentiation with respect to r, and $\boldsymbol{\mathcal{M}}$ is the 6×6 matrix

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} & M_{46} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ M_{61} & M_{62} & M_{63} & M_{64} & M_{65} & M_{66} \end{bmatrix}, \tag{104}$$

whose non-zero elements M_{4i}, $i \in \{1, \dots, 6\}$, are quite lengthy and therefore listed in Appendix A, while M_{6i}, $i \in \{1, \dots, 6\}$, are given by

$$\begin{aligned}
&M_{61} = -(m - 2)d'/f, \quad M_{62} = (me - 2rd')/f, \quad M_{63} = -(rd' + 4d)r/f, \\
&M_{64} = -dr^2/f, \quad M_{65} = mg/(r^2 f), \quad M_{66} = -f'/f.
\end{aligned} \tag{105}$$

The corresponding boundary conditions $(101)_{1,2}$ and (102) become, respectively,

$$(m - 2)y_1 + 2ry_2 + r^2 y_3 = 0, \quad y_6 = 0, \quad \sum_{i=1}^{6} b_i y_i = 0 \quad \text{on} \quad r = a, b, \tag{106}$$

where the coefficients b_i, $i \in \{1, \dots, 6\}$, are given by

$$\begin{aligned}
b_1 &= (m-2)(rc' - r\tau'_{33} + h - rdd'/f),\\
b_2 &= r[2rc' + r\tau'_{33} - h - 2mb - mc + 6c + d(me - 2rd')/f],\\
b_3 &= r^2[rc' + 6c - d(rd' + 4d)/f], \quad b_4 = r^3(cf - d^2)/f,\\
b_5 &= -m(e + d - gd/f)/r, \quad b_6 = rd' + 2d - rdf'/f.
\end{aligned} \tag{107}$$

6 Application to specific energy functions

The equations derived thus far are valid for any form of isotropic electroelastic model, but to illustrate the theory it is necessary to consider specific forms of energy function. For this purpose we now restrict attention to standard models that depend only on the invariants I_1 and I_5 in the form

$$\Omega^*(I_1, I_5) = W(I_1) + \frac{1}{2}\varepsilon^{-1} I_5, \tag{108}$$

where $W(I_1)$ is the strain energy of a purely elastic material in the absence of an electric field and the constant ε is the electric permittivity of the electroelastic material.

The particular forms of $W(I_1)$ that we consider here are the neo-Hookean and the Gent [8] models given by

$$W(I_1) = \frac{1}{2}\mu(I_1 - 3), \quad W(I_1) = -\frac{\mu G}{2}\log[1 - (I_1 - 3)/G], \tag{109}$$

respectively, where the constant μ is the shear modulus in the reference configuration and G is a non-dimensional material constant, known as the Gent constant.

Since, for equibiaxial deformations, $I_1 = 2\lambda^2 + \lambda^{-4}$ we may also consider $W(I_1)$ as a function of λ, which we denote by $\hat{W}(\lambda)$, so that $\hat{W}(\lambda) = W(I_1) = W(2\lambda^2 + \lambda^{-4})$, $\partial I_1/\partial\lambda = 4(\lambda - \lambda^{-5})$,

$$4W_1\lambda^{-4} = \frac{\lambda \hat{W}'}{\lambda^6 - 1}, \quad 16W_{11}\lambda^{-8} = \frac{\lambda^2 \hat{W}''}{(\lambda^6 - 1)^2} - (\lambda^6 + 5)\frac{\lambda \hat{W}'}{(\lambda^6 - 1)^3}, \tag{110}$$

$$64W_{111}\lambda^{-12} = \frac{\lambda^3 \hat{W}'''}{(\lambda^6 - 1)^3} - 3(\lambda^6 + 5)\frac{\lambda^2 \hat{W}''}{(\lambda^6 - 1)^4} + 3(\lambda^{12} + 20\lambda^6 + 15)\frac{\lambda \hat{W}'}{(\lambda^6 - 1)^5}, \tag{111}$$

provide the connections between the derivatives of $W(I_1)$ and $\hat{W}(\lambda)$ that are needed subsequently.

An example of $\hat{W}(\lambda)$ comes from the model introduced in [9] and known as the Ogden model, which, for equibiaxial deformations, has the form

$$\hat{W}(\lambda) = \sum_{n=1}^{N} \mu_n(2\lambda^{\alpha_n} + \lambda^{-2\alpha_n} - 3)/\alpha_n, \tag{112}$$

where $N = 1, 2, \dots,$ and μ_n and α_n are material constants.

By specializing the general results given in [4] to models of the type (108) the components of $\boldsymbol{\mathcal{A}}_0^*$, $\mathbb{A}_0^*$, A_0^* take on the simple forms

$$\begin{aligned}\mathcal{A}_{0piqj}^* &= 4W_{11}b_{ip}b_{jq} + 2W_1\delta_{ij}b_{pq} + \varepsilon^{-1}\delta_{ij}D_pD_q,\\ \mathbb{A}_{0pi|q}^* &= \varepsilon^{-1}(\delta_{pq}D_i + \delta_{iq}D_p), \quad \mathsf{A}_{0ij}^* = \varepsilon^{-1}\delta_{ij},\end{aligned} \tag{113}$$

and for the considered equibiaxial deformations the formulas (84) simplify to

$$\begin{aligned}&a = 2W_1\lambda^2, \quad 2b = 4W_{11}(\lambda^2 - \lambda^{-4})^2 + 2W_1(\lambda^2 + \lambda^{-4}) + \varepsilon^{-1}D^2,\\ &c = 2W_1\lambda^{-4} + \varepsilon^{-1}D^2, \quad d = e = \varepsilon^{-1}D, \quad f = g = \varepsilon^{-1}, \quad h = 0,\end{aligned} \tag{114}$$

where $D = D_3$. For the neo-Hookean model a, b, c specialize to

$$a = \mu\lambda^2, \quad 2b = a + c, \quad c = \mu\lambda^{-4} + \varepsilon^{-1}D^2, \tag{115}$$

while d, e, f, g, h are unchanged. For the Gent model W_1 and W_{11} are given in Appendix A, and for the model (112) are obtained on use of (110).

We also note that when the formula (57) for the potential difference is specialized for the model (108) it yields

$$V = \varepsilon^{-1}aD(a)\frac{(b-a)}{b} = \varepsilon^{-1}\frac{Q_a}{4\pi}\frac{(b-a)}{ab}. \tag{116}$$

6.1 Non-dimensionalization and numerical results

For numerical purposes we now rewrite the governing equations and the boundary conditions in non-dimensional form. The dimensions of the expressions (69) and $(97)_1$, (70) and $(97)_2$ suggest the non-dimensionalizations functions $\hat{f}_n(\hat{r})$ and $\hat{g}_n(\hat{r})$ defined by

$$\hat{f}_n(\hat{r}) = \frac{f_n(r)}{A}, \quad \hat{g}_n(\hat{r}) = \frac{g_n(r)}{D(a)A^2}. \tag{117}$$

where $\hat{r} = r/A$, and we also adopt the non-dimensional parameters

$$\hat{\sigma}_{\rm fa} = \sigma_{\rm fa}/\sqrt{\mu\varepsilon}, \quad \hat{Q} = Q_a/(4\pi A^2\sqrt{\mu\varepsilon}), \quad \hat{V} = V/(A\sqrt{\mu/\varepsilon}), \tag{118}$$

with the connections

$$\lambda_a^2\hat{\sigma}_{\rm fa} = \hat{Q}, \quad \hat{V} = [1/\lambda_a - A/(B\lambda_b)]\hat{Q}, \tag{119}$$

the latter obtained from (116) and the definitions $\lambda_a = a/A, \lambda_b = b/B$.

The dimensionless forms of the variables $y_1, \ldots, y_6$ are taken to be

$$\begin{aligned}&\hat{y}_1(\hat{r}) = \hat{f}_n(\hat{r}), \quad \hat{y}_2(\hat{r}) = \hat{f}_n'(\hat{r}), \quad \hat{y}_3(\hat{r}) = \hat{f}_n''(\hat{r}),\\ &\hat{y}_4(\hat{r}) = \hat{f}_n'''(\hat{r}), \quad \hat{y}_5(\hat{r}) = \hat{g}_n(\hat{r}), \quad \hat{y}_6(\hat{r}) = \hat{g}_n'(\hat{r}),\end{aligned} \tag{120}$$

where the prime represents differentiation with respect to the argument $\hat{r}$. The total Cauchy stress $\boldsymbol{\tau}$ and the updated electroelastic moduli tensors are non-dimensionalized according to

$$\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}/\mu, \quad \hat{\boldsymbol{\mathcal{A}}}_{\mathbf{0}}^* = \boldsymbol{\mathcal{A}}_0^*/\mu, \quad \hat{\mathbb{A}}_0^* = \mathbb{A}_0^*\varepsilon/\sigma_{\rm fa}, \quad \hat{\mathsf{A}}_0^* = \mathsf{A}_0^*\varepsilon. \tag{121}$$

6.1.1 Results for the neo-Hookean model

First we note that for the neo-Hookean electroelastic material the pressure P (in dimensionless form P/μ) is given by

$$P/\mu = \frac{\lambda_a - \lambda_b}{2\lambda_a^4\lambda_b^4}\left[4\lambda_a^3\lambda_b^3 + (\lambda_a + \lambda_b)(\lambda_a^2 + \lambda_b^2)\right] - \frac{1}{2}\frac{\hat{Q}^2\left[\lambda_b^4 - \lambda_a^4(A/B)^4\right]}{\lambda_a^4\lambda_b^4}. \tag{122}$$

This formula was given in [4] in a slightly different form and can be obtained from (63) with (59) and (108).

For the initial calculations we used the numerical scheme described in [5] and obtained values of λ_a and λ_b for which bifurcation first becomes possible and the associated mode numbers and values of the dimensionless pressure. The results of the calculations for the purely elastic case ($\hat{Q} = 0$) are given in Table 1. The final column in Table 1 provides the corresponding results for λ_b given in [1]. Clearly, our calculations are very close to those in [1], and we note that a different numerical scheme was used in [1], where it was reported that for thin shells their method became increasingly sensitive. In [1] it was found that for the neo-Hookean model no bifurcation solutions were possible for internally pressurized spherical shells ($P > 0$), only for an external pressure, as we have also found.

Table 1: Bifurcation values λ_a, λ_b and non-dimensional pressure P/μ for the purely elastic neo-Hookean model (108) of a spherical shell for different values A/B, with the mode number n for which bifurcation first becomes possible. The final column gives the results for λ_b from [1] for comparison.

A/B	n	Present results			Results from [1]
		λ_a	λ_b	P/μ	λ_b
0.95	7	0.9817	0.9844	−0.0115	0.985
0.9	5	0.9600	0.9712	−0.0507	0.971
0.85	4	0.9345	0.9608	−0.1275	0.961
0.8	3	0.9063	0.9543	−0.2492	0.955
0.7	2	0.8356	0.9499	−0.6995	0.950
0.6	2	0.7744	0.9598	−1.3000	0.960

For the remaining calculations we have used `NDSolve` in Mathematica [14]. Consistently with the values of λ_a and some of the mode numbers in Table 1, in Fig. 1(a) are plotted the values of λ_a versus A/B for which bifurcation becomes possible along with the curve for $P = 0$ from (122). The bifurcation curves lie entirely in the region where $P < 0$ and there is no solution for the $n = 1$ mode.

The purely mechanical case of axisymmetric bifurcations of inflated and compressed spherical shells was considered in a more recent work of deBotton et al. [2]. They used the same theory as in [1] with different strain-energy functions. They also reported that

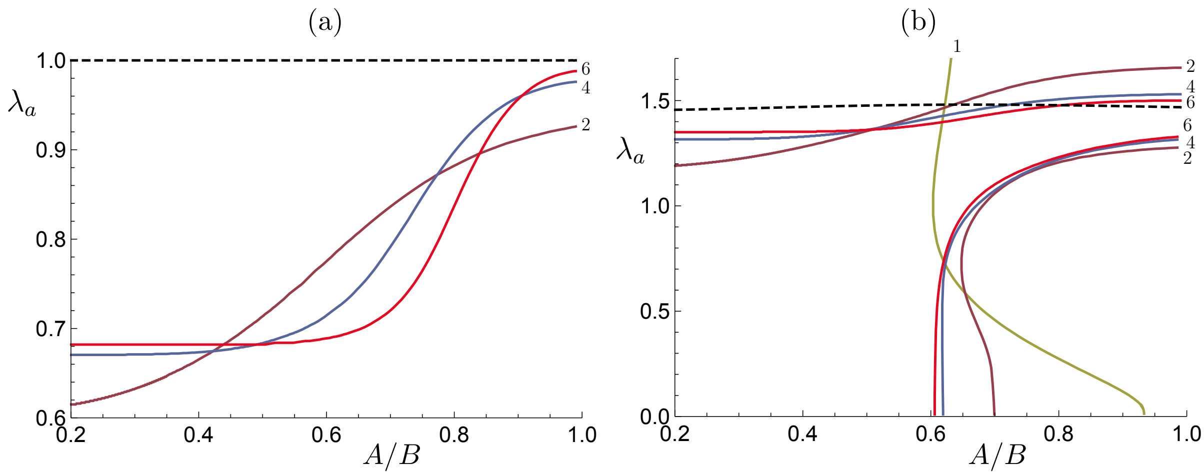

Figure 1: The critical stretch λ_a versus A/B for bifurcation modes $n = 1, 2, 4, 6$ for an electroelastic neo-Hookean model with (a) $\hat{Q} = 0$, (b) $\hat{Q} = 3$. The dashed curve corresponds to the values of λ_a for which $P = 0$.

additional solutions were found which had not previously been reported in the literature for a one-term Ogden model. They used a different numerical scheme from that in [1], the compound matrix method, details of which can be found in their paper and in references cited therein.

As the electric field is applied the situation changes and bifurcation can occur for $P > 0$. This is illustrated for $\hat{Q} = 3$ in Fig. 1(b) for mode numbers $1, 2, 4, 6$, where, for the thinner walled spheres (A/B nearer to 1), bifurcation is possible both for $P > 0$ and $P < 0$, while for thicker walled shells the upper curve (where $P < 0$) for each value of n has priority for bifurcation as λ_a decreases and the lower curve is not relevant. The curve for $P = 0$ is obtained from (122). When $P > 0$ there is only a small range of values of A/B for which bifurcation is possible in the $n = 1$ mode, while for $P < 0$ the other mode numbers have priority.

Note that for a fixed value of either $\hat{Q}$ or $\hat{V}$ it follows from the connections (119) that $\hat{\sigma}_{\text{f}a}$ depends on the deformation. It is not a practical proposition to impose a fixed value of $\hat{\sigma}_{\text{f}a}$, nor is it easy to fix a constant value of $\hat{Q}$ for the present geometry, and this leaves the only real practical option to prescribe the potential difference. For these reasons we now focus on results for fixed $\hat{V}$, but it is also of theoretical interest to compare these with results for fixed $\hat{Q}$, which are quite different.

In Fig. 2 results for the fixed value $\hat{V} = 0.2$ are shown for illustration, for each of the mode numbers $n = 1, 2, 4, 6$. Above and to the left of the dashed curve, which corresponds to $P = 0$, the pressure is positive, and in this region the $n = 1$ mode has priority over the other modes, while below curve $P = 0$ one or another of the other modes has priority. The curves above the $n = 1$ curve to the right of the dashed curve are in the region where $P < 0$ and here the value $\hat{V} = 0.2$ causes significant inflation in the presence of an external pressure, which is quite small since the curves are close to the $P = 0$ curve. For

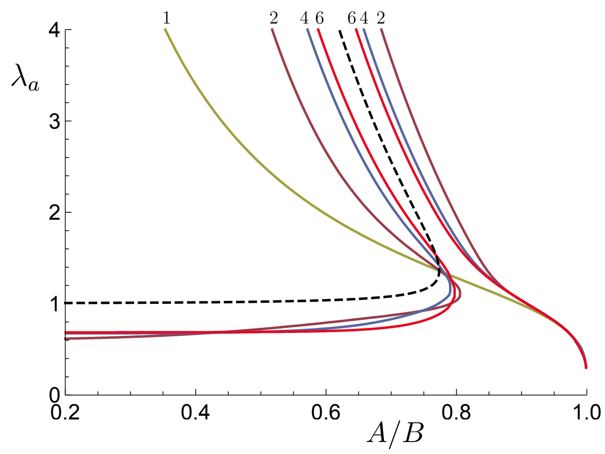

Figure 2: The critical stretch λ_a versus A/B for bifurcation modes $n = 1, 2, 4, 6$ for an electroelastic neo-Hookean model with $\hat{V} = 0.2$. The dashed curve corresponds to the values of λ_a for which $P = 0$.

thin-walled shells the different mode number curves merge where $\lambda_a < 1$, i.e. the external pressure reduces the shell radius in the presence of the potential difference. For $\hat{V} = 0$ the bifurcation curves are identical to those in Fig. 1(a).

Henceforth, since from Fig. 2 the $n = 1$ mode has priority when $P > 0$, we confine attention to the $n = 1$ mode. Staying with the neo-Hookean model, in Fig. 3 the values of λ_a corresponding to bifurcation in the $n = 1$ mode are shown as functions of the radii ratio A/B for several fixed values of $\hat{Q}$ and $\hat{V}$. Also shown are the curves corresponding to $P = 0$ as dashed curves. In Fig. 3(a) bifurcation is possible for $P > 0$ above the dashed curves for the appropriate value of $\hat{Q}$ in each case, i.e. under internal pressure, while below the dashed curves bifurcation is possible under external pressure. In Fig. 3(b), for fixed values of $\hat{V}$, the pressure is positive to the left and above the dashed curves and negative to the right and below, so bifurcation is possible under either internal or external pressure depending on the value of A/B. As already noted no bifurcation is possible under internal pressure for $\hat{Q} = \hat{V} = 0$, while for external pressure results have been presented in Table 1 as well as Fig. 1(a). Depending on the value of A/B, bifurcation is possible for zero pressure where a $P = 0$ curve intersects the corresponding bifurcation curve for the same value of $\hat{Q}$ or $\hat{V}$, as appropriate.

The neo-Hookean model for elastomeric materials has limited applicability for large deformations, as is well known, so the results shown in the above figures may not be very realistic for values of λ_a much beyond 2 or much below 0.7. We note that for the neo-Hookean model no snap-through is possible since the pressure decays to zero with increasing λ_a, and there is a maximum value of the voltage that the material can support, as exemplified in [7]. The values of $\hat{V}$ used here are below this maximum. We now consider two models which are more realistic for very large deformations.

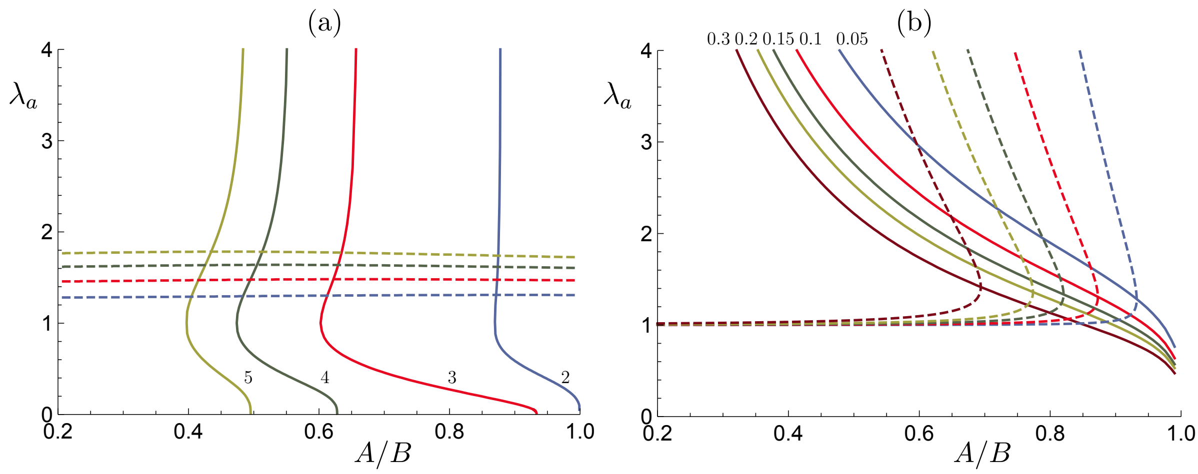

Figure 3: The critical stretch λ_a versus A/B for bifurcation mode $n = 1$ for a neo-Hookean model: (a) $\hat{Q} = 2, 3, 4, 5$; (b) $\hat{V} = 0.05, 0.1, 0.15, 0.2, 0.3$. The dashed curves identify the values of λ_a for which $P = 0$ for the specified values of $\hat{Q}$ and $\hat{V}$.

6.1.2 Results for the Gent model

For $P < 0$ the results for the Gent model are very similar to those for the neo-Hookean model shown in Fig. 1(a) and are not therefore displayed separately, and this is also the case for $\hat{Q} = 3$ in Fig. 1(b). There are some differences with respect to Fig. 2 for $\hat{V} = 0.2$, as shown in Fig. 4, the counterpart of Fig. 2, which provides the bifurcation curves for $\hat{V} = 0.2$ in respect of the Gent model with the Gent constant $G = 97.2$. A feature that is similar to that shown in Fig. 2 is that for $P > 0$ the $n = 1$ mode is dominant, so again we focus on this mode. Figure 5 shows the $n = 1$ results for $\hat{Q} = 2, 3, 4, 5$ in Fig. 5(a) and for $\hat{V} = 0.05, 0.1, 0.15.0.2, 0.3$ in Fig. 5(b). The bifurcation curves are quite different from those in Fig. 3, while the zero pressure curves have some similarities. The interpretation in terms of internal or external pressure follows the same pattern as for Fig. 3. Note that there are no bifurcation curves for the purely elastic case ($\hat{Q} = \hat{V} = 0$), as was also found for the Gent model with the same value of G in [2].

For either fixed $\hat{Q}$ or fixed $\hat{V}$ an increase in the value moves the bifurcation curves to the left so that bifurcation becomes possible for thicker-walled shells, but is confined to thinner-walled shells for the smaller values.

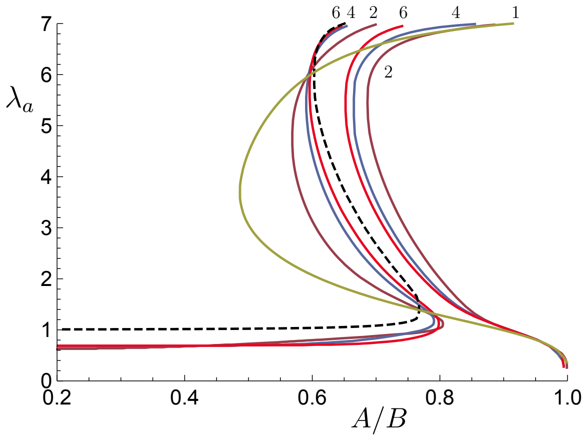

Figure 4: The critical stretch λ_a versus A/B for bifurcation modes $n = 1, 2, 4, 6$ for an electroelastic Gent model with $G = 97.2$ and $\hat{V} = 0.2$. The dashed curve corresponds to the values of λ_a for which $P = 0$.

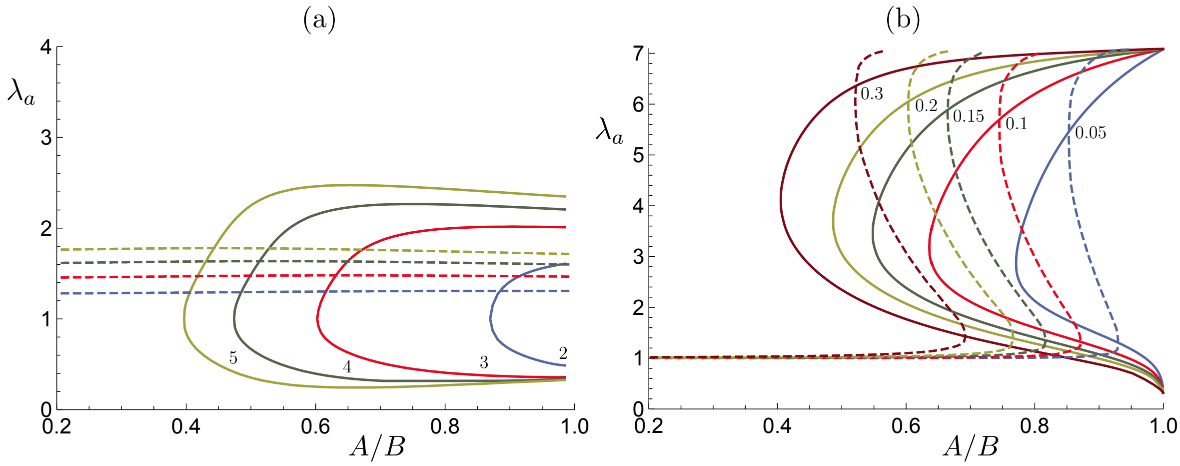

Figure 5: The critical stretch λ_a versus A/B for bifurcation mode $n = 1$ for a Gent model with $G = 97.2$: (a) $\hat{Q} = 2, 3, 4, 5$; (b) $\hat{V} = 0.05, 0.1, 0.15, 0.2, 0.3$. The dashed curves identify the values of λ_a for which $P = 0$ for the specified values of $\hat{Q}$ and $\hat{V}$.

6.1.3 Results for the 3-term Ogden model

The final illustration is for the three-term Ogden model (112) with the values of the material parameters as in [9], for which $\alpha_1 = 1.3, \alpha_2 = 5, \alpha_3 = -2$, $\mu_1/\mu = 1.491, \mu_2/\mu = 0.003, \mu_3/\mu = -0.023$ in dimensionless form. The results for the bifurcation curves have some differences from and similarities to those for the neo-Hookean and Gent models but the zero pressure curves are very similar. In Fig. 6 the bifurcation results are shown for the specific value $\hat{V} = 0.2$ and mode numbers $n = 1, 2, 4, 6$, along with the zero pressure curve. In the region $P < 0$ the results are very similar to those for the other two models, including for the case of fixed $\hat{Q}$. Once again the $n = 1$ mode has priority in the region

$P > 0$ and hence in Fig. 7 attention is restricted to $n = 1$ modes, for fixed $\hat{Q}$ in Fig. 7(a) and fixed $\hat{V}$ in Fig. 7(b). The curve corresponding to $\hat{Q} = \hat{V} = 0$ is for the purely elastic case and coincides with that obtained in [1] except that the horizontal scale used therein was the reverse of that used here. The purely elastic result is entirely in the region where $P > 0$. Referring to Fig. 7, it is clear that, as for the neo-Hookean and Gent models, the range of values of A/B for which bifurcation is possible increases with the magnitude of the applied $\hat{Q}$ and $\hat{V}$.

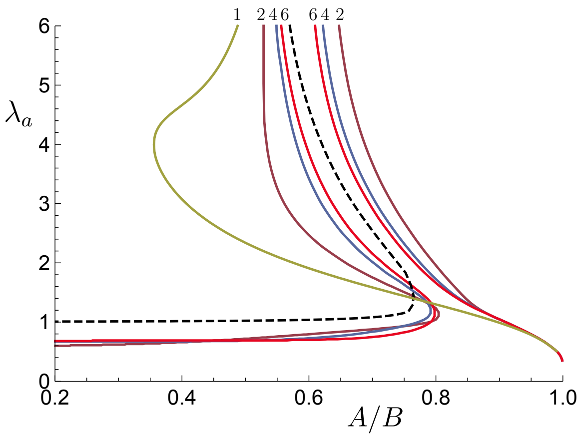

Figure 6: The critical stretch λ_a versus A/B for bifurcation modes $n = 1, 2, 4, 6$ for an electroelastic 3-term Ogden model with $\hat{V} = 0.2$. The dashed curve corresponds to the values of λ_a for which $P = 0$.

7 Concluding remarks

In this paper we have analyzed possible axisymmetric bifurcation modes for an electroelastic spherical shell subject to internal or external pressure and a radial electric field that is generated by compliant electrodes on its inner and outer boundaries. The theory of small incremental electroelastic deformations superimposed on a finitely deformed electroelastic thick-walled spherical shell has been used to determine the underlying configurations and ratios of inner to outer undeformed radii for which the superimposed deformations do not maintain the perfect spherical shape. Illustrative results have been obtained numerically for three different electroelastic energy functions based on the purely elastic neo-Hookean, Gent and Ogden models.

The results in the presence of the electric field are quite different from those in the purely elastic case. For the neo-Hookean model, in particular, depending on the geometrical parameters of the shell, bifurcations become possible in the presence of an electric field under internal pressure, which is not the case without the field. For the Gent and Ogden

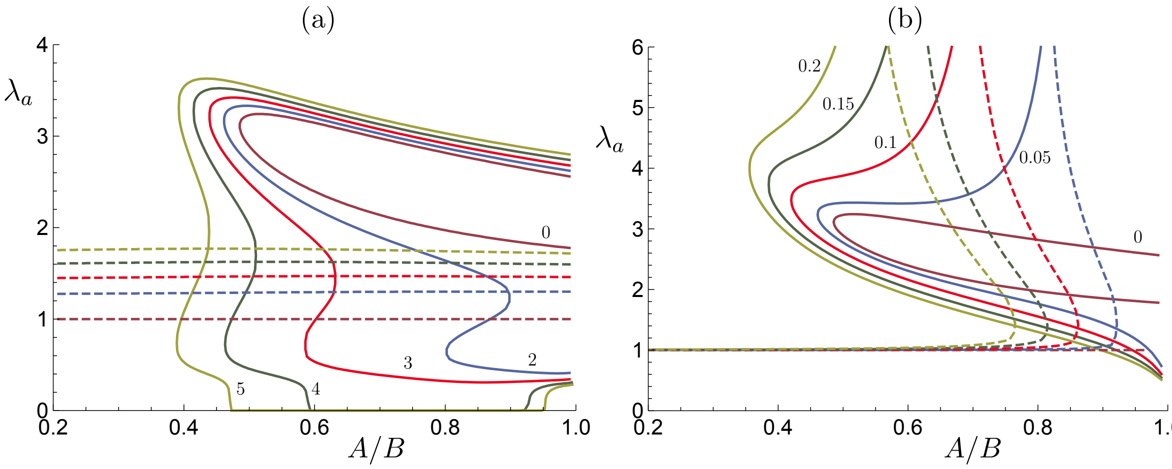

Figure 7: The critical stretch λ_a versus A/B for bifurcation mode $n = 1$ for a 3-term Ogden model with $\alpha_1 = 1.3, \alpha_2 = 5, \alpha_3 = -2$, $\mu_1/\mu = 1.491, \mu_2/\mu = 0.003, \mu_3/\mu = -0.023$: (a) $\hat{Q} = 2, 3, 4, 5$; (b) $\hat{V} = 0.05, 0.1, 0.15, 0.2$. The dashed curves identify the values of λ_a for which $P = 0$ for the specified values of $\hat{Q}$ and $\hat{V}$.

models there are significant differences from the results for the neo-Hookean model and between the Gent and Ogden models themselves.

Problems of the kind considered here, and similar problems for tubes considered elsewhere (for example, in [5] and [6]), provide a theoretical underpinning for practical applications to, for example, dielectric elastomer actuators and micropumps for which significant deformations are possible.

As far as the voltage is concerned, we note that the qualitative nature of the results is independent of the voltage and sphere wall thickness, as can be seen in Figs. 3(b), 5(b) and 7(b) for the different energy functions, so that a wide range of voltages and thickness ratios is accommodated by the analysis. For large voltages dielectric or material failure could occur prior to bifurcation, but such instabilities are outside the scope of the present work.

Acknowledgement

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A Expressions for the components M_{4i}, M_{6i} and b_i

Without specializing the isotropic electroelastic energy function the components M_{4i}, $i = 1, \dots, 6$, are given by

$$\begin{aligned}
M_{41} &= -(m-2)[(r^2c'' - r^2\tau''_{33} + rh' - h + ma)f^2 - (rdd'' + 2rd'^2 + 2dd')rf \\
&\quad + 2r^2dd'f']/[r^4f(cf-d^2)], \\
M_{42} &= -[(2r^2c'' + r^2\tau''_{33} + 6rc' + 3r\tau'_{33} - 4mb - 2h - 2mrb' - rh')f^2 \\
&\quad - (2r^2dd'' + mrdd' - mrde' + 4r^2d'^2 + 4rdd' - 2mred' - 2med)f \\
&\quad + 4r^2dd'f' - 2mrdef']/[r^3f(cf-d^2)], \\
M_{43} &= -[(r^2c'' + 10rc' + 12c + r\tau'_{33} - 2mb - h)f^2 \\
&\quad - (r^2dd'' + 18rdd' + 12d^2 - mde + 2r^2d'^2)f + 2rd(rd' + 4d)f']/[r^2f(cf-d^2)], \\
M_{44} &= -[2(rc' + 4c)f^2 - 4d(rd' + 2d)f + 2d^2rf']/[rf(cf-d^2)], \\
M_{45} &= -m[-(e' + d')f^2 + (dg' + 2gd')f - 2gdf']/[r^4f(cf-d^2)], \\
M_{46} &= -\{(r^2d'' + 2rd' - 2d - me)f^2 - [dr^2f'' - mgd + 2rf'(rd' + d)]f \\
&\quad + 2dr^2f'^2\}/[r^4f(cf-d^2)]
\end{aligned}$$

For the model (108) the above formulas require the expressions

$$\begin{aligned}
rc' &= -8W_{11}\lambda^{-8}(\lambda^3-1)^2(\lambda^3+1) + 8W_1\lambda^{-4}(\lambda^3-1) - 4\varepsilon^{-1}D^2, \\
r\tau'_{33} &= 4W_1\lambda^{-4}(\lambda^6-1) - 2\varepsilon^{-1}D^2, \\
r^2\tau''_{33} &= -16W_{11}\lambda^{-8}(\lambda^3-1)^3(\lambda^3+1)^2 - 4W_1\lambda^{-4}(\lambda^3-1)(2\lambda^6+\lambda^3+5) + 10\varepsilon^{-1}D^2, \\
r^2c'' &= 32W_{111}\lambda^{-12}(\lambda^3-1)^4(\lambda^3+1)^2 + 8W_{11}\lambda^{-8}(\lambda^3-1)^2(-3\lambda^6+4\lambda^3+13) \\
&\quad + 8W_1\lambda^{-4}(\lambda^3-1)(\lambda^3-5) + 20\varepsilon^{-1}D^2, \\
rb' &= -8W_{111}\lambda^{-12}(\lambda^3-1)^4(\lambda^3+1)^3 - 4W_{11}\lambda^{-8}(\lambda^3-1)^2(\lambda^3+1)(3\lambda^6+5) \\
&\quad - 2W_1\lambda^{-4}(\lambda^3-1)(\lambda^6-2) - 2\varepsilon^{-1}D^2
\end{aligned}$$

and then become

$$\begin{aligned}
M_{41} &= -(m-2)[16W_{111}\lambda^{-8}(\lambda^3-1)^4(\lambda^3+1)^2 + 4W_{11}\lambda^{-4}(\lambda^3-1)^2(2\lambda^9-\lambda^6+2\lambda^3+11) \\
&\quad + 2W_1(\lambda^3-1)^2(2\lambda^3+5) + mW_1\lambda^6]/(r^4W_1), \\
M_{42} &= -[32W_{111}\lambda^{-8}(\lambda^3-1)^4(\lambda^3+1)^2 + 8mW_{111}\lambda^{-8}(\lambda^3-1)^4(\lambda^3+1)^3 \\
&\quad - 8W_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^9+4\lambda^6-2\lambda^3-11) + 4mW_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^3+1)(3\lambda^6-\lambda^3+4) \\
&\quad - 4W_1(\lambda^3-1)(\lambda^6-3\lambda^3+5) + 2mW_1(\lambda^3+1)(\lambda^6-3\lambda^3+1)]/(r^3W_1), \\
M_{43} &= -[16W_{111}\lambda^{-8}(\lambda^3-1)^4(\lambda^3+1)^2 - 12W_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^6+2\lambda^3-1) \\
&\quad - 2mW_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^3+1)^2 + 2W_1(3\lambda^6+8\lambda^3-5) - mW_1(\lambda^6+1)]/(r^2W_1), \\
M_{44} &= 8[W_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^3+1) - W_1\lambda^3]/(rW_1),
\end{aligned}$$

with $M_{45} = M_{46} = 0$.

The corresponding expressions for M_{6i}, $i = 1, \dots, 6$, in (105) are

$$M_{61} = 2(m-2)D/r, \quad M_{62} = (m+4)D, \quad M_{63} = -2Dr, \quad M_{64} = -Dr^2, \quad M_{65} = m/r^2,$$

with $M_{66} = 0$, and the coefficients $b_1, \dots, b_6$ in (107) become

$$
\begin{aligned}
b_1 &= -4(m-2)\lambda^{-4}(\lambda^3-1)^2[2W_{11}\lambda^{-4}(\lambda^3+1)+W_1],\\
b_2 &= -r\{16W_{11}\lambda^{-8}(\lambda^3-1)^2(\lambda^3+1)-4W_1\lambda^{-4}(\lambda^6+4\lambda^3-2)\\
&\quad + m[4W_{11}\lambda^{-8}(\lambda^6-1)^2+2W_1\lambda^{-4}(\lambda^6+2)+\varepsilon^{-1}D^2]\},\\
b_3 &= -4r^2\lambda^{-4}[2W_{11}\lambda^{-4}(\lambda^3-1)^2(\lambda^3+1)-W_1(2\lambda^3+1)],\\
b_4 &= 2r^3W_1\lambda^{-4}, \quad b_5 = -md/r, \quad b_6 = 0.
\end{aligned}
$$

For the Gent model the first three derivatives of $W(I_1)$, given by

$$
W_1 = \frac{1}{2}\frac{\mu G}{(3+G-I_1)}, \quad W_{11} = \frac{1}{2}\frac{\mu G}{(3+G-I_1)^2}, \quad W_{111} = \frac{\mu G}{(3+G-I_1)^3},
$$

are required in the formulas above.

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