

Kent Academic Repository

Full text document (pdf)

Citation for published version

Roelands, Mark and Schwanke, C. (2019) Series and power series on universally complete complex vector lattices. *Journal of Mathematical Analysis and Applications*, 473 (2). pp. 680-694. ISSN 0022-247X.

DOI

<https://doi.org/10.1016/j.jmaa.2018.12.041>

Link to record in KAR

<https://kar.kent.ac.uk/80086/>

Document Version

Author's Accepted Manuscript

Copyright & reuse

Content in the Kent Academic Repository is made available for research purposes. Unless otherwise stated all content is protected by copyright and in the absence of an open licence (eg Creative Commons), permissions for further reuse of content should be sought from the publisher, author or other copyright holder.

Versions of research

The version in the Kent Academic Repository may differ from the final published version.

Users are advised to check <http://kar.kent.ac.uk> for the status of the paper. **Users should always cite the published version of record.**

Enquiries

For any further enquiries regarding the licence status of this document, please contact:

researchsupport@kent.ac.uk

If you believe this document infringes copyright then please contact the KAR admin team with the take-down information provided at <http://kar.kent.ac.uk/contact.html>

SERIES AND POWER SERIES ON UNIVERSALLY COMPLETE COMPLEX VECTOR LATTICES

M. ROELANDS AND C. SCHWANKE

ABSTRACT. In this paper we prove an n th root test for series as well as a Cauchy-Hadamard type formula and Abel's theorem for power series on universally complete Archimedean complex vector lattices. These results are aimed at developing an alternative approach to the classical theory of complex series and power series using the topology of order convergence.

1. INTRODUCTION

The theory of complex analysis on complex Banach spaces enjoys a long, well-known, and fruitful history, which is summarized in the text [13]. This theory generalizes the classical theory of functions of a complex variable by using norms as a generalization of the ordinary modulus on the complex plane. In this paper we investigate complex analysis via an alternative generalization of the complex modulus on \mathbb{C} : *the modulus on Archimedean complex vector lattices*. This complex vector lattice modulus naturally leads to the notion of order convergence, providing us with an order-theoretic perspective of classical complex analysis.

The idea of an abstract order-theoretic complex modulus, to the authors' best knowledge, first appeared on complex AL -spaces in a 1963 paper by Rieffel (see [15, 16]). Here Rieffel defined the modulus for any element z of a complex AL -space by

$$|z| := \sup\{Re(e^{i\theta}z) : \theta \in [0, 2\pi]\}. \quad (1)$$

In 1968, this formula for the modulus was defined more generally on Banach lattices by Lotz [9] and was further extended to vector space complexifications of uniformly complete Archimedean vector lattices in 1971 by Luxemburg and Zaanen [10]. The authors of [10] referred to these complexifications of uniformly complete Archimedean vector lattices as *complex Riesz spaces*.

Date: March 22, 2018.

2010 Mathematics Subject Classification. Primary: 46A40; Secondary: 40J05.

Key words and phrases. series, power series, complex vector lattice, order topology, complex analysis.

The notion of a complex Riesz space, or complex vector lattice, was extended slightly by de Schipper in 1973 (see [17]), who defined a complex vector lattice to be a complex vector space E of the form $E = E_\rho \oplus iE_\rho$, where E_ρ is a real vector lattice that is closed under the supremum in (1). This extension was not a trivial one, as there exist real vector lattices that are closed under the supremum in (1) that are not uniformly complete. An example provided by Azouzi in [3] is the space of all real-valued step functions. A trivial variation of the formula (1) was termed the *de Schipper formula* in [4].

Providing an alternative approach, Mittelmeyer and Wolff in [12] defined a modulus on a complex vector space E to be an idempotent mapping $|\cdot|: E \rightarrow E$ that satisfies

- (i) $|\alpha x| = |\alpha||x|$ for all $\alpha \in \mathbb{C}$ and each $x \in E$,
- (ii) $\left| \left| |x| + |y| \right| - |x + y| \right| = |x| + |y| - |x + y|$, and
- (iii) E is in the \mathbb{C} -linear hull of the range of $|\cdot|$.

The authors of [12] proved that any complex vector space E equipped with a modulus is a complex vector lattice (in the sense of de Schipper), and that the modulus on E was necessarily given by the de Schipper formula.

We focus here on using this complex vector lattice modulus to abstract the classical n th root test for series as well as the Cauchy-Hadamard formula and Abel's Theorem for power series on universally complete Archimedean complex vector lattices (see Theorem 3.6, Theorem 3.11, and Proposition 3.12). Indeed, universally complete Archimedean complex vector lattices provide a natural framework for order-theoretic generalizations of these classical results. In particular, these spaces possess a vector lattice complex modulus, giving us notions of magnitude and order convergence. Additionally, it is well known that every universally complete Archimedean (real) vector lattice is in fact a Φ -algebra, i.e. an f -algebra with a multiplicative identity. It follows that universally complete Archimedean complex vector lattices possess an algebraic structure which is compatible with their order structure. Moreover, we can define n th roots of positive elements inside every universally complete Archimedean complex vector lattice, allowing us to formulate an abstract n th root test in this setting. We stress that the assumption of universal completeness is essential to this paper. For example, Remark 3.7 illustrates that Lemma 3.5 and Theorem 3.6(i) fail to hold inside a non-universally complete Archimedean complex Φ -algebra.

To our best knowledge, this is the first paper that studies power series abstractly on universally complete Archimedean complex vector lattices. The main results here illustrate how fundamental aspects of vector lattice theory, such as weak order units and projection bands, play a significant yet almost hidden role in classical complex analysis.

2. PRELIMINARIES

We refer the reader to the standard texts [2, 11, 18] for any unexplained terminology or basic results regarding vector lattices, f -algebras, and Φ -algebras. As conventional, \mathbb{C} represents the field of complex numbers in this paper. The set of strictly positive integers is represented by \mathbb{N} , and we use the notation $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ throughout.

We say that an Archimedean (real) vector lattice E is *square mean closed* (see [4, page 482] or [17, page 356]) if $\sup\{(\cos \theta)f + (\sin \theta)g : \theta \in [0, 2\pi]\}$ exists in E for every $f, g \in E$, in which case we write

$$f \boxplus g := \sup\{(\cos \theta)f + (\sin \theta)g : \theta \in [0, 2\pi]\} = \sup\{Re(e^{-i\theta}(f + ig)) : \theta \in [0, 2\pi]\}.$$

For example, every uniformly complete Archimedean vector lattice is square mean closed (see [5, Section 2]). Given a square mean closed Archimedean (real) vector lattice E , we call the vector space complexification $E \oplus iE$ an *Archimedean complex vector lattice* [17, pages 356–357]. The vector lattice E is called the *real part* of $E \oplus iE$. Given an Archimedean complex vector lattice E , we denote its real part by E_ρ .

The *modulus* on an Archimedean complex vector lattice E is defined by

$$|f + ig| := f \boxplus g \quad (f, g \in E_\rho).$$

The notion of the positive cone of a vector lattice naturally extends to complex vector lattices. Indeed, the *positive cone* of an Archimedean complex vector lattice E is the set $E_+ := \{f \in E : |f| = f\}$. Note that E_+ is also the positive cone of E_ρ .

A key observation to be made is that every universally complete Archimedean (real) vector lattice is a Φ -algebra (see e.g. [8, Theorem 15.22]), allowing us to introduce the notions of power series and n th roots in these spaces. Recall that a (*real*) Φ -algebra is a (real) f -algebra possessing a multiplicative identity. The multiplication on a universally complete Archimedean complex vector lattice E will always be indicated by juxtaposition, and we denote the multiplicative identity of E by e throughout. Observe that the Dedekind completeness of E implies that E is square mean closed, and also note that the multiplication on E canonically extends to $E \oplus iE$. Therefore, $E \oplus iE$ is an Archimedean complex vector lattice endowed with a complex Φ -algebra structure. For more information on complex Φ -algebras, see [7].

Given an Archimedean (real) f -algebra A , $a \in A_+$, and $n \in \mathbb{N}$, if there exists $r \in A_+$ such that $r^n = a$, we call r an *n th root* of a . In a uniformly complete Archimedean complex Φ -algebra A , there exists a unique n th root of a for every $a \in A_+$ and all $n \in \mathbb{N}$ (see [6, Corollary 6]). This n th root of a is denoted by $a^{1/n}$.

We proceed by recording some fundamental properties of an Archimedean complex Φ -algebra A that we will repeatedly use throughout the next section. Real Φ -algebra analogues of statements (i)–(iv) can be found in [18, Section 142], and (v) is a consequence of (iv), which follows from [2, Theorem 2.37]. Finally, [2, Theorem 2.44] and [11, Theorem 18.13] yield (vi).

For all $a, b \in A$ and any order projection \mathbb{P} on A ,

- (i) $ab = ba$.
- (ii) $|ab| = |a||b|$.
- (iii) $|a| \wedge |b| = 0$ if and only if $ab = 0$.
- (iv) If a is invertible and positive then a^{-1} is positive.
- (iv) $\mathbb{P}(ab) = a\mathbb{P}(b)$.
- (v) $\mathbb{P}(ab) = \mathbb{P}(a)\mathbb{P}(b)$.
- (vi) \mathbb{P} is an order continuous normal Riesz homomorphism.

We conclude this section by introducing the order topology on Archimedean complex vector lattices, which we use to develop the notion of convergence of series on these spaces. Recall that a sequence $(x_n)_{n \geq 0}$ in a (real) vector lattice *converges in order* to x , which we denote by $x_n \rightarrow x$, if there exists a decreasing sequence $(p_n)_{n \geq 0}$ with $\inf_{n \geq 0} p_n = 0$ (in symbols $p_n \downarrow 0$) and $|x_n - x| \leq p_n$ for all $n \in \mathbb{N}_0$. As noted in [11, page 80], a (real) vector lattice can be endowed with the corresponding order topology. We call a sequence $(x_n)_{n \geq 0}$ an *order Cauchy sequence* if there exists a sequence $(p_n)_{n \geq 0}$ satisfying $p_n \downarrow 0$ for which $|x_n - x_m| \leq p_n$ for all $m, n \in \mathbb{N}_0$ with $n \leq m$. A vector lattice E with the property that every order Cauchy sequence converges in order in E is called *order Cauchy complete*. It is known that every Dedekind complete vector lattice is order Cauchy complete (see e.g. [19, Exercise 10.6]).

The modulus naturally lends itself to analogous notions of order convergence, order Cauchy sequences, and order Cauchy completeness on complex vector lattices. In this paper we equip Archimedean complex vector lattices with the natural complex analogue of the aforementioned order topology.

Given an order bounded sequence $(x_n)_{n \geq 0}$ in a Dedekind σ -complete (real) vector lattice E , we define

$$\limsup_{n \rightarrow \infty} x_n := \inf_{n \geq 0} \sup_{m \geq n} x_m \quad \text{and} \quad \liminf_{n \rightarrow \infty} x_n := \sup_{n \geq 0} \inf_{m \geq n} x_m.$$

Note that $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$ are in fact the order limits of the sequences $(\sup_{m \geq n} x_m)_{n \geq 0}$ and $(\inf_{m \geq n} x_m)_{n \geq 0}$, respectively.

3. SERIES AND POWER SERIES

A *series* in an Archimedean complex vector lattice E is an infinite sum $\sum_{n=0}^{\infty} a_n$, where $a_n \in E$ for all $n \in \mathbb{N}_0$. We say $\sum_{n=0}^{\infty} a_n$ *converges in order* if the sequence $(\sum_{n=0}^m a_n)_{m \geq 0}$ of partial sums has an order limit in E . Moreover, $\sum_{n=0}^{\infty} a_n$ is said to *converge absolutely in order* if $\sum_{n=0}^{\infty} |a_n|$ converges in order. It is readily verified that if E is order Cauchy complete then any series in E that converges absolutely in order also converges in order.

We proceed with the divergence test for series in Archimedean complex vector lattices, whose standard proof is left to the reader.

Lemma 3.1 (The Divergence Test). *Let E be an Archimedean complex vector lattice. If the series $\sum_{n=0}^{\infty} a_n$ in E converges in order then $a_n \rightarrow 0$.*

The following notation will be used frequently throughout the remainder of the paper.

Notation 3.2. Given an Archimedean (real) vector lattice E and $x, y \in E$, we write $x \ll y$ if $(y - x)^+$ is a weak order unit in E .

For example, in the space $\ell_{\infty}(\mathbb{N})$ of all bounded sequences, two elements $x, y \in \ell_{\infty}(\mathbb{N})$ satisfy $x \ll y$ if and only if $x_n < y_n$ for every $n \in \mathbb{N}$.

Remark 3.3. Every positive invertible element in an Archimedean (real) Φ -algebra is a weak order unit [18, Theorem 142.2(ii)]. The converse however is not true in general. For example, the sequence $(x_n)_{n \geq 1}$ defined by $x_n = \frac{1}{n}$ ($n \in \mathbb{N}$) is a weak order unit in the space of all convergent real-valued sequences c but is not invertible in c . However, if E is a universally complete Archimedean (real) vector lattice then E can be identified with $C^{\infty}(X)$ for some X that is an extremely disconnected compact Hausdorff space, see [8, Theorem 15.22]. Using this identification, we can easily see that every weak order unit u in a universally complete Archimedean complex vector lattice E is invertible. Indeed, if u is a weak order unit of $C^{\infty}(X)$, we have from [2, Theorem 1.38] that

$$\mathbf{1} = \sup_{n \geq 1} \{\mathbf{1} \wedge nu\},$$

where $\mathbf{1}$ is the constant function on X taking the value 1. It follows that there does not exist a dense subset Y of X for which $u(x) = 0$ for all $x \in Y$. Hence u has a continuous inverse in $C^{\infty}(X)$.

We proceed by providing a useful characterization of the relation \ll introduced in Notation 3.2 in the context of universally complete Archimedean (real) vector lattices.

Lemma 3.4. *Let E be a universally complete Archimedean (real) vector lattice, and let $x, y \in E$. Then $x \ll y$ if and only if $x \leq y$ and $y - x$ is invertible.*

Proof. Suppose $x \ll y$, so $(y - x)^+$ is a weak order unit. From Remark 3.3, $(y - x)^+$ is invertible. Moreover, we have from [2, Theorem 1.38] that

$$(y - x)^- = \sup_{n \in \mathbb{N}} (y - x)^- \wedge n(y - x)^+ = 0.$$

Thus $x \leq y$.

Conversely, assume $x \leq y$ and that $y - x$ is invertible. Then $(y - x)^+ = y - x$ is a weak order unit by Remark 3.3. \square

The following lemma regarding the convergence of the geometric series will play a crucial role in the proof of the n th root test Theorem 3.6 and Abel's theorem Proposition 3.12. In a universally complete Archimedean (real or complex) vector lattice E and $a \in E$ we adopt the standard convention that $a^0 = e$.

Lemma 3.5. *Let E be a universally complete Archimedean complex vector lattice. The geometric series $\sum_{n=0}^{\infty} a^n$ in E converges absolutely in order if and only if $|a| \ll e$. In this case, $\sum_{n=0}^{\infty} a^n = (e - a)^{-1}$.*

Proof. Suppose $\sum_{n=0}^{\infty} a^n$ converges absolutely in order. Note that for $m \in \mathbb{N}_0$,

$$(e - |a|) \sum_{n=0}^m |a|^n = e - |a|^{m+1}. \quad (2)$$

Moreover,

$$(e - |a|) \sum_{n=0}^m |a|^n \rightarrow (e - |a|) \sum_{n=0}^{\infty} |a|^n,$$

while by Lemma 3.1,

$$e - |a|^{m+1} \rightarrow e.$$

Since order limits are unique, we have

$$(e - |a|) \sum_{n=0}^{\infty} |a|^n = e.$$

It now follows from Lemma 3.4 that $|a| \ll e$.

On the other hand, if $|a| \ll e$ then $|a| \leq e$ and $e - |a|$ is invertible by Lemma 3.4. Thus for each $m \in \mathbb{N}_0$,

$$\sum_{n=0}^m |a|^n = (e - |a|)^{-1} (e - |a|^{m+1}) \leq (e - |a|)^{-1},$$

hence $\sum_{n=0}^{\infty} |a|^n$ converges in order, as E is Dedekind complete. It follows from Lemma 3.1 that $|a|^n \rightarrow 0$. Hence, in light of (2), we obtain $\sum_{n=0}^{\infty} |a|^n = (e - |a|)^{-1}$.

Finally, since E is order Cauchy complete, $\sum_{n=0}^{\infty} a^n$ also converges in order. A similar argument shows that $\sum_{n=0}^{\infty} a^n = (e - a)^{-1}$. \square

Next we provide an order-theoretic version of the classical n th root test.

Theorem 3.6 (nth Root Test). *Let E be a universally complete Archimedean complex vector lattice, and let $\sum_{n=0}^{\infty} a_n$ be a series in E .*

- (i) *If the sequence $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \ll e$ then the series $\sum_{n=0}^{\infty} a_n$ converges absolutely in order.*
- (ii) *If the series $\sum_{n=0}^{\infty} a_n$ converges in order then the sequence $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq e$.*
- (iii) *If $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and we have that both $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \not\ll e$ and $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq e$ hold then this test is inconclusive.*

Proof. (i) Suppose the sequence $(|a_n|^{1/n})_{n \geq 1}$ is order bounded, and also assume that

$$L := \limsup_{n \rightarrow \infty} |a_n|^{1/n} \ll e.$$

Put $b_m := \sup_{n \geq m} |a_n|^{1/n}$ for $m \in \mathbb{N}$, and set $b_0 := e$. For each $m \in \mathbb{N}_0$ define $B_{b_m < e}$ to be the principal band generated by $(e - b_m)^+$, and set $\mathbb{P}_{b_m < e}$ to be the corresponding order projection onto $B_{b_m < e}$. Moreover, set $B_{b_m \geq e}$ to be the disjoint complement of $B_{b_m < e}$ and denote the corresponding order projection by $\mathbb{P}_{b_m \geq e}$ for each $m \in \mathbb{N}_0$. Define

$$\mathbb{Q}_m := \mathbb{P}_{b_m < e} - \mathbb{P}_{b_{m-1} < e} \quad (m \in \mathbb{N}).$$

Then for $m, p \in \mathbb{N}$ with $m < p$ and $a \in E_+$ we have

$$\begin{aligned} 0 \leq \mathbb{Q}_m(a) \wedge \mathbb{Q}_p(a) &= (\mathbb{P}_{b_m < e}(a) - \mathbb{P}_{b_{m-1} < e}(a)) \wedge (\mathbb{P}_{b_p < e}(a) - \mathbb{P}_{b_{p-1} < e}(a)) \\ &\leq \mathbb{P}_{b_{p-1} < e}(a) \wedge (a - \mathbb{P}_{b_{p-1} < e}(a)) \\ &= 0. \end{aligned}$$

Thus $(\mathbb{Q}_m)_{m \geq 1}$ is a pairwise disjoint sequence of order projections. Next we claim that $\mathbb{P}_{b_m < e}(b_m) \ll e$ for all $m \in \mathbb{N}_0$. Indeed, for $m \in \mathbb{N}_0$ we have

$$e - \mathbb{P}_{b_m < e}(b_m) \geq \mathbb{P}_{b_m < e}(e - b_m) = (e - b_m)^+ \geq 0, \quad (3)$$

where the above equality follows from [19, page 215]. Also, note that

$$\begin{aligned} e - \mathbb{P}_{b_m < e}(b_m) &= \mathbb{P}_{b_m < e}(e - \mathbb{P}_{b_m < e}(b_m)) + \mathbb{P}_{b_m \geq e}(e - \mathbb{P}_{b_m < e}(b_m)) \\ &= \mathbb{P}_{b_m < e}(e - \mathbb{P}_{b_m < e}(b_m)) + \mathbb{P}_{b_m \geq e}(e). \end{aligned}$$

Note that by (3), we have that $\mathbb{P}_{b_m < e}(e - \mathbb{P}_{b_m < e}(b_m)) \geq \mathbb{P}_{b_m < e}(e - b_m)^+ = (e - b_m)^+$, which shows that $\mathbb{P}_{b_m < e}(e - \mathbb{P}_{b_m < e}(b_m))$ is a weak order unit in $B_{b_m < e}$. Since we also have that $\mathbb{P}_{b_m \geq e}(e)$ is a weak order unit in $B_{b_m \geq e}$, we conclude that $e - \mathbb{P}_{b_m < e}(b_m)$ is a weak order unit. Thus $\mathbb{P}_{b_m < e}(b_m) \ll e$, as claimed.

Next let $m, p \in \mathbb{N}$ with $m \leq p$. Recalling that $b_m := \sup_{n \geq m} |a_n|^{1/n}$ for $m \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{n=0}^p \mathbb{Q}_m(|a_n|) &\leq \sum_{n=0}^p \mathbb{P}_{b_m < e}(|a_n|) = \sum_{n=0}^{m-1} \mathbb{P}_{b_m < e}(|a_n|) + \sum_{n=m}^p \mathbb{P}_{b_m < e}(|a_n|) \\ &\leq \sum_{n=0}^{m-1} \mathbb{P}_{b_m < e}(|a_n|) + \sum_{n=m}^{\infty} (\mathbb{P}_{b_m < e}(b_m))^n, \end{aligned}$$

and we stress that $\sum_{n=m}^{\infty} (\mathbb{P}_{b_m < e}(b_m))^n$ converges in order by Lemma 3.5. Hence

$$\sup_{p \geq m} \sum_{n=0}^p \mathbb{Q}_m(|a_n|) \in E,$$

and it follows that $\sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|)$ converges in order for all $m \in \mathbb{N}$. Also, for all $m, p \in \mathbb{N}$ with $m \neq p$ and each $k, l \in \mathbb{N}_0$ we have $\mathbb{Q}_m(|a_k|) \perp \mathbb{Q}_p(|a_l|)$. Thus $\mathbb{Q}_m(|a_k|)\mathbb{Q}_p(|a_l|) = 0$ for all such k, l, m, p . It follows that for $t_1, t_2 \in \mathbb{N}$,

$$\sum_{n=0}^{t_1} \mathbb{Q}_m(|a_n|) \sum_{n=0}^{t_2} \mathbb{Q}_p(|a_n|) = 0,$$

and by the order continuity of f -algebra multiplication, with the limits below taken to be order limits, we have for $m \neq p$ that

$$\sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|) \sum_{n=0}^{\infty} \mathbb{Q}_p(|a_n|) = \lim_{t_1, t_2 \rightarrow \infty} \sum_{n=0}^{t_1} \mathbb{Q}_m(|a_n|) \sum_{n=0}^{t_2} \mathbb{Q}_p(|a_n|) = 0.$$

Thus $\sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|) \perp \sum_{n=0}^{\infty} \mathbb{Q}_p(|a_n|)$ holds for all $m \neq p$, and hence

$$\sup_{m \geq 1} \sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|) \in E.$$

Furthermore, for any $k, p \in \mathbb{N}$ we have $\mathbb{Q}_k(\sum_{n=0}^p |a_n|) \leq \sup_{m \geq 1} \sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|)$. Hence

$$\sup_{k \geq 1} \mathbb{Q}_k \left(\sum_{n=0}^p |a_n| \right) \leq \sup_{m \geq 1} \sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|). \quad (4)$$

Using [19, Theorem 8.2(1)] in the second equality below, we obtain, with the limits below taken to be order limits,

$$\begin{aligned} \sup_{k \geq 1} \mathbb{Q}_k \left(\sum_{n=0}^p |a_n| \right) &= \lim_{m \rightarrow \infty} \sup_{k \leq m} \mathbb{Q}_k \left(\sum_{n=0}^p |a_n| \right) = \lim_{m \rightarrow \infty} \sum_{k=1}^m \mathbb{Q}_k \left(\sum_{n=0}^p |a_n| \right) \\ &= \lim_{m \rightarrow \infty} \mathbb{P}_{b_m < e} \left(\sum_{n=0}^p |a_n| \right). \end{aligned}$$

Next note that $((e - b_m)^+)_{m \geq 1}$ is an increasing sequence with supremum $(e - L)^+ = e - L$, which is by assumption a weak order unit. Also, for each $m \in \mathbb{N}$ the band B generated by $\bigcup_{m=1}^{\infty} B_{b_m < e}$ contains $(e - b_m)^+$, so $e - L \in B$, as bands are order closed. Therefore, $B = E$. It now follows from [11, Theorems 30.4 and 30.5(ii)] that

$$\lim_{m \rightarrow \infty} \mathbb{P}_{b_m < e} \left(\sum_{n=0}^p |a_n| \right) = \sum_{n=0}^p |a_n|.$$

Therefore, by (4) and the string of equalities which follow, we have

$$\sum_{n=0}^p |a_n| \leq \sup_{m \geq 1} \sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|)$$

for every $p \in \mathbb{N}$. By the Dedekind completeness of E , we have that

$$\sup_{p \geq 1} \sum_{n=0}^p |a_n| \in E.$$

We conclude that $\sum_{n=0}^{\infty} |a_n|$ converges in order, and in particular,

$$\sum_{n=0}^{\infty} |a_n| = \sup_{m \geq 1} \sum_{n=0}^{\infty} \mathbb{Q}_m(|a_n|).$$

(ii) Suppose $\sum_{n=0}^{\infty} a_n$ converges in order. Then $a_n \rightarrow 0$ by Lemma 3.1. Thus there exists a sequence $p_n \downarrow 0$ such that $|a_n| \leq p_n$ for all $n \in \mathbb{N}_0$. By the proof of [7, Lemma 4.2], we have for any $n \in \mathbb{N}$,

$$\begin{aligned} |a_n|^{1/n} &= \inf \left\{ \frac{1}{n} \theta_1 |a_n| + \left(1 - \frac{1}{n}\right) \theta_2 e : \theta_1, \theta_2 \in (0, \infty), \theta_1^{1/n} \theta_2^{1-1/n} = 1 \right\} \\ &\leq \frac{1}{n} |a_n| + \left(1 - \frac{1}{n}\right) e. \end{aligned}$$

Hence for all $m, n \in \mathbb{N}$ with $m \geq n$ we have $\sup_{n \geq m} |a_n|^{1/n} \leq \frac{1}{m} p_m + e$. Taking $m \rightarrow \infty$ now yields $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq e$.

(iii) For any Archimedean complex Φ -algebra A , the series $\sum_{n=1}^{\infty} \frac{1}{n^2} e$ converges in order, while

$$\limsup_{n \rightarrow \infty} \left(\frac{1}{n^2} e \right)^{1/n} = e.$$

On the other hand, consider the universally complete Archimedean complex vector lattice $E := \mathbb{C}^{\mathbb{C}}$. For each $n \in \mathbb{N}$ define $f_n := \chi_{\bar{D}_{1/n}(0)}$, where $\chi_{\bar{D}_{1/n}(0)}$ is the characteristic function on the closed disk $\bar{D}_{1/n}(0)$ centered at 0 with radius $1/n$. Then $|f_n|^{1/n} = |f_n|$ for every $n \in \mathbb{N}$ and

$$L := \limsup_{n \rightarrow \infty} |f_n|^{1/n} = \chi_{\{0\}} \ll \mathbf{1}$$

and $L \leq \mathbf{1}$, where again $\mathbf{1}$ denotes the constant function on \mathbb{C} taking the value 1. Note that for each $m \in \mathbb{N}$ we have $\sum_{n=1}^m f_n(0) = m$, and therefore the series $\sum_{n=1}^{\infty} f_n$ diverges. \square

The following remark illustrates the necessity of the universal completeness assumption in both Lemma 3.5 and Theorem 3.6(i).

Remark 3.7. Consider the non-universally complete Archimedean complex Φ -algebra A of complex-valued bounded continuous functions defined on $(0, 1)$. Let $f \in A$ be the identity function on $(0, 1)$. Observe that $f = |f| \ll \mathbf{1}$ but the series $\sum_{n=0}^{\infty} f^n$ does not converge in order in A .

A *power series* on a universally complete Archimedean complex vector lattice E , centered at $c \in E$, is a series of the form $S(z) := \sum_{n=0}^{\infty} a_n(z - c)^n$, where $a_n \in E$ for each $n \in \mathbb{N}_0$ and z represents a variable in E . We say a power series $S(z)$ *converges uniformly in order* on a region $D \subseteq E$ if there exists a sequence $p_m \downarrow 0$ such that

$$\sup_{z \in D} \left| \sum_{n=0}^m a_n(z - c)^n - S(z) \right| \leq p_m$$

holds for each $m \in \mathbb{N}_0$.

Given $c \in E$ and $r \in E_+$, we use the notation $\bar{\Delta}(c, r) := \{z \in E : |z - c| \leq r\}$ to indicate the order closed ball centered at c with radius r . For a power series $S(z) = \sum_{n=0}^{\infty} a_n(z - c)^n$ we define

$$\Omega_S := \{r \in E_+ : S(z) \text{ converges uniformly in order on } \bar{\Delta}(c, r)\}.$$

If Ω_S is order bounded then $\sup \Omega_S$ exists in E_+ , and we call $\rho_S := \sup \Omega_S$ the *radius of convergence* of $S(z)$.

For example, if we consider the geometric series $G(z) = \sum_{n=0}^{\infty} z^n$ on a universally complete Archimedean complex vector lattice then for $r \ll e$ and $z \in \bar{\Delta}(0, r)$ we have that

$$\left| \sum_{n=0}^m z^n - G(z) \right| \leq \sum_{n=m+1}^{\infty} |z|^n \leq \sum_{n=m+1}^{\infty} r^n \downarrow_m 0.$$

Hence $G(z)$ converges uniformly in order on $\bar{\Delta}(0, r)$, and thus $r \in \Omega_G$. On the other hand, by Lemmas 3.4 and 3.5, we know $r \leq e$ for all $r \in \Omega_G$. Thus Ω_G is order bounded and $\rho_G \leq e$. Furthermore, it follows from the above argument that $\epsilon e \in \Omega_G$ for all $0 \leq \epsilon < 1$, showing that $\rho_G = e$.

The following lemma will be useful throughout the remainder of the paper.

Lemma 3.8. *Let A be a uniformly complete Archimedean complex Φ -algebra. For all $B \subseteq A_+$ such that $\sup B$ exists in A_+ , respectively, $\inf B$ exists in A_+ , we have*

$$\sup(aB) = a \sup B, \quad \text{respectively} \quad \inf(aB) = a \inf B \quad (a \in A_+).$$

Proof. Let $B \subseteq A_+$ be such that $\sup B$ exists in A_+ , and let $a \in A_+$. For every $b \in B$ we have $ab \leq a \sup B$. Next let $ab \leq u$ for every $b \in B$. Then $(a + e)b \leq u + b \leq u + \sup B$. Note that A is e -uniformly complete by [14, Theorem 11.4], and therefore it follows from [14, Theorem 11.1] that $a + e$ is invertible in A . Hence $b \leq (a + e)^{-1}(u + \sup B)$. Therefore, $\sup B \leq (a + e)^{-1}(u + \sup B)$, which implies $a \sup B \leq u$. A similar argument proves the second identity. \square

We next record some useful facts regarding the set Ω_S with respect to a power series S .

Proposition 3.9. *Let $S(z) := \sum_{n=0}^{\infty} a_n(z - c)^n$ be a power series on a universally complete Archimedean complex vector lattice E , and let Ω_S be defined as above.*

- (i) Ω_S is a solid sublattice of E_+ .
- (ii) *If the sequence $(|a_n|^{1/n})_{n \geq 1}$ is order bounded then, denoting the band generated by $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ by B_L and its disjoint complement by B_L^d , we have the inclusion $(B_L^d)_+ \subseteq \Omega_S$.*

Proof. (i) It is clear from its definition that Ω_S is a solid set, and thus Ω_S is closed under finite infima. To prove that Ω_S is closed under finite suprema, let $r, s \in \Omega_S$. By using a change of variables if necessary, we can assume that $c = 0$. Then $S(z)$ converges uniformly in order on $\bar{\Delta}(0, r)$ and also on $\bar{\Delta}(0, s)$. Using [19, page 215] in the second identity below, we obtain

$$\mathbb{P}_{r < s}(s) - \mathbb{P}_{r < s}(r) = \mathbb{P}_{r < s}(s - r) = (s - r)^+ \geq 0.$$

Thus

$$\mathbb{P}_{r < s}(r \vee s) = \mathbb{P}_{r < s}(r) \vee \mathbb{P}_{r < s}(s) = \mathbb{P}_{r < s}(s).$$

It follows that for $|z| \leq r \vee s$,

$$s - \mathbb{P}_{r < s}(|z|) \geq s - \mathbb{P}_{r < s}(r \vee s) = s - \mathbb{P}_{r < s}(s) \geq 0,$$

and thus $\mathbb{P}_{r < s}(|z|) \leq s$. Hence $\sum_{n=0}^{\infty} a_n \mathbb{P}_{r < s}(z)^n$ converges uniformly in order on $\bar{\Delta}(0, r \vee s)$. Similarly, $\sum_{n=0}^{\infty} a_n \mathbb{P}_{r \geq s}(z)^n$ also converges uniformly in order on $\bar{\Delta}(0, r \vee s)$. Finally, observe that for all $|z| \leq r \vee s$ and any $m \in \mathbb{N}_0$,

$$\sum_{n=0}^m a_n z^n = \sum_{n=0}^m a_n \mathbb{P}_{r < s}(z)^n + \sum_{n=0}^m a_n \mathbb{P}_{r \geq s}(z)^n.$$

It now follows that $S(z)$ converges uniformly in order on $\bar{\Delta}(0, r \vee s)$. Therefore, $r \vee s \in \Omega_S$.

(ii) Suppose $(|a_n|^{1/n})_{n \geq 1}$ is order bounded, and also set $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$. Let $r \in (B_L^d)_+$, and take $z \in E$ such that $|z - c| \leq r$. Since $L \perp |z - c|$, we have $L(z - c) = 0$. Using Lemma 3.8 in the second equality below we obtain

$$\limsup_{n \rightarrow \infty} (|a_n| |z - c|^n)^{1/n} = \limsup_{n \rightarrow \infty} (|a_n|^{1/n} |z - c|) = L |z - c| = 0 \ll e.$$

Hence $S(z)$ converges absolutely in order on $\bar{\Delta}(c, r)$ by Theorem 3.6(i). Moreover, for each $m \in \mathbb{N}_0$ and all $z \in \bar{\Delta}(0, r)$,

$$\left| \sum_{n=0}^m a_n (z - c)^n - S(z) \right| \leq \sum_{n=m+1}^{\infty} |a_n| r^n \downarrow_m 0,$$

and thus $S(z)$ converges uniformly in order on $\bar{\Delta}(c, r)$ as well. Therefore, $r \in \Omega_S$. \square

Given a universally complete Archimedean complex vector lattice E , we denote the principle band generated by the element a inside E by B_a and the associated band projection onto B_a by \mathbb{P}_a . We note that, for each $a \in E \setminus \{0\}$, it is readily checked that B_a is a universally complete f -subalgebra of E that has $\mathbb{P}_a(e)$ as its multiplicative identity. Moreover, if we have $a \in E \setminus \{0\}$ then a is a weak order unit in B_a and thus has a multiplicative inverse in B_a .

Notation 3.10. For a universally complete Archimedean complex vector lattice E and $a \in E$, $a \neq 0$, we denote the multiplicative inverse of a in B_a by a^* . Moreover, we set $0^* := 0$.

We proceed by proving a universally complete Archimedean complex vector lattice version of the classical Cauchy-Hadamard formula.

Theorem 3.11 (Cauchy-Hadamard). *Let E be a universally complete Archimedean complex vector lattice, and let $S(z) := \sum_{n=0}^{\infty} a_n(z-c)^n$ be a power series on E .*

- (i) *If $(|a_n|^{1/n})_{n \geq 1}$ is order bounded then for $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ we have that $\mathbb{P}_L(\Omega_S)$ is order bounded and $L^* = \sup_{r \in \Omega_S} \mathbb{P}_L(r)$.*
- (ii) *If Ω_S is order bounded then $(\mathbb{P}_{\rho_S}(|a_n|^{1/n}))_{n \geq 1}$ is order bounded and furthermore we have $\limsup_{n \rightarrow \infty} \mathbb{P}_{\rho_S}(|a_n|^{1/n}) = \rho_S^*$.*
- (iii) *$(|a_n|^{1/n})_{n \geq 1}$ is order bounded and $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ is a weak order unit if and only if Ω_S is order bounded and ρ_S is a weak order unit. In either case, we have $L^{-1} = \rho_S$.*
- (iv) *$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 0$ if and only if $\Omega_S = E_+$.*

Proof. (i) Suppose $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and set $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$. If $L = 0$ then statement (i) trivially holds, so we can assume $L \neq 0$. Let $0 \leq \epsilon < 1$, and suppose $z \in E$ satisfies $|z - c| \leq \epsilon L^*$. Using Lemma 3.8 in the second equality, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} |a_n(z-c)^n|^{1/n} &= \limsup_{n \rightarrow \infty} \left(|a_n|^{1/n} |z-c| \right) \leq \epsilon \limsup_{n \rightarrow \infty} \left(|a_n|^{1/n} L^* \right) \\ &= \epsilon L L^* = \epsilon \mathbb{P}_L(e) \ll e. \end{aligned}$$

It follows from Theorem 3.6(i) that $S(z)$ converges absolutely in order on $\bar{\Delta}(c, \epsilon L^*)$. Since E is order Cauchy complete, we know that $S(z)$ converges in order on $\bar{\Delta}(c, \epsilon L^*)$. Moreover, for any $z \in \bar{\Delta}(c, \epsilon L^*)$, we have

$$\left| \sum_{n=0}^m a_n(z-c)^n - S(z) \right| \leq \sum_{n=m+1}^{\infty} |a_n| (\epsilon L^*)^n \downarrow_m 0,$$

showing that $S(z)$ converges uniformly in order on $\bar{\Delta}(c, \epsilon L^*)$. Hence $\epsilon L^* \in \Omega_S$. Next let $r \in \Omega_S$ be arbitrary. Then by Lemma 3.8 we have

$$\limsup_{n \rightarrow \infty} \left(|a_n|^{1/n} r \right) = Lr,$$

and we also know that $Lr \leq e$ from Theorem 3.6(ii). Thus $L\mathbb{P}_L(r) \leq \mathbb{P}_L(e)$, and hence $\mathbb{P}_L(r) \leq L^*$ for all $r \in \Omega_S$. Hence $\mathbb{P}_L(\Omega_S)$ is order bounded. Set

$$\rho'_S := \sup_{r \in \Omega_S} \mathbb{P}_L(r).$$

Then $\rho'_S \leq L^*$. Moreover, since $\epsilon L^* \leq \rho'_S$ holds for all $0 \leq \epsilon < 1$, we also have $L^* \leq \rho'_S$.

(ii) Next suppose Ω_S is order bounded. If $\rho_S = 0$ then statement (ii) of the theorem is trivial, so we can assume that $\rho_S > 0$. Let $r \in \Omega_S$. By Theorem 3.6(ii), we know

$$\limsup_{n \rightarrow \infty} (|a_n|^{1/n} r) \leq e.$$

Thus there exists a sequence $p_m \downarrow 0$ such that

$$\begin{aligned} \sup_{n \geq m} (|a_n|^{1/n} r) &= \sup_{n \geq m} (|a_n|^{1/n} r) - \limsup_{n \rightarrow \infty} (|a_n|^{1/n} r) + \limsup_{n \rightarrow \infty} (|a_n|^{1/n} r) \\ &\leq p_m + e \end{aligned}$$

holds for each $m \in \mathbb{N}$. Thus for every $m \in \mathbb{N}$ and all $n \geq m$, it follows from Lemma 3.8 that

$$|a_n|^{1/n} \rho_S = \sup_{r \in \Omega_S} \{|a_n|^{1/n} r\} \leq p_m + e,$$

and hence

$$\mathbb{P}_{\rho_S}(|a_n|^{1/n}) \rho_S \leq \mathbb{P}_{\rho_S}(p_m) + \mathbb{P}_{\rho_S}(e).$$

Thus we obtain

$$\sup_{n \geq m} \mathbb{P}_{\rho_S}(|a_n|^{1/n}) \leq \mathbb{P}_{\rho_S}(p_m) \rho_S^* + \rho_S^*,$$

showing that $(\mathbb{P}_{\rho_S}(|a_n|^{1/n}))_{n \geq 1}$ is order bounded. Taking $m \rightarrow \infty$, we see that

$$L' := \limsup_{n \rightarrow \infty} \mathbb{P}_{\rho_S}(|a_n|^{1/n}) \leq \rho_S^*.$$

Next applying Proposition 3.9 (ii) to the power series

$$R(w) = \sum_{n=0}^{\infty} \mathbb{P}_{\rho_S}(a_n) (w - \mathbb{P}_{\rho_S}(c))^n$$

on B_{ρ_S} , we have $(B_{L'}^d)_+ \subseteq \Omega_R$, where $B_{L'}$ is the principal band generated by L' inside B_{ρ_S} . We next prove that $\Omega_R \subseteq \Omega_S$. To this end, let $r \in \Omega_R$. Then $R(w)$ converges uniformly in order on $\bar{\Delta}(\mathbb{P}_{\rho_S}(c), r)$. To show that $r \in \Omega_S$, suppose $z \in E$ satisfies $|z - c| \leq r$. Then $z - c \in B_{\rho_S}$, and thus for every $n \in \mathbb{N}$ we have that $a_n(z - c)^n \in B_{\rho_S}$ as well. Hence for each $m \in \mathbb{N}$ we have

$$\sum_{n=1}^m a_n (z - c)^n = \sum_{n=1}^m \mathbb{P}_{\rho_S}(a_n) (\mathbb{P}_{\rho_S}(z) - \mathbb{P}_{\rho_S}(c))^n.$$

Next let $p_m \downarrow 0$ satisfy

$$\sup_{w \in \bar{\Delta}(\mathbb{P}_{\rho_S}(c), r)} \left| \sum_{n=0}^m \mathbb{P}_{\rho_S}(a_n) (w - \mathbb{P}_{\rho_S}(c))^n - R(w) \right| \leq p_m \quad (m \in \mathbb{N}_0).$$

Then

$$\begin{aligned} \left| \sum_{n=1}^m a_n(z-c)^n - \left(R(\mathbb{P}_{\rho_S}(z)) - \mathbb{P}_{\rho_S}(a_0) \right) \right| &= \left| \sum_{n=0}^m \mathbb{P}_{\rho_S}(a_n) (\mathbb{P}_{\rho_S}(z) - \mathbb{P}_{\rho_S}(c))^n - R(\mathbb{P}_{\rho_S}(z)) \right| \\ &\leq \sup_{w \in \bar{\Delta}(\mathbb{P}_{\rho_S}(c), r)} \left| \sum_{n=0}^m \mathbb{P}_{\rho_S}(a_n) (w - \mathbb{P}_{\rho_S}(c))^n - R(w) \right| \leq p_m, \end{aligned}$$

and hence $S(z)$ converges uniformly in order on $\bar{\Delta}(c, r)$, where we have

$$S(z) = R(\mathbb{P}_{\rho_S}(z)) - \mathbb{P}_{\rho_S}(a_0) + a_0.$$

This shows that $r \in \Omega_S$, and thus we obtain $(B_{L'}^d)_+ \subseteq \Omega_S$. Noting that Ω_S is order bounded, we see that L' is a weak order unit in B_{ρ_S} . Applying (i) to the space B_{ρ_S} , we have that $L' = \rho_S^*$.

(iii) Suppose $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and that $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ is a weak order unit. It follows from (i) that Ω_S is order bounded, that ρ_S is a weak order unit and that $L^{-1} = \rho_S$. Conversely, if Ω_S is order bounded and ρ_S is a weak order unit then it follows from (ii) that the sequence $(|a_n|^{1/n})_{n \geq 1}$ is order bounded and that $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ is a weak order unit that satisfies $L^{-1} = \rho_S$.

(iv) Suppose $L := \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 0$. Then $E_+ = (B_L^d)_+ \subseteq \Omega_S$ by Proposition 3.9 (ii), and thus $\Omega_S = E_+$. Conversely, if $\Omega_S = E_+$ then Theorem 3.6(ii) and the fact that $ne \in \Omega_S$ for all $n \in \mathbb{N}$ imply that $S(ne + c)$ converges in order, that L exists in E , and that $nL \leq e$ for all $n \in \mathbb{N}$. By the Archimedean property of E , we conclude that $L = 0$. \square

In the classical theory of power series, Abel's theorem relates the limit of a power series to its series of coefficients. Given that the tools used in the classical theorem of Abel are elementary in nature, it is not surprising that the proof of this theorem (see e.g. [1, Section 2.5]) adapts to a more general version in universally complete Archimedean complex vector lattices.

Proposition 3.12 (Abel's Theorem). *Let E be a universally complete Archimedean complex vector lattice, and assume that $S(z) := \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence e . Suppose $\sum_{n=0}^{\infty} a_n$ converges in order. Then for any sequence $(z_k)_{k \geq 0}$ in E such that*

- (i) $z_k \rightarrow e$,
- (ii) for each $k \in \mathbb{N}_0$ we have $|z_k| \ll e$, and
- (iii) the set $\{|e - z_k|(e - |z_k|)^{-1} : k \in \mathbb{N}_0\}$ is order bounded,

we have $S(z_k) \rightarrow S(e)$.

Proof. First suppose that $\sum_{n=0}^{\infty} a_n = 0$. Set $s_m = \sum_{n=0}^m a_n$ for each $m \in \mathbb{N}_0$. Since $s_m \rightarrow 0$, there exists a sequence $p_m \downarrow 0$ such that $|s_m| \leq p_m$ for all $m \in \mathbb{N}_0$. Using summation by parts, we obtain

$$\sum_{n=0}^m a_n z^n = (e - z) \left(\sum_{n=0}^{m-1} s_n z^n \right) + s_m z^m.$$

Hence, for any $r \in \Omega_S$ and all $z \in \bar{\Delta}(0, r)$ we have

$$\left| \sum_{n=0}^m a_n z^n - (e - z) \left(\sum_{n=0}^{m-1} s_n z^n \right) \right| \leq |s_m| |z|^m \leq |s_m| \leq p_m,$$

and so

$$S(z) = (e - z) \sum_{n=0}^{\infty} s_n z^n.$$

Next let $(z_k)_{k \geq 0}$ be a sequence satisfying (i), (ii), and (iii) above, and say $|e - z_k| \leq q_k$ for each $k \in \mathbb{N}_0$, with $q_k \downarrow 0$. Let $v \in E_+$ be an upper bound of the set

$$\{|e - z_k|(e - |z_k|)^{-1} : k \in \mathbb{N}_0\},$$

and let $m \in \mathbb{N}_0$ be arbitrary. Then for each $k \in \mathbb{N}_0$, we obtain from Lemma 3.5 that

$$\begin{aligned} |S(z_k)| &\leq |e - z_k| \sum_{n=0}^m |s_n| |z_k|^n + |e - z_k| \sum_{n=m+1}^{\infty} |s_n| |z_k|^n \\ &\leq q_k \sum_{n=0}^m |s_n| + p_{m+1} |e - z_k| (e - |z_k|)^{-1} \\ &\leq q_k \sum_{n=0}^m |s_n| + p_{m+1} v. \end{aligned}$$

Thus $(|S(z_k)|)_{k \geq 1}$ is order bounded. Furthermore, using Lemma 3.8 in the last equality below, we obtain

$$0 \leq \limsup_{k \rightarrow \infty} |S(z_k)| = \inf_{p \geq 1} \left(\sup_{k \geq p} |S(z_k)| \right) \leq \inf_{p \geq 1} \left\{ q_p \left(\sum_{n=0}^m |s_n| \right) + p_{m+1} v \right\} = p_{m+1} v.$$

Letting $m \rightarrow \infty$ shows that $\limsup_{k \rightarrow \infty} |S(z_k)| = 0$. Hence $\liminf_{k \rightarrow \infty} |S(z_k)| = 0$, and by [19, Theorem 12.7], we have $S(z_k) \rightarrow 0 = S(e)$.

More generally, assume $\sum_{n=0}^{\infty} a_n = a$ for some $a \in E$. Define a power series on E by $S'(z) := \sum_{n=0}^{\infty} a'_n z^n$, where $a'_0 := a_0 - a$ and $a'_n := a_n$ for all $n \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} a'_n = 0$, and by the argument above, $S'(z_k) \rightarrow 0$. This however is equivalent to $S(z_k) \rightarrow a = S(e)$. \square

Note that e in Proposition 3.12 can be replaced by any positive weak order unit in E .

Corollary 3.13. *Let $S(z) := \sum_{n=0}^{\infty} a_n z^n$ be a power series on a universally complete Archimedean complex vector lattice E with radius of convergence ρ_S which is a weak order unit. Suppose $S(\rho_S)$ converges in order, and assume that $(z_k)_{k \geq 0}$ is a sequence in E such that*

- (i) $z_k \rightarrow \rho_S$,
- (ii) for each $k \in \mathbb{N}_0$ we have $|z_k| \ll \rho_S$, and
- (iii) the set $\{|\rho_S - z_k|(\rho_S - |z_k|)^{-1} : k \in \mathbb{N}_0\}$ is order bounded.

Then $S(z_k) \rightarrow S(\rho_S)$.

Proof. Consider the power series $S'(z) = \sum_{n=0}^{\infty} a_n \rho_S^n z^n$. By using Lemma 3.8 as well as Theorem 3.11(iii), we obtain

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} \rho_S = \left(\limsup_{n \rightarrow \infty} |a_n|^{1/n} \right) \rho_S = e.$$

Hence, by Theorem 3.11(iii) again, we conclude that $\rho_{S'} = e$. Furthermore, note that $S'(e)$ converges in order. Next note that the sequence $(\rho_S^{-1} z_k)_{k \geq 0}$ satisfies the hypotheses (i), (ii), and (iii) in the statement of Proposition 3.12. By Proposition 3.12, we obtain

$$S(z_k) = S'(\rho_S^{-1} z_k) \rightarrow S'(e) = S(\rho_S).$$

□

Acknowledgment. This research was partially supported by the Claude Leon Foundation and by the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) (second author). Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

REFERENCES

- [1] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.
- [2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, Orlando, 1985.
- [3] Y. Azouzi, *Square Mean Closed Real Riesz Spaces*, Thesis, Tunis, 2008.
- [4] Y. Azzouzi, K. Boulabiar, and G. Buskes, *The de Schipper formula and squares of Riesz spaces*, Indag. Math. (N.S.) **17** (2006), no. 4, 479–496.
- [5] F. Beukers, C. Huijsmans, and B. de Pagter, *Unital embedding and complexification of f -algebras*, Math Z. **183** (1983), no. 1, 131–144.
- [6] F. Beukers and C. B. Huijsmans, *Calculus in f -algebras*, J. Austral. Math. Soc. Ser. A **37** (1984), no. 1, 110–116.

- [7] G. Buskes and C. Schwanke, *Vector lattices and f -algebras: the classical inequalities*, Banach J. Math. Anal. **12** (2018), no. 1, 191–205.
- [8] E. de Jonge and A. C. M. van Rooij, *Introduction to Riesz spaces*, Mathematisch Centrum, Amsterdam, 1977, Mathematical Centre Tracts, No. 78.
- [9] H.P. Lotz, *Über das spektrum positiver operatoren*, Math. Z. **108** (1968), 15–32.
- [10] W. A. J. Luxemburg and A. C. Zaanen, *The linear modulus of an order bounded linear transformation. I*, Nederl. Akad. Wetensch. Proc. Ser. A **74**=Indag. Math. **33** (1971), 422–434.
- [11] ———, *Riesz Spaces Vol. I*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971.
- [12] G. Mittelmeyer and M. Wolff, *Über den Absolutbetrag auf komplexen Vektorverbänden*, Math. Z. **137** (1974), 87–92.
- [13] Jorge Mujica, *Complex analysis in Banach spaces*, North-Holland Mathematics Studies, vol. 120, North-Holland Publishing Co., Amsterdam, 1986, Holomorphic functions and domains of holomorphy in finite and infinite dimensions, Notas de Matemática [Mathematical Notes], 107.
- [14] B. de Pagter, *f -algebras and Orthomorphisms*, Thesis, Leiden, 1981.
- [15] M.A. Rieffel, *A characterization of commutative group algebras and measure algebras*, Bull. Amer. Math. Soc. **69** (1963), 812–814.
- [16] ———, *A characterization of commutative group algebras and measure algebras*, Trans. Amer. Math. Soc. **116** (1965), 32–65.
- [17] W. J. de Schipper, *A note on the modulus of an order bounded linear operator between complex vector lattices*, Nederl. Akad. Wetensch. Proc. Ser. A **76** (1973), 355–367.
- [18] A. C. Zaanen, *Riesz Spaces II*, North-Holland Mathematical Library, vol. 30, North-Holland Publishing Co., Amsterdam, 1983.
- [19] ———, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, 1997.

UNIT FOR BMI, NORTH-WEST UNIVERSITY, PRIVATE BAG X6001, POTCHEFSTROOM, 2520, SOUTH AFRICA

E-mail address: mark.roelands@gmail.com

UNIT FOR BMI, NORTH-WEST UNIVERSITY, PRIVATE BAG X6001, POTCHEFSTROOM, 2520, SOUTH AFRICA

E-mail address: cmschwanke26@gmail.com