Frobenius structures, Coxeter discriminants, and supersymmetric mechanics

by

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Abstract

This thesis contains two directions both related to Frobenius manifolds.

In the first part we deal with the orbit space $\mathcal{M}_W = V/W$ of a finite Coxeter group W acting in its reflection representation V. The orbit space $\mathcal{M}_W$ carries the structure of a Frobenius manifold and admits a pencil of flat metrics of which the Saito flat metric η, defined as the Lie derivative of the W-invariant form g on V is the key object. In the main result of the first part we find the determinant of Saito metric restricted on the Coxeter discriminant strata in $\mathcal{M}_W$. It is shown that this determinant in the flat coordinates of the form g is proportional to a product of linear factors. We also find multiplicities of these factors in terms of Coxeter geometry of the stratum.

In the second part we study $\mathcal{N} = 4$ supersymmetric extensions of quantum mechanical systems of Calogero–Moser type. We show that for any $\vee$-system, in particular, for any Coxeter root system, the corresponding Hamiltonian can be extended to the supersymmetric Hamiltonian with $D(2,1;\alpha)$ symmetry. We also obtain $\mathcal{N} = 4$ supersymmetric extensions of Calogero–Moser–Sutherland systems. Thus, we construct supersymmetric Hamiltonians for the root systems BC_N, F_4 and G_2.

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Many years ago I returned home from school one day and I emphatically told my parents, "No more maths, I don't want to study it any longer!" After all this time I now understand the significance of a great teacher-mentor in someone's life. I therefore would like to thank my supervisors Misha and Ian, firstly for the opportunity that they gave me to pursue this project, and also for their support, guidance and perspective, as well as the time they have dedicated to me. I am truly grateful to them.

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To my parents

Declaration

I declare that, except where explicit reference is made to the contribution of others, this thesis is the result of my own work and has not been submitted for any other degree at the University of Glasgow or any other institution.

Chapter 1

Introduction

In this thesis we address two problems from the areas of Frobenius manifolds and supersymmetry. The structures that we consider, despite being seemingly different, share the common ground of Coxeter groups and Witten–Dijkgraaf–Verlinde–Verlinde equations.

1.1 Frobenius structures

Frobenius manifolds have rich differential-geometric properties. Despite the name, general theory is local and is nontrivial, already in a vector space case. They were introduced in the early 90s by Dubrovin [22], who in particular provided differential-geometric context to the work of the physicists E. Witten, R. Dijkgraaf, E.Verlinde, and H.Verlinde on topological field theories (TFTs) [21, 87]. Key elements of Frobenius manifolds were already developed by Dubrovin and Novikov in their study of bi-Hamiltonian structures of hydrodynamic type (see [23] and references therein). In the framework of TFTs a remarkable system of nonlinear partial differential equations for a holomorphic function F emerged which are now known as Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. They appeared as a consequence of associativity of the operator algebra of primary fields in two-dimensional topological theories (see Chapter 2).

Let us recall some key elements of Frobenius manifolds. Let M be a complex smooth n-dimensional manifold (which may be just a complex domain) equipped with a holomorphic flat metric η, that is a non-degenerate symmetric bilinear form on the complex tangent bundle TM such that the associated Levi-Civita connection ∇ for this metric has zero curvature. A Frobenius manifold is such a manifold M which also possesses some additional properties. Thus, there should exist a symmetric tensor $c \in \Gamma^3(T^*M)$ such that an associative commutative multiplication $\circ$ is defined on TM by the formula

$$\eta(x \circ y, z) = c(x, y, z), \quad x, y, z \in \Gamma(TM), \tag{1.1}$$

and a flat vector field $e \in \Gamma(TM)$, namely $\nabla e = 0$, such that e is the unity for the multiplication. The metric and multiplication are assumed to be homogeneous with respect to an additional vector field E on M, which is called Euler vector field.

The multiplication (1.1) makes TM into a family of commutative associative algebras with unity e, which is a family of Frobenius algebras. The tensor c is required to have additional symmetry properties which lead to existence of a prepotential $F = F(t^1, t^2, \ldots, t^n)$, which is a function on M. The variables t^α $(1 \le \alpha \le n)$ are flat coordinates of the metric η. Then the associativity of the multiplication $\circ$ leads to WDVV equations for F:

$$c_{\alpha\beta\lambda}(t)\eta^{\lambda\mu}c_{\mu\gamma\nu}(t) = c_{\alpha\gamma\lambda}(t)\eta^{\lambda\mu}c_{\mu\beta\nu}(t), \quad c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}, \tag{1.2}$$

for any $1 \le \alpha, \beta, \gamma, \nu \le n$. Frobenius manifolds and related structures have been studied intensively over the last three decades. They appear to have surprising connections with many areas of mathematics, perhaps most prominently with singularity theory and quantum cohomology. Already in the early 80s, K. Saito found some key structures of Frobenius manifolds in interesting examples (see [46] and references therein). Based on the theory of primitive forms of K. Saito, such structures were realised on the base spaces of the semiuniversal unfoldings of the simple hypersurface singularities A_n $(n \ge 1)$, D_n $(n \ge 4)$, and E_6, E_7, E_8.

1.1.1 Singularity theory

Frobenius structures coming from singularity theory have their origins in the close relation of singularities of holomorphic functions with the geometry of Coxeter groups. More precisely, it is known that the complexified orbit space $\mathcal{M}_W$ of an irreducible (finite) Coxeter group W is biholomorphic to the semiuniversal unfolding of the corresponding singularity (see [85] and references therein).

K. Saito proved the existence of a flat structure on $\mathcal{M}_W$ [77], which can be viewed as a flat metric, now known as *Saito metric*. Dubrovin used this metric to establish that the orbit space $\mathcal{M}_W$ carries the structure of a Frobenius manifold [22, Lecture 4]. He also conjectured that these Frobenius manifolds (and their products) are the only semisimple Frobenius manifolds with polynomial prepotential F. This conjecture was later proved by Hertling based on the notion of an *F-manifold* which was introduced by himself and Manin [47,61]. An F-manifold is weaker than a Frobenius manifold as one only assumes the existence of a commutative and associative multiplication on the tangent bundle satisfying a certain integrability condition which is automatically satisfied by a Frobenius manifold [46].

An *isolated hypersurface singularity* is a holomorphic function germ $f : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at $x = 0$. The *multiplicity of the singularity* f is the dimension of its *local algebra*, that is the quotient $\mathbb{C}[x_1,\dots,x_m]/I_f$, where x_i, $i=1,\dots,m$ are coordinates in $\mathbb{C}^m$ and I_f is the ideal generated by $\partial_{x_i}f$, $i=1,\dots,m$. An *unfolding* (or deformation) of f is a holomorphic function germ $\mathcal{F}:(\mathbb{C}^m\times\mathbb{C}^l,0)\to(\mathbb{C},0)$ such that $\mathcal{F}(x,0)=f(x)$. The space $\mathbb{C}^l$ is called the *base space* of the unfolding $\mathcal{F}$. The deformation $\mathcal{F}(x,\nu)$, $\nu\in\mathbb{C}^l$ is *versal*, if any deformation $\widetilde{\mathcal{F}}(x,\mu)$, $\mu\in\mathbb{C}^k$ of the singularity is equivalent to the deformation induced by $\mathcal{F}$. There is a (unique) versal deformation such that the multiplicity of the singularity equals the dimension of the parameter space. This deformation is said to be *semiuniversal* [7,8].

In this framework, the simplest Frobenius structure is given by the semiuniversal unfolding of the simple singularity A_n, $f(x)=x^{n+1}$ [7,22]. The associated Coxeter group is the symmetric group S_{n+1} which acts on the space $\mathbb{C}^{n+1}$ by permutation of the coordinates $x_0,x_1,\dots,x_n\in\mathbb{C}$. The action is restricted onto the hyperplane $H:\sum_{i=0}^n x_i=0$. By Chevalley's Theorem the corresponding orbit space $\mathcal{M}_{S_{n+1}}$ maps isomorphically to the hyperplane $H\cong\mathbb{C}^n$. Let $\sigma_i(x_0,\dots,x_n)$ be the i-th elementary symmetric polynomial in $n+1$ variables. A semiuniversal deformation of the singularity $f(x)$ has the form

$$\mathcal{F}(x,a)=x^{n+1}+a_1x^{n-1}+\cdots+a_n,\quad a=(a_1,\dots,a_n), \tag{1.3}$$

where $a_i=a_i(x_0,\dots,x_n)=(-1)^{i+1}\sigma_{i+1}(x_0,\dots,x_n)$, $i=1,\dots,n$. The variables $x_0,\dots,x_n$ are identified with the roots of the polynomial $\mathcal{F}(x,a)$, they satisfy $\sum_{i=0}^n x_i=0$. That is,

$$\mathcal{F}(x,a)=\prod_{i=0}^n(x-x_i),\quad \sum_{i=0}^n x_i=0. \tag{1.4}$$

The set $\Sigma\subset\mathcal{M}_W$ given by

$$\Sigma=\{(a_1,\dots,a_n)\in\mathbb{C}^n|\ \mathcal{F}(x,a)|_{\sum_{i=0}^n x_i=0}\ \text{has multiple roots}\}$$

is isomorphic to the *discriminant* of the singularity A_n. In other words, the set Σ consists of those values of a for which the polynomial has a critical point with critical value equal to zero, that is it has multiple root at $x_i=x_j$. This condition defines the mirrors of the Coxeter group S_{n+1}. The Frobenius algebra Q_a is realised as the ring of complex polynomials modulo polynomials vanishing at the critical points of $\mathcal{F}$, namely $Q_a=\mathbb{C}[x]/\mathcal{F}'(x,a)$. This algebra coincides with the local algebra of the singularity A_n at the origin $a=0$. The Saito metric on $\mathcal{M}_W$ coincides (up to proportionality) with the inner product on Q_a defined as

$$\langle f,g\rangle_a=-\operatorname{res}|_{x=\infty}\frac{f(x)g(x)}{\mathcal{F}'(x,a)}dx. \tag{1.5}$$

The primitive form (up to a constant factor) is given by the differential dx [76].

More generally, (Frobenius) flat metric for singularities by the use of primitive forms is induced from a Grothendieck residue pairing [46]. Existence of a primitive form in general was proved by M. Saito (see [46] and references therein). In the case of hypersurface singularities D_n ($n \geq 4$) and boundary singularities B_n ($n \geq 2$) semiuniversal unfoldings can be brought into a form similar to (1.3) and thus construction is analogous to the case of type A singularity [7, 80]. We take this approach in our considerations in Chapter 3.

1.1.2 Quantum cohomology

Quantum cohomology can be viewed as a deformation of the ordinary cohomology (De Rham etc.) where the cup product is replaced by a certain 'quantum product'. Definition involves the intersections of cycles in the space of 'complex curves' in a manifold M. The (large) quantum product defined on the total cohomology group H^*M is directly related to the third-order derivatives of the generating function F of Gromov-Witten invariants. This product is commutative and associative and one can show that for any $x \in H^*M$ the (large) quantum cohomology algerba is a Frobenius algebra [44]. The function F satisfies WDVV equations and the (large) quantum cohomology of M can be equipped with the structure of a Frobenius manifold. The following example is due to Kontsevich and Manin [44, 54]. Consider the complex projective plane $M = \mathbb{C}P^2$. The starting point is to fix a constant metric η on the vector space $\mathbb{C}^3$ and a basis $e_1, e_2, e_3 \in \mathbb{C}^3$ such that

$$\eta(e_i, e_j) = \delta_{i+j,4}, \quad 1 \leq i, j \leq 3.$$

The cohomology algebra $H^*M = \bigoplus_{i=0}^2 H^{2i}(M; \mathbb{C})$ can be written in the form

$$H^*M \cong \mathbb{C}[z]/z^3 \cong \langle 1, z, z^2 \rangle \cong \langle e_1, e_2, e_3 \rangle,$$

where e_i is identified with the generator of the cohomology group $H^{2(i-1)}(M; \mathbb{C})$. One then considers a family of quantum products $*_x : H^*M \times H^*M \to H^*M$, where $x = x_1e_1 + x_2e_2 + x_3e_3$ and e_1 is fixed to be the identity element for these products. There exists a prepotential F such that

$$\frac{\partial^3 F}{\partial x^i \partial x^j \partial x^k} = \eta(e_i *_x e_j, e_k) := F_{ijk}.$$

In the special case when $x \in H^2M$, that is $x_1 = x_3 = 0$, $*_x$ determines the (small) quantum cohomology algebra given explicitly as

$$e_1 *_x e_i = e_i, \ 1 \leq i \leq 3, \quad e_2 *_x e_2 = e_3, \quad e_2 *_x e_3 = qe_1, \quad e_3 *_x e_3 = qe_2,$$

where the parameter $q = e^{x_2}$. This algebra is isomorphic to $\mathbb{C}[z, q]/(z^3 - q)$.

The associativity of the quantum products for the function F is equivalent to the single WDVV equation (cf. (1.2))

$$F_{333} = F_{223}^2 - F_{222}F_{233}.$$

One can show that there is a solution unique up to third-order terms of the form

$$F(x_1, x_2, x_3) = \frac{1}{2}(x_1x_2^2 + x_1^2x_3) + \sum_{d\geq 1} N(d)e^{dx_2}\frac{x_3^{3d-1}}{(3d-1)!},$$

where $N(d)$ are positive integers determined recursively by $N(1) = 1$ and

$$N(d) = \sum_{i+j=d} N(i)N(j)\left(\binom{3d-4}{3i-2}i^2j^2 - i^3j\binom{3d-4}{3i-1}\right), \quad d \geq 2.$$

The numbers $N(d)$ are directly related to the Gromov-Witten invariants of M and for a fixed d, $N(d)$ is the number of rational curves of degree d in M which hit $3d-1$ generic points. The first few values of $N(d)$ are

$$N(2) = 1,\ N(3) = 12,\ N(4) = 620,\ N(5) = 87304,\ N(6) = 26312976.$$

1.1.3 More instances of WDVV equations

Other places where (generalised) WDVV equations emerge include Seiberg-Witten theories and $\mathcal{N} = 4$ supersymmetric mechanics. These generalised WDVV equations are similar to (1.2) but there is no complete structure of Fronenius manifolds which may be associated with them, in general.

In Seiberg-Witten (effective) theory, the exact Seiberg-Witten prepotential F is defined in terms of a family of auxiliary Riemann surfaces, which are endowed with some special meromorphic differential dS [62, 63]. For example in the case of pure $\mathcal{N} = 2$ SUSY gauge theory with $SU(n+1)$ gauge group, the Riemann surfaces (genus $g = n$ hyperelliptic curves) and differential dS have the form

$$y^2 = \mathcal{F}(x, a)^2 - \Lambda^{2(n+1)}, \quad dS = x\frac{d\mathcal{F}(x, a)}{y},$$

where $\mathcal{F}(x, a)$ is the semiuniversal unfolding associated to the A_n singularity given by (1.3) and Λ is a complex parameter. The third-order derivatives of F are defined in terms of residues of some carefully chosen differentials. Then the leading perturbative approximation to the prepotential F as Λ tends to 0 after rescaling satisfies generalised WDVV equations and is given by

$$F^{pert} = \frac{1}{2}\sum_{i<j}^{n}(x_i - x_j)^2 \log(x_i - x_j), \quad \sum_{i=0}^{n} x_i = 0.$$

Motivated by the above construction, Martini and Gragert showed that functions of the form

$$F = \frac{\lambda}{4} \sum_{\gamma \in \mathcal{A}} (\gamma, x)^2 \log(\gamma, x), \quad \lambda \in \mathbb{C}, \tag{1.6}$$

where $\mathcal{A}$ is the root system associated to a semisimple Lie algebra also satisfy generalised WDVV equations [66]. Later, Veselov extended this class further to the so-called $\vee$-systems which in particular contain all Coxeter root systems [86].

$\vee$-systems form special collections of vectors in a linear space, which satisfy certain linear algebraic conditions. More precisely, let $V = \mathbb{C}^n$, $\mathcal{A} \subset V$ and define a non-degenerate bilinear form on V by $G_{\mathcal{A}}(u, v) = \sum_{\alpha \in \mathcal{A}} (\alpha, u)(\alpha, v)$, $u, v \in V$. Then $\mathcal{A}$ is said to be a $\vee$-system if for any $\gamma \in \mathcal{A}$ and for any two-dimensional plane $\pi \subset V$ such that $\gamma \in \pi$ one has

$$\sum_{\beta \in \mathcal{A} \cap \pi} G_{\mathcal{A}}(\beta, \gamma)\beta = \mu\gamma,$$

for some complex parameter $\mu = \mu(\gamma, \pi)$. A logarithmic prepotential (1.6) corresponding to a collection of vectors $\mathcal{A}$ satisfies generalised WDVV equations if and only if $\mathcal{A}$ is a $\vee$-system. The class of $\vee$–systems contains Coxeter root systems, deformations of generalized root systems of Lie superalgebras, special subsystems in and restrictions of such systems [36, 79]. A complete description of the class remains open (see [37] and references therein).

1.2 Supersymmetric mechanics

Calogero–Moser Hamiltonian is a famous example of an integrable system [20, 67, 82] which is related to a number of mathematical areas (see e.g. [28]). Generalised Calogero–Moser systems associated with an arbitrary root system were introduced by Olshanetsky and Perelomov [69], [70].

$\mathcal{N} = 2$ supersymmetric quantum Calogero–Moser systems were constructed in [39] and considered further in [18]. They were generalised to classical root systems in [19] and to an arbitrary root system in [16]. In such constructions one considers N quantum particles on a line with coordinates $x = (x_1, \ldots, x_N)$ and momenta $p = (p_1, \ldots, p_N)$, which satisfy canonical commutation relations. Additionally one takes $2N$ fermionic variables, $\psi = (\psi^1, \ldots, \psi^N)$, $\bar{\psi} = (\bar{\psi}^1, \ldots, \bar{\psi}^N)$ which satisfy the canonical (anti)-commutation relations,

$$\{\psi^i, \bar{\psi}^j\} = -\frac{1}{2}\delta^{ij}, \quad \{\psi^i, \psi^j\} = \{\bar{\psi}^i, \bar{\psi}^j\} = 0, \quad i, j = 1, \ldots, N.$$

The dynamics of the system are controlled by a potential U and there are two supercharges $Q(x, p, \psi, \bar{\psi})$, $\bar{Q}(x, p, \psi, \bar{\psi})$ which generate the $\mathcal{N} = 2$ supersymmetry algebra

$$Q^2 = \bar{Q}^2 = 0, \quad H_{SUSY} = -\frac{1}{2}(Q\bar{Q} + \bar{Q}Q),$$

where H_{SUSY} is the corresponding supersymmetric Hamiltonian [16]. In the case of (rational) generalised Calogero–Moser system one considers a potential of the form

$$U(x) = \frac{1}{2}\sum_{\alpha\in\mathcal{A}} c_\alpha \log(\alpha, x), \quad c_\alpha > 0,$$

where $\mathcal{A}$ is any Coxeter root system. Then the bosonic part of the Hamiltonian H_{SUSY} (up to rescaling) takes the form

$$H_B = -\Delta + \sum_{\alpha\in\mathcal{A}} \frac{c_\alpha(c_\alpha - 1)(\alpha,\alpha)}{(\alpha, x)^2}.$$

A motivation for construction of $\mathcal{N} = 4$ Calogero–Moser system goes back to the work [42] on a conjectural description of near–horizon limit of Reissner–Nordström black holes where appearance of $su(1, 1|2)$ superconformal Calogero–Moser model was suggested. Though we also note more recent different considerations of near extremal black holes in [60]. Another motivation to study supersymmetric (trigonometric) Calogero–Moser–Sutherland systems comes from the relation of these systems with conformal blocks and possible generalisation of these relations to the supersymmetric case [52]. It has been a long standing problem to construct $\mathcal{N} = 4$ supersymmetric extensions of Calogero–Moser systems.

Wyllard gave an ansatz for $\mathcal{N} = 4$ supercharges in [88]. In general his ansatz depends on two potentials W and F. In order to realise an $\mathcal{N} = 4$ supersymmetric mechanical system one can take $4N$ fermionic variables, thus to each particle four fermionic variables $\{\psi^{aj}, \bar\psi^j_a \,|\, a = 1, 2\}$ are associated. They satisfy canonical (anti)-commutation relations. Then the $\mathcal{N} = 4$ supercharges are defined by $(a, b = 1, 2)$

$$Q^a = p_r(\psi^{ar} + iW_r) + iF_{rjk}\Psi^{rjk}, \quad \bar Q_b = p_l(\bar\psi^l_b + iW_l) + iF_{lmn}\bar\Psi^{lmn}, \tag{1.7}$$

where Ψ^{rjk}, $\bar\Psi^{lmn}$ are particular cubic fermionic terms and $W_i = \partial_{x_i} W$, $F_{ijk} = \partial_{x_i}\partial_{x_j}\partial_{x_k} F$. The $\mathcal{N} = 4$ supersymmetry algebra has the form

$$\{Q^a, Q^b\} = \{\bar Q_a, \bar Q_b\} = 0, \quad H_{SUSY} = -\frac{1}{2}(Q^a\bar Q_a + \bar Q_a Q^a), \tag{1.8}$$

where H_{SUSY} is the supersymmetric Hamiltonian. Wyllard considered a $su(1, 1|2)$ superconformal extension of the $\mathcal{N} = 4$ supersymmetry algebra (1.8) by incorporating additional generators so that $su(1, 1|2)$ relations are satisfied, and derived necessary differential equations for F and W [88]. Thus prepotential F satisfies generalised WDVV equations (as it was pointed out in [11]) and potential W satisfies equations of the form

$$\partial_{kl}W + F_{klj}\partial_j W = 0. \tag{1.9}$$

Wyllard's prepotential F has the form (1.6), where $\mathcal{A}$ is the root system A_{N-1}. Then W is a twisted period (see Definition 2.2.18) of the Frobenius manifold on the space of orbits $\mathcal{M}_{S_N}$. Wyllard constructed $su(1,1|2)$ N particle Calogero–Moser Hamiltonian for a single value of the coupling parameter $c = 1/N$ as bosonic part of his supersymmetric Hamiltonian with $W = 0$. He argued that his ansatz does not produce superconformal Calogero–Moser Hamiltonians for general values of c. Examples based on root systems $\mathcal{A} = G_2, B_3$ were also considered in [88].

Wyllard's ansatz for $\mathcal{N} = 4$ supercharges was extended to other root systems in [40], [41] where solutions for a small number of particles were studied both for $W = 0$ and $W \neq 0$. In particular, $su(1,1|2)$ superconformal Calogero–Moser systems related to $\mathcal{A} = A_1 \oplus G_2, F_4$ and subsystems of F_4 were derived. Superconformal $su(1,1|2)$ Calogero–Moser systems for the rank two root systems were derived in [13] via suitable action in the superspace. For the WDVV equations arising in the superfield (Lagrangian) approach, which involves consideration of $\mathcal{N} = 4$ supersymmetric action, we refer to [56].

A many-body model with $D(2,1;\alpha)$ supersymmetry algebra with $\alpha = -\frac{1}{2}$ was considered in [29]. This model was obtained by a reduction from matrix model and it incorporates an extra set of bosonic variables ("$U(2)$ spin variables") which enter the bosonic potential of the corresponding Hamiltonian. One-dimensional version of such a model was considered in [30] and, for any α, in [12], [31]. A generalisation of the many-body classical spin superconformal model for any value of the parameter α was proposed in [55]. In the survey paper [32, p. 33] it is stated: "...it turns out that the realization of $D(2,1;\alpha)$ superconformal symmetry on the multi-particle phase space for $\alpha \neq -1$ or 0 requires at least one pair of (bosonic) isospin variables $\{u^i, \bar{u}_i | i = 1,2\}$ parametrizing an internal two-sphere...".

Within $D(2,1;\alpha)$ supersymmetry ansatz of [55] a class of bosonic potentials was obtained in [34]. The prepotential F has the form (1.6) for a root system $\mathcal{A}$. Then W is a twisted period of the Frobenius manifold on the space of orbits corresponding to the root system $\mathcal{A}$. Such polynomial twisted periods were described in [34], they exist for special values of parameter α. Although the corresponding bosonic potentials are algebraic this class does not seem to contain generalised Calogero–Moser potentials associated with $\mathcal{A}$.

Recently a construction of type A_{N-1} supersymmetric (classical) Calogero–Moser model with extra spin bosonic generators and $\mathcal{N}N^2$ fermionic variables (for any even $\mathcal{N}$) was presented in [57]. The ansatz for supercharges is more involved and extra fermionic variables appear due to reduction from a matrix model. A related quantum $\mathcal{N} = 4$ supersymmetric spin A_{N-1} Calogero–Moser system was studied recently in [33]. Furthermore, a simpler ansatz for supercharges for the spin classical A_{N-1} Calogero–Moser system was presented in [58]. This model has $\frac{1}{2}\mathcal{N}N(N+1)$ fermionic variables and the supersymmetry algebra is $osp(\mathcal{N}|2)$. Most recently classical supersymmetric $osp(\mathcal{N}|2)$ Calogero–Moser systems were presented in [59]; these models have nonlinear Hermitian conjugation property of matrix fermions and supercharges are cubic in fermions.

1.3 Present work and plan of this thesis

1.3.1 Main results

I. Determinant of restricted Saito metric

The first question that we address in this work is motivated by Frobenius structures arising in singularity theory (Chapter 3). Natural objects from the point of view of Coxeter geometry are the *discriminant* and the corresponding *discriminant strata* in $\mathcal{M}_W = V/W$ and V (see e.g. [46] for a discussion on the geometry of discriminants). The discriminant Σ in $\mathcal{M}_W$ is the union of *irregular orbits* of the action of the group W, that is Σ is the union of the orbits of W with cardinality less than the order of W. The preimage of Σ in V under the quotient map $\pi : V \to \mathcal{M}_W$ corresponds to the union of *mirrors* of W.

Let $V = \mathbb{C}^n$ and let $\mathcal{R} \subset V$ be a Coxeter root system corresponding to W. For any $\alpha \in \mathcal{R}$, the hyperplane $\Pi_\alpha = \{x \in V | (\alpha, x) = 0\}$ is called a *mirror*. Consider the subspace $D = \cap_{\beta \in S} \Pi_\beta \subset V$ for some subset $S \subset \mathcal{R}$. Its image $\pi(D)$ (and also $D \subset V$) in $\mathcal{M}_W$ under the map π is called a *discriminant stratum*. As an example, consider the case when $W = S_{n+1}$. An arbitrary discriminant stratum (up to the action of W) $D \subset V$ is given by the following equations:

$$\begin{aligned} x_0 = \ldots = x_{m_0} &= \xi_0, \\ x_{m_0+1} = \ldots = x_{m_0+m_1} &= \xi_1 \\ &\vdots \\ x_{\sum_{i=0}^{N-1} m_i + 1} = \ldots = x_{\sum_{i=0}^{N} m_i} &= \xi_N, \end{aligned} \tag{1.10}$$

where $\xi_1, \ldots, \xi_N$ can serve as coordinates on D and $\xi_0 = -\sum_{i=1}^N \frac{m_i}{m_0}\xi_i$, $N, m_i \in \mathbb{N}$.

The Saito metric η on $\mathcal{M}_W$ is defined as the Lie derivative along the unity field e of the intersection form. A natural question is to study the restriction η_D of the (covariant) metric η to discriminant strata D. In particular, to find the determinant of this metric (in a suitable coordinate system). In the case of A_n singularity this metric has a form similar to (1.5) where $\mathcal{F}$ is replaced with its restriction $\mathcal{F}_D$ on D, which we define as

$$\mathcal{F}_D(x, a) = \mathcal{F}(x, a)\big|_D = \prod_{i=0}^{N} (x - \xi_i)^{m_i}. \tag{1.11}$$

We obtain two structure theorems (Main Theorems 1, 2, Chapter 3) for this determinant $\det \eta_D$. More precisely, we show in Main Theorem 1 that $\det \eta_D$ (in linear coordinates on D) is proportional to a product of linear forms which define the restricted Coxeter arrangement $\mathcal{A}_D$. In the case where S is empty this theorem (essentially) reduces to a well-known statement that the Jacobian J of basic invariants of the group W is proportional to $\prod_{\alpha\in\mathcal{R}_+}\alpha$ (Proposition 2.4.20). Thus we obtain:

Main Theorem 1. *The following proportionality takes place*

$$\det \eta_D \sim \prod_{H\in\mathcal{A}_D} l_H^{k_H}, \quad k_H \in \mathbb{N}.$$

In Main Theorem 2 we explain that the multiplicities k_H of the linear forms l_H are related to the Coxeter numbers of certain parabolic subgroups of W. Let us illustrate these theorems by an example.

Example. Let us consider the case where $\mathcal{R} = A_4$ and consider a stratum D of type A_2 given by $D : x_0 = x_1 = x_2$ with the corresponding parabolic subgroup generated by (orthogonal) reflections in the mirrors $x_0 = x_1$ and $x_1 = x_2$. Coordinates on D are chosen as: $\xi_0 = x_0 = x_1 = x_2$, $\xi_1 = x_3$, and $\xi_2 = x_4$ subject to the condition $3\xi_0 + \xi_1 + \xi_2 = 0$. Then

$$\det \eta_D \sim (\xi_0 - \xi_1)^4(\xi_0 - \xi_2)^4(\xi_1 - \xi_2)^2.$$

The numbers $4, 4, 2$ are Coxeter numbers of certain parabolic subgroups. The multiplicity 4 of the linear form $\xi_0 - \xi_1$ is the Coxeter number of the parabolic subgroup of type A_3 generated by reflections in the mirrors $x_0 = x_1$, $x_1 = x_2$ and $x_2 = x_3$. The multiplicity 4 of the linear form $\xi_0 - \xi_2$ is the Coxeter number of the parabolic subgroup of type A_3 generated by reflections in the mirrors $x_0 = x_1$, $x_1 = x_2$ and $x_2 = x_4$. Finally, the multiplicity 2 of the linear form $\xi_1 - \xi_2$ is the Coxeter number of the parabolic subgroup of type A_1 generated by a reflection in the mirror $x_3 = x_4$.

In the case of classical Coxeter groups, namely the families A_n ($n \geq 1$), B_n ($n \geq 2$), D_n ($n \geq 4$) the proof of Main Theorem 1 relies on the use of Landau-Ginzburg superpotentials, which is function $\mathcal{F}(x, a)$ given by (1.4) for type A.

In the remaining cases, namely the dihedral groups $I_2(m)$ ($m \geq 5$) and the exceptional groups $E_6, E_7, E_8, H_3, H_4, F_4$ the proofs of Main Theorems 1 and 2 rely heavily on the geometry of the corresponding root systems and their subsystems.

II. Dubrovin's almost duality

Logarithmic solutions of generalised WDVV equations of the form (1.6) for any root system $\mathcal{A}$ associated to a finite Coxeter group are related to polynomial solutions of WDVV equations via the notion of almost duality introduced by Dubrovin [25].

Solutions (1.6) determine a multiplication structure $*$ on the tangent bundle of the complement to the discriminant Σ in $\mathcal{M}_W$. The space $\mathcal{M}_W \setminus \Sigma$ satisfies all the properties of a Frobenius manifold but flatness of the identity field for the 'new' multiplication $*$. The two multiplications are related by the formula

$$x * y = E^{-1} \circ x \circ y, \tag{1.12}$$

where $x, y \in T_p\mathcal{M}_W$ for a point $p \in \mathcal{M}_W$, and E^{-1} is the inverse of the Euler vector field associated to $\mathcal{M}_W$. Feigin and Veselov showed that almost duality admits a natural (in some suitable sense) restriction on discriminant strata in $\Sigma \subset \mathcal{M}_W$ [35]. In particular, they proved that the left-hand-side of (1.12) has a well-defined limit at generic points in a stratum.

On the other hand, the submanifolds of an arbitrary Frobenius manifold M which carry the structure of a Frobenius algebra on each tangent space were considered by Strachan and are called natural submanifolds. Key examples of natural submanifolds were expected to be discriminant strata in the orbit spaces $\mathcal{M}_W$ as well as caustics [81].

We confirm this to be the case for discriminant strata. Namely, we show that for vector fields $u, v \in T_{x_0}D$, where x_0 is a generic point in D, the product $u \circ v$ is well-defined and that $u \circ v \in T_{x_0}D$ (Proposition 3.8.2). Further to that, as a consequence of our considerations (see Remark 3.1.4) we also get that the restricted Saito metric η_D to any stratum D is generically non-degenerate. We apply our results to strengthen almost duality (1.12) on the discriminant strata (Section 3.8).

III. Superconformal extension of Calogero–Moser Hamiltonian

Several attempts have been made to construct supersymmetric mechanics such that the corresponding Hamiltonian has bosonic potential of Calogero–Moser type with a reasonably general coupling parameter(s). In the survey paper [32] various problems and obstacles in these constructions are mentioned.

In the current work (Chapter 4) we construct supersymmetric Calogero–Moser systems without extra isospin variables. In fact, we present two constructions of $\mathcal{N} = 4$ supersymmetric quantum mechanical systems, where the superconformal algebra is $D(2, 1; \alpha)$, starting with an arbitrary $\vee$-system. Thus, in the case of a Coxeter root system $\mathcal{A}$ the bosonic part of the Hamiltonian is the Calogero–Moser Hamiltonian associated with $\mathcal{A}$ introduced by Olshanetsky and Perelomov in [70], which we get in two different gauges: the potential and potential free ones. In the latter case the Hamiltonian is not formally self-adjoint; this gauge comes from the radial part of the Laplace–Beltrami operator on symmetric spaces [14, 45, 70]. The parameter α depends on the $\vee$-system and is ultimately related with the coupling parameter in the resulting Calogero–Moser type Hamiltonian.

We use original ansatz (1.7) for the supercharges [40], [88] based on two potentials F, W and we take $W = 0$. The algebra $D(2, 1; \alpha)$ contains the supersymmetry algebra as its subalgebra and has some additional generators and relations. We construct two representations of the algebra $D(2,1;\alpha)$ which crucially depend on the choice of the cubic fermionic terms Ψ^{rjk}, $\bar{\Psi}^{lmn}$ in (1.7). We use a prepotential of the form (1.6) where $\mathcal{A}$ is an arbitrary $\vee$-system and the parameters λ and α satisfy a linear relation. We obtain the following result.

Theorem (Theorems 4.4.4, 4.4.5, 4.4.8)**.** *The supersymmetric quantum Hamiltonians $H^{(i)}_{SUSY}$, $(i = 1, 2)$ take the form*

$$H^{(i)}_{SUSY} = H^{(i)}_B + \Phi,$$

where Φ is the fermionic part and the bosonic parts (up to rescaling) $H^{(i)}_B$ take the form

$$H^{(1)}_B = -\Delta + \frac{\lambda}{2}\sum_{\gamma\in\mathcal{A}}\frac{(\gamma,\gamma)^2}{(\gamma,x)^2} + \frac{\lambda^2}{4}\sum_{\gamma,\beta\in\mathcal{A}}\frac{(\gamma,\gamma)(\beta,\beta)(\gamma,\beta)}{(\gamma,x)(\beta,x)}, \quad H^{(2)}_B = -\Delta + \lambda\sum_{\gamma\in\mathcal{A}}\frac{(\gamma,\gamma)}{(\gamma,x)}\partial_\gamma.$$

In the case of a Coxeter root system $\mathcal{A} = \mathcal{R}$, the bosonic parts $H^{(i)}_B$ are the generalised Calogero–Moser Hamiltonians

$$H^{(1)}_B = -\Delta + \sum_{\gamma\in\mathcal{R}_+}\frac{2\lambda(\lambda+1)}{(\gamma,x)^2}, \quad H^{(2)}_B = -\Delta + \sum_{\gamma\in\mathcal{R}_+}\frac{2\lambda}{(\gamma,x)}\partial_\gamma,$$

where $(\gamma,\gamma) = 2$ for all $\gamma\in\mathcal{R}$.

In the special case when $\alpha = -1$ the superalgebra $D(2,1;-1)$ contains the superalgebra $su(1,1|2)$ as its subalebra, and our first ansatz on the $su(1,1|2)$ generators reduces to the one considered by Galajinsky, Lechtenfeld and Polovnikov in [40, 41]. It was emphasised in [41] that such quantum models with $\alpha = -1$ and $W = 0$ are non-trivial with bosonic potentials proportional to squared Planck constant, though they were not considered in many details in [41], in particular the explicit form of the Hamiltonian was not given. Thus we extend considerations in [41] for $W = 0$ to the case of superconformal algebra $D(2,1;\alpha)$ for any α, and we get in this framework quantum Calogero–Moser type systems associated with an arbitrary $\vee$–system, which includes Olshanetsky–Perelomov generalisations of the Calogero–Moser system with arbitrary invariant coupling parameters. The parameter α depends on these coupling parameters.

IV. Supersymmetric extension of Calogero–Moser–Sutherland Hamiltonian

We also consider generalised trigonometric Calogero–Moser–Sutherland systems related to a collection of vectors $\mathcal{A}$ with multiplicities (Section 4.5). We include these Hamiltonians in the supersymmetry algebra provided that extra assumptions on $\mathcal{A}$ are satisfied which are similar to WDVV equations for the trigonometric version of the prepotential F. We show that these assumptions can be satisfied when $\mathcal{A}$ is an irreducible root system with more than one orbit of the Weyl group, that is BC_N, F_4 and G_2 cases.

In the case when $\mathcal{A} = BC_N$ the corresponding bosonic parts of the supersymmetric Hamiltonians take the form (Theorem 4.5.8):

$$H_B^{(1)} = -\Delta + \sum_{i=1}^{N} \Big(\frac{(8s+2(N-2)q)(2(N-2)q-1)}{\sinh^2 x_i} + \frac{16s(4s+1)}{\sinh^2 2x_i} \Big) + \sum_{i<j}^{N} \frac{4q(2q+1)}{\sinh^2(x_i \pm x_j)},$$

$$H_B^{(2)} = -\Delta + 2\sum_{i=1}^{N} \big(8s\coth 2x_i - (8s+2(N-2)q)\coth x_i\big)\partial_i + 4q\sum_{i<j}^{N} \coth(x_i \pm x_j)(\partial_i \pm \partial_j),$$

where the multiplicity parameters r, s, q satisfy a linear relation.

It turns out that in this case one can show that the corresponding prepotential satisfies generalised WDVV equations which is a generalisation of the solution for the root system B_N obtained in [49] (we refer to [2] for this development).

1.3.2 Structure of the thesis

Chapter 2 In this chapter we provide an overview of notations, basic definitions and results one should be familiar with throughout the rest of the thesis. In Section 2.1 we review schematically the appearance of the WDVV equations in topological field theories and thus motivate the construction of a Frobenius manifold. In Section 2.2 we recall key notions from the theory of Frobenius manifolds including Landau-Ginzburg superpotentials and almost duality. We introduce generalised WDVV equations in Section 2.3. In Section 2.4 finite Coxeter groups and elements of their invariant theory are introduced. In particular, K. Saito's flat structure on Coxeter orbit spaces is discussed. In Section 2.5 we survey Dubrovin's realisation of Coxeter orbit spaces endowed with such flat structures as one of the main examples of Frobenius manifolds. These Frobenius structures are the central objects in our considerations in Chapter 3. The main sources which are used in the present chapter are references [22, 25] for the Frobenius manifold theory and references [17, 51] for the theory of Coxeter groups. All the results in this chapter are well-known.

Chapter 3 In this chapter we state and prove Main Theorems 1, 2 for the determinant of the restricted Saito metric on Coxeter discriminant strata. In Section 3.1 we formulate Main Theorem 1. We show that this determinant is a product of linear factors with some multiplicities. In Section 3.2 we formulate Main Theorem 2 on the multiplicities of these linear factors of Main Theorem 1.

We prove Main Theorems 1 and 2 for classical root systems in Section 3.3 and

3.4 respectively. Considerations are based on the use of superpotentials for the corresponding Frobenius manifolds and their discriminant strata. That is, in the case of type A singularity we define the Saito metric η_D on an arbitrary stratum $D \subset V$ (1.10) with the use of the corresponding Landau-Ginzburg superpotential defined on D (1.11). We find η_D in the variables ξ_i $(i = 0, \ldots, N)$ in Theorem 3.3.5 and show that the statement of Main Theorem 2 is true in Theorem 3.4.1.

In Section 3.5 we derive a general formula for the determinant of the restricted Saito metric on discriminant strata (Theorem 3.5.9), which is obtained by considering the Saito metric on $\mathcal{M}_W$ in the corresponding flat coordinates. This formula is given in terms of (some of the components) η^{ij} of the contravariant Saito metric on $\mathcal{M}_W$ which take the form (3.5.5):

$$\eta^{ik} = (-1)^{n+1+k}\partial_{\omega^i}\frac{J_k}{J} + (-1)^{n+1+i}\partial_{\omega^k}\frac{J_i}{J}, \tag{1.13}$$

where J_a are particular minors of the Jacobi matrix of basic invariants of W. We use this formula for the components of Saito metric to prove Main Theorems 1 and 2 for the strata of the exceptional root systems in dimension 1 and codimensions 1, 2, 3 and 4 in Section 3.6. Due to the use of Theorem 3.5.9 the cases of dimension 1 and codimension 1 are easier to handle, but in general the difficulty of the corresponding proofs increases with increase of codimension, as one has to deal with the determinant of a matrix of size $\operatorname{codim} D \times \operatorname{codim} D$. Our analysis in codimensions 3 and 4 is done by case by case considerations of the subgraphs of the Coxeter graph corresponding to the group W. This analysis covers all strata in the orbit spaces of the Coxeter groups $I_2(p)$, H_3, H_4, F_4.

In Section 3.7 we consider the remaining cases, namely strata of codimension 5 in E_7 and strata of codimensions 5 and 6 in E_8. In these cases we obtain explicit formulae for the determinant of the restricted Saito metric and analyse corresponding multiplicities with the help of Mathematica. This completes Main Theorems 1 and 2 for all the cases.

In Section 3.8 we revisit Dubrovin's duality on discriminant strata. Part of this chapter is joint work with M. Feigin and I. Strachan [4].

Chapter 4 We recall the definition of Lie superalgebra $D(2,1;\alpha)$ in Section 4.1. The dimension of this algebra is 17, and in particular the even part (dimension 9) comprises of three mutually commuting $sl(2)$ algebras. We give two types of representations of this superalgebra in Sections 4.2, 4.3. Odd generators include $\mathcal{N} = 4$ supercharges (1.7) defined in terms of a prepotential F of the form (1.6).

Starting with any $\vee$-system $\mathcal{A}$ we get two corresponding supersymmetric Hamiltonians. In Section 4.4 we present them explicitly. We consider supersymmetric trigonometric Calogero–Moser–Sutherland systems in Section 4.5. Part of Chapter 4 is joint work with M. Feigin [3].

For the reader's convenience we also include considerations for one particle systems in Appendix B which are particular cases of considerations from Sections 4.2 and 4.3.

Chapter 5 In the last chapter we summarise main results from this thesis and pose questions for further research.

Chapter 2

Frobenius manifolds and finite Coxeter groups

This chapter provides an introduction to the most important aspects from the theory of Frobenius manifolds and from that of finite Coxeter groups, which feature in this work.

2.1 WDVV equations in TQFTs

The *Witten-Dijkgraaf-Verlinde-Verlinde equations* (WDVV) of associativity constitute a famous example of a nonlinear integrable system which emerges in numerous areas of modern mathematical and theoretical physics. They form an over-determined system of PDEs for a function F which is defined locally in terms of some variables $t = (t^1, t^2, \ldots, t^n)$. Originally, WDVV equations appeared in the context of 2-dimensional topological quantum field theories (TQFTs) [21, 87].

Roughly speaking, a quantum field theory (QFT) is defined by specifying the properties of the physical correlation functions. Let us consider a QFT on a n-dimensional manifold Σ and let $\phi_\alpha(x)$ be a family of local fields (observables) on Σ. In general, correlators

$$\langle \phi_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\phi_{i_s}(x_s)\rangle_\Sigma, \quad x_i \in \Sigma,$$

depend on the geometric-topological properties of Σ. Here, we consider a particular class of 2-dimensional QFTs which exhibits topological invariance. In such topological theories ($2D$-TQFTs) the action is invariant with respect to arbitrary changes of the metric on Σ, which is a non-dynamic variable. Correlators depend only on the topology of Σ (i.e genus g) and the labels of the operators, but not on their positions, that is

$$\langle \phi_{i_1}(x_1)\phi_{i_2}(x_2)\ldots\phi_{i_s}(x_s)\rangle_\Sigma \equiv \langle \phi_{i_1}\phi_{i_2}\ldots\phi_{i_s}\rangle_g.$$

The set of all physical states $\mathcal{H} = \{|\phi_i\rangle\}$ is a Hilbert space. One assumes the existence of an identity operator $\phi_n := \mathbb{1}$ corresponding to the state $|0\rangle$. The inner product on $\mathcal{H}$ is defined by the following three-point function on the sphere

$$\eta_{\alpha\beta} = c_{1\alpha\beta} = \langle \mathbb{1}\phi_\alpha\phi_\beta\rangle_0 = \langle \phi_\alpha\phi_\beta\rangle_0. \tag{2.1}$$

A key feature in the study of a QFT is the operator algebra A of the local (primary) fields ϕ_α. We will assume that A is n-dimensional. The algebra A has the structure of a unital commutative and associative ring and formally the multiplication is defined as follows:

$$\phi_\alpha \cdot \phi_\beta = c^\gamma_{\alpha\beta}\phi_\gamma, \tag{2.2}$$

where $c^\gamma_{\alpha\beta}$ are the structure constants of the algebra and summation over γ is finite. These are described via three-point functions on the sphere,

$$c_{\alpha\beta\gamma} := \langle \phi_\alpha\phi_\beta\phi_\gamma\rangle_0 = c^\epsilon_{\alpha\beta}\langle \phi_n\phi_\epsilon\phi_\gamma\rangle_0 = \eta_{\epsilon\gamma}c^\epsilon_{\alpha\beta},$$

where the multiplication (2.2) is used in the second equality.

The requirement for TQFTs to satisfy certain factorisation theorems (crossing relations) leads to the following system of equations for the three point functions

$$c_{\alpha\beta\epsilon}\eta^{\epsilon\lambda}c_{\lambda\mu\nu} = c_{\alpha\mu\epsilon}\eta^{\epsilon\lambda}c_{\lambda\beta\nu}, \tag{2.3}$$

where $\eta^{\alpha\beta}$ is the inverse matrix (assuming non-degeneracy) of (2.1).

One can consider perturbations of TQFTs by introducing a suitable family of actions depending on a set of parameters $t = (t^1, \ldots, t^n)$ such that there is a one-to-one correspondence of the perturbed operators $\phi_\alpha(t)$ with ϕ_α. Similarly to the unperturbed theory crossing relations must hold and thus it can be shown that the correlation functions $c_{\alpha\beta\gamma}(t)$ satisfy relations (2.3) for any parameter t in this case as well.

Additional assumptions on the structure of the theory, namely conformal invariance lead to the following integrability equations for the correlations functions $c_{\alpha\beta\gamma}(t)$ for any parameter t:

$$\frac{\partial c_{\alpha\beta\gamma}(t)}{\partial t^\mu} = \frac{\partial c_{\alpha\beta\mu}(t)}{\partial t^\gamma}. \tag{2.4}$$

Poincaré's Lemma then implies that there exists (at least locally) a generating function $F(t)$ such that the three-point correlation functions of the operators ϕ_α coincide with the third-order derivatives of F, namely

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{2.5}$$

Further on, one can show that equations (2.4) imply that the matrix $\eta_{\alpha\beta}$ does not depend on the parameters t, that is

$$\frac{\partial \eta_{\alpha\beta}}{\partial t^{\gamma}} = 0, \tag{2.6}$$

for any α, β, γ. Finally, the function F must be quasi-homogeneous (see Section 2.2) in the variables t as a consequence of a scaling law of the theory. The WDVV equations are defined as the combination of formulae (2.3), (2.5) and (2.6).

Atiyah introduced a set of axioms which specify the properties of the correlators in the (matter sector) of a $2D$-TQFT [9, 10]. The operator algebra A carries the structure of a Frobenius algebra [22]. Indeed, it has been shown that the category of $2D$-TQFTs is in some sense equivalent to that of commutative Frobenius algebras [53].

Frobenius manifolds were constructed by Dubrovin [22] in the $90s$ in an effort to provide a rigorous geometric formalism for TQFTs as families of Frobenius algebras and to investigate possible connections with the theory of integrable systems, in particular integrable hierarchies of KdV type. In Section 2.2 we recall key aspects from the theory of these manifolds.

2.2 Frobenius manifolds

Frobenius manifolds have been in the centre of intensive study since their appearance due to their rich geometric structure as well as their surprising connections to different areas of mathematics.

2.2.1 Frobenius manifolds and WDVV equations

A key property of a Frobenius manifold is the existence of a Frobenius algebra structure on any tangent plane. Let us recall the following definition of a Frobenius algebra at first (see for example [53]).

Definition 2.2.1. Let A be a $\mathbb{C}$-algebra of finite dimension and let $<,>$ be a non-degenerate symmetric bilinear $\mathbb{C}$-valued form on A. Then A is a *Frobenius algebra* if

(i) A is an associative algebra with unity e;

(ii) the multiplication in the algebra, $\circ$, is compatible with the form $<,>$, namely

$$< a \circ b, c > = < a, b \circ c >, \quad \text{for any} \quad a, b, c, \in A.$$

Example 2.2.2. The ring of all square matrices over $\mathbb{C}$ forms a Frobenius algebra where the inner product is defined as the trace of the product.

In our considerations below we only deal with commutative Frobenius algebras.

Definition 2.2.3. A *Frobenius manifold* of charge $d \in \mathbb{C}$ is a (complex) smooth n-dimensional manifold M with (commutative) Frobenius algebra structure $(A, \eta :=<,>)$ on each tangent space satisfying the following axioms for any $x, y, z, w \in \Gamma(TM)$:

(i) the metric η on M is flat (η is a complex valued form);

(ii) the unity vector field e is constant with respect to the Levi-Civita connection ∇ of the metric η;

(iii) the $(0,4)$-tensor $\nabla_w c(x,y,z)$ is totally symmetric, where $c(x,y,z) := \eta(x \circ y, z)$;

(iv) there exists a vector field E which is covariantly linear $\nabla\nabla E = 0$, and

- $\mathcal{L}_E e = -e$,
- $(\mathcal{L}_E \eta)(x,y) = E(\eta(x,y)) - \eta(\mathcal{L}_E x, y) - \eta(x, \mathcal{L}_E y) = (2-d)\eta(x,y)$,
- $(\mathcal{L}_E c)(x,y,z) = E(c(x,y,z)) - c(\mathcal{L}_E x, y, z) - c(x, \mathcal{L}_E y, z) - c(x, y, \mathcal{L}_E z) = (3-d)c(x,y,z)$;

The last three properties mean that E generates conformal rescalings of the metric and of the Frobenius structure.

Note that the last two properties of axiom (iv) above imply that $\mathcal{L}_E(\circ) = \circ$, that is

$$\mathcal{L}_E(x \circ y) - (\mathcal{L}_E x) \circ y - x \circ (\mathcal{L}_E y) = x \circ y, \quad x, y \in \Gamma(TM). \tag{2.7}$$

Flatness of the metric η implies that locally there exist *flat coordinates* t^α, $1 \leq \alpha \leq n$, such that the metric η is constant and the components of the Levi-Civita connection ∇ vanish. Then locally in the basis $\frac{\partial}{\partial t^\alpha} = \partial_{t^\alpha}$, $1 \leq \alpha \leq n$ covariant derivatives become partial derivatives and the total symmetry of the tensor ∇c is equivalent to the total symmetry of $\partial_{t^\alpha} c_{\beta\gamma\delta}$, where $c_{\alpha\beta\gamma}(t) =< \partial_{t^\alpha} \circ \partial_{t^\beta}, \partial_{t^\gamma} >$. It follows that there exists (locally) a *prepotential* $F = F(t^1, \ldots, t^n)$ such that the tensor components $c_{\alpha\beta\gamma}(t)$ coincide with third-order partial derivatives of F (cf. (2.5)), that is

$$c_{\alpha\beta\gamma}(t) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{2.8}$$

We have from axiom (ii) of Definition 2.2.3 that $\nabla e = 0$. Thus a linear change of flat coordinates can be performed in such a way that e takes the form $e = \frac{\partial}{\partial t^n}$. Then the metric η takes the form

$$\eta_{\alpha\beta} = \eta(\frac{\partial}{\partial t^\alpha} \circ e, \frac{\partial}{\partial t^\beta}) = c(e, \frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}) = \frac{\partial^3 F}{\partial t^n \partial t^\alpha \partial t^\beta}.$$

The multiplication $\circ$ in the algebra A has the form $\partial_{t^\alpha} \circ \partial_{t^\beta} = c^\gamma_{\alpha\beta}(t)\partial_{t^\gamma}$, where the structure constants $c^\gamma_{\alpha\beta}(t)$ satisfy $c_{\alpha\beta\gamma}(t) = c^\lambda_{\alpha\beta}(t)\eta_{\lambda\gamma}$.

The vector field E is called *Euler vector field.* In general in flat coordinates it must take the form

$$E(t) = (q^\alpha_\beta t^\beta + b^\alpha)\frac{\partial}{\partial t^\alpha}, \tag{2.9}$$

for some scalars q^α_β, b^α satisfying $q^\alpha_n = \delta^\alpha_n$, $b^n = 0$, since $[E, e] = -e$. Let us define the *gradient operator* $Q = \nabla E : TM \to TM$, $x \mapsto \nabla_x E$. If $Q = (q^\alpha_\beta)$ is diagonalisable then the Euler vector field can be can be represented as

$$E(t) = \sum_{\alpha=1}^{n}(d_\alpha t^\alpha + b^\alpha)\frac{\partial}{\partial t^\alpha},$$

where d_α are the eigenvalues of the operator Q and are normalised such that $d_n = 1$. Moreover, up to a translation in the flat coordinates we have

$$E = \sum_\alpha d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \sum_{\alpha | d_\alpha = 0} b^\alpha \frac{\partial}{\partial t^\alpha}.$$

In this work we will only consider Frobenius manifolds where the numbers d_α are non-zero for all α, and thus we take an Euler field of the form:

$$E = \sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} = \sum_{\alpha=1}^{n}(1 - q_\alpha)t^\alpha \frac{\partial}{\partial t^\alpha}, \tag{2.10}$$

where $q_\alpha = 1 - d_\alpha$ are the eigenvalues[1] of the operator $\mathrm{id} - Q$. The degrees d_α of the variables t^α are called *scaling dimensions* of M. Let us now recall the following notion of quasi-homogeneity.

Definition 2.2.4. A function $f : M \to \mathbb{C}$ is said to be *quasi-homogeneous* of degree d_f if it is an eigenfunction of the Euler vector field,

$$E(f) = d_f f.$$

It follows from axiom (iv) of Definition 2.2.3 and formula (2.8) that

$$E(c_{\alpha\beta\gamma}(t)) - (3 - d)c_{\alpha\beta\gamma}(t) = 0.$$

Then we have by integrating

$$\mathcal{L}_E F = (3 - d)F + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C, \tag{2.11}$$

for some constants $A_{\alpha\beta}$, B_α, C. Thus, F is quasi-homogeneous function of degree $d_F := 3 - d$ modulo quadratic terms which are in the kernel of ∇^3.

[1] In TQFTs the numbers q_α are called *charges* of the primary fields (see Subsection 2.1).

Let us introduce an inner product on cotangent planes of M, $<,>: T_t^*M \times T_t^*M \to \mathbb{C}$ defined by the inverse of the metric η, namely $(\eta^{-1})^{\alpha\beta} = \eta^{\alpha\beta} =< dt^\alpha, dt^\beta >$, where dt^α, $1 \leq \alpha \leq n$ is a flat basis of T_t^*M. That is, we identify cotangent plane T_t^*M with tangent plane T_tM with the help of the metric η so that dt^α corresponds to $\eta^{\alpha\beta}\partial_{t^\beta}$. Then multiplication on T_tM induces multiplication on T_t^*M which takes the form

$$dt^\alpha \circ dt^\beta = \eta^{\alpha\gamma}\eta^{\beta\epsilon}c_{\gamma\epsilon\mu}(t)dt^\mu. \tag{2.12}$$

Let us now recall how associativity of Frobenius algebra A implies that F satisfies WDVV equations. We have

$$(\partial_{t^\alpha} \circ \partial_{t^\beta}) \circ \partial_{t^\gamma} = \partial_{t^\alpha} \circ (\partial_{t^\beta} \circ \partial_{t^\gamma}) \iff c^\mu_{\alpha\beta}(t)c^\rho_{\mu\gamma}(t)\partial_{t^\rho} = c^\lambda_{\beta\gamma}(t)c^\rho_{\lambda\alpha}(t)\partial_{t^\rho}.$$

The last equality implies that $c^\mu_{\alpha\beta}(t)c^\rho_{\mu\gamma}(t) = c^\lambda_{\beta\gamma}(t)c^\rho_{\lambda\alpha}(t)$. Since $c^\gamma_{\alpha\beta}(t) = \eta^{\gamma\mu}c_{\mu\alpha\beta}(t)$ we have that

$$\eta^{\mu\nu}c_{\nu\alpha\beta}(t)c^\rho_{\mu\gamma}(t) = \eta^{\lambda\nu}c_{\nu\beta\gamma}(t)c^\rho_{\lambda\alpha}(t),$$

which implies the following system of equations:

$$c_{\alpha\beta\nu}(t)\eta^{\nu\mu}c_{\mu\rho\gamma}(t) = c_{\gamma\beta\nu}(t)\eta^{\nu\mu}c_{\mu\rho\alpha}(t), \quad 1 \leq \alpha, \beta, \gamma, \rho \leq n. \tag{2.13}$$

Example 2.2.5. In dimension $n = 2$ the associativity of the algebra A is trivial since the algebra is unital. Thus, WDVV equations are empty. The form of the prepotential F is constrained only by quasi-homogeneity and the metric η. In particular, it can be checked that F takes the following form:

$$F(t^1, t^2) = \begin{cases} \frac{1}{2}(t^2)^2t^1 + f(t^1), & \text{if} \quad \eta_{22} = 0, \\ \frac{c}{6}(t^2)^3 + \frac{1}{2}(t^2)^2t^1 + (t^1)^3, & \text{if} \quad \eta_{22} \neq 0, \end{cases}$$

where $c \in \mathbb{C}^\times$ and f is a function of polynomial, logarithmic, or exponential type depending on the charge d of the Frobenius manifold.

In this work we will consider only Frobenius manifolds such that the metric η satisfies the condition $\eta_{nn} = 0$. In these cases and for an Euler vector field of the form (2.10), flat coordinates can be chosen in such a way that the matrix $\eta_{\alpha\beta}$ is anti-diagonal

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}, \quad 1 \leq \alpha, \beta \leq n. \tag{2.14}$$

Recall that from axiom (iv) of Definition 2.2.3 for $x = \partial_{t^\alpha}$ and $y = \partial_{t^\beta}$ we have that $(\mathcal{L}_E\eta)(\partial_{t^\alpha}, \partial_{t^\beta}) = (2-d)\eta_{\alpha\beta}$, which implies the condition $\eta_{\alpha\beta}(d_\alpha + d_\beta + d - 2) = 0$ for any α, β. Then in the coordinates such that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$, the numbers d_α must satisfy the following relation:

$$d_\alpha + d_{n+1-\alpha} = 2 - d, \quad 1 \le \alpha \le n. \tag{2.15}$$

2.2.2 Intersection form

On a Frobenius manifold one can define a second flat metric. It plays a key role in the theory and it is directly related to the metric η. Conventionally, it is defined as an inner product of 1-forms.

Definition 2.2.6. The (contravariant) metric $g \in \Gamma^2(TM)$ defined by

$$g(\theta, \omega) = E(\theta \circ \omega), \tag{2.16}$$

for any $\theta, \omega \in \Gamma(T^*M)$ is called the *intersection form* of the Frobenius manifold.

Let us recall the following statement.

Proposition 2.2.7. *The (covariant) metric $g \in \Gamma^2(T^*M)$ is related to the metric η by the following formula:*

$$g(E \circ u, v) = \eta(u, v), \quad u, v \in \Gamma(TM). \tag{2.17}$$

Proof. We have from Definition 2.2.6 and formula (2.12) that

$$g^{\alpha\beta}(t) = g(dt^\alpha, dt^\beta) = E^\mu (dt^\alpha \circ dt^\beta)_\mu = \eta^{\alpha\gamma}\eta^{\beta\epsilon} c_{\gamma\epsilon\mu}(t) E^\mu. \tag{2.18}$$

Hence, multiplying both sides of (2.18) with $g_{\beta\lambda}(t)$ we get

$$\delta^\alpha_\lambda = g^{\alpha\beta}(t) g_{\beta\lambda}(t) = \eta^{\alpha\gamma}\eta^{\beta\epsilon} g_{\beta\lambda}(t) c_{\gamma\epsilon\mu}(t) E^\mu.$$

Then multiplying by $\eta_{\alpha\rho}$ we obtain $\eta_{\lambda\rho} = g_{\lambda\beta}(t) c^\beta_{\rho\mu}(t) E^\mu$. The statement follows. □

Consider points $t \in M$ where there exists $E^{-1} \in T_tM$ such that $E^{-1} \circ E = e$. Then it follows by Proposition 2.2.7 that

$$g(u, v) = \eta(E^{-1}, u \circ v). \tag{2.19}$$

Thus the metric g is well-defined on the points of M where E is an invertible element of the algebra.

A key observation in the theory of Frobenius manifolds is that the prepotential F, and thus the Frobenius structure, can be reconstructed uniquely in general from the knowledge of the metric $g \in \Gamma^2(TM)$, the vector fields e, E as well as of the numbers d, d_α, $1 \le \alpha \le n$. Indeed, let us introduce the operator $\mathcal{V} : TM \to TM$ given by

$$\mathcal{V} = \frac{2-d}{2} - Q, \tag{2.20}$$

and let $F_{\alpha\beta}$ denote the components of the Hessian matrix of $F(t)$. It can be checked using formula (2.11) that for any α, β we have

$$E(F_{\alpha\beta}) = (3-d)F_{\alpha\beta} - Q_{\alpha}^{\gamma}F_{\gamma\beta} - F_{\alpha\gamma}Q_{\beta}^{\gamma} + A_{\alpha\beta}.$$

Then by Definition 2.2.6 we get [2]

$$g^{\alpha\beta}(t) = \eta^{\alpha\gamma}\eta^{\beta\epsilon}E^{\mu}c_{\mu\gamma\epsilon}(t) = F^{\alpha\beta} + \mathcal{V}_{\nu}^{\alpha}F^{\nu\beta} + F^{\alpha\nu}\mathcal{V}_{\nu}^{\beta} + A^{\alpha\beta}, \tag{2.21}$$

where

$$F^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}F_{\alpha'\beta'} \quad \text{and} \quad A^{\alpha\beta} = \eta^{\alpha\alpha'}\eta^{\beta\beta'}A_{\alpha'\beta'}. \tag{2.22}$$

If the matrix $\eta_{\alpha\beta}$ is anti-diagonal (see formula (2.14)) and Q is diagonal then equation (2.21) reads

$$g^{\alpha\beta}(t) = (d_\alpha + d_\beta + d - 1)F^{\alpha\beta} + A^{\alpha\beta}, \tag{2.23}$$

since the numbers d_α satisfy condition (2.15). Thus Frobenius structure can be recovered uniquely if $d_\alpha + d_\beta + d - 1 \neq 0$, for any $1 \leq \alpha, \beta \leq n$. The contravariant metric η can be defined directly in terms of the metric g by setting

$$\eta^{\alpha\beta} = \mathcal{L}_e g^{\alpha\beta}. \tag{2.24}$$

This is compatible with (2.22), (2.23) and $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$ which holds in the coordinate system t^α, $1 \leq \alpha \leq n$ since $d_\alpha + d_\beta + d - 1 = 1$, if $\alpha + \beta = n+1$.

Consider a metric $\widetilde{g} \in \Gamma^2(T^*M)$ with Levi-Civita connection $\nabla_{\widetilde{g}}$ and Christoffel symbols Γ_{jk}^i (in some basis). Let $\widetilde{g}^{ij} = (\widetilde{g}^{-1})^{ij}$. We define the *contravariant connection* of the metric $\widetilde{g}$ by $\nabla_{\widetilde{g}}^i := \widetilde{g}^{ij}\nabla_{\widetilde{g}j}$ with *contravariant Christoffel symbols* as $\Gamma_k^{ij} := -\widetilde{g}^{il}\Gamma_{lk}^j$.

Definition 2.2.8. Two non-proportional metrics $g_{(i)} \in \Gamma^2(TM)$, $i = 1, 2$ form a *flat pencil* if the following conditions hold:

(i) the metric $g_{(1)} + \lambda g_{(2)}$ is flat for any $\lambda \in \mathbb{C}$;

(ii) the Levi-Civita connection of the metric $g_{(1)} + \lambda g_{(2)}$ takes the form

$$\Gamma_{(1)\cdot}^{\cdot\cdot} + \lambda\Gamma_{(2)\cdot}^{\cdot\cdot},$$

where $\Gamma_{(i)\cdot}^{\cdot\cdot}$ are the contravariant Christoffel symbols of the metric $g_{(i)}^{-1}$, $i = 1, 2$.

The following proposition implies that the metrics η and g form a flat pencil on the Frobenius manifold.

[2]Note that there seem to be typos in the formula for $g^{\alpha\beta}(t)$ in [25, p. 9].

Proposition 2.2.9. *Let $g^{\alpha\beta}(t;\lambda)$ denote the matrix*

$$g^{\alpha\beta}(t;\lambda) = g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}, \quad \lambda \in \mathbb{C} \tag{2.25}$$

and let $\Sigma_\lambda \subset M$ be

$$\Sigma_\lambda = \{t \in M | \det g^{\alpha\beta}(t;\lambda) = 0\}. \tag{2.26}$$

Then the inverse matrix $g_{\alpha\beta}(t;\lambda) := (g^{\alpha\beta}(t;\lambda))^{-1}$ defines a flat metric, g_λ on $M \setminus \Sigma_\lambda$. Furthermore, the Christofell symbols for this metric take the form

$$\Gamma^a_{\beta\gamma}(t;\lambda) = -g_{\beta\nu}(t;\lambda)\Gamma^{\nu\alpha}_\gamma(t), \tag{2.27}$$

where

$$\Gamma^{\alpha\beta}_\gamma(t) := c^{\alpha\mu}_\gamma(t)\big(\frac{1}{2} - \mathcal{V}\big)^\beta_\mu. \tag{2.28}$$

In particular, $\Gamma^{\alpha\beta}_\gamma(t)$ are contravariant Christoffel symbols for the metric g given by (2.19).

Let $\overline{\nabla}$ denote the contravariant Levi-Civita connection of the metric g (2.19) with components defined by (2.28). A function $p^a_\lambda = p^a(t;\lambda)$ satisfying the conditions

$$(\overline{\nabla} - \lambda\nabla)dp^a_\lambda = 0 \tag{2.29}$$

is called a λ-*period* of the Frobenius manifold. Flatness of the metric g_λ implies the existence of n independent λ-periods on the universal covering of $M \times \mathbb{C} \setminus \bigcup_\lambda \Sigma_\lambda \times \lambda$. These define a system of flat coordinates for the metric g_λ on a small domain of $M \setminus \Sigma_\lambda$.

Let $\xi_\beta = \partial_{t^\beta} p^a_\lambda$, $\beta = 1, \ldots, n$. In the basis t^α conditions (2.29) read $(\overline{\nabla}^\alpha - \lambda\nabla^\alpha)\xi_\beta = 0$, for any $1 \leq \alpha, \beta \leq n$. Then we get

$$(\overline{\nabla}^\alpha - \lambda\nabla^\alpha)\xi_\beta = (g^{\alpha\gamma}(t)\overline{\nabla}_\gamma - \lambda\eta^{\alpha\gamma}\nabla_\gamma)\xi_\beta = (g^{\alpha\gamma}(t) - \lambda\eta^{\alpha\gamma})\partial_{t^\gamma}\xi_\beta - g^{\alpha\gamma}(t)\Gamma^\mu_{\gamma\beta}(t)\xi_\mu.$$

It follows by Proposition 2.2.9 that conditions (2.29) take the form

$$(g^{\alpha\gamma}(t) - \lambda\eta^{\alpha\gamma})\partial_{t^\gamma}\xi_\beta = c^{\alpha\rho}_\beta(t)(\mathcal{V} - \frac{1}{2})^\mu_\rho\xi_\mu, \quad 1 \leq \alpha, \beta \leq n. \tag{2.30}$$

Let $\mathcal{U}$ be the operator of multiplication by E, $\mathcal{U} = E\circ : TM \to TM$. That is

$$\mathcal{U}^\alpha_\beta(t) = (E\circ)^\alpha_\beta = g^{\alpha\epsilon}(t)\eta_{\epsilon\beta}. \tag{2.31}$$

Let $C_\alpha(t)$ be the $n \times n$ matrices defined by $(C_\alpha(t))^\beta_\gamma = c^\beta_{\alpha\gamma}(t)$ and consider the vector $\xi(t) = (\xi_1, \ldots, \xi_n)$. Then conditions (2.30) are equivalent in a matrix form to the following system of equations:

$$\partial_{t^\alpha}\xi(t)(\mathcal{U}-\lambda)=\xi(t)(\mathcal{V}-\frac{1}{2})C_\alpha, \qquad 1\le\alpha\le n. \tag{2.32}$$

By definition the intersection form degenerates precisely on $\Sigma := \Sigma_0$. A system of n independent periods $p^a := p^a(t;0)$ gives flat coordinates of the intersection form. They determine a local isometry between the space $M\setminus\Sigma$ and $\mathbb{C}^n$. More precisely, this map is defined by

$$\boldsymbol{p}: M\setminus\Sigma\to\mathbb{C}^n, \qquad t\mapsto \boldsymbol{p}(t):=(p^1,\dots,p^n), \tag{2.33}$$

and is called *period mapping*. Thus, the functions p^a $(a=1,\dots,n)$ can serve locally as coordinates on $\mathbb{C}^n$ and the intersection form $g^{ab}=(dp^a,dp^b)$ can also be viewed as an inner product on $\mathbb{C}^n$.

2.2.3 Semisimplicity and canonical coordinates

We will study a special class of Frobenius manifolds which possess additional structure. The Frobenius algebra at a generic point on the manifold is required to be semisimple as defined below.

Definition 2.2.10. [43] A module over an algebra is called *semisimple* if it is the direct sum of its simple submodules.

Definition 2.2.11. [43] An algebra A is *semisimple* if all non-zero A-modules are semisimple.

By Artin-Wedderburn Theorem any finite dimensional semisimple algebra over $\mathbb{C}$ is isomorphic to

$$\prod_{i\in\mathbb{N}} M_{n_i}(\mathbb{C}), \quad n_i\in\mathbb{N},$$

where $M_{n_i}(\mathbb{C})$ is the matrix algebra of $n_i\times n_i$ matrices over $\mathbb{C}$. Therefore, an n-dimensional commutative semisimple Frobenius algebra A is isomorphic to n copies of $\mathbb{C}$. There exists a basis $e_i\in A$, $1\le i\le n$ (idempotents) such that the multiplication becomes

$$e_i\circ e_j=\delta_{ij}e_j.$$

Definition 2.2.12. A Frobenius manifold M is called *semisimple* if the family of n-dimensional algebras T_tM is semisimple at any generic point $t\in M$. Such a point $t\in M$ is called *semisimple point.*

Locally, near a semisimple point $t\in M$ there exists a basis of vector fields δ_i, $1\le i\le n$, with the property

$$\delta_i \circ \delta_j = \delta_{ij}\delta_j. \tag{2.34}$$

These vector fields are called *idempotent vector fields* and are unique up to renumbering. It can be checked that δ_i commute pairwise, $[\delta_i, \delta_j] = 0$ for all $1 \leq i, j \leq n$ and hence δ_i determine a canonical coordinate system u_i such that $\delta_i = \frac{\partial}{\partial u_i}$, $1 \leq i \leq n$. *Canonical coordinates* u_i are unique up to shifts and permutations. Recall that the Euler field E satisfies $\mathcal{L}_E(\circ) = \circ$ (formula (2.7)) and hence takes the form[3] $E = \sum_{i=1}^n (u_i + c_i)\delta_i$ for some $c_i \in \mathbb{C}$. Therefore near a semisimple point t the eigenvalues of the operator $\mathcal{U}(= E\circ)$ are canonical coordinates (up to the mentioned non-uniqueness).

Canonical coordinates can be chosen such that the Euler vector field takes the form

$$E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}. \tag{2.35}$$

Similarly note that by (2.34) we have $(\sum_{i=1}^n \delta_i) \circ X = (\sum_{i,j=1}^n \delta_i \circ \delta_j) X^j = X$ for any $X = \sum_{j=1}^n X^j \delta_j \in T_u M$. Therefore the identity field can be represented as

$$e = \sum_{i=1}^n \frac{\partial}{\partial u_i}. \tag{2.36}$$

It follows by formula (2.34) that the metric $\eta(u)$ is diagonal

$$\eta_{ij}(u) =< \delta_i, \delta_j >= \eta_{ii}(u)\delta_{ij}, \quad 1 \leq i, j \leq n, \tag{2.37}$$

where $\eta_{ii}(u)$ are some non-zero functions. Similarly it is easy to see by Proposition 2.2.7 and formula (2.37) that

$$g_{ij}(u) = u_i^{-1}\eta_{ii}(u)\delta_{ij}, \quad 1 \leq i, j \leq n.$$

2.2.4 Natural submanifolds

It is natural to consider submanifolds of a Frobenius manifold which behave well with respect to the restricted Frobenius structure. Strachan considered submanifolds of a Frobenius manifold which carry a Frobenius algebra structure on each tangent space and studied their differential-geometric properties [81].

Definition 2.2.13. [81] A *natural submanifold* N of a Frobenius manifold M is a submanifold $N \subset M$ such that the Euler vector field at any $t \in N$ is tangential to N and induced Frobenius multiplication on N is closed, namely $TN \circ TN \subset TN$.

Definition 2.2.14. [46] The *caustic* $K \subset M$ is the set of points where M is not semisimple.

[3] Throughout we employ the convention of no summation over repeated indices when working with canonical coordinates.

The *discriminant* $\Sigma \subset M$ is the set $\Sigma = \{t \in M |\, \mathcal{U}$ is not invertible on $T_t M\}$.

Let I be the set $I = \{1, \ldots, n\}$. It is mentioned in [81] that the natural submanifolds of a semisimple Frobenius manifold may be obtained as the level sets

$$\{u_i = 0 |\, i \in \mathcal{I}\} \cap \{u_i = u_j \,|\, i \neq j, (i, j) \in \mathcal{J}\},$$

for some arbitrary subsets $\mathcal{I} \subset I$, $\mathcal{J} \subset I \times I$, provided that the 'unconstrained' variables u_i define a coordinate system on the submanifold, which requires further analysis.

2.2.5 Superpotential description of Frobenius manifolds

The theory of topological Landau-Ginzburg (LG) models involves a holomorphic function called superpotential, depending in general on several complex variables. In the simplest case this function is a polynomial depending on a variable p. The moduli space in the LG theory can be described via a family of actions which depend on an additional set of parameters $a = (a_1, \ldots, a_n)$ through the deformed LG superpotential $\lambda(p) = \lambda(p, a)$ [84] (see also [62]). Dubrovin showed that the space of parameters carries a Frobenius structure [22]. Moreover, any semisimple Frobenius manifold M admits a description through LG superpotential such that

$$u_i(a) = \lambda(q_i(a), a), \quad i = 1, \ldots, n,$$

where q_i are critical points of λ:

$$\lambda'(p) = \frac{d\lambda(p)}{dp}\bigg|_{p=q_i(a)} = 0,$$

that is the canonical coordinates on the Frobenius manifold are precisely the critical values of the superpotential λ. This assumes that the roots of $\lambda'(p)$ are generically distinct. The expressions for the metrics η, g and Frobenius multiplication are given by the following residue formulae:

$$\eta(\partial_i, \partial_j) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\lambda(p))\partial_j(\lambda(p))}{\lambda'(p)} dp, \tag{2.38}$$

$$g(\partial_i, \partial_j) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\log\lambda(p))\partial_j(\log\lambda(p))}{(\log\lambda)'(p)} dp, \tag{2.39}$$

$$\eta(\partial_i \circ \partial_j, \partial_k) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\lambda(p))\partial_j(\lambda(p))\partial_k(\lambda(p))}{\lambda'(p)} dp, \tag{2.40}$$

where ∂_i denote some vector fields on M and $\lambda'(p) = \frac{d\lambda(p)}{dp}$. The following example is analogous to Example 1.7 in [22] (see also references [48], [91] and Chapter 3 for details of similar calculations).

Example 2.2.15. Let M be the (affine) space of complex polynomials of the form

$$\lambda(p) = p^{2n} + a_1 p^{2n-2} + \cdots + a_n, \quad a_1, \ldots, a_n \in \mathbb{C}. \tag{2.41}$$

Tangent vectors to M at a point $a = (a_1, \ldots, a_n)$ take the form

$$\dot{\lambda}(p) = \dot{a}_1 p^{2n-2} + \cdots + \dot{a}_n,$$

where the 'dot' means derivative with respect to the parameter s on a curve passing through the point a. The algebra on the tangent space of M at any point $a \in M$ is determined by formula (2.40) and is commutative, associative with unity. Moreover it is isomorphic for any $\alpha \in M$ to the algebra of truncated polynomials

$$A_\lambda = \mathbb{C}[p]/\lambda^{'}(p).$$

Indeed, let us define a bilinear form $\eta = \eta_\lambda$ on A_λ given by

$$\eta(f(p), g(p)) = -\operatorname{res}|_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)} dp.$$

Let also define polynomials

$$k(p) = \partial_i \lambda(p), \quad l(p) = \partial_j \lambda(p), \quad h(p) = \partial_k \lambda(p),$$

for some vector fields $\partial_i, \partial_j, \partial_k$ on M. Note that we have

$$k(p)l(p) = \lambda^{'}(p)q(p) + r(p), \tag{2.42}$$

for some polynomials $q(p), r(p)$ with $\deg r(p) < 2n - 1$. In the algebra A_λ the product (2.42) takes the form $k(p)l(p) = r(p)$. Then using (2.42) formula (2.40) can be written as

$$\begin{aligned}
\eta(\partial_i \circ \partial_j, \partial_k) &= -\operatorname{res}|_{p=\infty} \frac{k(p)l(p)h(p)}{\lambda'(p)} dp \\
&= -\operatorname{res}|_{p=\infty} \frac{r(p)h(p)}{\lambda'(p)} dp - \operatorname{res}|_{p=\infty} q(p)h(p)dp \\
&= -\operatorname{res}|_{p=\infty} \frac{r(p)h(p)}{\lambda'(p)} dp.
\end{aligned}$$

Then $\eta(\partial_i \circ \partial_j, \partial_k)$ coincides with the bilinear form $\eta(r(p), h(p)) = \eta(k(p)l(p), h(p))$ and the multiplication $\circ$ is the same as the multiplication in the algebra A_λ.

The space M is a Frobenius manifold, $T_\lambda M = (A_\lambda, \eta)$ where the metric η and Frobenius multiplication are given by formulae (2.38) and (2.40) respectively. It follows by formula (2.41) that

$$\lambda^{'}(p) = 2np^{-1}\prod_{j=1}^{n}(p^2 - q_j^2), \tag{2.43}$$

for some points $q_i \in \mathbb{C}$ ($q_i = 0$ for some i). Then we define coordinates u_i by $u_i = \lambda(q_i)$, $i = 1, \dots, n$.

Let us now show that $\eta(u)$ is diagonal (cf. formula (2.37)) and that u_i are canonical coordinates for M. By definition we have

$$\delta_{ij} = \frac{\partial u_j}{\partial u_i} = \partial_{u_i}\lambda(q_j).$$

Then considering the Taylor expansion of $\lambda(p)$ centred at $p = q_j$ we have $\lambda(p) = \lambda(q_j) + \mathcal{O}$, where $\mathcal{O}$ denotes the rest of the terms, and $\mathcal{O}$ has zero of order at least two at $p = q_j$. Then

$$\partial_{u_i}\lambda(p)|_{p=q_j} = \partial_{u_i}\lambda(q_j) = \delta_{ij}. \tag{2.44}$$

It follows by Lagrange interpolation that

$$\partial_{u_i}\lambda(p) = \frac{2\epsilon_i p\lambda^{'}(p)}{(p^2 - q_i^2)\lambda''(q_i)}, \tag{2.45}$$

where $\epsilon_i = 1$ if $q_i \neq 0$ and $\epsilon_i = \frac{1}{2}$ if $q_i = 0$[4]. Let us now consider formula (2.38), $\eta(\partial_{u_i}, \partial_{u_j})$. In the case when $i \neq j$ the polynomial $\partial_{u_i}\lambda(p)\partial_{u_j}\lambda(p)$ is divisible by $\lambda^{'}(p)$ and the residues at the points q_i ($1 \leq i \leq n$) are trivial. Hence $\eta(\partial_{u_i}, \partial_{u_j}) = 0$. Let us now consider the case when $i = j$. We get

$$\eta(\partial_{u_i}, \partial_{u_i}) = \sum_{p_s:\lambda^{'}(p_s)=0} res|_{p=p_s} \frac{\partial_{u_i}(\lambda(p))^2}{\lambda^{'}(p)}dp = \sum_{p_s:\lambda^{'}(p_s)=0} res|_{p=p_s} \frac{4\epsilon_i^2 p^2}{(p^2 - q_i^2)^2}\frac{\lambda^{'}(p)}{\lambda''(q_i)^2}dp.$$

Let us note that

$$\lambda^{''}(q_i) = 4\epsilon_i n\prod_{\substack{j=1\\ j\neq i}}^{n}(q_i^2 - q_j^2). \tag{2.46}$$

Then

$$\eta(\partial_{u_i}, \partial_{u_i}) = 8\epsilon_i^2 n \sum_{p_s:\lambda^{'}(p_s)=0} res|_{p=p_s} \frac{p\prod_{j\neq i}(p^2 - q_j^2)}{p^2 - q_i^2}\frac{dp}{\lambda''(q_i)^2} = \frac{2\epsilon_i}{\lambda''(q_i)}. \tag{2.47}$$

It follows directly by formula (2.39) that $g(\partial_{u_i}, \partial_{u_j}) = u_i^{-1}\eta(\partial_{u_i}, \partial_{u_j})$, which implies

$$g(\partial_{u_i}, \partial_{u_j}) = \frac{2\epsilon_i\delta_{ij}}{u_i\lambda''(q_i)} \tag{2.48}$$

[4] Note that the case $q_i = 0$ is not considered in [91], thus formula (2.45) differs from formula (5.19) in [91] by a factor of ϵ_i.

by formula (2.47). Similarly, one can show that for the Frobenius multiplication (2.40) in the case when $i \neq j$ or $j \neq k$ the polynomial $\partial_i\lambda(p)\partial_j\lambda(p)\partial_k\lambda(p)$ is divisible by $\lambda^{'}(p)$. Therefore the residues at the points q_i $(1 \leq i \leq n)$ are trivial. Then it can be checked that the multiplication takes the form

$$\eta(\partial_{u_i} \circ \partial_{u_j}, \partial_{u_k}) = \frac{2\epsilon_i\delta_{ij}\delta_{jk}}{\lambda''(q_i)}. \tag{2.49}$$

Formulae (2.47) and (2.49) imply that $\partial_{u_i} \circ \partial_{u_j} = \delta_{ij}\partial_{u_j}$, as required.

As a final remark in this example let us note that the discriminant Σ corresponds to the set of polynomials $\lambda(p)$ in M which have zero as a critical value.

2.2.6 Almost dual Frobenius manifolds

Dubrovin showed that given a Frobenius manifold M one can associate a new structure on $M \setminus \Sigma$, called almost dual Frobenius manifold. For any $t \in M \setminus \Sigma$ a new multiplication of tangent vectors $u, v \in T_tM$ is defined by the following formula:

$$u * v = E^{-1} \circ u \circ v, \tag{2.50}$$

where E^{-1} is the inverse of the Euler vector field associated to M. Note that $E * u = u$, hence the Euler vector field E is the identity for the product $*$. The multiplication (2.50) together with the metric g (2.19) and the Euler vector field E satisfy all the axioms of the Frobenius manifold but constancy of the identity E.

Let us define the algebra $\overset{*}{A}_t = (T_tM, *)$, $t \in M \setminus \Sigma$. It follows from formula (2.50) that $\overset{*}{A}_t$ is unital, commutative and associative. Moreover, $\overset{*}{A}_t$ is a Frobenius algebra since the multiplication $*$ is compatible with the metric g, that is for any $u, v, w \in T_tM$ we have the following property by formula (2.19):

$$g(u * v, w) = \eta(E^{-1}, (u * v) \circ w) = \eta(E^{-1}, u \circ (v * w)) = g(u, v * w),$$

since $(u * v) \circ w = u \circ (v * w)$ by (2.50). At any point $t \in M \setminus \Sigma$, the map $\phi : \overset{*}{A}_t \to A_t$, $u \mapsto \phi(u) := E^{-1} \circ u$ is an algebra isomorphism. Indeed for any $u, v \in T_tM$ it follows from (2.50) that

$$\phi(u * v) = E^{-1} \circ (u * v) = E^{-1} \circ E^{-1} \circ u \circ v = \phi(u) \circ \phi(v). \tag{2.51}$$

Note that in the basis t^α, $(1 \leq \alpha \leq n)$ we have

$$\frac{\partial}{\partial t^\alpha} \mapsto \phi(\frac{\partial}{\partial t^\alpha}) = g_{\alpha\lambda}(t)\eta^{\lambda\gamma}\frac{\partial}{\partial t^\gamma}. \tag{2.52}$$

The following lemma takes place.

Lemma 2.2.16. *The multiplications* $*$, $\circ$ *coincide on the cotangent planes* T_t^*M, $t \in M \setminus \Sigma$.

Proof. Let $\overset{*}{c}{}^{\mu\nu}_{\lambda}(t)$ denote the structure constants of the dual of $\overset{*}{A}_t$. Let us consider equality (2.51) with $u = \partial_{t^\alpha}$, $v = \partial_{t^\beta}$. Using (2.52) we have

$$\overset{*}{c}{}^{\gamma}_{\alpha\beta}(t) g_{\gamma\theta}(t)\eta^{\theta\sigma}\partial_{t^\sigma} = c^{\sigma}_{\mu\nu}(t) g_{\alpha\epsilon}(t) g_{\beta\rho}(t)\eta^{\epsilon\mu}\eta^{\rho\nu}\partial_{t^\sigma},$$

which implies that $\overset{*}{c}{}^{\gamma}_{\alpha\beta}(t) g_{\gamma\theta}(t)\eta^{\theta\sigma} = c^{\sigma}_{\mu\nu}(t) g_{\alpha\epsilon}(t) g_{\beta\rho}(t)\eta^{\epsilon\mu}\eta^{\rho\nu}$. Multiplying by $\eta_{\sigma\lambda}$ we get $\overset{*}{c}{}^{\gamma}_{\alpha\beta}(t) g_{\gamma\lambda}(t) = c^{\epsilon\rho}_{\lambda}(t) g_{\alpha\epsilon}(t) g_{\beta\rho}(t)$. It follows that $\overset{*}{c}{}^{\mu\nu}_{\lambda}(t) = c^{\mu\nu}_{\lambda}(t)$ by multiplying with $g^{\mu\alpha}(t) g^{\nu\beta}(t)$, as required. $\square$

Using that $\mathcal{L}_E\eta^{\alpha\beta} = (d-2)\eta^{\alpha\beta}$, it follows that

$$\mathcal{L}_E c^{\alpha\beta}_{\gamma}(t) = (d-1) c^{\alpha\beta}_{\gamma}(t), \quad \text{and} \quad \mathcal{L}_E g^{\alpha\beta}(t) = (d-1) g^{\alpha\beta}(t).$$

Thus E is the Euler vector field for the $*$ multiplication as well. Let us define the tensor $\overset{*}{c}_{\alpha\beta\gamma}(t) := g_{\alpha\lambda}(t) g_{\beta\mu}(t)\overset{*}{c}{}^{\lambda\mu}_{\gamma}(t)$. It can be checked that the tensor $\overline{\nabla}^{\gamma} c^{\alpha\beta}_{\rho}(t) := g^{\gamma\mu}(t)\overline{\nabla}_{\mu} c^{\alpha\beta}_{\rho}(t)$ is symmetric for any ρ with respect to α, β and γ. Thus the covariant derivatives

$$\overline{\nabla}_{\rho}\overset{*}{c}_{\alpha\beta\gamma}(t) = g_{\alpha\mu}(t) g_{\beta\nu}(t) g_{\rho\lambda}(t)\overline{\nabla}^{\lambda} c^{\mu\nu}_{\gamma}(t)$$

are totally symmetric. In the flat coordinates p^a, $1 \le a \le n$ of the metric g this implies that locally there is a function F_* such that

$$\overset{*}{c}_{abc}(p) = \frac{\partial^3 F_*}{\partial p^a \partial p^b \partial p^c} = \frac{\partial t^\alpha}{\partial p^a}\frac{\partial t^\beta}{\partial p^b}\frac{\partial t^\gamma}{\partial p^c}\overset{*}{c}_{\alpha\beta\gamma}(t) = g_{aj} g_{bl}\frac{\partial t^\gamma}{\partial p^c}\frac{\partial p^j}{\partial t^\mu}\frac{\partial p^l}{\partial t^\nu} c^{\mu\nu}_{\gamma}(t), \tag{2.53}$$

where $g_{ab} = g(\frac{\partial}{\partial p^a}, \frac{\partial}{\partial p^b})$ is the Gram constant matrix of the metric g in the flat coordinates p^a. Associativity of the algebra $\overset{*}{A}_p$ implies that the function $F_*(p)$ satisfies the following system of equations:

$$\overset{*}{c}_{abl}(p) g^{lm}\overset{*}{c}_{mck}(p) = \overset{*}{c}_{acl}(p) g^{lm}\overset{*}{c}_{mbk}(p), \tag{2.54}$$

for any $1 \le a, b, c, k \le n$.

Definition 2.2.17. An *almost dual Frobenius manifold* of charge $d \neq 1$ is the manifold $M \setminus \Sigma$ with a smoothly varying (commutative) Frobenius algebra structure on each tangent space, $T_pM = (\overset{*}{A}_p, g)$, $p \in M \setminus \Sigma$ satisfying the following axioms:

(i) the metric g is flat;

(ii) in the flat coordinates p^a of the metric g the structure constants $\overset{*}{c}{}^{a}_{bc}(p)$ of the algebra

$\overset{*}{A}_p$ can be represented locally in the form

$$\overset{*}{c}{}^a_{bc}(p) = g^{al}\frac{\partial^3 F_*(p)}{\partial p^l \partial p^b \partial p^c}, \tag{2.55}$$

for some function $F_*(p)$ and $g^{ab} = (dp^a, dp^b)$.

(iii) the Euler vector field takes the form

$$E = \frac{1-d}{2}\sum_{\alpha=1}^{n} p^\alpha \frac{\partial}{\partial p^\alpha}, \tag{2.56}$$

and the function $F_*(p)$ must satisfy the following homogeneity condition:

$$\mathcal{L}_E F_*(p) = (1-d)F_*(p) + \text{quadratic terms in } p;$$

(iv) there exists a vector field $e(p)$ being an invertible element of $\overset{*}{A}_p$, $p \in M \setminus \Sigma$ such that it acts by shifts $\nu \mapsto \nu - 1$ on the solutions of the system of equations

$$\frac{\partial^2 \tilde{p}}{\partial p^a \partial p^b} = \nu \overset{*}{c}{}^c_{ab}(p)\frac{\partial \tilde{p}}{\partial p^c}, \tag{2.57}$$

for some function $\tilde{p} = \tilde{p}(p; \nu)$.

Definition 2.2.18. Any function $\tilde{p}(p; \nu)$ satisfying the system of equations (2.57) is called a *twisted period* of the Frobenius manifold.

Equations (2.57) arise from the vanishing of the torsion and curvature of the so-called *deformed flat connection* $\nabla^{(\nu)}$ defined on $(M \setminus \Sigma) \times \mathbb{C}$ for any $u, v \in T_tM$, $t \in M \setminus \Sigma$ by the formula

$$\nabla^{(\nu)}_u v = \overline{\nabla}_u v + \nu\, u * v, \quad \nu \in \mathbb{C}.$$

One can show that in the flat coordinates t^α equations (2.57) can be written in an equivalent matrix form

$$\frac{\partial \tilde{\xi}(t)}{\partial t^\alpha}\mathcal{U} = \tilde{\xi}(t)(\mathcal{V} + \nu - \frac{1}{2}), \quad 1 \le \alpha \le n, \tag{2.58}$$

where $\tilde{\xi}(t) = (\partial_{t^1}\tilde{p}(t;\nu), \ldots, \partial_{t^n}\tilde{p}(t;\nu))$. Note that for the special value $\nu = 0$ the system (2.58) coincides with (2.32) with $\lambda = 0$ for the flat coordinates of the metric g. The vector field e of axiom (iv) in Definition 2.2.17 coincides with the identity field $e = \partial_{t^n}$ of the Frobenius manifold. It can be checked that

$$\dot{\xi}(t) := \frac{\partial \tilde{\xi}(t)}{\partial t^n},$$

satisfies
$$\frac{\partial \dot{\xi}(t)}{\partial t^\alpha}\mathcal{U} = \dot{\xi}(t)(\mathcal{V} + \nu - \frac{3}{2}), \quad 1 \leq \alpha \leq n.$$
This implies that axiom (iv) of Definition 2.2.17 is satisfied.

2.3 Generalised WDVV Equations

Let F be a function defined locally in terms of some variables[5] $t = (t^1, t^2, \ldots, t^n)$. Let F_α be the $n \times n$ matrix constructed from the third-order derivatives of the function F:

$$(F_\alpha)_{\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} =: c_{\alpha\beta\gamma}(t).$$

Then *generalised WDVV equations* take the following form:

$$F_\alpha F_\beta^{-1} F_\gamma = F_\gamma F_\beta^{-1} F_\alpha, \tag{2.59}$$

for any indices α, β, γ. It is immediate that the system (2.59) is non-trivial only when F depends on at least three variables. These equations appeared in the works of Marshakov, Mironov and Morozov in the context of Seiberg-Witten theory [63–65]. The same authors indicated that generalised WDVV equations can be written in an equivalent form where F_β in (2.59) is replaced by any invertible linear combination of the matrices F_α (see also [66]). Let us explain this in the next two propositions. Let us define the matrix

$$G = \sum_{\beta=1}^{n} b_\beta(t) F_\beta, \tag{2.60}$$

where the coefficients $b_\beta(t)$ are some functions in t, and suppose that G^{-1} exists. Let us also assume below that the matrices F_α are invertible for any α.

Proposition 2.3.1. [66] *Suppose that a function F satisfies the system of equations*

$$F_\alpha F_{\epsilon_0}^{-1} F_\beta = F_\beta F_{\epsilon_0}^{-1} F_\alpha, \tag{2.61}$$

for some fixed index ϵ_0 and for any indices α, β. Then for any α, β the following equations hold:

$$F_\alpha G^{-1} F_\beta = F_\beta G^{-1} F_\alpha. \tag{2.62}$$

Proof. For any index α let $C_\alpha^{(\epsilon_0)}$ be the matrix $C_\alpha^{(\epsilon_0)} = F_{\epsilon_0}^{-1} F_\alpha$. Then equations (2.61) can be written in the equivalent form

$$[C_\alpha^{(\epsilon_0)}, C_\beta^{(\epsilon_0)}] = 0. \tag{2.63}$$

[5] See discussion in [62] for infinite dimensional situation.

It follows that for any invertible linear combination $H = \sum_{\beta=1}^{n} b_\beta(t) C_\beta^{(\epsilon_0)}$ and any index α we have $[(C_\alpha^{(\epsilon_0)})^{-1}, H] = 0$, and hence by taking inverse

$$[C_\alpha^{(\epsilon_0)}, H^{-1}] = 0. \tag{2.64}$$

Note that $H = F_{\epsilon_0}^{-1} G$, hence $G^{-1} = H^{-1} F_{\epsilon_0}^{-1}$. The left-hand-side of formula (2.62) takes the form

$$F_\alpha G^{-1} F_\beta = F_\alpha H^{-1} F_{\epsilon_0}^{-1} F_\beta = F_{\epsilon_0} C_\alpha^{(\epsilon_0)} H^{-1} C_\beta^{(\epsilon_0)}, \tag{2.65}$$

and it follows from relations (2.63), (2.64) that the right-hand-side of (2.65) is symmetric under the swap of α and β, as required. □

Note in particular that in the case where $G = F_\gamma$ for some γ we have from Proposition 2.3.1 that system (2.61) implies that F satisfies generalised WDVV equations. Now we prove a converse statement in the following form.

Proposition 2.3.2. [66] *Suppose that a function F satisfies the system of equations* (2.62) *for all $\alpha, \beta = 1, \dots, n$. Then F satisfies generalised WDVV equations* (2.59).

Proof. The system (2.62) is equivalent to

$$[G^{-1} F_\alpha, G^{-1} F_\beta] = 0, \quad \text{for any} \quad \alpha, \beta = 1, \dots, n. \tag{2.66}$$

Note that the left-hand-side of (2.59) can be written as

$$F_\alpha F_\beta^{-1} F_\gamma = G (G^{-1} F_\alpha)(G^{-1} F_\beta)^{-1} (G^{-1} F_\gamma). \tag{2.67}$$

Using (2.66) we have that the right-hand-side of (2.67) is symmetric under the swap of α and γ, and thus the statement follows. □

It follows from Propositions 2.3.1 and 2.3.2 that generalised WDVV equations are equivalent to the system (2.62) for any particular choice of matrix G of the form (2.60). The matrix G (resp. G^{-1}) which is usually referred to as the 'metric' can be used to lower (resp. raise) indices,

$$c_{\alpha\beta\gamma}(t) = G_{\alpha\epsilon}(t) c_{\beta\gamma}^{\epsilon}(t),$$

where $c_{\alpha\beta}^{\gamma}(t) = c_{\alpha\beta}^{\gamma}$ are the matrix entries of $C_\alpha : (C_\alpha)_\beta^\gamma = c_{\alpha\beta}^\gamma$, and $G_{\alpha\beta}(t)$ are the matrix entries of $G : G = (G_{\alpha\beta}(t))$.

Note that contrary to the formulation of WDVV equations in Section 2.1 in the case of generalised WDVV equations all indices are treated on an equal footing and no constancy of the metric is assumed.

2.4 Finite Coxeter groups and their orbit spaces

In this section we recall main properties from the theory of finite Coxeter groups [17, 50, 51] and the corresponding orbit spaces [77, 78]. Let V be a real n-dimensional Euclidean space endowed with a positive definite symmetric bilinear form[6] $(\,,\,) =: g$.

Definition 2.4.1. Let $u, \alpha \in V$. A *reflection* is a linear operator s_α on V defined by

$$u \mapsto s_\alpha u = u - 2\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha.$$

Definition 2.4.2. Let $\mathcal{R}$ be a finite set of non-zero vectors in V. The set $\mathcal{R}$ is called a *root system* if

(i) for every $\alpha \in \mathcal{R}$, the set $\mathcal{R}$ is closed under the reflection s_α, $s_\alpha\mathcal{R} = \mathcal{R}$;

(ii) the only colinear vectors to a *root* $\alpha \in \mathcal{R}$ are α and $-\alpha$.

For any $\alpha \in \mathcal{R}$ we define the corresponding *mirror* to be the hyperplane $\Pi_\alpha = \{x \in V | (\alpha, x) = 0\}$, then $s_\alpha\Pi_\alpha = \Pi_\alpha$. The group $W \subset O(V)$ defined by $W = \langle s_\alpha | \alpha \in \mathcal{R}\rangle$ is called *reflection group* and is associated to the root system $\mathcal{R}$. Note that W is finite. To see this, let $S_\mathcal{R}$ denote the symmetric group on the set $\mathcal{R}$ and define a group homomorphism $\phi : W \to S_\mathcal{R}$ by sending $w \in W$ to the element of $S_\mathcal{R}$ which permutes the roots in the same way as w. Then $\ker\phi = \{1\}$ since only the identity element of W can fix all elements of $\mathcal{R}$.

Definition 2.4.3. A root system $\mathcal{R}$ is called *crystallographic root system* if for all $\alpha, \beta \in \mathcal{R}$ the following constraint is satisfied:

$$2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}.$$

The associated reflection group W is called *Weyl group*.

The choice of a root system $\mathcal{R}$ is not unique: the map

$$\{\text{Root systems}\} \to \{\text{Reflection groups}\}$$

is not injective since different collections of vectors satisfying the geometric conditions (i), (ii) in Definition 2.4.2 can generate the same reflection group.

Given a root system $\mathcal{R}$ this completely determines W. However, $\mathcal{R}$ can be very large and thus it is more natural to define a subset of vectors in $\mathcal{R}$ which completely describes the set $\mathcal{R}$. Let H be a generic hyperplane in V and fix a vector $u \in V$ such that it is normal to the hyperplane H. Let V_+ denote the *open half-space* $V_+ = \{x \in V | (x, u) > 0\}$.

[6] The choice of notation g for this form will become apparent below.

Definition 2.4.4. A *positive root system* in a root system $\mathcal{R}$ is a subset $\mathcal{R}_+ \subset \mathcal{R}$ such that $\mathcal{R}_+ = \mathcal{R} \cap V_+$.

It is clear that $\mathcal{R}$ can be decomposed as $\mathcal{R} = \mathcal{R}_+ \sqcup (-\mathcal{R}_+)$.

Definition 2.4.5. The set $\Delta \subset \mathcal{R}_+$ is a *simple system* if

(i) it is a basis for the $\mathbb{R}$-span of $\mathcal{R}$ in V;

(ii) each $\alpha \in \mathcal{R}$ is a linear combination of elements of Δ with coefficients all of the same sign.

Note that for any $w \in W$, the subset $w\Delta$ is also a simple system with corresponding positive root system $w\mathcal{R}_+$.

Definition 2.4.6. Let $\mathcal{R}$ be a root system with associated reflection group W. Let $U = \langle R \rangle \subset V$. The *rank* of $\mathcal{R}$ (and of W) is the dimension of the vector space U.

Let $\mathcal{R}_1$, $\mathcal{R}_2 \subset \mathcal{R}$. We write $(\mathcal{R}_1, \mathcal{R}_2) = 0$ if $(\alpha, \beta) = 0$ for any $\alpha \in \mathcal{R}_1$ and $\beta \in \mathcal{R}_2$. Let us recall the following definition of a reducible root system.

Definition 2.4.7. Let $\mathcal{R}$ be a root system and let $\mathcal{R}_1$, $\mathcal{R}_2 \subset \mathcal{R}$ be such that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ and $(\mathcal{R}_1, \mathcal{R}_2) = 0$. Then we call $\mathcal{R}$ *reducible root system.*

Definition 2.4.8. A root system $\mathcal{R}$ is called *irreducible* if it is not reducible.

Definition 2.4.9. A subset $\mathcal{R}' \subset \mathcal{R}$ is called a *subsystem* of $\mathcal{R}$ if $\mathcal{R}' = \mathcal{R} \cap U$ for some vector subspace $U \subset V$.

Note that a subsystem is also a root system.

Proposition 2.4.10. [17, Ch. VI] *Let V_1 be a vector subspace of V. Let $\mathcal{R}'$ be the subsystem $\mathcal{R}' = \mathcal{R} \cap V_1$. Let V_2 be the vector subspace $V_2 = \langle \mathcal{R}' \rangle$. Then $V_2 \subseteq V_1$ and $\mathcal{R}'$ is a root system in both V_1 and V_2.*

Lemma 2.4.11. *Let $\mathcal{R}$ be a reducible root system, so that $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$ for some subsets $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ such that $(\mathcal{R}_1, \mathcal{R}_2) = 0$. Then $\mathcal{R}_i$ $(1 \leq i \leq 2)$ is a subsystem of $\mathcal{R}$.*

Proof. Let $U = \langle \mathcal{R} \rangle \subset V$ and consider the corresponding vector space decomposition $U = U_1 \oplus U_2$, where $U_i = \langle \mathcal{R}_i \rangle$, $1 \leq i \leq 2$. Then $\mathcal{R}_i = \mathcal{R} \cap U_i$, $(1 \leq i \leq 2)$ as required. $\square$

Note that every positive root system contains a unique simple system. Given a simple system Δ the group W is generated by s_α, $\alpha \in \Delta$. Indeed let us recall the following statement.

Theorem 2.4.12. *Let $\Delta \subset \mathcal{R}_+$ be a simple system. Then W is generated by the set $S = \{s_\alpha | \alpha \in \Delta\}$ subject to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \quad \alpha, \beta \in \Delta,$$

where $m(\alpha, \beta) \in \mathbb{Z}_{\geq 0}$ and $m(\alpha, \alpha) = 1$, for all $\alpha \in \Delta$.

Any (finite) group W having such a presentation relative to a generating set S is called a (finite) *Coxeter group* and the pair (W, S) is called a *Coxeter system.* In addition, W is determined up to an isomorphism by the collection of $m(\alpha, \beta)$. One can encode this information in a graph with vertex set in one-to-one correspondence with Δ, which gives rise to the notion of a *Coxeter graph.* These are constructed as follows. To each pair of simple roots α, β one associates the corresponding vertices which are connected by an edge only if the condition $m(\alpha, \beta) \geq 3$ is met. In addition, if $m(\alpha, \beta) \geq 4$ such an edge acquires the label m. Further on, a *Coxeter subgraph* is graph obtained from a Coxeter graph by either omitting some vertices or by decreasing the labels on one or more edges (if the label is 3 then it is not indicated).

Definition 2.4.13. A Coxeter system (W, S) is *irreducible* if the associated Coxeter graph is connected.

Equivalently, the above definition states that there exists no partition of S into two non-empty subsets S_1 and S_2 of S such that each element of S_1 commutes with each element of S_2. Thus irreducibility of Coxeter system (W, S) is equivalent to irreducibility of associated root system $\mathcal{R}$. Finite Coxeter groups were classified by Coxeter. The following theorem takes place.

Theorem 2.4.14. *The graph of any irreducible finite Coxeter system (W, S) is one of the following ones:*

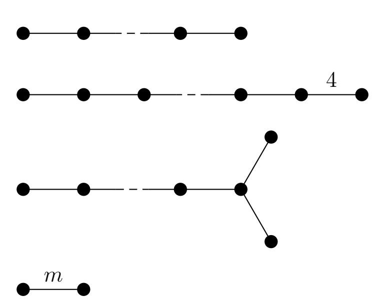
Classical series

$A_n, (n \geq 1)$:

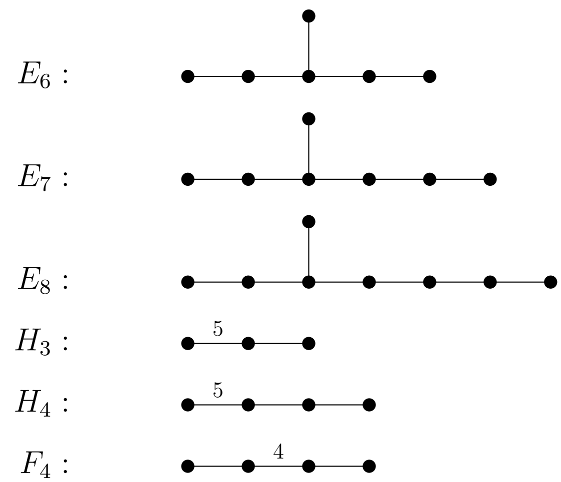
$B_n, (n \geq 2)$:

$D_n, (n \geq 4)$:

$I_2(m), (m \geq 5)$:



Exceptional groups



Example 2.4.15. Let $W = S_n$ be the symmetric group. Let ϵ_i, $i = 1, \dots, n$ be the standard orthonormal basis in V, then W acts on V by permutations of the standard basis. It fixes pointwise the line $L = \{\mathbb{R}\beta\}$, $\beta = \epsilon_1 + \cdots + \epsilon_n$. Hence, we usually denote W by A_{n-1}.The corresponding root and simple systems can be chosen as follows:

$$\mathcal{R} = \{\pm(\epsilon_i - \epsilon_j)\}, \quad 1 \leq i < j \leq n, \quad \Delta = \{\epsilon_i - \epsilon_{i+1}\}, \quad 1 \leq i \leq n-1.$$

Moreover, for distinct $\alpha, \beta \in \Delta$ we have that

$$m(\alpha, \beta) = \begin{cases} 2, & \text{disjoint vertices,} \\ 3, & \text{otherwise.} \end{cases}$$

Remark 2.4.16. It is worth noting that there is weaker notion of a root subsystem (see [74]) where a subset $\mathcal{R}' \subset \mathcal{R}$ is called a subsystem of the root system $\mathcal{R}$ if $w(\mathcal{R}') = \mathcal{R}'$ for all $w = s_\alpha$, $\alpha \in \mathcal{R}'$. For example $A_1 \times A_1 \subset B_2$ is a reducible subsystem of B_2 in this sense but it is not a subsystem in terms of Definition 2.4.9.

Finally, let us introduce the notion of *parabolic subgroups* of W. We fix a Coxeter system (W, S).

Definition 2.4.17. For any subset X of S the subgroup W_X of W generated by X is called a parabolic subgroup of W.

Note that under the action of W the group W_X is mapped to its conjugate $W_{wX} = wW_Xw^{-1}$, for any $w \in W$.

2.4.1 Chevalley's Theorem

Let us fix an irreducible finite Coxeter group W of rank n. Consider the dual space V^* and let $S = S(V^*)$ denote the symmetric algebra on V^*, namely the algebra of polynomials on V with real coefficients. Thus S has a natural graded ring structure $S = \bigoplus_{d=0}^{\infty} S^{(d)}$. An element f of $S^{(d)}$ is called *homogeneous* of degree d. The group W acts naturally on S as a group of automorphisms by defining

$$(w.f)(u) = f(w^{-1}u),$$

for any $w \in W$, $u \in V$, $f \in V^*$. We say $f \in S$ is W*-invariant* if $w.f = f$ for all $w \in W$ The subalgebra of W-invariants $R = S^W$ has also a graded ring structure $R = \mathbb{R} \oplus R^+$, where $R^+ = \bigoplus_{d>0}^{\infty} R^{(d)}$, with $R^{(d)} := R \cap S^{(d)}$ and elements of $\mathbb{R}$ correspond to constant polynomials. The structure of the algebra R is the subject of the following result of Chevalley.

Theorem 2.4.18. *The subalgebra of invariants R is generated by n algebraically independent homogeneous polynomials $f^1, \dots, f^n$ of positive degree, $d_i = \deg f^i$.*

Definition 2.4.19. A set of algebraically independent homogeneous polynomials $f^1, \dots, f^n$ of positive degrees is called a set of *basic invariants* of W.

Let x^i $(1 \le i \le n)$ be a generator system for the algebra S and let $J(f)$ be the Jacobian $J(f^1, \dots, f^n) = \det(\partial f^i / \partial x^j)_{i,j=1}^n$. The following well-known result is crucial for our considerations.

Proposition 2.4.20. [51] *There is a proportionality*

$$J(f) \sim \prod_{\alpha \in \mathcal{R}_+} (\alpha, x).$$

The basic invariants $f^1, \dots, f^n$ are not canonically determined. However, one can show that the corresponding degrees d_i are independent of the choice of these generators and are invariants of the group. The numbers d_i can be described explicitly for every group W and are related to the eigenvalues of a Coxeter element.

Definition 2.4.21. Let $c \in W$ be the product $c = \prod_{\alpha \in \Delta} s_\alpha$ with an assumed choice of ordering of the simple reflections s_α. Any such element c is called a *Coxeter element* and the order h of c in W is called the *Coxeter number.*

Although a Coxeter element depends on the choice of ordering of simple reflections and on the choice of simple system Δ it can be shown that all such elements are conjugate in W. Then all Coxeter elements (as elements of $\mathrm{GL}(V)$) have the same characteristic polynomial $P = P_c$ which takes the form

$$P(\lambda) = \prod_{j=1}^{n} (\lambda - e^{\frac{2i\pi m_j}{h}}),$$

where m_i $(1 \leq i \leq n)$ are integers which in fact satisfy

$$0 < m_1 = 1 < m_2 \leq ... \leq m_{n-1} < m_n = h - 1 < h. \tag{2.68}$$

Note in particular the strict inequalities $m_1 < m_2$ and $m_{n-1} < m_n$ [17, Ch. V, p. 127, Corollary 2]. The polynomial $P(\lambda)$ has real coefficients. Hence for all j the power of the term $\lambda - \exp \frac{2i\pi m_j}{h}$ in $P(\lambda)$ is equal to that of another factor which has to have the form $\lambda - \exp \frac{2i\pi(h-m_j)}{h}$. It follows that the numbers m_i satisfy the relation

$$m_j + m_{n+1-j} = h, \quad 1 \leq j \leq n. \tag{2.69}$$

A surprising fact [51, Theorem 3.19] is that the degrees d_j of W are related to the numbers m_j by the formula $d_j = m_j + 1$, for any $1 \leq j \leq n$. Then it follows from equalities (2.69) that

$$d_j + d_{n-j+1} = h + 2, \quad 1 \leq j \leq n.$$

In Table 2.1 we list the degrees of invariant polynomials for all the finite Coxeter groups.

Table 2.1: Degrees of basic invariants

Type	$d_1, \dots, d_n$
A_n	$2, 3, \dots, n+1$
B_n	$2, 4, 5, \dots 2n$
D_n	$2, 4, 6, \dots, 2n-2, n$
$I_2(m)$	$2, m$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
F_4	$2, 6, 8, 12$

2.4.2 Flat structure on Coxeter orbit spaces

K. Saito proved uniformly the existence of a *flat structure* on the orbit space of a (irreducible) finite Coxeter group W acting on the complexification of a real vector space [77]. Case by case construction of such structure was determined explicitly for all irreducible finite Coxeter groups except for the types of E_7 and E_8 in [78]. For E_7 and E_8 this was accomplished later in [1,83] (see also [89]). Let us recall this notion of flat structure.

Let x^i, $1 \le i \le n$ be some linear coordinates on V. The exterior derivative $f \mapsto df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ induces a bilinear map on $R^+ \times R^+ \to R^+$ defined by[7]

$$p \times q \mapsto (dp, dq) = \sum_{k,l} \frac{\partial p}{\partial x^k} \frac{\partial q}{\partial x^l} (dx^k, dx^l), \quad p, q \in R^+. \tag{2.70}$$

Note that this product is well-defined since the bilinear form (,) on V is W-invariant and is uniquely determined up to a non-zero constant multiple [17, Ch. V, p. 70, Proposition 1]. By Proposition 2.4.20, $\det(df^i, df^j)_{i,j=1}^n$ is proportional to $\prod_{\alpha \in \mathcal{R}_+} (\alpha, x)^2$, thus the form (dp, dq) for any $p, q \in R^+$ degenerates on the union of the mirrors of W. The following theorem takes place.

Theorem 2.4.22. [78] *The matrix $\frac{\partial}{\partial f^n}(df^i, df^j)_{i,j=1}^n$ is non-degenerate. Furthermore there exists a real n-dimensional subspace Ω of R^+ such that for any $p, q \in \Omega$ the form $\frac{\partial}{\partial f^n}(dp, dq)$ takes constant values and Ω generates R.*

Note that the operation $\frac{\partial}{\partial f^n}$ is defined uniquely up to constant factor due to the strict inequality $d_n > d_{n-1}$. A $\mathbb{R}$-basis of Ω is called a *flat generator system*. Thus the problem of determining a flat structure on the orbit space of W reduces into finding a set of basic invariants f^i of W which is a flat generator system, namely $\frac{\partial}{\partial f^n}(df^i, df^j) \in \mathbb{R}$.

Let us now consider the complexified vector space $V \otimes \mathbb{C} \cong \mathbb{C}^n$ with (complex) coordinates x^i. Let $\mathcal{M}_W$ be orbit space $\mathcal{M}_W = \mathbb{C}^n/W$. By Theorem 2.4.18, $\mathcal{M}_W$ is isomorphic to $\mathbb{C}^n$ as an affine variety with coordinate ring $R \otimes \mathbb{C}$ generated by basic invariants $y^1, \dots, y^n$. Namely,

$$R \otimes \mathbb{C} = \mathbb{C}[x^1, \dots, x^n]^W = \mathbb{C}[y^1, \dots, y^n]. \tag{2.71}$$

Let us recall the following important notion.

Definition 2.4.23. The set Σ called *discriminant* is defined as the image of the union of the (complexified) mirrors of W under the quotient map

$$\pi : \mathbb{C}^n \to \mathcal{M}_W. \tag{2.72}$$

Equivalently, Σ consists of the irregular orbits of W.

---

[7] This map is related to V. Arnold's convolution of invariants [6].

We will sometimes refer to the union of all hyperplanes Π_β, $\beta \in \mathcal{R}$ as discriminant as well. The quotient map on the complement to the discriminant,

$$\pi_\Sigma : \mathbb{C}^n \setminus \cup_{\alpha\in\mathcal{R}_+}\Pi_\alpha \to \mathcal{M}_W \setminus \Sigma, \quad x = (x^1, \ldots, x^n) \mapsto y(x) = (y^1, \ldots, y^n), \tag{2.73}$$

is a local diffeomorphism. Then, the linear coordinates x^i, $(1 \leq i \leq n)$ on $\mathbb{C}^n$ can be viewed as local coordinates on $\mathcal{M}_W \setminus \Sigma$. The bilinear form $(\,,\,)$ on V is extended to a complex quadratic form on $\mathbb{C}^n$. Note that this form is also defined on $\mathcal{M}_W \setminus \Sigma$ due to its W-invariance. Then the map (2.70) induces a (complex) metric $g(y)$ on the space of orbits given by

$$g^{ij}(y) = (dy^i, dy^j) = \sum_{k,l} \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^l}(dx^k, dx^l). \tag{2.74}$$

Then $\mathcal{M}_W$ can be regarded as a complex vector space with linear coordinates y^i $(1 \leq i \leq n)$ endowed with the form (2.74). As a consequence of Theorem 2.4.22 a flat structure is determined on $\mathcal{M}_W$. This flat structure can be thought of as a flat complex-valued metric. We will be taking this view in the next section.

2.5 Frobenius structures on Coxeter orbit spaces

Dubrovin using the flat structure introduced by K.Saito showed that the complexified orbit space of a finite irreducible Coxeter group W provides interesting examples of Frobenius manifolds. This somewhat surprising relation originated from an observation from Arnold that the degrees of certain polynomial prepotentials are related to degrees of basic invariants. Dubrovin conjectured that this construction is unique in the sense that all analytic (at the origin) solutions of WDVV equations ($d < 1$) which satisfy the semisimplicity condition arise in this way. This conjecture was proved later by Hertling [46] (see Theorem 2.5.7).

Let now V denote the complex vector space $\mathbb{C}^n$. Consider the action of W in V which is a complexification of the action of W in $\mathbb{R}^n$ by (composition of) reflections. Let $\mathcal{M}_W = V/W$ be the orbit space as before. We move to describing the Frobenius manifold structures on $\mathcal{M}_W$.

2.5.1 Saito metric and main constructions

The first key point is that the form (2.74) is the intersection form (2.16) of the Frobenius structure on $\mathcal{M}_W$. Let e_i, $i = 1, \ldots, n$ be the standard basis in V, namely $g(e_i, e_j) = (e_i, e_j) = \delta_{ij}$. Without loss of generality we will assume below that the coordinates x^i in

V are the corresponding orthonormal coordinates with respect to g, so that

$$g(dx^i, dx^j) = (dx^i, dx^j) = g^{ij} = \delta^{ij}. \tag{2.75}$$

Then we fix a metric $g(y)$ on $\mathcal{M}_W \setminus \Sigma$ with components

$$g^{ij}(y) = (dy^i, dy^j) = \sum_{k=1}^{n} \frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial x^k}. \tag{2.76}$$

Locally, the coordinates x^i on $\mathcal{M}_W \setminus \Sigma$ are flat coordinates of the metric (2.76) and the period mapping (2.33) is given by inverting the map (2.73), namely by solving the system of algebraic equations

$$y^1 = y^1(x^1, \dots, x^n), \quad y^2 = y^2(x^1, \dots, x^n), \quad \dots, \quad y^n = y^n(x^1, \dots, x^n).$$

The Euler vector field is defined as

$$E = \frac{1}{h}\sum_{i=1}^{n} d_i y^i \frac{\partial}{\partial y^i} = \frac{1}{h}x^i\frac{\partial}{\partial x^i}. \tag{2.77}$$

It is normalised such that it agrees with formulae (2.9), (2.10). By Theorem 2.4.22 the matrix $\partial_{y^n} g^{ij}(y)$ is non-degenerate and thus defines a metric on $\mathcal{M}_W$. We also have by Theorem 2.4.22 that there exists a system of basic invariants such that this matrix takes constant values. More specifically there exists a set of basic invariants t^α ($\alpha = 1, \dots, n$), $\deg t^\alpha = d_\alpha$ such that in these coordinates

$$\frac{\partial g^{\alpha\beta}(t)}{\partial t^n} = \delta^{\alpha+\beta,n+1}, \quad 1 \le \alpha, \beta \le n.$$

Definition 2.5.1. The *Saito metric* η on $\mathcal{M}_W$ is defined as

$$\eta^{\alpha\beta} = \mathcal{L}_e g^{\alpha\beta}(t) = \delta^{\alpha+\beta,n+1}, \quad 1 \le \alpha, \beta \le n. \tag{2.78}$$

where $\mathcal{L}_e$ is the Lie derivative along the vector field $e = \frac{\partial}{\partial t^n}$.

Note that the Saito metric is defined uniquely up to proportionality. The coordinates t^α are called *Saito polynomials* or *Saito flat coordinates.*

Example 2.5.2. [22,34] Let $W = A_{n-1}$ and let z be a complex parameter. Saito polynomials take the form

$$t^\alpha = \operatorname{res}_{z=\infty} \prod_{j=1}^{n} (z - x^j)^\nu \Big|_{\sum_{i=1}^n x^i = 0}, \quad \nu = \frac{\alpha}{n}, \quad \alpha = 1, \dots, n-1.$$

In fact the coordinates t^α are examples of polynomial twisted periods (see Definition 2.2.18). The existence of a Frobenius structure on $\mathcal{M}_W$ is established in the following theorem.

Theorem 2.5.3. *Let t^α be Saito flat coordinates and let $\eta^{\alpha\beta}$ be the corresponding Saito metric given by formula (2.78). Then there exists a quasi-homogeneous polynomial $F(t)$ of degree $2h+2$ defined (uniquely up to quadratic terms in t^α) by the following equations $(\alpha, \beta = 1, \dots, n)$*

$$g^{\alpha\beta}(t) = \frac{1}{h}(d_\alpha + d_\beta - 2)\eta^{\alpha\gamma}\eta^{\beta\epsilon}\partial_{t^\gamma}\partial_{t^\epsilon}F(t),$$

with the Euler vector field (2.77). Furthermore, the polynomial $F(t)$ determines a polynomial Frobenius structure on $\mathcal{M}_W$ with the structure constants

$$c^\alpha_{\beta\gamma}(t) = \eta^{\alpha\epsilon}\frac{\partial^3 F(t)}{\partial t^\epsilon \partial t^\beta \partial t^\gamma},$$

and the unity $e = \frac{\partial}{\partial t^n}$.

Example 2.5.4. [24] Suppose $\operatorname{rank} W = 3$. Then the polynomial $F(t)$ takes one of the following forms:

$$\begin{aligned}
F_{A_3}(t) &= \frac{(t^3)^2t^1 + t^3(t^2)^2}{2} + \frac{(t^2t^1)^2}{4} + \frac{(t^1)^5}{60},\\
F_{B_3}(t) &= \frac{(t^3)^2t^1 + t^3(t^2)^2}{2} + \frac{(t^2)^3t^1}{6} + \frac{(t^2)^2(t^1)^3}{6} + \frac{(t^1)^7}{210},\\
F_{H_3}(t) &= \frac{(t^3)^2t^1 + t^3(t^2)^2}{2} + \frac{(t^2)^3(t^1)^2}{6} + \frac{(t^2)^2(t^1)^5}{20} + \frac{(t^1)^{11}}{3960}.
\end{aligned}$$

Remark 2.5.5. Let $W = B_n$. The orbit space $\mathcal{M}_W$ can be identified with the space of complex polynomials of the form (2.41) in Example 2.2.15. The coefficients $a_1, \dots, a_n$ are coordinates on $\mathcal{M}_W$ with corresponding degrees $d_1 = 2$, $d_2 = 4$, $\dots$, $d_n = 2n$ (as functions in x^i). The superpotential $\lambda(p)$ can be represented as (see Chapter 3 for more general superpotentials and corresponding analysis)

$$\lambda(p) = \prod_{i=1}^n (p^2 - \frac{1}{2}(x^i)^2).$$

It can be checked that

$$\frac{\partial x^a}{\partial u_i} = \frac{2\epsilon_i x^a}{q_i^2 - \frac{1}{2}(x^a)^2}\frac{1}{\lambda''(q_i)}, \qquad 1 \le i, a \le n.$$

Then it follows by formula (2.48) that

$$g(dx^a, dx^b) = \sum_{i,j=1}^n \frac{\partial x^a}{\partial u_i}\frac{\partial x^b}{\partial u_j} g(du^i, du^j) = \sum_{i=1}^n \frac{2\epsilon_i x^a x^b u_i}{(q_i^2 - \frac{1}{2}(x^a)^2)(q_i^2 - \frac{1}{2}(x^b)^2)\lambda''(q_i)}.$$

Using formulae (2.43) and (2.46) we get

$$g(dx^a, dx^b) = \mathrm{res}|_{d\lambda=0} \frac{x^a x^b \lambda(p)}{(p^2 - \frac{1}{2}(x^a)^2)(p^2 - \frac{1}{2}(x^b)^2)\lambda'(p)}.$$

Then

$$g(dx^a, dx^b) = -(\mathrm{res}|_{p=\infty} + \mathrm{res}|_{p=\pm x^a} + \mathrm{res}|_{p=\pm x^b}) \frac{x^a x^b \lambda(p)}{(p^2 - \frac{1}{2}(x^a)^2)(p^2 - \frac{1}{2}(x^b)^2)\lambda'(p)} = -\delta^{ab}.$$

Therefore the metric (2.48) coincides (up to a sign) with the W-invariant metric on V defined by formula (2.75). The identity field e is proportional to $\frac{\partial}{\partial a_n}$ and the critical points q_i (hence also the values $\lambda''(q_i)$) do not depend on the coordinate a_n since $\deg a_n > \deg a_{n-1}$. It follows from (2.48) that

$$\mathcal{L}_e g^{ij}(u) = \mathcal{L}_e\big(\frac{1}{2\epsilon_i} u_i \lambda''(q_i)\delta^{ij}\big) = \frac{1}{2\epsilon_i}\lambda''(q_i)\delta^{ij}\mathcal{L}_e u_i = \frac{1}{2\epsilon_i}\lambda''(q_i)\delta^{ij} = \eta^{ij}(u),$$

since the identity e in canonical coordinates takes the form (2.36). Then the metric (2.38) coincides (up to proportionality) with the Saito metric on $\mathcal{M}_W$. Hence, the Frobenius structure on the space M of complex polynomials of the form (2.41) coincides (up to equivalence) with the Frobenius structure on the orbit space $\mathcal{M}_W$.

Example 2.5.6. Let $W = B_2$. Consider polynomials $y^1, y^2 \in \mathbb{C}[x]^W$ given by

$$y^1 = (x^1)^2 + (x^2)^2 \quad \text{and} \quad y^2 = a((x^1)^4 + (x^2)^4) + b(x^1x^2)^2,$$

for some $a, b \in \mathbb{C}^\times$. One can check that $t^1 = \frac{1}{8}y^1$ and $t^2 = y^2|_{b=-6a}$ are Saito polynomials, that is $\eta_{\alpha\beta} = \delta_{\alpha+\beta,3}$. The determinant of the intersection form is a homogeneous polynomial of degree 8 in the x^i coordinates and it vanishes precisely on the discriminant

$$\Delta(t) = -(t^1)^2 + 4096a^2(t^2)^4 = 16a^2\big(x^1x^2((x^1)^2 - (x^2)^2)\big)^2 = 0.$$

Note that canonical coordinates can be chosen as

$$u_1 = (x^1x^2)^2, \quad \text{and} \quad u_2 = -\frac{((x^1)^2 - (x^2)^2)^2}{4},$$

though in general they are not polynomial.

It was shown by Dubrovin that the Frobenius manifold $\mathcal{M}_W$ is in fact semisimple. The following theorem by Hertling establishes that the only polynomial and semisimple Frobenius manifolds are those constructed on the (complexified) orbit space of finite Coxeter groups.

Theorem 2.5.7. [46, Theorem 5.25] *Let $(M, \circ, e, E, \eta)$ be a semisimple Frobenius manifold with the following properties:*

(i) $\mathcal{L}_E(x \circ y) = x \circ y, \quad \mathcal{L}_E \eta(x, y) = (2-d)\eta(x, y), \quad$ *for any* $x, y \in \Gamma(TM)$*;*

(ii) the Euler field takes the form

$$E = \sum \widetilde{d}_\alpha \widetilde{t}^\alpha \frac{\partial}{\partial \widetilde{t}^\alpha},$$

for a basis of flat coordinates $\widetilde{t}^\alpha$ *and* $\widetilde{d}_\alpha > 0$ *for all* α*.*

Then M decomposes uniquely into a product of Frobenius manifolds $\mathcal{M}_W$, where W is an irreducible finite Coxeter group with Coxeter number $h = \frac{2}{1-d}$.

2.5.2 Almost duality on $\mathcal{M}_W$

It was shown [66, 86] that the for any root system $\mathcal{R} \subset \mathbb{R}^n$ of a finite Coxeter group the function

$$\mathcal{F}(x) = \frac{1}{4} \sum_{\gamma \in \mathcal{R}} (\gamma, x)^2 \log(\gamma, x), \tag{2.79}$$

satisfies generalised WDVV equations. Dubrovin related polynomial solutions to WDVV equations with logarithmic solutions of the form (2.79) through the concept of almost duality. The almost dual structure constructed on $\mathcal{M}_W \setminus \Sigma$ has a prepotential $F_*(x)$ which is of the form (2.79), where the roots are normalised so that $(\gamma, \gamma) = 2$.

Let us define the tensor $\overset{*}{c}_{abc}(x)$ by taking third order derivatives of $F_*(x)$,

$$\overset{*}{c}_{abc}(x) = \frac{\partial^3 F_*}{\partial x^a \partial x^b \partial x^c} = \sum_{\gamma \in \mathcal{R}_+} \frac{\gamma_a \gamma_b \gamma_c}{(\gamma, x)}, \tag{2.80}$$

where $\gamma_i = (e_i, \gamma)$. Let $\overset{*}{c}{}^d_{bc}(x) = g^{da} \overset{*}{c}_{abc}(x)$, where g^{ab} is defined in (2.75). Then for any $x \in \mathbb{C}^n \setminus \cup_{\alpha \in \mathcal{R}_+} \Pi_\alpha$, the tensor $\overset{*}{c}{}^d_{bc}(x)$ forms the structure constants of an associative n-dimensional algebra [25, Corollary 3.2]. The Euler/identity vector field of the almost dual Frobenius manifold is defined by formula (2.77). Note that this agrees with axiom (iii) of Definition 2.2.17 since $1 - d = \frac{2}{h}$. The vector field e in axiom (iv) coincides with the identity $e = \partial_{t^n}$ of the Frobenius manifold $\mathcal{M}_W$.

2.5.3 Almost duality on discriminant strata

Feigin and Veselov showed in [35] that almost dual Frobenius multiplication (2.50)

$$u * v = E^{-1} \circ u \circ v$$

has a natural restriction on discriminant strata. The corresponding prepotentials also have the form (2.79) with summation running over some projections of root systems. Below, we recall some of these results.

Let us recall the notion of a discriminant stratum. Let us fix a collection of roots $S \subset \mathcal{R}$ and let $D = \cap_{\beta \in S} \Pi_\beta$.

Definition 2.5.8. A *discriminant stratum* in the orbit space $\mathcal{M}_W$ is defined to be the image of D under the quotient map π given by (2.72).

Sometimes we will refer to the intersection of hyperplanes D as a discriminant stratum as well.

The left-hand-side of equality (2.50) can be restricted to any stratum D with u and v being tangential vectors to D. Let $\mathcal{R}_D = \mathcal{R} \cap \langle S \rangle$. Let Σ_D be the union of the hyperplanes $\Pi_\gamma \cap D$ in D, where $\gamma \in \mathcal{R} \setminus \mathcal{R}_D$ and consider point x_0 in $D \setminus \Sigma_D$. Let $u, v \in T_{x_0} D$ and consider extension of u and v to two local analytic vector fields $u(x), v(x) \in T_x V$ such that $u(x_0) = u$ and $v = v(x_0)$. Let us recall the following result.

Lemma 2.5.9. *The multiplication $u(x) * v(x)$ has a limit when x tends to x_0 which is proportional to*

$$\sum_{\alpha \in \mathcal{R} \setminus \mathcal{R}_D} \frac{(\alpha, u)(\alpha, v)}{(\alpha, x_0)} \alpha.$$

Furthermore, the product $u * v$ at x_0 is tangential to D.

As a corollary the following theorem takes place.

Theorem 2.5.10. *The almost dual Frobenius structure* (2.50), (2.79) *has a natural restriction to the space $D \setminus \Sigma_D$ with the prepotential*

$$\mathcal{F}_D(x) = \frac{1}{4} \sum_{\gamma \in \mathcal{R} \setminus \mathcal{R}_D} (\gamma, x)^2 \log(\gamma, x), \quad x \in D \setminus \Sigma_D. \tag{2.81}$$

which also satisfies the WDVV equations.

The above results establish that there is a limit of the formula $u * v = E^{-1} \circ u \circ v$, as x tends to x_0, and u, v are tangential to D in the limit. However it is not clear what happens with individual terms on the right hand side of this formula in this limit. We give more details on this in Chapter 3 thus clarifying the missing bits in the almost duality relation (2.50) on the discriminant strata.

Chapter 3

Saito Determinant for Coxeter discriminant strata

The Saito metric η defined on an orbit space of a finite Coxeter group induces a metric on the Coxeter discriminant strata which is given by restriction of the metric η to the strata. In this chapter we obtain the determinant of the induced metric. It is shown that this determinant in the flat coordinates of the intersection form is proportional to a product of linear factors. We also find multiplicities of these factors in the determinant in terms of Coxeter geometry of the stratum.

3.1 Main Theorem 1

Let us fix some notation. Let $V = \mathbb{C}^n$ with the standard metric g given by $g(e_i, e_j) = (e_i, e_j) = \delta_{ij}$, where e_i, $i = 1, \ldots, n$, is the standard basis in V. Let x^i, $i = 1, \ldots, n$ be the corresponding orthonormal coordinates in V. Let W be an irreducible finite Coxeter group of rank n which acts in V by orthogonal transformations such that V is the complexified reflection representation of W. Let $\mathcal{M}_W$ be the orbit space $\mathcal{M}_W = V/W$. Let $\mathcal{R} \subset V$ be the Coxeter root system associated with the group W.

Consider a collection of roots $\beta_1, \ldots, \beta_k \in \mathcal{R}$, let $D = \cap_{j=1}^{k} \Pi_{\beta_j}$ and let π be the projection map given by (2.72). The metric η on $\mathcal{M}_W$ induces a metric on the stratum $\pi(D)$ which is naturally given as the restriction of η to $\pi(D)$. We will denote this metric by η_D and its inverse by η^D. Let us recall that the map π is a local diffeomorphism on D near generic point $x_0 \in D$. This allows us to lift metrics η_D, η^D to the linear space D. Likewise the metric η can be lifted to V near a generic point.

Definition 3.1.1. [73] A finite set $\mathcal{A}$ of hyperplanes in a vector space is called an *arrangement*.

We will only be considering hyperplanes passing through the origin. Let $\mathcal{A}$ be an arrangement in V. Then we have its defining polynomial given (up to a scalar multiple) by

$$I(\mathcal{A}) = \prod_{\pi \in \mathcal{A}} \alpha_\pi, \tag{3.1}$$

where $\alpha_\pi \in V^*$ is such that $\pi = \{x \in V : \alpha_\pi(x) = 0\}$.

Definition 3.1.2. [73] A *Coxeter arrangement* is an arrangement of mirrors of the Coxeter group W.

Let $\mathcal{A}$ be the Coxeter arrangement corresponding to W. Then the determinant of the Saito metric is proportional to $I(\mathcal{A})^2$. To see this, let p^i $(i = 1, \ldots, n)$ be basic invariants for W and let $J(p)$ be the Jacobian $J(p^1, \ldots, p^n) = \det\left(\partial p^i / \partial x^j\right)_{i,j=1}^n$. We have from Proposition 2.4.20 that there is proportionality $J(p) \sim I(\mathcal{A})$. Let us take basis of Saito polynomials t^α, $1 \le \alpha \le n$ and fix $J = J(t^1, \ldots, t^n)$.

Proposition 3.1.3. *We have*

$$\det \eta(x) = -J^2. \tag{3.2}$$

This proposition follows by performing a coordinate transformation $t = t(x)$. Then $\eta_{ij}(x) = \frac{\partial t^k}{\partial x^i} \frac{\partial t^l}{\partial x^j} \eta_{kl}(t)$, which implies the statement due to Definition 2.5.1.

We are interested in the determinant of the restricted Saito metric η_D on the discriminant strata D. We will show that $\det \eta_D$ is a product of linear forms which can be viewed as a generalization of formula (3.2). Let $\mathcal{A}_D$ be the restriction of arrangement $\mathcal{A}$ to D, namely

$$\mathcal{A}_D = \{D \cap H | H \in \mathcal{A} \text{ and } D \not\subset H\}.$$

For each $H \in \mathcal{A}_D$ we choose $l_H \in D^*$ such that $H = \{x \in D | l_H(x) = 0\}$. We can identify vectors and covectors using bilinear form $(\ ,\ )$, so that $\beta \in \mathcal{R}$ also means a covector $\beta = \beta(x) = (\beta, x)$. Note that for any $H \in \mathcal{A}_D$ there is a root $\beta \in \mathcal{R}$ such that $\beta|_D$ is proportional to l_H.

Let us consider the determinant of η_D in some coordinates which are given as linear combinations of the coordinates x^i, $i = 1, \ldots, n$. In these coordinates the following theorem holds.

Main Theorem 1. *The determinant of the restricted Saito metric η_D is proportional to the product of linear forms*

$$\prod_{H \in \mathcal{A}_D} l_H^{k_H}, \tag{3.3}$$

where $k_H \in \mathbb{N}$.

Remark 3.1.4. In fact $\det \eta_D$ is generically non-zero. In the case of classical root systems this follows from Theorems 3.3.5, 3.3.14. In the case of strata of codimension 1, 2, 3 and 4 in exceptional root systems this follows from our corresponding analysis in Section 3.6.

Similarly, for the strata considered in Section 3.7. For the strata of dimension 1 see our discussion in Appendix A.

Let us consider a constant metric of the form $\widehat{\eta} = \sum_{i=1}^{n} dp^i dp^{n+1-i}$. A natural question is whether restriction of such metric to any stratum D satisfies the factorisation property as in Main Theorem 1. In other words, how special is the property of the metric η_D to have a factorised determinant with prescribed structure of linear factors? Let us consider the following example.

Example 3.1.5. Let $W = D_3 = A_3$ and consider the following basic invariants:

$$p^1 = \frac{1}{8}\sum_{i=1}^{3}(x^i)^2, \quad p^2 = \prod_{i=1}^{3} x^i, \quad p^3 = a\sum_{i=1}^{3}(x^i)^4 + b(p^1)^2,$$

for some $a, b \in \mathbb{C}$. Let $\alpha = e_2 - e_3$ and consider the corresponding stratum $D = \Pi_\alpha$. Then the determinant of metric $\widehat{\eta}$ restricted to D is proportional to

$$(x^3)^2\big((x^1)^2 - (x^3)^2\big)^2\big((x^1)^2(-64a + 32a^2 - b) - (x^3)^2(64a + 2b)\big).$$

Furthermore, $\det \widehat{\eta}_D$ is a product of linear factors which all vanish on the intersection of mirrors with D exactly when $a \neq 0$ and b takes one of the following values:

$$\begin{aligned} \det \widehat{\eta}_D &\sim \Big(x^3x^1\big((x^1)^2 - (x^3)^2\big)\Big)^2, \quad b = -32a, \\ \det \widehat{\eta}_D &\sim (x^3)^4\big((x^1)^2 - (x^3)^2\big)^2, \quad b = 32(-2a + a^2), \\ \det \widehat{\eta}_D &\sim (x^3)^2\big((x^1)^2 - (x^3)^2\big)^3, \quad b = \frac{32}{3}\left(a^2 - 4a\right). \end{aligned} \tag{3.4}$$

Note that p^i, $(i = 1, 2, 3)$ are Saito polynomials if $a = -\frac{1}{2}$ and $b = 24$. In this case $\widehat{\eta} = \eta$ and $\det \eta_D$ takes the form (3.4) as expected from Main Theorem 1. The degrees of linear factors in (3.4) are related below in Main Theorem 2 to some Coxeter numbers. More generally, metric $\widehat{\eta}$ in higher dimensions can have determinant of a restriction on a stratum D with nonlinear zero loci.

3.2 Degrees of linear factors

In this section we formulate a main theorem on the degrees k_H in Main Theorem 1.

Lemma 3.2.1. *Let $\mathcal{R}$ be a reducible root system and let $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{R}$ be such that $\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_2$. Consider the corresponding vector space decomposition $\langle \mathcal{R} \rangle = V_1 \oplus V_2$, where $V_i = \langle \mathcal{R}_i \rangle$. Let $\widetilde{\mathcal{R}} \subset \mathcal{R}$ be an irreducible subsystem. Then either $\widetilde{\mathcal{R}} \subset V_1$ or $\widetilde{\mathcal{R}} \subset V_2$.*

Let $u \in V$ and let B be a set of vectors in V. We will denote by $\langle B, u \rangle$ the vector space spanned by elements of B and u. Let S be a collection of linearly independent roots $S = \{\gamma_1, \dots, \gamma_k\} \subset \mathcal{R}$, $1 \le k < n$ and let D be the corresponding discriminant stratum $D = \cap_{\gamma \in S} \Pi_\gamma$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle S \rangle$ and consider its orthogonal decomposition into irreducible root systems

$$\mathcal{R}_D = \bigsqcup_{i=1}^{l} \mathcal{R}_D^{(i)}.$$

Below we will denote by $\mathcal{A}^D$ the corresponding Coxeter arrangement. For any $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ we define the root system $\mathcal{R}_{D,\beta} = \langle \mathcal{R}_D, \beta \rangle \cap \mathcal{R}$ which can be represented as a disjoint union of irreducible root systems $\mathcal{R}_{D,\beta}^{(i)}$, $(i = 0, \dots, p)$, as follows:

$$\mathcal{R}_{D,\beta} = \bigsqcup_{i=0}^{p} \mathcal{R}_{D,\beta}^{(i)}. \tag{3.5}$$

We will assume below that $\beta \in \mathcal{R}_{D,\beta}^{(0)}$. It follows from Lemma 3.2.1 that

$$\mathcal{R}_{D,\beta}^{(0)} \supset \bigsqcup_{i \in I} \mathcal{R}_D^{(i)}, \tag{3.6}$$

for some subset $I \subset \{1, \dots, l\}$ and

$$\mathcal{R}_{D,\beta}^{(j)} = \mathcal{R}_D^{(i_j)}, \tag{3.7}$$

where $1 \le j \le p$, $p = l - |I|$ and $i_j \in \{1, \dots, l\} \setminus I$.

Proposition 3.2.2. *Let $\mathcal{R}_{D,\beta}^{(0)}$ be root system from the decomposition (3.5). Let $\widetilde{\beta} \in \mathcal{R}$ be such that $\widetilde{\beta}\Big|_D$ is a non-zero multiple of $\beta|_D$. Then $\widetilde{\beta} \in \mathcal{R}_{D,\beta}^{(0)}$.*

Proof. Let $\widehat{V}$ be the vector space $\widehat{V} = \langle \mathcal{R}_D, \beta \rangle = \langle \mathcal{R}_D, \widetilde{\beta} \rangle$ and consider the root system $\widehat{\mathcal{R}} = \widehat{V} \cap \mathcal{R}$. Then $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = \langle \mathcal{R}_D, \beta \rangle \cap \mathcal{R} = \bigsqcup_{i=0}^{p} \mathcal{R}_{D,\beta}^{(i)}.$$

Let us now assume that $\widetilde{\beta} \notin \mathcal{R}_{D,\beta}^{(0)}$. Then $\widetilde{\beta} \in \mathcal{R}_D^{(i)}$ for some $i \in \{1, \dots, l\} \setminus I$, hence $\widetilde{\beta}\Big|_D = 0$, which is a contradiction. Thus the statement follows. □

Main Theorem 2. *Let $H \in \mathcal{A}_D$. Let $\beta \in \mathcal{R}$ be such that $\beta|_D$ is proportional to l_H and it is non-zero. The multiplicity of l_H in the expression (3.3) is $k_H = h(\mathcal{R}_{D,\beta}^{(0)})$, where $h(\mathcal{R}_{D,\beta}^{(0)})$ is the Coxeter number of the root system $\mathcal{R}_{D,\beta}^{(0)}$ from the decomposition (3.5).*

We are going to prove Main Theorems 1, 2 (in the case of exceptional Coxeter groups) for a subset of simple roots $L \subset \Delta$, $L = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$, $1 \le k < n$ and the corresponding stratum. Let us show how the statement of Main Theorem 1 then follows in general. Let $\widetilde{D} \subset V$ be a stratum such that there exists $w \in W$ satisfying $\widetilde{D} = wD$.

Lemma 3.2.3. *Let y^i and z^i $(i = 1, \dots, n-k)$ be some coordinates on D and $\widetilde{D}$ respectively. Then*

$$\det \eta_{\widetilde{D}}(z) = \det B^2 \det \eta_D(y), \tag{3.8}$$

where $B = (\frac{\partial y^i}{\partial z^j})_{i,j=1}^{n-k}$ is the Jacobi matrix of the transformation $w \in W$, $w : D \to \widetilde{D}$.

Proof. We note that η is W-invariant. Then we have

$$\eta_{\widetilde{D}} = w^{-1}\eta_D. \tag{3.9}$$

Using equality (3.9) the determinant of $\eta_{\widetilde{D}}$ is thus obtained from the determinant of η_D by replacing y coordinates with z coordinates, and the statement follows. □

This implies the following W-invariance of Main Theorem 1.

Proposition 3.2.4. *Suppose that the Main Theorem 1 is true for D. Then it is true for $\widetilde{D}$.*

Proof. Let $\widetilde{S}$ be such that $\widetilde{D} = \cap_{\gamma \in \widetilde{S}} \Pi_\gamma$ and let $\widetilde{\beta} \in \mathcal{R} \setminus \langle \widetilde{S} \rangle$. Then $\widetilde{\beta} = w\beta$ for some $\beta \in \mathcal{R}$ such that $\beta|_D \neq 0$. We therefore have $\mathcal{R}_{\widetilde{D},\widetilde{\beta}} = \langle \widetilde{S}, \widetilde{\beta} \rangle \cap \mathcal{R}$. This shows that

$$\mathcal{R}_{\widetilde{D},\widetilde{\beta}} = w\langle S, \beta \rangle \cap \mathcal{R} = w\mathcal{R}_{D,\beta} = w\mathcal{R}^{(0)}_{D,\beta} \sqcup \dots \sqcup w\mathcal{R}^{(p)}_{D,\beta},$$

where $\mathcal{R}_{D,\beta}$ is given by (3.5). If $\beta \in \mathcal{R}^{(0)}_{D,\beta}$ then $\widetilde{\beta} \in \mathcal{R}^{(0)}_{\widetilde{D},\widetilde{\beta}} = w\mathcal{R}^{(0)}_{D,\beta}$, and the Coxeter numbers of $\mathcal{R}^{(0)}_{D,\beta}$ and $\mathcal{R}^{(0)}_{\widetilde{D},\widetilde{\beta}}$ are equal. Therefore the statement follows by Lemma 3.2.3. □

Let simple system $\Delta \subset \mathcal{R}$. The following result establishes that it is sufficient to prove Main Theorem 1 for $L = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta$, $1 \leq k < n$ and $D = \cap_{\alpha \in L} \Pi_\alpha$. It follows from the simply transitive action of W on the family of alcoves and their closure.

Proposition 3.2.5. [51] *Let $\widetilde{L} = \{\gamma_1, \dots, \gamma_k\} \subset \mathcal{R}$ be a collection of linearly independent roots and let $\widetilde{D}$ be the corresponding stratum $\widetilde{D} = \cap_{\gamma \in \widetilde{L}} \Pi_\gamma$. Then there exists $w \in W$ such that $D = w^{-1}\widetilde{D}$ has the form $D = \cap_{\alpha \in L} \Pi_\alpha$, where $L = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta$.*

3.3 Classical series: Main Theorem 1

In this section we show that Main Theorem 1 holds for the determinant of the Saito metric restricted to a stratum of a classical Coxeter system. We use Landau-Ginzburg superpotential description of the Frobenius structures on the discriminant strata.

3.3.1 A_N discriminant strata

Let us recall that the Landau-Ginzburg superpotential is given by [22]

$$\lambda(p) = \prod_{i=1}^{N+1}(p - x^i), \tag{3.10}$$

where p is some auxiliary variable and x^i, $1 \leq i \leq N+1$, are the standard orthonormal coordinates in $\mathbb{C}^{N+1}$ with the additional assumption $\sum_{i=1}^{N+1} x^i = 0$. Then $\lambda(p)$ is a function on the orbit space $\mathbb{C}^{N+1}/S_{N+1}$ for any fixed p. Note that up to a sign the metric (2.39) coincides with the standard S_{N+1}-invariant metric g on $\mathbb{C}^N$ [22].

Let us consider an arbitrary discriminant stratum D given by the following equations:

$$\begin{aligned} x^1 = \ldots = x^{m_0} &= \xi_0, \\ x^{m_0+1} = \ldots = x^{m_0+m_1} &= \xi_1 \\ &\vdots \\ x^{\sum_{i=0}^{n-1} m_i+1} = \ldots = x^{\sum_{i=0}^{n} m_i} &= \xi_n, \end{aligned} \tag{3.11}$$

where $n, m_i \in \mathbb{N}$ and $\sum_{i=0}^{n} m_i = N+1$. Note that the dimension of this stratum is n, and $\xi_1, \ldots, \xi_n$ can be considered as coordinates on D, $\xi_0 = -\sum_{i=1}^{n} \frac{m_i}{m_0}\xi_i$.

Then, the superpotential for the stratum D is

$$\lambda_D(p) = \lambda(p)|_D = \prod_{i=0}^{n}(p - \xi_i)^{m_i}. \tag{3.12}$$

The expressions for the restricted Saito metric $\eta_D = \eta|_D$ and algebra multiplication are then given as follows (cf. [22] for the case $m_i = 1, \forall i$),

$$\eta_D(\zeta_i, \zeta_j) = \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{\zeta_i(\lambda_D)\zeta_j(\lambda_D)}{\lambda_D'(p)}dp, \tag{3.13}$$

$$\eta_D(\zeta_i \circ \zeta_j, \zeta_k) = \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{\zeta_i(\lambda_D)\zeta_j(\lambda_D)\zeta_k(\lambda_D)}{\lambda_D'(p)}dp, \tag{3.14}$$

where ζ_i denote some vector fields tangential to D and $\lambda_D'(p) = \frac{d\lambda_D(p)}{dp}$.

Proposition 3.3.1. *On the stratum D we have the following expression for λ':*

$$\lambda_D'(p) = (N+1)\prod_{i=0}^{n}(p - \xi_i)^{m_i-1}\prod_{i=1}^{n}(p - q_i), \tag{3.15}$$

for some points $q_1, \ldots, q_n \in \mathbb{C}$.

Proof. Starting from formula (3.12), we have

$$\lambda_D'(p) = \prod_{i=0}^{n}(p-\xi_i)^{m_i-1}Q(p),$$

for some $Q \in \mathbb{C}[p]$, $\deg Q = n$. Then formula (3.15) follows. □

The following formula which follows from Proposition 3.3.1 will be useful below

$$\lambda_D''(q_l) = (N+1)\prod_{j\neq l} q_{lj}\prod_{a=0}^{n}(q_l-\xi_a)^{m_a-1}, \tag{3.16}$$

where $q_{lj} = q_l - q_j$.

Let $u_i = \lambda_D(q_i)$, $i = 1, \ldots, n$. Similarly to the case $n = N$ (see [22]) we have the following statement.

Proposition 3.3.2. *We have*

$$\partial_{u_i}\lambda_D(p)|_{p=q_j} = \delta_{ij}. \tag{3.17}$$

Proof. By definition we have

$$\delta_{ij} = \frac{\partial u_j}{\partial u_i} = \partial_{u_i}\lambda_D(q_j).$$

Then considering the Taylor expansion of $\lambda(p)$ centred at $p = q_j$ we have

$$\lambda_D(p) = \lambda_D(q_j) + \mathcal{O},$$

where $\mathcal{O}$ denotes the rest of the terms, and $\mathcal{O}$ has zero of order at least two at $p = q_j$. Then

$$\partial_{u_i}\lambda_D(p)|_{p=q_j} = \partial_{u_i}\lambda_D(q_j), \tag{3.18}$$

and the statement follows. □

Analogous to the $n = N$ case we obtain the following result.

Lemma 3.3.3. *We have*

$$\partial_{u_l}\lambda_D(p) = \frac{\lambda_D'(p)}{(p-q_l)\lambda_D''(q_l)}, \tag{3.19}$$

and

$$\partial_{u_l}\xi_a = \frac{1}{(q_l-\xi_a)\lambda_D''(q_l)}, \tag{3.20}$$

where $1 \leq l, a \leq n$.

Proof. Starting from formula (3.12) we get

$$\partial_{u_l}\lambda_D(p)=\prod_{a=0}^{n}(p-\xi_a)^{m_a-1}F(p;l), \tag{3.21}$$

where $F\in\mathbb{C}[p]$ and $\deg F=n-1$. From Proposition 3.3.2 we have $\partial_{u_l}\lambda_D(p)|_{p=q_j}=\delta_{lj}$ and therefore

$$F(q_j;l)=\frac{\delta_{lj}}{\prod_{a=0}^{n}(q_j-\xi_a)^{m_a-1}}.$$

Since $\deg F=n-1$, the points $(q_i,F(q_i))$, $i=1,\dots,n$ completely determine the polynomial F and therefore by the Lagrange interpolation formula we have $F(p;l)=\sum_{k=1}^{n}F_k(p;l)$, where

$$F_k(p;l)=F(q_k;l)\prod_{i\neq k}\frac{p-q_i}{q_k-q_i}.$$

Hence

$$F_k(p;l)=\frac{\delta_{lk}}{\prod_{a=0}^{n}(q_k-\xi_a)^{m_a-1}}\prod_{i\neq k}\frac{p-q_i}{q_k-q_i}. \tag{3.22}$$

It follows that $F(p;l)=F_l(p;l)$. Therefore by considering $\frac{\lambda_D'(p)}{\lambda_D''(q_l)}$, where $\lambda_D''(q_l)$ is given by (3.16), the first statement follows from formulae (3.15), (3.21), (3.22).

Let us express $\lambda_D(p)$ as the product $\lambda_D(p)=(p-\xi_0)^{m_0}\prod_{a=1}^{n}(p-\xi_a)^{m_a}$. Then

$$\begin{aligned}\partial_{u_l}\lambda_D(p)&=\partial_{u_l}\big((p-\xi_0)^{m_0}\big)\prod_{a=1}^{n}(p-\xi_a)^{m_a}-(p-\xi_0)^{m_0}\prod_{a=1}^{n}(p-\xi_a)^{m_a}\big(\sum_{b=1}^{n}m_b\frac{\partial_{u_l}\xi_b}{p-\xi_b}\big)\\&=\partial_{u_l}\big((p-\xi_0)^{m_0}\big)\prod_{a=1}^{n}(p-\xi_a)^{m_a}-\lambda_D(p)\sum_{b=1}^{n}m_b\frac{\partial_{u_l}\xi_b}{p-\xi_b}.\end{aligned}$$

From the first statement of the lemma this equals to $\frac{\lambda_D'(p)}{(p-q_l)\lambda_D''(q_l)}$. Dividing both sides by $(p-\xi_k)^{m_k-1}$ for some k, $1\le k\le n$ we arrive at the following relation

$$\partial_{u_l}\big((p-\xi_0)^{m_0}\big)(p-\xi_k)\prod_{\substack{a=1\\a\neq k}}^{n}(p-\xi_a)^{m_a}-\frac{\lambda_D(p)}{(p-\xi_k)^{m_k-1}}\sum_{b=1}^{n}m_b\frac{\partial_{u_l}\xi_b}{(p-\xi_b)}=\frac{\lambda_D'(p)}{(p-q_l)(p-\xi_k)^{m_k-1}\lambda_D''(q_l)}. \tag{3.23}$$

Note that

$$\left.\frac{\lambda_D'(p)}{(p-\xi_k)^{m_k-1}}\right|_{p=\xi_k}=m_k\prod_{\substack{b=0\\b\neq k}}^{n}(\xi_k-\xi_b)^{m_b}. \tag{3.24}$$

We substitute $p = \xi_k$ in the relation (3.23) and we get with the help of (3.24) that

$$-m_k \prod_{\substack{a=0 \\ a\neq k}}^{n} (\xi_k - \xi_a)^{m_a} \partial_{u_l} \xi_k = m_k \frac{\prod_{\substack{b=0 \\ b\neq k}}^{n} (\xi_k - \xi_b)^{m_b}}{(\xi_k - q_l)\lambda_D''(q_i)}.$$

The statement follows. $\square$

Lemma 3.3.4. *The critical values $u_i = \lambda_D(q_i)$, $(i = 1, \ldots, n)$ are the canonical coordinates for the structures (3.13), (3.14) on the stratum D, that is*

$$\eta_D(\partial_{u_i}, \partial_{u_j}) = \frac{\delta_{ij}}{\lambda_D''(q_i)},$$
$$\partial_{u_i} \circ \partial_{u_j} = \delta_{ij}\partial_{u_j}.$$

Proof. We use formulae (3.13), (3.14) together with (3.19). We consider consider formulae (3.13), (3.14) with the vector fields $\zeta_i = \partial_{u_i}$, $\zeta_j = \partial_{u_j}$. Note that the residues are trivial in ξ_a $(0 \leq a \leq n)$ by Lemma 3.3.3.

Let us consider first formula (3.13). In the case when $i \neq j$ the residues at q_l $(1 \leq l \leq n)$ are trivial by Lemma 3.3.3, and hence $\eta_D(\partial_{u_i}, \partial_{u_j}) = 0$. Further on, by (3.13) and (3.19) we have

$$\eta_D(\partial_{u_i}, \partial_{u_i}) = \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{(\partial_{u_i}\lambda_D(p))^2}{\lambda_D'(p)} dp = \frac{1}{\lambda_D''(q_i)^2} \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{\lambda_D'(p)}{(p-q_i)^2} dp.$$

It then follows from Proposition 3.3.1 and formulae (3.12), (3.16) that

$$\eta_D(\partial_{u_i}, \partial_{u_i}) = \frac{N+1}{\lambda_D''(q_i)^2} \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{\prod_{j=0}^{n}(p-\xi_j)^{m_j-1}\prod_{j\neq i}^{n}(p-q_j)}{p-q_i} dp = \frac{1}{\lambda_D''(q_i)},$$

as required.

Let us now consider formula (3.14). In the case when $i \neq j$ or $j \neq k$ the residues at q_l $(1 \leq l \leq n)$ are trivial by Lemma 3.3.3, and hence $\eta_D(\partial_{u_i} \circ \partial_{u_j}, \partial_{u_k}) = 0$. Further on, by (3.14) and (3.19) we have

$$\eta_D(\partial_{u_i} \circ \partial_{u_i}, \partial_{u_i}) = \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{(\partial_{u_i}\lambda_D(p))^3}{\lambda_D'(p)} dp = \frac{1}{\lambda_D''(q_i)^3} \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{(\lambda_D'(p))^2}{(p-q_i)^3} dp.$$

It then follows from Proposition 3.3.1 and formula (3.12) that

$$\eta_D(\partial_{u_i} \circ \partial_{u_i}, \partial_{u_i}) = \frac{(N+1)^2}{\lambda_D''(q_i)^3} \sum_{p_s:\lambda_D'(p_s)=0} res|_{p=p_s} \frac{\prod_{j=0}^{n}(p-\xi_j)^{2(m_j-1)}\prod_{j\neq i}^{n}(p-q_j)^2}{p-q_i} dp = \frac{1}{\lambda_D''(q_i)}.$$

Therefore

$$\eta_D(\partial_{u_i} \circ \partial_{u_j}, \partial_{u_k}) = \frac{\delta_{ij}\delta_{jk}}{\lambda''_D(q_i)},$$

which implies the second statement of the lemma. □

This allows us to find $\det \eta_D$ in the coordinates ξ_i, $1 \leq i \leq n$.

Theorem 3.3.5. *The determinant of the restricted Saito metric η_D in the coordinates ξ_i, $1 \leq i \leq n$, is factorised into a product of linear forms given as follows,*

$$\det \eta_D = K \prod_{0 \leqslant i < j \leqslant n} \xi_{ij}^{m_i + m_j}, \tag{3.25}$$

where $\xi_{ij} = \xi_i - \xi_j$ and $K = (-1)^{\sum_{i=1}^n i m_i + nN}(N+1)^{-N} \prod_{a=1}^n m_a^2 \prod_{a=0}^n m_a^{m_a - 1}$.

Proof. We have by Lemma 3.3.4 that the determinant of the Saito metric η_D in the coordinates u_i, $1 \leq i \leq n$, is

$$\det \eta_D(u) = \prod_{i=1}^n \frac{1}{\lambda''_D(q_i)}. \tag{3.26}$$

Therefore we get with the help of formula (3.20) that

$$\det \eta_D(\xi) = (\det A)^{-2} \prod_{i=1}^n \lambda''_D(q_i), \tag{3.27}$$

where A is the $n \times n$ matrix with the matrix elements $\frac{1}{\xi_a - q_i}$, $1 \leqslant i, a \leqslant n$. We note that this is a Cauchy's determinant and can be expressed as

$$\det A = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{\substack{i=1 \\ i<j}}^n \xi_{ij} q_{ij}}{\prod_{i,a=1}^n (\xi_a - q_i)}, \tag{3.28}$$

where $q_{ij} = q_i - q_j$.

From Proposition 3.3.1 and formula (3.12) we get

$$\prod_{i=1}^n (\xi_a - q_i) = (N+1)^{-1} \left. \frac{\lambda'_D(p)}{(p - \xi_a)^{m_a - 1}} \right|_{p = \xi_a} \prod_{\substack{b=0 \\ b \neq a}}^n \xi_{ab}^{-m_b + 1} = \frac{m_a}{N+1} \prod_{\substack{b=0 \\ b \neq a}}^n \xi_{ab}. \tag{3.29}$$

Using formula (3.16) we have

$$\frac{\prod_{i=1}^n \lambda''_D(q_i)}{\prod_{i<j} q_{ij}^2} = (-1)^{\frac{n(n-1)}{2}} (N+1)^n \prod_{\substack{0 \leqslant a \leqslant n \\ 1 \leqslant i \leqslant n}} (q_i - \xi_a)^{m_a - 1}.$$

By formula (3.29) we get

$$\frac{\prod_{i=1}^{n} \lambda''_D(q_i)}{\prod_{i<j} q_{ij}^2} = c \prod_{a=0}^{n} \prod_{\substack{b=0 \\ b\neq a}}^{n} \xi_{ab}^{m_a-1}, \tag{3.30}$$

where $c = (-1)^{\frac{n(n-1)}{2}+n\sum_{a=0}^{n}(m_a-1)}(N+1)^n \prod_{a=0}^{n} \left((N+1)^{-1}m_a\right)^{m_a-1}$.

Combining formulae (3.27), (3.28), (3.30), we obtain the following expression for $\det \eta_D$

$$\det \eta_D = \frac{\prod_{i=1}^{n} \lambda''_D(q_i) \prod_{i,a=1}^{n}(\xi_a - q_i)^2}{\prod_{i<j} \xi_{ij}^2 q_{ij}^2} = c \prod_{a=1}^{n} \left((N+1)^{-1}m_a\right)^2 \prod_{0\leqslant a<b\leqslant n} \xi_{ab}^2 \prod_{a=0}^{n} \prod_{\substack{b=0 \\ b\neq a}}^{n} \xi_{ab}^{m_a-1}.$$

Finally, we note that

$$\prod_{a=0}^{n} \prod_{\substack{b=0 \\ b\neq a}}^{n} \xi_{ab}^{m_a-1} = (-1)^{\sum_{i=1}^{n} im_i - \frac{n(n+1)}{2}} \prod_{0\leqslant a<b\leqslant n} \xi_{ab}^{m_a+m_b-2},$$

which gives the required statement as $c\prod_{a=1}^{n}\left((N+1)^{-1}m_a\right)^2(-1)^{\sum_{i=1}^{n} im_i - \frac{n(n+1)}{2}} = K$. $\square$

3.3.2 B_N, D_N discriminant strata

We consider the Landau-Ginzburg superpotential

$$\lambda(p) = p^{2k} \prod_{i=1}^{N} (p^2 - (x^i)^2), \tag{3.31}$$

where p is some auxiliary variable and x^i, $1 \leq i \leq N$ are the standard orthonormal coordinates in $\mathbb{C}^N$ and $k = 0, -1$. In the case $k = 0$, λ is the superpotential for the B_N orbit space and in the case $k = -1$, λ is the superpotential for the D_N orbit space. Note that up to a scalar multiple (see Remark 2.5.5) the metric (2.39) for the superpotential (3.31) ($k = 0$) coincides with the standard B_N-invariant metric g on $\mathbb{C}^N$. Similarly for the case of D_N.

Let us consider a B_N/D_N stratum D in $\mathbb{C}^N$ given by the following equations:

$$\begin{aligned} x^1 &= \ldots = x^l = 0, \\ \varepsilon_1 x^{l+1} &= \ldots = \varepsilon_{m_1} x^{l+m_1} = \xi_1, \\ \varepsilon_{m_1+1} x^{l+m_1+1} &= \ldots = \varepsilon_{m_1+m_2} x^{l+m_1+m_2} = \xi_2, \\ &\vdots \\ \varepsilon_{\sum_{i=1}^{n-1} m_i+1} x^{l+\sum_{i=1}^{n-1} m_i+1} &= \ldots = \varepsilon_{\sum_{i=1}^{n} m_i} x^{l+\sum_{i=1}^{n} m_i} = \xi_n, \end{aligned} \tag{3.32}$$

where $l \in \mathbb{N} \cup \{0\}$, $\varepsilon_j = \pm 1$ ($j = 1, \dots, N-l$), $n, m_i \in \mathbb{N}$ ($i = 1, \dots, n$), $\sum_{i=1}^n m_i = N - l$, and $\xi_1, \dots, \xi_n$ are coordinates on D. Equations (3.32) define discriminant stratum for D_N provided $l \neq 1$, and they always define a discriminant stratum for B_N.

We then consider the following superpotential on D:

$$\lambda_D(p) = p^{2m} \prod_{i=1}^{n} (p^2 - \xi_i^2)^{m_i}, \tag{3.33}$$

where $m \in \mathbb{Z}$ and $\widehat{N} = m + \sum_{i=1}^n m_i \neq 0$. In the cases when $m \geq -1$ the superpotential $\lambda_D(p)$ corresponds to restriction of the superpotential $\lambda(p)$ to discriminant stratum D. Indeed, we get $\lambda_D(p)$ with $m = l - 1$, where $l = 0$ or $l \geq 2$ by restricting $\lambda(p)$ with $k = -1$ (type D_N) to the stratum (3.32). And we get $\lambda_D(p)$ with $m = l \geq 0$ as the restriction of $\lambda(p)$ with $k = 0$ (type B_N) to the stratum (3.32). Superpotentials (3.33) with $m_i = 1$ for all i and $-n + 1 \leq m \leq 0$ were considered in [15, 91]. The following statement follows from formula (3.33).

Proposition 3.3.6. *We have the following expression for the derivative $\lambda_D'(p)$:*

$$\lambda_D'(p) = 2\widehat{N} p^{2m-1} \prod_{i=1}^{n} (p^2 - \xi_i^2)^{m_i - 1} \prod_{i=1}^{n} (p^2 - q_i^2), \tag{3.34}$$

for some points $q_i \in \mathbb{C}$.

Let us define (canonical) coordinates $u_i = \lambda_D(q_i)$, $i = 1, \dots, n$.

Proposition 3.3.7. *We have the relation*

$$\partial_{u_i} \lambda_D(p)\big|_{p=q_j} = \delta_{ij}. \tag{3.35}$$

Proof. The proof is similar to the proof of Proposition 3.3.2. □

The following formula which follows from Proposition 3.3.6 will be useful below:

$$\lambda_D''(q_i) = 4\epsilon_i \widehat{N} q_i^{2m} \prod_{a=1}^{n} (q_i^2 - \xi_a^2)^{m_a - 1} \prod_{\substack{b=1 \\ b \neq i}}^{n} (q_i^2 - q_b^2), \tag{3.36}$$

where $\epsilon_i = 1$ if $q_i \neq 0$ and $\epsilon_i = \frac{1}{2}$ if $q_i = 0$. The latter case occurs if and only if $m = 0$. Let $\epsilon = \prod_{i=1}^n \epsilon_i$.

Lemma 3.3.8. *We have*

$$\partial_{u_i} \lambda_D(p) = \frac{2\epsilon_i p}{p^2 - q_i^2} \frac{\lambda_D'(p)}{\lambda_D''(q_i)}. \tag{3.37}$$

Proof. Let $U_i(p) = \frac{2\epsilon_i p}{p^2-q_i^2}\frac{\lambda'_D(p)}{\lambda''_D(q_i)}$. By Proposition 3.3.6 and formula (3.36) we get

$$U_i(p) = \frac{p^{2m}\prod_{a=1}^n (p^2-\xi_a^2)^{m_a-1}\prod_{\substack{b=1\\ b\neq i}}^n (p^2-q_b^2)}{q_i^{2m}\prod_{a=1}^n (q_i^2-\xi_a^2)^{m_a-1}\prod_{\substack{b=1\\ b\neq i}}^n (q_i^2-q_b^2)},$$

with $\deg U_i(p) = \deg \partial_{u_i}\lambda_D(p) = 2\widehat{N}-2$. It also follows that $U_i(p)|_{p=q_j} = \partial_{u_i}\lambda_D(q_j) = \delta_{ij}$ by Proposition 3.3.7. Note that the functions $U_i(p)$ and $\partial_{u_i}\lambda_D(p)$ have the form of a product of even polynomials of degree $2n-2$ and the function $p^{2m}\prod_{a=1}^n (p^2-\xi_a^2)^{m_a-1}$. This implies the lemma. □

Next we determine the Jacobi matrix between the coordinates ξ_i and u_i.

Lemma 3.3.9. *We have*

$$\partial_{u_i}\xi_a = \frac{2\epsilon_i\xi_a}{q_i^2-\xi_a^2}\frac{1}{\lambda''_D(q_i)}, \tag{3.38}$$

where $i, a = 1, \dots, n$.

Proof. From (3.33) we obtain

$$\partial_{u_i}\lambda_D(p) = -2\sum_{j=1}^n \frac{\lambda_D(p)}{p^2-\xi_j^2} m_j\xi_j\partial_{u_i}\xi_j. \tag{3.39}$$

By Lemma 3.3.8 we obtain the following identity from (3.39) for a fixed k ($k = 1, \dots, n$):

$$\frac{2\epsilon_i p}{(p^2-q_i^2)(p^2-\xi_k^2)^{m_k-1}}\frac{\lambda'_D(p)}{\lambda''_D(q_i)} = -2\sum_{j=1}^n \frac{\lambda_D(p)}{(p^2-\xi_j^2)(p^2-\xi_k^2)^{m_k-1}} m_j\xi_j\partial_{u_i}\xi_j. \tag{3.40}$$

We then consider the Taylor expansion of λ_D centred at $p = \xi_k$ observing that $\lambda_D^{(r)}(\xi_k) = 0$, $r = 1, \dots, m_k - 1$. Finally we substitute $p = \xi_k$ in the identity (3.40) and we obtain

$$\frac{2\epsilon_i\xi_k}{(\xi_k^2-q_i^2)\lambda''_D(q_i)}\left.\frac{\lambda'_D(p)}{(p^2-\xi_k^2)^{m_k-1}}\right|_{p=\xi_k} = -\frac{\lambda_D^{(m_k)}(\xi_k)}{(m_k-1)!(2\xi_k)^{m_k-1}}\partial_{u_i}\xi_k,$$

which implies the statement. □

Lemma 3.3.10. *The critical values* $u_i = \lambda_D(q_i)$, $i = 1, \dots, n$ *are the canonical coordinates for the structures (3.13), (3.14) on the stratum* D, *that is*

$$\eta_D(\partial_{u_i}, \partial_{u_j}) = \frac{2\epsilon_i\delta_{ij}}{\lambda''_D(q_i)},$$
$$\partial_{u_i}\circ\partial_{u_j} = \delta_{ij}\partial_{u_j}.$$

Proof. We use formulae (3.13), (3.14) together with (3.37). We consider formulae (3.13), (3.14) with the vector fields $\zeta_i = \partial_{u_i}$, $\zeta_j = \partial_{u_j}$. Note that the residues are trivial in ξ_a $(1 \leq a \leq n)$ by Lemma 3.3.8.

Let us consider first formula (3.13). In the case when $i \neq j$ the residues at q_l $(1 \leq l \leq n)$ are trivial by Lemma 3.3.8, and hence $\eta_D(\partial_{u_i}, \partial_{u_j}) = 0$. Further on, by (3.13) and (3.37) we have

$$\eta_D(\partial_{u_i}, \partial_{u_i}) = \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{(\partial_{u_i}\lambda_D(p))^2}{\lambda'_D(p)} dp = \frac{4\epsilon_i^2}{\lambda''_D(q_i)^2} \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{p^2\lambda'_D(p)}{(p^2-q_i^2)^2} dp.$$

It then follows from Proposition 3.3.6 and formulae (3.33), (3.36) that

$$\eta_D(\partial_{u_i}, \partial_{u_i}) = \frac{8\widehat{N}\epsilon_i^2}{\lambda''_D(q_i)^2} \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{p^{2m+1}\prod_{j=1}^n (p^2-\xi_j^2)^{m_j-1} \prod_{\substack{j=1 \\ j\neq i}}^n (p^2-q_j^2)}{p^2-q_i^2} dp = \frac{2\epsilon_i}{\lambda''_D(q_i)},$$

as required.

Let us now consider formula (3.14). In the case when $i \neq j$ or $j \neq k$ the residues at q_l $(1 \leq l \leq n)$ are trivial by Lemma 3.3.8, and hence $\eta_D(\partial_{u_i} \circ \partial_{u_j}, \partial_{u_k}) = 0$. Further on, by (3.14) and (3.37) we have

$$\eta_D(\partial_{u_i}\circ\partial_{u_i}, \partial_{u_i}) = \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{(\partial_{u_i}\lambda_D(p))^3}{\lambda'_D(p)} dp = \frac{8\epsilon_i^3}{\lambda''_D(q_i)^3} \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{p^3(\lambda'_D(p))^2}{(p^2-q_i^2)^3} dp.$$

It then follows from Proposition 3.3.6 that

$$\eta_D(\partial_{u_i}\circ\partial_{u_i}, \partial_{u_i}) = \frac{32\widehat{N}^2\epsilon_i^3}{\lambda''_D(q_i)^3} \sum_{p_s : \lambda'_D(p_s)=0} res|_{p=p_s} \frac{p^{4m+1}\prod_{j=1}^n (p^2-\xi_j^2)^{2(m_j-1)} \prod_{\substack{j=1 \\ j\neq i}}^n (p^2-q_j^2)^2}{p^2-q_i^2} dp.$$

Therefore we get using formula (3.36)

$$\eta_D(\partial_{u_i} \circ \partial_{u_i}, \partial_{u_i}) = \frac{2\epsilon_i}{\lambda''_D(q_i)}$$

and hence

$$\eta_D(\partial_{u_i} \circ \partial_{u_j}, \partial_{u_k}) = \frac{2\epsilon_i\delta_{ij}\delta_{jk}}{\lambda''_D(q_i)},$$

which implies the second statement of the lemma. ∎

The following lemmas will be useful below.

Lemma 3.3.11. *We have*

$$\prod_{i=1}^{n}(\xi_a^2 - q_i^2) = \frac{m_a}{\widehat{N}}\xi_a^2 \prod_{\substack{b=1\\ b\neq a}}^{n}(\xi_a^2 - \xi_b^2). \tag{3.41}$$

Proof. We have with the help of Proposition 3.3.6

$$\prod_{i=1}^{n}(\xi_a^2 - q_i^2) = \left.\frac{\lambda_D'(p)}{2\widehat{N}p^{2m-1}(p^2-\xi_a^2)^{m_a-1}}\right|_{p=\xi_a} \prod_{\substack{b=1\\ b\neq a}}^{n}(\xi_a^2 - \xi_b^2)^{-m_b+1}. \tag{3.42}$$

We note that

$$\left.\frac{\lambda_D'(p)}{(p^2-\xi_a^2)^{m_a-1}}\right|_{p=\xi_a} = 2m_a\xi_a^{2m+1}\prod_{\substack{b=1\\ b\neq a}}^{n}(\xi_a^2 - \xi_b^2)^{m_b},$$

by formula (3.33) and the statement follows. □

Lemma 3.3.12. *We have*

$$\prod_{a=1}^{n} q_a^2 = \frac{m}{\widehat{N}}\prod_{a=1}^{n}\xi_a^2.$$

Proof. The function λ_D' can be expressed as

$$\lambda_D'(p) = 2mp^{2m-1}\prod_{a=1}^{n}(p^2-\xi_a^2)^{m_a} + p^{2m}\frac{d}{dp}\prod_{a=1}^{n}(p^2-\xi_a^2)^{m_a}.$$

We equate the above formula to (3.34) and we divide both sides by p^{2m-1}. Finally we substitute $p=0$ to obtain

$$2\widehat{N}\prod_{a=1}^{n}(-\xi_a^2)^{m_a-1}\prod_{a=1}^{n}(-q_a^2) = 2m\prod_{a=1}^{n}(-\xi_a^2)^{m_a},$$

which implies the statement. □

Lemma 3.3.13. *Let*

$$z = \frac{\prod_{i=1}^{n}\lambda_D''(q_i)}{\prod_{\substack{i=1\\ i<j}}^{n}(q_i^2-q_j^2)^2}.$$

Then

$$z = c\prod_{a=1}^{n}\xi_a^{2(m+m_a-1)}\prod_{\substack{a,b=1\\ b\neq a}}^{n}(\xi_a^2-\xi_b^2)^{m_a-1},$$

where

$$c = (-1)^{n(\sum_{a=1}^{n} m_a - \frac{(n+1)}{2})}\epsilon 4^n \widehat{N}^{2n-m-\sum_{a=1}^{n} m_a}\prod_{a=1}^{n} m_a^{m_a-1}m^m. \tag{3.43}$$

Proof. Let us recall formula (3.36):

$$\lambda''_D(q_i) = 4\epsilon_i \widehat{N} q_i^{2m} \prod_{a=1}^{n} (q_i^2 - \xi_a^2)^{m_a-1} \prod_{\substack{b=1 \\ b\neq i}}^{n} (q_i^2 - q_b^2).$$

We have then

$$\begin{aligned} z &= \frac{(4\widehat{N})^n \epsilon \prod_{i=1}^n q_i^{2m} \prod_{a,i=1}^n (q_i^2 - \xi_a^2)^{m_a-1} \prod_{\substack{i,j=1 \\ j\neq i}}^n (q_i^2 - q_j^2)}{\prod_{\substack{i=1 \\ i<j}}^n (q_i^2 - q_j^2)^2} \\ &= (-1)^{\frac{n(n-1)}{2}} \epsilon (4\widehat{N})^n \prod_{i=1}^n q_i^{2m} \prod_{a,i=1}^n (q_i^2 - \xi_a^2)^{m_a-1}. \end{aligned}$$

By Lemma 3.3.11 we have

$$\prod_{a,i=1}^n (q_i^2 - \xi_a^2)^{m_a-1} = (-1)^{n\sum_{a=1}^n (m_a-1)} \widehat{N}^{-\sum_{a=1}^n (m_a-1)} \prod_{a=1}^n (m_a \xi_a^2)^{m_a-1} \prod_{\substack{a,b=1 \\ b\neq a}}^n (\xi_a^2 - \xi_b^2)^{m_a-1}.$$

Therefore the statement follows by Lemma 3.3.12. □

Theorem 3.3.14. *The determinant of the metric, η_D, given by* (3.13) *for the superpotential* (3.33) *in the coordinates ξ_i, $1 \leq i \leq n$, is factorised into a product of linear forms given as follows*

$$\det \eta_D = K \prod_{i=1}^n \xi_i^{2(m_i+m)} \prod_{1\leq i<j\leq n} (\xi_i^2 - \xi_j^2)^{m_i+m_j}, \tag{3.44}$$

where $K = (-1)^{n^2+n(\widehat{N}-m)+\sum_{i=1}^{n-1} i m_{i+1}} 2^n m^m \widehat{N}^{-\widehat{N}} \prod_{a=1}^n m_a^{m_a+1}$.

Proof. In the coordinates ξ_i the determinant of the metric η_D by Lemma 3.3.10 is

$$\det \eta_D = \epsilon (\det A)^{-2} \prod_{i=1}^n \frac{2}{\lambda''_D(q_i)}, \tag{3.45}$$

where A is the Jacobi matrix $(\partial_{u_i} \xi_a)_{i,a=1}^n$. By Lemma 3.3.9 we have

$$\det A = (-2)^n \epsilon \prod_{a=1}^n \xi_a \prod_{i=1}^n (\lambda''_D(q_i))^{-1} \det B, \tag{3.46}$$

where matrix $B = (\frac{1}{\xi_a^2 - q_i^2})_{i,a=1}^n$. The determinant of the matrix B is a Cauchy's determinant and can therefore be expressed as

$$\det B = (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{\substack{i=1 \\ i<j}}^n (\xi_i^2 - \xi_j^2)(q_i^2 - q_j^2)}{\prod_{i,a=1}^n (\xi_a^2 - q_i^2)}.$$

Hence $\det B$ can be expressed with the help of Lemma 3.3.11 as follows:

$$\det B = \frac{\widehat{N}^n}{\prod_{a=1}^n m_a \xi_a^2} \frac{\prod_{\substack{i=1 \\ i<j}}^n (q_i^2 - q_j^2)}{\prod_{1 \le a < b \le n} (\xi_a^2 - \xi_b^2)}. \tag{3.47}$$

Combining formulae (3.45), (3.46), (3.47) and Lemma 3.3.13 we then have

$$\det \eta_D = \epsilon^{-1} (2\widehat{N}^2)^{-n} \prod_{a=1}^n m_a^2 \xi_a^2 \prod_{1 \le a < b \le n} (\xi_a^2 - \xi_b^2)^2 z. \tag{3.48}$$

We get the required statement since

$$(-1)^{\sum_{i=1}^{n-1} i m_{i+1} - \frac{n(n-1)}{2}} \epsilon^{-1} (2\widehat{N}^2)^{-n} \prod_{a=1}^n m_a^2 c = K,$$

where c is given by formula (3.43). □

3.4 Classical series: Main Theorem 2

We show that the statement of Main Theorem 2 is true for the root systems A_N, B_N and D_N.

3.4.1 A_N discriminant strata

Theorem 3.4.1. *Suppose $\mathcal{R} = A_N$. Then the statement of Main Theorem 2 is true.*

Proof. Let $S \subset A_N$ be a collection of roots such that the discriminant stratum $D = \cap_{\gamma \in S} \Pi_\gamma$ is given by equations (3.11). Let $\xi_0, \ldots, \xi_n$ be the corresponding functions on D (see (3.11)).

Let $\mathcal{R}_D$ be the root system

$$\mathcal{R}_D = \langle S \rangle \cap A_N = \{\alpha \in A_N | \, \alpha|_D = 0\}.$$

Then $\mathcal{R}_D$ has the following structure

$$\mathcal{R}_D = \bigsqcup_{i: m_i > 1} A_{m_i - 1}.$$

We are interested in the multiplicities of the linear forms $l(\xi) = \xi_a - \xi_b$ $(0 \le a < b \le n)$ in the formula (3.25). We choose corresponding roots $\beta \in A_N$ such that $l = \beta|_D$ as follows:

$$\beta = e_{m_0 + \cdots + m_a} - e_{m_0 + \cdots + m_b}.$$

Let H be the hyperplane in D given by the kernel of l. Then we have

$$\langle S, \beta\rangle \cap A_N = \bigsqcup_{\substack{i:m_i>1 \\ i\neq a,b}} A_{m_i-1} \sqcup A_{m_a+m_b-1},$$

where the last root system $A_{m_a+m_b-1}$ contains β. Therefore $m_a+m_b = h(A_{m_a+m_b-1}) = k_H$ as required. This completes the proof for the root system A_N. □

3.4.2 B_N, D_N discriminant strata

Theorem 3.4.2. *Suppose $\mathcal{R} = B_N$. Then the statement of Main Theorem 2 is true.*

Proof. Let $S \subset B_N$ be a collection of roots such that the discriminant stratum $D = \cap_{\gamma\in S}\Pi_\gamma$ is given by equations (3.32). Let $\xi_1, \dots, \xi_n$ be the corresponding coordinates on D (see (3.32)).

Let $\mathcal{R}_D$ be the root system

$$\mathcal{R}_D = \langle S\rangle \cap B_N = \{\alpha \in B_N |\ \alpha|_D = 0\},$$

and consider root system A_{m_i-1} with corresponding simple system

$$\varepsilon_j e_{j+l} - \varepsilon_{j+1} e_{j+1+l}, \qquad \sum_{k=1}^{i-1} m_k + 1 \leq j \leq \sum_{k=1}^{i} m_k - 1.$$

Note that if $l = 0$, then $\mathcal{R}_D$ takes the form

$$\mathcal{R}_D = \bigsqcup_{i:m_i>1} A_{m_i-1},$$

and

$$\mathcal{R}_D = \bigsqcup_{i:m_i>1} A_{m_i-1} \sqcup \mathcal{R}^{(1)}, \tag{3.49}$$

where $\mathcal{R}^{(1)} = B_l$ if $l \geq 2$, $\mathcal{R}^{(1)} = A_1$, if $l = 1$.

We are interested in the multiplicities of the linear forms $\widehat{l}(\xi) = \xi_a$ $(1 \leq a \leq n)$ and $\widetilde{l}(\xi) = \xi_a \pm \xi_b$ $(1 \leq a < b \leq n)$ in (3.44). We choose corresponding roots $\widehat{\beta}, \widetilde{\beta} \in B_N$ such that $\widehat{l} = \widehat{\beta}\Big|_D$ and $\widetilde{l} = \widetilde{\beta}\Big|_D$, as follows:

$$\widehat{\beta} = e_{l+m_1+\dots+m_a}, \qquad \widetilde{\beta} = \varepsilon_{m_1+\dots+m_a} e_{l+m_1+\dots+m_a} \pm \varepsilon_{m_1+\dots+m_b} e_{l+m_1+\dots+m_b}.$$

Let $\widehat{H}$ and $\widetilde{H}$ be hyperplanes in D given by the kernels of $\widehat{l}$ and $\widetilde{l}$ respectively.

Let us consider firstly the form $\widehat{l}(\xi)$. If $m_a = 1$ then we have that

$$\langle S, \widehat{\beta} \rangle \cap B_N = \bigsqcup_{i:m_i>1} A_{m_i-1} \sqcup \mathcal{R}^{(2)},$$

where $\mathcal{R}^{(2)} = A_1$ when $l = 0$ and $\mathcal{R}^{(2)} = B_{l+1}$ when $l \geq 1$. The root system $\mathcal{R}^{(2)}$ contains $\widehat{\beta}$, and $2(l + m_a) = 2(l + 1) = h(\mathcal{R}^{(2)}) = k_{\widehat{H}}$ as required.

If $m_a > 1$ then we have

$$\langle S, \widehat{\beta} \rangle \cap B_N = \bigsqcup_{\substack{i:m_i>1 \\ i\neq a}} A_{m_i-1} \sqcup B_{l+m_a},$$

where the root system B_{l+m_a} contains $\widehat{\beta}$. Therefore $2(l+m_a) = h(B_{l+m_a}) = k_{\widehat{H}}$ as required.

Let us now consider the form $\widetilde{l}(\xi)$. Then

$$\langle S, \tilde{\beta} \rangle \cap B_N = \bigsqcup_{\substack{i:m_i>1 \\ i\neq a,b}} A_{m_i-1} \sqcup A_{m_a+m_b-1} \sqcup \mathcal{R}^{(1)},$$

where the root system $A_{m_a+m_b-1}$ contains $\widetilde{\beta}$ and $\mathcal{R}^{(1)}$ is the same as in (3.49). Therefore $m_a + m_b = h(A_{m_a+m_b-1}) = k_{\widetilde{H}}$ as required. □

Theorem 3.4.3. *Suppose $\mathcal{R} = D_N$. Then the statement of Main Theorem 2 is true.*

Proof. Let $S \subset D_N$ be a collection of roots such that the discriminant stratum $D = \cap_{\gamma\in S}\Pi_\gamma$ is given by equations (3.32), where $l \neq 1$. Let $\xi_1, \dots, \xi_n$ be the corresponding coordinates on D (see (3.32)).

Let $\mathcal{R}_D$ be the root system

$$\mathcal{R}_D = \langle S \rangle \cap D_N = \{\alpha \in D_N |\ \alpha|_D = 0\}.$$

Note that

$$\mathcal{R}_D = \bigsqcup_{i:m_i>1} A_{m_i-1}$$

if $l = 0$, and

$$\mathcal{R}_D = \bigsqcup_{i:m_i>1} A_{m_i-1} \sqcup \mathcal{R}^{(1)}, \tag{3.50}$$

where $\mathcal{R}^{(1)} = D_l$ if $l \geq 3$, and $\mathcal{R}^{(1)} = A_1 \times A_1$ if $l = 2$.

We are interested in the multiplicities of the linear forms $\widehat{l}(\xi) = \xi_a$ $(1 \leq a \leq n)$ and $\widetilde{l}(\xi) = \xi_a \pm \xi_b$ $(1 \leq a < b \leq n)$ in (3.44). We choose corresponding roots $\widehat{\beta}, \widetilde{\beta} \in D_N$ such that $\widehat{l} = \widehat{\beta}\big|_D$ and $\widetilde{l} = \widetilde{\beta}\big|_D$. Let $\widehat{H}$ and $\widetilde{H}$ be hyperplanes in D given by the kernels of $\widehat{l}$ and $\widetilde{l}$ respectively.

Let us consider firstly the form $\widehat{l}(\xi)$. This form has non-zero power in the formula (3.44) provided that $l \geq 2$ or $m_a \geq 2$. In the former case one can choose $\widehat{\beta} = e_l + e_{l+m_1+\dots+m_a}$ and in the latter one can choose

$$\widehat{\beta} = \varepsilon_{m_1+\dots+m_a-1} e_{l+m_1+\dots+m_a-1} + \varepsilon_{m_1+\dots+m_a} e_{l+m_1+\dots+m_a}.$$

If $m_a = 1$ then $l \geq 2$ and we have that

$$\langle S, \beta \rangle \cap D_N = \bigsqcup_{i:m_i>1} A_{m_i-1} \sqcup D_{l+1},$$

where the root system D_{l+1} contains $\widehat{\beta}$. Therefore,

$$2(m_a + m) = 2(m+1) = 2l = h(D_{l+1}) = k_{\widehat{H}},$$

as required.

If $m_a \geq 2$ then we have that

$$\langle S, \beta \rangle \cap D_N = \bigsqcup_{\substack{i:m_i>1 \\ i \neq a}} A_{m_i-1} \sqcup \mathcal{R}^{(2)},$$

where $\mathcal{R}^{(2)} = A_1 \times A_1$ if $m_a = 2$ and $l = 0$, and $\mathcal{R}^{(2)} = D_{l+m_a}$ if $l + m_a \geq 3$. The root system $\mathcal{R}^{(2)}$ contains $\widehat{\beta}$ and we have $2(m_a+m) = 2(m_a+l-1) = h(\mathcal{R}^{(2)}) = k_{\widehat{H}}$ as required.

Let us now consider the form $\widetilde{l}(\xi)$. The root $\widetilde{\beta}$ can be chosen as

$$\widetilde{\beta} = \varepsilon_{m_1+\dots+m_a} e_{l+m_1+\dots+m_a} \pm \varepsilon_{m_1+\dots+m_b} e_{l+m_1+\dots+m_b}.$$

Then

$$\langle S, \tilde{\beta} \rangle \cap D_N = \bigsqcup_{\substack{i:m_i>1 \\ i \neq a,b}} A_{m_i-1} \sqcup A_{m_a+m_b-1} \sqcup \mathcal{R}^{(1)},$$

where the root system $A_{m_a+m_b-1}$ contains $\widetilde{\beta}$ and $\mathcal{R}^{(1)}$ is the same as in (3.50). Therefore $m_a + m_b = h(A_{m_a+m_b-1}) = k_{\widetilde{H}}$ as required. $\square$

3.5 A general formula for the restricted Saito determinant

In what follows, let us fix a basis of simple roots α_i $(i = 1, \dots, n)$ for an n-dimensional system. We will formulate a general expression for the determinant of η_D. Let us define

$$\partial_{\alpha_k} = \sum_{i=1}^{n} \alpha_k^{(i)} \frac{\partial}{\partial x^i}, \tag{3.51}$$

where $\alpha_k = (\alpha_k^{(1)}, \dots, \alpha_k^{(n)})$. A basis of fundamental coweights, $\omega^i \in V$ $(i = 1, \dots, n)$ is defined by

$$(\omega^i, \alpha_j) = \delta^i_j.$$

Let us define a new coordinate system on V given by $\widetilde{x}^i = (\omega^i, x)$, $i = 1, \dots, n$.

Lemma 3.5.1. *In the coordinates $\widetilde{x}^i$, $1 \le i \le n$, we have $\frac{\partial}{\partial \widetilde{x}^i} = \partial_{\alpha_i}$.*

Proof. Let $x = (x^1, \dots, x^n)^\intercal$ and $\widetilde{x} = (\widetilde{x}^1, \dots, \widetilde{x}^n)^\intercal$. Then $\widetilde{x}^\intercal = \Omega x^\intercal$, where Ω is the $n \times n$ matrix $\Omega = (\Omega_{ij})_{i,j=1}^n$ with $\Omega_{ij} = \omega^i_{(j)}$ if $\omega^i = (\omega^i_{(1)}, \dots, \omega^i_{(n)})$. Then $x^\intercal = \Omega^{-1}\widetilde{x}^\intercal$, and it is easy to see that the (i,j)-th entry of Ω^{-1} equals $\alpha_j^{(i)}$. Therefore $\frac{\partial}{\partial \widetilde{x}^i} = \frac{\partial x^k}{\partial \widetilde{x}^i}\frac{\partial}{\partial x^k} = \alpha_i^{(k)} \frac{\partial}{\partial x^k} = \partial_{\alpha_i}$. ☐

For any basis of basic invariants p^i, $i = 1, \dots, n$, let $B(p)$ be the $(n-1) \times (n-1)$ matrix obtained from the Jacobi matrix $(\partial_{\alpha_i} p^j)_{i,j=1}^n$ by eliminating the k-th column and n-th row. Let $J_k(p) = J_k(p^1, \dots, p^{n-1}) = \det B(p)$ and let us fix $J_k = J_k(t^1, \dots, t^{n-1})$ for a basis of Saito polynomials. Note that the degree of J_k as a polynomial in x^i, is $|\mathcal{R}_+| - h + 1$ where h is the Coxeter number, since the entries of the n-th row consist of homogeneous polynomials of degrees $h - 1$, and $\deg J = |\mathcal{R}_+|$.

Proposition 3.5.2. [90] *The vector field $\frac{\partial}{\partial p^n}$ can be represented as*

$$\frac{\partial}{\partial p^n} = J^{-1}(p) \begin{vmatrix} \frac{\partial p^1}{\partial x^1} & \cdots & \frac{\partial p^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial p^{n-1}}{\partial x^1} & \cdots & \frac{\partial p^{n-1}}{\partial x^n} \\ \frac{\partial}{\partial x^1} & \cdots & \frac{\partial}{\partial x^n} \end{vmatrix}. \tag{3.52}$$

The above proposition can be checked easily by applying left-hand side and right-hand side of equality (3.52) to the polynomials p^i. Similarly, one can replace coordinates x^i in the right-hand-side of (3.52) with another coordinate system on V. This gives the following statement.

Proposition 3.5.3. *The identity field, $e = \frac{\partial}{\partial t^n}$, in the vector field basis ∂_{α_i}, $(i = 1, \dots, n)$ can be represented as*

$$e = J^{-1}\sum_{i=1}^{n}(-1)^{n+i}J_i\partial_{\alpha_i}. \tag{3.53}$$

Lemma 3.5.4. *Let $\mathcal{R} = B_n$. Then identity field e takes the following form:*

$$e = c\sum_{i=1}^{n}(x^i)^{-1}\prod_{\substack{j=1\\ j\neq i}}^{n}((x^i)^2-(x^j)^2)^{-1}\frac{\partial}{\partial x^i},$$

where $c \in \mathbb{C}^\times$.

Proof. Note that identity field e is proportional to $\frac{\partial}{\partial p^n}$, where $\frac{\partial}{\partial p^n} = J^{-1}(p)\sum_{i=1}^{n}(-1)^{n+i}J_i(p)\frac{\partial}{\partial x^i}$ by Proposition 3.5.2. The polynomial $J(p)$ is proportional to $I(\mathcal{A})$ where $\mathcal{A}$ is the arrangement corresponding to $\mathcal{R}$, namely

$$J(p) \sim \prod_{i=1}^{n}x^i\prod_{1\leq i<j\leq n}((x^i)^2-(x^j)^2).$$

For any i, $1 \leq i \leq n$ basic invariants can be chosen as

$$p^i = \sum_{j=1}^{n}(x^j)^{2i}$$

and thus one can show that $J_i(p)$ is proportional to the Vandermonde determinant

$$\prod_{\substack{j=1\\ j\neq i}}^{n}x^j\prod_{\substack{1\leq l<k\leq n\\ l,k\neq i}}((x^l)^2-(x^k)^2).$$

Then the statement follows. □

Proposition 3.5.3 can also be restated as follows. By formula (3.53) we can represent the identity field as

$$e = \sum_{\alpha\in\Delta}e^\alpha\partial_\alpha, \tag{3.54}$$

where

$$e^\alpha = \epsilon_\alpha\frac{J_\alpha}{J}, \quad J_\alpha = J_i, \tag{3.55}$$

with $\alpha = \alpha_i$ and ϵ_α the sign corresponding to the ordering of simple roots. More precisely, $\epsilon_\alpha = (-1)^{n+\sigma^{-1}(\alpha)}$ and σ is a bijection $\sigma : \{1,\ldots,n\} \to \Delta$.

Theorem 3.5.5. *In the coordinates $\widetilde{x}^i$ $(i = 1,\ldots,n)$ the contravariant Saito metric $\eta^{-1} = \eta^{ik}\frac{\partial}{\partial\widetilde{x}^i}\frac{\partial}{\partial\widetilde{x}^k}$ is given by*

$$\eta^{ik} = (-1)^{n+1+k}\partial_{\omega^i}\frac{J_k}{J} + (-1)^{n+1+i}\partial_{\omega^k}\frac{J_i}{J}. \tag{3.56}$$

Proof. For the Euclidean metric g in the coordinates $\widetilde{x}^i$ we have by Lemma 3.5.1

$$g_{ij} = (\partial_{\alpha_i}, \partial_{\alpha_j}) = (\alpha_i^{(k)} \frac{\partial}{\partial x^k}, \alpha_j^{(l)} \frac{\partial}{\partial x^l}) = \alpha_i^{(k)} \alpha_j^{(l)} \delta_{kl} = \sum_{k=1}^{n} \alpha_i^{(k)} \alpha_j^{(k)} = (\alpha_i, \alpha_j).$$

Hence, for the Saito metric 2.78 we have

$$\eta^{ij} = \mathcal{L}_e g^{ij} = -g^{kj} \partial_{\alpha_k} e^i - g^{ik} \partial_{\alpha_k} e^j = -\partial_{u^j} e^i - \partial_{u^i} e^j,$$

where vector $u^i = g^{ij} \alpha_j$. Therefore we have

$$(u^i, \alpha_j) = \sum_{k=1}^{n} g^{ik} (\alpha_k, \alpha_j) = g^{ik} g_{kj} = \delta^i_j.$$

Hence we can identify u^i with ω^i and rewrite ∂_{u^i} as ∂_{ω^i}. The result then follows immediately using Proposition 3.5.3. □

Proposition 3.5.6. *We have*

$$\partial_{\omega^i} \prod_{\alpha \in \Delta} \alpha = \prod_{\alpha \in \Delta \setminus \alpha_i} \alpha, \quad i = 1, \dots, n.$$

Proof. We have

$$\partial_{\omega^i} \prod_{\alpha \in \Delta} \alpha = \sum_{\alpha \in \Delta} \frac{(\omega^i, \alpha)}{\alpha} \prod_{\alpha \in \Delta} \alpha = \prod_{\alpha \in \Delta \setminus \alpha_i} \alpha,$$

by the definition of fundamental coweights. □

To get the determinant of the restricted Saito metric η_D we will need the following general result on determinants.

Proposition 3.5.7. [75] *Let* $A = (a_{ij})_{i,j=1}^n$ *be a square* $n \times n$ *matrix and let* $A_{ij} = (-1)^{i+j} M_{ij}$, *where* M_{ij} *is the* (i, j)*-th minor of* A. *Let* $1 \leqslant p < n$ *and let* $\sigma = \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ *be an arbitrary permutation from the symmetric group* S_n. *Then*

$$\begin{vmatrix} A_{i_1 j_1} & \dots & A_{i_1 j_p} \\ \vdots & \ddots & \vdots \\ A_{i_p j_1} & \dots & A_{i_p j_p} \end{vmatrix} = (-1)^{|\sigma|} (\det A)^{p-1} \begin{vmatrix} a_{i_{p+1}, j_{p+1}} & \dots & a_{i_{p+1}, j_n} \\ \vdots & \ddots & \vdots \\ a_{i_n, j_{p+1}} & \dots & a_{i_n j_n}, \end{vmatrix}$$

where $|\sigma|$ *is the sign of* σ.

We denote $\underline{n} = \{1, \dots, n\}$. Let $I \subset \underline{n}$ be a subset of cardinality $|I| = k$, $1 \leq k < n$.

Lemma 3.5.8. *Let* $D = \cap_{q \in I} \Pi_{\alpha_q}$ *be a discriminant stratum. Then* $\widetilde{x}^i$, $i \notin I$ *is a coordinate system on* D.

Proof. Note that $D = \langle \omega^i : i \notin I \rangle$. Let us consider a linear dependence of $\widetilde{x}^i$ $(i \notin I)$ on D:

$$\sum_{i \notin I} a_i \widetilde{x}^i = \sum_{i \notin I} a_i (\omega^i, x) = 0,$$

where $a_i \in \mathbb{C}$. Then $\sum_{i \notin I} a_i (\omega^i, \omega^j) = 0$ for all $j \notin I$. This is a system of $n - k$ linear equations and the matrix of this system is $\Omega = (\Omega_{ij})_{i,j \notin I}$, $\Omega_{ij} = (\omega^i, \omega^j)$. Since ω^i, $i \notin I$ are linearly independent the Gram matrix Ω is non-degenerate. Therefore the only solution to this system is the trivial one, $a_i = 0$ for all $i \notin I$. □

Let us now fix basic invariants to be Saito polynomials. We obtain the following result.

Theorem 3.5.9. *Let $I = \{i_1, \dots, i_k\}$, $1 \le k < n$, $1 \le i_1 < \dots < i_k \le n$ and let $D = \cap_{q \in I} \Pi_{\alpha_q}$. Then the determinant $\det \eta_D$ of the restricted Saito metric η_D in the coordinates $\widetilde{x}^i$ $(i \notin I)$ has the following form on D:*

$$\det \eta_D = -J^2 \begin{vmatrix} \eta^{i_1 i_1} & \dots & \eta^{i_1 i_k} \\ \vdots & \ddots & \vdots \\ \eta^{i_1 i_k} & \dots & \eta^{i_k i_k} \end{vmatrix}. \tag{3.57}$$

In particular, the right-hand-side of the expression (3.57) *has a well-defined limit as one tends to D.*

Proof. Let us consider the covariant Saito metric in the flat coordinates t^i, $(1 \le i \le n)$

$$\eta = \eta_{ij} dt^i dt^j = \sum_{i=1}^n dt^i dt^{n+1-i} = \sum_{i=1}^n \sum_{r=1}^n \partial_{\widetilde{x}^r} t^i d\widetilde{x}^r \sum_{l=1}^n \partial_{\widetilde{x}^l} t^{n+1-i} d\widetilde{x}^l. \tag{3.58}$$

Note that for any $p \in \mathbb{C}[x]^W$ one has that

$$\partial_{\widetilde{x}^i} p(x)\big|_{\alpha_i = 0} = \partial_{\alpha_i} p(x)\big|_{\alpha_i = 0} = 0. \tag{3.59}$$

Therefore, $\partial_{\widetilde{x}^l} t^i \big|_D = 0$, if $l = i_1, \dots, i_k$. Hence, using the property (3.59) and restricting formula (3.58) on D, we get that the Saito metric on D is given by

$$\eta_D = \sum_{i=1}^n \sum_{r \in \widehat{I}} \partial_{\widetilde{x}^r} t^i d\widetilde{x}^r \sum_{l \in \widehat{I}} \partial_{\widetilde{x}^l} t^{n+1-i} d\widetilde{x}^l = \sum_{r,l \in \widehat{I}} \eta_{rl} d\widetilde{x}^r d\widetilde{x}^l, \tag{3.60}$$

where $\eta_{rl} = \sum_{i=1}^n (\partial_{\widetilde{x}^r} t^i)(\partial_{\widetilde{x}^l} t^{n+1-i})$ and $\widehat{I} = \underline{n} \setminus I$.

Let $Q = (q_{ij})_{i,j=1}^n$ be the matrix of η^{-1} in the coordinates $\widetilde{x}^i$. Then

$$\eta_{rl} = \frac{(-1)^{r+l} Q_{rl}}{\det Q}, \quad r, l = 1, \dots, n,$$

where Q_{rl} is the (r,l)-th minor of Q. Consider the matrix $C = (\eta_{rl})_{r,l\in\widehat{I}}$. It follows from formula (3.60) that $C|_D$ is the matrix of η_D and $\det \eta_D = \det C|_D$. By Proposition 3.5.7 applied for $A = Q$, $p = |\widehat{I}| = n-k$ and $\sigma = \mathrm{Id}$ we have

$$\det C = (\det Q)^{-(n-k)}(\det Q)^{n-k-1}\det Q_I = \det Q^{-1}\det Q_I,$$

where Q_I is the matrix $(q_{ij})_{i,j\in I}$. Since $\det Q^{-1} = \det \eta$, which is equal to $-J^2$ by Proposition 3.1.3, the statement follows. □

Proposition 3.5.10. *Let J_k be as above. Then J_k is divisible by $\alpha(x)$ for all $\alpha \in \mathcal{R}\cap U$, where vector space $U = \langle \alpha_i : 1 \le i \le n, i \neq k\rangle$.*

Proof. Let $\alpha = \sum_{\substack{i=1\\ i\neq k}}^{n} c_i\alpha_i$ for some $c_i \in \mathbb{C}$. Consider the linear combination of columns of the matrix $B(t)$ such that the i-th column is taken with the coefficient c_i. The resulting entries are of the form $\partial_\alpha t^i$ hence they are divisible by $\alpha(x)$. One can assume that such a column appears in the matrix whose determinant is proportional to J_k, hence the statement follows. □

Proposition 3.5.11. *The identity field e is singular on every hyperplane of the discriminant of W.*

Proof. We have that $\deg J = |\mathcal{R}_+|$ and $\deg J_k = |\mathcal{R}_+| - h + 1$ for any k, $1 \le k \le n$. Hence by formula (3.53) $e^k = (-1)^{n+k}\frac{J_k}{J}$ is a rational function of degree $1-h$. Let us suppose that e is non-singular everywhere on V. Then we must have that $e^k = 0$ for all k and thus e is identically zero which is a contradiction. Therefore e is singular on Π_β for some $\beta \in \mathcal{R}$. Since e is W-invariant it follows that it is singular on Π_γ for any $\gamma \in \mathcal{R}$ such that $\gamma \in W\beta$.

In the case where $\mathcal{R}$ is an irreducible root system with a single orbit of the group W it follows that e is singular on Π_β for all $\beta \in \mathcal{R}$.

Let us now consider the cases where $\mathcal{R}$ is an irreducible root system with two orbits of W. Let the root system $\mathcal{R} = F_4$ and let $\Delta \subset \mathcal{R}$ be a simple system. Recall that the corresponding Coxeter number is $h = 12$. Let $\alpha \in \Delta$ be such that $e^\alpha = \epsilon_\alpha \frac{J_\alpha}{J}$ is non-zero. Let us now assume that e is non-singular on the hyperplane Π_α. By Proposition 3.5.10 J_β is divisible by (α, x) for any $\beta \in \Delta \setminus \{\alpha\}$. Hence e^β is non-singular on Π_α, and therefore J_α must also be divisible by (α, x). It follows that

$$\prod_{\beta\in\Delta}\beta \mid J_\alpha. \tag{3.61}$$

Since e^α is singular on at least $h-1 = 11$ different hyperplanes inside the discriminant and there are 12 short and 12 long positive roots, it follows from (3.61) that e^α has singularities on hyperplanes from both orbits. It follows by W-invariance of e that e is singular on Π_α, which is a contradiction. We therefore have that e^α is singular on Π_α and hence e is singular on Π_β for all $\beta \in \Delta$ such that $\beta \in W\alpha$. Thus by Proposition 3.5.10 we have that $e^\beta = \epsilon_\beta \frac{J_\beta}{J} \neq 0$. Let us now assume (see e.g. [51]) that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

If α is a long root then it follows from the previous that $J_{\alpha_1} \neq 0$. By Proposition 3.5.10 J_{α_1} is divisible by $\alpha_2\alpha_3\alpha_4(\alpha_2 + 2\alpha_3)$. Since this product has two long roots and two short roots and $\deg e^{\alpha_1} = -11$ it follows that e^{α_1} has singularities on both orbits. If α is a short root then it follows by the previous that $J_{\alpha_4} \neq 0$. By Proposition 3.5.10 J_{α_4} is divisible by $\alpha_1\alpha_2\alpha_3(\alpha_2 + \alpha_3)$. Since this product has two long and two short roots it follows similarly to the previous case that e^{α_4} is singular on both orbits. The statement follows for both cases due to W-invariance of e.

Let us now consider the case where $\mathcal{R} = I_2(2m)$, $m \geq 3$. By Proposition 3.5.2 identity field e is proportional to

$$J^{-1}(p)(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}),$$

and the statement follows. Finally, the case where $\mathcal{R} = B_n$ is a corollary of Lemma 3.5.4. □

Corollary 3.5.12. *The polynomial J_k is not identically zero on the hyperplane Π_{α_k}. In particular, J_k is not a zero polynomial.*

Proof. We have from Proposition 3.5.11 that the identity field e is singular on the hyperplane Π_γ for all $\gamma \in \mathcal{R}$ and thus it is singular on Π_{α_k}. Further on, we have from Proposition 3.5.10 that the polynomial J_i, $i \neq k$, is divisible by $\alpha_k(x)$. Since the degree of vanishing of J on Π_{α_k} is 1 and e is singular on Π_{α_k} it follows that J_k is not divisible by $\alpha_k(x)$. The statement follows. □

Proposition 3.5.13. *Let $\beta, \gamma \in \mathcal{R}$, $\beta \neq \pm\gamma$. Then $J_k(x) = 0$ if $\beta(x) = \gamma(x) = 0$.*

Proof. There exists a non-trivial linear combination of β and γ such that

$$a_1\beta + a_2\gamma = \sum_{\substack{i=1 \\ i \neq k}}^{n} b_i\alpha_i,$$

where $a_1, a_2, b_i \in \mathbb{C}$. Note that $\partial_\beta p|_D = \partial_\gamma p|_D = 0$ for $D = \Pi_\beta \cap \Pi_\gamma$ and any invariant polynomial p. Hence a linear combination of columns of the matrix $B(t)$ is zero on D. □

Let us recall the following statement on the cardinality of restricted Coxeter arrangement.

Proposition 3.5.14. [71] *Let $\mathcal{A}$ be a Coxeter arrangement for an irreducible Coxeter group W, and let $H \in \mathcal{A}$. Then the cardinality of $\mathcal{A}_H$ is*

$$|\mathcal{A}_H| = |\mathcal{A}| - h + 1, \tag{3.62}$$

where h is the Coxeter number of W. In particular, $|\mathcal{A}_H|$ does not depend on the choice of H.

Using Propositions 3.5.13 and 3.5.14 we get the following statement on the structure of J_k.

Corollary 3.5.15. *Let $\mathcal{A}$ be a Coxeter arrangement and $D = \Pi_{\alpha_k}$ for some k, $1 \leq k \leq n$. Then $J_k|_D$ is proportional to $I(\mathcal{A}_D)$.*

Proof. We have that $\deg J_k = |\mathcal{R}_+| - h + 1$, and hence $\deg J_k = |\mathcal{A}_\mathcal{D}|$ using Proposition 3.5.14. From Proposition 3.5.13, it follows that $J_k|_D$ is divisible by $\beta|_D$ for any $\beta \in \mathcal{R} \setminus \{\pm\alpha_k\}$. The statement follows. □

Let us define the following polynomials:

$$I := J \prod_{\alpha \in \Delta} \alpha(x)^{-1} \quad \text{and} \quad I_k := J_k \prod_{\alpha \in \Delta \setminus \alpha_k} \alpha(x)^{-1}, \tag{3.63}$$

where $1 \leq k \leq n$.

We denote the discriminant strata as $D_{i_1,\dots,i_k} = D_{\alpha_{i_1},\dots,\alpha_{i_k}}$, namely,

$$D_{i_1,\dots,i_k} = \cap_{j=1}^{k} \Pi_{\alpha_{i_j}}. \tag{3.64}$$

We obtain the following useful result on the relation between I_m and I_l on $D_{l,m}$.

Proposition 3.5.16. *Let $\alpha_l, \alpha_m \in \Delta$ be such that $|\mathcal{R}_+ \cap S| > 2$, where $S = \langle \alpha_l, \alpha_m \rangle$. Let $D = D_{l,m}$ be the corresponding stratum. Then*

$$I_m|_D = (-1)^{l-m-1} \, I_l|_D \, . \tag{3.65}$$

Proof. Let v_k denote the column vector

$$v_k = \left(\partial_{\alpha_k} p^1, \dots, \partial_{\alpha_k} p^{n-1}\right)^\intercal, \tag{3.66}$$

$k = 1, \dots, n$. We have

$$\partial_{\alpha_k} p^i = \alpha_k(x) Q_{ki}(x), \tag{3.67}$$

for some $Q_{ki}(x) \in \mathbb{C}[x]$. Denote the corresponding column vector $Q_k = (Q_{k1}, \dots, Q_{k,n-1})^\intercal$.

Let us firstly consider the case when $(\alpha_l, \alpha_m) \neq 0$. It follows from Equation (3.67) that

$$\partial_{\alpha_m}\partial_{\alpha_l} p^i = (\alpha_l, \alpha_m) Q_{li}(x) + \alpha_l(x)\partial_{\alpha_m} Q_{li}(x). \tag{3.68}$$

Similarly,

$$\partial_{\alpha_l}\partial_{\alpha_m} p^i = (\alpha_l, \alpha_m) Q_{mi}(x) + \alpha_m(x)\partial_{\alpha_l} Q_{mi}(x) \tag{3.69}$$

Restriction of equalities (3.68), (3.69) to D gives

$$Q_{li}(x)|_D = Q_{mi}(x)|_D \,. \tag{3.70}$$

Let $\widetilde{\Delta} = \Delta \setminus \{\alpha_m, \alpha_l\}$. Note that $I_m = a(x)\alpha_l(x)^{-1}J_m(x)$ and $I_l(x) = a(x)\alpha_m(x)^{-1}J_l(x)$, where $a(x) = \prod_{\alpha\in\widetilde{\Delta}} \alpha(x)^{-1}$. If $l < m$ then $\alpha_l(x)^{-1}J_m(x) = \det A_{lm}$, where the matrix A_{lm} has columns $v_1, \ldots, v_{l-1}, Q_l, v_{l+1}, \ldots, \widehat{v}_m, \ldots, v_n$. Similarly, $\alpha_m(x)^{-1}J_l(x) = \det A_{ml}$, where the matrix A_{ml} has columns $v_1, \ldots, \widehat{v}_l, \ldots, v_{m-1}, Q_m, v_{m+1}, \ldots, v_n$ and where $\widehat{v}_m$, $\widehat{v}_l$ means that the corresponding column is omitted. By the property (3.70) the matrices $A_{lm}|_D$, $A_{ml}|_D$ have the same columns up to a permutation. The case $m < l$ is similar. Now, let us suppose that $(\alpha_l, \alpha_m) = 0$. We are going to establish property (3.70). Since $|\mathcal{R}_+ \cap S| > 2$, there exists $\gamma \in \mathcal{R}_+$ such that $\gamma = c_1\alpha_l + c_2\alpha_m$ for some $c_1, c_2 \in \mathbb{C}^\times$. Then

$$\partial_\gamma p^i = \gamma(x) Q_{\gamma,i}, \tag{3.71}$$

for some $Q_{\gamma,i} \in \mathbb{C}[x]$, $i = 1, \ldots, n$. Therefore

$$\partial_{\alpha_l} p^i = c_1^{-1}(\partial_\gamma - c_2\partial_{\alpha_m})p^i = c_1^{-1}\big((c_1\alpha_l(x) + c_2\alpha_m(x))Q_{\gamma,i} - c_2\alpha_m(x)Q_{mi}\big).$$

Hence

$$Q_{li} = \alpha_l(x)^{-1}\partial_l p^i = Q_{\gamma i} + \frac{c_2}{c_1}\frac{\alpha_m(x)}{\alpha_l(x)}(Q_{\gamma i} - Q_{mi}),$$

which implies that

$$Q_{li}|_{D_m} = Q_{\gamma,i}|_{D_m} \,. \tag{3.72}$$

Further on by differentiating (3.67) for $k = m$ and by differentiating (3.71) we get

$$\partial_\gamma\partial_{\alpha_m} p^i = (\gamma, \alpha_m)Q_{mi} + \alpha_m(x)\partial_\gamma Q_{mi}(x) = (\gamma, \alpha_m)Q_{\gamma,i} + \gamma(x)\partial_{\alpha_m}Q_{\gamma,i}(x). \tag{3.73}$$

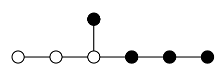
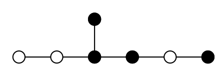
Since $(\gamma, \alpha_m) \neq 0$ the restriction of equality (3.73) to D gives

$$Q_{mi}|_D = Q_{\gamma,i}|_D \,. \tag{3.74}$$

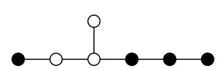
By formulae (3.72), (3.74) we have $Q_{li}|_D = Q_{mi}|_D$. The statement follows similarly to the case $(\alpha_l, \alpha_m) \neq 0$. $\square$

3.6 Exceptional groups: dimension 1 and codimensions 1, 2, 3, 4.

Orlik and Solomon [72] and Shcherbak [80] classified the strata in the Coxeter discriminants. We say that $x, y \in V$ are equivalent if their corresponding stabilizers in W are conjugate subgroups in W. Let Γ be a Coxeter subgraph of the graph associated to W and let W_Γ be the parabolic subgroup generated by reflections in the mirrors corresponding to the vertices of Γ. Then W_Γ is the stabilizer W_x of a generic point x on the stratum corresponding to the vertices of Γ. Thus up to equivalence there is a map from strata in the Coxeter discriminant to types of subgraphs. This map is surjective but in general not injective since it can be that there are different subgraphs of the Coxeter graph which have the same type and which are mapped to by different discriminant strata ([35], [72], [80]). For example there are two non-conjugate classes of subgroups of type $A_3 \times A_1$ in E_7: there are 11 subgraphs of type $A_3 \times A_1$, one stratum corresponding to the subgraphs

 or 

and one stratum corresponding to the remaining 9, for example

.

Let us stress that our analysis depends on the parabolic subgroup, that is the type of the Coxeter subgraph only and it does not depend on the particular choice of stratum for a given parabolic subgroup. We are going to prove Main Theorems 1, 2 for a subset of simple roots $L = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\} \subset \Delta$, $1 \leq k < n$ and the corresponding stratum D.

In this section, we obtain formulae for the determinant of the restricted Saito metric for exceptional groups in codimensions $1, 2, 3, 4$ and $n-1$ and thus show that the statement of Main Theorem 1 and Main Theorem 2 is true.

3.6.1 Dimension 1

Let us choose $n-1$ different elements $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$ and consider the stratum $D = D_{i_1,\ldots,i_{n-1}}$. Let $i_n \neq i_k$ for any $k = 1, \ldots, n-1$, $1 \leq i_n \leq n$.

Theorem 3.6.1. *The determinant of the restricted Saito metric η_D is proportional to $(\widetilde{x}^{i_n})^h$, where h is the Coxeter number of $\mathcal{R}$.*

Proof. The covariant Saito metric η can be expressed as

$$\eta = \sum_{k=1}^{n} \sum_{i=1}^{n} \partial_{\widetilde{x}^i} t^k d\widetilde{x}^i \sum_{j=1}^{n} \partial_{\widetilde{x}^j} t^{n+1-k} d\widetilde{x}^j. \tag{3.75}$$

Since $\partial_{\widetilde{x}^{i_j}} t^k \big|_D = \partial_{\alpha_{i_j}} t^k \Big|_D = 0$ for $j = 1, \ldots, n-1$ we get

$$\eta_D = \sum_{k=1}^{n} \partial_{\widetilde{x}^{i_n}} t^k \partial_{\widetilde{x}^{i_n}} t^{n+1-k} (d\widetilde{x}^{i_n})^2. \tag{3.76}$$

Note that $\partial_{\widetilde{x}^{i_n}} t^k \partial_{\widetilde{x}^{i_n}} t^{n+1-k}$ is proportional to $(\widetilde{x}^{i_n})^h$ since $\deg t^j = d_j$ for all j and $d_j + d_{n+1-j} = h+2$. This implies the statement. □

Corollary 3.6.2. *The statement of Main Theorems* 1, 2 *is true.*

3.6.2 Codimension 1

Fix m, $1 \leq m \leq n$ and consider the corresponding $(n-1)$-dimensional stratum $D = D_m$. Let $H \in \mathcal{A}_D$.

Theorem 3.6.3. *The determinant of the restricted Saito metric η_D is factorisable into a product of linear forms on D. Furthermore it is proportional to*

$$\prod_{H \in \mathcal{A}_D} l_H^{m_H}, \tag{3.77}$$

where $m_H = |\Sigma_H|$, with $\Sigma_H = \{X \in \mathcal{A} | H \subset X\}$.

Proof. By Theorem 3.5.9 we have that $\det \eta_D$ is proportional to $-\,\eta^{mm} J^2 \big|_D$ and therefore by Theorem 3.5.5 that

$$\det \eta_D = 2J^2 \partial_{\omega^m} \frac{J_m}{J} \bigg|_D = -\,2J_m \partial_{\omega^m} J \big|_D .$$

Recall that $J = c\alpha_m \prod_{\beta \in R_m} \beta$, where $R_m = \mathcal{R}_+ \setminus \{\alpha_m\}$, $c \in \mathbb{C}$. We thus note that $\partial_{\omega^m} J|_D = c \prod_{\beta \in R_m} \beta|_D$. Therefore, by Corollary 3.5.15, one has that $\det \eta_D$ is proportional to

$$I(\mathcal{A}_D) \prod_{\beta \in R_m} \beta|_D \,, \tag{3.78}$$

Note that the second product in equality (3.78) can be written as $\prod_{H \in \mathcal{A}_D} l_H^{q_H}$, where $q_H = |\widetilde{\Sigma}_H|$ with $\widetilde{\Sigma}_H = \{X \in \mathcal{A} | X \supset H, H \neq D\}$. That is, $q_H = m_H - 1$. Thus, we obtain the required result. □

Corollary 3.6.4. *The statement of Main Theorem* 1 *is true.*

Fix $H \in \mathcal{A}_D$. Let $\beta \in \mathcal{R}$ be such that $H = \{x \in D | \beta(x) = 0\}$ and consider $\mathcal{R}_{D,\beta} = \mathcal{R} \cap \langle \alpha_m, \beta \rangle$.

Proposition 3.6.5. *We have that*

$$m_H = h_{m,\beta}$$

where $h_{m,\beta}$ is the Coxeter number of the irreducible subsystem in $\mathcal{R}_{D,\beta}$ which contains β.

Proof. The root system $\mathcal{R}_{D,\beta}$ is a rank 2 subsystem of the root system $\mathcal{R}$ containing α_m and β. Note that $m_H = \frac{1}{2}|\mathcal{R}_{D,\beta}|$. If the dihedral root system $\mathcal{R}_{D,\beta}$ is irreducible then $\frac{1}{2}|\mathcal{R}_{D,\beta}|$ equals its Coxeter number, and the statement follows. If the root system $\mathcal{R}_{D,\beta}$ is reducible then

$$\mathcal{R}_{D,\beta} = \{\pm\alpha_m, \pm\beta\} = A_1 \times A_1$$

and $m_H = \frac{1}{2}|\mathcal{R}_{D,\beta}| = 2$. Since the Coxeter number of the root system A_1 equals 2 the statement holds in this case as well. □

Corollary 3.6.6. *The statement of Main Theorem* 2 *is true.*

3.6.3 Codimension 2

Let $\alpha_l, \alpha_m \in \Delta$, $1 \leq m < l \leq n$. Let us consider the $(n-2)$-dimensional stratum $D = D_{l,m}$. We note that restriction of the covariant Saito metric η to the stratum D is well-defined as the components of the metric η are polynomial expressions in the coordinates x^i $(i = 1, \dots, n)$. However, this is not necessarily true for the individual terms in the expansion (3.57) of $\det \eta_D$ as these terms can be singular on D. Below we will calculate limits of these terms as x tends to D in a prescribed way which will give us the value of $\det \eta_D$.

More specifically, by Theorem 3.5.9, the determinant $\det \eta_D$ is given by

$$\det \eta_D = -J^2 \begin{vmatrix} \eta^{mm} & \eta^{ml} \\ \eta^{ml} & \eta^{ll} \end{vmatrix}, \tag{3.79}$$

where the limit of the right-hand side as x tends to D is taken. Furthermore, recall that by Theorem 3.5.5 we have

$$\eta^{ik} = (-1)^{n+k+1}\partial_{\omega^i}\frac{J_k}{J} + (-1)^{n+i+1}\partial_{\omega^k}\frac{J_i}{J}, \tag{3.80}$$

$i, k = 1, \dots, n$. Therefore, using formulae (3.63) one has

$$\partial_{\omega^i}\frac{J_k}{J} = \partial_{\omega^i}\frac{I_k}{\alpha_k(x)I} = -\frac{1}{\alpha_k(x)^2}\frac{I_k}{I}\delta_{ik} + \frac{1}{\alpha_k(x)}\partial_{\omega^i}\frac{I_k}{I}. \tag{3.81}$$

Further on, we are interested in the structure of I. Let $d \in \mathbb{Z}_{\geq 0}$ be the degree of vanishing of I on D. Note that $d = |\mathcal{R}_+ \cap \langle \alpha_l, \alpha_m \rangle| - 2$. Let us represent I as

$$I = fg, \tag{3.82}$$

where $f \in \mathbb{C}[x]$ is a homogeneous polynomial of degree d in the variables $\alpha_m(x), \alpha_l(x)$ and $g \in \mathbb{C}[x]$ is not identically zero on D. Let d_0 be the degree of $f(x)$ as a polynomial in α_l, $d_0 \leq d$. We represent $f(x)$ as

$$f(x) = \sum_{i=0}^{d_0} a_i \alpha_l^i(x) \alpha_m^{d-i}(x) = \alpha_m^{d-d_0}(x) \sum_{i=0}^{d_0} a_i \alpha_l^i(x) \alpha_m^{d_0-i}(x), \tag{3.83}$$

where $a_i \in \mathbb{C}$, $a_{d_0} \neq 0$. We have the following result.

Let $\alpha, \beta \in \mathcal{R}$. In what follows, we will mean by $F|_{\substack{\alpha=0\\ \beta=0}}$ the restriction of a function F onto $\alpha = \beta = 0$ in the order $\alpha = 0$ first followed by taking the limit $\beta \to 0$.

Lemma 3.6.7. *We have*

$$\alpha_l(x) I(x)^{-1} \partial_{\omega^l} I(x)\big|_{\substack{\alpha_m=0\\ \alpha_l=0}} = d_0. \tag{3.84}$$

Proof. Let f and g be as defined in (3.82), (3.83). Then

$$\alpha_l(x) I(x)^{-1} \partial_{\omega^l} I(x) = \alpha_l(x) g(x)^{-1} \partial_{\omega^l} g(x) + \alpha_l(x) f(x)^{-1} \partial_{\omega^l} f(x). \tag{3.85}$$

We note that by formula (3.83) one has

$$\alpha_l(x) f(x)^{-1} \partial_{\omega^l} f(x)\big|_{\alpha_m=0} = \frac{\sum_{i=1}^{d_0} i a_i \alpha_l^i \alpha_m^{d_0-i}}{\sum_{i=0}^{d_0} a_i \alpha_l^i \alpha_m^{d_0-i}}\Bigg|_{\alpha_m=0} = d_0.$$

Therefore restricting expression (3.85) onto $\alpha_m = 0$ first followed by the restriction onto $\alpha_l = 0$, the statement follows. □

Let $\widetilde{\Delta} = \Delta \setminus \{\alpha_l, \alpha_m\}$. Let us consider the diagonal and anti-diagonal terms in the determinant (3.79) separately.

Lemma 3.6.8. *Let $A = J^2 \eta^{mm} \eta^{ll}$. Then*

$$A\big|_{\substack{\alpha_m=0\\ \alpha_l=0}} = (-1)^{m+l} 4(d_0+1) I_l I_m \prod_{\alpha \in \widetilde{\Delta}} \alpha^2\big|_{\substack{\alpha_m=0\\ \alpha_l=0}}. \tag{3.86}$$

Proof. By formulae (3.80), (3.81) we have that

$$\eta^{kk} = (-1)^{n+k+1} 2 \partial_{\omega^k} \frac{J_k}{J} = (-1)^{n+k+1} 2\Big(-\frac{1}{\alpha_k^2}\frac{I_k}{I} + \frac{1}{\alpha_k}\partial_{\omega^k}\frac{I_k}{I}\Big),$$

for any $k = 1, \dots, n$. Then

$$J\eta^{kk} = (-1)^{n+k+1} 2\big(-\alpha_k^{-1} I_k + \partial_{\omega^k} I_k - I_k I^{-1} \partial_{\omega^k} I\big) \prod_{\alpha \in \Delta \setminus \{\alpha_k\}} \alpha.$$

Then

$$\begin{aligned} A &= (-1)^{m+l} 4\big(-\alpha_m^{-1} I_m + \partial_{\omega^m} I_m - I_m I^{-1} \partial_{\omega^m} I\big)\big(-\alpha_l^{-1} I_l + \partial_{\omega^l} I_l - I_l I^{-1} \partial_{\omega^l} I\big) \prod_{\alpha \in \Delta} \alpha \prod_{\alpha \in \widetilde{\Delta}} \alpha \\ &= (-1)^{m+l} 4\Big(-I_m + \alpha_m \big(\partial_{\omega^m} I_m - I_m I^{-1} \partial_{\omega^m} I\big)\Big)\Big(-I_l + \alpha_l \big(\partial_{\omega^l} I_l - I_l I^{-1} \partial_{\omega^l} I\big)\Big) \prod_{\alpha \in \widetilde{\Delta}} \alpha^2. \end{aligned}$$

We consider the restriction of A on D_m first. This gives,

$$A\big|_{\alpha_m=0} = (-1)^{m+l} 4 I_m \Big(I_l - \alpha_l(x) \big(\partial_{\omega^l} I_l - I_l I^{-1} \partial_{\omega^l} I\big)\Big) \prod_{\alpha \in \widetilde{\Delta}} \alpha^2\big|_{\alpha_m=0}.$$

Therefore, restricting A further on D_l and using Lemma 3.6.7, we obtain the required result. □

Let us now consider the anti-diagonal terms.

Lemma 3.6.9. *Let $B = \eta^{ml} J$. Then*

$$B|_{\substack{\alpha_m=0 \\ \alpha_l=0}} = d_0 (-1)^{n+m} I_m \prod_{\alpha \in \widetilde{\Delta}} \alpha|_{\substack{\alpha_m=0 \\ \alpha_l=0}}, \tag{3.87}$$

Proof. Using formulae (3.80), (3.81) we have

$$B = J\Big((-1)^{n+l+1} \partial_{\omega^m} \frac{J_l}{J} + (-1)^{n+m+1} \partial_{\omega^l} \frac{J_m}{J} \Big),$$

that is

$$B = \Big((-1)^{n+l+1} \alpha_m \big(\partial_{\omega^m} I_l - I_l I^{-1} \partial_{\omega^m} I\big) + (-1)^{n+m+1} \alpha_l(x) \big(\partial_{\omega^l} I_m - I_m I^{-1} \partial_{\omega^l} I\big)\Big) \prod_{\alpha \in \widetilde{\Delta}} \alpha.$$

We consider the restriction of B on D_m at first. This gives

$$B\big|_{\alpha_m=0} = (-1)^{n+m+1} \alpha_l \big(\partial_{\omega^l} I_m - I_m I^{-1} \partial_{\omega^l} I\big) \prod_{\alpha \in \widetilde{\Delta}} \alpha\big|_{\alpha_m=0}. \tag{3.88}$$

Then restricting B further on D_l and using Lemma 3.6.7 we obtain the required result. □

Using the above results we obtain a general expression for the determinant $\det \eta_D$.

Theorem 3.6.10. *The determinant of the metric η_D is proportional to*

$$I_m I_l \prod_{\alpha\in\widetilde{\Delta}} \alpha^2\big|_D . \tag{3.89}$$

Proof. In the notation of Lemmas 3.6.8, 3.6.9 we have

$$\det \eta_D = B^2\big|_{\substack{\alpha_m=0\\ \alpha_l=0}} - A\big|_{\substack{\alpha_m=0\\ \alpha_l=0}} .$$

By these lemmas we get

$$\det \eta_D = d_0^2 I_m^2 \prod_{\alpha\in\widetilde{\Delta}} \alpha^2\big|_D + (-1)^{m+l+1} 4(d_0+1) I_l I_m \prod_{\alpha\in\widetilde{\Delta}} \alpha^2\big|_D . \tag{3.90}$$

Let us first consider the case where $d>0$. Then $|\mathcal{R}_+ \cap \langle \alpha_l, \alpha_m\rangle| > 2$ and from Proposition 3.5.16, we know that $I_m = (-1)^{l-m-1} I_l$ on D. Therefore

$$\det \eta_D = (-1)^{l+m+1}(d_0+2)^2 I_m I_l \prod_{\alpha\in\widetilde{\Delta}} \alpha^2\big|_D , \tag{3.91}$$

as required. Let us now suppose that $d=0$. Then f is constant and $d_0 = 0$. Therefore (3.90) implies (3.91) as well. □

Let us reformulate Theorem 3.6.10 in terms of defining polynomials of some arrangements. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \alpha_m, \alpha_l\rangle$. Note that $\mathcal{A}^D_{D_m} = \mathcal{A}^D_{D_l} = \{D\}$.

Theorem 3.6.11. *The statement of Main Theorem 1 is true. Furthermore, the determinant of the metric η_D is proportional to*

$$I(\mathcal{A}_{D_m} \setminus \mathcal{A}^D_{D_m}) I(\mathcal{A}_{D_l} \setminus \mathcal{A}^D_{D_l}), \tag{3.92}$$

on D.

Proof. For any $\widehat{H} \in \mathcal{A}_{D_m}$ let $\alpha_{\widehat{H}} \in \mathcal{R}$ be the corresponding covector such that $\widehat{H} = \{x \in D_m | \alpha_{\widehat{H}}(x) = 0\}$. Similarly for any $\widehat{H} \in \mathcal{A}_{D_l}$ we choose $\alpha_{\widehat{H}} \in \mathcal{R}$ such that $\widehat{H} = \{x \in D_l | \alpha_{\widehat{H}}(x) = 0\}$. We note that from Theorem 3.6.10 and Corollary 3.5.15 $\det \eta_D$ is proportional to

$$\alpha_l^{-1} J_m\big|_D \, \alpha_m^{-1} J_l\big|_D \sim \prod_{\substack{\widehat{H}\in\mathcal{A}_{D_m}\\ \widehat{H}\neq D}} \alpha_{\widehat{H}}\big|_D \prod_{\substack{\widehat{H}\in\mathcal{A}_{D_l}\\ \widehat{H}\neq D}} \alpha_{\widehat{H}}\big|_D \sim \prod_{H\in\mathcal{A}_D} l_H^{k_H}, \tag{3.93}$$

where $l_H = \alpha_{\widehat{H}}\big|_D$ for $\widehat{H} \in \mathcal{A}_{D_m} \cup \mathcal{A}_{D_l}$ such that $H = \widehat{H} \cap D$, $k_H \in \mathbb{N}$. Thus Main Theorem 1 holds. Formula (3.92) follows from (3.93). □

Theorem 3.6.12. *The statement of Main Theorem 2 is true.*

Proof. Let us now fix $H \in \mathcal{A}_D$. We have to show that the multiplicity k_H in Theorem 3.6.11 (formula (3.93)) takes the required form. Let $\beta \in \mathcal{R}$ be such that $H = \{x \in D | \beta(x) = 0\}$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \alpha_m, \alpha_l, \beta \rangle$. Let $\widehat{\mathcal{A}}$ be the corresponding arrangement. Note that the multiplicity k_H is given by

$$k_H = |\widehat{\mathcal{A}}_{D_m} \setminus D| + |\widehat{\mathcal{A}}_{D_l} \setminus D|. \tag{3.94}$$

If the root system $\widehat{\mathcal{R}}$ is irreducible then

$$|\widehat{\mathcal{A}}_{D_m} \setminus D| = |\widehat{\mathcal{A}}_{D_l} \setminus D| = |\widehat{\mathcal{A}}| - h = \frac{3h}{2} - h = \frac{h}{2},$$

where h is the Coxeter number of $\widehat{\mathcal{R}}$, and the statement of Main Theorem 2 follows.

Let us now consider the case where $\widehat{\mathcal{R}}$ is reducible. Suppose firstly that $\mathcal{R}_D$ is an irreducible rank 2 system. Then

$$\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = \mathcal{R}_D \times A_1, \tag{3.95}$$

and

$$|\widehat{\mathcal{A}}_{D_m} \setminus D| = |\widehat{\mathcal{A}}_{D_l} \setminus D| = 1. \tag{3.96}$$

Then $k_H = 2$ equals the Coxeter number of A_1, and the statement holds in this case as well.

Let us now consider the case where $\mathcal{R}_D$ is reducible. Suppose firstly that $\mathcal{R}_{\alpha_l,\beta} = \mathcal{R} \cap \langle \alpha_l, \beta \rangle$ is an irreducible rank 2 system and let $\widetilde{\mathcal{A}}$ be the corresponding arrangement. Note that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = \mathcal{R}_{\alpha_l,\beta} \sqcup \{\pm\alpha_m\} = \mathcal{R}_{\alpha_l,\beta} \times A_1. \tag{3.97}$$

Then $|\widehat{\mathcal{A}}_{D_m} \setminus D| = |\widetilde{\mathcal{A}}| - 1 = h - 1$ and $|\widehat{\mathcal{A}}_{D_l} \setminus D| = |\widetilde{\mathcal{A}}_{D_l}| = 1$, where h is the Coxeter number of $\mathcal{R}_{\alpha_l,\beta}$, and the statement of Main Theorem 2 follows. The case where $\mathcal{R} \cap \langle \alpha_m, \beta \rangle$ is irreducible is similar. The final case to consider is when $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = \{\pm\alpha_m\} \sqcup \{\pm\alpha_l\} \sqcup \{\pm\beta\} = A_1 \times A_1 \times A_1.$$

Then equalities (3.96) hold, and $k_H = 2$ as required. □

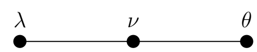
The above analysis shows that the statement of Main Theorems 1 and 2 for the the determinant of the restricted Saito metric in codimensions 1, 2 and $n-1$ is true. This covers all strata in finite Coxeter groups $I_2(p), H_3, H_4, F_4$. This leaves us to study simply laced cases E_6, E_7, E_8 only. The analysis becomes more involved and it will depend on the parabolic subgroups which we consider.

3.6.4 Codimension 3

We consider $(n-3)$-dimensional strata D for simply laced Coxeter groups. Thus we obtain factorisation formulae for the determinant of the metric η_D for strata D of type A_3, $A_2 \times A_1$ and A_1^3.

Let $\mathcal{R}_+$ be the positive root system of the root systems E_n, $n = 6, 7, 8$, although the presented analysis below works for any irreducible simply laced root system. Let λ, ν, θ be simple roots and consider the corresponding stratum $D = D_{\lambda,\nu,\theta}$ of codimension 3.

Stratum $\mathbf{A_3}$. Let us assume that $\mathcal{R}_D = \mathcal{R} \cap \langle \lambda, \nu, \theta \rangle$ is a subsystem of $\mathcal{R}$ of type A_3 and consider the corresponding Coxeter graph

A_3 : 

Note that $\lambda + \nu$, $\nu + \theta$, $\lambda + \nu + \theta \in \mathcal{R}_+$. The Jacobian J can be represented as

$$J = \lambda\nu\theta(\lambda + \nu)(\nu + \theta)(\lambda + \nu + \theta)\Pi, \tag{3.98}$$

where Π is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$ and Π is non-zero on D. By Proposition 3.5.10 we have in the notation of (3.55)

$$J_\lambda = \nu\theta(\nu + \theta)K_\lambda, \tag{3.99}$$

$$J_\nu = \lambda\theta K_\nu, \tag{3.100}$$

$$J_\theta = \lambda\nu(\lambda + \nu)K_\theta, \tag{3.101}$$

for some polynomials $K_\lambda, K_\theta, K_\nu \in \mathbb{C}[x]$.

We assume without loss of generality that the ordering of simple roots $\sigma : \{1, \dots, n\} \to \Delta$ is such that $n + \sigma^{-1}(\lambda)$ is odd, and that $\sigma^{-1}(\nu) = \sigma^{-1}(\lambda) + 1$, $\sigma^{-1}(\theta) = \sigma^{-1}(\lambda) + 2$. The following statement follows from Proposition 3.5.3 and formulae (3.98)-(3.101).

Proposition 3.6.13. *The λ, ν, and θ components of the identity field e are given by*

$$e^\lambda = -\frac{K_\lambda}{\lambda(\lambda + \nu)(\lambda + \nu + \theta)\Pi}, \tag{3.102}$$

$$e^\nu = \frac{K_\nu}{\nu(\lambda + \nu)(\nu + \theta)(\lambda + \nu + \theta)\Pi}, \tag{3.103}$$

$$e^\theta = -\frac{K_\theta}{\theta(\nu + \theta)(\lambda + \nu + \theta)\Pi}. \tag{3.104}$$

In what follows, we deal with the restricted metric η_D by restricting η on D_ν firstly, then on $D_{\nu,\theta}$ and finally on D. Firstly, we derive relations between $K_\lambda, K_\theta, K_\nu$.

Lemma 3.6.14. *We have*

$$K_\nu|_{D_\nu} = \lambda K_\theta + \theta B|_{D_\nu}\,, \tag{3.105}$$

for some polynomial $B \in \mathbb{C}[x]$ such that

$$B|_D = K_\theta|_D\,. \tag{3.106}$$

Proof. By Proposition 3.5.16, $\frac{J_\nu}{\theta} = \frac{J_\theta}{\nu}$ on $D_{\nu,\theta}$. Then by (3.100), (3.101), $K_\nu = \lambda K_\theta$ on $D_{\nu,\theta}$. Consider $K_\nu - \lambda K_\theta$ on the hyperplane D_ν. This polynomial vanishes if $\theta = 0$. Therefore we can represent K_ν on D_ν as

$$K_\nu|_{D_\nu} = \lambda K_\theta + \theta B|_{D_\nu}\,, \tag{3.107}$$

for some $B \in \mathbb{C}[x]$ as required.

Furthermore, we note that K_ν is divisible by $(\lambda+\theta)$ on D_ν since by Corollary 3.5.15 $J_\nu|_{D_\nu}$ is divisible by $\lambda+\nu+\theta|_{D_\nu}$. Hence,

$$K_\nu|_{D_\nu} = \lambda K_\theta + \theta B|_{D_\nu} = (\lambda+\theta)P|_{D_\nu}\,, \tag{3.108}$$

for some $P \in \mathbb{C}[x]$. Moreover by restricting equality (3.108) further on $D_{\nu,\lambda}$, we get that

$$B|_{D_{\nu,\lambda}} = P|_{D_{\nu,\lambda}}\,. \tag{3.109}$$

Similarly, restricting equality (3.108) further on $D_{\nu,\theta}$, we get that $P|_{D_{\nu,\theta}} = K_\theta|_{D_{\nu,\theta}}$. It follows from equality (3.109) that

$$B|_D = K_\theta|_D\,,$$

as required. □

Lemma 3.6.15. *We have*

$$K_\lambda|_D = K_\theta|_D\,. \tag{3.110}$$

Proof. By Proposition 3.5.16, we have $\frac{J_\nu}{\lambda} = \frac{J_\lambda}{\nu}$ on $D_{\nu,\lambda}$ and hence $K_\nu = \theta K_\lambda$ on $D_{\nu,\lambda}$. It follows from equality (3.105) that $K_\nu|_{D_{\nu,\lambda}} = \theta B|_{D_{\nu,\lambda}}$, hence $K_\lambda|_{D_{\nu,\lambda}} = B|_{D_{\nu,\lambda}}$. The statement now follows from formula (3.106). □

Theorem 3.5.9 gives a general formula for the determinant of the Saito metric η_D which we now specialize to the case of codimension 3 stratum. Let us represent J, given by formula (3.98) as $J = \lambda\nu\theta\bar{J}$, where

$$\bar{J} = (\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)\Pi. \tag{3.111}$$

We will write components of Saito metric $\eta^{\sigma^{-1}(\alpha)\sigma^{-1}(\beta)}$ as $\eta^{\alpha\beta}$, $\alpha, \beta \in \Delta$. We rearrange $\det \eta_D$ as

$$\det \eta_D = - \begin{vmatrix} \eta^{\lambda\lambda} & \eta^{\lambda\nu} & \eta^{\lambda\theta} \\ \eta^{\lambda\nu} & \eta^{\nu\nu} & \eta^{\nu\theta} \\ \eta^{\lambda\theta} & \eta^{\nu\theta} & \eta^{\theta\theta} \end{vmatrix} J^2\Big|_D = - \begin{vmatrix} \lambda^2\eta^{\lambda\lambda} & \lambda\nu\eta^{\lambda\nu} & \lambda\theta\eta^{\lambda\theta} \\ \lambda\nu\eta^{\lambda\nu} & \nu^2\eta^{\nu\nu} & \nu\theta\eta^{\nu\theta} \\ \lambda\theta\eta^{\lambda\theta} & \nu\theta\eta^{\nu\theta} & \theta^2\eta^{\theta\theta} \end{vmatrix} \bar{J}^2\Big|_D. \tag{3.112}$$

Let $A = (a_{ij})_{i,j=1}^3$ be the matrix

$$A = \begin{pmatrix} \lambda^2\eta^{\lambda\lambda} & \lambda\nu\eta^{\lambda\nu} & \lambda\theta\eta^{\lambda\theta} \\ \lambda\nu\eta^{\lambda\nu} & \nu^2\eta^{\nu\nu} & \nu\theta\eta^{\nu\theta} \\ \lambda\theta\eta^{\lambda\theta} & \nu\theta\eta^{\nu\theta} & \theta^2\eta^{\theta\theta} \end{pmatrix}. \tag{3.113}$$

Let us recall the basis of fundamental coweights ω^i $(i = 1, \dots, n)$, we will also write ω^λ for $\omega^{\sigma^{-1}(\lambda)}$, $\lambda \in \Delta$.

Proposition 3.6.16. *The matrix entries a_{ij} $(1 \le i, j \le 3)$ are well-defined generically on D_ν, and they have the following form on D_ν:*

$$a_{11} = \lambda^2\eta^{\lambda\lambda} = 2\lambda^2\partial_{\omega^\lambda}\Big(\frac{K_\lambda}{\lambda^2(\lambda+\theta)\Pi}\Big), \tag{3.114}$$

$$a_{22} = \nu^2\eta^{\nu\nu} = \frac{2K_\nu}{\lambda\theta(\lambda+\theta)\Pi}, \tag{3.115}$$

$$a_{33} = \theta^2\eta^{\theta\theta} = 2\theta^2\partial_{\omega^\theta}\Big(\frac{K_\theta}{\theta^2(\lambda+\theta)\Pi}\Big), \tag{3.116}$$

$$a_{12} = \lambda\nu\eta^{\lambda\nu} = -\frac{\lambda}{\theta}\partial_{\omega^\lambda}\Big(\frac{K_\nu}{\lambda(\lambda+\theta)\Pi}\Big), \tag{3.117}$$

$$a_{13} = \lambda\theta\eta^{\lambda\theta} = \frac{\lambda}{\theta}\partial_{\omega^\lambda}\Big(\frac{K_\theta}{(\lambda+\theta)\Pi}\Big) + \frac{\theta}{\lambda}\partial_{\omega^\theta}\Big(\frac{K_\lambda}{(\lambda+\theta)\Pi}\Big), \tag{3.118}$$

$$a_{23} = \nu\theta\eta^{\nu\theta} = -\frac{\theta}{\lambda}\partial_{\omega^\theta}\Big(\frac{K_\nu}{\theta(\lambda+\theta)\Pi}\Big). \tag{3.119}$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\lambda, \mu, \nu\}$. Formulae (3.114), (3.116), (3.118) follow from Proposition 3.6.13 immediately. Let us prove formula (3.115). We have

$$\nu^2\eta^{\nu\nu} = -2\nu^2\partial_{\omega^\nu}\Big(\frac{K_\nu}{\nu(\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)\Pi}\Big).$$

By Leibniz rule and taking the limit $\nu(x) \to 0$ we obtain the formula. Formulae (3.117), (3.119) follow similarly. □

By Proposition 3.6.16 we see that the entries of A may be singular on $D_{\nu,\theta}$. Therefore, in order to restrict $\bar{J}^2 \det A$ on $D_{\nu,\theta}$ we consider the expansion of $\det A$ and collect the terms with the same order of poles at $\theta = 0$. Let $\det A$ be

$$\det A = C + E, \tag{3.120}$$

where

$$C = -a_{12}^2 a_{33} + 2a_{12}a_{23}a_{13} - a_{13}^2 a_{22},$$

and

$$E = a_{11}\big(a_{22}a_{33} - a_{23}^2\big).$$

Note that E has a pole at $\theta = 0$ of order at most 2. Now we study the term C near $\theta = 0$.

Lemma 3.6.17. *We have*

$$C = \frac{1}{\theta^3}C_1 + \frac{1}{\theta^2}C_2, \tag{3.121}$$

where C_1, C_2 are well-defined generically on $D_{\nu,\theta}$ and have the following form on D_ν:

$$\begin{aligned}
C_1 = (\lambda+\theta)^{-3}\Bigg(&\frac{4K_\theta}{\Pi}\Big(\frac{K_\nu}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big) - \partial_{\omega^\lambda}\frac{K_\nu}{\Pi}\Big)^2 \\
&- \frac{2K_\nu}{\Pi}\Big(\frac{K_\nu}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big) - \partial_{\omega^\lambda}\frac{K_\nu}{\Pi}\Big)\times \\
&\times\Big(\frac{K_\theta}{\Pi}(\lambda+\theta)^{-1} - \partial_{\omega^\lambda}\frac{K_\theta}{\Pi}\Big) - \frac{2\lambda K_\nu}{\Pi}\Big(\frac{K_\theta}{\Pi}(\lambda+\theta)^{-1} - \partial_{\omega^\lambda}\frac{K_\theta}{\Pi}\Big)^2\Bigg),
\end{aligned}$$

$$\begin{aligned}
C_2 = (\lambda+\theta)^{-3}\Bigg(&2\Big(\partial_{\omega^\lambda}\frac{K_\nu}{\Pi} - \frac{K_\nu}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big)\Big)^2\Big(\frac{K_\theta}{\Pi}(\lambda+\theta)^{-1} - \partial_{\omega^\theta}\frac{K_\theta}{\Pi}\Big) \\
&+ 2\Big(\partial_{\omega^\theta}\frac{K_\nu}{\Pi} - \frac{K_\nu}{\Pi}(\lambda+\theta)^{-1}\Big)\Big(\partial_{\omega^\lambda}\frac{K_\nu}{\Pi} - \frac{K_\nu}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big)\Big)\times \\
&\times\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}(\lambda+\theta)^{-1}\Big) + 2\theta^2\lambda^{-2}\Big(\partial_{\omega^\theta}\frac{K_\lambda}{\Pi} - \frac{K_\lambda}{\Pi}(\lambda+\theta)^{-1}\Big)\times \\
&\times\Big(\frac{K_\nu}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big) - \partial_{\omega^\lambda}\frac{K_\nu}{\Pi}\Big)\Big(\frac{K_\nu}{\Pi}\Big(\theta^{-1} + (\lambda+\theta)^{-1}\Big) - \partial_{\omega^\theta}\frac{K_\nu}{\Pi}\Big) - \\
&- 2\theta\lambda^{-1}\frac{K_\nu}{\Pi}\Bigg((\theta\lambda^{-1})^2\Big(\partial_{\omega^\theta}\frac{K_\lambda}{\Pi} - \frac{K_\lambda}{\Pi}(\lambda+\theta)^{-1}\Big)^2 + \\
&+ 2\Big(\partial_{\omega^\theta}\frac{K_\lambda}{\Pi} - \frac{K_\lambda}{\Pi}(\lambda+\theta)^{-1}\Big)\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}(\lambda+\theta)^{-1}\Big)\Bigg)\Bigg).
\end{aligned} \tag{3.122}$$

Proof. We expand formulae (3.116), (3.119) as

$$\begin{aligned}
a_{33} &= 2\partial_{\omega^\theta}\Big(\frac{K_\theta}{(\lambda+\theta)\Pi}\Big) - \frac{4K_\theta}{\theta(\lambda+\theta)\Pi}, \\
a_{23} &= -\frac{1}{\lambda}\partial_{\omega^\theta}\Big(\frac{K_\nu}{(\lambda+\theta)\Pi}\Big) + \frac{K_\nu}{\lambda\theta(\lambda+\theta)\Pi}.
\end{aligned}$$

Then expressions for C_1, C_2 follow by Proposition 3.6.16 using Leibniz rule and by collecting terms with the same degree of the pole at $\theta = 0$. □

By Lemma 3.6.14 we have

$$K_\nu|_{D_\nu} = \lambda K_\theta + \theta B|_{D_\nu}, \tag{3.123}$$

and hence we can represent K_ν as

$$K_\nu = \lambda K_\theta + \theta B + \nu Q, \tag{3.124}$$

for some polynomial $Q \in \mathbb{C}[x]$. Therefore we get

$$\partial_{\omega^\lambda} \frac{K_\nu}{\Pi}\Big|_{D_\nu} = \partial_{\omega^\lambda} \frac{\lambda K_\theta + \theta B}{\Pi}\Big|_{D_\nu}, \tag{3.125}$$

since for $\lambda, \mu \in \Delta$ we have

$$(\omega^\lambda, \mu) = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu. \end{cases}$$

Lemma 3.6.18. *The expression $C_1|_{D_\nu}$ is divisible by θ, that is we can represent it as*

$$C_1 = (\widetilde{C}_1 + F\theta)\theta\Big|_{D_\nu},$$

where $\widetilde{C}_1$, F are well-defined generically on $D_{\nu,\theta}$ and have the following form on D_ν:

$$\widetilde{C}_1 = (\lambda+\theta)^{-3}\Bigg(-4\lambda \frac{B}{\Pi}\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}(\lambda+\theta)^{-1}\Big)^2 + 6\lambda\frac{K_\theta}{\Pi}\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}(\lambda+\theta)^{-1}\Big)\times \\ \times\Big(\widehat{B} - \frac{B}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big)\Big)\Bigg), \tag{3.126}$$

and

$$F = (\lambda+\theta)^{-3}\Bigg(4\Big(\frac{K_\theta}{\Pi}\Big)\Big(-\widehat{B} + \frac{B}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big)\Big)^2 - \\ - \frac{2B}{\Pi}\Big(-\widehat{B} + \frac{B}{\Pi}\Big(\lambda^{-1} + (\lambda+\theta)^{-1}\Big)\Big)\Big(-\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} + \frac{K_\theta}{\Pi}(\lambda+\theta)^{-1}\Big)\Bigg),$$

where $\widehat{B} = \partial_{\omega^\lambda}\frac{B}{\Pi}$.

Proof. Note that

$$\frac{\lambda K_\theta}{\Pi}\big(\lambda^{-1} + (\lambda+\theta)^{-1}\big) - \partial_{\omega^\lambda}\frac{\lambda K_\theta}{\Pi} = \lambda\Big(\frac{K_\theta}{\Pi}(\lambda+\theta)^{-1} - \partial_{\omega^\lambda}\frac{K_\theta}{\Pi}\Big). \tag{3.127}$$

The statement follows by substituting formulae (3.123), (3.125) into C_1, collecting equal powers of θ and making use of (3.127). $\square$

Lemma 3.6.19. *We have*

$$\widetilde{C}_1\Big|_{D_{\nu,\theta}} = 2\lambda^{-4}\left(4B\frac{K_\theta^2}{\Pi^3} - 2B\lambda\frac{K_\theta}{\Pi^2}\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - 3\widehat{B}\lambda\frac{K_\theta^2}{\Pi^2} - 2\frac{B\lambda^2}{\Pi}(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi})^2 + 3\widehat{B}\lambda^2\frac{K_\theta}{\Pi}\partial_{\omega^\lambda}\frac{K_\theta}{\Pi}\right)\Bigg|_{D_{\nu,\theta}}.$$

Proof. The statement follows immediately from the restriction of formula (3.126) to the stratum $D_{\nu,\theta}$. $\square$

Let us now consider the term C_2 in equality (3.121). The restriction of C_2 to $D_{\nu,\theta}$ is given in the following lemma.

Lemma 3.6.20. *We have*

$$C_2|_{D_{\nu,\theta}} = 2\lambda^{-2}\frac{B}{\Pi}\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}\lambda^{-1}\Big)^2\Bigg|_{D_{\nu,\theta}}.$$

Proof. Restricting formula (3.122) to $D_{\nu,\theta}$ we get

$$C_2|_{D_{\nu,\theta}} = 2\lambda^{-3}\Bigg(\Big(\partial_{\omega^\lambda}\frac{K_\nu}{\Pi} - \frac{2K_\nu}{\lambda\Pi}\Big)^2\Big(-\partial_{\omega^\theta}\frac{K_\theta}{\Pi} + \frac{K_\theta}{\lambda\Pi}\Big) + \Big(\partial_{\omega^\theta}\frac{K_\nu}{\Pi} - \frac{K_\nu}{\lambda\Pi}\Big)\Big(\partial_{\omega^\lambda}\frac{K_\nu}{\Pi} - \frac{2K_\nu}{\lambda\Pi}\Big)\times$$
$$\times\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\lambda\Pi}\Big)\Bigg)\Bigg|_{D_{\nu,\theta}}.$$

It follows from (3.123) that

$$K_\nu|_{D_{\nu,\theta}} = \lambda K_\theta|_{D_{\nu,\theta}}, \tag{3.128}$$

and it follows from (3.125) that

$$\partial_{\omega^\lambda}\frac{K_\nu}{\Pi}\Bigg|_{D_{\nu,\theta}} = \partial_{\omega^\lambda}\frac{\lambda K_\theta}{\Pi}\Bigg|_{D_{\nu,\theta}}. \tag{3.129}$$

We also have from (3.124) that

$$\partial_{\omega^\theta}\frac{K_\nu}{\Pi}\Bigg|_{D_{\nu,\theta}} = \lambda\partial_{\omega^\theta}\frac{K_\theta}{\Pi} + \frac{B}{\Pi}\Bigg|_{D_{\nu,\theta}}. \tag{3.130}$$

By using (3.128), (3.129) we get

$$\partial_{\omega^\lambda}\frac{K_\nu}{\Pi} - \frac{2K_\nu}{\lambda\Pi}\Bigg|_{D_{\nu,\theta}} = \lambda\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}\Bigg|_{D_{\nu,\theta}}.$$

Hence,

$$C_2|_{D_{\nu,\theta}} = 2\lambda^{-3}\big(\lambda\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}\big)^2\Big(- \partial_{\omega^\theta}\frac{K_\theta}{\Pi} + \frac{K_\theta}{\lambda\Pi} + \lambda^{-1}(\partial_{\omega^\theta}\frac{K_\nu}{\Pi} - \frac{K_\nu}{\lambda\Pi})\Big)\Big|_{D_{\nu,\theta}}. \tag{3.131}$$

The statement follows from the formula (3.131) after substituting expressions (3.128) and (3.130). □

Lemma 3.6.21. *Let $z = \theta^2 C$. Then*

$$z|_{D_{\nu,\theta}} = 2\lambda^{-4}\Bigg(4B\frac{K_\theta^2}{\Pi^3} - 2B\lambda\frac{K_\theta}{\Pi^2}(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi}) - 3\widehat{B}\lambda\frac{K_\theta^2}{\Pi^2} - 2\frac{B\lambda^2}{\Pi}(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi})^2 + 3\widehat{B}\lambda^2\frac{K_\theta}{\Pi}\partial_{\omega^\lambda}\frac{K_\theta}{\Pi}\Bigg)\Bigg|_{D_{\nu,\theta}}$$
$$+ 2\lambda^{-2}\frac{B}{\Pi}\Big(\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} - \frac{K_\theta}{\Pi}\lambda^{-1}\Big)^2\Big|_{D_{\nu,\theta}}, \tag{3.132}$$

where $\widehat{B} = \partial_{\omega^\lambda}\frac{B}{\Pi}$. Further to that,

$$\lambda^4 z\big|_D = 10\,\Big(\frac{K_\theta}{\Pi}\Big)^3\Big|_D. \tag{3.133}$$

Proof. By Lemmas 3.6.19, 3.6.20 we have

$$z|_{D_{\nu,\theta}} = (\frac{1}{\theta}C_1 + C_2)\Big|_{D_{\nu,\theta}} = (\widetilde{C}_1 + C_2)\Big|_{D_{\nu,\theta}},$$

which implies (3.132). Therefore

$$\lambda^4 z\big|_D = 10B\frac{K_\theta^2}{\Pi^3} = 10\big(\frac{K_\theta}{\Pi}\big)^3\Big|_D,$$

since $B\big|_D = K_\theta\big|_D$, by Lemma 3.6.14. □

Finally, we consider the term E in $\det A$. Note that $\theta^2 E$ is well-defined generically at $\theta = 0$. Furthermore, we obtain the following result.

Lemma 3.6.22. *We have*

$$\theta^2 E\big|_{D_{\nu,\theta}} = -\frac{18K_\theta^2}{\Pi^2}\partial_{\omega^\lambda}\Big(\frac{K_\lambda}{\Pi}\lambda^{-3}\Big)\Big|_{D_{\nu,\theta}}. \tag{3.134}$$

and furthermore,

$$\lambda^4\theta^2 E\big|_D = 54\big(\frac{K_\theta}{\Pi}\big)^3\Big|_D, \tag{3.135}$$

where we take restrictions on $\nu = 0$ at first, then on $\theta = 0$ and then on $\lambda = 0$.

Proof. By Proposition 3.6.16 we have

$$\theta^2 E\big|_{D_{\nu,\theta}} = 2\lambda^2\partial_{\omega^\lambda}\Big(\frac{K_\lambda}{\Pi}\lambda^{-3}\Big)\Big(- 8\frac{K_\nu K_\theta}{\Pi^2}\lambda^{-3} - \big(\frac{K_\nu}{\Pi}\big)^2\lambda^{-4}\Big)\Big|_{D_{\nu,\theta}},$$

which implies (3.134) since $K_\nu|_{D_{\nu,\theta}} = \lambda K_\theta|_{D_{\nu,\theta}}$ by Lemma 3.6.14. Therefore

$$\lambda^4\theta^2 E\big|_D = 54\frac{K_\theta^2 K_\lambda}{\Pi^3} = 54\big(\frac{K_\theta}{\Pi}\big)^3\Big|_D,$$

by Lemma 3.6.15. □

Using the above we have the following result.

Theorem 3.6.23. *The determinant of the metric η_D is proportional to $\Pi^{-1}K_\theta^3$ on D.*

Proof. We have $\det\eta_D = -\bar{J}^2\det A\big|_D = -\bar{J}^2(C+E)\big|_D$. Note that $\theta^2 C$ and $\theta^2 E$ are well-defined generically on $D_{\nu,\theta}$, and $\bar{J}\big|_{D_\nu} = \lambda\theta(\lambda+\theta)\Pi|_{D_\nu}$. Hence we have

$$\bar{J}^2 E\big|_{D_{\nu,\theta}} = \lambda^2\theta^2(\lambda+\theta)^2 E\Pi^2\big|_{D_{\nu,\theta}} = \lambda^4(\theta^2 E)\Pi^2\big|_{D_{\nu,\theta}}.$$

By Lemma 3.6.22 it follows that

$$\bar{J}^2 E\big|_D = 54\frac{K_\theta^3}{\Pi}\Big|_D.$$

Similarly we have

$$\bar{J}^2 C\big|_{D_{\nu,\theta}} = \lambda^4(\theta^2 C)\Pi^2\big|_{D_{\nu,\theta}}.$$

By Lemma 3.6.21 we get

$$\bar{J}^2 C\big|_D = 10\frac{K_\theta^3}{\Pi}\Big|_D,$$

and the statement follows. □

For any $H\in\mathcal{A}$ let $\alpha_H\in\mathcal{R}$ be such that $H=\{x\in V|\alpha_H(x)=0\}$. Similarly for any $H\in\mathcal{A}_{D_\theta}$ we choose $\alpha_H\in\mathcal{R}$ such that $H=\{x\in D_\theta|\alpha_H(x)=0\}$. It follows from Corollary 3.5.15 and formula (3.101) that

$$K_\theta|_D \sim \prod_{\substack{H\in\mathcal{A}_{D_\theta}\\ H\notin\mathcal{A}^D_{D_\theta}}}\alpha_H|_D \sim I(\mathcal{A}_{D_\theta}\setminus\mathcal{A}^D_{D_\theta})\big|_D. \tag{3.136}$$

Moreover we have that

$$\Pi|_D \sim \prod_{\substack{H\in\mathcal{A}\\ H\notin\mathcal{A}^D}}\alpha_H|_D \sim I(\mathcal{A}\setminus\mathcal{A}^D)\big|_D. \tag{3.137}$$

The above considerations produce the following reformulation of Theorem 3.6.23.

Theorem 3.6.24. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})^3 I(\mathcal{A} \setminus \mathcal{A}^D)^{-1} \tag{3.138}$$

on D.

Now we have to show that powers of distinct linear factors in (3.138) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.25. *The statement of Main Theorems 1 and 2 is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. Note that the root system $\widehat{\mathcal{R}}$ is a rank 4 subsystem of $\mathcal{R}$. We have from formulae (3.136), (3.137) that the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})\big|_D$ is

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 3, \tag{3.139}$$

and the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 6. \tag{3.140}$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{\mathcal{R}} = A_4$ or $\widehat{\mathcal{R}} = D_4$. Then by Proposition 3.5.14 we have $|\widehat{\mathcal{A}}_{D_\theta}| = |\widehat{\mathcal{A}}| - h + 1$, where h is the Coxeter number of $\widehat{\mathcal{R}}$. Hence formula (3.139) implies that

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}| - h - 2 = h - 2,$$

and formula (3.140) implies

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = 2h - 6.$$

Then it follows from Theorem 3.6.23 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now suppose that $\widehat{\mathcal{R}}$ is reducible, that is $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_3 \times A_1$. Then we get from formulae (3.139), (3.140) that

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 3 = 1,$$

and

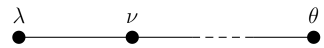
$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 6 = 1,$$

Then it follows from Theorem 3.6.23 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. □

Let us now consider the cases where the root system $\mathcal{R}_D = \mathcal{R} \cap \langle \lambda, \nu, \theta \rangle$ is reducible, that is $\mathcal{R}_D = A_2 \times A_1$ or $\mathcal{R}_D = A_1^3$.

Stratum $\mathbf{A_2} \times \mathbf{A_1}$. Let us assume that $\mathcal{R}_D$ is a root subsystem of $\mathcal{R}$ of type $A_2 \times A_1$ and consider the corresponding Coxeter graph

$A_2 \times A_1$:



Note that $\lambda + \nu \in \mathcal{R}_+$. The Jacobian can be represented as

$$J = \lambda\nu\theta(\lambda+\nu)\Pi, \tag{3.141}$$

where Π is is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$. Note that Π is non-zero on D. By Proposition 3.5.10, we have

$$J_\lambda = \nu\theta K_\lambda, \tag{3.142}$$

$$J_\nu = \lambda\theta K_\nu, \tag{3.143}$$

$$J_\theta = \lambda\nu(\lambda+\nu)K_\theta \tag{3.144}$$

for some polynomials $K_\lambda, K_\nu, K_\theta \in \mathbb{C}[x]$. We assume without loss of generality that $n + \sigma^{-1}(\lambda)$ is even, $\sigma^{-1}(\nu) = \sigma^{-1}(\lambda) + 1$ and $\sigma^{-1}(\theta) - \sigma^{-1}(\lambda)$ is even. This leads to the following expressions of components of the identity field e by Proposition 3.5.3.

Proposition 3.6.26. *The λ, ν and θ components of the identity field e are given by*

$$e^\lambda = \frac{K_\lambda}{\lambda(\lambda+\nu)\Pi}, \quad e^\nu = -\frac{K_\nu}{\nu(\lambda+\nu)\Pi}, \quad \text{and} \quad e^\theta = \frac{K_\theta}{\theta\Pi}. \tag{3.145}$$

Let us introduce $\bar{J} = (\lambda+\nu)\Pi$ so that $J = \lambda\nu\theta\bar{J}$. Recall that in these notations $\det \eta_D$ is given by formula (3.112). The entries of the matrix $A = (a_{ij})_{i,j=1}^3$ defined in (3.113) are given as follows.

Proposition 3.6.27. *All the matrix entries a_{ij} are well-defined generically on $D_{\lambda,\theta}$. They have the following form on $D_{\lambda,\theta}$:*

$$a_{11} = \frac{2K_\lambda}{\nu\Pi}, \quad a_{22} = 2\partial_{\omega^\nu}\frac{K_\nu}{\Pi} - \frac{4K_\nu}{\nu\Pi}, \quad a_{33} = \frac{2K_\theta}{\Pi},$$

$$a_{12} = \frac{K_\lambda}{\nu\Pi} - \partial_{\omega^\nu}\frac{K_\lambda}{\Pi}, \quad a_{23} = -\nu\partial_{\omega^\nu}\frac{K_\theta}{\Pi}, \quad a_{13} = 0.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\lambda, \nu, \theta\}$. Therefore by Proposition 3.6.26 the statement follows. $\square$

For any $H \in \mathcal{A}$ let $\alpha_H \in \mathcal{R}$ be such that $H = \{x \in V | \alpha_H(x) = 0\}$. Similarly for any

$H \in \mathcal{A}_{D_\gamma}$, $\gamma \in \{\theta, \nu\}$ we choose $\alpha_H \in \mathcal{R}$ such that $H = \{x \in D_\gamma | \alpha_H(x) = 0\}$. It follows from Corollary 3.5.15 and formulae (3.143), (3.144) that

$$K_\nu\big|_D \sim \prod_{\substack{H \in \mathcal{A}_{D_\nu} \\ H \notin \mathcal{A}^D_{D_\nu}}} \alpha_H\big|_D \sim I(\mathcal{A}_{D_\nu} \setminus \mathcal{A}^D_{D_\nu})\big|_D, \tag{3.146}$$

and

$$K_\theta\big|_D \sim \prod_{\substack{H \in \mathcal{A}_{D_\theta} \\ H \notin \mathcal{A}^D_{D_\theta}}} \alpha_H\big|_D \sim I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})\big|_D \tag{3.147}$$

Moreover we have that

$$\Pi\big|_D \sim \prod_{\substack{H \in \mathcal{A} \\ H \notin \mathcal{A}^D}} \alpha_H\big|_D \sim I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D. \tag{3.148}$$

We obtain the following statement on $\det \eta_D$.

Theorem 3.6.28. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}_{D_\nu} \setminus \mathcal{A}^D_{D_\nu})^2 I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}) I(\mathcal{A} \setminus \mathcal{A}^D)^{-1} \tag{3.149}$$

on D. The same is true with ν replaced with λ in (3.149).

Proof. By formula (3.112) we have $\det \eta_D = -\bar{J}^2 \det A\big|_D$, where A is given by (3.113). Therefore by Proposition 3.6.27

$$\det \eta_D = \big((a_{12}^2 - a_{11}a_{22})a_{33} + a_{11}a_{23}^2\big)(\lambda+\nu)^2\Pi^2\big|_D = 16\frac{K_\lambda K_\nu K_\theta}{\Pi}\bigg|_D + 2\frac{K_\lambda^2 K_\theta}{\Pi}\bigg|_D.$$

By Proposition 3.5.16, we have $\frac{J_\lambda}{\nu} = \frac{J_\nu}{\lambda}$ on $D_{\lambda,\nu}$ and hence $K_\nu\big|_{D_{\lambda,\nu}} = K_\lambda\big|_{D_{\lambda,\nu}}$. Therefore $\det \eta_D$ is proportional to $\Pi^{-1}K_\nu^2 K_\theta$ on D. The statement follows by formulae (3.146), (3.147) and (3.148). □

Let us now show that powers of distinct linear factors in (3.149) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.29. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the rank 4 root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. Let h be the Coxeter number of $\widehat{\mathcal{R}}$. We have that the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\nu} \setminus \mathcal{A}^D_{D_\nu})\big|_D$ is

$$|\widehat{\mathcal{A}}_{D_\nu} \setminus \mathcal{A}^D_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\nu}| - 2, \tag{3.150}$$

since $\mathcal{A}^D_{D_\nu} = \{D_{\lambda,\nu}, D_{\theta,\nu}\}$, and the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})\big|_D$

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 3, \tag{3.151}$$

since $\mathcal{A}^D_{D_\theta} = \{D_{\lambda,\theta}, D_{\nu,\theta}, D_{\lambda+\nu,\theta}\}$. Similarly, the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 4. \tag{3.152}$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{\mathcal{R}} = A_4$ or $\widehat{\mathcal{R}} = D_4$. Then $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = |\widehat{\mathcal{A}}| - h + 1$ [71]. Hence formula (3.150) implies that

$$|\widehat{\mathcal{A}}_{D_\nu} \setminus \mathcal{A}^D_{D_\nu}| = |\widehat{\mathcal{A}}| - h - 1 = h - 1,$$

and formula (3.151) implies that

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}| - h - 2 = h - 2.$$

Similarly formula (3.152) implies that

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = 2h - 4.$$

Then it follows from Theorem 3.6.28 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now suppose that $\widehat{\mathcal{R}}$ is reducible. If $\widehat{\mathcal{R}} = A_2 \times A_2$ then we get that $\widehat{\mathcal{A}}_{D_\nu} = \{D_{\lambda,\nu}, D_{\theta,\nu}, D_{\theta+\epsilon\beta,\nu}, D_{\beta,\nu}\}$, where either $\epsilon = 1$ or $\epsilon = -1$, $\widehat{\mathcal{A}}_{D_\theta} = \{D_{\lambda,\theta}, D_{\nu,\theta}, D_{\lambda+\nu,\theta}, D_{\beta,\theta}\}$ and $|\widehat{\mathcal{A}}| = 6$. Therefore, $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = 4$. It follows by formulae (3.150), (3.151), (3.152) and Theorem 3.6.28 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 3, which is the Coxeter number of A_2, as required. Let us now consider the case where $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \beta, \lambda, \nu \rangle) \sqcup \{\pm\theta\} = A_3 \times A_1,$$

and let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to A_3. Hence $|\widehat{\mathcal{A}}_{D_\nu}| = |\widetilde{\mathcal{A}}_{D_\nu}| + 1 = 4$, $|\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}| = 6$ and $|\widehat{\mathcal{A}}| = 7$. It follows by formulae (3.150), (3.151), (3.152) and Theorem 3.6.28 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required. Finally, let us consider the case where

$$\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_2 \times A_1 \times A_1.$$

Then we have that $\widehat{\mathcal{A}}_{D_\nu} = \{D_{\lambda,\nu}, D_{\theta,\nu}, D_{\beta,\nu}\}$, $\widehat{\mathcal{A}}_{D_\theta} = \{D_{\lambda,\theta}, D_{\nu,\theta}, D_{\lambda+\nu,\theta}, D_{\beta,\theta}\}$ and $|\widehat{\mathcal{A}}| = 5$. Then it follows by formulae (3.150), (3.151), (3.152) and Theorem 3.6.28 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. The statement follows. $\square$

Stratum $\mathbf{A_1^3}$. Let us assume that $\mathcal{R}_D$ is a root subsystem of $\mathcal{R}$ of type A_1^3 and consider the corresponding Coxeter graph

$$A_1^3: \quad \overset{\lambda}{\bullet} \text{———} - - - \text{———} \overset{\nu}{\bullet} \text{———} - - - \text{———} \overset{\theta}{\bullet}$$

The Jacobian J can be represented as

$$J = \lambda\nu\theta\Pi, \tag{3.153}$$

where Π is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$ and it is non-zero on D. By Proposition 3.5.10 we have

$$J_\lambda = \nu\theta K_\lambda, \tag{3.154}$$
$$J_\nu = \lambda\theta K_\nu, \tag{3.155}$$
$$J_\theta = \lambda\nu K_\theta, \tag{3.156}$$

for some polynomials $K_\lambda, K_\nu, K_\theta \in \mathbb{C}[x]$. We assume without loss of generality that $n + \sigma^{-1}(\gamma)$ is even for any $\gamma \in \{\lambda, \nu, \theta\}$. This leads to the following expressions of components of the identity field e by Proposition 3.5.3.

Proposition 3.6.30. *The λ, ν and θ components of the identity field e are given by*

$$e^\lambda = \frac{K_\lambda}{\lambda\Pi}, \quad e^\nu = \frac{K_\nu}{\nu\Pi}, \quad \text{and} \quad e^\theta = \frac{K_\theta}{\theta\Pi}. \tag{3.157}$$

Let us introduce $\bar{J} = \Pi$ so that $J = \lambda\nu\theta\bar{J}$. Recall that in these notations $\det\eta_D$ is given by formula (3.112). The entries of the matrix $A = (a_{ij})_{i,j=1}^3$ defined in (3.113) are given as follows.

Proposition 3.6.31. *All the matrix entries a_{ij} $(1 \le i, j \le 3)$ are well-defined generically on D. They have the following form on D:*

$$a_{11} = \frac{2K_\lambda}{\Pi}, \qquad a_{22} = \frac{2K_\nu}{\Pi}, \qquad a_{33} = \frac{2K_\theta}{\Pi}, \qquad a_{ij} = 0, \quad \text{if} \quad i \neq j.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\lambda, \nu, \theta\}$. Therefore by Proposition 3.6.30 the statement follows. $\square$

For any $H \in \mathcal{A}$ let $\alpha_H \in \mathcal{R}$ be such that $H = \{x \in V | \alpha_H(x) = 0\}$. Similarly for any $H \in \mathcal{A}_{D_\gamma}$, $\gamma \in \{\lambda, \nu, \theta\}$ we choose $\alpha_H \in \mathcal{R}$ such that $H = \{x \in D_\gamma | \alpha_H(x) = 0\}$. It follows from Corollary 3.5.15 and formulae (3.154), (3.155), (3.156) that

$$K_\gamma|_D \sim \prod_{\substack{H \in \mathcal{A}_{D_\gamma} \\ H \notin \mathcal{A}^D_{D_\gamma}}} \alpha_H|_D \sim I(\mathcal{A}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma})\Big|_D. \tag{3.158}$$

Moreover we have that

$$\Pi\big|_D \sim \prod_{\substack{H\in\mathcal{A}\\ H\notin\mathcal{A}^D}} \alpha_H\big|_D \sim I(\mathcal{A}\setminus\mathcal{A}^D)\big|_D. \tag{3.159}$$

We obtain the following statement on $\det\eta_D$.

Theorem 3.6.32. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}\setminus\mathcal{A}^D)^{-1}\prod_{\gamma\in\{\lambda,\nu,\theta\}} I(\mathcal{A}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma}). \tag{3.160}$$

Proof. By formula (3.112) we have $\det\eta_D = -\bar{J}^2\det A\big|_D$, where A is given by formula (3.113). Therefore by Proposition 3.6.31 we get

$$\det\eta_D = -\bar{J}^2\det A\big|_D = -a_{11}a_{22}a_{33}\Pi^2\big|_D = -\left.\frac{8K_\lambda K_\nu K_\theta}{\Pi}\right|_D,$$

and the statement follows by formulae (3.158) and (3.159). □

We now show that powers of distinct linear factors in formula (3.160) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.33. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta\in\mathcal{R}\setminus\mathcal{R}_D$ and let $\widehat{\mathcal{R}}=\mathcal{R}_{D,\beta}$ be the rank 4 root system $\widehat{\mathcal{R}}=\mathcal{R}\cap\langle\lambda,\nu,\theta,\beta\rangle$. Let $\widehat{\mathcal{A}}$ be the corresponding arrangement. Note that $|\mathcal{A}^D_{D_\lambda}|=|\mathcal{A}^D_{D_\nu}|=|\mathcal{A}^D_{D_\theta}|=2$. Then we have from formulae (3.158) that the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma})\Big|_D$ for any $\gamma\in\{\lambda,\nu,\theta\}$ is

$$|\widehat{\mathcal{A}}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma}| = |\widehat{\mathcal{A}}_{D_\gamma}|-2. \tag{3.161}$$

Similarly, we have from formula (3.159) that the multiplicity of $\beta|_D$ in $I(\mathcal{A}\setminus\mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}}\setminus\mathcal{A}^D| = |\widehat{\mathcal{A}}|-3. \tag{3.162}$$

Let us consider firstly the case where $\widehat{\mathcal{R}}$ is irreducible. Then $|\widehat{\mathcal{A}}_{D_\gamma}|=|\widehat{\mathcal{A}}|-h+1$ for any $\gamma\in\{\lambda,\nu,\theta\}$, where h is the Coxeter number of $\widehat{\mathcal{R}}$. Hence formula (3.161) implies that

$$|\widehat{\mathcal{A}}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma}| = h-1$$

and formula (3.162) implies that

$$|\widehat{\mathcal{A}}\setminus\mathcal{A}^D| = 2h-3.$$

Then it follows from Theorem 3.6.32 that the multiplicity of $\beta|_D$ in $\det\eta_D$ is h, as required.

Let us now consider the case where $\widehat{\mathcal{R}} = A_3 \times A_1$. We can assume without loss of generality that

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \lambda, \nu, \beta \rangle) \sqcup \{\pm\theta\} = A_3 \times A_1.$$

Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to A_3. Then we have $|\widehat{\mathcal{A}}_{D_\gamma}| = |\widetilde{\mathcal{A}}_{D_\gamma}| + 1 = 4$, for any $\gamma \in \{\lambda, \nu\}$ and $|\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}| = 6$. Note that $|\widehat{\mathcal{A}}| = 7$. Then it follows from Theorem 3.6.32 and formulae (3.161), (3.162) that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required.

Let us now suppose that $\widehat{\mathcal{R}} = A_2 \times A_1^2$ and assume without loss of generality that

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \lambda, \beta \rangle) \sqcup \{\pm\nu\} \sqcup \{\pm\theta\} = A_2 \times A_1 \times A_1.$$

Then it follows that $\widehat{\mathcal{A}}_{D_\lambda} = \{D_{\beta,\lambda}, D_{\nu,\lambda}, D_{\theta,\lambda}\}$, $\widehat{\mathcal{A}}_{D_\nu} = \{D_{\beta,\nu}, D_{\lambda,\nu}, D_{\theta,\nu}, D_{\lambda+\epsilon\beta,\nu}\}$ and $\widehat{\mathcal{A}}_{D_\theta} = \{D_{\beta,\theta}, D_{\lambda,\theta}, D_{\nu,\theta}, D_{\lambda+\epsilon\beta,\theta}\}$, where either $\epsilon = 1$ or $\epsilon = -1$. Hence $|\widehat{\mathcal{A}}_{D_\lambda}| = 3$ and $|\widehat{\mathcal{A}}_{D_\gamma}| = 4$, for any $\gamma \in \{\nu, \theta\}$. Note that $|\widehat{\mathcal{A}}| = 5$. Then it follows from Theorem 3.6.32 and formulae (3.161), (3.162) that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 3, which is the Coxeter number of A_2, as required.

Finally, let us consider the case where $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_1^4$. Then we get $|\widehat{\mathcal{A}}_{D_\lambda}| = |\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = 3$ and $|\widehat{\mathcal{A}}| = 4$. It follows from Theorem 3.6.32 and formulae (3.161), (3.162) that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. Thus the statement follows. □

3.6.5 Codimension 4

In this section we consider $(n-4)$-dimensional strata for simply laced Coxeter groups. Thus, we obtain factorisation formulae for the determinant of the restricted Saito metric for strata of type A_4, D_4, $A_3 \times A_1$, $A_2 \times A_2$, $A_2 \times A_1^2$ and A_1^4.

Let $\mathcal{R}_+$ be the positive root system of the root systems E_n, $n = 6, 7, 8$. Note that the following analysis works in fact for any irreducible simply laced root system. Let $\mu, \lambda, \nu, \theta$ be simple roots and consider the corresponding stratum $D = D_{\mu,\lambda,\nu,\theta}$. We have a number of cases depending on the type of stratum D.

Stratum $\mathbf{A_4}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type A_4. Let us consider the corresponding Coxeter graph

A_4 :  (3.163)

By Proposition 3.5.10, in the notation of (3.55) we have

$$J_\mu = \lambda\nu\theta(\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)\widetilde{K}_\mu \tag{3.164}$$

for some $\widetilde{K}_\mu \in \mathbb{C}[x]$. Note that the polynomials J_λ, J_ν, J_θ and J are still given by formulae (3.98)-(3.101) and thus it follows from Proposition 3.5.10 and the form of the graph (3.163) that

$$\Pi = \mu(\mu+\lambda)(\mu+\lambda+\nu)(\mu+\lambda+\nu+\theta)\widetilde{\Pi}, \tag{3.165}$$

$$K_\lambda = \mu\widetilde{K}_\lambda, \tag{3.166}$$

$$K_\nu = \mu(\mu+\lambda)\widetilde{K}_\nu, \tag{3.167}$$

$$K_\theta = \mu(\mu+\lambda)(\mu+\lambda+\nu)\widetilde{K}_\theta, \tag{3.168}$$

for some $\widetilde{K}_\lambda, \widetilde{K}_\nu, \widetilde{K}_\theta, \widetilde{\Pi} \in \mathbb{C}[x]$. Note that the polynomial $\widetilde{\Pi}$ is proportional to $I(\mathcal{A}\setminus\mathcal{A}^D)$ and is non-zero on D, and $\widetilde{K}_\alpha$ is proportional to $I(\mathcal{A}_{D_\alpha})I(\mathcal{A}^D_{D_\alpha})^{-1}$ on D_α, for $\alpha = \mu, \theta$. The ordering of the simple roots λ, ν and θ is assumed to be the same as in the case $\mathcal{R}_D = A_3$. We also assume without loss of generality that $\sigma^{-1}(\mu) = \sigma^{-1}(\lambda) - 1$ and that simple roots $\mu, \lambda, \nu, \theta$ are taken consecutively in this order in the Jacobi matrix.

In the following Lemmas 3.6.34–3.6.38 we study the structure of the polynomials $\widetilde{K}_\nu$ and $\widetilde{K}_\lambda$.

Lemma 3.6.34. *We have*

$$\widetilde{K}_\nu\Big|_{D_\nu} = \lambda(\mu+\lambda)\widetilde{K}_\theta + \theta\widetilde{B}\Big|_{D_\nu}, \tag{3.169}$$

for some polynomial $\widetilde{B}$ such that

$$\widetilde{B}\Big|_{D_{\mu,\nu}} = \lambda F_1 + \theta F_2|_{D_{\mu,\nu}}, \quad F_1, F_2 \in \mathbb{C}[x]. \tag{3.170}$$

One also has

$$F_2|_{D_{\mu,\nu,\theta}} = \widetilde{K}_\theta\Big|_{D_{\mu,\nu,\theta}}, \tag{3.171}$$

and

$$F_1|_D = 2\widetilde{K}_\theta\Big|_D. \tag{3.172}$$

Proof. By Proposition 3.5.16, we have $\frac{J_\nu}{\theta} = \frac{J_\theta}{\nu}$ on $D_{\nu,\theta}$, hence $K_\nu|_{D_{\nu,\theta}} = \lambda K_\theta|_{D_{\nu,\theta}}$. Thus using equalities (3.167), (3.168) we have

$$\widetilde{K}_\nu\Big|_{D_{\nu,\theta}} = \lambda(\mu+\lambda)\widetilde{K}_\theta\Big|_{D_{\nu,\theta}}. \tag{3.173}$$

Therefore, relation (3.169) follows.

To derive formula (3.170), let $\gamma = \mu+\lambda+\nu+\theta$ and $\delta = \lambda+\nu+\theta$. By Corollary 3.5.15 we get

$$J_\nu|_{D_\nu} = \lambda\theta\mu(\mu+\lambda)\widetilde{K}_\nu\Big|_{D_\nu} = \gamma\delta F|_{D_\nu} = (\mu+\lambda+\theta)(\lambda+\theta)F|_{D_\nu},$$

for some polynomial F. It follows that $\widetilde{K}_\nu$ is divisible by $(\mu+\lambda+\theta)(\lambda+\theta)$ on D_ν, that is

$$\widetilde{K}_\nu\Big|_{D_\nu} = (\mu+\lambda+\theta)(\lambda+\theta)\widehat{F}\Big|_{D_\nu},$$

for some polynomial $\widehat{F}$. Hence, $\widetilde{K}_\nu\Big|_{D_{\mu,\nu}} = (\lambda+\theta)^2\widehat{F}\Big|_{D_{\mu,\nu}}$. By (3.169) we get that

$$\theta\widetilde{B}\Big|_{D_{\mu,\nu}} = \lambda^2(\widehat{F}-\widetilde{K}_\theta) + 2\lambda\theta\widehat{F} + \theta^2\widehat{F}\Big|_{D_{\mu,\nu}}. \tag{3.174}$$

Hence $\widetilde{K}_\theta - \widehat{F}$ is divisible by θ on $D_{\mu,\nu}$, so we let

$$\widetilde{K}_\theta\Big|_{D_{\mu,\nu}} = \widehat{F} + \theta G\Big|_{D_{\mu,\nu}}, \tag{3.175}$$

where $G \in \mathbb{C}[x]$. Equality (3.170) then follows from (3.174), (3.175) with $F_1 = \lambda G + 2\widehat{F}$ and $F_2 = \widehat{F}$. Relations (3.171) and (3.172) now follow by further restrictions to D. □

We relate the polynomial B given by (3.105) and the polynomial $\widetilde{B}$ in the following lemma.

Lemma 3.6.35. *We have*

$$B|_{D_\nu} = \mu(\mu+\lambda)\widetilde{B}\Big|_{D_\nu}, \tag{3.176}$$

and furthermore,

$$\frac{B}{\Pi}\Big|_{D_{\nu,\mu,\theta}} = \lambda^{-1}\frac{F_1}{\widetilde{\Pi}}\Big|_{D_{\nu,\mu,\theta}}, \tag{3.177}$$

where we restrict on $D_{\nu,\mu,\theta}$ by first restricting to $\nu=0$, then to $\mu=0$, and then on $\theta=0$.

Proof. Combining formulae (3.105), (3.167) and (3.168) we have

$$\mu(\mu+\lambda)\widetilde{K}_\nu\Big|_{D_\nu} = \lambda\mu(\mu+\lambda)^2\widetilde{K}_\theta + \theta B\Big|_{D_\nu}. \tag{3.178}$$

Relation (3.176) follows from the relations (3.169), (3.178). To obtain formula (3.177) we first note that $\frac{B}{\Pi}\big|_{D_\nu} = \frac{B|_{D_\nu}}{\Pi|_{D_\nu}}$. Thus, using formula (3.165) we have

$$\frac{B}{\Pi}\Big|_{\substack{\nu=0\\ \mu=0}} = \frac{\widetilde{B}}{\lambda(\lambda+\theta)\widetilde{\Pi}}\Bigg|_{\substack{\nu=0\\ \mu=0}}, \tag{3.179}$$

where we first restrict to $\nu=0$, and then to $\mu=0$. Formula (3.177) follows from (3.170) and (3.179). □

In what follows fix $(\alpha,\alpha)=2$ for all $\alpha\in\mathcal{R}$. Let $\xi\in\mathcal{R}$ and let s_ξ denote the orthogonal reflection with respect to the hyperplane $\xi=0$. We have $s_\xi : \alpha \mapsto \widehat{\alpha} = \alpha - (\alpha,\xi)\xi$ and hence

$$s_\xi : \partial_\alpha \mapsto \partial_{\widehat{\alpha}} = \partial_\alpha - (\alpha, \xi)\partial_\xi.$$

We thus have

$$s_\xi : \partial_\alpha p(x) \mapsto \partial_{\widehat{\alpha}} p(s_\xi(x)) = \partial_{\widehat{\alpha}} p(x), \tag{3.180}$$

for $p \in \mathbb{C}[x]^W$. By Corollary 3.5.15, $\widetilde{K}_\lambda$ can be represented as

$$\widetilde{K}_\lambda = P + \lambda R, \tag{3.181}$$

where $P, R \in \mathbb{C}[x]$ and P is divisible by $\gamma = \mu + \lambda + \nu + \theta$ and $\beta = \mu + \lambda + \nu$ that is,

$$P = \gamma\beta S, \tag{3.182}$$

for some $S \in \mathbb{C}[x]$. In the next few statements we study the behaviour of polynomials P, R and $\widetilde{K}_\lambda$ as well as $\widetilde{K}_\mu, \widetilde{K}_\theta$ as one restricts to the strata $D_{\mu,\nu,\theta}$ and D. The required result is formulated in Proposition 3.6.38. We need a few lemmas in order to establish this proposition.

Lemma 3.6.36. *We have*

$$R|_{s_\lambda(D_\mu)} = \lambda^{-1}\big[(\nu + \lambda + \theta)(\nu + \lambda)s_\lambda(\widetilde{K}_\mu) - \nu(\nu + \theta)S\big]\Big|_{s_\lambda(D_\mu)}, \tag{3.183}$$

and

$$R|_{s_\lambda(D_\nu)} = \lambda^{-1}\big[(\mu + \lambda)(\mu + \theta)s_\lambda(Q_1) - \mu(\mu + \theta)S\big]\big|_{s_\lambda(D_\nu)}, \tag{3.184}$$

where $Q_1 \in \mathbb{C}[x]$.

Proof. Let us recall that $J_\lambda = \mu\nu\theta(\nu + \theta)\widetilde{K}_\lambda$. By applying orthogonal reflectionn s_λ we have

$$s_\lambda(J_\lambda) = (\mu + \lambda)(\nu + \lambda)\theta(\nu + \lambda + \theta)s_\lambda(\widetilde{K}_\lambda). \tag{3.185}$$

Note that J_λ is the determinant of a matrix with entries of the form $\partial_\alpha p$ for some simple roots α and $p \in \mathbb{C}[x]^W$. We can assume without loss of generality that there exists a simple root $\beta \neq \mu, \lambda, \nu, \theta$ such that

$$(\beta, \alpha) = \begin{cases} 0, & \alpha = \mu, \nu, \theta, \\ -1 & \alpha = \lambda. \end{cases}$$

Then for any $\alpha \in \Delta$

$$s_\lambda \alpha = \begin{cases} \alpha, & \alpha \neq \mu, \lambda, \nu, \beta \\ \lambda + \alpha, & \alpha = \mu, \nu, \beta \\ -\lambda, & \alpha = \lambda. \end{cases}$$

Formula (3.180) and linearity of determinants implies that

$$s_\lambda(J_\lambda) = J_\mu + J_\lambda + J_\nu + J_\beta.$$

Then from above and (3.185) we get

$$J_\mu + J_\lambda + J_\nu + J_\beta = (\mu+\lambda)(\nu+\lambda)\theta(\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\lambda). \tag{3.186}$$

Restricting equality (3.186) on D_μ we have by Proposition 3.5.10

$$J_\mu|_{D_\mu} = \lambda\theta(\nu+\lambda)(\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\lambda)\Big|_{D_\mu}, \tag{3.187}$$

where J_μ is given by formula (3.164). Therefore

$$\nu(\nu+\theta)\widetilde{K}_\mu\Big|_{D_\mu} = s_\lambda(\widetilde{K}_\lambda)\Big|_{D_\mu}. \tag{3.188}$$

Applying s_λ to equality (3.188) we obtain

$$\widetilde{K}_\lambda\Big|_{s_\lambda(D_\mu)} = (\nu+\lambda)(\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\mu)\Big|_{s_\lambda(D_\mu)}. \tag{3.189}$$

Notice that $s_\lambda(D_\mu) = D_{\mu+\lambda}$ and $\gamma\beta|_{s_\lambda(D_\mu)} = \nu(\nu+\theta)|_{s_\lambda(D_\mu)}$. Therefore using (3.189) and (3.182), we solve for R to obtain (3.183). Similarly, restricting equality (3.186) on D_ν we obtain

$$J_\nu|_{D_\nu} = \lambda\theta(\mu+\lambda)(\lambda+\theta)s_\lambda(\widetilde{K}_\lambda)\Big|_{D_\nu}. \tag{3.190}$$

Recall that $J_\nu = \lambda\theta\mu(\mu+\lambda)\widetilde{K}_\nu$. It follows from (3.190) that

$$\mu\widetilde{K}_\nu\Big|_{D_\nu} = (\lambda+\theta)s_\lambda(\widetilde{K}_\lambda)\Big|_{D_\nu}. \tag{3.191}$$

By Corollary 3.5.15, $\widetilde{K}_\nu$ can be represented as

$$\widetilde{K}_\nu = (\mu+\lambda+\nu+\theta)(\lambda+\nu+\theta)Q_1 + \nu Q_2, \tag{3.192}$$

for some $Q_1, Q_2 \in \mathbb{C}[x]$. Applying s_λ to equality (3.191), we find

$$\widetilde{K}_\lambda\Big|_{s_\lambda(D_\nu)} = (\mu+\lambda)(\theta-\lambda)^{-1}s_\lambda(\widetilde{K}_\nu)\Big|_{s_\lambda(D_\nu)}. \tag{3.193}$$

We get from (3.192) that

$$s_\lambda(\widetilde{K}_\nu)\Big|_{s_\lambda(D_\nu)} = (\mu+\theta)(\theta-\lambda)s_\lambda(Q_1)|_{s_\lambda(D_\nu)}. \tag{3.194}$$

It follows from (3.193), (3.194) that

$$\widetilde{K}_\lambda\Big|_{s_\lambda(D_\nu)} = (\mu+\lambda)(\mu+\theta)s_\lambda(Q_1)|_{s_\lambda(D_\nu)}. \tag{3.195}$$

Using (3.181), (3.189) and (3.195) we solve for R to obtain (3.184). □

Let us consider an orthonormal coordinate system y_i, $(1 \le i \le n)$ where a vector $y \in \mathbb{C}^n$ has coordinates $y_1 = \frac{1}{\sqrt{2}}(\mu+\lambda)(y)$, $y_2 = \frac{1}{\sqrt{2}}(\nu+\lambda)(y)$, $y_3 = \frac{1}{2}(\mu+\nu)(y)$, $y_4 = \frac{1}{2\sqrt{5}}(\mu+2\lambda+3\nu+4\theta)(y)$. In the next lemma we will consider the Taylor expansion of the polynomial R in the variables y_i $(1 \le i \le 4)$. Note that $D = \{y | y_1 = y_2 = y_3 = y_4 = 0\}$.

Lemma 3.6.37. *We have*

$$R|_{D_{\mu,\nu,\theta}} = 2S\lambda|_{D_{\mu,\nu,\theta}} + \widetilde{\mathcal{O}}, \tag{3.196}$$

where $\widetilde{\mathcal{O}}$ is a polynomial in $\lambda, y_5, \dots, y_n$ which is divisible by λ^2. Furthermore,

$$\left(\lambda^{-1}R|_{D_{\mu,\nu,\theta}}\right)\Big|_D = 2S|_D = 2\widetilde{K}_\mu\Big|_D. \tag{3.197}$$

Proof. Consider restriction of the polynomial R on D by taking first $y_2 = 0$. It follows from (3.183) that

$$R|_D = 0. \tag{3.198}$$

This, together with (3.181) implies that $\widetilde{K}_\lambda$ is divisible by λ^2 on $D_{\mu,\nu,\theta}$. Let us now compute the first order terms in the Taylor expansion of R. We have

$$\partial_{y_1} R|_{y_2=0} = \frac{1}{\sqrt{2}}\partial_{\mu+\lambda}\, R|_{s_\lambda(D_\nu)}. \tag{3.199}$$

Note that $(\mu+\lambda, \lambda) = (\mu+\lambda, \mu) = 1$. Therefore by formula (3.184) we have

$$\begin{aligned}\partial_{\mu+\lambda}\, R|_{s_\lambda(D_\nu)} =& -\lambda^{-2}\big[(\mu+\lambda)(\mu+\theta)s_\lambda(Q_1) - \mu(\mu+\theta)S\big]\big|_{s_\lambda(D_\nu)}\\ &+\lambda^{-1}\Big[2(\mu+\theta)s_\lambda(Q_1) + (\mu+\lambda)\partial_{\mu+\lambda}\Big((\mu+\theta)s_\lambda(Q_1)\Big) - (\mu+\theta)S - \mu S\\ &-\mu(\mu+\theta)\partial_{\mu+\lambda}S\Big]\Big|_{s_\lambda(D_\nu)}.\end{aligned} \tag{3.200}$$

We are going to restrict equality (3.200) onto $\{y_1 = y_2 = y_4 = 0\}$ which is equivalent to $\mu = \nu = -2\theta = -\lambda$. We get

$$\partial_{\mu+\lambda}R|_{y_1=y_2=y_4=0} = \frac{S}{2} + \lambda^{-1}\Big[-\lambda s_\lambda(Q_1) + \frac{3S}{2}\lambda - \frac{\lambda^2}{2}\partial_{\mu+\lambda}S\Big]\Big|_{y_1=y_2=y_4=0} \tag{3.201}$$

$$= 2S - s_\lambda(Q_1) - \frac{\lambda}{2}\partial_{\mu+\lambda}S\Big|_{y_1=y_2=y_4=0}. \tag{3.202}$$

Finally, restricting on $\{y_3 = 0\}$ and applying formula (3.199) we obtain

$$\partial_{y_1} R|_D = \frac{1}{\sqrt{2}}(2S - Q_1)\Big|_D . \tag{3.203}$$

Further on, we have

$$\partial_{y_2}\ R|_{y_1=0} = \frac{1}{\sqrt{2}}\partial_{\nu+\lambda}\ R|_{s_\lambda(D_\mu)} . \tag{3.204}$$

Note that $(\nu+\lambda,\lambda) = (\nu+\lambda,\nu) = 1$, and $(\nu+\lambda,\nu+\theta) = 0$. Then by formula (3.183) we have

$$\begin{aligned}\partial_{\nu+\lambda}\ R|_{s_\lambda(D_\mu)} =& -\lambda^{-2}\big[(\nu+\lambda+\theta)(\nu+\lambda)s_\lambda(\widetilde{K}_\mu) - \nu(\nu+\theta)S\big]\Big|_{s_\lambda(D_\mu)} \\ &+ \lambda^{-1}\big[2(\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\mu) + (\nu+\lambda)\partial_{\nu+\lambda}\Big((\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\mu)\Big) \\ &-(\nu+\theta)S - \nu(\nu+\theta)\partial_{\nu+\lambda}S\big]\big|_{s_\lambda(D_\mu)} .\end{aligned} \tag{3.205}$$

Restricting equality (3.205) onto $\{y_1 = y_2 = y_4 = 0\}$, we obtain

$$\begin{aligned}\partial_{\nu+\lambda}R|_{y_1=y_2=y_4=0} &= \frac{S}{2} + \lambda^{-1}\big[\lambda s_\lambda(\widetilde{K}_\mu) + \frac{S}{2}\lambda - \frac{1}{2}\lambda^2\partial_{\nu+\lambda}S\big]\Big|_{y_1=y_2=y_4=0} \\ &= S + s_\lambda(\widetilde{K}_\mu) - \frac{1}{2}\lambda\partial_{\nu+\lambda}S\Big|_{y_1=y_2=y_4=0} .\end{aligned} \tag{3.206}$$

Finally, restricting on $\{y_3 = 0\}$ and applying formula (3.204) we have

$$\partial_{y_2} R|_D = \frac{1}{\sqrt{2}}(S + \widetilde{K}_\mu)\Big|_D . \tag{3.207}$$

Further on, we have

$$\partial_{y_4}\ R|_{y_1=0} = \frac{1}{2\sqrt{5}}(\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)\ R|_{s_\lambda D_\mu} .$$

Note that

$$(\mu + 2\lambda + 3\nu + 4\theta, \alpha) = \begin{cases} 0, & \alpha = \nu, \lambda, \\ 5, & \alpha = \theta. \end{cases}$$

Then by formula (3.183) we have

$$
\begin{aligned}
2\sqrt{5}\,\partial_{y_4}R\big|_{y_1=0} &= (\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)\,\lambda^{-1}\big[(\nu+\lambda+\theta)(\nu+\lambda)s_\lambda(\widetilde{K}_\mu) - \nu(\nu+\theta)S\big]\Big|_{s_\lambda(D_\mu)} \\
&= (\nu+\lambda)\lambda^{-1}(\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)\big[(\nu+\lambda+\theta)s_\lambda(\widetilde{K}_\mu)\big]\Big|_{s_\lambda(D_\mu)} - \\
&- \nu\lambda^{-1}(\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)(\nu+\theta)S\big|_{s_\lambda(D_\mu)} = \\
&= (\nu+\lambda)\lambda^{-1}\big[5s_\lambda(\widetilde{K}_\mu) + (\nu+\lambda+\theta)(\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)s_\lambda(\widetilde{K}_\mu)\big]\Big|_{s_\lambda(D_\mu)} - \\
&- \nu\lambda^{-1}\big[5S + (\nu+\theta)(\partial_\mu + 2\partial_\lambda + 3\partial_\nu + 4\partial_\theta)S\big]\big|_{s_\lambda(D_\mu)}\,. \qquad (3.208)
\end{aligned}
$$

By restricting (3.208) on $y_2 = 0$ and then further on D we get

$$\partial_{y_4}R\big|_D = \frac{\sqrt{5}}{2}S\Big|_D\,. \qquad (3.209)$$

Let us now study the polynomial Q_1 on D. From (3.200) we get

$$\partial_{y_1}\;R\big|_{s_\lambda(D_\nu)}\Big|_{\mu+\lambda=\mu+\theta=0} = \frac{1}{\sqrt{2}}S\Big|_{\mu+\lambda=\mu+\theta=\nu+\lambda=0}\,. \qquad (3.210)$$

Combining this formula with (3.203) on D we obtain

$$Q_1\big|_D = S\big|_D\,. \qquad (3.211)$$

Similarly, restricting (3.205) on $\nu+\lambda=\theta=0$ we have

$$\partial_{\nu+\lambda}\;R\big|_{s_\lambda(D_\mu)}\Big|_{\nu+\lambda=\theta=0} = 2S - \lambda\partial_{\nu+\lambda}S\big|_{\nu+\lambda=\theta=\lambda+\mu=0}\,. \qquad (3.212)$$

Combining this with formula (3.206) we obtain that

$$S\big|_D = \widetilde{K}_\mu\Big|_D\,, \qquad (3.213)$$

which gives the second required equality in (3.197).

Let us now find R on $D_{\mu,\nu,\theta}$ using its Taylor series on D. Coordinates $y_1,\ldots,y_4$ on the space $D_{\mu,\nu,\theta}$ satisfy equations

$$
\begin{cases}
y_1 = y_2 = \dfrac{\lambda}{\sqrt{2}} \\
y_4 = \sqrt{\dfrac{2}{5}}y_1 = \dfrac{\lambda}{\sqrt{5}} \\
y_3 = 0.
\end{cases}
$$

We have that

$$R = R\big|_D + y_1\partial_{y_1}R\big|_D + y_2\partial_{y_2}R\big|_D + y_3\,\partial_{y_3}R\big|_D + y_4\,\partial_{y_4}R\big|_D + \mathcal{O}, \tag{3.214}$$

where $\mathcal{O}$ denotes terms of degree at least 2 in the variables $y_1, \dots, y_4$. From equations (3.203), (3.207), (3.209), (3.211) and (3.213) we have

$$\begin{aligned} \partial_{y_1}\, R\big|_D &= \tfrac{1}{\sqrt{2}}\, S\big|_D\,, \\ \partial_{y_2}\, R\big|_D &= \sqrt{2}\, S\big|_D\,, \\ \partial_{y_4}\, R\big|_D &= \tfrac{\sqrt{5}}{2}\, S\big|_D\,. \end{aligned} \tag{3.215}$$

By (3.198), (3.215) Taylor expansion (3.214) takes the required form (3.196) on the space (3.6.5). The first equality in formula (3.197) follows. □

Proposition 3.6.38. *We have that*

$$\widetilde{K}_\lambda\Big|_{D_{\mu,\nu,\theta}} = \lambda^2 Q\big|_{D_{\mu,\nu,\theta}}\,, \tag{3.216}$$

for some $Q \in \mathbb{C}[x]$ such that

$$Q\big|_D = 3\widetilde{K}_\theta\Big|_D = 3\widetilde{K}_\mu\Big|_D\,. \tag{3.217}$$

Proof. By equalities (3.181), (3.182) and (3.196) we have

$$\widetilde{K}_\lambda\Big|_{D_{\mu,\nu,\theta}} = 3S\lambda^2\big|_{D_{\mu,\nu,\theta}} + \widehat{\mathcal{O}},$$

where $\widehat{\mathcal{O}}$ denotes a polynomial divisible by λ^3. We are now going to show that $S\big|_D = \widetilde{K}_\theta\Big|_D$. By Proposition 3.5.16 we have

$$\theta\widetilde{K}_\lambda\Big|_{D_{\nu,\lambda}} = \mu\widetilde{K}_\nu\Big|_{D_{\nu,\lambda}}\,. \tag{3.218}$$

By formula (3.181) we have $\widetilde{K}_\lambda\Big|_{D_{\nu,\lambda}} = \mu(\mu+\theta)S\big|_{D_{\nu,\lambda}}$, therefore (3.218) gives

$$\theta(\mu+\theta)S\big|_{D_{\nu,\lambda}} = \widetilde{K}_\nu\Big|_{D_{\nu,\lambda}}\,. \tag{3.219}$$

By Lemma 3.6.34 $\widetilde{K}_\nu\Big|_{D_{\nu,\lambda,\mu}} = \theta^2 F_2\big|_{D_{\nu,\lambda,\mu}}$. Hence (3.219) implies that $S\big|_{D_{\nu,\lambda,\mu}} = F_2\big|_{D_{\nu,\lambda,\mu}}$. It follows by formula (3.171) that

$$S\big|_D = \widetilde{K}_\theta\Big|_D\,. \tag{3.220}$$

Therefore, taking into account Lemma 3.6.37 the statement follows. □

The following proposition follows from formulae (3.164) and (3.165).

Proposition 3.6.39. *The μ component of the identity field e, is given by*

$$e^{\mu} = \frac{\widetilde{K}_{\mu}}{\mu(\mu+\lambda)(\mu+\lambda+\nu)(\mu+\lambda+\nu+\theta)\widetilde{\Pi}}, \tag{3.221}$$

where $\widetilde{K}_{\mu}$, $\widetilde{\Pi}$ are given by formulae (3.164), (3.165).

Let us define the polynomial $\widehat{J} = (\mu\lambda\nu\theta)^{-1}J$, where J is given by (3.98). We specialize the formula for the determinant of the restricted Saito metric given by Theorem 3.5.9 to the case of codimension 4 strata. We rearrange $\det\eta_D$ as

$$\det\eta_D = -\begin{vmatrix} \eta^{\mu\mu} & \eta^{\mu\lambda} & \eta^{\mu\nu} & \eta^{\mu\theta} \\ \eta^{\mu\lambda} & \eta^{\lambda\lambda} & \eta^{\lambda\nu} & \eta^{\lambda\theta} \\ \eta^{\mu\nu} & \eta^{\lambda\nu} & \eta^{\nu\nu} & \eta^{\nu\theta} \\ \eta^{\mu\theta} & \eta^{\lambda\theta} & \eta^{\nu\theta} & \eta^{\theta\theta} \end{vmatrix} J^2\Big|_D = -\begin{vmatrix} \mu^2\eta^{\mu\mu} & \mu\lambda\eta^{\mu\lambda} & \mu\nu\eta^{\mu\nu} & \mu\theta\eta^{\mu\theta} \\ \mu\lambda\eta^{\mu\lambda} & \lambda^2\eta^{\lambda\lambda} & \lambda\nu\eta^{\lambda\nu} & \lambda\theta\eta^{\lambda\theta} \\ \mu\nu\eta^{\mu\nu} & \lambda\nu\eta^{\lambda\nu} & \nu^2\eta^{\nu\nu} & \nu\theta\eta^{\nu\theta} \\ \mu\theta\eta^{\mu\theta} & \lambda\theta\eta^{\lambda\theta} & \nu\theta\eta^{\nu\theta} & \theta^2\eta^{\theta\theta} \end{vmatrix} \widehat{J}^2\Big|_D.$$

Let $\widehat{A} = (\widehat{a}_{ij})_{i,j=0}^3$ be the matrix

$$\widehat{A} = \begin{pmatrix} \mu^2\eta^{\mu\mu} & \mu\lambda\eta^{\mu\lambda} & \mu\nu\eta^{\mu\nu} & \mu\theta\eta^{\mu\theta} \\ \mu\lambda\eta^{\mu\lambda} & \lambda^2\eta^{\lambda\lambda} & \lambda\nu\eta^{\lambda\nu} & \lambda\theta\eta^{\lambda\theta} \\ \mu\nu\eta^{\mu\nu} & \lambda\nu\eta^{\lambda\nu} & \nu^2\eta^{\nu\nu} & \nu\theta\eta^{\nu\theta} \\ \mu\theta\eta^{\mu\theta} & \lambda\theta\eta^{\lambda\theta} & \nu\theta\eta^{\nu\theta} & \theta^2\eta^{\theta\theta} \end{pmatrix}. \tag{3.222}$$

Thus

$$\det\eta_D = -\widehat{J}^2 \det\widehat{A}\Big|_D. \tag{3.223}$$

Proposition 3.6.40. *The matrix entries $\widehat{a}_{0j}$ have the following form on $D_{\mu,\nu}$:*

$$\widehat{a}_{00} = \mu^2\eta^{\mu\mu} = \frac{2\widetilde{K}_{\mu}}{\lambda^2(\lambda+\theta)\widetilde{\Pi}}, \tag{3.224}$$

$$\widehat{a}_{01} = \mu\lambda\eta^{\mu\lambda} = -\lambda\partial_{\omega^{\lambda}}\Big(\frac{\widetilde{K}_{\mu}}{\lambda^2(\lambda+\theta)\widetilde{\Pi}}\Big), \tag{3.225}$$

$$\widehat{a}_{02} = \mu\nu\eta^{\mu\nu} = 0, \tag{3.226}$$

$$\widehat{a}_{03} = \mu\theta\eta^{\mu\theta} = -\frac{\theta}{\lambda^2}\partial_{\omega^{\theta}}\Big(\frac{\widetilde{K}_{\mu}}{(\lambda+\theta)\widetilde{\Pi}}\Big). \tag{3.227}$$

Furthermore, $\widehat{a}_{ij} = a_{ij}$ on $D_{\mu,\nu}$, for $i,j = 1,2,3$ where a_{ij} are given by Proposition 3.6.16. In particular, the entries $\widehat{a}_{ij}$ are well-defined generically on $D_{\mu,\nu}$ for $0 \le i,j \le 3$.

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha} e^\beta - \partial_{\omega^\beta} e^\alpha$ for $\alpha, \beta \in \{\mu, \lambda, \nu, \theta\}$. Then formulae (3.225) and (3.227) follow by Propositions 3.6.13, 3.6.39 immediately. Similarly, it is easy to show that $\widehat{a}_{02} = 0$. Let us prove formula (3.224). We have

$$\mu^2 \eta^{\mu\mu} = -2\mu^2 \partial_{\omega^\mu} \Big(\frac{\widetilde{K}_\mu}{\mu(\mu+\lambda)(\mu+\lambda+\nu)(\mu+\lambda+\nu+\theta)\widetilde{\Pi}} \Big).$$

By Leibniz rule and taking the limit $\mu(x), \nu(x) \to 0$ we obtain the required formula. It follows from established formulae (3.224)–(3.227) that entries $\widehat{a}_{0j}$, $0 \le j \le 3$, are well-defined generically on $D_{\mu,\nu}$. Note that polynomials K_λ, K_ν, K_θ given by formulae (3.165)–(3.168) are divisible by μ. Hence the entries $\widehat{a}_{ij} = a_{ij}$, $1 \le i, j \le 3$, are also well-defined generically on $D_{\mu,\nu}$ by Proposition 3.6.16. □

We are going to find $\det \eta_D$ given by formula (3.223) by restricting the right-hand side to $D_{\mu,\nu}$ first, then to $\theta = 0$, and then to $\lambda = 0$. Let us denote by M_{ij} the (i, j) minor of $\widehat{A}$ and consider a row expansion for $\det \widehat{A}$,

$$\det \widehat{A} = \widehat{a}_{00} M_{00} - \widehat{a}_{01} M_{01} + \widehat{a}_{02} M_{02} - \widehat{a}_{03} M_{03}, \tag{3.228}$$

where $M_{00} = \det A$ and A is given by (3.113). By Proposition 3.6.40 $\widehat{a}_{02}|_{D_{\mu,\nu}} = 0$ and M_{0j} $(0 \le j \le 3)$ is regular on $D_{\mu,\nu}$. Hence

$$\det \widehat{A}\Big|_{D_{\mu,\nu}} = \widehat{a}_{00} \det A - \widehat{a}_{01} M_{01} - \widehat{a}_{03} M_{03}|_{D_{\mu,\nu}}. \tag{3.229}$$

Let us note that $\widehat{J}\Big|_{D_{\mu,\nu}}$ is divisible by θ. Further on, we observe that $\widehat{a}_{03}|_{D_{\mu,\nu,\theta}} = 0$ by Proposition 3.6.40 and that $\theta^2 M_{03}$ is well-defined generically on $D_{\mu,\nu,\theta}$ by Propositions 3.6.16, 3.6.40. Therefore we have

$$\theta^2 \det \widehat{A}\Big|_{D_{\mu,\nu,\theta}} = \theta^2 \big(\widehat{a}_{00} \det A - \widehat{a}_{01} M_{01}\big)\big|_{D_{\mu,\nu,\theta}}. \tag{3.230}$$

By Propositions 3.6.16, 3.6.40 we get that

$$\theta^2 M_{01}\big|_{D_{\mu,\nu,\theta}} = \theta^2 \widehat{a}_{01} (\widehat{a}_{22} \widehat{a}_{33} - \widehat{a}_{23}^2)\big|_{D_{\mu,\nu,\theta}}. \tag{3.231}$$

Lemma 3.6.41. *We have*

$$\frac{K_\lambda}{\Pi}\Big|_{D_{\mu,\nu,\theta}} = \frac{Q}{\lambda \widetilde{\Pi}}\Big|_{D_{\mu,\nu,\theta}}, \tag{3.232}$$

where Q is given by Lemma 3.6.38.

Proof. By formulae (3.165), (3.166) we have

$$\frac{K_\lambda}{\Pi} = \frac{\widetilde{K}_\lambda}{(\mu+\lambda)(\mu+\lambda+\nu)(\mu+\lambda+\nu+\theta)\widetilde{\Pi}}.$$

Hence, $\frac{K_\lambda}{\Pi}\big|_{D_{\mu,\nu,\theta}} = \frac{\widetilde{K}_\lambda}{\lambda^3\widetilde{\Pi}}\Big|_{D_{\mu,\nu,\theta}}$ and formula (3.232) follows by Lemma 3.6.38. □

Now, let us observe that $(\nu+\theta)^{-1}\,\widehat{J}\Big|_{D_{\mu,\nu,\theta}}$ is divisible by λ^5 and that $\widehat{a}_{00}|_{D_{\mu,\nu,\theta}}$ has third order pole at $\lambda = 0$.

Lemma 3.6.42. *We have*

$$\lambda^{10}\theta^2 \det \widehat{A}\Big|_D = c\frac{\widetilde{K}_\theta^4}{\widetilde{\Pi}^4}\Bigg|_D. \tag{3.233}$$

where $c \in \mathbb{C}^\times$*.*

Proof. We calculate $\theta^2\lambda^{10}\det \widehat{A}\Big|_D$ by making use of expression (3.230). Restrictions of $\det A$ on D_ν and $\theta^2 \det A$ on $D_{\nu,\theta}$ were found in Subsection 3.6.4 on codimension 3 strata (type A_3 case). The corresponding terms z and $\theta^2 E$ as well as their restrictions on D_ν and $D_{\nu,\theta}$ are regular at $\mu = 0$. Therefore we will be using results of Subsection 3.6.4 on type A_3 with further restriction to $\mu = 0$, which we will be doing after restriction to $\nu = 0$ and before restriction to $\theta = 0$. Using formula (3.120) we have

$$\theta^2 \det A\Big|_{D_{\nu,\mu,\theta}} = z + \theta^2 E\big|_{D_{\nu,\mu,\theta}}. \tag{3.234}$$

Let us find an expression for the restriction $z|_{D_{\nu,\mu,\theta}}$. By formulae (3.165), (3.168) we have $\frac{K_\theta}{\Pi}\big|_{D_{\nu,\mu,\theta}} = \frac{\widetilde{K}_\theta}{\lambda\widetilde{\Pi}}\Big|_{D_{\nu,\mu,\theta}}$. By Lemma 3.6.35 we have $\frac{B}{\Pi}\big|_{D_{\nu,\mu,\theta}} = \frac{F_1}{\lambda\widetilde{\Pi}}\Big|_{D_{\nu,\mu,\theta}}$, and hence $\partial_{\omega^\lambda}\frac{B}{\Pi}\big|_{D_{\nu,\mu,\theta}} = \partial_{\omega^\lambda}\frac{F_1}{\lambda\widetilde{\Pi}}\Big|_{D_{\nu,\mu,\theta}}$. Therefore by Lemma 3.6.21 we get

$$\begin{aligned} z|_{D_{\nu,\mu,\theta}} = \frac{2}{\lambda^7}\Bigg(&\frac{4F_1\widetilde{K}_\theta^2}{\widetilde{\Pi}^3} - \frac{2\lambda^2F_1\widetilde{K}_\theta}{\widetilde{\Pi}^2}\partial_{\omega^\lambda}\frac{\widetilde{K}_\theta}{\lambda\widetilde{\Pi}} - \frac{3\lambda^2\widetilde{K}_\theta^2}{\widetilde{\Pi}^2}\partial_{\omega^\lambda}\frac{F_1}{\lambda\widetilde{\Pi}} \\ &- \frac{2\lambda^4F_1}{\widetilde{\Pi}}\Big(\partial_{\omega^\lambda}\frac{\widetilde{K}_\theta}{\lambda\widetilde{\Pi}}\Big)^2 + \frac{3\lambda^4\widetilde{K}_\theta}{\widetilde{\Pi}}\Big(\partial_{\omega^\lambda}\frac{F_1}{\lambda\widetilde{\Pi}}\Big)\Big(\partial_{\omega^\lambda}\frac{\widetilde{K}_\theta}{\lambda\widetilde{\Pi}}\Big)\Bigg)\Bigg|_{D_{\nu,\mu,\theta}} + \frac{2F_1}{\lambda^3\widetilde{\Pi}}\Big(\partial_{\omega^\lambda}\frac{\widetilde{K}_\theta}{\lambda\widetilde{\Pi}} - \frac{\widetilde{K}_\theta}{\lambda^2\widetilde{\Pi}}\Big)^2\Bigg|_{D_{\nu,\mu,\theta}}. \end{aligned} \tag{3.235}$$

By Lemmas 3.6.22 and 3.6.41 we get

$$\theta^2 E\big|_{D_{\nu,\mu,\theta}} = -\frac{18\widetilde{K}_\theta^2}{\lambda^2\widetilde{\Pi}^2}\partial_{\omega^\lambda}\Big(\frac{Q}{\lambda^4\widetilde{\Pi}}\Big)\Bigg|_{D_{\nu,\mu,\theta}}. \tag{3.236}$$

Note that formulae (3.235), (3.236) lead to the following expressions on D:

$$\lambda^7 z\big|_D = 36\frac{F_1\widetilde{K}_\theta^2}{\widetilde{\Pi}^3}\bigg|_D, \quad \lambda^7\theta^2 E\big|_D = 72\frac{Q\widetilde{K}_\theta^2}{\widetilde{\Pi}^3}. \tag{3.237}$$

It follows from (3.172), (3.217) that

$$\lambda^7(z+\theta^2 E)\big|_D = 288\frac{\widetilde{K}_\theta^3}{\widetilde{\Pi}^3}\bigg|_D. \tag{3.238}$$

By Proposition 3.6.40 we have $\lambda^3\widehat{a}_{00}|_{D_{\mu,\nu,\theta}} = 2\frac{\widetilde{K}_\mu}{\widetilde{\Pi}}\Big|_{D_{\mu,\nu,\theta}}$. Therefore formulae (3.238) and (3.217) imply that

$$\theta^2\lambda^{10}\widehat{a}_{00}\det A\big|_D = 576\Big(\frac{\widetilde{K}_\theta}{\widetilde{\Pi}}\Big)^4\bigg|_D. \tag{3.239}$$

Now we would like to simplify remaining terms in (3.230), see also (3.231). By Proposition 3.6.40 we get

$$\widehat{a}_{01}|_{D_{\mu,\nu,\theta}} = \frac{3}{\lambda^3}\frac{\widetilde{K}_\mu}{\widetilde{\Pi}} - \frac{1}{\lambda^2}\partial_{\omega^\lambda}\frac{\widetilde{K}_\mu}{\widetilde{\Pi}}\bigg|_{D_{\mu,\nu,\theta}}.$$

Hence

$$\lambda^6\widehat{a}_{01}^2\big|_D = 9\frac{\widetilde{K}_\mu^2}{\widetilde{\Pi}^2}\bigg|_D. \tag{3.240}$$

By Proposition 3.6.16 (in the notations of Proposition 3.6.40) and making use of formulae (3.165), (3.167), (3.168) we get

$$\theta^2(\widehat{a}_{22}\widehat{a}_{33}-\widehat{a}_{23}^2)\big|_{D_{\mu,\nu,\theta}} = -\frac{8}{\lambda^6}\frac{\widetilde{K}_\nu\widetilde{K}_\theta}{\widetilde{\Pi}^2} - \frac{1}{\lambda^8}\frac{\widetilde{K}_\nu^2}{\widetilde{\Pi}^2}\bigg|_{D_{\mu,\nu,\theta}}.$$

By (3.169) we get $\widetilde{K}_\nu\Big|_{D_{\mu,\nu,\theta}} = \lambda^2\widetilde{K}_\theta\Big|_{D_{\mu,\nu,\theta}}$, therefore

$$\lambda^4\theta^2(\widehat{a}_{22}\widehat{a}_{33}-\widehat{a}_{23}^2)\big|_D = -9\frac{\widetilde{K}_\theta^2}{\widetilde{\Pi}^2}\bigg|_D. \tag{3.241}$$

Since $\widetilde{K}_\mu\Big|_D = \widetilde{K}_\theta\Big|_D$ by Lemma 3.6.38 it follows by multiplying (3.240) with (3.241) that

$$\lambda^{10}\theta^2\widehat{a}_{01}^2(\widehat{a}_{22}\widehat{a}_{33}-\widehat{a}_{23}^2)\big|_D = -81\frac{\widetilde{K}_\theta^4}{\widetilde{\Pi}^4}\bigg|_D. \tag{3.242}$$

Substituting formulae (3.239) and (3.242) into the expression (3.230) for $\det\widehat{A}$ we get the statement. □

Theorem 3.6.43. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})^4 I(\mathcal{A} \setminus \mathcal{A}^D)^{-2} \tag{3.243}$$

on D. The same is true with θ replaced by μ in (3.243).

Proof. We have by formula (3.223)

$$\det \eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D = -\left((\lambda^3(\lambda+\theta)^2\theta\widetilde{\Pi})^2 \det \widehat{A}\right)\Big|_{D_{\mu,\nu,\theta}}\Big|_{\lambda=0}. \tag{3.244}$$

By Lemma 3.6.42 we have

$$\theta^2 \det \widehat{A}\Big|_{D_{\mu,\nu,\theta}} = \frac{c\widetilde{K}_\theta^4}{\lambda^{10}\widetilde{\Pi}^4} + \frac{f}{\lambda^9}\Big|_{D_{\mu,\nu,\theta}},$$

for some rational function f regular generically on $D_{\mu,\nu,\theta}$. It follows from (3.244) that

$$\det \eta_D = -\left(\lambda^{10}\widetilde{\Pi}^2\Big(\frac{c}{\lambda^{10}}\frac{\widetilde{K}_\theta^4}{\widetilde{\Pi}^4} + \frac{1}{\lambda^9}f\Big)\big|_{D_{\mu,\nu,\theta}}\right)\Big|_{\lambda=0}$$

and thus $\det \eta_D$ is proportional to $\widetilde{\Pi}^{-2}\widetilde{K}_\theta^4$ on D. Then the statement follows by Corollary 3.5.15. Replacement of θ with μ is possible by (3.217). □

Let us now show that the powers of distinct linear factors in (3.243) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.44. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. Note that the root system $\widehat{\mathcal{R}}$ is a rank 5 subsystem of $\mathcal{R}$. The multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})\big|_D$ is

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 6, \tag{3.245}$$

and the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 10. \tag{3.246}$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{\mathcal{R}} = A_5$ or $\widehat{\mathcal{R}} = D_5$. Then $|\widehat{\mathcal{A}}_{D_\theta}| = |\widehat{\mathcal{A}}| - h + 1$, where h is the Coxeter number of $\widehat{\mathcal{R}}$ [71]. Hence formula (3.245) implies that

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}| - h - 5 = \frac{3h}{2} - 5,$$

and formula (3.246) implies

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = \frac{5h}{2} - 10.$$

Then it follows from Theorem 3.6.43 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now suppose that $\widehat{\mathcal{R}}$ is reducible, that is $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_4 \times A_1$. Then we get from formulae (3.245), (3.246) that

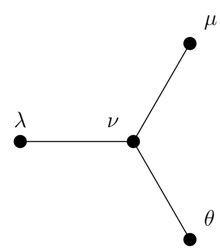
$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 6 = 1,$$

and

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 10 = 1.$$

Then it follows from Theorem 3.6.43 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. $\square$

Stratum $\mathbf{D_4}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type D_4. Let us consider the corresponding Coxeter graph

D_4 : 

(3.247)

Notice that

$$\lambda+\nu,\ \nu+\theta,\ \nu+\mu,\ \lambda+\nu+\mu,\ \lambda+\nu+\theta,\ \mu+\nu+\theta,\ \mu+\lambda+\nu+\theta,\ \mu+\lambda+2\nu+\theta\ \in \mathcal{R}_+.$$

The Jacobian J can be represented as

$$\begin{aligned} J = {} & \lambda\nu\theta\mu(\lambda+\nu)(\nu+\theta)(\nu+\mu)(\lambda+\nu+\theta)(\lambda+\nu+\mu)\times \\ & \times (\nu+\theta+\mu)(\lambda+\nu+\theta+\mu)(\lambda+2\nu+\theta+\mu)\Pi, \end{aligned} \tag{3.248}$$

where $\Pi \in \mathbb{C}[x]$ is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$ and is non-zero on D. By Proposition 3.5.10 and the form of the graph (3.247) we get

$$J_\lambda = \nu\theta\mu(\nu+\mu)(\nu+\theta)(\nu+\theta+\mu)K_\lambda, \tag{3.249}$$
$$J_\nu = \lambda\theta\mu K_\nu, \tag{3.250}$$
$$J_\theta = \lambda\nu\mu(\lambda+\nu)(\nu+\mu)(\lambda+\nu+\mu)K_\theta, \tag{3.251}$$
$$J_\mu = \lambda\nu\theta(\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)K_\mu, \tag{3.252}$$

for some $K_\lambda, K_\nu, K_\theta, K_\mu \in \mathbb{C}[x]$. We assume without loss of generality that the ordering of simple roots is such that $n+\sigma^{-1}(\alpha)$ is odd if $\alpha \in \{\nu, \mu\}$ and that $n+\sigma^{-1}(\alpha)$ is even if $\alpha \in \{\lambda, \theta\}$. Furthermore, we assume that simple roots $\lambda, \nu, \theta, \mu$ are taken consecutively in this order in the Jacobi matrix. It is convenient to define polynomials $S_\alpha \in \mathbb{C}[x]$, $\alpha = \lambda, \nu, \theta, \mu$ as follows:

$$\begin{aligned}
S_\lambda &= \lambda(\lambda+\nu)(\lambda+\nu+\theta)(\lambda+\nu+\mu)(\lambda+\nu+\theta+\mu)(\lambda+2\nu+\theta+\mu),\\
S_\nu &= \nu(\lambda+\nu)(\nu+\theta)(\nu+\mu)(\lambda+\nu+\theta)(\lambda+\nu+\mu)(\nu+\theta+\mu)(\lambda+\nu+\theta+\mu)\times\\
&\quad\times(\lambda+2\nu+\theta+\mu),\\
S_\theta &= \theta(\nu+\theta)(\lambda+\nu+\theta)(\nu+\theta+\mu)(\lambda+\nu+\theta+\mu)(\lambda+2\nu+\theta+\mu),\\
S_\mu &= \mu(\nu+\mu)(\lambda+\nu+\mu)(\nu+\theta+\mu)(\lambda+\nu+\theta+\mu)(\lambda+2\nu+\theta+\mu).
\end{aligned}$$

The following statement follows from formulae (3.248)–(3.252) and Proposition 3.5.3.

Proposition 3.6.45. *The $\lambda, \nu, \theta, \mu$ components of the identity field e are given by*

$$e^\alpha = (-1)^{n+\sigma^{-1}(\alpha)}\frac{K_\alpha}{S_\alpha \Pi}, \qquad \alpha = \lambda, \nu, \theta, \mu.$$

By Corollary 3.5.15 we can represent polynomials K_ν, K_λ, K_θ and K_μ as follows:

$$K_\nu = (\lambda+\nu+\mu)(\lambda+\nu+\theta)(\nu+\theta+\mu)(\lambda+2\nu+\theta+\mu)A_\nu + \nu R, \tag{3.253}$$

for some $A_\nu, R \in \mathbb{C}[x]$, and

$$K_\alpha = (\lambda+2\nu+\theta+\mu)A_\alpha + \alpha Q_\alpha, \tag{3.254}$$

for some $A_\alpha, Q_\alpha \in \mathbb{C}[x]$, $\alpha = \lambda, \theta, \mu$. Moreover, note that for any $\alpha \in \Delta$

$$s_\nu \alpha = \begin{cases} \alpha, & \alpha \neq \lambda, \theta, \mu,\\ \nu+\alpha, & \alpha = \lambda, \theta, \mu,\\ -\nu, & \alpha = \nu. \end{cases} \tag{3.255}$$

In the following Lemmas 3.6.46–3.6.49 we study the structure of the polynomials K_α, $\alpha = \nu, \lambda, \theta, \mu$.

Lemma 3.6.46. *The polynomial R defined in* (3.253) *satisfies conditions*

$$\begin{aligned}
R|_{s_\nu(D_\lambda)} &= \nu^{-1}(\nu+\mu)(\theta+\nu)(\nu+\theta+\mu)(\theta+\mu)s_\nu(A_\lambda)\big|_{s_\nu(D_\lambda)}\\
&\quad - \nu^{-1}\mu\theta(\nu+\theta+\mu)^2 A_\nu\big|_{s_\nu(D_\lambda)},
\end{aligned} \tag{3.256}$$

$$R|_{s_\nu(D_\theta)} = \nu^{-1}(\nu+\mu)(\lambda+\nu)(\nu+\lambda+\mu)(\lambda+\mu)s_\nu(A_\theta)\big|_{s_\nu(D_\theta)} \\ -\nu^{-1}\mu\lambda(\nu+\lambda+\mu)^2 A_\nu\big|_{s_\nu(D_\theta)}, \tag{3.257}$$

and

$$R|_{s_\nu(D_\mu)} = -\nu^{-1}(\nu+\theta)(\lambda+\nu)(\nu+\lambda+\theta)(\lambda+\theta)s_\nu(A_\mu)\big|_{s_\nu(D_\mu)} \\ -\nu^{-1}\theta\lambda(\nu+\lambda+\theta)^2 A_\nu\big|_{s_\nu(D_\mu)}. \tag{3.258}$$

Proof. By applying orthogonal reflection s_ν to J_ν we get

$$s_\nu(J_\nu) = (\lambda+\nu)(\theta+\nu)(\mu+\nu)s_\nu(K_\nu). \tag{3.259}$$

By formulae (3.180), (3.255) and linearity of determinants we get

$$s_\nu(J_\nu) = J_\lambda + J_\nu + J_\theta - J_\mu. \tag{3.260}$$

Then restricting equality (3.259) on D_λ we have by Proposition 3.5.10

$$J_\lambda|_{D_\lambda} = \nu(\theta+\nu)(\mu+\nu)s_\nu(K_\nu)|_{D_\lambda}, \tag{3.261}$$

where J_λ is given by (3.249). Therefore we get by formulae (3.249), (3.261)

$$s_\nu(K_\nu)|_{D_\lambda} = \theta\mu(\nu+\theta+\mu)K_\lambda|_{D_\lambda}. \tag{3.262}$$

Then by applying s_ν to equality (3.262) we obtain,

$$K_\nu|_{s_\nu(D_\lambda)} = (\nu+\mu)(\theta+\nu)(\nu+\theta+\mu)s_\nu(K_\lambda)|_{s_\nu(D_\lambda)}, \tag{3.263}$$

and $s_\nu(K_\lambda)|_{s_\nu(D_\lambda)} = (\theta+\mu)s_\nu(A_\lambda)|_{s_\nu(D_\lambda)}$ using (3.254). Using (3.253), (3.263) we solve for R to obtain (3.256). Formula (3.257) follows by symmetry which allows to swap λ and θ. Similarly formula (3.258) follows by the symmetry of the graph (3.247) and taking into account the sign in (3.260). □

Lemma 3.6.47. *Let $\beta = \lambda + 2\nu + \theta + \mu$. We have*

$$R|_{s_\nu\beta=0} = (\lambda+\nu)(\theta+\nu)(\mu+\nu)s_\nu(Q_\lambda) - \theta\mu(\nu+\theta+\mu)A_\nu|_{s_\nu\beta=0}. \tag{3.264}$$

Proof. It follows from (3.250) and (3.260) that

$$s_\nu(J_\nu) = J_\lambda + J_\nu + J_\theta - J_\mu = (\lambda+\nu)(\theta+\nu)(\mu+\nu)s_\nu(K_\nu). \tag{3.265}$$

Let us express λ as $\lambda = \beta - 2\nu - \theta - \mu$ and substitute it in the determinants J_ν, J_θ, and J_μ. By linearity of determinants we obtain

$$J_\nu = -2J_\lambda, \quad J_\theta = J_\lambda, \quad J_\mu = -J_\lambda.$$

on D_β. Therefore by restricting $s_\nu(J_\nu)$ to $\beta = 0$ we have that

$$s_\nu(J_\nu)|_{\beta=0} = J_\lambda|_{\beta=0}\,. \tag{3.266}$$

From the definition of R in (3.253) we get that

$$R|_{s_\nu\beta=0} = \nu^{-1}K_\nu\big|_{s_\nu\beta=0} + \theta\mu\lambda A_\nu|_{s_\nu\beta=0}\,. \tag{3.267}$$

By restricting (3.265) to $\beta = 0$ we get with the help of (3.266) that

$$J_\lambda|_{\beta=0} = (\lambda+\nu)(\theta+\nu)(\mu+\nu)s_\nu(K_\nu)|_{\beta=0}\,. \tag{3.268}$$

By formulae (3.249) and (3.254) we get

$$J_\lambda|_{\beta=0} = \nu\theta\mu\lambda(\nu+\mu)(\theta+\nu)(\nu+\theta+\mu)Q_\lambda|_{\beta=0}\,. \tag{3.269}$$

It follows from (3.268) and (3.269) by applying s_ν that

$$K_\nu|_{s_\nu\beta=0} = \nu(\theta+\nu)(\mu+\nu)(\lambda+\nu)s_\nu Q_\lambda|_{s_\nu\beta=0}\,. \tag{3.270}$$

Substituting (3.270) into (3.267) we get the required statement. $\square$

In the following lemma we are going to study the structure of the polynomials Q_λ, A_μ, A_θ and A_λ.

Lemma 3.6.48. *We have*

$$Q_\lambda|_D = A_\lambda|_D = A_\nu|_D = A_\theta|_D = -\,A_\mu|_D\,. \tag{3.271}$$

Proof. Applying s_λ to equality (3.254) we have

$$s_\lambda(K_\lambda) = (\lambda+2\nu+\theta+\mu)s_\lambda(A_\lambda) - \lambda s_\lambda(Q_\lambda). \tag{3.272}$$

We can assume that there is a simple root $\beta \neq \lambda, \nu, \theta, \mu$ such that

$$(\beta,\alpha) = \begin{cases} 0, & \alpha = \nu, \theta, \mu, \\ -1 & \alpha = \lambda. \end{cases}$$

Then by applying s_λ to equality (3.249) we obtain

$$s_\lambda(J_\lambda) = J_\beta + J_\lambda + J_\nu = \theta\mu(\nu+\lambda)(\nu+\mu+\lambda)(\nu+\lambda+\theta)(\nu+\lambda+\theta+\mu)s_\lambda(K_\lambda), \quad (3.273)$$

and restricting (3.273) to D_ν we thus have by Proposition 3.5.10 that

$$J_\nu|_{D_\nu} = \mu\lambda\theta(\theta+\lambda)(\lambda+\mu)(\lambda+\theta+\mu)s_\lambda(K_\lambda)|_{D_\nu},$$

where the left-hand side is given by $J_\nu|_{D_\nu} = \mu\lambda\theta(\theta+\lambda)(\lambda+\mu)(\theta+\mu)(\lambda+\theta+\mu)A_\nu|_{D_\nu}$ using formulae (3.250) and (3.253). Therefore,we have

$$s_\lambda(K_\lambda)|_{D_\nu} = (\theta+\mu)A_\nu|_{D_\nu}.$$

Comparing with formula (3.272) we obtain

$$(\lambda+\theta+\mu)s_\lambda(A_\lambda) - \lambda s_\lambda(Q_\lambda)|_{D_\nu} = (\theta+\mu)A_\nu|_{D_\nu}, \quad (3.274)$$

and restricting equality (3.274) to $\theta+\mu=0$ we obtain that $Q_\lambda = A_\lambda$ on $\{\theta+\mu=0\}\cap D_{\lambda,\nu}$. Further to that restricting equality (3.274) on D_λ we obtain that $A_\lambda|_{D_{\lambda,\nu}} = A_\nu|_{D_{\lambda,\nu}}$.

By Proposition 3.5.16 we have that $\frac{J_\nu}{\theta} = \frac{J_\theta}{\nu}$ on $D_{\nu,\theta}$. Therefore using formulae (3.250), (3.251) we have

$$\lambda\mu(\lambda+\mu)K_\theta|_{D_{\nu,\theta}} = K_\nu|_{D_{\nu,\theta}},$$

which implies that $A_\theta|_{D_{\nu,\theta}} = A_\nu|_{D_{\nu,\theta}}$ using formulae (3.253), (3.254). Further on, by Proposition 3.5.16 we have $\frac{J_\nu}{\mu} = -\frac{J_\mu}{\nu}$ on $D_{\mu,\nu}$. Similarly to above we obtain $A_\mu|_{D_{\mu,\nu}} = -A_\nu|_{D_{\mu,\nu}}$. Therefore the statement follows. $\square$

Lemma 3.6.49. *The polynomial R from formula* (3.253) *satisfies*

$$\left(\nu^{-3}R|_{D_{\lambda,\theta,\mu}}\right)\Big|_D = 4A_\nu|_D. \quad (3.275)$$

Proof. Let us consider an orthonormal coordinate system y_i, $(1 \le i \le n)$, where a vector $y \in \mathbb{C}^n$ has coordinates

$$y_1 = \frac{1}{\sqrt{2}}(\lambda+\nu)(y),\ y_2 = \frac{1}{\sqrt{2}}(\theta+\nu)(y),\ y_3 = \frac{1}{\sqrt{2}}(\mu+\nu)(y),\ y_4 = \frac{1}{\sqrt{2}}(\lambda+\nu+\theta+\mu)(y).$$

We consider the Taylor expansion of R in the variables y_i, $(1 \le i \le 4)$. Let us note that

$$D = \{x | y_1 = y_2 = y_3 = y_4 = 0\}.$$

Consider restriction of the polynomial R on D by taking first $y_2 = 0$. It follows from

(3.256) that
$$R\big|_D = 0. \tag{3.276}$$

Further to that, let us apply formula (3.264) where we note that $s_\nu\beta = 0$ can be written as $y_4 = 0$. The polynomial $R\big|_{y_4=0}$ has the form of a cubic polynomial in y_1, y_2, y_3 times another polynomial. Hence
$$\partial_{y_i} R\big|_D = \partial_{y_i}\partial_{y_j} R\big|_D = 0,$$

for any $i, j = 1, 2, 3$. Let us now use formula (3.258) for $R\big|_{y_3=0}$. It is easy to see that derivatives $\partial_{y_k}\partial_{y_4} R\big|_{y_3=0}$, $\partial^k_{y_4} R\big|_{y_3=0}$, $(k = 1, 2)$ after further restriction on $\lambda = \theta = 0$ have the form of a polynomial of order $3 - k$ in y_1, y_2, y_4 variables times another polynomial. Hence
$$\partial_{y_k}\partial_{y_4} R\big|_D = \partial^k_{y_4} R\big|_D = 0.$$

Similarly, using formula (3.256) it follows that $\partial_{y_3}\partial_{y_4} R\big|_D = 0$.

Let us now compute the third order terms in the Taylor expansion for R. We present some of these calculations while the other terms are similarly computed. By formula (3.257) we have

$$\begin{aligned} 2\sqrt{2}\,\partial^3_{y_1} R\big|_D = \partial^3_{\lambda+\nu} R\big|_D = \partial^3_{\lambda+\nu}\ R|_{s_\nu(D_\theta)}\Big|_D &= \partial^3_{\lambda+\nu}\big[-\lambda\mu\nu^{-1}(\lambda+\nu+\mu)^2 A_\nu\big]\big|_D \\ &+ (\mu+\nu)\partial^3_{\lambda+\nu}\big[\nu^{-1}(\lambda+\nu)(\lambda+\nu+\mu)(\lambda+\mu)s_\nu(A_\theta)\big]\big|_D\,, \end{aligned}$$

since $(\lambda+\nu, \mu+\nu) = 0$. Let us rearrange the function inside the first derivative by using relation $\lambda\mu\nu^{-1} = \lambda(\mu+\nu)\nu^{-1} - \lambda$. It follows by restricting at $\mu + \nu = 0$ at first that

$$\partial^3_{\lambda+\nu} R\big|_D = 6A_\nu|_D\,, \tag{3.277}$$

since $(\lambda+\nu, \lambda) = 1$. Similarly, using formulae (3.256), (3.258), (3.264) we obtain $\partial^3_{y_i} R$, $(i = 2, 3, 4)$ on D:
$$\partial^3_{\theta+\nu} R\big|_D = \partial^3_{\mu+\nu} R\big|_D = \partial^3_{s_\nu(\beta)} R\big|_D = 6A_\nu|_D\,. \tag{3.278}$$

Let us now consider mixed partial derivatives of R. Using (3.264) we have

$$2\sqrt{2}\,\partial^2_{y_1}\partial_{y_2} R\big|_D = \partial^2_{\lambda+\nu}\partial_{\theta+\nu} R\big|_D = \partial^2_{\lambda+\nu}\partial_{\theta+\nu}\ R|_{s_\nu(\beta)=0}\Big|_D\,,$$

that is

$$\begin{aligned} 2\sqrt{2}\,\partial^2_{y_1}\partial_{y_2} R\big|_D &= \partial^2_{\lambda+\nu}\partial_{\theta+\nu}\big[-\theta\mu(\nu+\theta+\mu)A_\nu\big]\big|_D \\ &+ (\mu+\nu)\partial^2_{\lambda+\nu}\partial_{\theta+\nu}\big[(\lambda+\nu)(\theta+\nu)s_\nu(Q_\lambda)\big]\big|_D \\ &= -\,2A_\nu|_D\,, \end{aligned}$$

since $(\theta+\nu, \mu+\nu) = 0$. Similarly, we obtain all other derivatives of the form $\partial^2_{y_i}\partial_{y_j} R\big|_D$

$(1 \leq i, j \leq 3)$,

$$\begin{aligned}\partial^2_{\lambda+\nu}\partial_{\mu+\nu}R\big|_D &= \partial^2_{\theta+\nu}\partial_{\lambda+\nu}R\big|_D = \partial^2_{\theta+\nu}\partial_{\mu+\nu}R\big|_D \\ &= \partial^2_{\mu+\nu}\partial_{\lambda+\nu}R\big|_D = \partial^2_{\mu+\nu}\partial_{\theta+\nu}R\big|_D \\ &= -\,2A_\nu|_D\,.\end{aligned}$$

Further to that we obtain derivatives $\partial^2_{y_1}\partial_{y_4}R\big|_D$ from (3.258) by specialising it at $\theta+\nu = 0 = y_2$, and similarly for $\partial^2_{y_i}\partial_{y_4}R\big|_D$, $(i = 2, 3)$:

$$\partial^2_{\lambda+\nu}\partial_{s_\nu(\beta)}R\big|_D = \partial^2_{\theta+\nu}\partial_{s_\nu(\beta)}R\big|_D = \partial^2_{\mu+\nu}\partial_{s_\nu(\beta)}R\big|_D = 6\ A_\nu|_D\,.$$

In the same way we find derivatives $\partial^2_{y_4}\partial_{y_i}R\big|_D$, $1 \leq i \leq 3$:

$$\partial^2_{s_\nu(\beta)}\partial_{\lambda+\nu}R\big|_D = \partial^2_{s_\nu(\beta)}\partial_{\theta+\nu}R\big|_D = \partial^2_{s_\nu(\beta)}\partial_{\mu+\nu}R\big|_D = 6\ A_\nu|_D\,.$$

Furthermore, using formula (3.264) we find $\partial_{y_1}\partial_{y_2}\partial_{y_3}R|_D$:

$$\partial_{\lambda+\nu}\partial_{\theta+\nu}\partial_{\mu+\nu}R|_D = \partial_{\lambda+\nu}\partial_{\theta+\nu}\partial_{\mu+\nu}\ R|_{s_\nu(\beta)=0}\Big|_D = 8Q_\lambda + 2A_\nu|_D\,.$$

And using formula (3.258) we find $\partial_{y_1}\partial_{y_2}\partial_{y_4}R|_D$:

$$\partial_{\lambda+\nu}\partial_{\theta+\nu}\partial_{s_\nu(\beta)}R\big|_D = \partial_{\lambda+\nu}\partial_{\theta+\nu}\partial_{s_\nu(\beta)}\ R|_{s_\nu(D_\mu)}\Big|_D = 10A_\nu + 8A_\mu|_D\,.$$

Finally, symmetry considerations $(\theta \leftrightarrow \mu), (\lambda \leftrightarrow \mu)$ give by Lemma 3.6.48 $\partial_{y_1}\partial_{y_3}\partial_{y_4}R|_D$, $\partial_{y_2}\partial_{y_3}\partial_{y_4}R|_D$:

$$\partial_{\lambda+\nu}\partial_{\mu+\nu}\partial_{s_\nu(\beta)}R\big|_D = 10A_\nu - 8A_\theta|_D\,,$$

$$\partial_{\theta+\nu}\partial_{\mu+\nu}\partial_{s_\nu(\beta)}R\big|_D = 10A_\nu - 8A_\lambda|_D\,.$$

Let us find R on $D_{\lambda,\mu,\theta}$ using its Taylor series near D. We have that

$$R = \frac{1}{3!}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} y_iy_jy_k\partial_{y_i}\partial_{y_j}\partial_{y_k}R\big|_D + \mathcal{O}, \tag{3.279}$$

where $\mathcal{O}$ denotes the higher order terms in y_1, y_2, y_3, y_4. Note that D is the subspace of $D_{\lambda,\mu,\theta}$ which is given by

$$y_1 = y_2 = y_3 = y_4 = \frac{\nu}{\sqrt{2}}. \tag{3.280}$$

Then by Lemma 3.6.48 and collecting the derivatives found above we get that the Taylor expansion (3.279) takes the following form on the space (3.280):

$$R|_{D_{\lambda,\mu,\theta}} = 4A_\nu\nu^3\big|_{D_{\lambda,\mu,\theta}} + \widetilde{\mathcal{O}}, \tag{3.281}$$

where $\widetilde{\mathcal{O}} = \mathcal{O}|_{D_{\lambda,\mu,\theta}}$ is a polynomial in $\nu, y_5, \ldots, y_n$ which is divisible by ν^4. Therefore the statement follows. $\square$

By Theorem 3.5.9 the determinant of the restricted Saito metric is given by

$$\det \eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D , \tag{3.282}$$

where $\widehat{A}$ is given by (3.222) and $\widehat{J}$ is defined by $J = \mu\lambda\nu\theta\widehat{J}$. We are going to find $\det \eta_D$ given by (3.282) by restricting the right-hand side to $D_{\lambda,\theta,\mu}$ first, and then to $\nu = 0$.

Proposition 3.6.50. *The matrix entries $\widehat{a}_{ij}$, $(0 \leq i, j \leq 3)$ are well-defined generically on $D_{\lambda,\theta,\mu}$. Furthermore, the entries $\widehat{a}_{ij}$ which are non-zero on $D_{\lambda,\mu,\theta}$ have the following form on $D_{\lambda,\mu,\theta}$:*

$$\widehat{a}_{00} = -\frac{K_\mu}{\Pi}\nu^{-5}, \qquad \widehat{a}_{11} = \frac{K_\lambda}{\Pi}\nu^{-5}, \qquad \widehat{a}_{33} = \frac{K_\theta}{\Pi}\nu^{-5},$$

and

$$\widehat{a}_{22} = \nu^{-7}\partial_{\omega^\nu}\frac{K_\nu}{\Pi} - \nu^{-8}\frac{9K_\nu}{\Pi}, \qquad \widehat{a}_{12} = \widehat{a}_{21} = -\frac{1}{2}\nu^{-4}\partial_{\omega^\nu}\frac{K_\lambda}{\Pi} + \nu^{-5}\frac{5K_\lambda}{2\Pi},$$
$$\widehat{a}_{23} = \widehat{a}_{32} = -\frac{1}{2}\nu^{-4}\partial_{\omega^\nu}\frac{K_\theta}{\Pi} + \nu^{-5}\frac{5K_\theta}{2\Pi}, \qquad \widehat{a}_{02} = \widehat{a}_{20} = \frac{1}{2}\nu^{-4}\partial_{\omega^\nu}\frac{K_\mu}{\Pi} - \nu^{-5}\frac{5K_\mu}{2\Pi}.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\mu, \lambda, \nu, \theta\}$. It is easy to see that $\nu\alpha\partial_{\omega^\alpha}\frac{J_\nu}{J}$ vanishes on D_α, $\alpha = \mu, \theta, \lambda$. Also

$$\nu\alpha\partial_{\omega^\nu}\frac{J_\alpha}{J}\Big|_{D_{\lambda,\mu,\theta}} = \frac{1}{2}\nu\partial_{\omega^\nu}\frac{K_\alpha}{\nu^5\Pi}\Big|_{D_{\lambda,\mu,\theta}}.$$

Then formulae for non-zero matrix entries follow by Proposition 3.6.45. Note also that

$$\alpha\beta\partial_{\omega^\alpha}\frac{J_\beta}{J}\Big|_{D_{\alpha,\beta}} = \alpha\partial_{\omega^\alpha}\frac{\beta J_\beta}{J}\Big|_{D_{\alpha,\beta}} = 0,$$

for all $\alpha, \beta \in \{\lambda, \mu, \theta\}$, $\alpha \neq \beta$ since $\frac{\beta J_\beta}{J}$ is non-singular on $D_{\alpha,\beta}$, which implies that all other matrix entries $\widehat{a}_{01}, \widehat{a}_{03}, \widehat{a}_{13}$ vanish on $D_{\alpha,\beta}$. $\square$

Formula (3.282), Proposition 3.6.50 and row expansion of $\det \widehat{A}$ imply that

$$\det \eta_D = -4\,\nu^{16}\Pi^2\big(\widehat{a}_{33}M_{33} - \widehat{a}_{23}M_{23}\,\big)\big|_{D_{\lambda,\theta,\mu}}\Big|_{D_\nu}, \tag{3.283}$$

where M_{23}, M_{33} are the minors of the matrix $\widehat{A}$. Note that by formula (3.253) and Lemma

3.6.49 we have

$$K_\nu|_{D_{\lambda,\mu,\theta}} = 2\nu^4 A_\nu + \nu R\Big|_{D_{\lambda,\mu,\theta}} = 6\nu^4 A_\nu + \nu\widetilde{\mathcal{O}}\Big|_{D_{\lambda,\mu,\theta}}.$$

It follows that

$$(\nu^{-4}K_\nu|_{D_{\lambda,\mu,\theta}})\Big|_D = 6A_\nu|_D\,, \tag{3.284}$$

and

$$(\nu^{-3}\partial_{\omega^\nu}\frac{K_\nu}{\Pi}\Big|_{D_{\lambda,\mu,\theta}})\Big|_D = 24\frac{A_\nu}{\Pi}\Big|_D. \tag{3.285}$$

In the following lemma we calculate terms from (3.283).

Lemma 3.6.51. *We have*

$$(\nu^{16}\widehat{a}_{33}M_{33}|_{D_{\lambda,\mu,\theta}})\Big|_D = -368\,\frac{A_\nu^4}{\Pi^4}\Big|_D, \tag{3.286}$$

and

$$(\nu^{16}\widehat{a}_{23}M_{23}|_{D_{\lambda,\mu,\theta}})\Big|_D = 64\,\frac{A_\nu^4}{\Pi^4}\Big|_D.$$

Proof. We have that $\widehat{a}_{33}M_{33}|_{D_{\lambda,\mu,\theta}} = \widehat{a}_{33}\big(\widehat{a}_{00}\widehat{a}_{11}\widehat{a}_{22} - \widehat{a}_{00}\widehat{a}_{12}^2 - \widehat{a}_{02}^2\widehat{a}_{11}\big)\big|_{D_{\lambda,\mu,\theta}}$. Let us recall that by (3.254) we have $K_\alpha|_{D_{\lambda,\mu,\theta}} = 2\nu A_\alpha|_{D_{\lambda,\mu,\theta}}$, where $\alpha = \lambda,\mu,\theta$. Then by Proposition 3.6.50, Lemma 3.6.49 and formulae (3.284), (3.285) we obtain

$$\prod_{i=0}^{3}\widehat{a}_{ii}|_{D_{\lambda,\mu,\theta}} = \frac{240A_\mu A_\lambda A_\theta A_\nu}{\Pi^4}\nu^{-16}\Big|_{D_{\lambda,\mu,\theta}} + \mathcal{O}|_{D_{\lambda,\mu,\theta}}\,,$$

where $\mathcal{O}$ is a rational function in $\nu, y_5,\dots,y_n$ with poles of order at most 15 at $\nu = 0$. Then by Lemma 3.6.48 we have

$$(\nu^{16}\prod_{i=1}^{4}\widehat{a}_{ii}|_{D_{\lambda,\mu,\theta}})\Big|_D = \frac{-240A_\nu^4}{\Pi^4}\Big|_D.$$

Similarly, it can be shown that

$$\nu^{16}(\widehat{a}_{33}\big(\widehat{a}_{12}^2\widehat{a}_{00} + \widehat{a}_{02}^2\widehat{a}_{11}\big)|_{D_{\lambda,\mu,\theta}})\Big|_D = \frac{128A_\nu^4}{\Pi^4}\Big|_D,$$

hence (3.286) follows. Moreover,

$$\nu^{16}(\widehat{a}_{23}M_{23}|_{D_{\lambda,\mu,\theta}})\Big|_D = (\nu^{16}\,\widehat{a}_{00}\widehat{a}_{11}\widehat{a}_{23}^2|_{D_{\lambda,\mu,\theta}})\Big|_D = \frac{64A_\nu^4}{\Pi^4}\Big|_D.$$

Therefore the statement follows. ∎

Theorem 3.6.52. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}_{D_\gamma} \setminus \mathcal{A}_{D_\gamma}^D)^4 I(\mathcal{A} \setminus \mathcal{A}^D)^{-2} \tag{3.287}$$

on D for any $\gamma = \lambda, \mu, \theta, \nu$.

Proof. Let us recall that $\det \eta_D$ is given by (3.283). By Lemma 3.6.51 we get that

$$\det \eta_D = 1728 \frac{A_\nu^4}{\Pi^2}\bigg|_D.$$

Then formula (3.287) follows by Corollary 3.5.15 and formulae (3.250), (3.253) for $\gamma = \nu$. Similarly (3.287) follows for $\gamma = \lambda, \mu, \theta$ by Lemma 3.6.48. □

Theorem 3.6.53. *The statement of Main Theorems 1 and 2 is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. The root system $\widehat{\mathcal{R}}$ is a rank 5 subsystem of $\mathcal{R}$.

By Theorem 3.6.52 the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\nu} \setminus \mathcal{A}_{D_\nu}^D)\big|_D$ is

$$|\widehat{\mathcal{A}}_{D_\nu} \setminus \mathcal{A}_{D_\nu}^D| = |\widehat{\mathcal{A}}_{D_\nu}| - 7, \tag{3.288}$$

and the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 12. \tag{3.289}$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{\mathcal{R}} = A_5$ or $\widehat{\mathcal{R}} = D_5$. Note that $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}| - h + 1$, where h is the Coxeter number of $\widehat{\mathcal{R}}$ [71]. Hence formula (3.288) implies that

$$|\widehat{\mathcal{A}}_{D_\nu} \setminus \mathcal{A}_{D_\nu}^D| = \frac{3h}{2} - 6,$$

and formula (3.289) implies that

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = \frac{5h}{2} - 12.$$

Then it follows from Theorem 3.6.52 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now suppose that $\widehat{R}$ is reducible, that is $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = D_4 \times A_1$. Then we get

$$|\widehat{\mathcal{A}}_{D_\nu} \setminus \mathcal{A}_{D_\nu}^D| = |\widehat{\mathcal{A}}_{D_\nu}| - 7 = 1,$$

since $|\widehat{\mathcal{A}}_{D_\nu}| = |\mathcal{A}_{D_\nu}^D| + 1 = 8$, and

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 12 = 1.$$

Then it follows from Theorem 3.6.52 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. $\square$

Let us now consider the cases where $\mathcal{R}_D$ is a reducible rank 4 root system. We consider first strata of type $A_2 \times A_2$.

Stratum $\mathbf{A_2^2}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type $A_2 \times A_2$. Let us consider the corresponding Coxeter graph

$$A_2^2 : \quad \overset{\mu}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\overset{\lambda}{\bullet} \;\cdots\; \overset{\nu}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\overset{\theta}{\bullet} \tag{3.290}$$

In this case, the Jacobian J can be represented as

$$J = \mu\lambda\nu\theta(\mu+\lambda)(\nu+\theta)\Pi, \tag{3.291}$$

where Π is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$ and Π is non-zero on D. Further on, by Proposition 3.5.15 we have

$$J_\mu = \lambda\nu\theta(\nu+\theta)K_\mu, \tag{3.292}$$

$$J_\lambda = \mu\nu\theta(\nu+\theta)K_\lambda, \tag{3.293}$$

$$J_\nu = \mu\lambda\theta(\mu+\lambda)K_\nu, \tag{3.294}$$

$$J_\theta = \mu\lambda\nu(\mu+\lambda)K_\theta, \tag{3.295}$$

for $K_\alpha \in \mathbb{C}[x]$, $\alpha = \mu, \lambda, \nu, \theta$. We assume without loss of generality that the ordering of simple roots $\mu, \lambda, \nu, \theta$ is such that $n + \sigma^{-1}(\alpha)$ is odd if $\alpha \in \{\lambda, \nu\}$ and that $n + \sigma^{-1}(\alpha)$ is even if $\alpha \in \{\mu, \theta\}$. The following proposition follows from formulae (3.291)-(3.295) and Proposition 3.5.3.

Proposition 3.6.54. *The μ, λ, ν and θ components of the identity field e are given by*

$$e^\mu = \frac{K_\mu}{\mu(\mu+\lambda)\Pi}, \qquad e^\lambda = -\frac{K_\lambda}{\lambda(\mu+\lambda)\Pi},$$
$$e^\nu = -\frac{K_\nu}{\nu(\nu+\theta)\Pi}, \qquad e^\theta = \frac{K_\theta}{\theta(\nu+\theta)\Pi}.$$

By Theorem 3.5.9 the determinant of the restricted Saito metric is given by

$$\det \eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D, \tag{3.296}$$

where $\widehat{A}$ is given by (3.222) and $\widehat{J}$ is defined by $\widehat{J} = (\mu\lambda\nu\theta)^{-1}J$. We are going to find (3.296) by restricting the right-hand side to $D_{\mu,\nu}$ first, then to $\lambda = 0$ and finally to $\theta = 0$.

Proposition 3.6.55. *The matrix entries $\widehat{a}_{ij}$ $(0 \le i \le 3)$ are well-defined generically on $D_{\mu,\nu}$. Furthermore, the entries $\widehat{a}_{ij}$ which are non-zero on $D_{\mu,\nu}$ have the following form on*

$D_{\mu,\nu}$:

$$\widehat{a}_{00} = 2\lambda^{-1}\frac{K_\mu}{\Pi}, \quad \widehat{a}_{01} = \widehat{a}_{10} = -\partial_{\omega^\lambda}\frac{K_\mu}{\Pi} + \lambda^{-1}\frac{K_\mu}{\Pi}, \quad \widehat{a}_{03} = \widehat{a}_{30} = -\theta\lambda^{-1}\partial_{\omega^\theta}\frac{K_\mu}{\Pi},$$

$$\widehat{a}_{11} = 2\partial_{\omega^\lambda}\frac{K_\lambda}{\Pi} - 4\lambda^{-1}\frac{K_\lambda}{\Pi}, \quad \widehat{a}_{12} = \widehat{a}_{21} = \lambda\theta^{-1}\partial_{\omega^\lambda}\frac{K_\nu}{\Pi},$$

$$\widehat{a}_{13} = \widehat{a}_{31} = -\lambda\theta^{-1}\partial_{\omega^\lambda}\frac{K_\theta}{\Pi} + \theta\lambda^{-1}\partial_{\omega^\theta}\frac{K_\lambda}{\Pi},$$

and

$$\widehat{a}_{22} = -2\theta^{-1}\frac{K_\nu}{\Pi}, \quad \widehat{a}_{23} = \widehat{a}_{32} = \partial_{\omega^\theta}\frac{K_\nu}{\Pi} - \theta^{-1}\frac{K_\nu}{\Pi}, \quad \widehat{a}_{33} = -2\partial_{\omega^\theta}\frac{K_\theta}{\Pi} + 4\theta^{-1}\frac{K_\theta}{\Pi}.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\mu, \lambda, \nu, \theta\}$. Thus by Proposition 3.6.54 the statement follows. □

Theorem 3.6.56. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}\setminus\mathcal{A}^D)^{-2}\prod_{\gamma\in\{\mu,\nu\}} I(\mathcal{A}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma})^2 \tag{3.297}$$

on D. The same is true with μ replaced with λ and ν replaced with θ in (3.297)*.*

Proof. Let us collect terms with third-order poles at $\lambda = 0$ in the expansion of $\det \widehat{A}$. By Proposition 3.6.55 such terms are part of the following expression on $D_{\mu,\nu}$:

$$C = \widehat{a}_{22}\big(\widehat{a}_{03}(\widehat{a}_{01}\widehat{a}_{13} - \widehat{a}_{11}\widehat{a}_{03}) - \widehat{a}_{13}(\widehat{a}_{00}\widehat{a}_{13} - \widehat{a}_{01}\widehat{a}_{03})\big). \tag{3.298}$$

By Proposition 3.5.16 we have that $K_\mu|_{D_\mu} = K_\lambda + \lambda P|_{D_\mu}$ for some $P \in \mathbb{C}[x]$. It follows that terms with third-order poles at $\lambda = 0$ in (3.298) cancel each other, thus the function $\lambda^2 C$ is regular on $D_{\mu,\nu,\lambda}$. Moreover, it is easy to see that $\theta^2(\lambda^2\, C|_{D_{\mu,\nu,\lambda}})|_{\theta=0} = 0$. Therefore it follows by formula (3.296) and Proposition 3.6.55 that $\det \widehat{A}$ takes the form

$$\det \eta_D = -\theta^2\lambda^2\Pi^2 \det \widehat{A}\Big|_{D_{\mu,\nu}}\Big|_D = -\theta^2\lambda^2\Pi^2 \det B_1 \det B_2|_{D_{\mu,\nu}}\Big|_D,$$

where $B_1 = (\widehat{a}_{ij})^1_{i,j=0}$ and $B_2 = (\widehat{a}_{ij})^3_{i,j=2}$, since the remaining terms in the expression $\det \widehat{A}\Big|_{D_{\mu,\nu}}$ have poles at $\lambda = 0$ of order at most 1. Note also that $K_\theta|_D = K_\nu|_D$ by Proposition 3.5.16. It follows that

$$\det \eta_D = -\Pi^2\Big(-8\frac{K_\mu K_\lambda}{\Pi^2} - \frac{K_\mu^2}{\Pi^2}\Big)\Big(-8\frac{K_\nu K_\theta}{\Pi^2} - \frac{K_\nu^2}{\Pi^2}\Big)\Big|_D = -81\Pi^{-2}K_\mu^2K_\nu^2|_D,$$

Then the statement follows by Corollary 3.5.15 and formulae (3.292), (3.294). □

Let us now show that the powers of distinct linear forms in (3.297) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.57. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. The root system $\widehat{\mathcal{R}}$ is a rank 5 subsystem of $\mathcal{R}$. The multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma})\Big|_D$, $(\gamma = \mu, \nu)$ is

$$|\widehat{\mathcal{A}}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma}| = |\widehat{\mathcal{A}}_{D_\gamma}| - 4. \tag{3.299}$$

Similarly, the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 6. \tag{3.300}$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{R} = A_5$ or $\widehat{\mathcal{R}} = D_5$. Note that $|\widehat{\mathcal{A}}_{D_\gamma}| = \frac{3h}{2} + 1$ and $|\widehat{\mathcal{A}}| = \frac{5h}{2}$, where h is the Coxeter number of $\widehat{\mathcal{R}}$. Then

$$|\widehat{\mathcal{A}}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma}| = \frac{3h}{2} - 3, \quad \text{and} \quad |\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = \frac{5h}{2} - 6,$$

and it follows from Theorem 3.6.56 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now consider the case where $\widehat{\mathcal{R}}$ is reducible. If $\widehat{\mathcal{R}} = A_3 \times A_2$ we can assume without loss of generality that

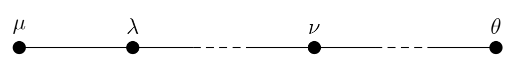
$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda, \beta \rangle) \sqcup (\mathcal{R} \cap \langle \nu, \theta \rangle) = A_3 \times A_2.$$

Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to A_3. Then $|\widehat{\mathcal{A}}_{D_\nu}| = |\widetilde{\mathcal{A}}| + 1$ and $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}_{D_\mu}| + 3$. It follows by formulae (3.299), (3.300) and Theorem 3.6.56 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required.

Finally, consider the case when $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_2^2 \times A_1$. Then $|\widehat{\mathcal{A}}_{D_\mu}| = |\widehat{\mathcal{A}}_{D_\nu}| = 5$ and $|\widehat{\mathcal{A}}| = 7$. Thus the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. □

Stratum $\mathbf{A_2 \times A_1^2}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type $A_2 \times A_1^2$. Let us consider the corresponding Coxeter graph

$A_2 \times A_1^2$:



The Jacobian J can be represented as

$$J = \mu\lambda\nu\theta(\mu + \lambda)\Pi, \tag{3.301}$$

where Π is proportional to $I(\mathcal{A}\setminus\mathcal{A}^D)$ and Π is non-zero on D. By Proposition 3.5.15 we get

$$J_\mu = \lambda\nu\theta K_\mu, \tag{3.302}$$
$$J_\lambda = \mu\nu\theta K_\lambda, \tag{3.303}$$
$$J_\nu = \mu\lambda\theta(\mu+\lambda)K_\nu, \tag{3.304}$$
$$J_\theta = \mu\lambda\nu(\mu+\lambda)K_\theta \tag{3.305}$$

for $K_\alpha \in \mathbb{C}[x]$, $\alpha = \mu, \lambda, \nu, \theta$. We assume without loss of generality that the ordering is such that $n+\sigma^{-1}(\alpha)$ is even if $\alpha \in \{\mu, \nu, \theta\}$ and that $n+\sigma^{-1}(\lambda)$ is odd. The following proposition follows using formulae (3.301)-(3.305) and Proposition 3.5.3.

Proposition 3.6.58. *The μ, λ, ν and θ components of the identity field e are given by*

$$e^\mu = \frac{K_\mu}{\mu(\mu+\lambda)\Pi}, \qquad e^\nu = \frac{K_\nu}{\nu\Pi},$$
$$e^\lambda = -\frac{K_\lambda}{\lambda(\mu+\lambda)\Pi}, \qquad e^\theta = \frac{K_\theta}{\theta\Pi}.$$

By Theorem 3.5.9 the determinant of the restricted Saito metric is given by

$$\det\eta_D = -\widehat{J}^2 \det\widehat{A}\Big|_D, \tag{3.306}$$

where $\widehat{A}$ is given by (3.222) and $\widehat{J}$ is defined by $\widehat{J} = (\mu\lambda\nu\theta)^{-1}J$. We find (3.306) by restricting the right-hand side first on $D_{\mu,\nu,\theta}$ and finally to $\lambda = 0$.

Proposition 3.6.59. *The matrix entries $\widehat{a}_{ij}$ $(0 \le i, j \le 3)$ are well-defined generically on $D_{\mu,\nu,\theta}$. Furthermore, the entries $\widehat{a}_{ij}$ which are non-zero on $D_{\mu,\nu,\theta}$ have the following form on $D_{\mu,\nu,\theta}$:*

$$\widehat{a}_{00} = 2\frac{K_\mu}{\Pi}\lambda^{-1}, \quad \widehat{a}_{01} = \widehat{a}_{10} = -\partial_{\omega^\lambda}\frac{K_\mu}{\Pi} + \frac{K_\mu}{\Pi}\lambda^{-1}, \quad \widehat{a}_{11} = 2\partial_{\omega^\lambda}\frac{K_\lambda}{\Pi} - 4\frac{K_\lambda}{\Pi}\lambda^{-1},$$
$$\widehat{a}_{12} = \widehat{a}_{21} = -\lambda\partial_{\omega^\lambda}\frac{K_\nu}{\Pi}, \quad \widehat{a}_{13} = \widehat{a}_{31} = -\lambda\partial_{\omega^\lambda}\frac{K_\theta}{\Pi}, \quad \widehat{a}_{22} = 2\frac{K_\nu}{\Pi}, \quad \widehat{a}_{33} = 2\frac{K_\theta}{\Pi}.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha, \beta \in \{\mu, \lambda, \nu, \theta\}$. Thus by Proposition 3.6.58 the statement follows. □

We have the following statement.

Theorem 3.6.60. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}\setminus\mathcal{A}^D)^{-2} I(\mathcal{A}_{D_\mu}\setminus\mathcal{A}^D_{D_\mu})^2 \prod_{\gamma\in\{\nu,\theta\}} I(\mathcal{A}_{D_\gamma}\setminus\mathcal{A}^D_{D_\gamma}) \tag{3.307}$$

on D. The same is true if μ is replaced with λ in (3.307).

Proof. By formula (3.306), Proposition 3.6.59 and row expanding $\det \widehat{A}$ we obtain

$$\det \eta_D = -\lambda^2 \Pi^2 \ \det B_1 \det B_2|_{D_{\mu,\nu,\theta}}\Big|_D ,$$

where $B_1 = (\widehat{a}_{ij})_{i,j=0}^1$ and $B_2 = (\widehat{a}_{ij})_{i,j=2}^3$. By Proposition 3.5.16 we have $K_\mu|_D = K_\lambda|_D$. Therefore

$$\det \eta_D = 36\Pi^{-2} K_\mu^2 K_\nu K_\theta\big|_D .$$

Then the statement follows by Corollary 3.5.15 and formulae (3.302), (3.304), (3.305). □

Let us now show that the powers of distinct linear forms in (3.307) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.61. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. The multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\mu} \setminus \mathcal{A}^D_{D_\mu})\Big|_D$ is

$$|\widehat{\mathcal{A}}_{D_\mu} \setminus \mathcal{A}^D_{D_\mu}| = |\widehat{\mathcal{A}}_{D_\mu}| - 3, \tag{3.308}$$

and the multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\alpha} \setminus \mathcal{A}^D_{D_\alpha})\big|_D$, for $\alpha = \nu, \theta$ is

$$|\widehat{\mathcal{A}}_{D_\alpha} \setminus \mathcal{A}^D_{D_\alpha}| = |\widehat{\mathcal{A}}_{D_\alpha}| - 4. \tag{3.309}$$

Similarly, the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 5. \tag{3.310}$$

Let us consider first the case where $\widehat{\mathcal{R}}$ is irreducible. Then $|\widehat{\mathcal{A}}_{D_\alpha}| = |\widehat{\mathcal{A}}| - h + 1$, for any $\alpha = \mu, \nu, \theta$, where h is the Coxeter number of $\widehat{\mathcal{R}}$. We also have $|\widehat{\mathcal{A}}| = \frac{5h}{2}$. It follows from formulae (3.308)–(3.310) and Theorem 3.6.60 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now consider the case where $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda, \beta \rangle) \sqcup \{\pm\nu\} \sqcup \{\pm\theta\} = A_3 \times A_1 \times A_1.$$

Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to A_3. Then $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}| + 1 = 7$ and $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}_{D_\mu}| + 2 = 5$. We also have $|\widehat{\mathcal{A}}| = 8$. It follows by formulae (3.308)–(3.310) and Theorem 3.6.60 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required.

Let us suppose that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda \rangle) \sqcup (\mathcal{R} \cap \langle \nu, \theta, \beta \rangle) = A_2 \times A_3,$$

and let $\mathcal{A}'$, $\widetilde{\mathcal{A}}$ be the arrangements corresponding to A_2 and A_3 respectively. Then $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}| + 1 = 7$ and $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}_{D_\nu}| + |\mathcal{A}'| = 6$. Further to that, $|\widehat{\mathcal{A}}| = 9$. It follows from Theorem 3.6.60 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required.

Let us now assume without loss of generality that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda, \nu, \beta \rangle) \sqcup \{\pm\theta\} = \widetilde{\mathcal{R}} \times A_1, \tag{3.311}$$

where $\widetilde{\mathcal{R}} = A_4$ or $\widetilde{\mathcal{R}} = D_4$. Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to $\widetilde{\mathcal{R}}$. We have $|\widehat{\mathcal{A}}_{D_\mu}| = |\widehat{\mathcal{A}}_{D_\nu}| = |\widetilde{\mathcal{A}}_{D_\mu}| + 1 = h + 2$, where h is the Coxeter number of $\widetilde{\mathcal{R}}$, and $|\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}| = 2h$. Note also that $|\widehat{\mathcal{A}}| = 2h+1$. It follows from Theorem 3.6.60 that the multiplicity m_β of $\beta|_D$ in $\det \eta_D$ is

$$m_\beta = 2(h-1) + (h-2) + (2h-4) - 2(2h-4) = h,$$

as required.

Let us now suppose that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda \rangle) \sqcup (\mathcal{R} \cap \langle \nu, \beta \rangle) \sqcup \{\pm\theta\} = A_2 \times A_2 \times A_1. \tag{3.312}$$

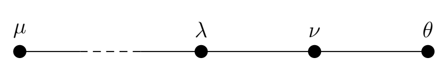
We have $\widehat{\mathcal{A}}_{D_\mu} = \{D_{\lambda,\mu}, D_{\nu,\mu}, D_{\beta,\mu}, D_{\nu+\epsilon\beta,\mu}, D_{\theta,\mu}\}$, where either $\epsilon = 1$ or $\epsilon = -1$, $\widehat{\mathcal{A}}_{D_\nu} = \{D_{\mu,\nu}, D_{\lambda,\nu}, D_{\mu+\lambda,\nu}, D_{\beta,\nu}, D_{\theta,\nu}\}$. Note also that $|\widehat{\mathcal{A}}_{D_\theta}| = 6$ and $|\widehat{\mathcal{A}}| = 7$. It follows from Theorem 3.6.60 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 3, which is the Coxeter number of A_2, as required. Note that the case where θ and ν are swapped in (3.312) is similar.

Finally, let us consider the case where $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_2 \times A_1^3$. Then $\widehat{\mathcal{A}}_{D_\mu} = \{D_{\lambda,\mu}, D_{\nu,\mu}, D_{\theta,\mu}, D_{\beta,\mu}\}$ and

$$\widehat{\mathcal{A}}_{D_\nu} = \{D_{\mu,\nu}, D_{\lambda,\nu}, D_{\mu+\lambda,\nu}, D_{\beta,\nu}, D_{\theta,\nu}\}. \tag{3.313}$$

Note that $\widehat{\mathcal{A}}_{D_\theta}$ is given by (3.313) where ν is swapped with θ and that $|\widehat{\mathcal{A}}| = 6$. Then it follows from Theorem 3.6.60 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. $\square$

Stratum $\mathbf{A_1 \times A_3}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type $A_1 \times A_3$. Let us consider the corresponding Coxeter graph

$A_1 \times A_3$: 

(3.314)

In this case the Jacobian J can be represented as

$$J = \mu\lambda\nu\theta(\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)\Pi, \tag{3.315}$$

where Π is proportional to $I(\mathcal{A}\setminus\mathcal{A}^D)$ and Π is non-zero on D. By Proposition 3.5.15 we get

$$J_\mu = \lambda\nu\theta(\lambda+\nu)(\nu+\theta)(\lambda+\nu+\theta)K_\mu, \tag{3.316}$$
$$J_\lambda = \mu\nu\theta(\nu+\theta)K_\lambda, \tag{3.317}$$
$$J_\nu = \mu\lambda\theta K_\nu, \tag{3.318}$$
$$J_\theta = \mu\lambda\nu(\lambda+\nu)K_\theta, \tag{3.319}$$

for $K_\mu, K_\lambda, K_\nu, K_\theta \in \mathbb{C}[x]$. We assume that the ordering of the simple roots λ, ν, θ is the same as in our considerations for type A_3 strata in Subsection 3.6.4. Further to that we assume without loss of generality that $n+\sigma^{-1}(\mu)$ is even. The following statement follows by formulae (3.315)–(3.319) and Proposition 3.5.3.

Proposition 3.6.62. *The μ component of the identity field e is given by $e^\mu = (\mu\Pi)^{-1}K_\mu$. Furthermore, the λ, ν and θ components are given in Proposition 3.6.13, where K_α, $\alpha = \lambda, \nu, \theta$ and Π are defined by formulae* (3.315)–(3.319).

By Theorem 3.5.9 the determinant of the restricted Saito metric is given by

$$\det\eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D, \tag{3.320}$$

where the matrix $\widehat{A}$ is given by (3.222) and $\widehat{J}$ is defined by $\widehat{J} = (\mu\lambda\nu\theta)^{-1}J$. We are going to find $\det\eta_D$ by restricting the right-hand side of (3.320) first on $D_{\mu,\nu}$ followed by restriction to $\theta=0$ and then to $\lambda=0$.

Proposition 3.6.63. *The matrix entries $\widehat{a}_{ij}$ $(0\le i,j\le 3)$ are well-defined generically on $D_{\mu,\nu}$. Furthermore, the non-zero entries $\widehat{a}_{0i}$, $(i=0,\dots,3)$ have the following form on $D_{\mu,\nu}$:*

$$\widehat{a}_{00} = 2\frac{K_\mu}{\Pi}, \quad \widehat{a}_{01} = \widehat{a}_{10} = -\lambda\partial_{\omega^\lambda}\frac{K_\mu}{\Pi}, \quad \widehat{a}_{03} = \widehat{a}_{30} = -\theta\partial_{\omega^\theta}\frac{K_\mu}{\Pi},$$

and the remaining matrix entries $\widehat{a}_{ij}$ $(1\le i,j\le 3)$ on $D_{\mu,\nu}$ are given by formulae (3.114)-(3.119) *restricted on $D_{\mu,\nu}$.*

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha}e^\beta - \partial_{\omega^\beta}e^\alpha$ for $\alpha,\beta\in\{\mu,\lambda,\nu,\theta\}$. By Proposition 3.6.62 the statement follows. □

Theorem 3.6.64. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A}\setminus\mathcal{A}^D)^{-2}I(\mathcal{A}_{D_\theta}\setminus\mathcal{A}^D_{D_\theta})^3 I(\mathcal{A}_{D_\mu}\setminus\mathcal{A}^D_{D_\mu}) \tag{3.321}$$

on D.

Proof. Let us consider row expansion for $\det \widehat{A}$. Since $\widehat{a}_{02}|_{D_{\mu,\nu}} = 0$ we have on $D_{\mu,\nu}$ that

$$\det \widehat{A} = \widehat{a}_{00} M_{00} - \widehat{a}_{01} M_{01} - \widehat{a}_{03} M_{03},$$

where M_{ij} is the (i,j) minor of $\widehat{A}$ and $M_{00} = \det A$ is given by formula (3.120). It follows by Proposition 3.6.63 that $\theta^2 M_{0j}$, $j = 1, 3$ is well-defined generically on $D_{\mu,\nu,\theta}$ and has poles at $\lambda = 0$ of order at most 3 on $D_{\mu,\nu,\theta}$. Then by Proposition 3.6.63 and formula (3.320) we have

$$\det \eta_D = -\left.\widehat{J}^2 \det \widehat{A}\right|_D = -(\lambda^2\theta^2(\lambda+\theta)^2\Pi^2 \left.\det \widehat{A}\right|_{D_{\mu,\nu,\theta}})|_D = -\widehat{a}_{00}\Pi^2\lambda^4\theta^2 \left.\det A\right|_D .$$

Note that $\lambda^4\theta^2 \det A|_D$ is found in Theorem 3.6.23. Then $\det \eta_D = -128\Pi^{-2}K_\theta^3 K_\mu|_D$. The statement follows by Corollary 3.5.15 and formulae (3.316), (3.319). □

Let us now show that the powers of distinct linear forms in (3.307) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.65. *The statement of Main Theorems* 1 *and* 2 *is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. The root system $\widehat{\mathcal{R}}$ is a rank 5 subsystem of $\mathcal{R}$. The multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta})|_D$ and in $\left.I(\mathcal{A}_{D_\mu} \setminus \mathcal{A}^D_{D_\mu})\right|_D$ is given respectively by

$$|\widehat{\mathcal{A}}_{D_\theta} \setminus \mathcal{A}^D_{D_\theta}| = |\widehat{\mathcal{A}}_{D_\theta}| - 4,$$

and

$$|\widehat{\mathcal{A}}_{D_\mu} \setminus \mathcal{A}^D_{D_\mu}| = |\widehat{\mathcal{A}}_{D_\mu}| - 6.$$

Similarly, the multiplicity of $\beta|_D$ in $\left.I(\mathcal{A} \setminus \mathcal{A}^D)\right|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 7.$$

Let us suppose firstly that $\widehat{\mathcal{R}}$ is irreducible, that is $\widehat{\mathcal{R}} = A_5$ or $\widehat{\mathcal{R}} = D_5$. Then $|\widehat{\mathcal{A}}_{D_\mu}| = |\widehat{\mathcal{A}}_{D_\theta}| = \frac{3h}{2} + 1$ and $|\widehat{\mathcal{A}}| = \frac{5h}{2}$, where h the Coxeter number of $\widehat{\mathcal{R}}$. Thus, it follows from Theorem 3.6.64 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, which is the Coxeter number of $\widehat{\mathcal{R}}$, as required.

Let us now consider the case where $\widehat{\mathcal{R}}$ is reducible and suppose firstly that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \lambda, \nu, \theta, \beta \rangle) \sqcup \{\pm\mu\} = \widetilde{\mathcal{R}} \times A_1,$$

where $\widetilde{\mathcal{R}} = A_4$ or $\widetilde{\mathcal{R}} = D_4$, with Coxeter number h. Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to $\widetilde{\mathcal{R}}$. Notice that $|\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}_{D_\theta}| + 1 = h + 2$, $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}| = 2h$ and $|\widehat{\mathcal{A}}| = 2h + 1$. It follows from Theorem 3.6.64 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h as required. Let us now assume that $\widehat{\mathcal{R}}$ is given by

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \beta \rangle) \sqcup (\mathcal{R} \cap \langle \lambda, \nu, \theta \rangle) = A_2 \times A_3,$$

and let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to the root system A_3. Then $|\widehat{\mathcal{A}}_{D_\theta}| = 6$, $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}| + 1 = 7$ and $|\widehat{\mathcal{A}}| = 9$. Thus, it follows from Theorem 3.6.64 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 3, which is the Coxeter number of A_2, as required.

Finally, let us consider the case where

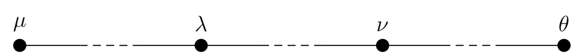
$$\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_1^2 \times A_3.$$

We have $|\widehat{\mathcal{A}}_{D_\theta}| = 5$, $|\widehat{\mathcal{A}}_{D_\mu}| = 7$ and $|\widehat{\mathcal{A}}| = 8$. It follows from Theorem 3.6.64 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. □

Finally, we consider a stratum of type A_1^4.

Stratum $\mathbf{A_1^4}$. Let $\mathcal{R}_D = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta \rangle$ be a subsystem of $\mathcal{R}$ of type A_1^4. Let us consider the corresponding Coxeter graph

A_1^4 :



The Jacobian can be represented as

$$J = \mu\lambda\nu\theta\Pi, \tag{3.322}$$

where Π is proportional to $I(\mathcal{A} \setminus \mathcal{A}^D)$ and Π is non-zero on D. Let $S = \{\mu, \lambda, \nu, \theta\} \subset \Delta$. By Proposition 3.5.15 we have for any $\alpha \in S$ that

$$J_\alpha = K_\alpha \prod_{\substack{\gamma \in S \\ \gamma \neq \alpha}} \gamma, \tag{3.323}$$

for $K_\alpha \in \mathbb{C}[x]$. We assume without loss of generality that the ordering of simple roots $\mu, \lambda, \nu, \theta$ is such that $n + \sigma^{-1}(\alpha)$ is even for any $\alpha \in S$. The following proposition follows using formulae (3.322), (3.323) and Proposition 3.5.3.

Proposition 3.6.66. *The μ, λ, ν and θ components of the identity field e are given by $e^\alpha = (\alpha\Pi)^{-1}K_\alpha$, $\alpha \in S$.*

By Theorem 3.5.9 the determinant of the restricted Saito metric is given by

$$\det \eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D, \tag{3.324}$$

where the matrix $\widehat{A}$ is given by (3.222) and $\widehat{J}$ is defined by $\widehat{J} = (\mu\lambda\nu\theta)^{-1}J$.

Proposition 3.6.67. *The matrix entries $\widehat{a}_{ij}$, $(0 \le i, j \le 3)$ are well-defined generically on D. In particular, the entries $\widehat{a}_{ij}$ which are non-zero on D have the following form on D:*

$$\widehat{a}_{00} = 2\Pi^{-1}K_\mu, \quad \widehat{a}_{11} = 2\Pi^{-1}K_\lambda, \quad \widehat{a}_{22} = 2\Pi^{-1}K_\nu, \quad \widehat{a}_{33} = 2\Pi^{-1}K_\theta.$$

Proof. By Theorem 3.5.5 we have $\eta^{\alpha\beta} = -\partial_{\omega^\alpha} e^\beta - \partial_{\omega^\beta} e^\alpha$ for $\alpha, \beta \in \{\mu, \lambda, \nu, \theta\}$. By Proposition 3.6.66 the statement follows. □

Theorem 3.6.68. *The determinant of the metric η_D is proportional to*

$$I(\mathcal{A} \setminus \mathcal{A}^D)^{-2} \prod_{\gamma \in S} I(\mathcal{A}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma}) \tag{3.325}$$

on D.

Proof. It follows from formula (3.324) and Proposition (3.6.67) that

$$\det \eta_D = -\widehat{J}^2 \det \widehat{A}\Big|_D = -16\Pi^{-2} \prod_{\gamma \in S} K_\gamma|_D .$$

The statement follows by Corollary 3.5.15 and formulae (3.322), (3.323). □

Let us now show that the powers of distinct linear forms in (3.325) are non-negative and are equal to the corresponding Coxeter numbers.

Theorem 3.6.69. *The statement of Main Theorems 1 and 2 is true.*

Proof. Let $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ and let $\widehat{\mathcal{R}} = \mathcal{R}_{D,\beta}$ be the root system $\widehat{\mathcal{R}} = \mathcal{R} \cap \langle \mu, \lambda, \nu, \theta, \beta \rangle$ with the corresponding arrangement $\widehat{\mathcal{A}}$. The multiplicity of $\beta|_D$ in $I(\mathcal{A}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma})\Big|_D$, for any $\gamma \in S$ is given by

$$|\widehat{\mathcal{A}}_{D_\gamma} \setminus \mathcal{A}^D_{D_\gamma}| = |\widehat{\mathcal{A}}_{D_\gamma}| - 3. \tag{3.326}$$

Similarly, the multiplicity of $\beta|_D$ in $I(\mathcal{A} \setminus \mathcal{A}^D)\big|_D$ is

$$|\widehat{\mathcal{A}} \setminus \mathcal{A}^D| = |\widehat{\mathcal{A}}| - 4. \tag{3.327}$$

Let us consider first the case where $\widehat{\mathcal{R}}$ is irreducible. Then $|\widehat{\mathcal{A}}_{D_\gamma}| = |\widehat{\mathcal{A}}| - h + 1$, for any $\gamma \in S$, where h is the Coxeter number of $\widehat{\mathcal{R}}$. Note also that $|\widehat{\mathcal{A}}| = \frac{5h}{2}$. It follows from formulae (3.326), (3.327) and Theorem 3.6.68 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us now assume without loss of generality that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \lambda, \nu, \theta, \beta \rangle) \sqcup \{\pm\mu\} = \widetilde{\mathcal{R}} \times A_1,$$

where $\widetilde{\mathcal{R}} = A_4$ or $\widetilde{\mathcal{R}} = D_4$, with Coxeter number h. Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to $\widetilde{\mathcal{R}}$. Notice that $|\widehat{\mathcal{A}}_{D_\gamma}| = |\widetilde{\mathcal{A}}_{D_\gamma}| + 1 = h + 2$, for any $\gamma \in S \setminus \{\mu\}$, $|\widehat{\mathcal{A}}_{D_\mu}| = |\widetilde{\mathcal{A}}| = 2h$ and $|\widehat{\mathcal{A}}| = 2h + 1$. It follows from Theorem 3.6.68 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is h, as required.

Let us assume without loss of generality that $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \beta \rangle) \sqcup \{\pm\lambda\} \sqcup \{\pm\nu\} \sqcup \{\pm\theta\} = A_2 \times A_1^3.$$

Then $|\widehat{\mathcal{A}}_{D_\mu}| = 4$ and $|\widehat{\mathcal{A}}_{D_\gamma}| = 5$, for $\gamma \in S \setminus \{\mu\}$. We also have $|\widehat{\mathcal{A}}| = 6$. It follows from Theorem 3.6.68 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 3, which is the Coxeter number of A_2, as required.

Let us now suppose that without loss of generality $\widehat{\mathcal{R}}$ takes the form

$$\widehat{\mathcal{R}} = (\mathcal{R} \cap \langle \mu, \lambda, \beta \rangle) \sqcup \{\pm\nu\} \sqcup \{\pm\theta\} = A_3 \times A_1^2. \tag{3.328}$$

Let $\widetilde{\mathcal{A}}$ be the arrangement corresponding to A_3. Then $|\widehat{\mathcal{A}}_{D_\mu}| = \widehat{\mathcal{A}}_{D_\lambda}| = 5$, $|\widehat{\mathcal{A}}_{D_\nu}| = |\widehat{\mathcal{A}}_{D_\theta}| = |\widetilde{\mathcal{A}}| + 1 = 7$ and $|\widehat{\mathcal{A}}| = 8$. It follows from Theorem 3.6.68 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 4, which is the Coxeter number of A_3, as required.

Finally, let us consider the case where $\widehat{\mathcal{R}} = \mathcal{R}_D \sqcup \{\pm\beta\} = A_1^5$. For any $\gamma \in S$ we have $|\widehat{\mathcal{A}}_{D_\gamma}| = 4$ and $|\widehat{\mathcal{A}}| = 5$. It follows from Theorem 3.6.68 that the multiplicity of $\beta|_D$ in $\det \eta_D$ is 2, which is the Coxeter number of A_1, as required. □

3.7 Exceptional groups: the remaining cases

In this section we obtain formulae for the determinant of the restricted Saito metric and analyse the corresponding multiplicities for the remaining cases with the help of Mathematica [5]. Thus we consider codimension 5 strata for $\mathcal{R} = E_7$ and codimension 5 and 6 strata for $\mathcal{R} = E_8$. We consider Saito metric and use Saito polynomials for these root systems $\mathcal{R} = E_n$, $n = 7, 8$. These are explicitly constructed in [83] and also in [1].

Let us start with the case $n = 8$, $\mathcal{R} = E_8$. We use Saito polynomials from [83] which are written in terms of coordinates y_i $(i = 1, \ldots, 8)$ (denoted as x_i in [83]) defined by

$$y_i = \begin{cases} \dfrac{1}{2}(x_i + x_{i+1}), & i \text{ odd} \\ \dfrac{1}{2}(x_{i-1} - x_i), & i \text{ even}. \end{cases} \tag{3.329}$$

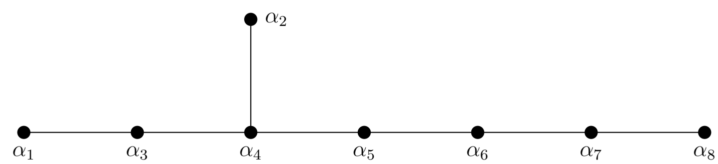
Let us recall the positive part of the root system $E_8 \subset V = \mathbb{C}^8$ (see for example [51]):

$$e_i \pm e_j, \quad 1 \leq i < j \leq 8, \quad \frac{1}{2}(e_1 \pm e_2 \pm \cdots \pm e_8) \quad \text{(even number of + signs)}.$$

Let us fix the following simple system $\Delta \subset E_8$:

$$\begin{aligned} &\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \\ &\alpha_2 = e_1 + e_2, \\ &\alpha_i = e_{i-1} - e_{i-2}, \quad 3 \leqslant i \leqslant 8. \end{aligned} \tag{3.330}$$

and consider the corresponding Coxeter graph:



Let us also introduce coordinates $z_i = (\alpha_i, x)$, $1 \leq i \leq 8$. Note that $z_i = A^{(8)}_{ij} y_j$, where $A = A^{(8)} = (A^{(8)}_{ij})^8_{i,j=1}$ is the following matrix:

$$A^{(8)} = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

We have

$$\eta = \sum_{i=1}^{n} dt^i dt^{n+1-i} = \sum_{i=1}^{n} \sum_{r=1}^{n} \sum_{l=1}^{n} \frac{\partial t^i}{\partial y_r} \frac{\partial t^{n+1-i}}{\partial y_l} dy_r dy_l = \sum_{r,l=1}^{n} \eta_{rl} dy_r dy_l.$$

In z-coordinates we have $\eta(z) = \sum_{i,j=1}^{n} \eta_{ij}(z) dz_i dz_j$, where

$$\eta_{ij}(z) = \sum_{k,l=1}^{n} (A^{(n)})^{-1}_{ki} (A^{(n)})^{-1}_{lj} \eta_{kl}. \tag{3.331}$$

Let $I = \{i_1, \ldots, i_k\}$, $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and let $J = \{1, \ldots, n\} \setminus I$. Consider the corresponding stratum $D = D_{i_1,\ldots,i_k}$. It follows that the restriction of $\eta(z)$ on D takes the form

$$\eta_D = \sum_{l,k\in J} \eta_{lk}(z)|_D \, dz_l dz_k. \qquad (3.332)$$

We use formula (3.332) to find the determinant of the restricted Saito metric with the help of Mathematica [5]. Tables 3.1 and 3.2 below give $\det \eta_D$ up to a non-zero proportionality factor for all three- and two-dimensional strata D in E_8 respectively. We list types of strata $\mathcal{R}_D = \mathcal{R} \cap \langle S \rangle$, where $S = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \mathcal{R}$ in the first column of these tables. We use the notation $\{i_1, \dots, i_k\} = \{1, \dots, k\}$ to denote the stratum D. We get the following statement.

Theorem 3.7.1. *Let D be any two- or three-dimensional stratum in $\mathcal{R} = E_8$. Then the statement of Main Theorem 1 is true.*

Table 3.1: Determinant of restricted Saito metric, $\dim D = 3$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\det \eta_D$
A_5, $\{4,5,6,7,8\}$	$\alpha_1^2\alpha_2^7\alpha_3^7(\alpha_1+\alpha_3)^7(\alpha_2+\alpha_3)^{10}(\alpha_1+\alpha_2+\alpha_3)^{10}(\alpha_1+\alpha_2+2\alpha_3)^{12}\times$ $\times(\alpha_1+2\alpha_2+3\alpha_3)^7(2\alpha_1+2\alpha_2+3\alpha_3)^7(2\alpha_1+3\alpha_2+4\alpha_3)^7(\alpha_1+2(\alpha_2+\alpha_3))^{10}\times$ $\times(\alpha_1+3(\alpha_2+\alpha_3))^2(2\alpha_1+3(\alpha_2+\alpha_3))^2$
D_5, $\{1,2,3,4,5\}$	$\alpha_6^{12}\alpha_7^2(\alpha_6+\alpha_7)^{12}(2\alpha_6+\alpha_7)^{10}\alpha_8^2(\alpha_7+\alpha_8)^2(\alpha_6+\alpha_7+\alpha_8)^{12}(2\alpha_6+\alpha_7+\alpha_8)^{10}\times$ $\times(2\alpha_6+2\alpha_7+\alpha_8)^{10}(3\alpha_6+2\alpha_7+\alpha_8)^{12}(4\alpha_6+2\alpha_7+\alpha_8)^2(4\alpha_6+3\alpha_7+\alpha_8)^2\times$ $\times(4\alpha_6+3\alpha_7+2\alpha_8)^2$
$D_4\times A_1$, $\{2,3,4,5,7\}$	$\alpha_1^8\alpha_6^{10}(\alpha_1+\alpha_6)^{10}(\alpha_1+2\alpha_6)^8\alpha_8^3(\alpha_6+\alpha_8)^8(\alpha_1+\alpha_6+\alpha_8)^8(2\alpha_6+\alpha_8)^3\times$ $\times(\alpha_1+2\alpha_6+\alpha_8)^{10}(2\alpha_1+2\alpha_6+\alpha_8)^3(\alpha_1+3\alpha_6+\alpha_8)^8(2\alpha_1+3\alpha_6+\alpha_8)^8\times$ $\times(2\alpha_1+4\alpha_6+\alpha_8)^3$
$A_4\times A_1$, $\{1,3,4,5,7\}$	$\alpha_2^8\alpha_6^7(\alpha_2+\alpha_6)^{12}(2\alpha_2+\alpha_6)^3(\alpha_2+2\alpha_6)^6\alpha_8^3(\alpha_6+\alpha_8)^6(\alpha_2+\alpha_6+\alpha_8)^8\times$ $\times(2\alpha_2+\alpha_6+\alpha_8)^2(\alpha_2+2\alpha_6+\alpha_8)^7(2\alpha_2+2\alpha_6+\alpha_8)^7(\alpha_2+3\alpha_6+\alpha_8)^2\times$ $\times(2\alpha_2+3\alpha_6+\alpha_8)^8(3\alpha_2+3\alpha_6+\alpha_8)^6(3\alpha_2+4\alpha_6+\alpha_8)^3(3\alpha_2+4\alpha_6+2\alpha_8)^2$
$A_3\times A_2$, $\{2,3,4,6,7\}$	$\alpha_1^5\alpha_5^{10}(\alpha_1+\alpha_5)^7(\alpha_1+2\alpha_5)^7(\alpha_1+3\alpha_5)^5(2\alpha_1+3\alpha_5)^2\alpha_8^4(\alpha_5+\alpha_8)^6\times$ $\times(\alpha_1+\alpha_5+\alpha_8)^5(2\alpha_5+\alpha_8)^4(\alpha_1+2\alpha_5+\alpha_8)^7(\alpha_1+3\alpha_5+\alpha_8)^7\times$ $\times(2\alpha_1+3\alpha_5+\alpha_8)^4(\alpha_1+4\alpha_5+\alpha_8)^5(2\alpha_1+4\alpha_5+\alpha_8)^6(2\alpha_1+5\alpha_5+\alpha_8)^4\times$ $\times(2\alpha_1+5\alpha_5+2\alpha_8)^2$
$A_3\times A_1^2$, $\{2,3,5,6,7\}$	$\alpha_1^3\alpha_4^{10}(\alpha_1+\alpha_4)^6(\alpha_1+2\alpha_4)^8(\alpha_1+3\alpha_4)^6(\alpha_1+4\alpha_4)^3\alpha_8^5(\alpha_4+\alpha_8)^4\times$ $\times(\alpha_1+\alpha_4+\alpha_8)^3(2\alpha_4+\alpha_8)^5(\alpha_1+2\alpha_4+\alpha_8)^6(\alpha_1+3\alpha_4+\alpha_8)^8(\alpha_1+4\alpha_4+\alpha_8)^6\times$ $\times(2\alpha_1+4\alpha_4+\alpha_8)^5(\alpha_1+5\alpha_4+\alpha_8)^3(2\alpha_1+5\alpha_4+\alpha_8)^4(2\alpha_1+6\alpha_4+\alpha_8)^5$
$A_2^2\times A_1$, $\{1,2,3,5,6\}$	$\alpha_4^{12}\alpha_7^4(\alpha_4+\alpha_7)^5(2\alpha_4+\alpha_7)^6(3\alpha_4+\alpha_7)^5(4\alpha_4+\alpha_7)^4\alpha_8^2(\alpha_7+\alpha_8)^4(\alpha_4+\alpha_7+\alpha_8)^5\times$ $\times(2\alpha_4+\alpha_7+\alpha_8)^6(3\alpha_4+\alpha_7+\alpha_8)^5(4\alpha_4+\alpha_7+\alpha_8)^4(2\alpha_4+2\alpha_7+\alpha_8)^4\times$ $\times(3\alpha_4+2\alpha_7+\alpha_8)^5(4\alpha_4+2\alpha_7+\alpha_8)^6(5\alpha_4+2\alpha_7+\alpha_8)^5(6\alpha_4+2\alpha_7+\alpha_8)^4\times$ $\times(6\alpha_4+3\alpha_7+\alpha_8)^2(6\alpha_4+3\alpha_7+2\alpha_8)^2$
$A_2\times A_1^3$, $\{2,3,5,7,8\}$	$\alpha_1^3\alpha_4^6(\alpha_1+\alpha_4)^4(\alpha_1+2\alpha_4)^3\alpha_6^5(\alpha_4+\alpha_6)^8(\alpha_1+\alpha_4+\alpha_6)^5(2\alpha_4+\alpha_6)^5\times$ $\times(\alpha_1+2\alpha_4+\alpha_6)^8(\alpha_1+3\alpha_4+\alpha_6)^5(\alpha_1+3\alpha_4+2\alpha_6)^8(\alpha_1+4\alpha_4+2\alpha_6)^5\times$ $\times(\alpha_1+4\alpha_4+3\alpha_6)^4(2\alpha_1+4\alpha_4+3\alpha_6)^3(\alpha_1+5\alpha_4+3\alpha_6)^3(2\alpha_1+5\alpha_4+3\alpha_6)^4\times$ $\times(2\alpha_1+6\alpha_4+3\alpha_6)^3(\alpha_1+2(\alpha_4+\alpha_6))^5(\alpha_1+3(\alpha_4+\alpha_6))^3$

Let us now consider the case $n=7$, $\mathcal{R}=E_7\subset V=\mathbb{C}^8$. Recall the positive part of E_7

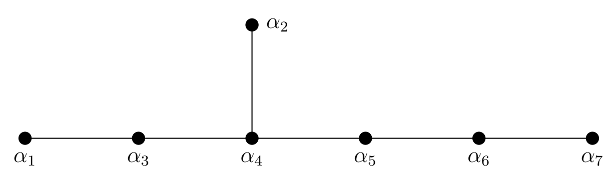
Table 3.2: Determinant of restricted Saito metric, $\dim D = 2$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\det \eta_D$
A_6, $\{2,4,5,6,7,8\}$	$\alpha_1^2\alpha_3^{12}\,(\alpha_1+\alpha_3)^{12}\,(\alpha_1+2\alpha_3)^{18}\,(\alpha_1+3\alpha_3)^{8}\,(2\alpha_1+3\alpha_3)^{8}$
D_6, $\{2,3,4,5,6,7\}$	$\alpha_1^{18}\alpha_8^{12}\,(\alpha_1+\alpha_8)^{18}\,(2\alpha_1+\alpha_8)^{12}$
E_6, $\{1,2,3,4,5,6\}$	$\alpha_7^{18}\alpha_8^{2}\,(\alpha_7+\alpha_8)^{18}\,(2\alpha_7+\alpha_8)^{18}\,(3\alpha_7+\alpha_8)^{2}\,(3\alpha_7+2\alpha_8)^{2}$
$A_5\times A_1$, $\{1,3,4,5,6,8\}$	$\alpha_2^{12}\alpha_7^{8}\,(\alpha_2+\alpha_7)^{18}\,(2\alpha_2+\alpha_7)^{8}\,(\alpha_2+2\alpha_7)^{7}\,(3\alpha_2+2\alpha_7)^{7}$
$D_5\times A_1$, $\{2,3,4,5,6,8\}$	$\alpha_1^{12}\alpha_7^{12}\,(\alpha_1+\alpha_7)^{18}\,(2\alpha_1+\alpha_7)^{3}\,(\alpha_1+2\alpha_7)^{12}\,(2\alpha_1+3\alpha_7)^{3}$
$A_4\times A_2$, $\{1,3,4,5,7,8\}$	$\alpha_2^{8}\alpha_6^{8}\,(\alpha_2+\alpha_6)^{18}\,(2\alpha_2+\alpha_6)^{4}\,(\alpha_2+2\alpha_6)^{8}\,(\alpha_2+3\alpha_6)^{2}\,(2\alpha_2+3\alpha_6)^{8}\,(3\alpha_2+4\alpha_6)^{4}$
$D_4\times A_2$, $\{2,3,4,5,7,8\}$	$\alpha_1^{8}\alpha_6^{12}\,(\alpha_1+\alpha_6)^{12}\,(\alpha_1+2\alpha_6)^{12}\,(\alpha_1+3\alpha_6)^{8}\,(2\alpha_1+3\alpha_6)^{8}$
$A_4\times A_1^2$, $\{2,3,5,6,7,8\}$	$\alpha_1^{3}\alpha_4^{12}\,(\alpha_1+\alpha_4)^{7}\,(\alpha_1+2\alpha_4)^{12}\,(\alpha_1+3\alpha_4)^{12}\,(\alpha_1+4\alpha_4)^{7}\,(\alpha_1+5\alpha_4)^{3}\,(2\alpha_1+5\alpha_4)^{4}$
A_3^2, $\{2,3,4,6,7,8\}$	$\alpha_1^{5}\alpha_5^{12}\,(\alpha_1+\alpha_5)^{8}\,(\alpha_1+2\alpha_5)^{12}\,(\alpha_1+3\alpha_5)^{8}\,(2\alpha_1+3\alpha_5)^{5}\,(\alpha_1+4\alpha_5)^{5}\,(2\alpha_1+5\alpha_5)^{5}$
$A_3\times A_2\times A_1$, $\{1,2,4,6,7,8\}$	$\alpha_3^{5}\alpha_5^{7}\,(\alpha_3+\alpha_5)^{18}\,(2\alpha_3+\alpha_5)^{5}\,(\alpha_3+2\alpha_5)^{8}\,(2\alpha_3+3\alpha_5)^{7}\,(3\alpha_3+4\alpha_5)^{5}\,(4\alpha_3+5\alpha_5)^{5}$
$A_2^2\times A_1^2$, $\{1,2,3,5,6,8\}$	$\alpha_4^{12}\alpha_7^{5}\,(\alpha_4+\alpha_7)^{8}\,(2\alpha_4+\alpha_7)^{12}\,(3\alpha_4+\alpha_7)^{8}\,(4\alpha_4+\alpha_7)^{5}\,(3\alpha_4+2\alpha_7)^{5}\,(5\alpha_4+2\alpha_7)^{5}$

(see for example [51]):

$$e_i \pm e_j, \quad 1 \le i < j \le 6, \quad e_7 - e_8, \quad \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} \pm e_i),$$

where the number of minus signs in the sum is odd. Let us fix simple system $\Delta = \{\alpha_1, \dots, \alpha_7\}$, where α_i, $i = 1, \dots, 7$ are defined in (3.330) and consider the corresponding Coxeter graph:



We use Saito polynomials from [83] which are written in terms of coordinates y_i defined by formulae (3.329) for any $1 \le i \le 4$ and defined by the following formulae for $i = 5, 6, 7$:

$$y_i = \begin{cases} \frac{1}{2}(x_i - x_{i+1}), & i = 5, 7 \\ \frac{1}{2}(x_{i-1} + x_i), & i = 6, \end{cases} \tag{3.333}$$

Let us also introduce coordinates $z_i = (\alpha_i, x)$, $1 \le i \le 7$. Note that $z_i = A^{(7)}_{ij} y_j$, where $A = A^{(7)} = (A^{(7)}_{ij})^7_{i,j=1}$ is the following matrix:

$$A^{(7)} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & -1 & -1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix}.$$

We use formulae (3.331), (3.332) ($n = 7$) to find the determinant of the restricted Saito metric with the help of Mathematica [5]. Table 3.3 gives $\det \eta_D$ up to a non-zero proportionality factor for any two-dimensional stratum D in E_7. We list types of strata $\mathcal{R}_D = \mathcal{R} \cap \langle S \rangle$, where $S = \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \mathcal{R}$ in the first column of this table. We use the notation $\{i_1, \dots, i_k\} = \{1, \dots, k\}$. Note that there are two non-equivalent strata of type A_5 [80]. The following statement is a direct corollary of this table.

Theorem 3.7.2. *Let D be any two-dimensional stratum in $\mathcal{R} = E_7$. Then the statement of Main Theorem 1 is true.*

Table 3.3: Determinant of restricted Saito metric, $\dim D = 2$, $\mathcal{R} = E_7$

$\mathcal{R}_D$, S	$\det \eta_D$
A_5, $\{2, 4, 5, 6, 7\}$	$\alpha_1^2 \alpha_3^{10} (\alpha_1 + \alpha_3)^{10} (\alpha_1 + 2\alpha_3)^{10} (\alpha_1 + 3\alpha_3)^2 (2\alpha_1 + 3\alpha_3)^2$
A_5', $\{3, 4, 5, 6, 7\}$	$\alpha_1^7 \alpha_2^{10} (\alpha_1 + \alpha_2)^{12} (\alpha_1 + 2\alpha_2)^7$
D_5, $\{1, 2, 3, 4, 5\}$	$\alpha_6^{12} \alpha_7^2 (\alpha_6 + \alpha_7)^{12} (2\alpha_6 + \alpha_7)^{10}$
$A_4 \times A_1$, $\{1, 2, 3, 4, 7\}$	$\alpha_5^8 \alpha_6^3 (\alpha_5 + \alpha_6)^{12} (2\alpha_5 + \alpha_6)^7 (3\alpha_5 + 2\alpha_6)^6$
$D_4 \times A_1$, $\{2, 3, 4, 5, 7\}$	$\alpha_1^8 \alpha_6^{10} (\alpha_1 + \alpha_6)^{10} (\alpha_1 + 2\alpha_6)^8$
$A_3 \times A_2$, $\{1, 3, 5, 6, 7\}$	$\alpha_2^2 \alpha_4^7 (\alpha_2 + \alpha_4)^7 (\alpha_2 + 2\alpha_4)^{10} (\alpha_2 + 3\alpha_4)^5 (2\alpha_2 + 3\alpha_4)^5$
$A_3 \times A_1^2$, $\{1, 2, 4, 5, 7\}$	$\alpha_3^8 \alpha_6^6 (\alpha_3 + \alpha_6)^{10} (2\alpha_3 + \alpha_6)^6 (\alpha_3 + 2\alpha_6)^3 (3\alpha_3 + 2\alpha_6)^3$
$A_2^2 \times A_1$, $\{1, 2, 4, 6, 7\}$	$\alpha_3^5 \alpha_5^6 (\alpha_3 + \alpha_5)^{12} (2\alpha_3 + \alpha_5)^4 (\alpha_3 + 2\alpha_5)^5 (2\alpha_3 + 3\alpha_5)^4$
$A_2 \times A_1^3$, $\{1, 2, 3, 5, 7\}$	$\alpha_4^8 \alpha_6^4 (\alpha_4 + \alpha_6)^8 (2\alpha_4 + \alpha_6)^8 (3\alpha_4 + \alpha_6)^4 (3\alpha_4 + 2\alpha_6)^4$

Now we are going to establish Main Theorem 2 for these strata in E_n. Recall that for any stratum D and $\beta \in \mathcal{R} \setminus \mathcal{R}_D$ we define the root system $\mathcal{R}_{D,\beta} = \langle \mathcal{R}_D, \beta \rangle \cap \mathcal{R}$ which has the decomposition (3.5) and that we have $\beta \in \mathcal{R}^{(0)}_{D,\beta}$. The approach to finding $\mathcal{R}^{(0)}_{D,\beta}$ is as follows. We compute the size $|\mathcal{R}_{D,\beta}|$ of the root system $\mathcal{R}_{D,\beta}$ using Mathematica [5]. In most cases considerations of subgraphs of the Coxeter graph of E_n allow to determine

$\mathcal{R}_{D,\beta}$ from its size uniquely (see also [74] for classification of all subsets of a root system which are irreducible root systems). We find the type of the root system $\mathcal{R}_{D,\beta}$ and we consider embedding of root systems $\mathcal{R}_D \subset \mathcal{R}_{D,\beta}$. Using Lemma 3.2.1 and relations (3.6), (3.7) we identify irreducible component $\mathcal{R}^{(0)}_{D,\beta}$. We give these results in Tables 3.5, 3.6 for the root system $\mathcal{R} = E_8$ and in Table 3.4 for the root system $\mathcal{R} = E_7$.

The cases when knowledge of $|\mathcal{R}_{D,\beta}|$ does not immediately lead to the type of $\mathcal{R}_{D,\beta}$ are as follows:

(i) $\mathcal{R} = E_8$, $\dim D = 3$, $|\mathcal{R}_{D,\beta}| = 42$ in which case $\mathcal{R}_{D,\beta} = A_6$ or $\mathcal{R}_{D,\beta} = D_5 \times A_1$,

(ii) $\mathcal{R} = E_8$, $\dim D = 3$, $|\mathcal{R}_{D,\beta}| = 24$ in which case $\mathcal{R}_{D,\beta} = A_4 \times A_1^2$ or $\mathcal{R}_{D,\beta} = A_3^2$,

(iii) $\mathcal{R} = E_7$, $\dim D = 2$, $|\mathcal{R}_{D,\beta}| = 42$ in which case $\mathcal{R}_{D,\beta} = A_6$ or $\mathcal{R}_{D,\beta} = D_5 \times A_1$.

Let us consider these remaining cases in detail.

(i) Considerations of Coxeter graphs and their subgraphs for D_5 and A_6 allow to determine $\mathcal{R}_{D,\beta}$ in all the cases except for when $\mathcal{R}_D = A_4 \times A_1$.

Let us consider firstly $\beta \in \mathcal{R}$ such that $\beta|_D = \alpha_6|_D$. Then it is immediate from the Coxeter graph of E_8 that $\mathcal{R}_{D,\beta} = A_6$.

Let us now consider $\beta|_D = \alpha_2 + 2\alpha_6 + \alpha_8|_D$. Suppose that $\mathcal{R}_{D,\beta} = D_5 \times A_1$. Note that $A_4 \times A_1$ is not a subsystem of D_5. Therefore it has to be that $\beta \in D_5$ and $\langle A_4, \beta\rangle \cap \mathcal{R} = D_5$. One can choose

$$\beta = \alpha_1 + \alpha_2 + \alpha_8 + 2(\alpha_3 + \alpha_6 + \alpha_7) + 3(\alpha_4 + \alpha_5) \in \mathcal{R}$$

so that $\beta|_D$ has the required form. Then one can check by Mathematica that $|\langle A_4, \beta\rangle \cap \mathcal{R}| = 30 \neq 40 = |D_5|$. This contradiction implies that $\mathcal{R}_{D,\beta} = A_6$.

The case $\beta|_D = 2\alpha_2 + 2\alpha_6 + \alpha_8|_D$ is similar. One can choose

$$\beta = \alpha_1 + \alpha_7 + \alpha_8 + 2(\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6) + 3\alpha_4 \in \mathcal{R}$$

so that $\beta|_D$ has the required form.

Now let us consider the case when $\beta|_D = \alpha_2|_D$. It is immediate from the Coxeter graph of E_8 that $\mathcal{R}_{D,\beta} = D_5 \times A_1$.

Consider the case when $\beta|_D = 2\alpha_2 + 3\alpha_6 + \alpha_8|_D$. One can choose

$$\beta = \alpha_1 + \alpha_8 + 2(\alpha_2 + \alpha_3 + \alpha_7) + 3(\alpha_4 + \alpha_5 + \alpha_6) \in \mathcal{R}$$

so that $\beta|_D$ has the required form. One can check by Mathematica that $|\langle A_4, \beta\rangle \cap \mathcal{R}| = 40 = |D_5|$. Note that $\pm\alpha_7 \in \mathcal{R}_{D,\beta}$. Since $|\mathcal{R}_{D,\beta}| = 42$ it follows that the root system $\mathcal{R}_{D,\beta}$ is reducible which implies that $\mathcal{R}_{D,\beta} = D_5 \times A_1$.

The case $\beta|_D = \alpha_2 + \alpha_6 + \alpha_8|_D$ is similar. One can choose $\beta = \sum_{i=1}^{8} \alpha_i \in \mathcal{R}$ so that $\beta|_D$ has the required form.

(ii) Considerations of Coxeter graphs and their subgraphs for A_3 and A_4 allow to determine $\mathcal{R}_{D,\beta}$ in all the cases except for when $\mathcal{R}_D = A_3 \times A_1^2$.

Consider firstly $\beta \in \mathcal{R}$ such that $\beta|_D = \alpha_4 + \alpha_8|_D$. Suppose that $\mathcal{R}_{D,\beta} = A_4 \times A_1^2$. Note that $A_3 \times A_1$ is not a subsystem of A_4. Therefore it has to be that $\beta \in A_4$ and $\langle A_3, \beta \rangle \cap \mathcal{R} = A_4$. One can choose $\beta = \sum_{\substack{i=2 \\ i \neq 3}}^{8} \alpha_i \in \mathcal{R}$ so that $\beta|_D$ has the required form. Then one can check by Mathematica that $|\langle A_3, \beta \rangle \cap \mathcal{R}| = 12 \neq 20 = |A_4|$. This contradiction implies that $\mathcal{R}_{D,\beta} = A_3^2$.

The case $\beta|_D = 2\alpha_1 + 5\alpha_4 + \alpha_8|_D$ is similar. In this case one can choose

$$\beta = \alpha_8 + 2(\alpha_1 + \alpha_7) + 3(\alpha_2 + \alpha_3 + \alpha_6) + 4\alpha_5 + 5\alpha_4 \in \mathcal{R}$$

so that $\beta|_D$ has the required form.

Now let us consider the case when $\beta|_D = \alpha_8|_D$. Then it is immediate from the Coxeter graph of E_8 that $\mathcal{R}_{D,\beta} = A_4 \times A_1^2$.

Consider now the case when $\beta|_D = 2\alpha_1 + 6\alpha_4 + \alpha_8|_D$. One can choose

$$\beta = \alpha_8 + 2(\alpha_1 + \alpha_7) + 3\alpha_2 + 4(\alpha_3 + \alpha_6) + 5\alpha_5 + 6\alpha_4 \in \mathcal{R}$$

so that $\beta|_D$ has the required form. Suppose that $\mathcal{R}_{D,\beta} = A_3^2$. Then it has to be that $\langle A_1^2, \beta \rangle \cap \mathcal{R} = A_3$. One can check by Mathematica that $|\langle A_1^2, \beta \rangle \cap \mathcal{R}| = 6 \neq 12 = |A_3|$. This contradiction implies that $\mathcal{R}_{D,\beta} = A_4 \times A_1^2$.

The cases $\beta|_D = 2\alpha_1 + 4\alpha_4 + \alpha_8|_D$ and $\beta|_D = 2\alpha_4 + \alpha_8$ are similar. One can choose

$$\beta = \alpha_7 + \alpha_8 + 2(\alpha_1 + \alpha_2 + \alpha_6) + 3(\alpha_3 + \alpha_5) + 4\alpha_4 \in \mathcal{R}$$

and

$$\beta = \alpha_7 + \alpha_8 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5 + \alpha_6) \in \mathcal{R}$$

respectively, so that $\beta|_D$ have the required forms.

(iii) Considerations of Coxeter graphs and their subgraphs for D_5 and A_6 allow to determine $\mathcal{R}_{D,\beta}$ in all the cases except for when $\mathcal{R}_D = A_4 \times A_1$.

Consider firstly $\beta \in \mathcal{R}$ such that $\beta|_D = \alpha_5|_D$. Then it is immediate from the Coxeter graph of E_7 that $\mathcal{R}_{D,\beta} = D_5 \times A_1$. Let us now consider the case when $\beta|_D = 2\alpha_5 + \alpha_6|_D$. Suppose that $\mathcal{R}_{D,\beta} = D_5 \times A_1$. Then it has to be that $\beta \in D_5$

and $\langle A_4, \beta\rangle \cap \mathcal{R} = D_5$. One can choose

$$\beta = \alpha_2 + \alpha_3 + \alpha_6 + 2(\alpha_4 + \alpha_5) \in \mathcal{R}$$

so that $\beta|_D$ has the required form. One can check by Mathematica that $|\langle A_4, \beta\rangle \cap \mathcal{R}| = 30 \neq 40 = |D_5|$. This contradiction implies that $\mathcal{R}_{D,\beta} = A_6$.

We get the following statement as a direct corollary of Table 3.4.

Theorem 3.7.3. *Let D be any two-dimensional stratum in $\mathcal{R} = E_7$. Then the statement of Main Theorem 2 is true.*

Table 3.4: $\mathcal{R}_{D,\beta}$, $\dim D = 2$, $\mathcal{R} = E_7$

$\mathcal{R}_D$, S	$\beta\|_D$	$\|\mathcal{R}_{D,\beta}\|$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_5, $\{2,4,5,6,7\}$	α_3, $\alpha_1+\alpha_3$, $\alpha_1+2\alpha_3$	60	D_6	D_6	10
	α_1, $2\alpha_1+3\alpha_3$, $\alpha_1+3\alpha_3$	32	$A_5\times A_1$	A_1	2
A_5', $\{3,4,5,6,7\}$	$\alpha_1+\alpha_2$	72	E_6	E_6	12
	α_2	60	D_6	D_6	10
	α_1, $\alpha_1+2\alpha_2$	42	A_6	A_6	7
D_5, $\{1,2,3,4,5\}$	α_6, $\alpha_6+\alpha_7$	72	E_6	E_6	12
	$2\alpha_6+\alpha_7$	60	D_6	D_6	10
	α_7	42	$D_5\times A_1$	A_1	2
$A_4\times A_1$, $\{1,2,3,4,7\}$	$\alpha_5+\alpha_6$	72	E_6	E_6	12
	$2\alpha_5+\alpha_6$	42	A_6	A_6	7
	α_5	42	$D_5\times A_1$	D_5	8
	$3\alpha_5+2\alpha_6$	32	$A_5\times A_1$	A_5	6
	α_6	26	$A_4\times A_2$	A_2	3
$D_4\times A_1$, $\{2,3,4,5,7\}$	α_6, $\alpha_1+\alpha_6$	60	D_6	D_6	10
	α_1, $\alpha_1+2\alpha_6$	42	$D_5\times A_1$	D_5	8
$A_3\times A_2$, $\{1,3,5,6,7\}$	$\alpha_2+2\alpha_4$	60	D_6	D_6	10
	α_4, $\alpha_2+\alpha_4$	42	A_6	A_6	7
	$2\alpha_2+3\alpha_4$, $\alpha_2+3\alpha_4$	26	$A_4\times A_2$	A_4	5
	α_2	20	$A_3\times A_2\times A_1$	A_1	2
$A_3\times A_1^2$, $\{1,2,4,5,7\}$	$\alpha_3+\alpha_6$	60	D_6	D_6	10
	α_3	42	$D_5\times A_1$	D_5	8
	α_6, $2\alpha_3+\alpha_6$	32	$A_5\times A_1$	A_5	6
	$\alpha_3+2\alpha_6$, $3\alpha_3+2\alpha_6$	20	$A_3\times A_2\times A_1$	A_2	3
$A_2^2\times A_1$, $\{1,2,4,6,7\}$	$\alpha_3+\alpha_5$	72	E_6	E_6	12
	α_5	32	$A_5\times A_1$	A_5	6
	α_3, $\alpha_3+2\alpha_5$	26	$A_4\times A_2$	A_4	5
	$2\alpha_3+\alpha_5$, $2\alpha_3+3\alpha_5$	20	$A_3\times A_2\times A_1$	A_3	4
$A_2\times A_1^3$, $\{1,2,3,5,7\}$	α_4, $\alpha_4+\alpha_6$, $2\alpha_4+\alpha_6$	42	$D_5\times A_1$	D_5	8
	α_6, $3\alpha_4+\alpha_6$, $3\alpha_4+2\alpha_6$	20	$A_3\times A_2\times A_1$	A_3	4

As a direct corollary of Tables 3.5, 3.6 we get the following statement.

Theorem 3.7.4. *Let D be any two- or three-dimensional stratum in $\mathcal{R} = E_8$. Then the statement of Main Theorem 2 is true.*

Table 3.5: $\mathcal{R}_{D,\beta}$, $\dim D = 3$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\beta\|_D$	$\|\mathcal{R}_{D,\beta}\|$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_5, $\{4,5,6,7,8\}$	$\alpha_1+\alpha_2+2\alpha_3$	72	E_6	E_6	12
	$\alpha_2+\alpha_3$, $\alpha_1+\alpha_2+\alpha_3$, $\alpha_1+2\alpha_2+2\alpha_3$	60	D_6	D_6	10
	α_2, α_3, $2\alpha_1+3\alpha_2+4\alpha_3$, $2\alpha_1+2\alpha_2+3\alpha_3$, $\alpha_1+2\alpha_2+3\alpha_3$, $\alpha_1+\alpha_3$	42	A_6	A_6	7
	α_1, $2\alpha_1+3\alpha_2+3\alpha_3$, $\alpha_1+3\alpha_2+3\alpha_3$	32	$A_5\times A_1$	A_1	2
D_5, $\{1,2,3,4,5\}$	α_6, $\alpha_6+\alpha_7+\alpha_8$, $3\alpha_6+2\alpha_7+\alpha_8$, $\alpha_6+\alpha_7$	72	E_6	E_6	12
	$2\alpha_6+\alpha_7+\alpha_8$, $2\alpha_6+2\alpha_7+\alpha_8$, $2\alpha_6+\alpha_7$	60	D_6	D_6	10
	α_7, α_8, $4\alpha_6+3\alpha_7+2\alpha_8$, $\alpha_7+\alpha_8$, $4\alpha_6+3\alpha_7+\alpha_8$, $4\alpha_6+2\alpha_7+\alpha_8$	42	$D_5\times A_1$	A_1	2
$D_4\times A_1$, $\{2,3,4,5,7\}$	α_6, $\alpha_1+2\alpha_6+\alpha_8$, $\alpha_1+\alpha_6$	60	D_6	D_6	10
	α_1, $\alpha_1+3\alpha_6+\alpha_8$, $\alpha_1+\alpha_6+\alpha_8$, $\alpha_6+\alpha_8$, $2\alpha_1+3\alpha_6+\alpha_8$, $\alpha_1+2\alpha_6$	42	$D_5\times A_1$	D_5	8
	α_8, $2\alpha_1+4\alpha_6+\alpha_8$, $2\alpha_1+2\alpha_6+\alpha_8$, $2\alpha_6+\alpha_8$	30	$D_4\times A_2$	A_2	3
$A_4\times A_1$, $\{1,3,4,5,7\}$	$\alpha_2+\alpha_6$	72	E_6	E_6	12
	α_6, $\alpha_2+2\alpha_6+\alpha_8$, $2\alpha_2+2\alpha_6+\alpha_8$	42	A_6	A_6	7
	α_2, $2\alpha_2+3\alpha_6+\alpha_8$, $\alpha_2+\alpha_6+\alpha_8$	42	$D_5\times A_1$	D_5	8
	$\alpha_2+2\alpha_6$, $\alpha_6+\alpha_8$, $3\alpha_2+3\alpha_6+\alpha_8$	32	$A_5\times A_1$	A_5	6
	α_8, $2\alpha_2+\alpha_6$, $3\alpha_2+4\alpha_6+\alpha_8$	26	$A_4\times A_2$	A_2	3
	$3\alpha_2+4\alpha_6+2\alpha_8$, $2\alpha_2+\alpha_6+\alpha_8$, $\alpha_2+3\alpha_6+\alpha_8$	24	$A_4\times A_1^2$	A_1	2
$A_3\times A_2$, $\{2,3,4,6,7\}$	α_5	60	D_6	D_6	10
	$\alpha_1+2\alpha_5+\alpha_8$, $\alpha_1+3\alpha_5+\alpha_8$, $\alpha_1+\alpha_5$, $\alpha_1+2\alpha_5$	42	A_6	A_6	7
	$\alpha_5+\alpha_8$, $2\alpha_1+4\alpha_5+\alpha_8$	30	$D_4\times A_2$	D_4	6
	α_1, $\alpha_1+4\alpha_5+\alpha_8$, $\alpha_1+\alpha_5+\alpha_8$, $\alpha_1+3\alpha_5$	26	$A_4\times A_2$	A_4	5
	α_8, $2\alpha_1+5\alpha_5+\alpha_8$, $2\alpha_1+3\alpha_5+\alpha_8$, $2\alpha_5+\alpha_8$	24	A_3^2	A_3	4
	$2\alpha_1+5\alpha_5+2\alpha_8$, $2\alpha_1+3\alpha_5$	20	$A_3\times A_2\times A_1$	A_1	2
$A_3\times A_1^2$, $\{2,3,5,6,7\}$	α_4	60	D_6	D_6	10
	$\alpha_1+3\alpha_4+\alpha_8$, $\alpha_1+2\alpha_4$	42	$D_5\times A_1$	D_5	8
	$\alpha_1+2\alpha_4+\alpha_8$, $\alpha_1+4\alpha_4+\alpha_8$, $\alpha_1+3\alpha_4$, $\alpha_1+\alpha_4$	32	$A_5\times A_1$	A_5	6
	$\alpha_4+\alpha_8$, $2\alpha_1+5\alpha_4+\alpha_8$	24	A_3^2	A_3	4
	α_8, $2\alpha_1+6\alpha_4+\alpha_8$, $2\alpha_1+4\alpha_4+\alpha_8$, $2\alpha_4+\alpha_8$	24	$A_4\times A_1^2$	A_4	5
	α_1, $\alpha_1+5\alpha_4+\alpha_8$, $\alpha_1+\alpha_4+\alpha_8$, $\alpha_1+4\alpha_4$	20	$A_3\times A_2\times A_1$	A_2	3
$A_2^2\times A_1$, $\{1,2,3,5,6\}$	α_4	72	E_6	E_6	12
	$2\alpha_4+\alpha_7$, $2\alpha_4+\alpha_7+\alpha_8$, $4\alpha_4+2\alpha_7+\alpha_8$	32	$A_5\times A_1$	A_5	6
	$\alpha_4+\alpha_7+\alpha_8$, $5\alpha_4+2\alpha_7+\alpha_8$, $3\alpha_4+\alpha_7$, $3\alpha_4+\alpha_7+\alpha_8$, $3\alpha_4+2\alpha_7+\alpha_8$, $\alpha_4+\alpha_7$	26	$A_4\times A_2$	A_4	5
	$4\alpha_4+\alpha_7+\alpha_8$, $6\alpha_4+2\alpha_7+\alpha_8$, α_7, $4\alpha_4+\alpha_7$, $\alpha_7+\alpha_8$, $2\alpha_4+2\alpha_7+\alpha_8$	20	$A_3\times A_2\times A_1$	A_3	4
	α_8, $6\alpha_4+3\alpha_7+2\alpha_8$, $6\alpha_4+3\alpha_7+\alpha_8$	16	$A_2^2\times A_1^2$	A_1	2
$A_2\times A_1^3$, $\{2,3,5,7,8\}$	$\alpha_4+\alpha_6$, $\alpha_1+2\alpha_4+\alpha_6$, $\alpha_1+3\alpha_4+2\alpha_6$	42	$D_5\times A_1$	D_5	8
	α_4	30	$D_4\times A_2$	D_4	6
	α_6, $2\alpha_4+\alpha_6$, $\alpha_1+\alpha_4+\alpha_6$, $\alpha_1+3\alpha_4+\alpha_6$, $\alpha_1+4\alpha_4+2\alpha_6$, $\alpha_1+2\alpha_4+2\alpha_6$	24	$A_4\times A_1^2$	A_4	5
	$2\alpha_1+5\alpha_4+3\alpha_6$, $\alpha_1+4\alpha_4+3\alpha_6$, $\alpha_1+\alpha_4$	20	$A_3\times A_2\times A_1$	A_3	4
	α_1, $2\alpha_1+4\alpha_4+3\alpha_6$, $2\alpha_1+6\alpha_4+3\alpha_6$, $\alpha_1+3\alpha_4+3\alpha_6$, $\alpha_1+5\alpha_4+3\alpha_6$, $\alpha_1+2\alpha_4$	16	$A_2^2\times A_1^2$	A_2	3

Table 3.6: $\mathcal{R}_{D,\beta}$, $\dim D = 2$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\beta\|_D$	$\|\mathcal{R}_{D,\beta}\|$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_6, $\{2,4,5,6,7,8\}$	$\alpha_1 + 2\alpha_3$	126	E_7	E_7	18
	α_3, $\alpha_1 + \alpha_3$	84	D_7	D_7	12
	$2\alpha_1 + 3\alpha_3$, $\alpha_1 + 3\alpha_3$	56	A_7	A_7	8
	α_1	44	$A_6 \times A_1$	A_1	2
D_6, $\{2,3,4,5,6,7\}$	α_1, $\alpha_1 + \alpha_8$	126	E_7	E_7	18
	α_8, $2\alpha_1 + \alpha_8$	84	D_7	D_7	12
E_6, $\{1,2,3,4,5,6\}$	α_7, $\alpha_7 + \alpha_8$, $2\alpha_7 + \alpha_8$	126	E_7	E_7	18
	α_8, $3\alpha_7 + \alpha_8$, $3\alpha_7 + 2\alpha_8$	74	$E_6 \times A_1$	A_1	2
$A_5 \times A_1$, $\{1,3,4,5,6,8\}$	$\alpha_2 + \alpha_7$	126	E_7	E_7	18
	α_7, $2\alpha_2 + \alpha_7$	56	A_7	A_7	8
	α_2	74	$E_6 \times A_1$	E_6	12
	$3\alpha_2 + 2\alpha_7$, $\alpha_2 + 2\alpha_7$	44	$A_6 \times A_1$	A_6	7
$D_5 \times A_1$, $\{2,3,4,5,6,8\}$	$\alpha_1 + \alpha_7$	126	E_7	E_7	18
	α_7	84	D_7	D_7	12
	α_1, $\alpha_1 + 2\alpha_7$	74	$E_6 \times A_1$	E_6	12
	$2\alpha_1 + 3\alpha_7$, $2\alpha_1 + \alpha_7$	46	$D_5 \times A_2$	A_2	3
$A_4 \times A_2$, $\{1,3,4,5,7,8\}$	$\alpha_2 + \alpha_6$	126	E_7	E_7	18
	α_6, $\alpha_2 + 2\alpha_6$	56	A_7	A_7	8
	α_2, $2\alpha_2 + 3\alpha_6$	46	$D_5 \times A_2$	D_5	8
	$2\alpha_2 + \alpha_6$, $3\alpha_2 + 4\alpha_6$	32	$A_4 \times A_3$	A_3	4
	$\alpha_2 + 3\alpha_6$	28	$A_4 \times A_2 \times A_1$	A_1	2
$D_4 \times A_2$, $\{2,3,4,5,7,8\}$	α_6, $\alpha_1 + 2\alpha_6$, $\alpha_1 + \alpha_6$	84	D_7	D_7	12
	α_1, $2\alpha_1 + 3\alpha_6$, $\alpha_1 + 3\alpha_6$	46	$D_5 \times A_2$	D_5	8
$A_4 \times A_1^2$, $\{2,3,5,6,7,8\}$	α_4	84	D_7	D_7	12
	$\alpha_1 + 3\alpha_4$, $\alpha_1 + 2\alpha_4$	74	$E_6 \times A_1$	E_6	12
	$\alpha_1 + 4\alpha_4$, $\alpha_1 + \alpha_4$	44	$A_6 \times A_1$	A_6	7
	$2\alpha_1 + 5\alpha_4$	32	$A_4 \times A_3$	A_3	4
	α_1, $\alpha_1 + 5\alpha_4$	28	$A_4 \times A_2 \times A_1$	A_2	3
A_3^2, $\{2,3,4,6,7,8\}$	α_5, $\alpha_1 + 2\alpha_5$	84	D_7	D_7	12
	$\alpha_1 + 3\alpha_5$, $\alpha_1 + \alpha_5$	56	A_7	A_7	8
	α_1, $2\alpha_1 + 5\alpha_5$, $\alpha_1 + 4\alpha_5$, $2\alpha_1 + 3\alpha_5$	32	$A_4 \times A_3$	A_4	5
$A_3 \times A_2 \times A_1$, $\{1,2,4,6,7,8\}$	$\alpha_3 + \alpha_5$	126	E_7	E_7	18
	α_5, $2\alpha_3 + 3\alpha_5$	44	$A_6 \times A_1$	A_6	7
	$\alpha_3 + 2\alpha_5$	46	$D_5 \times A_2$	D_5	8
	α_3, $3\alpha_3 + 4\alpha_5$	32	$A_4 \times A_3$	A_4	5
	$2\alpha_3 + \alpha_5$, $4\alpha_3 + 5\alpha_5$	28	$A_4 \times A_2 \times A_1$	A_4	5
$A_2^2 \times A_1^2$, $\{1,2,3,5,6,8\}$	α_4, $2\alpha_4 + \alpha_7$	74	$E_6 \times A_1$	E_6	12
	$3\alpha_4 + \alpha_7$, $\alpha_4 + \alpha_7$	46	$D_5 \times A_2$	D_5	8
	α_7, $4\alpha_4 + \alpha_7$, $5\alpha_4 + 2\alpha_7$, $3\alpha_4 + 2\alpha_7$	28	$A_4 \times A_2 \times A_1$	A_4	5

Below we give Tables 3.1, 3.2, 3.5, 3.6 for the root system $\mathcal{R} = E_8$ in terms of the coordinates y_i defined by formulae (3.329) and we also give Tables 3.3, 3.4 for the root system $\mathcal{R} = E_7$ in terms of the coordinates y_i defined by formulae (3.333).

Table 3.1a: Determinant of restricted Saito metric, $\dim D = 3$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\det \eta_D$
A_5, $\{4,5,6,7,8\}$	$y_1^7 (y_1 - y_5)^7 y_5^{10} (y_1 - y_8)^2 (y_1 - 3y_5 - y_8)^2 (y_1 - y_5 - y_8)^{10} (y_5 - y_8)^7 \times$ $\times (y_1 + y_5 - y_8)^{10} (y_1 + 3y_5 - y_8)^2 y_8^7 (y_1 - 3y_5 + y_8)^7 (y_1 - y_5 + y_8)^{12} (y_1 + y_5 + y_8)^7$
D_5, $\{1,2,3,4,5\}$	$y_6^2 y_7^2 (y_7 - y_6)^{10} (y_6 + y_7)^{10} (-y_6 + y_7 - 2y_8)^2 (y_6 + y_7 - 2y_8)^2 (y_7 - y_8)^{12} y_8^{10} \times$ $\times (y_8 - y_6)^{12} (y_6 + y_8)^{12} (y_7 + y_8)^{12} (-y_6 + y_7 + 2y_8)^2 (y_6 + y_7 + 2y_8)^2$
$D_4 \times A_1$, $\{2,3,4,5,7\}$	$y_5^{10} y_7^{10} (y_7 - y_5)^8 (y_5 + y_7)^8 (y_7 - y_8)^8 (-y_5 + y_7 - y_8)^3 (y_5 + y_7 - y_8)^3 y_8^{10} \times$ $\times (y_8 - y_5)^8 (y_5 + y_8)^8 (y_7 + y_8)^8 (-y_5 + y_7 + y_8)^3 (y_5 + y_7 + y_8)^3$
$A_4 \times A_1$, $\{1,3,4,5,7\}$	$y_3^8 (y_3 - y_5)^7 y_5^6 (y_3 + y_5)^{12} (3y_3 + y_5)^3 (y_3 - y_7)^7 (2y_3 - y_5 - y_7)^2 (2y_3 + y_5 - y_7) \times$ $\times^6 (y_3 + 2y_5 - y_7)^3 y_7^2 (y_3 + y_7)^7 (y_7 - y_5)^8 (2y_3 - y_5 + y_7)^2 (y_5 + y_7)^8 \times$ $\times (2y_3 + y_5 + y_7)^6 (y_3 + 2y_5 + y_7)^3$
$A_3 \times A_2$, $\{2,3,4,6,7\}$	$y_3^{10} y_7^2 (y_7 - 3y_3)^5 (y_7 - y_3)^7 (y_3 + y_7)^7 (3y_3 + y_7)^5 (y_7 - y_8)^6 (-2y_3 + y_7 - y_8)^4 \times$ $\times (2y_3 + y_7 - y_8)^4 y_8^2 (y_8 - 3y_3)^5 (y_8 - y_3)^7 (y_3 + y_8)^7 (3y_3 + y_8)^5 (y_7 + y_8)^6 \times$ $\times (-2y_3 + y_7 + y_8)^4 (2y_3 + y_7 + y_8)^4$
$A_3 \times A_1^2$, $\{2,3,5,6,7\}$	$y_5^{10} y_7^8 (y_7 - 2y_5)^3 (y_7 - y_5)^6 (y_5 + y_7)^6 (2y_5 + y_7)^3 (y_7 - y_8)^4 (-y_5 + y_7 - y_8)^5 \times$ $\times (y_5 + y_7 - y_8)^5 y_8^8 (y_8 - 2y_5)^3 (y_8 - y_5)^6 (y_5 + y_8)^6 (2y_5 + y_8)^3 (y_7 + y_8)^4 \times$ $\times (-y_5 + y_7 + y_8)^5 (y_5 + y_7 + y_8)^5$
$A_2^2 \times A_1$, $\{1,2,3,5,6\}$	$y_5^6 y_7^2 (y_7 - 3y_5)^2 (y_7 - y_5)^6 (y_5 + y_7)^6 (3y_5 + y_7)^2 (-3y_5 + y_7 - 2y_8)^4 \times$ $\times (-y_5 + y_7 - 2y_8)^4 (y_7 - y_8)^5 (-2y_5 + y_7 - y_8)^5 y_8^4 (y_8 - y_5)^5 (y_5 + y_8)^{12} \times$ $\times (2y_5 + y_8)^4 (3y_5 + y_8)^5 (y_7 + y_8)^5 (2y_5 + y_7 + y_8)^5 (y_5 + y_7 + 2y_8)^4 (3y_5 + y_7 + 2y_8)^4$
$A_2 \times A_1^3$, $\{2,3,5,7,8\}$	$y_3^6 y_5^8 (y_5 - y_3)^5 (y_3 + y_5)^5 (y_5 - 2y_8)^4 (-y_3 + y_5 - 2y_8)^3 (y_3 + y_5 - 2y_8)^3 (y_5 - y_8)^8 \times$ $\times (-y_3 + y_5 - y_8)^5 (y_3 + y_5 - y_8)^5 (2y_5 - y_8)^4 (-y_3 + 2y_5 - y_8)^3 (y_3 + 2y_5 - y_8)^3 y_8^8 \times$ $\times (y_8 - y_3)^5 (y_3 + y_8)^5 (y_5 + y_8)^4 (-y_3 + y_5 + y_8)^3 (y_3 + y_5 + y_8)^3$

Table 3.2a: Determinant of restricted Saito metric, $\dim D = 2$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\det \eta_D$
A_6, $\{2,4,5,6,7,8\}$	$y_3^{12}\left(y_3-y_7\right)^8\left(2y_3-y_7\right)^{12}\left(4y_3-y_7\right)^2 y_7^{18}\left(2y_3+y_7\right)^8$
D_6, $\{2,3,4,5,6,7\}$	$y_7^{18}\left(y_7-y_8\right)^{12} y_8^{18}\left(y_7+y_8\right)^{12}$
E_6, $\{1,2,3,4,5,6\}$	$y_6^{18}\left(y_6-y_7\right)^{18}\left(3y_6-y_7\right)^2 y_7^2\left(y_6+y_7\right)^{18}\left(3y_6+y_7\right)^2$
$A_5\times A_1$, $\{1,3,4,5,6,8\}$	$y_3^{12} y_8^8\left(y_3+y_8\right)^{18}\left(2y_3+y_8\right)^8\left(y_3+2y_8\right)^7\left(3y_3+2y_8\right)^7$
$D_5\times A_1$, $\{2,3,4,5,6,8\}$	$y_6^{12}\left(y_6-y_7\right)^{12} y_7^3\left(y_6+y_7\right)^{18}\left(2y_6+y_7\right)^3\left(3y_6+y_7\right)^{12}$
$A_4\times A_2$, $\{1,3,4,5,7,8\}$	$y_3^8\left(2y_3-y_8\right)^4 y_8^{18}\left(y_3+y_8\right)^8\left(2y_3+y_8\right)^8\left(y_3+2y_8\right)^4\left(2y_3+3y_8\right)^8\left(4y_3+3y_8\right)^2$
$D_4\times A_2$, $\{2,3,4,5,7,8\}$	$y_7^{12}\left(y_7-y_8\right)^8 y_8^{12}\left(y_7+y_8\right)^{12}\left(2y_7+y_8\right)^8\left(y_7+2y_8\right)^8$
$A_4\times A_1^2$, $\{2,3,5,6,7,8\}$	$y_3^{12}\left(y_3-2y_8\right)^4\left(y_3-y_8\right)^{12}\left(2y_3-y_8\right)^7\left(3y_3-y_8\right)^3 y_8^{12}\left(y_3+y_8\right)^7\left(2y_3+y_8\right)^3$
A_3^2, $\{2,3,4,6,7,8\}$	$y_3^{12}\left(y_3-y_8\right)^{12}\left(2y_3-y_8\right)^5\left(3y_3-y_8\right)^8\left(5y_3-y_8\right)^5 y_8^5\left(y_3+y_8\right)^8\left(3y_3+y_8\right)^5$
$A_3\times A_2\times A_1$, $\{1,2,4,6,7,8\}$	$y_5^8\left(3y_5-2y_8\right)^5\left(y_5-y_8\right)^5\left(3y_5-y_8\right)^7 y_8^{18}\left(y_5+y_8\right)^5\left(3y_5+y_8\right)^7\left(3y_5+2y_8\right)^5$
$A_2^2\times A_1^2$, $\{1,2,3,5,6,8\}$	$y_5^{12}\left(y_5-y_8\right)^8\left(3y_5-y_8\right)^5 y_8^5\left(y_5+y_8\right)^{12}\left(2y_5+y_8\right)^5\left(3y_5+y_8\right)^8\left(5y_5+y_8\right)^5$

Table 3.3a: Determinant of restricted Saito metric, $\dim D = 2$, $\mathcal{R} = E_7$

$\mathcal{R}_D$, S	$\det \eta_D$
A_5, $\{2,4,5,6,7\}$	$y_2^{10}\left(y_2-y_7\right)^{10}\left(3y_2-y_7\right)^2 y_7^2\left(y_2+y_7\right)^{10}\left(3y_2+y_7\right)^2$
A_5', $\{3,4,5,6,7\}$	$y_1^{10}\left(2y_1-y_7\right)^7 y_7^{12}\left(2y_1+y_7\right)^7$
D_5, $\{1,2,3,4,5\}$	$y_5^2\left(y_5-y_7\right)^{12} y_7^{10}\left(y_5+y_7\right)^{12}$
$A_4\times A_1$, $\{1,2,3,4,7\}$	$y_3^8\left(2y_3-y_6\right)^3 y_6^{12}\left(y_3+y_6\right)^6\left(2y_3+y_6\right)^7$
$D_4\times A_1$, $\{2,3,4,5,7\}$	$y_6^{10}\left(y_6-y_7\right)^8 y_7^{10}\left(y_6+y_7\right)^8$
$A_3\times A_2$, $\{1,3,5,6,7\}$	$y_1^2\left(y_1-3y_3\right)^5\left(y_1-y_3\right)^7 y_3^{10}\left(y_1+y_3\right)^7\left(y_1+3y_3\right)^5$
$A_3\times A_1^2$, $\{1,2,4,5,7\}$	$y_3^8\left(y_3-y_6\right)^6 y_6^3\left(y_3+y_6\right)^{10}\left(2y_3+y_6\right)^3\left(3y_3+y_6\right)^6$
$A_2^2\times A_1$, $\{1,2,4,6,7\}$	$y_3^{12}\left(y_3-y_4\right)^5\left(2y_3-y_4\right)^4 y_4^6\left(y_3+y_4\right)^5\left(2y_3+y_4\right)^4$
$A_2\times A_1^3$, $\{1,2,3,5,7\}$	$y_3^8\left(y_3-y_6\right)^4 y_6^8\left(y_3+y_6\right)^8\left(2y_3+y_6\right)^4\left(y_3+2y_6\right)^4$

Table 3.4a: $\mathcal{R}_{D,\beta}$, $\dim D = 2$, $\mathcal{R} = E_7$

$\mathcal{R}_D$, S	$\beta\|_D$	$\|\mathcal{R}_{D,\beta}\|$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_5, $\{2,4,5,6,7\}$	y_2, $y_2 \pm y_7$	60	D_6	D_6	10
	y_7, $3y_2 \pm y_7$	32	$A_5 \times A_1$	A_1	2
A_5', $\{3,4,5,6,7\}$	y_7	72	E_6	E_6	12
	y_1	60	D_6	D_6	10
	$2y_1 \pm y_7$	42	A_6	A_6	7
D_5, $\{1,2,3,4,5\}$	$y_5 \pm y_7$	72	E_6	E_6	12
	y_7	60	D_6	D_6	10
	y_5	42	$D_5 \times A_1$	A_1	2
$A_4 \times A_1$, $\{1,2,3,4,7\}$	y_6	72	E_6	E_6	12
	$2y_3 + y_6$	42	A_6	A_6	7
	y_3	42	$D_5 \times A_1$	D_5	8
	$y_3 + y_6$	32	$A_5 \times A_1$	A_5	6
	$2y_3 - y_6$	26	$A_4 \times A_2$	A_2	3
$D_4 \times A_1$, $\{2,3,4,5,7\}$	y_6, y_7	60	D_6	D_6	10
	$y_6 \pm y_7$	42	$D_5 \times A_1$	D_5	8
$A_3 \times A_2$, $\{1,3,5,6,7\}$	y_3	60	D_6	D_6	10
	$y_1 \pm y_3$	42	A_6	A_6	7
	$y_1 \pm 3y_3$	26	$A_4 \times A_2$	A_4	5
	y_1	20	$A_3 \times A_2 \times A_1$	A_1	2
$A_3 \times A_1^2$, $\{1,2,4,5,7\}$	$y_3 + y_6$	60	D_6	D_6	10
	y_3	42	$D_5 \times A_1$	D_5	8
	$y_3 - y_6$, $3y_3 + y_6$	32	$A_5 \times A_1$	A_5	6
	y_6, $2y_3 + y_6$	20	$A_3 \times A_2 \times A_1$	A_2	3
$A_2^2 \times A_1$, $\{1,2,4,6,7\}$	y_3	72	E_6	E_6	12
	y_4	32	$A_5 \times A_1$	A_5	6
	$y_3 \pm y_4$	26	$A_4 \times A_2$	A_4	5
	$2y_3 \pm y_4$	20	$A_3 \times A_2 \times A_1$	A_3	4
$A_2 \times A_1^3$, $\{1,2,3,5,7\}$	y_3, y_6, $y_3 + y_6$	42	$D_5 \times A_1$	D_5	8
	$y_3 - y_6$, $2y_3 + y_6$, $y_3 + 2y_6$	20	$A_3 \times A_2 \times A_1$	A_3	4

Table 3.5a: $\mathcal{R}_{D,\beta}$, $\dim D = 3$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\beta\|_D$	$\|\mathcal{R}_{D,\beta}\|$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_5, $\{4,5,6,7,8\}$	$y_1-y_5+y_8$	72	E_6	E_6	12
	y_5, $y_1-y_5-y_8$, $y_1+y_5-y_8$	60	D_6	D_6	10
	y_1, y_1-y_5, y_5-y_8, y_8, $y_1-3y_5+y_8$, $y_1+y_5+y_8$	42	A_6	A_6	7
	y_1-y_8, $y_1-3y_5-y_8$, $y_1+3y_5-y_8$	32	$A_5\times A_1$	A_1	2
D_5, $\{1,2,3,4,5\}$	$y_7\pm y_8$, $y_8\pm y_6$	72	E_6	E_6	12
	$y_6\pm y_7$, y_8	60	D_6	D_6	10
	y_6, y_7, $\pm y_6+y_7-2y_8$, $\pm y_6+y_7+2y_8$	42	$D_5\times A_1$	A_1	2
$D_4\times A_1$, $\{2,3,4,5,7\}$	y_5, y_7, y_8	60	D_6	D_6	10
	$y_5\pm y_7$, $y_7\pm y_8$, $y_5\pm y_8$	42	$D_5\times A_1$	D_5	8
	$\pm y_5+y_7-y_8$, $\pm y_5+y_7+y_8$	30	$D_4\times A_2$	A_2	3
$A_4\times A_1$, $\{1,3,4,5,7\}$	y_3+y_5	72	E_6	E_6	12
	y_3-y_5, $y_3\pm y_7$	42	A_6	A_6	7
	y_3, $y_7\pm y_5$	42	$D_5\times A_1$	D_5	8
	y_5, $2y_3+y_5-y_7$, $2y_3+y_5+y_7$	32	$A_5\times A_1$	A_5	6
	$3y_3+y_5$, $y_3+2y_5\pm y_7$	26	$A_4\times A_2$	A_2	3
	y_7, $2y_3-y_5\pm y_7$	24	$A_4\times A_1^2$	A_1	2
$A_3\times A_2$, $\{2,3,4,6,7\}$	y_3	60	D_6	D_6	10
	$y_7\pm y_3$, $y_8\pm y_3$	42	A_6	A_6	7
	$y_7\pm y_8$	30	$D_4\times A_2$	D_4	6
	$y_7\pm 3y_3$, $y_8\pm 3y_3$	26	$A_4\times A_2$	A_4	5
	$\pm 2y_3+y_7-y_8$, $\pm 2y_3+y_7+y_8$	24	A_3^2	A_3	4
	y_7, y_8	20	$A_3\times A_2\times A_1$	A_1	2
$A_3\times A_1^2$, $\{2,3,5,6,7\}$	y_5	60	D_6	D_6	10
	y_7, y_8	42	$D_5\times A_1$	D_5	8
	$y_7\pm y_5$, $y_8\pm y_5$	32	$A_5\times A_1$	A_5	6
	$y_7\pm y_8$	24	A_3^2	A_3	4
	$\pm y_5+y_7-y_8$, $\pm y_5+y_7+y_8$	24	$A_4\times A_1^2$	A_4	5
	$\pm 2y_5+y_7$, $\pm 2y_5+y_8$	20	$A_3\times A_2\times A_1$	A_2	3
$A_2^2\times A_1$, $\{1,2,3,5,6\}$	y_5+y_8	72	E_6	E_6	12
	y_5, $y_7\pm y_5$	32	$A_5\times A_1$	A_5	6
	$y_7\pm y_8$, y_8-y_5, $-2y_5+y_7-y_8$, $2y_5+y_7+y_8$, $3y_5+y_8$	26	$A_4\times A_2$	A_4	5
	$-3y_5+y_7-2y_8$, $-y_5+y_7-2y_8$, y_8, $2y_5+y_8$, $y_5+y_7+2y_8$, $3y_5+y_7+2y_8$	20	$A_3\times A_2\times A_1$	A_3	4
	y_7, $\pm 3y_5+y_7$	16	$A_2^2\times A_1^2$	A_1	2
$A_2\times A_1^3$, $\{2,3,5,7,8\}$	y_5, y_8, y_5-y_8	42	$D_5\times A_1$	D_5	8
	y_3	30	$D_4\times A_2$	D_4	6
	$y_5\pm y_3$,$y_8\pm y_3$, $\pm y_3+y_5-y_8$	24	$A_4\times A_1^2$	A_4	5
	y_5-2y_8, $2y_5-y_8$, y_5+y_8	20	$A_3\times A_2\times A_1$	A_3	4
	$\pm y_3+y_5-2y_8$, $\pm y_3+2y_5-y_8$, $\pm y_3+y_5+y_8$	16	$A_2^2\times A_1^2$	A_2	3

Table 3.6a: $\mathcal{R}_{D,\beta}$, $\dim D = 2$, $\mathcal{R} = E_8$

$\mathcal{R}_D$, S	$\beta\vert_D$	$\vert\mathcal{R}_{D,\beta}\vert$	$\mathcal{R}_{D,\beta}$	$\mathcal{R}^{(0)}_{D,\beta}$	$h(\mathcal{R}^{(0)}_{D,\beta})$
A_6, $\{2,4,5,6,7,8\}$	y_7	126	E_7	E_7	18
	y_3, $2y_3 - y_7$	84	D_7	D_7	12
	$y_3 - y_7$, $2y_3 + y_7$	56	A_7	A_7	8
	$4y_3 - y_7$	44	$A_6 \times A_1$	A_1	2
D_6, $\{2,3,4,5,6,7\}$	y_8, y_7	126	E_7	E_7	18
	$y_7 \pm y_8$	84	D_7	D_7	12
E_6, $\{1,2,3,4,5,6\}$	y_6, $y_6 \pm y_7$	126	E_7	E_7	18
	$3y_6 \pm y_7$, y_7	74	$E_6 \times A_1$	A_1	2
$A_5 \times A_1$, $\{1,3,4,5,6,8\}$	$y_3 + y_8$	126	E_7	E_7	18
	y_8, $2y_3 + y_8$	56	A_7	A_7	8
	y_3	74	$E_6 \times A_1$	E_6	12
	$y_3 + 2y_8$, $3y_3 + 2y_8$	44	$A_6 \times A_1$	A_6	7
$D_5 \times A_1$, $\{2,3,4,5,6,8\}$	$y_6 + y_7$	126	E_7	E_7	18
	y_6	84	D_7	D_7	12
	$y_6 - y_7$, $3y_6 + y_7$	74	$E_6 \times A_1$	E_6	12
	y_7, $2y_6 + y_7$	46	$D_5 \times A_2$	A_2	3
$A_4 \times A_2$, $\{1,3,4,5,7,8\}$	y_8	126	E_7	E_7	18
	$y_3 + y_8$, $2y_3 + y_8$	56	A_7	A_7	8
	y_3, $2y_3 + 3y_8$	46	$D_5 \times A_2$	D_5	8
	$2y_3 - y_8$, $y_3 + 2y_8$	32	$A_4 \times A_3$	A_3	4
	$4y_3 + 3y_8$	28	$A_4 \times A_2 \times A_1$	A_1	2
$D_4 \times A_2$, $\{2,3,4,5,7,8\}$	y_7, y_8, $y_7 + y_8$	84	D_7	D_7	12
	$y_7 - y_8$, $2y_7 + y_8$, $y_7 + 2y_8$	46	$D_5 \times A_2$	D_5	8
$A_4 \times A_1^2$, $\{2,3,5,6,7,8\}$	y_3	84	D_7	D_7	12
	$y_3 - y_8$, y_8	74	$E_6 \times A_1$	E_6	12
	$2y_3 - y_8$, $y_3 + y_8$	44	$A_6 \times A_1$	A_6	7
	$y_3 - 2y_8$	32	$A_4 \times A_3$	A_3	4
	$3y_3 - y_8$, $2y_3 + y_8$	28	$A_4 \times A_2 \times A_1$	A_2	3
A_3^2, $\{2,3,4,6,7,8\}$	y_3, $y_3 - y_8$	84	D_7	D_7	12
	$3y_3 - y_8$, $y_3 + y_8$	56	A_7	A_7	8
	$2y_3 - y_8$, $5y_3 - y_8$, y_8, $3y_3 + y_8$	32	$A_4 \times A_3$	A_4	5
$A_3 \times A_2 \times A_1$, $\{1,2,4,6,7,8\}$	y_8	126	E_7	E_7	18
	$3y_5 \pm y_8$	44	$A_6 \times A_1$	A_6	7
	y_5	46	$D_5 \times A_2$	D_5	8
	$3y_5 \pm 2y_8$	32	$A_4 \times A_3$	A_4	5
	$y_5 \pm y_8$	28	$A_4 \times A_2 \times A_1$	A_4	5
$A_2^2 \times A_1^2$, $\{1,2,3,5,6,8\}$	y_5, $y_5 + y_8$	74	$E_6 \times A_1$	E_6	12
	$y_5 - y_8$, $3y_5 + y_8$	46	$D_5 \times A_2$	D_5	8
	y_8, $3y_5 - y_8$, $2y_5 + y_8$, $5y_5 + y_8$	28	$A_4 \times A_2 \times A_1$	A_4	5

3.8 Dubrovin's duality on discriminant strata revisited

In this section we revisit almost duality of Frobenius manifolds on discriminant strata (see Chapter 2). Such a duality was considered in [35], and it was suggested in [81] that discriminant strata are natural submanifolds. We can now give all the details to complete this study proving that multiplication of tangential vectors from each stratum belongs to the stratum.

Let us recall that for or any $x \in \mathcal{M}_W \setminus \Sigma$ the almost dual Frobenius multiplication is defined by the formula

$$u * v = E^{-1} \circ u \circ v, \tag{3.334}$$

where $u, v \in T_x\mathcal{M}_W$ and E is the Euler vector field

$$E = \frac{1}{h} x^i \frac{\partial}{\partial x^i} = \frac{1}{h} \sum_{\alpha} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}.$$

Recall also that E is the identity field of the almost dual multiplication $*$. Let e^{-1} denote the inverse field of e with respect to the almost dual multiplication, namely $e^{-1} * e = E$. It follows by formula (3.334) that $E = E^{-1} \circ e^{-1}$, and hence e^{-1} can be represented as

$$e^{-1} = E \circ E. \tag{3.335}$$

Note that we also have by formulae (3.334), (3.335) that

$$e^{-1} * u * v = E^{-1} \circ (e^{-1} * u) \circ v = E^{-1} \circ (E^{-1} \circ e^{-1} \circ u) \circ v = u \circ v. \tag{3.336}$$

Let us now recall that Saito metric η and metric g are related as follows:

$$\eta(u, v) = g(E \circ u, v). \tag{3.337}$$

Let us consider the vector field $e^{-1} = e^{-1}(x)$ as a vector field on V, $x \in V$.

Lemma 3.8.1. *The vector field $e^{-1}(x)$ is well-defined at $x_0 \in D$. Moreover, $e^{-1}(x_0) \in T_{x_0}D$.*

Proof. We have by formulae (3.336) and (3.337) that

$$\eta(u, v) = g(E \circ u, v) = g(e^{-1} * u, v). \tag{3.338}$$

For the components $(e^{-1})^j$ $(1 \leq j \leq n)$ of the vector field e^{-1} we have

$$(e^{-1})^j = g(e^{-1}, \frac{\partial}{\partial x^j}) = g(e^{-1} * E, \frac{\partial}{\partial x^j}) = \eta(E, \frac{\partial}{\partial x^j}),$$

where the last equality follows by (3.338). Then

$$(e^{-1})^j = \frac{1}{h}\sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial t^\beta}{\partial x^j}\eta(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}) = \frac{1}{h}\sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial t^\beta}{\partial x^j}\eta_{\alpha\beta} = \frac{1}{h}\sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial t^{n+1-\alpha}}{\partial x^j}, \quad (3.339)$$

since $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$. Thus the first part of the statement follows. Let $\gamma \in \mathcal{R}_D$, and let ∂_γ be the corresponding vector field orthogonal to D. Then we have by formula (3.339) that $(e^{-1}(x), \partial_\gamma) = 0$, as x tends to x_0. Therefore $e^{-1}(x_0) \in T_{x_0}D$, as required. □

Let Σ_D be the union of the hyperplanes $\Pi_\gamma \cap D$ in D, where $\gamma \in \mathcal{R} \setminus \mathcal{R}_D$ and consider point x_0 in $D \setminus \Sigma_D$. Let $u, v \in T_{x_0}D$ and consider extension of u and v to two local analytic vector fields $u(x), v(x) \in T_xV$ such that $u(x_0) = u$ and $v = v(x_0)$. Recall that the multiplication $u(x) * v(x)$ has a limit when x tends to x_0 and furthermore that the product $u * v$ at x_0 is tangential to D (Lemma 2.5.9). As a result we get the following statement using Lemma 3.8.1 and formula (3.336).

Proposition 3.8.2. *Let $u, v \in T_{x_0}D$, $x_0 \in D \setminus \Sigma_D$. Then $u \circ v \in T_{x_0}D$. Furthermore, $u \circ v = e^{-1} * u * v$.*

The proposition is the strengthening of the results and observations from [35, 81]. Namely, in Dubrovin's duality formula (3.336) both sides are well-defined if $u, v \in T_{x_0}D$ and equality remains to be satisfied on D.

Chapter 4

Supersymmetric $\vee$–Systems

We construct $\mathcal{N}=4$ $D(2,1;\alpha)$ superconformal quantum mechanical system for any configuration of vectors forming a $\vee$-system. In the case of a Coxeter root system the bosonic potential of the supersymmetric Hamiltonian is the corresponding generalised Calogero–Moser potential. We also construct supersymmetric generalised trigonometric Calogero–Moser–Sutherland Hamiltonians for some root systems including BC_N.

4.1 The $D(2,1;\alpha)$ Lie superalgebra

Let us recall the definition of the family of Lie superalgebras $D(2,1;\alpha)$ which depends on a parameter $\alpha \in \mathbb{C}$ (see e.g. [38, p. 29]). This algebra has 8 odd generators F_{abc}, and 9 even generators $T_i^{(j)}$ $(i,j=1,2,3)$ so that for each fixed j they generate mutually commuting $sl(2)$ algebras. In the context of supersymmetry the generators of the algebra usually appear in a slighty different form (see e.g. [32]). We relate the forms of the algebra $D(2,1;\alpha)$ given in [38] and [32] as follows.

We take $Q^{abc} \equiv F_{abc}$, $s_1=-2i$, $s_2=2i(1+\alpha)$, $s_3=-2i\alpha$ and introduce different generators in the $sl(2)$

$$ {}^{(j)}\Sigma^{11} = -T_2^{(j)} + iT_1^{(j)}, \quad {}^{(j)}\Sigma^{22} = -T_2^{(j)} - iT_1^{(j)}, \quad {}^{(j)}\Sigma^{12} ={}^{(j)} \Sigma^{21} = -iT_3^{(j)}. \tag{4.1} $$

Further on, we re-denote generators ${}^{(j)}\Sigma^{ab}$ as follows: ${}^{(1)}\Sigma^{ab}=T^{ab}$, ${}^{(2)}\Sigma^{ab}=I^{ab}$ and ${}^{(3)}\Sigma^{ab}=J^{ab}$, $a,b=1,2$. Let ϵ_{ab}, ϵ^{ab} be the fully anti-symmetric tensors in two dimensions such that $\epsilon_{12}=\epsilon^{21}=1$. Then all the relations of the superalgebra $D(2,1;\alpha)$ take the following form:

$$ \{Q^{ace}, Q^{bdf}\} = -2\big(\epsilon^{ef}\epsilon^{cd}T^{ab} + \alpha\epsilon^{ab}\epsilon^{cd}J^{ef} - (\alpha+1)\epsilon^{ab}\epsilon^{ef}I^{cd}\big), \tag{4.2} $$

$$ [T^{ab}, T^{cd}] = -i\big(\epsilon^{ac}T^{bd} + \epsilon^{bd}T^{ac}\big), \tag{4.3} $$

$$\mathbf{a})\,[J^{ab},J^{cd}]=-i\big(\epsilon^{ac}J^{bd}+\epsilon^{bd}J^{ac}\big),\quad \mathbf{b})\,[I^{ab},I^{cd}]=-i\big(\epsilon^{ac}I^{bd}+\epsilon^{bd}I^{ac}\big),\tag{4.4}$$

$$\mathbf{a})\,[T^{ab},Q^{cdf}]=i\epsilon^{c(a}Q^{b)df},\quad \mathbf{b})\,[J^{ab},Q^{cdf}]=i\epsilon^{f(a}Q^{|cd|b)},\quad \mathbf{c})\,[I^{ab},Q^{cdf}]=i\epsilon^{d(a}Q^{|c|b)f},\tag{4.5}$$

where we symmetrise over two indices inside $(\dots)$ with indices inside $|\dots|$ being unchanged. For example, $\epsilon^{f(a}Q^{|cd|b)}=\frac{1}{2}\big(\epsilon^{fa}Q^{cdb}+\epsilon^{fb}Q^{cda}\big)$. We also have relations

$$[T^{ab},I^{cd}]=[I^{cd},J^{ef}]=[T^{ab},J^{ef}]=0,\tag{4.6}$$

for all $a,b,c,d,e,f=1,2$. Let us rename generators as follows:

$$Q^a=-Q^{21a},\quad \bar{Q}^a=-Q^{22a},\quad S^a=Q^{11a},\quad \bar{S}^a=Q^{12a},\quad a=1,2,$$

$$K=T^{11},\quad H=T^{22},\quad D=-T^{12}=-T^{21}.$$

We will use ϵ_{ab} and ϵ^{ab} to lower and raise indices, for example $Q^a=\epsilon^{ab}Q_b$, $\bar{Q}^a=\epsilon^{ab}\bar{Q}_b$. We consider N (quantum) particles on a line with coordinates and momenta (x_j,p_j), $j=1,\dots,N$ to each of which we associate four fermionic variables $\{\psi^{aj},\bar{\psi}^j_a|a=1,2\}$. We will also write $x=(x_1,\dots,x_N)$, $p=(p_1,\dots,p_N)$. We realise these variables as operators acting on wavefunctions which lie on the tensor product of the Hilbert space of functions of x and a 4^N-dimensional fermionic Fock space.

We assume the following (anti)-commutation relations ($a,b=1,2;j,k=1,\dots,N$):

$$[x_j,p_k]=i\delta_{jk},\quad \{\psi^{aj},\bar{\psi}^k_b\}=-\frac{1}{2}\delta^{jk}\delta^a_b,\quad \{\psi^{aj},\psi^{bk}\}=\{\bar{\psi}^j_a,\bar{\psi}^k_b\}=0.\tag{4.7}$$

Thus one can think of p_k as $p_k=-i\frac{\partial}{\partial x_k}$. We introduce further fermionic variables by

$$\psi^j_a=\epsilon_{ab}\psi^{bj},\quad \bar{\psi}^{aj}=\epsilon^{ab}\bar{\psi}^j_b.\tag{4.8}$$

They satisfy the following useful relations:

$$\{\psi^j_a,\bar{\psi}^{bk}\}=\frac{1}{2}\delta^{jk}\delta^b_a,\quad \{\psi^{aj},\bar{\psi}^{bk}\}=\frac{1}{2}\delta^{jk}\epsilon^{ab},\quad \{\psi^j_a,\bar{\psi}^k_b\}=\frac{1}{2}\delta^{jk}\epsilon_{ba}.\tag{4.9}$$

We will be assuming throughout that summation over repeated indices takes place (even when both indices are either low or upper indices) unless it is indicated that no summation is applied.

In addition it is convenient to define an involutive operation on any operator which we denote by '∼' and with the property that $\widetilde{AB}=\widetilde{A}\widetilde{B}$ for any operators A, B and

$$\widetilde{\psi^j_a}=\bar{\psi}^{aj},\quad \widetilde{\psi^{aj}}=\bar{\psi}^j_a,\quad \widetilde{p_j}=-p_j,\quad \widetilde{x_j}=x_j,\quad \tilde{i}=-i,\quad \widetilde{\epsilon_{ab}}=\epsilon^{ab},\quad \widetilde{\alpha}=\alpha.\tag{4.10}$$

Note that this is compatible with (4.8), and that one has to keep record of ϵ^{ab} when dealing with fermions with upper and lower indices and applying $\sim$.

Let $F = F(x_1, \ldots, x_N)$ be a function such that

$$x_r F_{rjk} = -(2\alpha+1)\delta_{jk}, \tag{4.11}$$

where $F_{rjk} = \frac{\partial^3 F}{\partial x_r \partial x_j \partial x_k}$ for any $r, j, k = 1, \ldots, N$. We assume that all the derivatives F_{rjk} are homogeneous in x of degree -1. Furthermore, we assume that F satisfies the following generalised WDVV equations (cf. (2.59))

$$F_{rjk}F_{kmn} = F_{rmk}F_{kjn}, \tag{4.12}$$

for any $r, j, m, n = 1, \ldots, N$.

The following relations for arbitrary operators A, B, C will be useful:

$$[AB, C] = A[B, C] + [A, C]B, \tag{4.13}$$
$$[AB, C] = A\{B, C\} - \{A, C\}B, \tag{4.14}$$
$$\{AB, C\} = A[B, C] + \{A, C\}B. \tag{4.15}$$

We are going to present two representations of $D(2, 1; \alpha)$ algebra using F.

4.2 The first representation

Let the supercharges be of the form

$$Q^a = p_r\psi^{ar} + iF_{rjk}\langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle, \tag{4.16}$$
$$\bar{Q}_c = p_l\bar{\psi}^l_c + iF_{lmn}\langle\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c\rangle, \tag{4.17}$$

where the symbol $\langle\ldots\rangle$ stands for the anti-symmetrisation. That is given N operators A_i, $(i = 1, \ldots, N)$ we define

$$\langle A_1 \ldots A_N\rangle = \frac{1}{N!}\sum_{\sigma\in S_N} \mathrm{sgn}(\sigma)A_{\sigma(1)}\ldots A_{\sigma(N)}. \tag{4.18}$$

Note that we have by (4.7), (4.9) and (4.18)

$$\begin{aligned}\langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle &= \frac{1}{6}(2\psi^{br}\psi^j_b\bar{\psi}^{ak} + 2\bar{\psi}^{ak}\psi^{br}\psi^j_b - \psi^{br}\bar{\psi}^{ak}\psi^j_b + \psi^j_b\bar{\psi}^{ak}\psi^{br})\\ &= \frac{1}{3}(\psi^{br}\psi^j_b\bar{\psi}^{ak} + \bar{\psi}^{ak}\psi^{br}\psi^j_b - \psi^{br}\bar{\psi}^{ak}\psi^j_b) + \frac{1}{12}(\delta^{jk}\psi^{ar} - \delta^{rk}\psi^{aj})\\ &= \psi^{br}\psi^j_b\bar{\psi}^{ak} - \frac{1}{6}\delta^{rk}\psi^{aj} - \frac{1}{3}\delta^{jk}\psi^{ar} + \frac{1}{12}(\delta^{jk}\psi^{ar} - \delta^{rk}\psi^{aj}).\end{aligned}$$

Note that $F_{rjk}(\delta^{jk}\psi^{ar} - \delta^{rk}\psi^{aj}) = 0$ since $\delta^{jk}\psi^{ar} - \delta^{rk}\psi^{aj}$ is anti-symmetric under the interchange of j and r. Therefore

$$F_{rjk}\langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle = F_{rjk}(\psi^{br}\psi^j_b\bar{\psi}^{ak} - \frac{1}{2}\psi^{ar}\delta^{jk}). \tag{4.19}$$

Similarly,

$$F_{lmn}\langle\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c\rangle = F_{lmn}(\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c - \frac{1}{2}\bar{\psi}^l_c\delta^{nm}). \tag{4.20}$$

Let also

$$K = x^2 = \sum_{j=1}^N x_j^2, \tag{4.21}$$

$$D = -\frac{1}{4}\{x_j, p_j\} = -\frac{1}{2}x_j p_j + \frac{iN}{2}, \tag{4.22}$$

$$I^{11} = -i\psi^j_a\psi^{aj}, \quad I^{22} = i\bar{\psi}^{aj}\bar{\psi}^j_a, \quad I^{12} = -\frac{i}{2}[\psi^j_a, \bar{\psi}^{aj}], \tag{4.23}$$

$$J^{ab} = J^{ba} = 2i\psi^{(aj}\bar{\psi}^{bj)}, \tag{4.24}$$

$$S^a = -2x_j\psi^{aj}, \quad \bar{S}_a = -2x_j\bar{\psi}^j_a. \tag{4.25}$$

Remark 4.2.1. Ansatz for the supercharges (4.16), (4.17) is the same as the ansatz introduced by Wyllard (see formula (2.15) in [88]) under a suitable scaling of the variables x, p and the following identification of fermionic variables:

$$\theta_k = i\sqrt{2}\psi^{1k}, \quad \tilde{\theta}_k = i\sqrt{2}\psi^{2k}, \quad \frac{\partial}{\partial\theta_k} = i\sqrt{2}\bar{\psi}^{1k}, \quad \frac{\partial}{\partial\tilde{\theta}_k} = i\sqrt{2}\bar{\psi}^{2k}.$$

Remark 4.2.2. Ansatz (4.16), (4.17), (4.21), (4.22), (4.24), (4.25) with F satisfying (4.11) at $\alpha = -1$ matches considerations in [40], [41] (see also [88]), where $su(1,1|2)$ superconformal mechanics were considered. Note that the superalgebra $su(1,1|2)$ generated by Q^{abc}, T^{ab}, J^{ab} is a subalgebra in the superalgebra $D(2,1;-1)$. Thus Lemmas 4.2.3, 4.2.5 below can be deduced from considerations in [40], [41]. We include these lemmas so that to have complete derivations for reader's convenience. Let us give some details on this correspondence.

For the sake of clarity let us denote generators from [41] with a 'hat'. In this case we deal with algebra generated by supercharges $\widehat{Q_a}$, $\widehat{\bar{Q}^a}$, their superconformal partners $\widehat{S_a}$, $\widehat{\bar{S}^a}$ $(a = 1, 2)$ and two $sl(2)$ algebras with generators J_a, $(a = 1, 2, 3)$ and $\widehat{K}$, $\widehat{H}$, $\widehat{D}$. Then bosonic and fermionic variables in [41] are related to the same variables defined in the present work as follows:

$$\widehat{x^k} = \sqrt{2}x_k, \quad \widehat{p_k} = \frac{1}{\sqrt{2}}p_k, \quad \widehat{\psi^k_a} = \sqrt{2}\psi^k_a, \quad \widehat{\bar{\psi}^{ak}} = \sqrt{2}\bar{\psi}^{ak}.$$

Let us then consider the generators J_a from [41] which are defined as $J_a = \frac{1}{2}\widehat{\bar\psi^{bk}}(\sigma_a)_b^c\widehat{\psi_c^k} = \bar\psi^{bk}(\sigma_a)_b^c\psi_c^k$, where σ_a $(a = 1,2,3)$ denote the Pauli matrices (see (3.5) in [41]). The generators J_a satisfy the relations $[J_a, J_b] = i\epsilon_{abc}J_c$. Then we have

$$J_1 = \bar\psi^{2k}\psi^{2k} - \bar\psi^{1k}\psi^{1k}, \quad J_2 = i(\bar\psi^{1k}\psi^{1k} + \bar\psi^{2k}\psi^{2k}), \quad J_3 = \bar\psi^{1k}\psi^{2k} + \bar\psi^{2k}\psi^{1k}.$$

Hence we obtain the following correspondence between the generators J^{ab} and J_a using (4.9):

$$J^{11} = 2i\psi^{1k}\bar\psi^{1k} = -J_2 + iJ_1, \quad J^{22} = 2i\psi^{2k}\bar\psi^{2k} = -J_2 - iJ_1,$$

$$J^{12} = i(\psi^{1k}\bar\psi^{2k} + \psi^{2k}\bar\psi^{1k}) = -iJ_3.$$

Then it is easy to to recover relations (4.4a). Finally we take

$$\widehat{S_a} = -S_a, \quad \widehat{\bar S^a} = -\bar S^a, \quad \widehat{\bar Q^a} = \bar Q^a, \quad \widehat{Q_a} = Q_a, \quad \widehat{K} = K, \quad \widehat{H} = H, \quad \widehat{D} = D.$$

Let us firstly check relations (4.4), (4.5) involving generators J^{ab} and I^{ab}.

Lemma 4.2.3 (cf. [40], [41])**.** *Let J^{ab} be given by* (4.24)*. Then relations (4.4a) hold.*

Proof. We consider the commutator

$$\begin{aligned}[][\psi^{aj}\bar\psi^{bj}, \psi^{ck}\bar\psi^{dk}] &= \psi^{aj}[\bar\psi^{bj}, \psi^{ck}\bar\psi^{dk}] + [\psi^{aj}, \psi^{ck}\bar\psi^{dk}]\bar\psi^{bj} \\ &= \frac{1}{2}\epsilon^{cb}\psi^{aj}\bar\psi^{dj} + \frac{1}{2}\epsilon^{da}\psi^{cj}\bar\psi^{bj},\end{aligned}$$

which implies the statement. □

We will use the following relations:

$$[\bar\psi^{bk}, \psi_a^j\psi^{aj}] = \psi^{bk}, \quad [\bar\psi^{aj}\bar\psi_a^j, \psi_b^k] = -\bar\psi_b^k. \tag{4.26}$$

Lemma 4.2.4. *Let I^{ab} be given by* (4.23)*. Then relations (4.4b) hold.*

Proof. The relations (4.4b) read

$$[I^{11}, I^{22}] = 2iI^{12}, \quad [I^{11}, I^{12}] = iI^{11}, \quad [I^{22}, I^{12}] = -iI^{22}.$$

We have

$$[I^{11}, I^{22}] = [\psi_a^j\psi^{aj}, \bar\psi^{bk}\bar\psi_b^k]. \tag{4.27}$$

By applying (4.13), (4.14) we rearrange expression (4.27) as

$$\begin{aligned}[I^{11}, I^{22}] &= \psi_a^j[\psi^{aj}, \bar{\psi}^{bk}\bar{\psi}_b^k] + [\psi_a^j, \bar{\psi}^{bk}\bar{\psi}_b^k]\psi^{aj}\\ &= \psi_a^j\bar{\psi}^{aj} + \bar{\psi}_a^j\psi^{aj} = \psi_a^j\bar{\psi}^{aj} - \bar{\psi}^{aj}\psi_a^j\\ &= 2iI^{12},\end{aligned}$$

as required. Moreover, using the Jacobi identity we have

$$[I^{11}, I^{12}] = -\frac{1}{2}[\psi_a^j\psi^{aj}, [\psi_b^k, \bar{\psi}^{bk}]] = \frac{1}{2}[\psi_b^k, [\bar{\psi}^{bk}, \psi_a^j\psi^{aj}]].$$

Thus by using the first relation in (4.26)

$$[I^{11}, I^{12}] = \psi_b^k\psi^{bk} = iI^{11}.$$

Similarly,

$$[I^{22}, I^{12}] = \frac{1}{2}[\bar{\psi}^{aj}\bar{\psi}_a^j, [\psi_b^k, \bar{\psi}^{bk}]] = -\frac{1}{2}[\bar{\psi}^{bk}, [\bar{\psi}^{aj}\bar{\psi}_a^j, \psi_b^k]].$$

Hence, by using the latter relation in (4.26)

$$[I^{22}, I^{12}] = \bar{\psi}^{bk}\bar{\psi}_b^k = -iI^{22},$$

and hence the statement follows. □

In what follows, we will use the following relation:

$$[\psi^{aj}\bar{\psi}^{bj}, \psi^{cl}] = -\frac{1}{2}\epsilon^{bc}\psi^{al}. \tag{4.28}$$

By formulae (4.13), (4.14) we also have

$$\begin{aligned}[\psi^{aj}\bar{\psi}^{bj}, \psi^{dl}\psi_d^m\bar{\psi}^{cn}] &= \psi^{dl}\psi_d^m[\psi^{aj}\bar{\psi}^{bj}, \bar{\psi}^{cn}] + [\psi^{aj}\bar{\psi}^{bj}, \psi^{dl}\psi_d^m]\bar{\psi}^{cn}\\ &= -\psi^{dl}\psi_d^m\bar{\psi}^{bj}\{\bar{\psi}^{cn}, \psi^{aj}\} + \psi^{dl}[\psi^{aj}\bar{\psi}^{bj}, \psi_d^m]\bar{\psi}^{cn} + [\psi^{aj}\bar{\psi}^{bj}, \psi^{dl}]\psi_d^m\bar{\psi}^{cn}\\ &= \frac{1}{2}\epsilon^{ca}\psi^{dl}\psi_d^m\bar{\psi}^{bn} + \frac{1}{2}\psi^{bl}\psi^{am}\bar{\psi}^{cn} + \frac{1}{2}\psi^{bm}\psi^{al}\bar{\psi}^{cn}.\end{aligned} \tag{4.29}$$

Lemma 4.2.5 (cf. [40], [41])**.** *Let Q^{abc}, J^{ab} be as above. Then the relations (4.5b) hold.*

Proof. Firstly let us note that the sum of the last two terms in (4.29) is anti-symmetric in a and b and $J^{ab} = J^{ba}$. Therefore we have by applying (4.29)

$$[J^{ab}, F_{lmn}\psi^{dl}\psi_d^m\bar{\psi}^{cn}] = \frac{i}{2}\epsilon^{ca}F_{lmn}\psi^{dl}\psi_d^m\bar{\psi}^{bn} + \frac{i}{2}\epsilon^{cb}F_{lmn}\psi^{dl}\psi_d^m\bar{\psi}^{an}. \tag{4.30}$$

Then

$$[J^{ab}, Q^{21c}] = -[J^{ab}, Q^c] = -[J^{ab}, p_l\psi^{cl}] - iF_{lmn}[J^{ab}, \langle \psi^{dl}\psi_d^m\bar{\psi}^{cn}\rangle].$$

Therefore we get from (4.28) and (4.30) that

$$\begin{aligned}[J^{ab}, Q^{21c}] &= \frac{i}{2}\big(\epsilon^{bc}p_l\psi^{al} + \epsilon^{ac}p_l\psi^{bl} - i\epsilon^{ca}F_{lmn}\langle \psi^{dl}\psi_d^m\bar{\psi}^{bn}\rangle - i\epsilon^{cb}F_{lmn}\langle \psi^{dl}\psi_d^m\bar{\psi}^{an}\rangle\big) \\ &= -\frac{i}{2}(\epsilon^{cb}Q^a + \epsilon^{ca}Q^b) = i\epsilon^{c(a}Q^{|21|b)},\end{aligned} \tag{4.31}$$

as required in (4.5b). Further, we consider

$$[J^{ab}, S^c] = -2x_l[J^{ab}, \psi^{cl}] = ix_l(\epsilon^{bc}\psi^{al} + \epsilon^{ac}\psi^{bl}) = \frac{i}{2}(\epsilon^{cb}S^a + \epsilon^{ca}S^b) = i\epsilon^{c(a}Q^{|11|b)},$$

which coincides with the corresponding relation in (4.5b). Finally, applying $\sim$ to $[J^{ab}, Q^{21c}]$ and $[J^{ab}, S^c]$ we obtain the remaining relations (see also Lemma B.1.3). □

Lemma 4.2.6. *Let Q^{abc}, I^{ab} be as above. Then relations (4.5c) hold.*

Proof. Let us first consider $[I^{11}, Q^{21a}]$. Using formulae (4.13), (4.14) we have

$$[\psi_d^r\psi^{dr}, \psi^{bl}\psi_b^m\bar{\psi}^{an}] = \psi^{bl}\psi_b^m[\psi_d^r\psi^{dr}, \bar{\psi}^{an}] = -\psi^{bl}\psi_b^m\psi^{an}. \tag{4.32}$$

It follows that $F_{lmn}[\psi_d^r\psi^{dr}, \psi^{bl}\psi_b^m\bar{\psi}^{an}] = 0$ and hence

$$[I^{11}, Q^{21a}] = i[\psi_d^r\psi^{dr}, Q^a] = i[\psi_d^r\psi^{dr}, p_l\psi^{al}] = 0, \tag{4.33}$$

as required for (4.5c). Let us now consider $[I^{22}, Q^{21a}]$. We have

$$[I^{22}, \psi^{al}] = i[\bar{\psi}^{dr}\bar{\psi}_d^r, \psi^{al}] = -i\bar{\psi}^{al}, \tag{4.34}$$

and hence

$$\begin{aligned}[\bar{\psi}^{dr}\bar{\psi}_d^r, \psi^{bl}\psi_b^m\bar{\psi}^{an}] &= -[\psi^{bl}\psi_b^m, \bar{\psi}^{dr}\bar{\psi}_d^r]\bar{\psi}^{an} \\ &= (\psi^{bl}[\bar{\psi}^{dr}\bar{\psi}_d^r, \psi_b^m] + [\bar{\psi}^{dr}\bar{\psi}_d^r, \psi^{bl}]\psi_b^m)\bar{\psi}^{an} \\ &= -\psi^{bl}\bar{\psi}_b^m\bar{\psi}^{an} - \bar{\psi}^{bl}\psi_b^m\bar{\psi}^{an}.\end{aligned} \tag{4.35}$$

By reordering terms in (4.35) we obtain

$$\begin{aligned}[\bar{\psi}^{dr}\bar{\psi}_d^r, \psi^{bl}\psi_b^m\bar{\psi}^{an}] &= (\bar{\psi}_b^m\psi^{bl} + \delta^{lm})\bar{\psi}^{an} + \bar{\psi}^{bl}(\bar{\psi}^{an}\psi_b^m - \frac{1}{2}\delta_b^a\delta^{nm}) \\ &= -\bar{\psi}_b^m\bar{\psi}^{an}\psi^{bl} - \frac{1}{2}\bar{\psi}^{am}\delta^{ln} + \delta^{lm}\bar{\psi}^{an} - \bar{\psi}_b^l\bar{\psi}^{an}\psi^{bm} - \frac{1}{2}\bar{\psi}^{al}\delta^{nm}.\end{aligned}$$

Therefore

$$F_{lmn}[\bar\psi^{dr}\bar\psi^r_d, \psi^{bl}\psi^m_b\bar\psi^{an}] = -2F_{lmn}\bar\psi^l_b\bar\psi^{an}\psi^{bm}. \tag{4.36}$$

Note that $F_{lmn}\bar\psi^l_c\bar\psi^{an}\psi^{cm} = 0$ if c is fixed such that $c \neq a$. Hence (4.36) can be rearranged as $-2F_{lmn}\bar\psi^l_a\bar\psi^{am}\psi^{an}$ which is also equal to $-F_{lmn}\bar\psi^l_b\bar\psi^{bm}\psi^{an}$. Therefore

$$[I^{22}, Q^{21a}] = -i[\bar\psi^{dr}\bar\psi^r_d, Q^a] = -i\big(-p_l\bar\psi^{al} + iF_{lmn}(-\bar\psi^l_b\bar\psi^{bm}\psi^{an} + \frac{1}{2}\bar\psi^{al}\delta^{nm})\big) = i\bar Q^a, \tag{4.37}$$

as required for (4.5c).

Further, let us consider $[I^{12}, Q^{21a}] = i[\psi^r_d\bar\psi^{dr}, Q^a]$. Then by (4.29) we have

$$[\psi^r_d\bar\psi^{dr}, \psi^{bl}\psi^m_b\bar\psi^{an}] = \frac{1}{2}\psi^{bl}\psi^m_b\bar\psi^{an}.$$

Therefore, with the help of (4.28) we get

$$[I^{12}, Q^{21a}] = \frac{i}{2}\big(p_l\psi^{al} + iF_{lmn}(\psi^{bl}\psi^m_b\bar\psi^{an} - \frac{1}{2}\psi^{al}\delta^{mn})\big) = \frac{i}{2}Q^a, \tag{4.38}$$

which matches with (4.5c).

Let us now consider the generator Q^{11a}. Firstly, it is immediate that $[I^{11}, Q^{11a}] = 0$, as required. In addition, we have by (4.34) that

$$[I^{22}, Q^{11a}] = i[\bar\psi^{dr}\bar\psi^r_d, S^a] = -2ix_j[\bar\psi^{dr}\bar\psi^r_d, \psi^{aj}] = -i\bar S^a,$$

and

$$[I^{12}, S^a] = -i[\psi^r_d\bar\psi^{dr}, S^a] = ix_j\psi^{aj} = -\frac{i}{2}S^a,$$

as required for (4.5c). The remaining relations in (4.5c) can be checked similarly by applying $\sim$. □

Let A_i, B_i $(i = 1, 2, 3)$ be operators. In the following theorem we will use the identity

$$\begin{aligned}\{A_1A_2A_3, B_1B_2B_3\} = {} & A_1A_2\{A_3, B_1\}B_2B_3 + A_1A_2B_1B_2\{B_3, A_3\} - A_1A_2B_1\{B_2, A_3\}B_3 - \\ & - A_1\{A_2, B_1\}B_2B_3A_3 - A_1B_1B_2\{B_3, A_2\}A_3 + A_1B_1\{B_2, A_2\}B_2A_3 \\ & + \{A_1, B_1\}B_2B_3A_2A_3 + B_1B_2\{B_3, A_1\}A_2A_3 - B_1\{B_2, A_1\}B_3A_2A_3.\end{aligned} \tag{4.39}$$

We will also use the following relations. We have by (4.13) and (4.15)

$$\{\psi^{ar}, \bar\psi^l_d\bar\psi^{dm}\psi^n_c\} = \bar\psi^l_d[\bar\psi^{dm}\psi^n_c, \psi^{ar}] + \bar\psi^{dm}\psi^n_c\{\bar\psi^l_d, \psi^{ar}\} = -\frac{1}{2}\bar\psi^{al}\psi^n_c\delta^{rm} - \frac{1}{2}\bar\psi^{am}\psi^n_c\delta^{rl}, \tag{4.40}$$

and similarly,

$$\{\bar\psi^l_c, \psi^{br}\psi^j_b\bar\psi^{ak}\} = -\frac{1}{2}\psi^r_c\bar\psi^{ak}\delta^{jl} - \frac{1}{2}\psi^j_c\bar\psi^{ak}\delta^{rl}. \tag{4.41}$$

Theorem 4.2.7. *For all $a, b = 1, 2$ we have $\{Q^a, \bar Q_b\} = -2H\delta^a_b$, where the Hamiltonian H is given by*

$$H = \frac{p^2}{4} - \frac{\partial_i F_{jlk}}{2}(\psi^{bi}\psi^j_b\bar\psi^l_d\bar\psi^{dk} - \psi^i_b\bar\psi^{bj}\delta^{lk} + \frac{1}{4}\delta^{ij}\delta^{lk}) + \frac{1}{16}F_{ijk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{ik} \tag{4.42}$$

with $p^2 = \sum_{i=1}^N p_i^2$.

Proof. Let us consider $\{Q^a, \bar Q_c\}$, where

$$Q^a = \overbrace{p_r\psi^{ar}}^{A} + \overbrace{iF_{rjk}\langle\psi^{br}\psi^j_b\bar\psi^{ak}\rangle}^{B}, \quad \bar Q_c = \overbrace{p_l\bar\psi^l_c}^{A'} + \overbrace{iF_{lmn}\langle\bar\psi^l_d\bar\psi^{dm}\psi^n_c\rangle}^{B'}.$$

We have

$$\{A, A'\} = -\frac{1}{2}\delta^a_c p^2.$$

Further on, by (4.20) we have

$$\begin{aligned}\{A, B'\} &= i\{\psi^{ar}p_r, F_{lmn}\langle\bar\psi^l_d\bar\psi^{dm}\psi^n_c\rangle\}\\ &= i\{\psi^{ar}p_r, F_{lmn}\bar\psi^l_d\bar\psi^{dm}\psi^n_c\} - \frac{i}{2}\delta^{nm}\{p_r\psi^{ar}, F_{lmn}\bar\psi^l_c\}\\ &= i\psi^{ar}\bar\psi^l_d\bar\psi^{dm}\psi^n_c[p_r, F_{lmn}] + i\{\psi^{ar}, \bar\psi^l_d\bar\psi^{dm}\psi^n_c\}F_{lmn}p_r -\\ &- \frac{i}{2}\delta^{nm}\psi^{ar}\bar\psi^l_c[p_r, F_{lmn}] + \frac{i}{4}\delta^{nm}\delta^a_c F_{rmn}p_r.\end{aligned}$$

By (4.40) we have

$$F_{lmn}\{\psi^{ar}, \bar\psi^l_d\bar\psi^{dm}\psi^n_c\} = -F_{lmn}\bar\psi^{al}\psi^n_c\delta^{rm}.$$

Therefore,

$$\{A, B'\} = i\psi^{ar}\bar\psi^l_d\bar\psi^{dm}\psi^n_c[p_r, F_{lmn}] - i\bar\psi^{al}\psi^n_c F_{lnr}p_r - \frac{i}{2}\delta^{nm}\psi^{ar}\bar\psi^l_c[p_r, F_{lmn}] + \frac{i}{4}\delta^{nm}\delta^a_c F_{rmn}p_r. \tag{4.43}$$

Similarly, using (4.41) we obtain

$$\{B, A'\} = i\bar\psi^l_c\psi^{br}\psi^j_b\bar\psi^{ak}[p_l, F_{rjk}] - i\psi^r_c\bar\psi^{ak}F_{rkj}p_j - \frac{i}{2}\delta^{jk}\bar\psi^l_c\psi^{ar}[p_l, F_{rjk}] + \frac{i}{4}\delta^{jk}\delta^a_c F_{rjk}p_r. \tag{4.44}$$

Note that $\bar\psi^{al}\psi^n_c F_{lnr}p_r + \psi^r_c\bar\psi^{ak}F_{rkj}p_j = \frac{1}{2}\delta^{ln}\delta^{ac}F_{lnr}p_r$. Then, after cancelling out terms and simplifying we have

$$\{A, B'\} + \{B, A'\} = \partial_r F_{ljk}(\psi^{ar}\bar\psi^l_d\bar\psi^{dk}\psi^j_c + \bar\psi^l_c\psi^{br}\psi^j_b\bar\psi^{ak}) + \frac{1}{4}\partial_r F_{lmn}\delta^{nm}\delta^{rl}\delta^a_c. \tag{4.45}$$

In particular, we note that using the symmetry of F_{ljk} we have that

$$\partial_r F_{ljk}\psi^{ar}\bar{\psi}^l_d\bar{\psi}^{dk}\psi^j_c = \partial_r F_{ljk}(\psi^{ar}\psi^j_c\bar{\psi}^l_d\bar{\psi}^{dk} + \psi^{ar}\bar{\psi}^k_c\delta^{lj}), \tag{4.46}$$

and

$$\partial_r F_{ljk}\bar{\psi}^l_c\psi^{br}\psi^j_b\bar{\psi}^{ak} = \partial_r F_{ljk}(\psi^{br}\psi^j_b\bar{\psi}^l_c\bar{\psi}^{ak} - \psi^r_c\bar{\psi}^{ak}\delta^{lj}). \tag{4.47}$$

Note that if $a \neq c$, we have

$$\psi^{ar}\bar{\psi}^k_c = \psi^r_c\bar{\psi}^{ak}, \quad \text{and} \quad \psi^{ar}\psi^j_c = -\psi^{aj}\psi^r_c, \quad \bar{\psi}^l_c\bar{\psi}^{ak} = -\bar{\psi}^k_a\bar{\psi}^{cl}. \tag{4.48}$$

Using the symmetry $\partial_r F_{ljk} = \partial_l F_{rjk}$ and $F_{ljk} = F_{kjl}$ it follows from (4.46), (4.47) and (4.48) that the sum of expressions in (4.46) and (4.47) vanishes if $a \neq c$. Therefore we get from (4.46), (4.47), (4.48) that

$$\partial_r F_{ljk}(\psi^{ar}\bar{\psi}^l_d\bar{\psi}^{dk}\psi^j_c + \bar{\psi}^l_c\psi^{br}\psi^j_b\bar{\psi}^{ak}) = \partial_r F_{ljk}(\psi^{ar}\psi^j_a\bar{\psi}^l_d\bar{\psi}^{dk} + \psi^{br}\psi^j_b\bar{\psi}^l_a\bar{\psi}^{ak} - \psi^r_d\bar{\psi}^{dk}\delta^{lj})\delta^a_c. \tag{4.49}$$

Note that

$$\psi^{ar}\psi^j_a = \psi^{\widehat{a}j}\psi^r_{\widehat{a}}, \quad \text{and} \quad \bar{\psi}^{ar}\bar{\psi}^j_a = \bar{\psi}^{\widehat{a}j}\bar{\psi}^r_{\widehat{a}}, \tag{4.50}$$

here $\widehat{a} \neq a$. Therefore the right-hand side of (4.49) equals

$$\partial_r F_{ljk}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_d\bar{\psi}^{dk}\delta^{lj})\delta^a_c. \tag{4.51}$$

Therefore in total expression (4.45) becomes

$$\{A, B'\} + \{B, A'\} = \partial_r F_{ljk}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_d\bar{\psi}^{dk}\delta^{lj} + \frac{1}{4}\delta^{rl}\delta^{jk})\delta^a_c.$$

Finally, let us consider the term $\{B, B'\}$. We first show that

$$C := F_{rjk}F_{lmn}\{\psi^{br}\psi^j_b\bar{\psi}^{ak}, \bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c\} = 0. \tag{4.52}$$

By using (4.39) we obtain

$$\begin{aligned} C = {} & F_{rjk}F_{lmn}\big(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm}\{\psi^n_c, \bar{\psi}^{ak}\} - \psi^{br}\bar{\psi}^{dm}\psi^n_c\bar{\psi}^{ak}\{\psi^j_b, \bar{\psi}^l_d\} \\ & + \psi^{br}\bar{\psi}^l_d\psi^n_c\bar{\psi}^{ak}\{\bar{\psi}^{dm}, \psi^j_b\} + \bar{\psi}^{dm}\psi^n_c\psi^j_b\bar{\psi}^{ak}\{\psi^{br}, \bar{\psi}^l_d\} - \bar{\psi}^l_d\psi^n_c\psi^j_b\bar{\psi}^{ak}\{\psi^{br}, \bar{\psi}^{dm}\}\big) \\ = {} & F_{rjk}F_{lmn}\big(\frac{1}{2}\delta^a_c\delta^{nk}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm} + \frac{1}{2}\delta^{lj}\psi^{br}\bar{\psi}^m_b\psi^n_c\bar{\psi}^{ak} + \frac{1}{2}\delta^{mj}\psi^{br}\bar{\psi}^l_b\psi^n_c\bar{\psi}^{ak} \\ & + \frac{1}{2}\delta^{rl}\bar{\psi}^{bm}\psi^j_b\psi^n_c\bar{\psi}^{ak} + \frac{1}{2}\delta^{rm}\bar{\psi}^{bl}\psi^j_b\psi^n_c\bar{\psi}^{ak}\big). \end{aligned}$$

Then using the symmetry of F_{lmn} under the swap of l and m we obtain

$$C = F_{rjk}F_{lmn}\big(\frac{1}{2}\delta^a_c\delta^{nk}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm} + \delta^{jl}\psi^{br}\bar{\psi}^m_b\psi^n_c\bar{\psi}^{ak} + \delta^{rl}\bar{\psi}^{bm}\psi^j_b\psi^n_c\bar{\psi}^{ak}\big).$$

Note that by (4.7), (4.9) we have

$$\psi^{br}\bar{\psi}^m_b\psi^n_c\bar{\psi}^{ak} = -\psi^{br}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak} - \frac{1}{2}\psi^r_c\bar{\psi}^{ak}\delta^{nm}, \tag{4.53}$$

and

$$\begin{aligned}\bar{\psi}^{bm}\psi^j_b\psi^n_c\bar{\psi}^{ak} &= -\psi^j_b\bar{\psi}^{bm}\psi^n_c\bar{\psi}^{ak} + \psi^n_c\bar{\psi}^{ak}\delta^{mj}\\ &= -\psi^{bj}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak} - \frac{1}{2}\psi^j_c\bar{\psi}^{ak}\delta^{nm} + \psi^n_c\bar{\psi}^{ak}\delta^{mj}.\end{aligned} \tag{4.54}$$

Further on by (4.12) we have $F_{rjk}F_{rmn} = F_{rnk}F_{rmj}$ and therefore some terms in the right-hand side of (4.53), (4.54) enter the relation

$$F_{rjk}F_{rmn}\psi^n_c\bar{\psi}^{ak}\delta^{mj} = \frac{1}{2}F_{rjk}F_{jmn}\psi^r_c\bar{\psi}^{ak}\delta^{mn} + \frac{1}{2}F_{rjk}F_{rmn}\psi^j_c\bar{\psi}^{ak}\delta^{mn}. \tag{4.55}$$

Then by using (4.53)-(4.55) and the symmetry of F_{rjk} under the swap of r and j we obtain

$$\begin{aligned}C &= F_{rjk}F_{lmn}\big(\frac{1}{2}\delta^a_c\delta^{nk}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm} - \delta^{jl}\psi^{br}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak} - \delta^{rl}\psi^{bj}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak}\big)\\ &= F_{rjk}F_{lmn}\big(\frac{1}{2}\delta^a_c\delta^{nk}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm} - 2\delta^{jl}\psi^{br}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak}\big).\end{aligned}$$

Note that for $c \neq a$ we have $C = 0$, since $F_{rjk}F_{lmn}\delta^{jl}\psi^{br}\psi^n_c\bar{\psi}^m_b\bar{\psi}^{ak} = 0$ by using (4.12). Further on, if $c = a$ then by using (4.12) we have

$$C = F_{rjk}F_{klm}\big(\frac{1}{2}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm} - 2\psi^{br}\psi^j_a\bar{\psi}^l_b\bar{\psi}^{am}\big). \tag{4.56}$$

Note that for $b \neq a$, $F_{rjk}\psi^{br}\psi^j_a = 0$. Hence

$$F_{rjk}F_{klm}\psi^{br}\psi^j_a\bar{\psi}^l_b\bar{\psi}^{am} = F_{rjk}F_{klm}\psi^{ar}\psi^j_a\bar{\psi}^l_a\bar{\psi}^{am}, \tag{4.57}$$

which is equal to $\frac{1}{4}F_{rjk}F_{klm}\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dm}$ because of relations (4.50). This proves that $C = 0$. Then the term $\{B, B'\}$ takes the following form:

$$\{B, B'\} = F_{rjk}F_{lmn}\big(\frac{1}{2}\delta^{nm}\{\psi^{br}\psi^j_b\bar{\psi}^{ak}, \bar{\psi}^l_c\} + \frac{1}{2}\delta^{jk}\{\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c, \psi^{ar}\} - \frac{1}{4}\delta^{jk}\delta^{nm}\{\psi^{ar}, \bar{\psi}^l_c\}\big).$$

By using formulae (4.40), (4.41) and (4.12) we obtain

$$\begin{aligned}\{B, B'\} &= -\frac{1}{2}F_{rjk}F_{lmn}\big(\psi^r_c\bar{\psi}^{ak}\delta^{nm}\delta^{jl} + \bar{\psi}^{al}\psi^n_c\delta^{mr}\delta^{jk} - \frac{1}{4}\delta^{jk}\delta^{nm}\delta^{rl}\delta^a_c\big)\\ &= -\frac{1}{2}F_{rjk}F_{lmn}\delta^{nm}\delta^{jl}\{\psi^r_c, \bar{\psi}^{ak}\} + \frac{1}{8}F_{rjk}F_{lmn}\delta^{jk}\delta^{nm}\delta^{rl}\delta^a_c\\ &= -\frac{1}{8}F_{rjk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{rk}\delta^a_c.\end{aligned}$$

Therefore, the statement follows. □

Lemma 4.2.8. *Let $T^{22} = H$ be given by Theorem 4.2.7. Let $T^{11} = K$ and $T^{12} = -D$ be given by* (4.21), (4.22). *Then relations (4.3) hold.*

Proof. Firstly, we have that $[K, H] = \frac{1}{4}[x^2, p^2] = \frac{i}{2}\{x_r, p_r\} = -2iD$, as required. Moreover, since H is homogeneous in x of degree -2 it follows that $[H, D] = iH$ as required. Further on, $[K, D] = -\frac{1}{2}[x_k^2, x_j p_j] = iK$, which is the corresponding relation (4.3). □

Lemma 4.2.9. *Let Q^{abc}, I^{ab}, T^{ab}, J^{ab} be as above. Then relations (4.2) hold.*

Proof. Firstly let us consider

$$\{Q^{21a}, Q^{11f}\} = -\{Q^a, S^f\}.$$

Note that

$$\{p_r\psi^{ar}, x_l\psi^{fl}\} = -i\psi^{ar}\psi^{fr} = -i\epsilon^{a\widehat{a}}\psi^r_{\widehat{a}}\psi^{fr},$$

where $\widehat{a}$ is complimentary to a. Note that we can assume now that $\widehat{a} = f$. Therefore

$$\{p_r\psi^{ar}, x_l\psi^{fl}\} = -i\epsilon^{af}\psi^r_f\psi^{fr} = -\frac{i}{2}\epsilon^{af}\psi^r_d\psi^{dr}.$$

Further,

$$F_{rjk}\{\psi^{br}\psi^j_b\bar{\psi}^{ak}, x_l\psi^{fl}\} = \frac{1}{2}\epsilon^{af}x_kF_{krj}\psi^r_d\psi^{dj}.$$

Therefore by formula (4.11)

$$\{Q^{21a}, Q^{11f}\} = -i\epsilon^{af}\psi^r_d\psi^{dr} + i\epsilon^{af}x_kF_{krj}\psi^r_d\psi^{dj} = 2(\alpha+1)\epsilon^{af}I^{11}, \tag{4.58}$$

as required for the corresponding relation (4.2).

Further on, consider $\{Q^{21a}, Q^{12b}\} = -\epsilon^{bd}\{Q^a, \bar{S}_d\}$. Now, by using formula (4.41) we have

$$\{Q^a, \bar{S}_d\} = -2\{p_r\psi^{ar}, x_l\bar{\psi}^l_d\} - 2ix_lF_{rjk}(\{\psi^{br}\psi^j_b\bar{\psi}^{ak}, \bar{\psi}^l_d\} - \frac{1}{2}\delta^{jk}\{\psi^{ar}, \bar{\psi}^l_d\})$$
$$= 2i\psi^{ar}\bar{\psi}^r_d + x_rp_r\delta^a_d + 2ix_jF_{jrk}\psi^r_d\bar{\psi}^{ak} - \frac{i}{2}\delta^{jk}\delta^a_dx_rF_{rjk}$$
$$= 2i\psi^{ar}\bar{\psi}^r_d + x_rp_r\delta^a_d - 2i(2\alpha+1)\psi^r_d\bar{\psi}^{ar} + \frac{i\delta^a_d}{2}N(2\alpha+1).$$

Therefore

$$\{Q^{21a}, Q^{12b}\} = -2i\psi^{ar}\bar{\psi}^{br} + x_rp_r\epsilon^{ab} + 2i(2\alpha+1)\psi^{br}\bar{\psi}^{ar} + \frac{i\epsilon^{ab}}{2}N(2\alpha+1). \tag{4.59}$$

Let us now note that

$$I^{12} = -\frac{i}{2}[\psi^j_a, \bar{\psi}^{aj}] = -i(\psi^{2j}\bar{\psi}^{1j} - \psi^{1j}\bar{\psi}^{2j} - \frac{N}{2}).$$

Hence the right-hand side of (4.2) for $\{Q^{21a}, Q^{12b}\}$ is

$$x_rp_r\epsilon^{ab} - \frac{iN}{2}\epsilon^{ab} + 4i\alpha\psi^{(ar}\bar{\psi}^{br)} - 2i(1+\alpha)\epsilon^{ab}(\psi^{2j}\bar{\psi}^{1j} - \psi^{1j}\bar{\psi}^{2j} - \frac{N}{2}). \tag{4.60}$$

By considering various values of $a, b \in \{1, 2\}$, expression (4.60) takes the form

$$x_rp_r\epsilon^{ab} + \frac{i\epsilon^{ab}}{2}N(2\alpha+1) - 2i\psi^{ar}\bar{\psi}^{br} + 2i(2\alpha+1)\psi^{br}\bar{\psi}^{ar}, \tag{4.61}$$

which is equal to (4.59) as required, so the corresponding relation (4.2) follows.

Further on, let us consider relation $\{Q^{21a}, Q^{21b}\} = \{Q^a, Q^b\}$. By using (4.7) and (4.9) we have

$$\{Q^a, Q^c\} = i\{p_r\psi^{ar}, F_{lmn}\psi^{dl}\psi^m_d\bar{\psi}^{cn}\} + i\{p_l\psi^{cl}, F_{rjk}\psi^{br}\psi^j_b\bar{\psi}^{ak}\} -$$
$$- F_{lmn}F_{rjk}\{\langle\psi^{dl}\psi^m_d\bar{\psi}^{cn}\rangle, \langle\psi^{br}\psi^j_b\bar{\psi}^{ak}\rangle\}.$$

Note that by (4.14), (4.15) we have

$$\{p_r\psi^{ar}, F_{lmn}\psi^{dl}\psi^m_d\bar{\psi}^{cn}\} = \psi^{ar}\psi^{dl}\psi^m_d\bar{\psi}^{cn}[p_r, F_{lmn}] + \{\psi^{ar}, \psi^{dl}\psi^m_d\bar{\psi}^{cn}\}F_{lmn}p_r$$
$$= -i\psi^{ar}\psi^{dl}\psi^m_d\bar{\psi}^{cn}\partial_rF_{lmn} + \{\psi^{ar}, \bar{\psi}^{cn}\}\psi^{dl}\psi^m_dF_{lmn}p_r$$
$$= -i\psi^{ar}\psi^{dl}\psi^m_d\bar{\psi}^{cn}\partial_rF_{lmn} - \frac{1}{2}\epsilon^{ca}\psi^{dl}\psi^m_dF_{lmr}p_r.$$

Note also that $\psi^{ar}\psi^{al}\partial_rF_{lmn} = 0$ using the symmetry of ∂_rF_{lmn} under the swap of r and

l. Then $\psi^{ar}\psi^{dl}\psi^m_d\bar\psi^{cn}\partial_r F_{lmn}=0$ and hence

$$\{p_r\psi^{ar}, F_{lmn}\psi^{dl}\psi^m_d\bar\psi^{cn}\} = -\frac{1}{2}\epsilon^{ca}F_{lmr}p_r\psi^{dl}\psi^m_d. \tag{4.62}$$

Similarly,

$$\begin{aligned}\{p_l\psi^{cl}, F_{rjk}\psi^{br}\psi^j_b\bar\psi^{ak}\} &= -i\psi^{cl}\psi^{br}\psi^j_b\bar\psi^{ak}\partial_l F_{rjk} - \frac{1}{2}\epsilon^{ac}F_{rjk}p_k\psi^{br}\psi^j_b\\ &= -\frac{1}{2}\epsilon^{ac}F_{rjk}p_k\psi^{br}\psi^j_b.\end{aligned} \tag{4.63}$$

Note that terms in (4.62) and (4.63) cancel. Further, we have

$$\begin{aligned}F_{lmn}F_{rjk}\{\langle\psi^{dl}\psi^m_d\bar\psi^{cn}\rangle,\langle\psi^{br}\psi^j_b\bar\psi^{ak}\rangle\} &= F_{lmn}F_{rjk}\{\psi^{dl}\psi^m_d\bar\psi^{cn},\psi^{br}\psi^j_b\bar\psi^{ak}\}\\ &+\frac{1}{4}\epsilon^{ca}F_{lmr}F_{rjj}\psi^{dl}\psi^m_d+\frac{1}{4}\epsilon^{ac}F_{rjk}F_{kmm}\psi^{br}\psi^j_b\\ &= F_{lmn}F_{rjk}\{\psi^{dl}\psi^m_d\bar\psi^{cn},\psi^{br}\psi^j_b\bar\psi^{ak}\},\end{aligned} \tag{4.64}$$

since the last two terms in (4.64) cancel. Note that by (4.39) we have

$$\begin{aligned}\{\psi^{dl}\psi^m_d\bar\psi^{cn},\psi^{br}\psi^j_b\bar\psi^{ak}\} &= \psi^{dl}\psi^m_d\big(\psi^j_b\bar\psi^{ak}\{\bar\psi^{cn},\psi^{br}\}-\psi^{br}\bar\psi^{ak}\{\psi^j_b,\bar\psi^{cn}\}\big)\\ &+\psi^{br}\psi^j_b\big(\psi^m_d\bar\psi^{cn}\{\bar\psi^{ak},\psi^{dl}\}-\psi^{dl}\bar\psi^{cn}\{\bar\psi^{ak},\psi^m_d\}\big)\\ &= -\frac{1}{2}\psi^{dl}\psi^m_d\big(\psi^{cj}\delta^{nr}+\psi^{cr}\delta^{jn}\big)\bar\psi^{ak}-\frac{1}{2}\psi^{br}\psi^j_b\big(\psi^{al}\delta^{km}+\psi^{am}\delta^{kl}\big)\bar\psi^{cn}.\end{aligned}$$

Therefore using the symmetry of F_{rjk} under the swap of j and r, and that of F_{lmn} under the swap of l and m we obtain

$$F_{lmn}F_{rjk}\{\psi^{dl}\psi^m_d\bar\psi^{cn},\psi^{br}\psi^j_b\bar\psi^{ak}\} = -F_{lmn}F_{rjk}\big(\psi^{dl}\psi^m_d\psi^{cj}\bar\psi^{ak}\delta^{nr}+\psi^{br}\psi^j_b\psi^{al}\bar\psi^{cn}\delta^{km}\big). \tag{4.65}$$

Further, note that for any $b\in\{1,2\}$ we have by using (4.12) that $F_{lmr}F_{rjk}\psi^{dl}\psi^m_d\psi^{bj}=0$. Hence the right-hand side of (4.65) vanishes. Therefore it follows that

$$F_{lmn}F_{rjk}\{\langle\psi^{dl}\psi^m_d\bar\psi^{cn}\rangle,\langle\psi^{br}\psi^j_b\bar\psi^{ak}\rangle\}=0$$

and hence that $\{Q^a,Q^b\}=0$ as required.

Further on it is easy to see that $\{Q^{11a},Q^{11b}\}=\{Q^{12a},Q^{12b}\}=0$. By Theorem 4.2.7 we have $\{Q^{21a},Q^{22b}\}=-2H\epsilon^{ba}$. The remaining relations (4.2) can be shown to hold by applying $\sim$ (see also Lemma B.1.7). $\square$

Remark 4.2.10. Let the supercharges Q^a and $\bar Q_b$ be of the form (4.16), (4.17) for a potential F. Then the $\mathcal{N}=4$ supersymmetry algebra

$$\{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 \quad \text{and} \quad \{Q^a, \bar{Q}_b\} = -2H\delta^a_b, \tag{4.66}$$

is satisfied if and only if the function F is solution to equations (4.12). Thus relations (4.66) do not imply WDVV equations for the potential F. Indeed, if F satisfies equations (4.12), then the statement follows by Theorem 4.2.7 and Lemma 4.2.9. The converse follows from the proof of Theorem 4.2.7. More precisely, we should have that $\{Q^a, \bar{Q}_b\} = 0$ for $a \neq b$. This implies that the term $\{B, B'\}$ should vanish. Imposing this constraint implies the statement.

Lemma 4.2.11. *Let T^{ab}, Q^{abc} be as above. Then relations (4.5a) hold.*

Proof. Firstly, it is easy to see that $[T^{11}, Q^{21a}] = -[K, Q^a] = -2ix_r\psi^{ar} = iS^a$, and $[T^{11}, Q^{11a}] = [K, S^a] = 0$, and $[T^{12}, Q^{11a}] = -[D, S^a] = -\frac{i}{2}Q^{11a}$. Moreover, we have $[T^{12}, Q^{21a}] = [D, Q^a] = \frac{i}{2}Q^{21a}$ as Q^a is homogeneous in x of degree -1. This gives relations (4.5a) for commutators between K, D and Q^a, S^a.

Further, we have

$$[\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk}, \psi^{am}] = \frac{1}{2}\psi^{br}\psi^j_b(\bar{\psi}^{al}\delta^{km} + \bar{\psi}^{ak}\delta^{lm}),$$

therefore

$$\partial_r F_{jlk}[\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk}, \psi^{am}] = \partial_r F_{jlm}\psi^{br}\psi^j_b\bar{\psi}^{al}. \tag{4.67}$$

Note also that

$$\partial_r F_{jlk}[\psi^r_b\bar{\psi}^{bj}\delta^{lk}, \psi^{am}] = \frac{1}{2}\partial_r F_{lmk}\psi^{ar}\delta^{lk}. \tag{4.68}$$

Hence we get from (4.67) and (4.68) that

$$\begin{aligned} \partial_r F_{jlk}[\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk}, \psi^{am}] &= \partial_r F_{jlm}\psi^{br}\psi^j_b\bar{\psi}^{al} - \frac{1}{2}\partial_r F_{lmk}\psi^{ar}\delta^{lk} \\ &= \partial_m F_{rjl}\langle\psi^{br}\psi^j_b\bar{\psi}^{al}\rangle, \end{aligned} \tag{4.69}$$

in view of (4.19). Therefore

$$\begin{aligned} [H, S^a] &= ip_r\psi^{ar} + x_m\partial_m F_{rjl}\langle\psi^{br}\psi^j_b\bar{\psi}^{al}\rangle \\ &= ip_r\psi^{ar} - F_{rjl}\langle\psi^{br}\psi^j_b\bar{\psi}^{al}\rangle \\ &= iQ^a, \end{aligned} \tag{4.70}$$

as required for (4.5a). Further on, by Theorem 4.2.7 we have $T^{22} = H = -\frac{1}{2}\{Q^a, \bar{Q}_a\}$. Since $(Q^a)^2 = 0$ we get that $[H, Q^a] = 0$ as required. The remaining relations (4.5a) can be shown using $\sim$ operation (cf. Lemma B.1.8). $\square$

Lemma 4.2.12. *Let T^{ab}, I^{ab}, J^{ab} be as above. Then relations (4.6) hold.*

Proof. Let us firstly consider $[I^{ab}, J^{cd}]$. We have by (4.13) and (4.26) that

$$[\psi^j_a\psi^{aj}, \psi^{ck}\bar{\psi}^{dk}] = \psi^{dk}\psi^{ck}.$$

Therefore

$$[I^{11}, J^{cd}] = 2[\psi^j_a\psi^{aj}, \psi^{(ck}\bar{\psi}^{dk)}] = 0,$$

as required. Further, we have by (4.13), (4.14) that

$$\begin{aligned}
[[\psi^j_a, \bar{\psi}^{aj}], \psi^{ck}\bar{\psi}^{dk}] &= 2[\psi^j_a\bar{\psi}^{aj}, \psi^{ck}\bar{\psi}^{dk}] \\
&= 2(\psi^j_a[\bar{\psi}^{aj}, \psi^{ck}\bar{\psi}^{dk}] + [\psi^j_a, \psi^{ck}\bar{\psi}^{dk}]\bar{\psi}^{aj}) \\
&= 2(\psi^j_a\bar{\psi}^{dk}\{\psi^{ck}, \bar{\psi}^{aj}\} - \psi^{ck}\bar{\psi}^{aj}\{\bar{\psi}^{dk}, \psi^j_a\}) = 0.
\end{aligned}$$

Therefore,

$$[I^{12}, J^{cd}] = [[\psi^j_a, \bar{\psi}^{aj}], \psi^{(ck}\bar{\psi}^{dk)}] = 0,$$

which is the corresponding relation (4.6). In addition we have by (4.13) and (4.26) that

$$[\bar{\psi}^{aj}\bar{\psi}^j_a, \psi^{ck}\bar{\psi}^{dk}] = -\bar{\psi}^{ck}\bar{\psi}^{dk}.$$

Therefore,

$$[I^{22}, J^{cd}] = -2[\bar{\psi}^{aj}\bar{\psi}^j_a, \psi^{(ck}\bar{\psi}^{dk)}] = 0,$$

as required.

Let us now consider relations $[I^{ab}, T^{cd}]$, $(a, b, c, d = 1, 2)$. It is easy to see that for $T^{12} = -D$ and $T^{11} = K$ relations (4.6) hold. Further, we have $T^{22} = H = -\frac{1}{2}\{Q^c, \bar{Q}_c\}$. Then by (4.13) we obtain

$$\begin{aligned}
[I^{ab}, H] &= -\frac{1}{2}([I^{ab}, Q^c\bar{Q}_c] + [I^{ab}, \bar{Q}_cQ^c]) \\
&= -\frac{1}{2}(Q^c[I^{ab}, \bar{Q}_c] + [I^{ab}, Q^c]\bar{Q}_c + \bar{Q}_c[I^{ab}, Q^c] + [I^{ab}, \bar{Q}_c]Q^c) \\
&= -\frac{1}{2}(-Q_{\widehat{c}}[I^{ab}, \bar{Q}^{\widehat{c}}] + [I^{ab}, Q^c]\bar{Q}_c + \bar{Q}_c[I^{ab}, Q^c] - [I^{ab}, \bar{Q}^{\widehat{c}}]Q_{\widehat{c}}),
\end{aligned}$$

where $\widehat{c}$ is complimentary to c. Then by Lemma 4.2.6 we have

$$[I^{ab}, Q^c] = -[I^{ab}, Q^{21c}] = -\frac{i}{2}(\epsilon^{1a}Q^{2bc} + \epsilon^{1b}Q^{2ac}) \quad \text{and} \quad [I^{ab}, \bar{Q}^c] = -\frac{i}{2}(\epsilon^{2a}Q^{2bc} + \epsilon^{2b}Q^{2ac}).$$

Therefore by considering various values of $a, b \in \{1, 2\}$ and by using Lemma 4.2.9 and

Theorem 4.2.7 we obtain the following:

$$[I^{11}, H] = \frac{i}{2}(Q_{\widehat{c}}Q^{\widehat{c}} + Q^{\widehat{c}}Q_{\widehat{c}}) = 0,$$
$$[I^{22}, H] = \frac{i}{2}(\bar{Q}^{c}\bar{Q}_{c} + \bar{Q}_{c}\bar{Q}^{c}) = 0,$$
$$[I^{12}, H] = \frac{i}{2}(Q_{\widehat{c}}\bar{Q}^{\widehat{c}} + Q^{c}\bar{Q}_{c} + \bar{Q}_{c}Q^{c} + \bar{Q}^{\widehat{c}}Q_{\widehat{c}}) = 0,$$

which are the corresponding relations (4.6).

Similarly we have

$$[J^{ab}, H] = -\frac{1}{2}(-Q_{\widehat{c}}[J^{ab}, \bar{Q}^{\widehat{c}}] + [J^{ab}, Q^{c}]\bar{Q}_{c} + \bar{Q}_{c}[J^{ab}, Q^{c}] - [J^{ab}, \bar{Q}^{\widehat{c}}]Q_{\widehat{c}}).$$

By Lemma 4.2.5 we have

$$[J^{ab}, Q^{c}] = \frac{i}{2}(\epsilon^{ca}Q^{b} + \epsilon^{cb}Q^{a}) \quad \text{and} \quad [J^{ab}, \bar{Q}^{c}] = \frac{i}{2}(\epsilon^{ca}\bar{Q}^{b} + \epsilon^{cb}\bar{Q}^{a}).$$

Therefore by considering various values of $a, b \in \{1, 2\}$ we obtain:

$$[J^{11}, H] = -\frac{i}{2}(-\epsilon^{\widehat{c}1}Q_{\widehat{c}}\bar{Q}^{1} + \epsilon^{c1}Q^{1}\bar{Q}_{c} + \epsilon^{c1}\bar{Q}_{c}Q^{1} - \epsilon^{\widehat{c}1}\bar{Q}^{1}Q_{\widehat{c}}), \tag{4.71}$$
$$\begin{aligned}[J^{12}, H] = -\frac{i}{4}(&-\epsilon^{\widehat{c}1}Q_{\widehat{c}}\bar{Q}^{2} - \epsilon^{\widehat{c}2}Q_{\widehat{c}}\bar{Q}^{1} + \epsilon^{c1}Q^{2}\bar{Q}_{c} + \epsilon^{c2}Q^{1}\bar{Q}_{c} \\ &+ \epsilon^{c1}\bar{Q}_{c}Q^{2} + \epsilon^{c2}\bar{Q}_{c}Q^{1} - \epsilon^{\widehat{c}1}\bar{Q}^{2}Q_{\widehat{c}} - \epsilon^{\widehat{c}2}\bar{Q}^{1}Q_{\widehat{c}}),\end{aligned} \tag{4.72}$$
$$[J^{22}, H] = -\frac{i}{2}(-\epsilon^{\widehat{c}2}Q_{\widehat{c}}\bar{Q}^{2} + \epsilon^{c2}Q^{2}\bar{Q}_{c} + \epsilon^{c2}\bar{Q}_{c}Q^{2} - \epsilon^{\widehat{c}2}\bar{Q}^{2}Q_{\widehat{c}}). \tag{4.73}$$

Then by considering various values of $c \in \{1, 2\}$ in (4.71)–(4.73) and by using Lemma 4.2.9 and Theorem 4.2.7 we obtain that

$$[J^{11}, H] = [J^{12}, H] = [J^{22}, H] = 0,$$

as required for (4.6). □

4.3 The second representation

Let now the supercharges be of the form

$$Q^{a} = p_{r}\psi^{ar} + iF_{rjk}\psi^{br}\psi_{b}^{j}\bar{\psi}^{ak}, \tag{4.74}$$
$$\bar{Q}_{c} = p_{l}\bar{\psi}_{c}^{l} + iF_{lmn}\bar{\psi}_{d}^{l}\bar{\psi}^{dm}\psi_{c}^{n}, \tag{4.75}$$

so we do not have anti-symmetrisation in the cubic fermionic terms. Let generators K, I^{ab}, J^{ab}, and $S^a, \bar{S}_a$ be given by formulas (4.21), (4.23), (4.24), (4.25) same as in the first representation, while the generator D is now given by

$$D = -\frac{1}{2}x_j p_j + \frac{i}{2}(\alpha + 1)N. \tag{4.76}$$

Theorem 4.3.1. *For all $a, b = 1, 2$ we have $\{Q^a, \bar{Q}_b\} = -2H\delta^a_b$, where the Hamiltonian H is*

$$H = \frac{p^2}{4} - \frac{\partial_r F_{jlk}}{2}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk}) + \frac{i}{4}\delta^{nm}F_{rmn}p_r. \tag{4.77}$$

Proof. Let us denote terms in (4.74), (4.75) as follows:

$$Q^a = \overbrace{p_r\psi^{ar}}^{A} + \overbrace{iF_{rjk}\psi^{br}\psi^j_b\bar{\psi}^{ak}}^{B}, \quad \bar{Q}_c = \overbrace{p_l\bar{\psi}^l_c}^{A'} + \overbrace{iF_{lmn}\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c}^{B'}.$$

Then, analogues of relations (4.43), (4.44) are

$$\{A, B'\} = i\psi^{ar}\bar{\psi}^l_d\bar{\psi}^{dm}\psi^n_c[p_r, F_{lmn}] - i\bar{\psi}^{al}\psi^n_c F_{lnr}p_r, \tag{4.78}$$
$$\{B, A'\} = i\bar{\psi}^l_c\psi^{br}\psi^j_b\bar{\psi}^{ak}[p_l, F_{rjk}] - i\psi^r_c\bar{\psi}^{ak}F_{rkj}p_j. \tag{4.79}$$

Then using (4.78) and (4.79) an analogue of equality (4.45) is (cf. (4.51))

$$\begin{aligned}\{A, B'\} + \{B, A'\} &= \partial_r F_{ljk}(\psi^{ar}\bar{\psi}^l_d\bar{\psi}^{dk}\psi^j_c + \bar{\psi}^l_c\psi^{br}\psi^j_b\bar{\psi}^{ak}) - \frac{i}{2}\delta^{nl}F_{lnr}p_r\delta^a_c \\ &= \partial_r F_{jlk}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk})\delta^a_c - \frac{i}{2}\delta^{nl}F_{lnr}p_r\delta^a_c\end{aligned}$$

Further on we have $\{B, B'\} = 0$ (cf. (4.52)). Therefore in total, we get that

$$\begin{aligned}\{Q^a, \bar{Q}_c\} &= -\frac{p^2}{2}\delta^a_c + \{A, B'\} + \{B, A'\} \\ &= -\frac{p^2}{2}\delta^a_c + \partial_r F_{jlk}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk})\delta^a_c - \frac{i}{2}\delta^{nm}F_{rmn}p_r\delta^a_c,\end{aligned} \tag{4.80}$$

and hence the statement follows. □

Lemma 4.3.2. *Let T^{ab} be given by* (4.76), (4.77) *and* (4.21). *Then relations (4.3) hold.*

Proof. Firstly, we have that

$$[K, H] = \frac{1}{4}[x^2, p^2] + \frac{i}{4}\delta^{nm}F_{rmn}[x^2, p_r] = \frac{i}{2}\{x_r, p_r\} + \frac{N}{2}(2\alpha + 1) = -2iD,$$

as required. Moreover we have $[F_{rmn}p_r, x_j p_j] = -iF_{rmn}p_r + ix_j\partial_j F_{rmn}p_r = -2iF_{rmn}p_r$. Then it is easy to see that $[H, D] = iH$, as required. Further on, $[K, D] = -\frac{1}{2}[x^2, x_j p_j] = iK$, which is the corresponding relation (4.3). □

We note that since generators J and I keep the same form as in the first representation, the statement of the Lemmas 4.2.3, 4.2.4 hold.

Lemma 4.3.3. *Let* Q^{abc}, I^{ab}, J^{ab} *be given by* (4.74), (4.75), (4.23), (4.24), (4.25). *Then relations (4.5b), (4.5c) hold.*

Proof. Relations (4.5b),(4.5c) are easy to verify by an adaptation of the proof of Lemmas 4.2.5 and 4.2.6 respectively. Indeed let us consider first relations (4.5b) for $[J^{ab}, Q^{21c}]$, which now takes the form (cf. (4.31))

$$\begin{aligned}[J^{ab}, Q^{21c}] &= \frac{i}{2}\big(\epsilon^{bc}p_l\psi^{al} + \epsilon^{ac}p_l\psi^{bl} - i\epsilon^{ca}F_{lmn}\psi^{dl}\psi_d^m\bar{\psi}^{bn} - i\epsilon^{cb}F_{lmn}\psi^{dl}\psi_d^m\bar{\psi}^{an}\big)\\ &= -\frac{i}{2}(\epsilon^{cb}Q^a + \epsilon^{ca}Q^b) = i\epsilon^{c(a}Q^{|21|b)},\end{aligned}$$

as required for (4.5b).

Further on, let us consider relations (4.5c) for $[I^{ab}, Q^{21c}]$. Expression (4.37) now takes the form

$$[I^{22}, Q^{21a}] = -i[\bar{\psi}^{dr}\bar{\psi}_d^r, Q^a] = i\big(p_l\bar{\psi}^{al} + iF_{lmn}\bar{\psi}_b^l\bar{\psi}^{bm}\psi^{an}\big) = i\bar{Q}^a,$$

as required. The analogue of (4.38) is

$$[I^{12}, Q^{21a}] = \frac{i}{2}\big(p_l\psi^{al} + iF_{lmn}\psi^{bl}\psi_b^m\bar{\psi}^{an}\big) = \frac{i}{2}Q^a,$$

which matches (4.5c). Finally, it is easy to see that $[I^{11}, Q^{21a}] = 0$ (see (4.32), (4.33) in Lemma 4.2.6). Relations (4.5) for S^a take the same form as in Lemmas 4.2.5 and 4.2.6. The remaining relations can be checked in a similar way. $\square$

Lemma 4.3.4. *Let* Q^{abc}, I^{ab}, J^{ab}, T^{ab} *be given by formulae* (4.74), (4.75), (4.21), (4.23), (4.24), (4.25), (4.76), (4.77). *Then relations (4.2) hold.*

Proof. We first note that by Theorem 4.3.1 we have $\{Q^a, \bar{Q}^c\} = \epsilon^{cb}\{Q^a, \bar{Q}_b\} = -2H\epsilon^{ca}$ which is the corresponding relation (4.2). The anticommutator $\{Q^{21a}, Q^{21b}\}$ vanishes since the terms (4.62), (4.63) cancel each other and the right-hand side of (4.65) vanishes. Further on it is immediate that $\{Q^{21a}, Q^{11f}\}$ is the same as in the first representation. Similarly for $\{Q^{22a}, Q^{22b}\}$, $\{Q^{22a}, Q^{12f}\}$. Note also that $\{Q^{11a}, Q^{11b}\}$, $\{Q^{12a}, Q^{12b}\}$, $\{Q^{11a}, Q^{12b}\}$ take the same form as in Lemma 4.2.9.

Further on, let us consider $\{Q^{21a}, Q^{12b}\}$. The left-hand side of (4.2) now takes the form (cf.(4.59) and the change in the generator D)

$$\{Q^{21a}, Q^{12b}\} = -2i\psi^{ar}\bar{\psi}^{br} + x_r p_r\epsilon^{ab} + 2i(1+2\alpha)\psi^{br}\bar{\psi}^{ar}, \tag{4.81}$$

and the right-hand side of (4.2) becomes (cf. (4.61))

$$\begin{aligned}\{Q^{21a}, Q^{12b}\} &= x_r p_r \epsilon^{ab} + 4i\alpha\psi^{(ar}\bar{\psi}^{br)} - 2i(1+\alpha)\epsilon^{ab}(\psi^{2r}\bar{\psi}^{1r} - \psi^{1r}\bar{\psi}^{2r}) \\ &= -2i\psi^{ar}\bar{\psi}^{br} + x_r p_r \epsilon^{ab} + 2i(1+2\alpha)\psi^{br}\bar{\psi}^{ar},\end{aligned}$$

which is equal to (4.81) as required. The remaining relations can be checked by applying $\sim$. □

Lemma 4.3.5. *Let T^{ab} and Q^{abc} be given by* (4.21)*,* (4.25)*,* (4.74)*,* (4.75)*,* (4.76)*,* (4.77)*. Then relations (4.5a) hold.*

Proof. Firstly, it is easy to see that $[T^{11}, Q^{21a}] = -[K, Q^a] = -2ix_r\psi^{ar} = iS^a$, and $[T^{11}, Q^{11a}] = [K, S^a] = 0$, and $[T^{12}, Q^{11a}] = -[D, S^a] = -\frac{i}{2}Q^{11a}$. Moreover, we have $[T^{12}, Q^{21a}] = \frac{i}{2}Q^{21a}$ as Q^a is homogeneous in x of degree -1.

Let us recall that from the proof of Lemma 4.2.11 (formula (4.69)) we have

$$\partial_r F_{jlk}[\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk}, \psi^{am}] = \delta^{km}\partial_k F_{rjl}(\psi^{br}\psi^j_b\bar{\psi}^{al} - \frac{1}{2}\delta^{jl}\psi^{ar}).$$

Therefore an analogue of (4.70) takes the form

$$\begin{aligned}[H, S^a] &= -\frac{1}{2}[p_r^2, x_m\psi^{am}] + x_m\partial_r F_{jlk}[\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk}, \psi^{am}] - \frac{1}{2}\delta^{nm}F_{rnm}\psi^{ar} \\ &= ip_r\psi^{ar} - F_{rjl}\psi^{br}\psi^j_b\bar{\psi}^{al} = iQ^a,\end{aligned}$$

as required for the corresponding relation (4.5a). Further on, we have that $[T^{22}, Q^a] = 0$ and similarly, $[T^{22}, \bar{Q}_a] = 0$, (cf. Lemma 4.2.11). The remaining relations can be checked by applying $\sim$. □

Lemma 4.3.6. *Let T^{ab}, I^{ab}, J^{ab} be given by* (4.21)*,* (4.23)*,* (4.24)*,* (4.76)*,* (4.77)*. Then relations (4.6) hold.*

The proof of the lemma is the same as the proof of Lemma B.1.9 for the first representation since I^{ab} and J^{ab} keep the same form, and the proof of commutation relations with H in Lemma B.1.9 relies only on relations (4.2) which express H as the anticommutator of the supercharges Q^a and $\bar{Q}_a$.

4.4 Hamiltonians

We now proceed to explicit calculations of Hamiltonians appearing in Theorem 4.2.7 and Theorem 4.3.1. We start with a Coxeter root system case.

4.4.1 Coxeter systems

In this case we take $\mathcal{R}$ to be a Coxeter root system in $V \cong \mathbb{R}^N$ (see Chapter 2). More exactly, let $\mathcal{R}$ be a collection of vectors which spans V and is invariant under orthogonal reflections about all the hyperplanes $(\gamma, x) = 0$, $\gamma \in \mathcal{R}$, where $(\cdot, \cdot)$ is the standard scalar product in V. Furthermore, let us assume that squared length $(\gamma, \gamma) = 2$ for any $\gamma \in \mathcal{R}$, and that $\mathcal{R}$ is irreducible. Non-equal choices of length of roots in the cases when the Coxeter group has two orbits on $\mathcal{R}$ are covered by considerations in Subsection 4.4.2 below.

The corresponding function F has the form

$$F(x_1, \dots, x_N) = \frac{\lambda}{2} \sum_{\gamma \in \mathcal{R}_+} (\gamma, x)^2 \log(\gamma, x). \tag{4.82}$$

Recall that F satisfies generalised WDVV equations (2.59) (see Subsection 2.5.2). Recall the following property.

Lemma 4.4.1. [17, Ch. V, p. 125] *For any $u, v \in V$*

$$\sum_{\gamma \in \mathcal{R}_+} (\gamma, u)(\gamma, v) = h(u, v),$$

where h is the Coxeter number of $\mathcal{R}$.

Lemma 4.4.1 has the following corollary.

Lemma 4.4.2. *Let F be given by (4.82). Then*

$$x_i F_{ijk} = \lambda h \delta_{jk}.$$

Proof. Let $\gamma \in \mathcal{R}$ have coordinates $\gamma = (\gamma_1, \dots, \gamma_N)$. By Lemma 4.4.1 we have

$$x_i F_{ijk} = \lambda \sum_{\gamma \in \mathcal{R}_+} \frac{x_i \gamma_i \gamma_j \gamma_k}{(\gamma, x)} = \lambda \sum_{\gamma \in \mathcal{R}_+} \gamma_j \gamma_k = \lambda h(e_j, e_k) = \lambda h \delta_{jk}.$$ □

The identity stated in the next lemma will be useful below.

Lemma 4.4.3. *We have*

$$\sum_{\substack{\beta, \gamma \in \mathcal{R}_+ \\ \beta \neq \gamma}} \frac{(\beta, \gamma)}{(\beta, x)(\gamma, x)} = 0. \tag{4.83}$$

Proof. Let us consider a pair of roots β, γ. If γ and β are orthogonal, then their contribution in equality (4.83) is zero. Hence, assume that $(\gamma, \beta) \neq 0$. Let $\gamma' = s_\beta(\gamma) \in R$. We have

$$(s_\beta(\gamma), x) = (\gamma, x) - (\beta, \gamma)(\beta, x).$$

Therefore

$$\frac{1}{(\beta,x)}\Big(\frac{(\gamma,\beta)}{(\gamma,x)}+\frac{(\gamma',\beta)}{(\gamma',x)}\Big)=-\frac{(\gamma,\beta)^2}{(\gamma,x)^2-(\beta,\gamma)(\beta,x)(\gamma,x)}. \tag{4.84}$$

Hence, the term (4.84) is non-singular at all the hyperplanes $(\beta,x)=0$, $\beta\in\mathcal{R}_+$. This implies the statement. □

Let us choose now

$$\alpha=-\frac{h\lambda+1}{2}. \tag{4.85}$$

Then $h\lambda=-(2\alpha+1)$, so by Lemma 4.4.2 function F satisfies the required condition (4.11). Thus it leads to $D(2,1;\alpha)$ superconformal mechanics with the Hamiltonians given by Theorems 4.2.7, 4.3.1. We now simplify these Hamiltonians for the root system case.

Theorem 4.4.4. *Let function F be given by* (4.82)*. Then the Hamiltonian H given by (4.42) is supersymmetric with the superconformal algebra $D(2,1;\alpha)$, where α is given by* (4.85)*. The rescaled Hamiltonian $H_1=4H$ has the form*

$$H_1=-\Delta+\sum_{\gamma\in\mathcal{R}_+}\frac{2\lambda(\lambda+1)}{(\gamma,x)^2}+\Phi,$$

where $\Delta=-p^2$ is the Laplacian in V and the fermionic term

$$\Phi=2\lambda\sum_{\gamma\in\mathcal{R}_+}\frac{\gamma_i\gamma_j\gamma_k\gamma_l}{(\gamma,x)^2}\psi^{bi}\psi^j_b\bar\psi^l_d\bar\psi^{dk}-4\lambda\sum_{\gamma\in\mathcal{R}_+}\frac{\gamma_i\gamma_j}{(\gamma,x)^2}\psi^i_b\bar\psi^{bj}. \tag{4.86}$$

Proof. By formula (4.42) we have that

$$H=\frac{p^2}{4}+\Psi+U,$$

where potential

$$U=-\frac{1}{8}\partial_iF_{jlk}\delta^{ij}\delta^{lk}+\frac{1}{16}F_{ijk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{ik}$$

and

$$\Psi=-\frac{1}{2}\partial_iF_{jlk}(\psi^{bi}\psi^j_b\bar\psi^l_d\bar\psi^{dk}-\psi^i_b\bar\psi^{bj}\delta^{lk}).$$

Let us firstly simplify U. We have

$$F_{jlk}=\lambda\sum_{\gamma\in\mathcal{R}_+}\frac{\gamma_j\gamma_l\gamma_k}{(\gamma,x)}.$$

Then

$$\partial_iF_{jlk}\delta^{ij}\delta^{lk}=-\lambda\sum_{\gamma\in\mathcal{R}_+}\frac{\gamma_i\gamma_j\gamma_l\gamma_k}{(\gamma,x)^2}\delta^{ij}\delta^{lk}=-4\lambda\sum_{\gamma\in\mathcal{R}_+}\frac{1}{(\gamma,x)^2} \tag{4.87}$$

and
$$F_{ijk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{ik} = 4\lambda^2 \sum_{\beta,\gamma\in\mathcal{R}_+} \frac{(\beta,\gamma)}{(\beta,x)(\gamma,x)} = \sum_{\gamma\in\mathcal{R}_+} \frac{8\lambda^2}{(\gamma,x)^2} \tag{4.88}$$

because of identity (4.83). The statement follows from formulae (4.87), (4.88). □

The following theorem can be easily checked directly.

Theorem 4.4.5. *For the function F given by* (4.82) *the Hamiltonian H given by (4.77) is supersymmetric with the superconformal algebra $D(2,1;\alpha)$, where α is given by* (4.85). *The rescaled Hamiltonian $H_2 = 4H$ has the form*

$$H_2 = -\Delta + \sum_{\gamma\in\mathcal{R}_+} \frac{2\lambda}{(\gamma,x)}\partial_\gamma + \Phi,$$

where Φ is defined by (4.86).

Proposition 4.4.6. *Hamiltonians H_1, H_2 from Theorems 4.4.4, 4.4.5 satisfy gauge relation $\delta^{-1}\circ H_2 \circ \delta = H_1$, where $\delta = \prod_{\beta\in\mathcal{R}_+}(\beta,x)^\lambda$.*

The proof follows immediately by making use of the identity (4.83).

Remark 4.4.7. We note that the Hamiltonian H_2 is not self-adjoint under hermitian involution defined by

$$\psi^{aj\,\dagger} = \bar\psi^j_a, \quad p_j^\dagger = p_j, \quad x_j^\dagger = x_j, \quad i^\dagger = -i, \quad \text{and} \quad (AB)^\dagger = B^\dagger A^\dagger$$

for any two operators A, B. One could have considered another ansatz for $\bar Q_a$ so that to obtain self-adjoint Hamiltonian. Namely, let Q^a be as in (4.74) and consider hermitian conjugate $(Q^a)^\dagger$. Let Q^a, $(Q^a)^\dagger$ $(a=1,2)$ be the ansatz for the supercharges. Then

$$(Q^a)^\dagger = p_r\bar\psi^r_a + iF_{rjk}\psi^k_a\bar\psi^r_b\bar\psi^{bj}.$$

Note that since $F_{rjk}\psi^k_a\bar\psi^r_b\bar\psi^{bj} = F_{rjk}(\bar\psi^r_b\bar\psi^{bj}\psi^k_a - \bar\psi^r_a\delta^{kj})$ we may express $(Q^a)^\dagger$ in terms of $\bar Q_a$ (see (4.75)) as follows
$$(Q^a)^\dagger = \bar Q_a - iF_{lmn}\bar\psi^l_a\delta^{nm}.$$

We then have

$$\begin{aligned}\{Q^a,(Q^c)^\dagger\} &= \{Q^a,\bar Q_c\} - i\{Q^a, F_{lmn}\bar\psi^l_c\}\delta^{nm}\\ &= \{Q^a,\bar Q_c\} - \psi^{ar}\bar\psi^l_c\partial_r F_{lmn}\delta^{nm} - \psi^r_c\bar\psi^{ak}F_{rkl}F_{lmn}\delta^{nm} + \frac{i}{2}F_{rmn}p_r\delta^a_c\delta^{nm},\end{aligned}$$

with $\{Q^a,\bar Q_c\}$ defined by (4.80). Then supersymmetry algebra constraint $\{Q^a,(Q^c)^\dagger\} = -2\delta^a_c H$ leads to restrictions $\alpha = -\frac{1}{2}$, or $\alpha = -\frac{h+2}{4}$. In both cases the bosonic part of the

Hamiltonian H can be seen to be zero. Note that for $\alpha = -\frac{h+2}{4}$ the rescaled Hamiltonian $4H$ is given by

$$4H = p^2 + \sum_{\gamma\in\mathcal{R}_+} \frac{\gamma_r\gamma_j\gamma_l\gamma_k}{(\gamma,x)^2}\psi^{br}\psi_b^j\bar\psi_d^l\bar\psi^{dk}.$$

Then bosonic part can appear only by reordering of fermionic terms.

4.4.2 General ∨-systems

Let us consider a finite collection of covectors $\mathcal{A}$ on $V \cong \mathbb{C}^N$ such that the corresponding bilinear form

$$G_{\mathcal{A}}(u,v) = \sum_{\gamma\in\mathcal{A}}(\gamma,u)(\gamma,v), \quad u,v\in V$$

is non-degenerate.

Let f be any linear transformation on V, $f : V \to V$ and let f^* denote the dual map $f^* : V^* \to V^*$ defined by $\rho \mapsto \rho\circ f$. Then we have that

$$G_{f^*(\mathcal{A})}(u,v) = \sum_{\gamma\in\mathcal{A}}(f^*(\gamma),u)(f^*(\gamma),v) = \sum_{\gamma\in\mathcal{A}}(\gamma,f(u))(\gamma,f(v)) = G_{\mathcal{A}}(f(u),f(v)).$$

Then it is easy to see that $f^*(\mathcal{A})$ satisfies the ∨–conditions.

Furthermore, since $G_{\mathcal{A}}$ is non-degenerate we can assume by applying a suitable linear transformation f that

$$G_{\mathcal{A}}(u,v) = (u,v)$$

for any $u,v\in V$. We can then identify vectors and covectors. In particular, in this case $\mathcal{A}$ is a ∨-system if for any $\gamma\in\mathcal{A}$ and for any two-dimensional plane $\pi\subset V$ such that $\gamma\in\pi$ one has

$$\sum_{\beta\in\mathcal{A}\cap\pi}(\beta,\gamma)\beta = \mu\gamma,$$

for some $\mu = \mu(\gamma,\pi)\in\mathbb{C}$.

Let $F = F_{\mathcal{A}}(x_1,\ldots,x_N)$ be the corresponding function

$$F = \frac{\lambda}{2}\sum_{\gamma\in\mathcal{A}}(\gamma,x)^2\log(\gamma,x). \tag{4.89}$$

Then F satisfies generalised WDVV equations (4.12). Furthermore, the condition

$$x_iF_{ijk} = -(2\alpha+1)\delta_{jk}$$

is satisfied if

$$\alpha = -\frac{1}{2}(\lambda+1).$$

Therefore this leads to $D(2,1;\alpha)$ superconformal mechanics with the Hamiltonians given by Theorems 4.2.7, 4.3.1, which we present explicitly in the following theorem.

Theorem 4.4.8. *Let function F be given by (4.89). Then the Hamiltonian H given by (4.42) is supersymmetric with the superconformal algebra $D(2,1;\alpha)$, where $\alpha=-\frac{1}{2}(\lambda+1)$. The rescaled Hamiltonian $H_1=4H$ has the form*

$$H_1=-\Delta+\frac{\lambda}{2}\sum_{\gamma\in\mathcal{A}}\frac{(\gamma,\gamma)^2}{(\gamma,x)^2}+\frac{\lambda^2}{4}\sum_{\gamma,\beta\in\mathcal{A}}\frac{(\gamma,\gamma)(\beta,\beta)(\gamma,\beta)}{(\gamma,x)(\beta,x)}+\Phi,$$

where $\Delta=-p^2$ is the Laplacian in V and the fermionic term

$$\Phi=\sum_{\gamma\in\mathcal{A}}\frac{2\lambda\gamma_r\gamma_j\gamma_l\gamma_k}{(\gamma,x)^2}\psi^{br}\psi^j_b\bar\psi^l_d\bar\psi^{dk}-\sum_{\gamma\in\mathcal{A}}\frac{2\lambda\gamma_r\gamma_j(\gamma,\gamma)}{(\gamma,x)^2}\psi^r_b\bar\psi^{bj}. \tag{4.90}$$

Furthermore, the Hamiltonian H given by (4.77) is also supersymmetric with the superconformal algebra $D(2,1;\alpha)$, where $\alpha=-\frac{1}{2}(\lambda+1)$ and the rescaled Hamiltonian $H_2=4H$ has the form

$$H_2=-\Delta+\lambda\sum_{\gamma\in\mathcal{A}}\frac{(\gamma,\gamma)}{(\gamma,x)}\partial_\gamma+\Phi.$$

The proof is similar to the one in the Coxeter case. The following proposition can also be checked directly.

Proposition 4.4.9. *Hamiltonians H_1,H_2 from Theorem 4.4.8 satisfy the gauge relation $\delta^{-1}\circ H_2\circ\delta=H_1$, where $\delta=\prod_{\beta\in\mathcal{A}}(\beta,x)^{\frac{\lambda}{2}(\beta,\beta)}$.*

4.5 Trigonometric version

In this section we consider prepotential functions $F=F(x_1,\dots,x_N)$ of the form

$$F=\sum_{\alpha\in\mathcal{A}}c_\alpha f((\alpha,x)), \tag{4.91}$$

where $\mathcal{A}$ is a finite set of vectors in $V\cong\mathbb{C}^N$, $c_\alpha\in\mathbb{C}$ are some multiplicities of these vectors, and function f is given by

$$f(z)=\frac{1}{6}z^3-\frac{1}{4}\mathrm{Li}_3(e^{-2z})$$

so that $f'''(z)=\coth z$.

We are interested in the supercharges of the form

$$Q^a = p_r\psi^{ar} + iF_{rjk}\langle\psi^{br}\psi_b^j\bar\psi^{ak}\rangle,$$
$$\bar Q_c = p_l\bar\psi_c^l + iF_{lmn}\langle\bar\psi_d^l\bar\psi^{dm}\psi_c^n\rangle,$$

$a, c = 1, 2$, which is analogous to the first representation considered in Section 4.2.

Function F should satisfy conditions

$$F_{rjk}F_{kmn} = F_{rmk}F_{kjn}, \tag{4.92}$$

for all $r, j, m, n = 1, \dots, N$ but we no longer assume conditions (4.11). Then we have the following statement on supersymmetry algebra.

Theorem 4.5.1. *Let us assume that F satisfies conditions* (4.92)*. Then for all $a, b = 1, 2$ we have*

$$\{Q^a, Q^b\} = \{\bar Q_a, \bar Q_b\} = 0 \quad \textit{and} \quad \{Q^a, \bar Q_b\} = -2H\delta_b^a,$$

where the Hamiltonian H is given by

$$H = \frac{p^2}{4} - \frac{\partial_i F_{jlk}}{2}(\psi^{bi}\psi_b^j\bar\psi_d^l\bar\psi^{dk} - \psi_b^i\bar\psi^{bj}\delta^{lk} + \frac{1}{4}\delta^{ij}\delta^{lk}) + \frac{1}{16}F_{ijk}F_{lmn}\delta^{nm}\delta^{jl}\delta^{ik}.$$

Furthermore, the rescaled Hamiltonian $H_1 = 4H$ has the form

$$H_1 = -\Delta + \frac{1}{2}\sum_{\alpha\in\mathcal{A}}\frac{c_\alpha(\alpha,\alpha)^2}{\sinh^2(\alpha,x)} + \frac{1}{4}\sum_{\alpha,\beta\in\mathcal{A}}c_\alpha c_\beta(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)\coth(\alpha,x)\coth(\beta,x) + \Phi, \tag{4.93}$$

where $\Delta = -p^2$ is the Laplacian in V and the fermionic term

$$\Phi = \sum_{\alpha\in\mathcal{A}}\frac{2c_\alpha\alpha_i\alpha_j}{\sinh^2(\alpha,x)}\big(\alpha_l\alpha_k\psi^{bi}\psi_b^j\bar\psi_d^l\bar\psi^{dk} - (\alpha,\alpha)\psi_b^i\bar\psi^{bj}\big). \tag{4.94}$$

The proof of the first part of the theorem is the same as the proof of Theorem 4.2.7 together with the proof of the relevant part of Lemma 4.2.9. The proof of formula (4.93) is similar to the proof of Theorem 4.4.4.

Let us now consider supercharges of the form

$$Q^a = p_r\psi^{ar} + iF_{rjk}\psi^{br}\psi_b^j\bar\psi^{ak},$$
$$\bar Q_c = p_l\bar\psi_c^l + iF_{lmn}\bar\psi_d^l\bar\psi^{dm}\psi_c^n,$$

$a, c = 1, 2$, which is analogous to the second representation considered in Section 4.3. Then we have the following statement on supersymmetry algebra.

Theorem 4.5.2. *Let us assume that F satisfies conditions* (4.92)*. Then for all $a, b = 1, 2$ we have*

$$\{Q^a, Q^b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 \quad \textit{and} \quad \{Q^a, \bar{Q}_b\} = -2H\delta^a_b,$$

where the Hamiltonian H is given by

$$H = \frac{p^2}{4} - \frac{\partial_r F_{jlk}}{2}(\psi^{br}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - \psi^r_b\bar{\psi}^{bj}\delta^{lk}) + \frac{i}{4}\delta^{nm}F_{rmn}p_r. \tag{4.95}$$

Furthermore, the rescaled Hamiltonian $H_2 = 4H$, has the form

$$H_2 = -\Delta + \sum_{\alpha\in\mathcal{A}} c_\alpha(\alpha,\alpha)\coth(\alpha,x)\partial_\alpha + \Phi, \tag{4.96}$$

where Φ is the fermionic term defined by (4.94)*.*

The proof of the first part of the theorem is the same as the proof of Theorem 4.3.1 together with the proof of the relevant part of Lemma 4.3.4. Then formula (4.96) can be easily derived from the form (4.95) of H.

Proposition 4.5.3. *Hamiltonians H_1, H_2 from Theorems 4.5.1, 4.5.2 respectively satisfy gauge relation*

$$\delta^{-1}\circ H_2\circ\delta = H_1,$$

where $\delta = \prod_{\alpha\in\mathcal{A}}\sinh(\alpha,x)^{\frac{c_\alpha}{2}(\alpha,\alpha)}$.

Let us now assume that $\mathcal{A} = \mathcal{R}$ is a crystallographic root system, and that the multiplicity function $c(\alpha) = c_\alpha$, $\alpha\in\mathcal{R}$ is invariant under the corresponding Weyl group W. For a general root system $\mathcal{R}$ the corresponding function F does not satisfy equations (4.92). For example, if $\mathcal{R} = A_{N-1}$ then relations (4.92) do not hold (see Remark 4.5.7) . But for some root systems and collections of multiplicities relations (4.92) are satisfied.

In the rest of this section we consider such cases when prepotential F satisfying (4.92) does exist. The corresponding root systems $\mathcal{R}$ have more than one orbit under the action of the Weyl group W. We start by simplifying the corresponding Hamiltonians H_1 given by (4.93).

Proposition 4.5.4. *Let us assume that prepotential F given by* (4.91) *for a root system $\mathcal{R}$ with invariant multiplicity function c satisfies* (4.92)*. Then Hamiltonian* (4.93) *can be rearranged as*

$$H_1 = -\Delta + \sum_{\alpha\in\mathcal{R}_+}\frac{\widetilde{c_\alpha}}{\sinh^2(\alpha,x)} + \widetilde{\Phi}, \tag{4.97}$$

where

$$\widetilde{c_\alpha} = \begin{cases} c_\alpha(\alpha,\alpha)^2\big(1 + c_\alpha(\alpha,\alpha)\big), & \textit{if} \quad 2\alpha\notin\mathcal{R}, \\ c_\alpha(\alpha,\alpha)^2\big(1 + (\alpha,\alpha)(c_\alpha + 8c_{2\alpha})\big), & \textit{if} \quad 2\alpha\in\mathcal{R}, \end{cases}$$

$\widetilde{\Phi} = \Phi + const$, *with* Φ *given by* (4.94) *and* $\mathcal{R}_+$ *is a positive subsystem in* $\mathcal{R}$.

Indeed, it is easy to see that for the crystallographic root system $\mathcal{R}$ the term

$$\sum_{\substack{\beta,\alpha\in\mathcal{R}\\ \beta\not\sim\alpha}} c_\alpha c_\beta(\alpha,\alpha)(\beta,\beta)(\alpha,\beta)\coth(\alpha,x)\coth(\beta,x)$$

is non-singular at $\tanh(\alpha,x)=0$ for all $\alpha\in\mathcal{R}$, hence it is constant. One can show that the Hamiltonian H_1 given by (4.93) simplifies to the required form.

We now show that solutions to equations (4.92) exist for the root systems $\mathcal{R}=BC_N$, $\mathcal{R}=F_4$ and $\mathcal{R}=G_2$, with special collections of invariant multiplicities.

Let $\mathcal{R}_+$ be a positive subsystem in the root system $\mathcal{R}$. For a pair of vectors $a,b\in V$ we define a 2-form $\mathcal{B}_{\mathcal{R}_+}^{(a,b)}$ by

$$\mathcal{B}_{\mathcal{R}_+}^{(a,b)} = \sum_{\beta,\gamma\in\mathcal{R}_+} c_\beta c_\gamma(\beta,\gamma)B_{\beta,\gamma}(a,b)\beta\wedge\gamma, \tag{4.98}$$

where $B_{\alpha,\beta}(a,b) = \alpha\wedge\beta(a,b) = (\alpha,a)(\beta,b)-(\alpha,b)(\beta,a)$. The form $\mathcal{B}_{\mathcal{R}_+}^{(a,b)}$ has good properties with regard to the action of the corresponding Weyl group W. Namely, the following statement takes place.

Proposition 4.5.5. *The* 2*-form (4.98) is* W*-invariant, that is*

$$w\mathcal{B}_{\mathcal{R}_+}^{(a,b)} = \mathcal{B}_{\mathcal{R}_+}^{(wa,wb)} = \mathcal{B}_{w\mathcal{R}_+}^{(wa,wb)}, \tag{4.99}$$

for any $w\in W$.

Proof. Let us choose a simple root $\alpha\in\mathcal{R}_+$. It is sufficient to prove the statement for $w=s_\alpha$. Let us rewrite $\mathcal{B}_{\mathcal{R}_+}^{(a,b)}$ as

$$\mathcal{B}_{\mathcal{R}_+}^{(a,b)} = 2c_\alpha\sum_{\beta\in\mathcal{R}_+} c_\beta(\alpha,\beta)B_{\alpha,\beta}(a,b)\alpha\wedge\beta + \sum_{\beta,\gamma\in\mathcal{R}_+\setminus\{\alpha\}} c_\beta c_\gamma(\beta,\gamma)B_{\beta,\gamma}(a,b)\beta\wedge\gamma.$$

It is easy to see that for any $\beta,\gamma\in\mathcal{R}$

$$B_{\beta,\gamma}(s_\alpha a, s_\alpha b) = B_{s_\alpha\beta,s_\alpha\gamma}(a,b) \tag{4.100}$$

since $(u,s_\alpha v)=(s_\alpha u,v)$ for any $u,v\in V$. Let us now apply s_α to equality (4.98). Since

$s_\alpha(\mathcal{R}_+ \setminus \{\alpha\}) = \mathcal{R}_+ \setminus \{\alpha\}$ we have

$$\begin{aligned} s_\alpha \mathcal{B}^{(a,b)}_{\mathcal{R}_+} &= -2c_\alpha \sum_{\beta\in\mathcal{R}_+} c_\beta(\alpha,\beta)B_{\alpha,\beta}(a,b)\alpha\wedge\beta + \sum_{\beta,\gamma\in\mathcal{R}_+\setminus\{\alpha)} c_\beta c_\gamma(\beta,\gamma)B_{\beta,\gamma}(a,b)s_\alpha\beta\wedge s_\alpha\gamma \\ &= 2c_\alpha \sum_{\beta\in\mathcal{R}_+} c_\beta(\alpha,\beta)B_{s_\alpha\alpha,s_\alpha\beta}(a,b)\alpha\wedge\beta + \sum_{\beta,\gamma\in\mathcal{R}_+\setminus\{\alpha)} c_\beta c_\gamma(\beta,\gamma)B_{s_\alpha\beta,s_\alpha\gamma}(a,b)\beta\wedge\gamma \\ &= \mathcal{B}^{(s_\alpha a,s_\alpha b)}_{\mathcal{R}_+}, \end{aligned}$$

by the relation (4.100). This proves the first equality in (4.99). In order to prove the second equality (4.99) let us notice that in fact

$$\sum_{\beta\in\mathcal{R}_+} c_\beta(\alpha,\beta)B_{\alpha,\beta}(a,b)\alpha\wedge\beta = 0.$$

Hence $s_\alpha \mathcal{B}^{(a,b)}_{\mathcal{R}_+} = \mathcal{B}^{(s_\alpha a,s_\alpha b)}_{s_\alpha\mathcal{R}_+}$. □

Below we will denote $\mathcal{B}^{(a,b)}_{\mathcal{R}_+} = \mathcal{B}^{(a,b)}$ since by Proposition 4.5.5, $\mathcal{B}^{(a,b)}_{\mathcal{R}_+}$ does not depend on the choice of root system. Let us derive some conditions for a function F to satisfy equations of the form (4.92). Let F_i be the $N\times N$ matrices of third derivatives of F, $(F_i)_{lm} = \frac{\partial^3 F}{\partial x_i \partial x_l \partial x_m}$, and for any vector $a = (a_1,\dots,a_N)\in V$ let us denote $F_a = \sum_{i=1}^N a_i F_i$.

Theorem 4.5.6. *Let $a, b\in V$. Then the equations*

$$F_aF_b = F_bF_a$$

are satisfied if and only if

$$\mathcal{B}^{(a,b)} = 0. \tag{4.101}$$

Proof. We have

$$(F_a)_{lk} = \sum_{\alpha\in\mathcal{R}} c_\alpha(\alpha,a)\alpha_l\alpha_k \coth(\alpha,x),$$

and therefore

$$F_aF_b = \sum_{\alpha,\beta\in\mathcal{R}} c_\alpha c_\beta(\alpha,a)(\beta,b)(\alpha,\beta)\coth(\alpha,x)\coth(\beta,x)\alpha\otimes\beta.$$

Hence the equations $[F_a,F_b]=0$ are equivalent to

$$\sum_{\alpha,\beta\in\mathcal{R}} c_\alpha c_\beta B_{\alpha,\beta}(a,b)(\alpha,\beta)\coth(\alpha,x)\coth(\beta,x)\alpha\otimes\beta = 0,$$

which can be easily checked to be equivalent to

$$\sum_{\alpha,\beta\in\mathcal{R}_+} c_\alpha c_\beta B_{\alpha,\beta}(a,b)(\alpha,\beta)\coth(\alpha,x)\coth(\beta,x)\alpha\wedge\beta = 0. \tag{4.102}$$

It is easy to see that the sum in the left-hand side of the equality (4.102) is non-singular at $\tanh(\alpha,x)=0$ for all $\alpha\in\mathcal{R}_+$, hence this sum is always constant. In an appropriate limit in a cone $\coth(\alpha,x)\to 1$ for all $\alpha\in\mathcal{R}_+$, and therefore the equality (4.102) is equivalent to the equality

$$\sum_{\alpha,\beta\in\mathcal{R}_+} c_\alpha c_\beta B_{\alpha,\beta}(a,b)(\alpha,\beta)\alpha\wedge\beta = 0,$$

as required. □

Let e_i, $i=1,\ldots,N$ be the standard orthonormal basis in V. We may express $\mathcal{B}^{(a,b)}$ in the basis $e_i\wedge e_j$ of $\Lambda^2 V$,

$$\mathcal{B}^{(a,b)} = \sum_{1\le i<j\le N} g_{ij} e_i\wedge e_j, \tag{4.103}$$

for some scalars $g_{ij}=g_{ij}(a,b)$. Then linear independence of the basis vectors and condition (4.101) give rise to $\binom{N}{2}$ equations $g_{ij}(a,b)=0$. If $A_{N-1}\subset\mathcal{R}$ then by Proposition 4.5.5 we should have that $g_{ij}(a,b)=\pm g_{\sigma(i)\sigma(j)}(\sigma(a),\sigma(b))$ for any transposition $\sigma\in S_N$ which acts on vectors a,b by the corresponding permutation of coordinates. This shows that the condition (4.101) reduces to a single equation $g_{ij}=0$ for any fixed i,j and general $a,b\in V$. For convenience we will write below $B_{e_i,e_j}(a,b)$ as $B_{ij}(a,b)$.

Remark 4.5.7. Let $\mathcal{R}=A_{N-1}$. Then relations (4.92) do not hold.

Proof. Let the positive half of the root system A_{N-1} be

$$e_i-e_j,\quad 1\le i<j\le N.$$

Let s be the multiplicity of the vectors in A_{N-1} and let us use Theorem 4.5.6 in order to deal with conditions (4.92). We consider the coefficient $g_{12}(a,b)$ at $e_1\wedge e_2$ by collecting respective terms in the corresponding form $\mathcal{B}^{(a,b)}$ given by (4.98), (4.103). The non-trivial contribution to g_{12} comes only from the following pairs of vectors $\{\beta,\gamma\}$ in the expansion (4.98):

$$\{e_1-e_2,e_1-e_j\},\quad \{e_1-e_2,e_2-e_j\},\quad \{e_1-e_j,e_2-e_j\},\quad 3\le j\le N.$$

Let $\alpha_k = (N-2)e_k - \sum_{i=3}^N e_i$, $k = 1, 2$. Then since $\alpha_1 - \alpha_2 = (N-2)(e_1 - e_2)$ we have

$$
\begin{aligned}
g_{12}(a,b) &= 2s^2\big(B_{e_1-e_2,\alpha_1}(a,b) - B_{e_1-e_2,\alpha_2}(a,b) + \sum_{j=3}^N B_{e_1-e_j,e_2-e_j}(a,b)\big) \\
&= 2s^2 \sum_{j=3}^N B_{e_1-e_j,e_2-e_j}(a,b).
\end{aligned} \tag{4.104}
$$

Then it is easy to see that the right-hand side of g_{12} (4.104) is generically non-zero. The statement then follows. □

Theorem 4.5.8. *Let $\mathcal{R} = BC_N$. Let the positive half of the root system BC_N be*

$$
\eta e_i,\ 2\eta e_i,\ 1 \le i \le N; \quad \eta(e_i \pm e_j),\ 1 \le i < j \le N,
$$

where $\eta \in \mathbb{C}^\times$ is a parameter. Let r be the multiplicity of vectors ηe_i, and let s be the multiplicity of vectors $2\eta e_i$. Let q be the multiplicity of vectors $\eta(e_i \pm e_j)$. Then the function

$$
F = \sum_{i=1}^N (rf(\eta x_i) + sf(2\eta x_i)) + q\sum_{i<j}^N f(\eta(x_i \pm x_j)) \tag{4.105}
$$

satisfies conditions (4.92) *if and only if $r = -8s - 2(N-2)q$. The corresponding supersymmetric Hamiltonians given by* (4.96), (4.97) *take the form*

$$
\begin{aligned}
H_1 = -\Delta + \eta^4 \sum_{i=1}^N \Big(&\frac{(8s + 2(N-2)q)(2(N-2)q\eta^2 - 1)}{\sinh^2 \eta x_i} + \frac{16s(4s\eta^2+1)}{\sinh^2 2\eta x_i}\Big) \\
&+ \eta^4 \sum_{i<j}^N \frac{4q(2q\eta^2+1)}{\sinh^2(\eta(x_i \pm x_j))} + \widetilde{\Phi},
\end{aligned} \tag{4.106}
$$

and

$$
\begin{aligned}
H_2 = -\Delta + 2\eta^3 &\sum_{i=1}^N \big(8s \coth 2\eta x_i - (8s + 2(N-2)q)\coth \eta x_i\big)\partial_i \\
&+ 4q\eta^3 \sum_{i<j}^N \coth(\eta(x_i \pm x_j))(\partial_i \pm \partial_j) + \Phi,
\end{aligned} \tag{4.107}
$$

with Φ given by

$$\Phi = 4\eta^4 \sum_{i=1}^{N} \Big(\frac{-(8s+2(N-2)q)}{\sinh^2 \eta x_i} + \frac{16s}{\sinh^2 2\eta x_i} \Big) \Big(\psi^{bi}\psi^i_b \bar{\psi}^i_d \bar{\psi}^{di} - \psi^i_b \bar{\psi}^{bi} \Big)$$
$$+ 4\eta^4 \sum_{\epsilon \in \{1,-1\}} \sum_{m<t}^{N} \sum_{i,j,l,k} \frac{q d_{mti} d_{mtj}}{\sinh^2(\eta(x_m + \epsilon x_t))} \Big(d_{mtl} d_{mtk} \psi^{bi}\psi^j_b \bar{\psi}^l_d \bar{\psi}^{dk} - 2\psi^i_b \bar{\psi}^{bj} \Big),$$

where $d_{mtk} = d_{mtk}(\epsilon) = \delta_{mk} + \epsilon\delta_{tk}$, and $\widetilde{\Phi} = \Phi + const$.

Proof. Let us use Theorem 4.5.6 in order to deal with conditions (4.92). Let us consider the coefficient $g_{12}(a,b)$ at $e_1 \wedge e_2$ by collecting respective terms in the corresponding form $\mathcal{B}^{(a,b)}$ given by (4.98), (4.103). The non-trivial contribution to g_{12} comes only from the following pairs of vectors $\{\beta, \gamma\}$ in the expansion (4.98):

$$(1)\ \{\eta e_1, \eta(e_1 \pm e_2)\}, \quad (2)\ \{2\eta e_1, \eta(e_1 \pm e_2)\}, \quad (3)\ \{\eta(e_1 \pm e_2), \eta(e_1 \pm e_j)\},\ 3 \le j \le N,$$

since contributions from pairs $\{\eta(e_1 \pm e_2), \eta(e_2 \pm e_j)\}$ and $\{\eta(e_1 \pm e_j), \eta(e_2 \pm e_j)\}$ is zero each. Pairs (1) contribute $4rq\eta^6 B_{12}(a,b)$, pairs (2) contribute $32sq\eta^6 B_{12}(a,b)$ and pairs (3) contribute $8q^2(N-2)\eta^6 B_{12}(a,b)$. Therefore

$$g_{12}(a,b) = 4q(r + 8s + 2(N-2)q)\eta^6 B_{12}(a,b).$$

By Proposition 4.5.5, $g_{ij} = 0$ for all $1 \le i < j \le N$ if and only if $r = -8s - 2(N-2)q$. The form of the Hamiltonians H_2, H_1 follows from Theorem 4.5.2 and Proposition 4.5.4 respectively. Then the statement follows. □

Remark 4.5.9. We note that for the multiplicity $s = 0$ Theorem 4.5.8 is contained in [49]. Indeed, Theorem 2.3 in [49] states that the function F given by formula (4.105) with root system $\mathcal{R} = B_N$ satisfies WDVV equations. It also follows from the proof of Theorem 2.3 in [49] that the corresponding metric is proportional to the standard metric δ_{ij}. Therefore WDVV equations are equivalent to equations (4.92).

More generally, it is shown in [2] that the function (4.105) satisfies WDVV equations if and only if the relation $r = -8s - 2(N-2)q$ for the multiplicities r, s, q in Theorem 4.5.8 takes place. Thus, a metric from the third derivatives of F is constructed which is proportional to the metric δ_{ij}, and in this way a generalisation of Theorem 2.3 in [49] is obtained. In fact, one can consider a generalisation of the configuration BC_N and show that the corresponding function also satisfies WDVV equations (see [2] for details).

Theorem 4.5.10. *Let $\mathcal{R} = F_4$. Let the positive half of the root system F_4 be*

$$\eta e_i,\ 1 \le i \le 4; \quad \eta(e_i \pm e_j),\ 1 \le i < j \le 4; \quad \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4),$$

where $\eta \in \mathbb{C}^\times$ is a parameter. Let r be the multiplicity of short roots ηe_i, $\frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$ and let q be the multiplicity of long roots $\eta(e_i \pm e_j)$. The function

$$F = r\sum_{i=1}^{4} f(\eta x_i) + r\sum_{\epsilon_i\in\{1,-1\}} f(\frac{\eta}{2}(\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + x_4)) + q\sum_{i<j}^{4} f(\eta(x_i \pm x_j))$$

satisfies conditions (4.92) *if and only if $r = -2q$ or $r = -4q$. The corresponding supersymmetric Hamiltonians* (4.97), (4.96) *take the form*

$$H_1 = -\Delta + r(1+r\eta^2)\eta^4\Big(\sum_{i=1}^{4}\frac{1}{\sinh^2 \eta x_i} + \sum_{\epsilon_i\in\{1,-1\}}\frac{1}{\sinh^2(\frac{\eta}{2}(\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + x_4))}\Big)$$
$$+ \eta^4\sum_{i<j}^{4}\frac{4q(1+2q\eta^2)}{\sinh^2(\eta(x_i \pm x_j))} + \widetilde{\Phi},$$

and

$$H_2 = -\Delta + r\eta^3\sum_{\epsilon_i\in\{1,-1\}}\coth(\frac{\eta}{2}(\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + x_4))(\epsilon_1\partial_1 + \epsilon_2\partial_2 + \epsilon_3\partial_3 + \partial_4)$$
$$+ 2r\eta^3\sum_{i=1}^{4}\coth \eta x_i \partial_i + 4q\eta^3\sum_{i<j}^{4}\coth(\eta(x_i \pm x_j))(\partial_i \pm \partial_j) + \Phi$$

with Φ given by

$$\Phi = 4\eta^4\sum_{i=1}^{4}\frac{r}{\sinh^2 \eta x_i}\Big(\psi^{bi}\psi^i_b\bar{\psi}^i_d\bar{\psi}^{di} - \psi^i_b\bar{\psi}^{bi}\Big)$$
$$+ 4\eta^4\sum_{\epsilon\in\{1,-1\}}\sum_{m<t}^{4}\sum_{i,j,l,k}\frac{qd_{mti}d_{mtj}}{\sinh^2 \eta((x_m + \epsilon x_t))}\Big(d_{mtl}d_{mtk}\psi^{bi}\psi^j_b\bar{\psi}^l_d\bar{\psi}^{dk} - 2\psi^i_b\bar{\psi}^{bj}\Big)$$
$$+ 4\eta^4\sum_{\epsilon_i\in\{1,-1\}}\sum_{i,j,l,k}\frac{rd_id_j}{\sinh^2(\frac{\eta}{2}(\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + x_4))}\Big(d_ld_k\psi^{bi}\psi^i_b\bar{\psi}^i_d\bar{\psi}^{di} - \psi^i_b\bar{\psi}^{bi}\Big),$$

where $r = -2q$ or $r = -4q$, $d_i = d_i(\epsilon_1, \epsilon_2, \epsilon_3) = \epsilon_1\delta_{1i} + \epsilon_2\delta_{2i} + \epsilon_3\delta_{3i} + \delta_{4i}$ and $\widetilde{\Phi} = \Phi + const.$

Proof. Since $B_4 \subset F_4$ we have the contribution to the coefficient g_{12} of the form (4.98), (4.103) from the pairs of vectors $\{\beta, \gamma\} \in B_4$ which is equal to $4q(4q+r)\eta^6 B_{12}(a,b)$. The remaining contribution to the coefficient g_{12} comes from the following pairs of vectors $\{\beta, \gamma\}$ in the expansion (4.98):

$$(1)\ \{\eta e_1, \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}, \quad (2)\ \{\eta(e_1 \pm e_3), \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\},$$

$$(3)\ \{\eta(e_1 \pm e_4), \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.$$

Indeed, let us demonstrate why pairs of vectors of the form

$$\frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \tag{4.108}$$

contribute trivially to the coefficient g_{12} of the form (4.98), (4.103). Let $\beta = \frac{\eta}{2}(e_1 + \lambda e_2 + \mu e_3 + \nu e_4)$ and $\widetilde{\beta} = \frac{\eta}{2}(e_1 + \lambda e_2 - \mu e_3 - \nu e_4)$, where $\lambda, \mu, \nu = \pm 1$. Non-trivial contribution with this β to g_{12} can only come from the two pairs $\{\beta, \pm\gamma\}$, where $\gamma_\pm = \frac{\eta}{2}(e_1 - \lambda e_2 \pm (\mu e_3 + \nu e_4))$. The same holds for $\widetilde{\beta}$. The contribution from the two pairs $\{\beta, \gamma_\pm\}$ is $-\frac{\lambda r^2}{4}\eta^6 B_{e_1+\lambda e_2, \mu e_3 + \nu e_4}$ while the contribution from the two pairs $\{\widetilde{\beta}, \gamma_\pm\}$ is $\frac{\lambda r^2}{4}\eta^6 B_{e_1+\lambda e_2, \mu e_3+\nu e_4}$. Hence altogether contributions to g_{12} from pairs of vectors of the form (4.108) cancel. Similarly, one can check that contributions from pairs $\{\eta e_2, \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$ and $\{\eta(e_1 \pm e_2), \frac{\eta}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$ is zero.

Then pairs (1) contribute $2r^2\eta^6 B_{12}(a, b)$ and pairs $(2), (3)$ contribute $4rq\eta^6 B_{12}(a, b)$ each. Therefore in total

$$g_{12}(a, b) = 2(8q^2 + 6rq + r^2)\eta^6 B_{12}(a, b).$$

By Proposition 4.5.5, $g_{ij} = 0$ for all $1 \le i < j \le 4$ if and only if $r = -2q$ or $r = -4q$. The form of the Hamiltonians H_2, H_1 follows from Theorem 4.5.2 and Proposition 4.5.4. Then the statement follows. $\square$

Theorem 4.5.11. *Let $\mathcal{R} = G_2$. Let the positive half of the root system G_2 considered in three dimensional space be*

$$\alpha_1 = \eta(e_1 - e_2), \quad \alpha_2 = \eta(e_1 - e_3), \quad \alpha_3 = \eta(e_2 - e_3),$$

$$\alpha_4 = \eta(2e_1 - e_2 - e_3), \quad \alpha_5 = \eta(e_1 + e_2 - 2e_3), \quad \alpha_6 = \eta(e_1 - 2e_2 + e_3),$$

where $\eta \in \mathbb{C}^\times$ is a parameter. Let s be the multiplicity of the short roots α_i, $i = 1, 2, 3$ and let r be the multiplicity of the long roots α_j, $j = 4, 5, 6$. Then the function

$$F = s\sum_{i<j}^{3} f(\eta(x_i - x_j)) + \frac{r}{2}\sum_{\sigma \in S_3} f(\eta(2x_{\sigma(1)} - x_{\sigma(2)} - x_{\sigma(3)}))$$

satisfies conditions (4.92) if and only if $s = -3r$ or $s = -9r$. The corresponding supersymmetric Hamiltonians (4.97), (4.96) take the form

$$H_1 = -\Delta + \eta^4 \sum_{i<j}^{3} \frac{4s(1 + 2s\eta^2)}{\sinh^2(\eta(x_i - x_j))} + \eta^4 \sum_{\sigma \in S_3} \frac{18r(1 + 6r\eta^2)}{\sinh^2(\eta(2x_{\sigma(1)} - x_{\sigma(2)} - x_{\sigma(3)}))} + \widetilde{\Phi},$$

and

$$H_2 = -\Delta + 4s\eta^3 \sum_{i<j}^{3} \coth(\eta(x_i - x_j))(\partial_i - \partial_j)$$
$$+ 6r\eta^3 \sum_{\sigma\in S_3} \coth(\eta(2x_{\sigma(1)} - x_{\sigma(2)} - x_{\sigma(3)}))(2\partial_{\sigma(1)} - \partial_{\sigma(2)} - \partial_{\sigma(3)}) + \Phi,$$

with Φ *given by*

$$\Phi = 4\eta^4 \sum_{m<t}^{3} \sum_{i,j,l,k} \frac{s d^-_{mti} d^-_{mtj}}{\sinh^2(\eta(x_m - x_t))} \Big(d^-_{mtl} d^-_{mtk} \psi^{bi} \psi^j_b \bar{\psi}^l_d \bar{\psi}^{dk} - 2\psi^i_b \bar{\psi}^{bj} \Big)$$
$$+ 2\eta^4 \sum_{\sigma\in S_3} \sum_{i,j,l,k} \frac{r d^\sigma_i d^\sigma_j}{\sinh^2(\eta(2x_{\sigma(1)} - x_{\sigma(2)} - x_{\sigma(3)}))} \Big(d^\sigma_l d^\sigma_k \psi^{bi} \psi^j_b \bar{\psi}^l_d \bar{\psi}^{dk} - 6\psi^i_b \bar{\psi}^{bj} \Big)$$

where $s = -3r$ *or* $s = -9r$*,* $d^-_{mti} = \delta_{mi} - \delta_{ti}$*,* $d^\sigma_i = 2\delta_{\sigma(1)i} - \delta_{\sigma(2)i} - \delta_{\sigma(3)i}$*, and* $\widetilde{\Phi} = \Phi + const.$

Proof. The coefficient at $e_1 \wedge e_2$ in the form $\mathcal{B}^{(a,b)}$ given by (4.98), (4.103) is

$$g_{12}(a,b) = \sum_{i<j}^{6} 2c_{\alpha_i} c_{\alpha_j} (\alpha_i, \alpha_j) B_{\alpha_i,\alpha_j}(a,b)(\alpha_i \wedge \alpha_j, e_1 \wedge e_2) = \sum_{i=1}^{5} A_i,$$

where $(\alpha_i \wedge \alpha_j, e_1 \wedge e_2) = \det(c_1, c_2)$ where c_k are the column vectors $c_k = ((\alpha_i, e_k), (\alpha_j, e_k))^\intercal$, $k = 1, 2$, and

$$A_i = \sum_{j=i+1}^{6} 2c_{\alpha_i} c_{\alpha_j} (\alpha_i, \alpha_j) B_{\alpha_i,\alpha_j}(a,b)(\alpha_i \wedge \alpha_j, e_1 \wedge e_2).$$

We have

$$\begin{aligned} A_1 &= 6sr\eta^6 B_{\alpha_1,\alpha_5}(a,b), \\ A_2 &= 2s\eta^6 \big(sB_{\alpha_2,\alpha_3}(a,b) - 3rB_{\alpha_2,\alpha_6}(a,b) \big), \\ A_3 &= 0, \\ A_4 &= 18r^2\eta^6 B_{\alpha_4,3\alpha_3}(a,b), \\ A_5 &= 18r^2\eta^6 B_{\alpha_5,\alpha_6}(a,b). \end{aligned}$$

Simplifying we obtain

$$g_{12}(a,b) = 2\eta^6(27r^2 + 12rs + s^2)(B_{12}(a,b) - B_{13}(a,b) + B_{23}(a,b)).$$

By Proposition 4.5.5, $g_{ij} = 0$ for all $1 \le i < j \le 3$ if and only if $s = -3r$ or $s = -9r$. The form of the Hamiltonians H_2, H_1 follows from Theorem 4.5.2 and Proposition 4.5.4 respectively. Then the statement follows. □

Remark 4.5.12. The bosonic part of the supersymmetric Hamiltonians (4.96), (4.97) becomes Calogero–Moser Hamiltonian in the rational limit. For example let us consider the case of the root system BC_N and let us introduce rescaled multiplicities $\widehat{s} = \eta^2 s$, $\widehat{q} = \eta^2 q$ and $\widehat{r} = \eta^2 r$ in Theorem 4.5.8. Then in the limit $\eta \to 0$ bosonic parts of Hamiltonians H_1 and H_2 given by (4.106), (4.107) become the rational B_N Hamiltonians $H_1^{b,r}, H_2^{b,r}$ with two independent coupling parameters, namely,

$$H_1^{b,r} = -\Delta + \sum_{i<j}^{N} \frac{4\widehat{q}(2\widehat{q}+1)}{(x_i \pm x_j)^2} + \sum_{i=1}^{N} \frac{l(l-1)}{x_i^2},$$

and

$$H_2^{b,r} = -\Delta + \sum_{i<j}^{N} \frac{4\widehat{q}}{x_i \pm x_j}(\partial_i \pm \partial_j) - \sum_{i=1}^{N} \frac{2l}{x_i}\partial_i,$$

where $l = 2((N-2)\widehat{q} + 2\widehat{s})$. Thus supersymmetric Hamiltonians (4.106), (4.107) can be viewed as η-deformation of the rational superconformal Hamiltonians considered in Theorems 4.4.4, 4.4.5 for the root system $\mathcal{R} = B_N$.

Chapter 5

Concluding remarks and open questions

5.1 Determinant of restricted Saito metric

In this work we considered the Saito metric η defined on the Coxeter orbit space $\mathcal{M}_W$. We studied the restriction of this metric on discriminant strata inside the discriminant of $\mathcal{M}_W$ and obtained a formula for its determinant which is described in terms of the underlying Coxeter geometry of root systems and corresponding arrangements of hyperplanes.

Comment 1. Main Theorems 1 and 2 are proved for exceptional Coxeter groups by case by case considerations (except for codimensions one and two, and dimension one). It would be more illuminating if a proof for all Coxeter groups can be obtained uniformly, perhaps via a different route.

Comment 2. It would be interesting to study other properties of the metric η_D on the strata D. For example, the scalar curvature may be of interest. Initial considerations suggest that it may have a factorised form, similarly to the determinant of η_D.

Comment 3. Frobenius manifold structures on the orbit spaces $\mathcal{M}_{\widetilde{W}}$ of extended affine Weyl groups $\widetilde{W}$ were considered firstly by Dubrovin and Zhang [26]. A non-degenerate flat metric (analogous to the Saito metric on $\mathcal{M}_W$) can be defined on these orbit spaces. It would be interesting to see if the restriction of this metric to the corresponding strata inside the discriminant of $\mathcal{M}_{\widetilde{W}}$ has a similar property to the restricted Saito metric, that is whether determinant has a nice form in suitable coordinate system.

Comment 4. T. Dourvopoulos recently informed us about their conjecture with C. Stump on freeness of a new class of multi-arrangements (see [73] book for the theory of free arrangements). These are restricted Coxeter arrangements and multiplicities come from the multiplicities of factors of the determinant of the restricted Saito metric considered in this work. It would be interesting to analyse this conjecture possibly, in relation with methods developed in this thesis.

5.2 Supersymmetric $\vee$–systems

Since work [88] there were extensive attempts to define superconformal $\mathcal{N} = 4$ Calogero–Moser type systems for sufficiently general coupling parameters and suitable superconformal algebras. Some low rank cases were treated in [40], [41]. A number of works were devoted to the superconformal extensions of Calogero–Moser systems where extra spin type variables had to be present (see [32] for a discussion and the review). In the current work we presented superconformal extensions of the ordinary Calogero–Moser system with scalar potential as well as its generalisations for an arbitrary $\vee$-system, which includes Olshanetsky–Perelomov generalisations of Calogero–Moser systems with arbitrary invariant coupling parameters, and without introduction of extra bosonic variables. The superconformal algebra is $D(2, 1; \alpha)$ where parameter α is related to the coupling parameter(s). It is crucial for our considerations that we deal with quantum rather than classical Calogero–Moser type systems.

We also presented supersymmetric non-conformal deformations of the trigonometric Calogero–Moser–Sutherland type systems related with the root system B_N (which may be thought of as the Calogero–Moser–Sutherland system with boundary terms) as well as with some other exceptional root systems.

Comment 5. It would be very interesting to see if there are any relations of considered systems with black holes (cf. [42] for the conjectural relation with supersymmetric Calogero–Moser systems and e.g. [60], [68] and references therein for non–conformal deformations of AdS_2 black hole geometry). We note that it is also suggested that the superconformal algerba $D(2, 1; \alpha)$ may be relevant to multi-black hole systems (see [32] and references therein).

Comment 6. We dealt with BC_N trigonometric prepotentials which were recently shown in [2] to satisfy generalised WDVV equations. It would be interesting to see whether there are more Frobenius manifold structures associated to this solution of WDVV equations.

Comment 7. It may also be interesting to clarify integrability of considered supersymmetric Hamiltonians.

Appendix A

One-dimensional strata: the cases F_4, H_4

It can be checked directly with the help of Mathematica (see [5]) that the metric η_D is non-zero for any one-dimensional stratum in the following Coxeter groups: F_4, H_4, E_6, E_7, E_8. All the remaining cases are considered in Chapter 3.

Let us give some details here only for the cases when $\mathcal{R} = F_4$ and $\mathcal{R} = H_4$. The cases when $\mathcal{R} = E_6, E_7, E_8$ are similar. Basic invariants and Saito polynomials for the groups of type E_6 and E_7, E_8 can be found in [1, 78] and [1, 83] respectively. Let us introduce coordinates $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$.

Let $\mathcal{R} = F_4 \subset \mathbb{R}^4$. The polynomials

$$I_k = \sum_{i<j}^{3} (x^i - x^j)^k + (x^i + x^j)^k, \quad k = 2, 6, 8, 12,$$

are basic invariants. Saito polynomials can be chosen as follows [35, 78]:

$$t^1 = \frac{1}{144} I_2, \quad t^2 = -\frac{1}{6}\Big(-\frac{1}{8} I_6 + \frac{15}{16}\Big(\frac{I_2}{6}\Big)^3\Big), \quad t^3 = \frac{1}{6}\Big(-\frac{3}{40} I_8 + \frac{21}{80}\Big(\frac{I_2}{6}\Big) I_6 - \frac{77}{64}\Big(\frac{I_2}{6}\Big)^4\Big),$$

$$t^4 = -\frac{1}{60} I_{12} + \frac{209}{960} I_8 \Big(\frac{I_2}{6}\Big)^2 + \frac{77}{480}\Big(\frac{I_6}{6}\Big)^2 - \frac{2959}{960}\Big(\frac{I_6}{6}\Big)\Big(\frac{I_2}{6}\Big)^3 + \frac{2211}{1280}\Big(\frac{I_2}{6}\Big)^6.$$

Note that normalisation of these polynomials is chosen such that $\eta_{\alpha\beta}(t) = \delta_{\alpha+\beta,5}$. There are two non-equivalent one-dimensional strata in F_4 which have types B_3 and $A_2 \times A_1$. In the former case we obtain $\eta_D = -2x_1^{12} dx_1^2$, with $D : x_0 = x_1,\ x_2 = x_3 = 0$ and in the latter $\eta_D = -576 x_1^{12} dx_1^2$, with $D : \frac{1}{3} x_0 = x_1 = x_2 = x_3$.

Let us now consider the case $\mathcal{R} = H_4 \subset \mathbb{R}^4$. The corresponding root system is given by cyclic permutations of the vectors $(\pm 2, 0, 0, 0)$ and $(\pm\tau, \pm(\tau - 1), \pm 1, 0)$, where $\tau = \frac{1}{2}(1 + \sqrt{5})$ (independent choices of signs) [27, 78]. A simple system can be chosen as follows:

$$\alpha_1 = (0,2,0,0), \quad \alpha_2 = (0,-\tau,\tau-1,-1), \quad \alpha_3 = (0,0,0,2), \quad \alpha_4 = (\tau-1,0,-\tau,-1)$$

with corresponding Coxeter graph

In the notations of [78, p. 405] Saito polynomials are defined as

$$\begin{aligned}
t^1 &= \frac{1}{60} z_2, \\
t^2 &= -\frac{1}{\sqrt{30}}(z_{12} + \frac{4}{45} z_2^6), \\
t^3 &= \frac{1}{\sqrt{30}}(z_{20} + \frac{1}{3} z_2^4 z_{12} + \frac{8}{405} z_2^{10}), \\
t^4 &= z_{30} - \frac{4}{15} z_2^5 z_{20} - \frac{2}{3} z_2^3 z_{12}^2 - \frac{56}{405} z_2^9 z_{12} - \frac{104}{18225} z_2^{15},
\end{aligned}$$

where z_2, z_{12}, z_{20}, z_{30} are particular polynomials of degrees $2, 12, 20, 30$ respectively, in the variables x_0, x_1, x_2, x_3. They are defined explicitly in [78, p. 403] but there seem to be typos in the expressions for the polynomials z_{12} and z_{30}. We give correct expressions below. We also normalised the polynomials t^1, t^2, t^3 so that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,5}$.

There are four non-equivalent one-dimensional strata D in H_4 which have type H_3, A_3, $I_2(5) \times A_1$ and $A_1 \times A_2$. In the following table we give the determinant $\det \eta_D(x)$ of the restricted Saito metric η_D for these strata. We use the notation $\{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$ to denote the stratum $D = D_{i_1,i_2,i_3}$.

Table A.1: Restricted Saito metric η_D, $\dim D = 1$, $\mathcal{R} = H_4$

Type of stratum	$\det \eta_D(x)$
H_3, $\{\alpha_1, \alpha_2, \alpha_3\}$	$2^3 \cdot 3^{-4} \cdot 5^{-1} \left(\sqrt{5} - 3\right) x_0^{30}$
A_3, $\{\alpha_2, \alpha_3, \alpha_4\}$	$2^{38} \cdot 3^{-4} \cdot 5^{-1} \left(7 - 3\sqrt{5}\right) x_2^{30}$
$I_2(5) \times A_1$, $\{\alpha_1, \alpha_2, \alpha_4\}$	$2^{38} \cdot 3^{-4} \cdot 5 \left(72\sqrt{5} + 161\right) x_2^{30}$
$A_1 \times A_2$, $\{\alpha_1, \alpha_3, \alpha_4\}$	$-2^7 \cdot 3 \cdot 5^{-1} \left(121393\sqrt{5} + 271443\right) x_2^{30}$

Let us now give expressions for the polynomials z_{12} and z_{30}. They are defined in terms of polynomials X_2, X_6, X_{10} (which are denoted by x_2, x_6, x_{10} in [78, p. 401]). These polynomials have the form

$$\begin{aligned}
X_2 &= p_1, \\
X_6 &= \sqrt{5}p_4 + p_1p_2 - 11p_3, \\
X_{10} &= 3\sqrt{5}p_4p_2 + 2p_1^3p_2 - 32p_1^2p_3 - 5p_1p_2^2 + 95p_2p_3,
\end{aligned}$$

where

$$p_1 = \sum_{i=1}^{3} x_i^2, \quad p_2 = \sum_{\substack{i=1\\ i<j}}^{3} x_i^2x_j^2, \quad p_3 = \prod_{i=1}^{3} x_i^2, \quad p_4 = \prod_{\substack{i=1\\ i<j}}^{3} (x_i^2 - x_j^2).$$

In these notations one gets

$$\begin{aligned}
z_{12} = &- 2x_0^{10}X_2 + 6x_0^8{X_2}^2 + x_0^6\big(33X_6 - 14{X_2}^3\big) - x_0^4\big(33X_2X_6 - 6{X_2}^4\big) \\
&+ x_0^2\big(11X_{10} - 2{X_2}^5\big) - X_{10}X_2 + \frac{3}{2}{X_6}^2,
\end{aligned}$$

$$\begin{aligned}
z_{30} = &\frac{32}{3}x_0^{24}X_2^3 - x_0^{22}\big(80X_2^4 + 120X_2X_6\big) + x_0^{20}\big(360X_{10} + \frac{1344}{5}X_2^5 + 672X_2^2X_6\big) \\
&+ x_0^{18}\big(- 2880X_{10}X_2 - \frac{1328}{3}X_2^6 - 1608X_2^3X_6 + 1080X_6^2\big) + x_0^{16}\big(10024X_{10}X_2^2 \\
&+ 272X_2^7 + 1248X_2^4X_6 - 5628X_2X_6^2\big) + x_0^{14}\big(- 16856X_{10}X_2^3 - 7620X_{10}X_6 \\
&+ 272X_2^8 + 18588X_2^2X_6^2\big) + x_0^{12}\big(14216X_{10}X_2^4 + 23508X_{10}X_2X_6 - \frac{1328}{3}X_2^9 \\
&- 1248X_2^6X_6 - 27396X_2^3X_6^2 - 5796X_6^3\big) + x_0^{10}\big(3240X_{10}^2 - 7160X_{10}X_2^5 - 25332X_{10}X_2^2X_6 \\
&+ \frac{1344}{5}X_2^{10} + 1608X_2^7X_6 + 19968X_2^4X_6^2 + 7350X_2X_6^3\big) + x_0^8\big(- 3232X_{10}^2X_2 + 2144X_{10}X_2^6 \\
&+ 10908X_{10}X_2^3X_6 - 906X_{10}X_6^2 - 80X_2^{11} - 672X_2^8X_6 - 6924X_2^5X_6^2 - 1956X_2^2X_6^3\big) \\
&+ x_0^6\big(1168X_{10}^2X_2^2 - 344X_{10}X_2^7 - 2172X_{10}X_2^4X_6 - 1908X_{10}X_2X_6^2 + \frac{32}{3}X_2^{12} \\
&+ 120X_2^9X_6 + 1332X_2^6X_6^2 + 288X_2^3X_6^3 + 2394X_6^4\big) + x_0^4\big(- 152X_{10}^2X_2^3 + 348X_{10}^2X_6 \\
&+ 16X_{10}X_2^8 + 60X_{10}X_2^5X_6 + 408X_{10}X_2^2X_6^2 - 84X_2^7X_6^2 + 84X_2^4X_6^3 - 909X_2X_6^4\big) \\
&+ x_0^2\big(8X_{10}^2X_2^4 - 42X_{10}X_2^3X_6^2 - 87X_{10}X_6^3 - 6X_2^5X_6^3 + 135X_2^2X_6^4\big) \\
&+ \frac{4}{3}X_{10}^3 - 3X_{10}X_2X_6^3 + \frac{9}{5}X_6^5.
\end{aligned}$$

Polynomials z_2 and z_{20} are the same as in [78, p. 403]. They have the form

$$z_2 = x_0^2 + X_2,$$

$$\begin{aligned}
z_{20} &= 4x_0^{16}X_2^2 - x_0^{14}\left(20X_2^3 + 30X_6\right) + x_0^{12}\left(44X_2^4 + 138X_2X_6\right)\\
&\quad + x_0^{10}\left(180X_{10} - 44X_2^5 - 402X_2^2X_6\right) + x_0^{8}\left(-464X_{10}X_2 + 44X_2^6 + 402X_2^3X_6 + 294X_6^2\right)\\
&\quad + x_0^{6}\left(296X_{10}X_2^2 - 20X_2^7 - 138X_2^4X_6 - 306X_2X_6^2\right)\\
&\quad + x_0^{4}\left(-76X_{10}X_2^3 - 114X_{10}X_6 + 4X_2^8 + 30X_2^5X_6 + 168X_2^2X_6^2\right)\\
&\quad + x_0^{2}\left(4X_{10}X_2^4 - 21X_2^3X_6^2 + \frac{57}{2}X_6^3\right) + X_{10}^2 - \frac{3}{2}X_2X_6^3.
\end{aligned}$$

Appendix B

One particle systems

In this appendix we include our considerations for one particle systems for reader's convenience. Namely, we construct two representations of the algebra $D(2,1;\alpha)$ which are particular cases of considerations from Sections 4.2, 4.3. Relations (4.7) for one particle take the form $(j,k=1,2)$

$$[x,p]=i, \quad \{\psi^k,\bar{\psi}_j\}=-\frac{1}{2}\delta^k_j, \quad \text{and} \quad \{\psi^j,\psi^k\}=\{\bar{\psi}_j,\bar{\psi}_k\}=0. \tag{B.1}$$

Relations (4.9) take the form

$$\{\psi_k,\bar{\psi}^j\}=\frac{1}{2}\delta^j_k, \quad \{\psi^k,\bar{\psi}^j\}=\frac{1}{2}\epsilon^{kj}, \quad \{\psi_k,\bar{\psi}_j\}=\frac{1}{2}\epsilon_{jk}, \tag{B.2}$$

since $\epsilon_{jl}\epsilon^{lk}=\delta^k_j$. We consider a potential of the form (4.82) with $\lambda=-\frac{2\alpha+1}{2}$.

B.1 The first representation

Let the supercharges be of the form (4.16), (4.17), namely

$$Q^a=p\psi^a+i\frac{2\alpha+1}{x}\langle\psi_k\psi^k\bar{\psi}^a\rangle, \tag{B.3}$$

$$\bar{Q}_b=p\bar{\psi}_b+i\frac{2\alpha+1}{x}\langle\bar{\psi}^k\bar{\psi}_k\psi_b\rangle. \tag{B.4}$$

Note that by (4.18) we have that

$$\begin{aligned}\langle\psi_k\psi^k\bar{\psi}^a\rangle &= \frac{1}{6}(\psi_k\psi^k\bar{\psi}^a+\bar{\psi}^a\psi_k\psi^k+\psi^k\bar{\psi}^a\psi_k-\psi_k\bar{\psi}^a\psi^k-\bar{\psi}^a\psi^k\psi_k-\psi^k\psi_k\bar{\psi}^a)\\ &= \frac{1}{3}(\psi_k\psi^k\bar{\psi}^a+\bar{\psi}^a\psi_k\psi^k-\psi_k\bar{\psi}^a\psi^k),\end{aligned} \tag{B.5}$$

since for any ϕ we have that

$$\psi^k\phi\psi_k=-\psi_k\phi\psi^k. \tag{B.6}$$

Note also that by (B.1), (B.2) we get

$$\bar{\psi}^a \psi_k \psi^k = \psi_k \psi^k \bar{\psi}^a + \psi^a, \quad \psi_k \bar{\psi}^a \psi^k = -\psi_k \psi^k \bar{\psi}^a - \frac{1}{2}\psi^a. \tag{B.7}$$

Hence it follows from (B.5) and (B.7) that

$$\langle \psi_k \psi^k \bar{\psi}^a \rangle = \psi_k \psi^k \bar{\psi}^a + \frac{1}{2}\psi^a. \tag{B.8}$$

Similarly,

$$\langle \bar{\psi}^k \bar{\psi}_k \psi_b \rangle = \bar{\psi}^k \bar{\psi}_k \psi_b + \frac{1}{2}\bar{\psi}_b.$$

Let also

$$K = x^2, \quad D = -\frac{1}{4}\{x, p\}, \tag{B.9}$$

$$I^{11} = -i\psi_m \psi^m, \quad I^{22} = i\bar{\psi}^m \bar{\psi}_m, \quad I^{12} = I^{21} = -\frac{i}{2}[\psi_m, \bar{\psi}^m], \tag{B.10}$$

$$J^{ab} = J^{ba} = 2i\psi^{(a}\bar{\psi}^{b)}, \tag{B.11}$$

$$S^a = -2x\psi^a, \quad \bar{S}_a = -2x\bar{\psi}_a. \tag{B.12}$$

Note that under the operation $\sim$ defined by (4.10) we obtain the following relations:

$$\widetilde{Q^a} = -\bar{Q}_a, \quad \widetilde{\bar{Q}^a} = -Q_a, \quad \widetilde{S^a} = \bar{S}_a, \quad \widetilde{\bar{S}^a} = S_a, \tag{B.13}$$

$$\widetilde{I^{11}} = I^{22}, \quad \widetilde{I^{12}} = I^{12}, \tag{B.14}$$

$$\widetilde{H} = H, \quad \widetilde{D} = -D, \quad \widetilde{K} = K. \tag{B.15}$$

Note also that

$$J_{ab} = \epsilon_{a\widehat{a}}\epsilon_{b\widehat{b}} J^{\widehat{a}\widehat{b}} = \widetilde{J^{ab}}, \tag{B.16}$$

since $\bar{\psi}_a \psi_b + \bar{\psi}_b \psi_a = -(\psi_a \bar{\psi}_b + \psi_b \bar{\psi}_a)$ by (B.2), where $\widehat{a}$, $\widehat{b}$ are complimentary to a and b respectively. Let us first check relations (4.4) involving generators J^{ab} and I^{ab}. Note that the statement of Lemma B.1.2 appears also in [31] and we include it here for completeness.

Lemma B.1.1. *Let J^{ab} be as above. Then the relations (4.4a) hold.*

Proof. We have the following commutator by applying (4.13), (4.14)

$$[\psi^a \bar{\psi}^b, \psi^c \bar{\psi}^d] = \frac{1}{2}\epsilon^{cb}\psi^a \bar{\psi}^d + \frac{1}{2}\epsilon^{da}\psi^c \bar{\psi}^b.$$

Thus,

$$
\begin{aligned}
[J^{ab}, J^{cd}] = & \frac{1}{2}\epsilon^{bc}(\psi^a\bar{\psi}^d + \psi^d\bar{\psi}^a) + \frac{1}{2}\epsilon^{ad}(\psi^b\bar{\psi}^c + \psi^c\bar{\psi}^b) \\
& + \frac{1}{2}\epsilon^{bd}(\psi^a\bar{\psi}^c + \psi^c\bar{\psi}^a) + \frac{1}{2}\epsilon^{ac}(\psi^b\bar{\psi}^d + \psi^d\bar{\psi}^b),
\end{aligned}
$$

which implies the lemma. □

Let us now check relations involving generators I^{ab}.

Lemma B.1.2 (cf. [31])**.** *Let I^{ab} be as above. Then relations (4.4b) hold.*

Proof. Relations (4.4b) read as follows:

$$
[I^{11}, I^{22}] = 2iI^{12}, \quad [I^{11}, I^{12}] = iI^{11}, \quad [I^{22}, I^{12}] = -iI^{22}.
$$

We have

$$
[I^{11}, I^{22}] = [\psi_a\psi^a, \bar{\psi}^b\bar{\psi}_b]. \tag{B.17}
$$

By applying (4.13), (4.14) we rearrange expression (B.17) as

$$
\begin{aligned}
[I^{11}, I^{22}] &= \psi_a[\psi^a, \bar{\psi}^b\bar{\psi}_b] + [\psi_a, \bar{\psi}^b\bar{\psi}_b]\psi^a \\
&= \psi_a\bar{\psi}^a + \bar{\psi}_a\psi^a = \psi_a\bar{\psi}^a - \bar{\psi}^a\psi_a \\
&= 2iI^{12},
\end{aligned}
$$

as required. Moreover, using the Jacobi identity we have

$$
[I^{11}, I^{12}] = -\frac{1}{2}[\psi_a\psi^a, [\psi_b, \bar{\psi}^b]] = \frac{1}{2}[\psi_b, [\bar{\psi}^b, \psi_a\psi^a]].
$$

We have $[\bar{\psi}^b, \psi_a\psi^a] = \psi^b$. Thus,

$$
[I^{11}, I^{12}] = \psi_b\psi^b = iI^{11}.
$$

Similarly,

$$
[I^{22}, I^{12}] = \frac{1}{2}[\bar{\psi}^a\bar{\psi}_a, [\psi_b, \bar{\psi}^b]] = -\frac{1}{2}[\bar{\psi}^b, [\bar{\psi}^a\bar{\psi}_a, \psi_b]],
$$

and $[\bar{\psi}^a\bar{\psi}_a, \psi_b] = -\bar{\psi}_b$. Thus,

$$
[I^{22}, I^{12}] = \bar{\psi}^b\bar{\psi}_b = -iI^{22},
$$

which is the corresponding relation (4.4b) and hence the statement follows. □

Lemma B.1.3. *Let Q^{abc}, J^{ab} be as above. Then relations (4.5b) hold.*

Proof. We have by (B.2) and (4.14) that

$$[\psi^a\bar{\psi}^b, \psi^j] = \psi^a\{\bar{\psi}^b, \psi^j\} = \frac{1}{2}\epsilon^{jb}\psi^a. \tag{B.18}$$

By (4.13), (4.14) and (B.2) we have that

$$\begin{aligned}
[\psi^a\bar{\psi}^b, \psi_k\psi^k\bar{\psi}^j] &= -\psi_k\psi^k[\bar{\psi}^j, \psi^a\bar{\psi}^b] - [\psi_k\psi^k, \psi^a\bar{\psi}^b]\bar{\psi}^j \\
&= -\psi_k\psi^k\bar{\psi}^b\{\bar{\psi}^j, \psi^a\} - \psi_k[\psi^k, \psi^a\bar{\psi}^b]\bar{\psi}^j - [\psi_k, \psi^a\bar{\psi}^b]\psi^k\bar{\psi}^j \\
&= \frac{1}{2}\epsilon^{ja}\psi_k\psi^k\bar{\psi}^b - \psi^b\psi^a\bar{\psi}^j.
\end{aligned} \tag{B.19}$$

Note that since J^{ab} is symmetric in a and b, it follows from (B.18), (B.19) that

$$[J^{ab}, \psi^c] = \frac{i}{2}\epsilon^{cb}\psi^a + \frac{i}{2}\epsilon^{ca}\psi^b, \tag{B.20}$$

$$[J^{ab}, \psi_k\psi^k\bar{\psi}^c] = \frac{i}{2}\epsilon^{ca}\psi_k\psi^k\bar{\psi}^b + \frac{i}{2}\epsilon^{cb}\psi_k\psi^k\bar{\psi}^a. \tag{B.21}$$

Therefore, the left-hand side of (4.5b) for $[J^{ab}, Q^{21c}]$ is

$$[J^{ab}, Q^{21c}] = -[J^{ab}, Q^c] = -[J^{ab}, \psi^c]p - \frac{i(2\alpha+1)}{x}[J^{ab}, \psi_k\psi^k\bar{\psi}^c + \frac{1}{2}\psi^c]. \tag{B.22}$$

Then by formulae (B.8), (B.20) and (B.21) we obtain

$$\begin{aligned}
[J^{ab}, Q^{21c}] &= \frac{i}{2}\epsilon^{bc}p\psi^a + \frac{i}{2}p\epsilon^{ac} + \frac{i(2\alpha+1)}{x}(\frac{i}{2}\epsilon^{ac}\psi_k\psi^k\bar{\psi}^b + \frac{i}{2}\epsilon^{bc}\psi_k\psi^k\bar{\psi}^a + \frac{i}{4}\epsilon^{bc}\psi^a + \frac{i}{4}\epsilon^{ac}\psi^b) \\
&= \frac{i\epsilon^{bc}}{2}\big(p\psi^a + \frac{i(2\alpha+1)}{x}\langle\psi_k\psi^k\bar{\psi}^a\rangle\big) + \frac{i\epsilon^{ac}}{2}\big(p\psi^b + \frac{i(2\alpha+1)}{x}\langle\psi_k\psi^k\bar{\psi}^b\rangle\big) \\
&= \frac{i}{2}(\epsilon^{bc}Q^a + \epsilon^{ac}Q^b) = \frac{i}{2}(\epsilon^{cb}Q^{21a} + \epsilon^{ca}Q^{21b}),
\end{aligned} \tag{B.23}$$

that is

$$[J^{ab}, Q^{21c}] = i\epsilon^{c(a}Q^{|21|b)}, \tag{B.24}$$

as required. Applying $\sim$ to (B.24) and by (B.13), (B.16) we obtain

$$[J_{ab}, \bar{Q}_c] = -\frac{i}{2}\epsilon_{cb}\bar{Q}_a - \frac{i}{2}\epsilon_{ca}\bar{Q}_b.$$

Therefore we have

$$[J^{ab}, Q^{22c}] = -[J^{ab}, \bar{Q}^c] = -\epsilon^{a\widehat{a}}\epsilon^{b\widehat{b}}\epsilon^{c\widehat{c}}[J_{\widehat{a}\widehat{b}}, \bar{Q}_{\widehat{c}}] = \frac{i}{2}(\epsilon^{bc}\bar{Q}^a + \epsilon^{ac}\bar{Q}^b) = i\epsilon^{c(a}Q^{|22|b)},$$

as required. Further on using (B.18) we obtain

$$[J^{ab}, Q^{11c}] = [J^{ab}, S^c] = -2x[J^{ab}, \psi^c] = i\epsilon^{bc}x\psi^a + i\epsilon^{ac}x\psi^b$$
$$= \frac{i}{2}(\epsilon^{cb}S^a + \epsilon^{ca}S^b) = i\epsilon^{c(a}Q^{|11|b)}. \tag{B.25}$$

Applying $\sim$ to $[J^{ab}, S^c]$ and using (B.13) we get that $[J_{ab}, \bar{S}_c] = \frac{i}{2}(\epsilon_{bc}\bar{S}_a + \epsilon_{ac}\bar{S}_b)$ from which it follows that

$$[J^{ab}, Q^{12c}] = [J^{ab}, \bar{S}^c] = \epsilon^{a\widehat{a}}\epsilon^{b\widehat{b}}\epsilon^{c\widehat{c}}[J_{\widehat{a}\widehat{b}}, \bar{S}_{\widehat{c}}] = \frac{i}{2}(\epsilon^{ca}\bar{S}^b + \epsilon^{cb}\bar{S}^a) = i\epsilon^{c(a}Q^{|12|b)}.$$

The statement follows. □

Lemma B.1.4. *Let Q^{abc}, I^{ab} be as above. Then relations (4.5c) hold.*

Proof. Let us first consider the commutator $[I^{11}, Q^{21f}] = i[\psi_m\psi^m, Q^f]$. Note that

$$[\psi_m\psi^m, \psi^f] = 0, \quad \text{and} \quad [\psi_m\psi^m, \psi_k\psi^k\bar{\psi}^f] = -\psi_k\psi^k[\bar{\psi}^f, \psi_m\psi^m] = -\psi^f\psi_k\psi^k = 0.$$

Therefore $[I^{11}, Q^{21f}] = 0$, as required. Now let us consider the commutator $[I^{22}, Q^{21f}]$. By (4.14) and (B.1), (B.2) we get

$$[\bar{\psi}^m\bar{\psi}_m, \psi^f] = -\bar{\psi}^f. \tag{B.26}$$

By (4.13) we get

$$[\bar{\psi}^m\bar{\psi}_m, \psi_k\psi^k\bar{\psi}^f] = -[\psi_k\psi^k, \bar{\psi}^m\bar{\psi}_m]\bar{\psi}^f. \tag{B.27}$$

We have by (4.13) and (B.26)

$$[\psi_k\psi^k, \bar{\psi}^m\bar{\psi}_m] = \psi_k[\psi^k, \bar{\psi}^m\bar{\psi}_m] + [\psi_k, \bar{\psi}^m\bar{\psi}_m]\psi^k = \psi_k\bar{\psi}^k - \bar{\psi}^k\psi_k, \tag{B.28}$$

where we also used (B.6). Note that by (B.2) we obtain

$$\psi_k\bar{\psi}^k\bar{\psi}^f = \bar{\psi}^k\bar{\psi}^f\psi_k + \frac{1}{2}\bar{\psi}^f, \tag{B.29}$$

and

$$\bar{\psi}^k\psi_k\bar{\psi}^f = -\bar{\psi}^k\bar{\psi}^f\psi_k + \frac{1}{2}\bar{\psi}^f. \tag{B.30}$$

Therefore we get from (B.28), (B.29) and (B.30) that

$$[\psi_k\psi^k, \bar{\psi}^m\bar{\psi}_m]\bar{\psi}^f = 2\bar{\psi}^k\bar{\psi}^f\psi_k. \tag{B.31}$$

Note that

$$\bar{\psi}^k\bar{\psi}^f\psi_k = \begin{cases} \bar{\psi}^2\bar{\psi}_2\psi^1, & \text{if} \quad f = 1, \\ \bar{\psi}^1\bar{\psi}_1\psi^2, & \text{if} \quad f = 2. \end{cases} \tag{B.32}$$

Since $\bar\psi^1\bar\psi_1 = \bar\psi^2\bar\psi_2$, we get from (B.32) that

$$\bar\psi^k\bar\psi^f\psi_k = \frac{1}{2}\bar\psi^k\bar\psi_k\psi^f,$$

for any $f = 1, 2$. Therefore it follows that expression (B.31) takes the form

$$[\psi_k\psi^k, \bar\psi^m\bar\psi_m]\bar\psi^f = \bar\psi^k\bar\psi_k\psi^f. \tag{B.33}$$

From the forms of I^{22} and Q^{21f} and using (B.26), (B.27), (B.33) we obtain

$$\begin{aligned}[I^{22}, Q^{21f}] &= -i[\bar\psi^m\bar\psi_m, Q^f] \\ &= -i\big(-p\bar\psi^f + i\frac{(2\alpha+1)}{x}(-\bar\psi^k\bar\psi_k\psi^f - \frac{1}{2}\bar\psi^f)\big) \\ &= i\bar Q^f,\end{aligned} \tag{B.34}$$

which is the corresponding relation (4.5c).

Further on, let us consider $[I^{12}, Q^{21f}]$. Note that $[\psi_m, \bar\psi^m] = 2\psi_m\bar\psi^m - 1$ and

$$[\psi_m\bar\psi^m, \psi^f] = \frac{1}{2}\psi^f, \quad [\bar\psi^f, \psi_m\psi^m] = \psi^f, \tag{B.35}$$

where we use (4.14) to get the latter expression. Now we have by (4.13)

$$\begin{aligned}[\psi_m\bar\psi^m, \psi_k\psi^k\bar\psi^f] &= [\psi_m\bar\psi^m, \psi_k\psi^k]\bar\psi^f + \psi_k\psi^k[\psi_m\bar\psi^m, \bar\psi^f] \\ &= \psi_m[\bar\psi^m, \psi_k\psi^k]\bar\psi^f + \psi_k\psi^k[\psi_m\bar\psi^m, \bar\psi^f] \\ &= \frac{1}{2}\psi_k\psi^k\bar\psi^f,\end{aligned} \tag{B.36}$$

where in the last equality we applied (B.35) and the formula $[\psi_m\bar\psi^m, \bar\psi^f] = -\frac{1}{2}\bar\psi^f$. Since Q^f is a linear combination of ψ^f and $\psi_k\psi^k\bar\psi^f$ with bosonic coefficients it follows from the first formula in (B.35) and (B.36) that $[\psi_m\bar\psi^m, Q^f] = \frac{1}{2}Q^f$, therefore $[I^{12}, Q^{21f}] = -\frac{i}{2}Q^{21f}$, as required. Using formulae (B.13), (B.14) and $\sim$ operation, it follows that relations (4.5c) hold for the supercharge Q^{22f} as well.

Finally, let us consider $[I^{ab}, Q^{11f}]$. Firstly, note that

$$[I^{11}, Q^{11f}] = -i[\psi_m\psi^m, S^f] = 2ix[\psi_m\psi^m, \psi^f] = 0,$$

as required. Moreover, we get from (B.26) that

$$[I^{22}, Q^{11f}] = i[\bar\psi^m\bar\psi_m, S^f] = -2ix[\bar\psi^m\bar\psi_m, \psi^f] = 2ix\bar\psi^f = -iQ^{12f},$$

and from the first formula in (B.35) that

$$[I^{12}, Q^{11f}] = -i[\psi_m \bar{\psi}^m, S^f] = 2ix[\psi_m \bar{\psi}^m, \psi^f] = ix\psi^f = -\frac{i}{2} Q^{11f},$$

which are the corresponding relations (4.5c). Using formulae (B.13), (B.14) and $\sim$ operation, it follows that relations (4.5c) for $[I^{ab}, Q^{12f}]$ $(a, b, f = 1, 2)$ hold as well. The statement follows. □

We will use the following relations below. By (4.15) we have

$$\begin{aligned} \{p\psi^a, \frac{1}{x}\bar{\psi}_b\} &= \psi^a \bar{\psi}_b [p, \frac{1}{x}] + \frac{1}{x}\{\psi^a, \bar{\psi}_b\} p \\ &= \frac{i}{x^2}\psi^a \bar{\psi}_b - \frac{\delta^a_b}{2}\frac{1}{x} p, \end{aligned} \tag{B.37}$$

and similarly

$$\{p\bar{\psi}_b, \frac{1}{x}\psi^a\} = \frac{i}{x^2}\bar{\psi}_b \psi^a - \frac{\delta^a_b}{2}\frac{1}{x} p.$$

Further on, by (4.14), (4.15) we have that

$$\begin{aligned} \{\bar{\psi}_b, \psi_m \psi^m \bar{\psi}^a\} &= -\psi_m \bar{\psi}^a \{\psi^m, \bar{\psi}_b\} + \psi^m \bar{\psi}^a \{\psi_m, \bar{\psi}_b\} \\ &= \frac{1}{2}\delta^m_b \psi_m \bar{\psi}^a - \frac{1}{2}\epsilon_{mb}\psi^m \bar{\psi}^a \\ &= \psi_b \bar{\psi}^a, \end{aligned} \tag{B.38}$$

and similarly

$$\{\psi^a, \bar{\psi}^m \bar{\psi}_m \psi_b\} = \bar{\psi}^a \psi_b. \tag{B.39}$$

Let us now compute the Hamiltonian of the system. We will use the following formulae below

$$\langle \psi_m \psi^m \bar{\psi}^k \bar{\psi}_k \rangle = \psi_m \psi^m \bar{\psi}^k \bar{\psi}_k - \psi_m \bar{\psi}^m + \frac{1}{4}. \tag{B.40}$$

Theorem B.1.5. *For $a, b = 1, 2$ we have $\{Q^a, \bar{Q}_b\} = -2H\delta^a_b$, where the Hamiltonian H is given by*

$$H = \frac{p^2}{4} - \frac{(2\alpha+1)}{2x^2}\langle \psi_m \psi^m \bar{\psi}^k \bar{\psi}_k \rangle + \frac{(2\alpha+1)^2}{16x^2}. \tag{B.41}$$

Proof. By (4.15) and (B.38), (B.39) we have

$$\{p\psi^a, \frac{i(2\alpha+1)}{x}\langle \bar{\psi}^m \bar{\psi}_m \psi_b \rangle\} = i(2\alpha+1)\Big(\frac{i}{x^2}\psi^a \bar{\psi}^m \bar{\psi}_m \psi_b + \bar{\psi}^a \psi_b \frac{1}{x} p + \frac{i}{2x^2}\psi^a \bar{\psi}_b - \frac{\delta^a_b}{4}\frac{1}{x} p\Big), \tag{B.42}$$

and similarly,

$$\{p\bar\psi_b, \frac{i(2\alpha+1)}{x}\langle\psi_k\psi^k\bar\psi^a\rangle\} = i(2\alpha+1)\Big(\frac{i}{x^2}\bar\psi_b\psi_k\psi^k\bar\psi^a + \psi_b\bar\psi^a\frac{1}{x}p + \frac{i}{2x^2}\bar\psi_b\psi^a - \frac{\delta^a_b}{4}\frac{1}{x}p\Big) \tag{B.43}$$

Finally, we have

$$\{\langle\bar\psi^m\bar\psi_m\psi_b\rangle, \langle\psi_k\psi^k\bar\psi^a\rangle\} = \frac{1}{8}\delta^a_b. \tag{B.44}$$

Indeed,

$$\begin{aligned}\{\langle\bar\psi^m\bar\psi_m\psi_b\rangle, \langle\psi_k\psi^k\bar\psi^a\rangle\} &= \{\bar\psi^m\bar\psi_m\psi_b, \psi_k\psi^k\bar\psi^a\} + \frac{1}{2}\{\bar\psi^m\bar\psi_m\psi_b, \psi^a\} + \frac{1}{2}\{\psi_k\psi^k\bar\psi^a, \bar\psi_b\}\\ &+ \frac{1}{4}\{\psi^a, \bar\psi_b\},\end{aligned} \tag{B.45}$$

where it is easy to see that the first term in equality (B.45) is zero and then by (B.38), (B.39) formula (B.44) follows.

Therefore in total, we get from (B.42)-(B.44) that

$$\{Q^a, \bar Q_b\} = -\frac{\delta^a_b}{2}p^2 + \frac{(2\alpha+1)}{x^2}\Big(\frac{\delta^a_b}{4} - \psi^a\bar\psi^m\bar\psi_m\psi_b - \bar\psi_b\psi_k\psi^k\bar\psi^a\Big) - \frac{(2\alpha+1)^2}{x^2}\frac{\delta^a_b}{8}.$$

Now let us try to simplify the above expression. We have by applying (B.2) that

$$\bar\psi^m\bar\psi_m\psi_b = \psi_b\bar\psi^m\bar\psi_m - \bar\psi_b, \tag{B.46}$$

and by (B.1), (B.2) that

$$\bar\psi_b\psi_k\psi^k = \psi_k\psi^k\bar\psi_b + \psi_b. \tag{B.47}$$

It follows from (B.46), (B.47) that

$$\psi^a\bar\psi^m\bar\psi_m\psi_b + \bar\psi_b\psi_k\psi^k\bar\psi^a = (\psi^a\bar\psi^m\bar\psi_m\psi_a + \bar\psi_a\psi_k\psi^k\bar\psi^a)\delta^a_b,$$

where there is no summation over a in the right-hand side. Note that $\psi^a\psi_a = \psi^{\widehat a}\psi_{\widehat a}$ and $\bar\psi^a\bar\psi_a = \bar\psi^{\widehat a}\bar\psi_{\widehat a}$ (no summation). Let us fix a and let $b = a$. Then we get from (B.46), (B.47) that

$$\psi^a\bar\psi^m\bar\psi_m\psi_a = -\psi_{\widehat a}\psi^{\widehat a}\bar\psi^m\bar\psi_m - \psi^a\bar\psi_a, \tag{B.48}$$

and

$$\bar\psi_a\psi_k\psi^k\bar\psi^a = -\psi_k\psi^k\bar\psi^{\widehat a}\bar\psi_{\widehat a} + \psi_a\bar\psi^a, \tag{B.49}$$

where $\widehat{a}$ is complementary to a. With summation over m only we have

$$\begin{aligned}\psi_m\psi^m\bar{\psi}^a\bar{\psi}_a &= \psi_a\psi^a\bar{\psi}^a\bar{\psi}_a + \psi_{\widehat{a}}\psi^{\widehat{a}}\bar{\psi}^a\bar{\psi}_a \\ &= \psi_{\widehat{a}}\psi^{\widehat{a}}\bar{\psi}^a\bar{\psi}_a + \psi_{\widehat{a}}\psi^{\widehat{a}}\bar{\psi}^{\widehat{a}}\bar{\psi}_{\widehat{a}} \\ &= \psi_{\widehat{a}}\psi^{\widehat{a}}\bar{\psi}^m\bar{\psi}_m. \end{aligned} \tag{B.50}$$

Hence we get from (B.48)-(B.50) that for any fixed a,

$$\psi^a\bar{\psi}^m\bar{\psi}_m\psi_a + \bar{\psi}_a\psi_k\psi^k\bar{\psi}^a = -\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k + \psi_m\bar{\psi}^m, \tag{B.51}$$

which is equal to $\frac{1}{4} - \langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle$ by (B.40). Hence,

$$\{Q^a, \bar{Q}_b\} = -\frac{\delta^a_b}{2}p^2 + \frac{(2\alpha+1)}{x^2}\langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle\delta^a_b - \frac{(2\alpha+1)^2}{x^2}\frac{\delta^a_b}{8}. \tag{B.52}$$

This concludes the proof. □

It is easy to check that the following lemma holds.

Lemma B.1.6. *Let H, K, D be as above. Then relations (4.3) hold.*

The following relations will be useful below. Firstly, by (B.40) it is easy to see that

$$\psi^a\langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle = -\psi^a\psi_m\bar{\psi}^m + \frac{1}{4}\psi^a. \tag{B.53}$$

Further, by using (B.1), (B.2) and (B.40) we obtain (for any fixed a) that

$$\begin{aligned}\langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle\psi^a &= \psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\psi^a - \psi_m\bar{\psi}^m\psi^a + \frac{1}{4}\psi^a \\ &= -\psi_m\psi^m\bar{\psi}^a - \psi^a\psi_m\bar{\psi}^m - \frac{1}{4}\psi^a. \end{aligned} \tag{B.54}$$

Hence by (B.8), (B.53) and (B.54) we have that

$$[\psi^a, \langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle] = \langle\psi_m\psi^m\bar{\psi}^a\rangle. \tag{B.55}$$

Lemma B.1.7. *Let Q^{abc}, I^{ab}, J^{ab}, T^{ab} be as above. Then relations (4.2) hold.*

Proof. Let us check at first that $\{Q^a, Q^b\} = 0$. We have

$$\{\psi^a, \psi_k\psi^k\bar{\psi}^b\} + \{\psi^b, \psi_k\psi^k\bar{\psi}^a\} = 0, \tag{B.56}$$

since

$$\{\psi^a, \psi_k\psi^k\bar{\psi}^b\} = \frac{1}{2}\epsilon^{ab}\psi_k\psi^k. \tag{B.57}$$

Hence (B.56) follows. Similarly, $\{\bar{Q}^a, \bar{Q}^b\} = 0$. Moreover, by Theorem B.1.5 we have $\{Q^{21a}, Q^{22c}\} = \{Q^a, \bar{Q}^c\} = \epsilon^{cb}\{Q^a, \bar{Q}_b\} = -2H\epsilon^{ca}$ as required.

Now, let us consider $\{Q^{21a}, Q^{11f}\}$. Then

$$\{p\psi^a, x\psi^f\} = -i\psi^a\psi^f = -\frac{i}{2}\epsilon^{af}\psi_k\psi^k,$$

since $\psi^1\psi_1 = \psi^2\psi_2$. Further on we have

$$\{\langle\psi_k\psi^k\bar{\psi}^a\rangle, \psi^f\} = -\frac{1}{2}\epsilon^{af}\psi_k\psi^k,$$

by formulae (B.8) and (B.57). Therefore,

$$\{Q^{21a}, Q^{11f}\} = -\{Q^a, S^f\} = -2i(1+\alpha)\epsilon^{af}\psi_k\psi^k = 2(1+\alpha)\epsilon^{af}I^{11},$$

as required by (4.2). By applying $\sim$ to $\{Q^{21a}, Q^{11f}\}$ and using (B.13), (B.14) we obtain

$$\{\bar{Q}_a, \bar{S}_f\} = 2(1+\alpha)\epsilon_{af}I^{22},$$

which matches with (4.2). Furthermore, it is easy to see that $\{Q^{11a}, Q^{11b}\} = \{S^a, S^b\} = 0$, $\{Q^{12a}, Q^{12b}\} = \{\bar{S}^a, \bar{S}^b\} = 0$ and that $\{Q^{11a}, Q^{12b}\} = \{S^a, \bar{S}^b\} = 2\epsilon^{ab}T^{11}$ as required by (4.2).

Finally, we consider the anti-commutator $\{Q^{21a}, Q^{12b}\}$. The left-hand side of (4.2) takes the following form in view of relations (B.38) and (4.15):

$$\begin{aligned}\{Q^a, \bar{S}_d\} &= -2\{p\psi^a, x\bar{\psi}_d\} - 2i(2\alpha+1)(\{\psi_k\psi^k\bar{\psi}^a, \bar{\psi}_d\} + \frac{1}{2}\{\psi^a, \bar{\psi}_d\}) \\ &= 2i\psi^a\bar{\psi}_d + xp\delta^a_d - 2i(2\alpha+1)(\psi_d\bar{\psi}^a - \frac{\delta^a_d}{4}).\end{aligned} \tag{B.58}$$

Therefore

$$-\epsilon^{bd}\{Q^a, \bar{S}_d\} = -2i\psi^a\bar{\psi}^b + xp\epsilon^{ab} + 2i(2\alpha+1)\psi^b\bar{\psi}^a + \frac{i(2\alpha+1)}{2}\epsilon^{ab}. \tag{B.59}$$

We have

$$\{Q^{21a}, Q^{12b}\} = -\{Q^a, \bar{S}^b\} = -\epsilon^{bd}\{Q^a, \bar{S}_d\}.$$

The right-hand side of (4.2) equals

$$\begin{aligned}2\big(\epsilon^{ab}T^{12} + \alpha J^{ab} + (1+\alpha)\epsilon^{ab}I^{12}\big) &= xp\epsilon^{ab} - \epsilon^{ab}\frac{i}{2} + 2\alpha J^{ab} + 2(1+\alpha)\epsilon^{ab}I^{12} \\ &= xp\epsilon^{ab} - \epsilon^{ab}\frac{i}{2} + 2\alpha i(\psi^a\bar{\psi}^b + \psi^b\bar{\psi}^a) - i(1+\alpha)\epsilon^{ab}[\psi_m, \bar{\psi}^m].\end{aligned} \tag{B.60}$$

Note that

$$[\psi_m, \bar{\psi}^m] = 2\psi^2\bar{\psi}^1 - 2\psi^1\bar{\psi}^2 - 1.$$

By considering various values of a and b one can see that the expression (B.60) takes the form

$$xp\epsilon^{ab} + \frac{i(2\alpha+1)}{2}\epsilon^{ab} - 2i\psi^a\bar{\psi}^b + 2i(1+2\alpha)\psi^b\bar{\psi}^a, \tag{B.61}$$

which is equal to expression (B.59) as required.

Finally, applying $\sim$ to $\{Q^a, \bar{S}^b\}$ we obtain

$$\begin{aligned}\{\bar{Q}_a, S_b\} &= 2(\epsilon_{ab}\widetilde{T^{12}} + \alpha\widetilde{J^{ab}} + (1+\alpha)\epsilon_{ab}\widetilde{I^{12}}) \\ &= 2(-\epsilon_{ab}T^{12} + \alpha J_{ab} + (1+\alpha)\epsilon_{ab}I^{12}),\end{aligned}$$

as required by (4.2). The statement thus follows. □

Lemma B.1.8. *Let T^{ab} be as above. Then relations (4.5a) hold.*

Proof. It is easy to see that relations (4.5a) hold for $T^{12} = -D$ and for $T^{11} = K$. Let us consider relations (4.5a) with $T^{22} = H$, and Q^a. Note that by Theorem B.1.5

$$T^{22} = -\frac{1}{2}\{Q^a, \bar{Q}_a\}.$$

Since $(Q^a)^2 = 0$ we get that $[H, Q^a] = 0$ as required. Similarly, $[H, \bar{Q}_a] = 0$.

Let us now consider relations (4.5a) with H and S^a. We have by (B.55) that

$$\begin{aligned}[H, S^f] &= -\frac{1}{2}[p^2, x\psi^f] + \frac{(2\alpha+1)}{x}[\langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle, \psi^f] \\ &= ip\psi^f - \frac{(2\alpha+1)}{x}\langle\psi_k\psi^k\bar{\psi}^f\rangle = iQ^f,\end{aligned} \tag{B.62}$$

which is the corresponding relation (4.5a). Using (B.13), (B.15) and $\sim$ operation, it follows that relations (4.5a) hold for $\bar{S}^a$. Hence the statement follows. □

Let us now check relations (4.6).

Lemma B.1.9. *Let T^{ab}, I^{ab}, J^{ab} be as above. Then relations (4.6) hold.*

Proof. Let us first consider $[I^{ab}, J^{cd}]$. We have by (4.13) and (4.14) that

$$[\psi_a\psi^a, \psi^c\bar{\psi}^d] = \psi^c[\psi_a\psi^a, \bar{\psi}^d] = \psi^d\psi^c.$$

Therefore

$$[I^{11}, J^{cd}] = 2[\psi_a\psi^a, \psi^{(c}\bar{\psi}^{d)}] = 0,$$

as required. Further we have by (4.13), (4.14) that

$$\begin{aligned}[][\psi_a, \bar{\psi}^a], \psi^c\bar{\psi}^d] &= 2[\psi_a\bar{\psi}^a, \psi^c\bar{\psi}^d] \\ &= 2(\psi_a[\bar{\psi}^a, \psi^c\bar{\psi}^d] + [\psi_a, \psi^c\bar{\psi}^d]\bar{\psi}^a) \\ &= 2(\psi_a\bar{\psi}^d\{\psi^c, \bar{\psi}^a\} - \psi^c\bar{\psi}^a\{\bar{\psi}^d, \psi_a\}) = 0.\end{aligned}$$

Therefore,

$$[I^{12}, J^{cd}] = [[\psi_a, \bar{\psi}^a], \psi^{(c}\bar{\psi}^{d)}] = 0,$$

which is the corresponding relation (4.6). In addition we have by (4.13) and (4.14) that

$$[\bar{\psi}^a\bar{\psi}_a, \psi^c\bar{\psi}^d] = [\bar{\psi}^a\bar{\psi}_a, \psi^c]\bar{\psi}^d = -\bar{\psi}^c\bar{\psi}^d.$$

Therefore,

$$[I^{22}, J^{cd}] = -2[\bar{\psi}^a\bar{\psi}_a, \psi^{(c}\bar{\psi}^{d)}] = 0,$$

as required.

Let us now consider relations $[I^{ab}, T^{cd}]$ $(a, b, c, d = 1, 2)$. It is easy to see that for $T^{12} = -D$ and $T^{11} = K$ relations (4.6) hold. We have $T^{22} = H = -\frac{1}{2}\{Q^c, \bar{Q}_c\}$. Then by (4.13) we obtain

$$\begin{aligned}[][I^{ab}, H] &= -\frac{1}{2}([I^{ab}, Q^c\bar{Q}_c] + [I^{ab}, \bar{Q}_cQ^c]) \\ &= -\frac{1}{2}(Q^c[I^{ab}, \bar{Q}_c] + [I^{ab}, Q^c]\bar{Q}_c + \bar{Q}_c[I^{ab}, Q^c] + [I^{ab}, \bar{Q}_c]Q^c) \\ &= -\frac{1}{2}(-Q_{\widehat{c}}[I^{ab}, \bar{Q}^{\widehat{c}}] + [I^{ab}, Q^c]\bar{Q}_c + \bar{Q}_c[I^{ab}, Q^c] - [I^{ab}, \bar{Q}^{\widehat{c}}]Q_{\widehat{c}}),\end{aligned}$$

where $\widehat{c}$ is complimentary to c. By Lemma B.1.4 we have

$$[I^{ab}, Q^c] = -[I^{ab}, Q^{21c}] = -\frac{i}{2}(\epsilon^{1a}Q^{2bc} + \epsilon^{1b}Q^{2ac}) \quad \text{and} \quad [I^{ab}, \bar{Q}^c] = -\frac{i}{2}(\epsilon^{2a}Q^{2bc} + \epsilon^{2b}Q^{2ac}).$$

Therefore by considering various values of $a, b \in \{1, 2\}$ and by using Lemma B.1.7 and Theorem B.1.5 we obtain the following:

$$\begin{aligned}[][I^{11}, H] &= \frac{i}{2}(Q_{\widehat{c}}Q^{\widehat{c}} + Q^{\widehat{c}}Q_{\widehat{c}}) = 0, \\ [I^{22}, H] &= \frac{i}{2}(\bar{Q}^c\bar{Q}_c + \bar{Q}_c\bar{Q}^c) = 0, \\ [I^{12}, H] &= \frac{i}{2}(Q_{\widehat{c}}\bar{Q}^{\widehat{c}} + Q^c\bar{Q}_c + \bar{Q}_cQ^c + \bar{Q}^{\widehat{c}}Q_{\widehat{c}}) = 0,\end{aligned}$$

which are the corresponding relations (4.6).

Similarly we have

$$[J^{ab}, H] = -\frac{1}{2}(-Q_{\widehat{c}}[J^{ab}, \bar{Q}^{\widehat{c}}] + [J^{ab}, Q^c]\bar{Q}_c + \bar{Q}_c[J^{ab}, Q^c] - [J^{ab}, \bar{Q}^{\widehat{c}}]Q_{\widehat{c}}).$$

By Lemma B.1.3 we have

$$[J^{ab}, Q^c] = \frac{i}{2}(\epsilon^{ca}Q^b + \epsilon^{cb}Q^a) \quad \text{and} \quad [J^{ab}, \bar{Q}^c] = \frac{i}{2}(\epsilon^{ca}\bar{Q}^b + \epsilon^{cb}\bar{Q}^a).$$

Therefore by considering various values of $a, b \in \{1, 2\}$ we obtain:

$$[J^{11}, H] = -\frac{i}{2}(-\epsilon^{\widehat{c}1}Q_{\widehat{c}}\bar{Q}^1 + \epsilon^{c1}Q^1\bar{Q}_c + \epsilon^{c1}\bar{Q}_cQ^1 - \epsilon^{\widehat{c}1}\bar{Q}^1Q_{\widehat{c}}b), \tag{B.63}$$

$$\begin{aligned}[J^{12}, H] = -\frac{i}{4}(&-\epsilon^{\widehat{c}1}Q_{\widehat{c}}\bar{Q}^2 - \epsilon^{\widehat{c}2}Q_{\widehat{c}}\bar{Q}^1 + \epsilon^{c1}Q^2\bar{Q}_c + \epsilon^{c2}Q^1\bar{Q}_c \\ &+ \epsilon^{c1}\bar{Q}_cQ^2 + \epsilon^{c2}\bar{Q}_cQ^1 - \epsilon^{\widehat{c}1}\bar{Q}^2Q_{\widehat{c}} - \epsilon^{\widehat{c}2}\bar{Q}^1Q_{\widehat{c}}),\end{aligned} \tag{B.64}$$

$$[J^{22}, H] = -\frac{i}{2}(-\epsilon^{\widehat{c}2}Q_{\widehat{c}}\bar{Q}^2 + \epsilon^{c2}Q^2\bar{Q}_c + \epsilon^{c2}\bar{Q}_cQ^2 - \epsilon^{\widehat{c}2}\bar{Q}^2Q_{\widehat{c}}b). \tag{B.65}$$

Then by considering various values of $c \in \{1, 2\}$ in (B.63)-(B.65) and by using Lemma B.1.7 and Theorem B.1.5 we obtain

$$[J^{11}, H] = [J^{12}, H] = [J^{22}, H] = 0,$$

as required. □

B.2 The second representation

Let the supercharges be of the form

$$Q^a = p\psi^a + i\frac{2\alpha + 1}{x}\psi_k\psi^k\bar{\psi}^a, \quad \bar{Q}_b = p\bar{\psi}_b + i\frac{2\alpha + 1}{x}\bar{\psi}^k\bar{\psi}_k\psi_b. \tag{B.66}$$

Let generators K, I^{ab}, J^{ab} and S^a, $\bar{S}_a$ be given by the corresponding formulae (B.9), (B.10), (B.11) and (B.12) same as in the first representation, while the generator D is now given by

$$D = -\frac{xp}{2} + \frac{i(1 + \alpha)}{2}.$$

Let us also note that

$$\widetilde{Q^a} = -\bar{Q}_a, \quad \widetilde{\bar{Q}^a} = -Q_a, \quad \widetilde{D} = -D.$$

Theorem B.2.1. *For any $a, b \in \{1, 2\}$ we have $\{Q^a, \bar{Q}_b\} = -2H\delta^a_b$, where the Hamilto-*

nian H is given by

$$H = \frac{p^2}{4} - \frac{i(2\alpha+1)}{4}\frac{1}{x}p + \frac{(2\alpha+1)}{2x^2}\Theta,$$

with $\Theta = \frac{1}{4} - \langle \psi_m \psi^m \bar{\psi}^k \bar{\psi}_k \rangle$.

Proof. Analogues of expressions (B.42), (B.43) are

$$\begin{aligned}\{p\psi^a, \frac{i(2\alpha+1)}{x}\bar{\psi}^m\bar{\psi}_m\psi_b\} &= i(2\alpha+1)\Big(\psi^a\bar{\psi}^m\bar{\psi}_m\psi_b[p,\frac{1}{x}] + \{\psi^a, \bar{\psi}^m\bar{\psi}_m\psi_b\}\frac{1}{x}p\Big) \\ &= i(2\alpha+1)\Big(\frac{i}{x^2}\psi^a\bar{\psi}^m\bar{\psi}_m\psi_b + \bar{\psi}^a\psi_b\frac{1}{x}p\Big),\end{aligned} \tag{B.67}$$

and

$$\{p\bar{\psi}_b, \frac{i(2\alpha+1)}{x}\psi_k\psi^k\bar{\psi}^a\} = i(2\alpha+1)\Big(\frac{i}{x^2}\bar{\psi}_b\psi_k\psi^k\bar{\psi}^a + \psi_b\bar{\psi}^a\frac{1}{x}p\Big), \tag{B.68}$$

respectively. Further on, the analogue of (B.45) is

$$\{\bar{\psi}^m\bar{\psi}_m\psi_b, \psi_k\psi^k\bar{\psi}^a\} = 0. \tag{B.69}$$

Therefore in total, we get from (B.67)-(B.69) (see also (B.51)) that

$$\{Q^a, \bar{Q}_b\} = -\frac{p^2}{2}\delta^a_b + \frac{i(2\alpha+1)}{2}\frac{1}{x}p\delta^a_b - \frac{(2\alpha+1)}{x^2}(\frac{1}{4} - \langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle)\delta^a_b, \tag{B.70}$$

and hence the statement follows. □

It is easy to check that the following lemma holds.

Lemma B.2.2. *Let T^{ab} be as above. Then relations (4.3) hold.*

Lemma B.2.3. *Let Q^{abc}, J^{ab}, I^{ab} be as above. Then relations (4.5b), (4.5c) hold.*

Proof. Relations (4.5b), (4.5c) can be shown to hold by an adaptation of the proofs of Lemmas B.1.3 and B.1.4 respectively. Indeed let us first consider $[J^{ab}, Q^{21c}]$. It now takes the form (cf. (B.22))

$$[J^{ab}, Q^{21c}] = -[J^{ab}, Q^c] = -[J^{ab}, \psi^c]p - \frac{i(2\alpha+1)}{x}[J^{ab}, \psi_k\psi^k\bar{\psi}^c].$$

Therefore similarly to (B.23) we have

$$\begin{aligned}[J^{ab}, Q^{21c}] &= \frac{i}{2}\epsilon^{bc}p\psi^a + \frac{i}{2}p\epsilon^{ac} + \frac{i(2\alpha+1)}{x}(\frac{i}{2}\epsilon^{ac}\psi_k\psi^k\bar{\psi}^b + \frac{i}{2}\epsilon^{bc}\psi_k\psi^k\bar{\psi}^a) \\ &= \frac{i\epsilon^{bc}}{2}\big(p\psi^a + \frac{i(2\alpha+1)}{x}\psi_k\psi^k\bar{\psi}^a\big) + \frac{i\epsilon^{ac}}{2}\big(p\psi^b + \frac{i(2\alpha+1)}{x}\psi_k\psi^k\bar{\psi}^b\big) \\ &= \frac{i}{2}(\epsilon^{bc}Q^a + \epsilon^{ac}Q^b) = i\epsilon^{c(a}Q^{|21|b)},\end{aligned} \tag{B.71}$$

as required. Applying $\sim$ to (B.71) we obtain the corresponding relation for $[J^{ab}, Q^{22c}]$. The rest of the relations in (4.4a) are shown to hold in Lemma B.1.3.

Further on, let us consider $[I^{ab}, Q^f]$. Relation (B.34) now takes the form

$$\begin{aligned}[I^{22}, Q^{21f}] &= -i[\bar{\psi}^m\bar{\psi}_m, Q^f] \\ &= -i\big(-p\bar{\psi}^f - i\frac{(2\alpha+1)}{x}\bar{\psi}^k\bar{\psi}_k\psi^f\big) \\ &= i\bar{Q}^f,\end{aligned}$$

which is the corresponding relation (4.5c). Similarly, it is easy to see that $[I^{12}, Q^{21f}] = \frac{i}{2}Q^f$ and that $[I^{11}, Q^f] = 0$, as required. Applying $\sim$ to $[I^{ab}, Q^f]$ we obtain the corresponding relation for $[I^{ab}, \bar{Q}^f]$. The rest of the relations in (4.4b) are shown to hold in Lemma B.1.3 and therefore the statement follows. □

Lemma B.2.4. *Let T^{ab}, Q^{abc} be as above. Then relations (4.5a) hold.*

Proof. It is easy to see that relations (4.5a) hold for $T^{12} = -D$ and for $T^{11} = K$. Similarly it is easy to see that expressions $[T^{22}, Q^a]$ and $[T^{22}, \bar{Q}_a]$ take the required form.

Let us now consider relations (4.5a) with H and S^a. In view of (B.55) relation $[T^{22}, Q^{11f}]$ now takes the following form (cf. (B.62))

$$\begin{aligned}[H, S^f] &= -\frac{1}{2}[p^2, x\psi^f] + \frac{i(2\alpha+1)}{2}\psi^f[\frac{1}{x}p, x] + \frac{(2\alpha+1)}{x}[\langle\psi_m\psi^m\bar{\psi}^k\bar{\psi}_k\rangle, \psi^f] \\ &= ip\psi^f + \frac{(2\alpha+1)}{2x}\psi^f - \frac{(2\alpha+1)}{x}(\psi_k\psi^k\bar{\psi}^f + \frac{1}{2}\psi^f).\end{aligned}$$

Therefore, $[H, S^f] = iQ^f$, as required. Applying $\sim$ to $[H, S^f]$ we obtain the corresponding relation for $[H, \bar{S}^f] = i\bar{Q}^f$ in (4.5a). Hence the statement follows. □

Lemma B.2.5. *Let Q^{abc}, I^{ab}, T^{ab}, J^{ab} be as above. Then the relations (4.2) hold.*

Proof. We first note that by Theorem B.2.1 we have $\{Q^a, \bar{Q}^c\} = \epsilon^{cb}\{Q^a, \bar{Q}_b\} = -2H\epsilon^{ca}$ which is the corresponding relation (4.2).

Moreover the expressions $\{Q^{21a}, Q^{21b}\}$, $\{Q^{22a}, Q^{22b}\}$, $\{Q^{21a}, Q^{11f}\}$, $\{Q^{22a}, Q^{12f}\}$, $\{Q^{11a}, Q^{11b}\}$, $\{Q^{12a}, Q^{12b}\}$, $\{Q^{11a}, Q^{12b}\}$ take the same form as in Lemma B.1.7.

Let us consider the anti-commutator $\{Q^{21a}, Q^{12b}\}$. The left-hand side of (4.2) now takes the form (cf. (B.58), (B.59))

$$\{Q^{21a}, Q^{12b}\} = xp\epsilon^{ab} - 2i\psi^a\bar{\psi}^b + 2i(2\alpha+1)\psi^b\bar{\psi}^a. \tag{B.72}$$

The right-hand side of (4.2) equals (cf. (B.60) which has different constant in the right-hand side)

$$-2\epsilon^{ab}D + 2\alpha J^{ab} - i(1+\alpha)\epsilon^{ab}[\psi_m, \bar{\psi}^m] = xp\epsilon^{ab} + 2\alpha J^{ab} - 2i(1+\alpha)\epsilon^{ab}(\psi^2\bar{\psi}^1 - \psi^1\bar{\psi}^2),$$

which takes the form

$$xp\epsilon^{ab} + 2\alpha i(\psi^a\bar{\psi}^b + \psi^b\bar{\psi}^a) - 2i(1+\alpha)\epsilon^{ab}(\psi^2\bar{\psi}^1 - \psi^1\bar{\psi}^2). \tag{B.73}$$

By considering various values of a and b one can see that the expression (B.73) can be rearranged (cf. (B.61)) as

$$xp\epsilon^{ab} - 2i\psi^a\bar{\psi}^b + 2i(2\alpha+1)\psi^b\bar{\psi}^a,$$

which is equal to expression (B.72) as required. Using $\sim$ operation we obtain the remaining relations. This concludes the proof. $\square$

Finally, we note that the statement of Lemma B.1.9 holds in this case as well, and the corresponding proof keeps the same form.

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