Rational Cherednik algebras, quiver Schur algebras and cohomological Hall algebras

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Submitted in fulfilment of the requirements for the
Degree of Doctor of Philosophy

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August 2019

Abstract

This thesis is devoted to three interrelated problems in representation theory. The first problem concerns the combinatorial aspects of the connection between rational Cherednik algebras at $t = 0$ and Hilbert schemes. The second problem concerns the critical-level limit of the Suzuki functor, which connects the representation theory of affine Lie algebras to that of rational Cherednik algebras. The third problem concerns the properties of certain generalizations of Khovanov-Lauda-Rouquier algebras, called quiver Schur algebras, and their relationship to cohomological Hall algebras. Let us describe our results in more detail.

In chapter 3, we study the combinatorial consequences of the relationship between rational Cherednik algebras of type $G(l,1,n)$, cyclic quiver varieties and Hilbert schemes. We classify and explicitly construct $\mathbb{C}^*$-fixed points in cyclic quiver varieties and calculate the corresponding characters of tautological bundles. We give a combinatorial description of the bijections between $\mathbb{C}^*$-fixed points induced by the Etingof-Ginzburg isomorphism and Nakajima reflection functors. We apply our results to obtain a new proof as well as a generalization of a well known combinatorial identity, called the q-hook formula. We also explain the connection between our results and Bezrukavnikov and Finkelberg's, as well as Losev's, proofs of Haiman's wreath Macdonald positivity conjecture.

In chapter 4, we define and study a critical-level generalization of the Suzuki functor, relating the affine general linear Lie algebra to the rational Cherednik algebra of type A. Our main result states that this functor induces a surjective algebra homomorphism from the centre of the completed universal enveloping algebra at the critical level to the centre of the rational Cherednik algebra at $t = 0$. We use this homomorphism to obtain several results about the functor. We compute it on Verma modules, Weyl modules, and their restricted versions. We describe the maps between endomorphism rings induced by the functor and deduce that every simple module over the rational Cherednik algebra lies in its image. Our homomorphism between the two centres gives rise to a closed embedding of the Calogero-Moser space into the space of opers on the punctured disc. We give a partial geometric description of this embedding.

In chapter 5, we establish a connection between a generalization of KLR algebras, called quiver Schur algebras, and the cohomological Hall algebras of Kontsevich and Soibelman. More specifically, we realize quiver Schur algebras as algebras of multiplication and comultiplication operators on the CoHA, and reinterpret the shuffle description of the CoHA in terms of Demazure operators. We introduce "mixed quiver Schur algebras" associated to quivers with a contravariant involution, and show that they are related, in an analogous way, to the cohomological Hall modules defined by Young. Furthermore, we obtain a geometric realization of the modified quiver Schur algebra, which appeared in a version of the Brundan-Kleshchev-Rouquier isomorphism for the affine q-Schur algebra due to Miemietz and Stroppel.

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Acknowledgements

I am deeply grateful to my advisors Dr Gwyn Bellamy and Prof. Catharina Stroppel for their guidance and unwavering support throughout my PhD, as well as inspiring mathematical discussions and countless comments on draft versions of this thesis. I would like to thank Prof. Iain Gordon, Prof. Alexander Molev and Dr Matthew Young for discussing their work with me. I am grateful to the College of Science & Engineering at the University of Glasgow and the Max Planck Institute for Mathematics in Bonn for financial support and excellent working conditions. Finally, I would like to thank my family for their support throughout my mathematical studies, and all my friends, without whom my time as a PhD student would never have been so enjoyable.

Declaration

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I certify that the thesis presented here for examination for a PhD degree of the University of Glasgow is solely my own work other than where I have clearly indicated that it is the work of others and that the thesis has not been edited by a third party beyond what is permitted by the University's PGR Code of Practice.

I declare that the thesis does not include work forming part of a thesis presented successfully for another degree.

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I acknowledge that if any issues are raised regarding good research practice based on review of the thesis, the examination may be postponed pending the outcome of any investigation of the issues.

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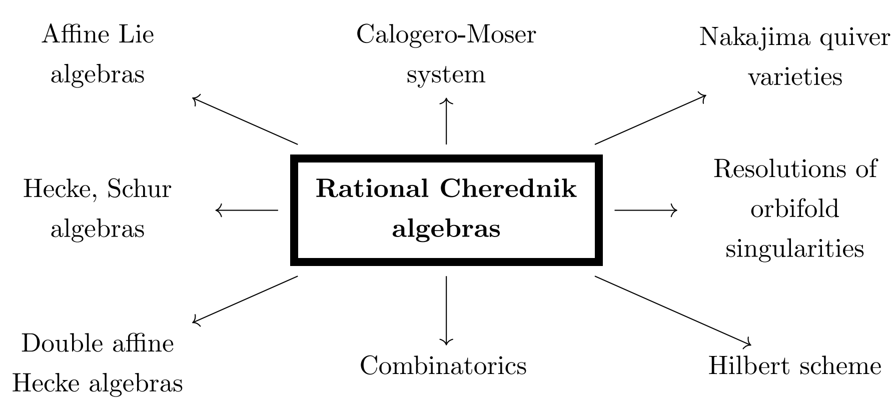
Chapter 1

Introduction

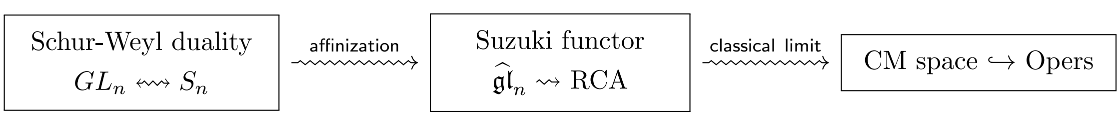
Below we sketch the background and the principal motivations behind the problems considered in this thesis. A more precise, and technical, summary of our main results can be found in the introductions to the three main chapters: §3.1, §4.1 and §5.1.

General overview. In this thesis we consider three distinct but interrelated problems in representation theory. The first one revolves around an affine algebraic variety called the Calogero-Moser space. This variety was introduced in [142] as the completed phase space of the Calogero-Moser (CM) integrable system, and was used to relate the collisions of the Calogero-Moser particles to solutions of the Kadomtsev-Petviashvili (KP) hierarchy. It later turned out that the Calogero-Moser space admits a more algebraic construction, which can be generalized to any complex reflection group. Etingof and Ginzburg [48] associated to any such group an algebra, called the rational Cherednik algebra (RCA), whose centre, in the symmetric group case, is isomorphic to the ring of functions on the Calogero-Moser space. Rational Cherednik algebras play an important role in symplectic geometry, representation theory, and combinatorics. For example, they have been instrumental in classifying symplectic resolutions of orbifold singularities, and generalizing the famous Macdonald positivity conjecture. They also have fascinating links to other objects such as Hecke algebras, Schur algebras and affine Lie algebras, some of which we explore in this thesis.

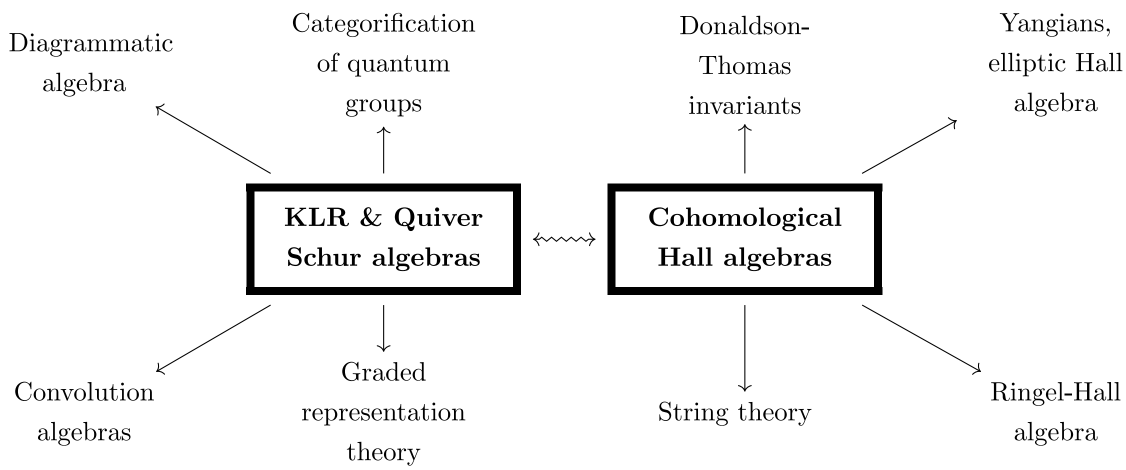


The Calogero-Moser space associated to the complex reflection group $G(l,1,n)$, i.e., the semi-direct product $S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$, admits yet another, more geometric, construction, as a Nakajima quiver variety. The interplay between the algebraic and geometric constructions of the Calogero-Moser space is the main topic of chapter 3. The rational Cherednik algebra is naturally graded, and this grading induces a $\mathbb{C}^*$-action on the Calogero-Moser space, which corresponds to the scaling action on the quiver variety. By studying the fixed points of these actions, we obtain a new combinatorial formula, which can be seen as a generalization and a quantization of the classical hook-length formula, expressing the dimension of a simple module over the symmetric group in terms of hooks in a Young diagram.

The main topic of chapter 4 is a variation on another classical theme, namely, Schur-Weyl duality. It was a great insight, which has, in different guises, guided the development of representation theory to this day, that the general linear group controls the representation theory of the symmetric group. We study an affine analogue of this relationship, with the general linear group replaced by the corresponding affine Lie algebra, and the symmetric group replaced by the rational Cherednik algebra. More specifically, we study the classical limit of a functor defined by Suzuki [132], and realize the centre of the rational Cherednik algebra as a quotient of the centre of the completed universal enveloping algebra of the affine Lie algebra at the critical level. This allows us to exhibit the Calogero-Moser space as an explicit closed subset inside a certain moduli space of G-bundles called opers.



Chapter 5 is devoted to a somewhat different problem, which is not directly related to rational Cherednik algebras. As our main result, we establish a connection between two well known algebras, which, historically, appeared in very different mathematical contexts and were introduced with rather different motivations in mind, namely: quiver Schur algebras and cohomological Hall algebras (CoHA's). The former are a generalization of Khovanov-Lauda-Rouquier (KLR) algebras, which play an important role in the categorification of quantum groups. We study them from a geometric point of view, focussing on their realization as convolution algebras associated to a certain Steinberg-type variety. This construction generalizes the well known geometric construction of the (degenerate) affine Hecke algebra. Our second main player, the CoHA, is, on the other hand, motivated by questions from physics, including string theory and Donaldson-Thomas invariants. The CoHA can be seen as a variation on the famous Ringel-Hall algebra, with a finite field replaced by the field of complex numbers and convolution of functions replaced by convolution of cohomology classes. Our main result gives, roughly speaking, an action of the quiver Schur algebra on the CoHA, which allows us to realize the quiver Schur algebra as an algebra of certain explicit operators on the CoHA.



Rational Cherednik algebras. We will now describe the main ideas of the thesis in more detail. We start by defining rational Cherednik algebras, and explaining the main motivations behind them. Let V be a finite dimensional complex vector space with an action of a finite group $G \subset GL(V)$. The well known Shephard Todd theorem (see, e.g., [24]) says that the variety $V/G = \operatorname{Spec} \mathbb{C}[V]^G$ is smooth if and only if G is a complex reflection group. In general, the space V/G is singular. For example, if $V = \mathbb{C}^2$ and G is a finite subgroup of $SL_2(\mathbb{C})$ then the space $\mathbb{C}^2/G$ is called a Kleinian (or Du Val) singularity. The finite subgroups of $SL_2(\mathbb{C})$ are classified, via McKay correspondence, by ADE Dynkin diagrams, and belong to a large class of groups called symplectic reflection groups.

Suppose that (V, ω) is a symplectic vector space and $G \subset Sp(V)$ is a finite subgroup. One may ask whether the quotient singularity V/G can be resolved via a symplectic resolution. A symplectic resolution is a birational morphism $\pi \colon X \to V/G$ from a smooth symplectic variety X, projective over V/G, such that the restriction of π to the preimage of the smooth locus of V/G is an isomorphism of symplectic varieties. Verbitsky [140] showed that if such a resolution exists then G is a symplectic reflection group, i.e., it is generated by elements s satisfying $\operatorname{rk}(1-s) = 2$.

From now on assume that G is a symplectic reflection group. Namikawa [105] showed that V/G admits a symplectic resolution if and only if it admits a smooth Poisson deformation. To study such deformations, it is convenient to pass to noncommutative geometry, i.e., replace the coordinate ring $\mathbb{C}[V/G] = \mathbb{C}[V]^G$ with the noncommutative skew group ring $\mathbb{C}[V] \rtimes G$. The two algebras are Morita equivalent and $\mathbb{C}[V]^G$ is equal to the centre of $\mathbb{C}[V] \rtimes G$. The Poisson structure on $\mathbb{C}[V] \rtimes G$ comes from its natural quantization, namely the skew group ring $\mathtt{A}_t(V) \rtimes G$ associated to the Weyl algebra $\mathtt{A}_t(V)$. Explicitly, $\mathtt{A}_t(V) \rtimes G$ is the quotient of $T(V^*) \rtimes G$ by the relation $[u, v] = t\omega(u, v)$ (for $u, v \in V^*$).

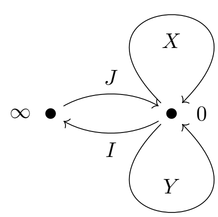
A Poisson deformation of the algebra $\mathbb{C}[V] \rtimes G$, depending on an extra parameter $\mathbf{c}$ associated to conjugacy classes of symplectic reflections, was defined by Etingof and Ginzburg [48]. This Poisson deformation is known as the *symplectic reflection algebra* $\mathtt{H}_{t,\mathbf{c}}(G)$ associated to G. It is the quotient of $T(V^*) \rtimes G$ by the relation

$$[u, v] = t\omega(u, v) - 2 \sum_{s \in \mathcal{S}} \mathbf{c}(s)\omega_s(u, v) \cdot s,$$

where $\mathcal{S}$ is the set of symplectic reflections and ω_s is the 2-form which equals ω on $\operatorname{Im}(1-s)$ and is trivial on $\ker(1-s)$. We see directly that $\mathtt{H}_{t,\mathbf{c}}(G)$ specializes to $\mathtt{A}_t(V) \rtimes G$ when $\mathbf{c} = 0$, and to $\mathbb{C}[V] \rtimes G$ if, additionally, $t = 0$. As we have already mentioned, the existence of a symplectic resolution of V/G is equivalent to the existence of a smooth Poisson deformation of V/G. It was proven by Ginzburg and Kaledin [62] that such a Poisson deformation exists if and only if Spec $Z(\mathtt{H}_{0,\mathbf{c}}(G))$ is smooth for generic $\mathbf{c}$. This result was used in, e.g., [10,62,63] to classify quotient singularities admitting symplectic resolutions.

We are interested in a special class of symplectic reflection algebras, associated to symplectic reflection groups constructed from complex reflection groups. Suppose that $G \subset GL(\mathfrak{h})$ is a complex reflection group. If we endow $V = \mathfrak{h} \oplus \mathfrak{h}^*$ with the natural symplectic structure and the diagonal G-action, then G, considered as a subgroup of $Sp(V)$, becomes a symplectic reflection group. The corresponding symplectic reflection algebra $\mathtt{H}_{t,\mathbf{c}}(G)$ is called a *rational Cherednik algebra.* This terminology is motivated by the fact that, when G is a Weyl group associated to a (finite) root system, $\mathtt{H}_{t,\mathbf{c}}(G)$ is a degeneration of Cherednik's double affine Hecke algebra.

Quiver varieties. The rational Cherednik algebras associated to complex reflection groups of type $G(l,1,n)$, i.e., $G = S_n \ltimes (\mathbb{Z}/l\mathbb{Z})^n$, are especially interesting. As we have already mentioned, the affine variety defined by the centre of $\mathtt{H}_{0,\mathbf{c}}(G)$ plays an important role in the study of symplectic resolutions. It turns out that when G is of type $G(l,1,n)$, the variety Spec $Z(\mathtt{H}_{0,\mathbf{c}}(G))$ can be realized as a Nakajima quiver variety [48, Theorem 11.16]. When G is the symmetric group, this quiver variety coincides with a well known object from the theory of integrable systems, namely the Calogero-Moser space [142]. Let us consider this example in more detail. We take the double of the framed Jordan quiver, as pictured below:



(1.1)

The space $\mathrm{Rep}(Q,\mathbf{d})$ of representations of this quiver with dimension vector $\mathbf{d} = (d_0 = n, d_\infty = 1)$ is isomorphic to $\mathrm{Mat}_{n\times n}(\mathbb{C})^{\oplus 2} \oplus \mathrm{Mat}_{n\times 1}(\mathbb{C}) \oplus \mathrm{Mat}_{1\times n}(\mathbb{C})$ and carries a natural action of $GL_n(\mathbb{C})$ by conjugation. Since $\mathrm{Rep}(Q,\mathbf{d})$ is isomorphic to the cotangent bundle on the space of representations of the (non-doubled) framed Jordan quiver, it carries a natural symplectic structure. The GL_n-action is Hamiltonian and the corresponding moment map is given by the formula

$$\mu\colon \mathrm{Rep}(Q,\mathbf{d}) \to \mathfrak{gl}_n, \quad (X,Y,I,J) \mapsto [X,Y] + JI.$$

The *Calogero-Moser space* is the quotient

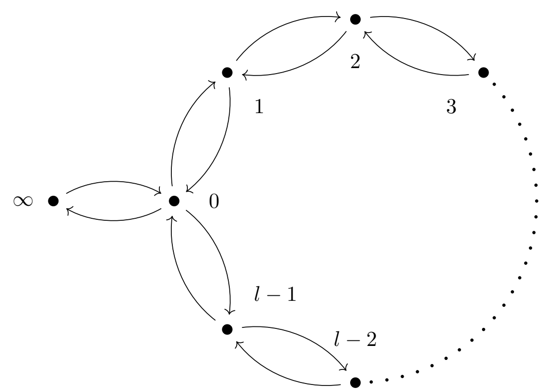
$$\mathcal{X}(\mathbf{d}) = \mu^{-1}(-\mathrm{id}) /\!\!/ GL_n = \mathrm{Spec}\, \mathbb{C}[\mu^{-1}(-\mathrm{id})]^{GL_n}. \tag{1.2}$$

To the same quiver, but different parameters, one can associate another well known variety, namely the Hilbert scheme of n points in the plane. We take the fibre of the moment map at 0 (rather than $-\mathrm{id}$), and replace the naive quotient (1.2) with the GIT quotient

$$\mathrm{Hilb}_n = \mu^{-1}(0) /\!\!/_{-1} GL_n.$$

The Hilbert scheme is a smooth irreducible variety parametrizing ideals in $\mathbb{C}[x,y]$ of colength n. Even though the Calogero-Moser space and the Hilbert scheme are very different as algebraic varieties, they are diffeomorphic if we consider them as hyper-Kähler manifolds.

The connection between rational Cherednik algebras, quiver varieties and Hilbert schemes has been generalized by Etingof and Ginzburg [48] as well as Gordon [64] to complex reflection groups of type $G(l, 1, n)$. The quiver (1.1) gets replaced by the following quiver



Consider the space of representations of this quiver with dimension vector $\mathbf{d} = (d_0 = n, \ldots, d_{l-1} = n, d_\infty = 1)$, equipped with the conjugation action of $G(\mathbf{d}) = \prod_{i=0}^{l-1} GL_{d_i}$. This action is Hamiltonian and gives rise to the moment map

$$\mu\colon \operatorname{Rep}(Q, \mathbf{d}) \to \operatorname{Lie} G(\mathbf{d}).$$

Given a parameter $\theta \in \mathbb{Q}^l$, we obtain two quiver varieties

$$\mathcal{X}_\theta(\mathbf{d}) = \mu^{-1}(\theta) /\!\!/ G(\mathbf{d}), \quad \mathcal{M}_\theta(\mathbf{d}) = \mu^{-1}(0) /\!\!/_\theta\, G(\mathbf{d}).$$

We will always assume that the parameter θ is chosen in such a way that these quiver varieties are smooth. Etingof and Ginzburg (and later Martino, in the non-smooth case) showed that there is a natural isomorphism of algebraic varieties

$$\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G)) \cong \mathcal{X}_{\theta_\mathbf{c}}(\mathbf{d}),$$

where $\theta_\mathbf{c}$ is a parameter depending on $\mathbf{c}$. In view of this isomorphism, the spectrum of the centre of a rational Cherednik algebra is often called a *generalized Calogero-Moser space*. The variety $\mathcal{X}_\theta(\mathbf{d})$ turns out to be diffeomorphic to the GIT quotient $\mathcal{M}_{2\theta}(\mathbf{d})$. Moreover, by simultaneously changing the dimension vector $\mathbf{d}$ and the parameter θ in an appropriate way, one obtains diffeomorphisms $\mathcal{M}_\theta(\mathbf{d}) \cong \mathcal{M}_{\theta'}(\mathbf{d}')$ called Nakajima reflection functors. Gordon combined these diffeomorphisms to produce a map from $\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G))$ to a certain closed subvariety of the Hilbert scheme of $K = nl+m$ points in the plane, where m depends on the parameter $\mathbf{c}$. To summarize, we have maps

$$\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G)) \to \mathcal{X}_{\theta_\mathbf{c}}(\mathbf{d}) \to \mathcal{M}_{2\theta_\mathbf{c}}(\mathbf{d}) \to \mathcal{X}_{\theta'}(\mathbf{d}') \hookrightarrow \operatorname{Hilb}_K. \tag{1.3}$$

Combinatorics. We are interested in the combinatorial applications of the connection between rational Cherednik algebras and Hilbert schemes. We pass from geometry to combinatorics by considering the fixed points of certain $\mathbb{C}^*$-actions: the action on $\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G))$ induced by the grading on the rational Cherednik algebra, and the natural scaling action on the quiver varieties. The fixed points in $\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G))$ are parametrized by l-multipartitions of n, while the fixed points in Hilb_K are parametrized by partitions of K. Since all the maps in (1.3) are equivariant, we obtain an injective map

$$\mathcal{P}(l,n) \hookrightarrow \mathcal{P}(K). \tag{1.4}$$

The image of this map consists of partitions with a certain fixed l-core, depending on the parameter $\mathbf{c}$. We remark that an l-core is a partition from which one cannot remove any hooks of length l. The main goal of chapter 3 is to give an explicit combinatorial description of (1.4). Our main result (Theorem D and Corollary E in §3.1) says that (the inverse of) (1.4) can be characterized as a twist of a well known map from combinatorics, which assigns to each partition its l-quotient. Since this notion is somewhat technical, we refer the reader to §3.3.5 for a precise definition.

Theorem 1. *The bijection*

$$\mathcal{P}(l,n) \longleftrightarrow \mathcal{P}_\nu(K) \tag{1.5}$$

from the set of l-multipartitions of n to the set of partitions of K with l-core ν (depending on $\mathbf{c}$*), induced by* (1.3)*, is given by a twist of the classical l-quotient bijection.*

Our result has several interesting applications. The first one has to do with the q-analogue of the well known hook length formula

$$d_\mu = \frac{n!}{\prod_{\square\in\mu} h_\mu(\square)}, \tag{1.6}$$

which expresses the dimension of the Specht module corresponding to the Young diagram of shape μ in terms of the product of the lengths of all the hooks in this diagram. The q-analogue of this formula, known as the *q-hook formula*, relates a certain polynomial depending on the contents of cells in the Young diagram to so-called fake degree polynomials (which can also be expressed in terms of hook polynomials):

$$\sum_{\square\in\mu} q^{c(\square)} = [n]_q \sum_{\lambda\uparrow\mu} \frac{f_\lambda(q)}{f_\mu(q)}. \tag{1.7}$$

As an application of Theorem 1, we obtain a new geometric proof and a "higher level" generalization of the q-hook formula (Theorem G in §3.1):

$$\sum_{\square\in\mu} q^{c(\square)} = [nl]_t \sum_{\underline{\lambda}\uparrow\underline{\mathsf{Quot}}(\mu)^\flat} \frac{f_{\underline{\lambda}}(q)}{f_{\underline{\mathsf{Quot}}(\mu)^\flat}(q)}. \tag{1.8}$$

Theorem 1 is also an important ingredient in the proofs [16, 95] of Haiman's wreath Macdonald positivity conjecture, which is a generalization of the original conjecture concerning the coefficients of Kostka-Macdonald polynomials, with the ring of symmetric functions replaced by the space of virtual characters of the complex reflection group of type $G(l,1,n)$. The role our result plays in these proofs is described in more detail in §3.1.6.

Category $\mathcal{O}$. So far we have focussed mainly on rational Cherednik algebras at $t=0$. In order to explain our next set of results, we also need to mention a few facts about rational Cherednik algebras at $t=1$. Their representation theory has a somewhat Lie-theoretic flavour. The triangular decomposition

$$\mathtt{H}_{1,\mathbf{c}}(G) \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[G] \otimes \mathbb{C}[\mathfrak{h}^*]$$

was used in [61] to define a category $\mathcal{O}$ for rational Cherednik algebras, reminiscent of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for semisimple Lie algebras. It is the full subcategory of $\mathtt{H}_{1,\mathbf{c}}(G)$-mod consisting of finitely generated modules such that the action of $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$ is locally nilpotent.

Category $\mathcal{O}$ is a highest weight category with standard objects given by certain Verma modules, constructed by induction from representations of the subalgebra $\mathbb{C}[\mathfrak{h}^*] \rtimes G \subset \mathtt{H}_{1,\mathbf{c}}(G)$. The standard objects are in one-to-one correspondence with irreducible representations of the group G. In particular, if G is of type $G(l,1,n)$, they are labelled by l-multipartitions of n. In order to define a highest weight category, one also needs to specify a partial order on the labelling set of standard modules. One such partial order is given by the so-called $\mathbf{c}$-function. When G is of type $G(l,1,n)$, the $\mathbf{c}$-function admits an explicit combinatorial description, see (3.59). One may ask whether there exist finer partial orders which still make category $\mathcal{O}$ into a highest weight category. This question was answered positively by Dunkl and Griffeth [42], who showed that $\mathcal{O}$ is a highest weight category with respect to a certain "combinatorial" order $\prec_{\mathbf{c}}^{\mathsf{com}}$. This partial order is induced by the dominance order on partitions via the bijection (1.5). Theorem 1 implies that the combinatorial order has a (partial) geometric interpretation (Corollary F in §3.1), i.e., it can be related to the "geometric" order $\prec_{\mathbf{c}}^{\mathsf{geo}}$ on $\mathcal{P}(l,n)$ defined by the closure relations between the attracting sets of $\mathbb{C}^*$-fixed points in $\mathcal{M}_{2\theta}(\mathbf{d})$. More precisely, given $\underline{\mu}, \underline{\lambda} \in \mathcal{P}(l,n)$, we have

$$\underline{\mu} \prec_{\mathbf{c}}^{\mathsf{geo}} \underline{\lambda} \;\Rightarrow\; \underline{\mu} \prec_{\mathbf{c}}^{\mathsf{com}} \underline{\lambda}.$$

Suzuki functor. Important information about the structure of a highest weight category is contained in the so-called decomposition numbers, which express the multiplicities with which simple modules occur in standard modules. This information can, for example, be used to calculate the characters of simple modules. In the case of semisimple Lie algebras, this problem was the subject of the famous Kazhdan-Lusztig conjectures, proven by Beilinson-Bernstein and Brylinski-Kashiwara, using D-modules and perverse sheaves techniques (see, e.g., [74]). In the context of rational Cherednik algebras, there are several approaches to the multiplicity problem. When G is the symmetric group, it was proven by Rouquier [115] that category $\mathcal{O}$ is equivalent to the category of modules over a certain q-Schur algebra. Multiplicities in the latter category can, by the work of Varagnolo and Vasserot [139], be described in terms of the q-Fock space.

There is another approach to the multiplicity problem, via a certain coinvariants functor defined by Suzuki [132]. This functor can be seen as a generalization of the classical Schur-Weyl duality relating the representation theory of $\mathfrak{gl}_n$ to the representation theory of the symmetric group S_n. According to Schur-Weyl duality, the natural actions

$$\mathfrak{gl}_n \curvearrowright (\mathbb{C}^n)^{\otimes n} \curvearrowleft S_n, \tag{1.9}$$

centralize each other. Let us abbreviate $\mathbf{V} = \mathbb{C}^n$. Tensoring with $(\mathbf{V}^*)^{\otimes n}$ and taking coinvariants gives rise to a functor

$$M \mapsto H_0(\mathfrak{gl}_n, (\mathbf{V}^*)^{\otimes n} \otimes M) = (\mathbf{V}^*)^{\otimes n} \otimes M / (\mathfrak{gl}_n \cdot ((\mathbf{V}^*)^{\otimes n} \otimes M))$$

from the category of $\mathfrak{gl}_n$-modules to the category of S_n-modules, sending the simple module of highest weight λ to the Specht module labelled by the corresponding partition.

We are interested in the affine analogue of this functor. One can replace $\mathbf{V}$ with the space $\mathbf{V}[z]$ of polynomials with coefficients in $\mathbf{V}$, and replace $\mathfrak{gl}_n$ with the current Lie algebra $\mathfrak{gl}_n[z]$. It was shown in [132] that, for each smooth $\widehat{\mathfrak{gl}}_n$-module M of level κ, there are actions

$$\mathfrak{gl}_n[z] \curvearrowright (\mathbf{V}^*[z])^{\otimes n} \otimes M \curvearrowleft \mathtt{H}_{n+\kappa,1}(S_n).$$

These actions no longer centralize each other, but the $\mathtt{H}_{n+\kappa,1}(S_n)$-action normalizes the $\mathfrak{gl}_n[z]$-action, and induces an action on the space of coinvariants. Therefore, we get a functor (called the *Suzuki functor*)

$$\mathsf{F}_\kappa\colon \widehat{\mathfrak{gl}}_n\text{-}\mathsf{mod}_{\kappa,sm} \to \mathtt{H}_{n+\kappa,1}(S_n)\text{-}\mathsf{mod}, \quad M \mapsto H_0(\mathfrak{gl}_n[z], (\mathbf{V}^*[z])^{\otimes n} \otimes M)$$

from the category of smooth modules over the affine Lie algebra $\widehat{\mathfrak{gl}}_n$ of level κ to the category of modules for the rational Cherednik algebra $\mathtt{H}_{n+\kappa,1}(S_n)$. It was shown in [138] that, under some mild assumptions, Suzuki's functor restricts to an equivalence of highest weight categories between category $\mathcal{O}$ for $\widehat{\mathfrak{gl}}_n$ and category $\mathcal{O}$ for $\mathtt{H}_{n+\kappa,1}(S_n)$.

Suzuki made a crucial restriction on the value of the parameter κ - he assumed that κ is not critical, i.e., that it is different from $-n$, which implies that the parameter t for the rational Cherednik algebra is different from 0. The main purpose of chapter 4 is to define a limit of the Suzuki functor as

$$\kappa \to c = -n, \quad t \to 0$$

and study its properties.

The representation theory of the rational Cherednik algebra at $t = 0$ differs radically from its representation theory at $t \neq 0$, mainly due to the fact that $\mathtt{H}_{0,1}(S_n)$ has a large centre $\mathtt{Z}$. An analogous pattern occurs in the representation theory of $\widehat{\mathfrak{gl}}_n$ - the centre of the completed universal enveloping algebra $\widehat{\mathbf{U}}_\kappa$ of $\widehat{\mathfrak{gl}}_n$ is trivial unless the level is critical. In the latter case, the centre $\mathfrak{Z}$ of $\widehat{\mathbf{U}}_c$ is a completion of a polynomial algebra in infinitely many variables, and, by a theorem of Feigin and Frenkel [50], it can be identified with the algebra of functions on a certain moduli space of G-bundles on the punctured disc (decorated with some additional data), called opers.

In general, a functor of abelian categories does not induce a homomorphism between their centres. In §4.7, we propose various ways to circumvent this problem. The following theorem (Theorems D and E in §4.1) is the main result of chapter 4.

Theorem 2. *There exists a surjective algebra homomorphism*

$$\Theta\colon\ \mathfrak{Z} \longrightarrow \mathtt{Z} \tag{1.10}$$

and a "reasonably big" subcategory $\mathcal{C}$ of $\widehat{\mathbf{U}}_c$-mod *such that the diagram*

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\Theta} & \mathtt{Z} \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \mathrm{End}_{\widehat{\mathbf{U}}_c}(M) & \xrightarrow{\mathsf{F}_c} & \mathrm{End}_{\mathtt{H}_{0,1}(S_n)}(\mathsf{F}_c(M)) \end{array} \tag{1.11}$$

commutes for all $M \in \mathcal{C}$.

The above theorem has many applications. For example, it allows us to compute the maps between endomorphism rings of certain Verma-type modules induced by the Suzuki functor (Corollary A in §4.1.3). We deduce that every simple module over the rational Cherednik algebra $\mathtt{H}_{0,1}(S_n)$ is in the image of the functor (Corollary B in §4.1.3). We remark that these modules are parametrized by the Calogero-Moser space - in particular, there are uncountably many of them. We also describe the behaviour of the functor on Arakawa and Fiebig's restricted category $\mathcal{O}$ (Corollary C in §4.1.3). Finally, we use (1.10) to construct an embedding of the Calogero-Moser space into the moduli space of opers on the punctured disc, and give a geometric description of this embedding (Corollary D in §4.1.3). Since the precise statements of some of the above mentioned results are rather technical, we refer the reader to §4.1 for details.

Convolution algebras. The main topic of chapter 5 is somewhat different from, though not unrelated to, the problems we have discussed so far. Rational Cherednik algebras arise as Poisson deformations of quotient singularities. The algebras to which we now turn our attention are a generalization of the convolution algebras associated to another famous singularity, the nilpotent cone, and the Springer resolution.

Let $\mathcal{N} \subset \mathfrak{gl}_n$ be the variety of nilpotent $n \times n$ matrices, and let G/B be the variety of complete flags in $\mathbb{C}^n$. The Springer resolution is the proper map

$$T^*(G/B) \to \mathcal{N}.$$

To this map one can associate a certain variety of triples

$$Z = T^*(G/B) \times_{\mathcal{N}} T^*(G/B)$$

called the Steinberg variety. The G-equivariant Borel-Moore homology $H_\bullet^G(Z)$ of the Steinberg variety can be endowed with a convolution product, which turns it into an associative algebra, isomorphic to the degenerate affine Hecke algebra. It has a faithful representation on the equivariant Borel-Moore homology of the cotangent bundle to the flag variety, which is isomorphic to the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The degenerate affine Hecke algebra is generated by elements which act by polynomial multiplication as well as certain divided difference operators called Demazure-Lusztig operators.

It turns out that the geometric construction of the degenerate affine Hecke algebra can be generalized in the context of quiver representations, so that the construction sketched above is the special case corresponding to the Jordan quiver. More precisely, given a quiver Q and a dimension vector $\mathbf{d}$, we replace the nilpotent cone $\mathcal{N}$ with the space $\operatorname{Rep}(Q, \mathbf{d})$ of representations of the quiver, and the cotangent bundle $T^*(G/B)$ with the space $\mathfrak{Q}_\mathbf{d}$ of complete flags of quiver representations. The forgetful map

$$\mathfrak{Q}_\mathbf{d} \to \operatorname{Rep}(Q, \mathbf{d}) \tag{1.12}$$

plays the role of the Springer resolution. The $G(\mathbf{d})$-equivariant Borel-Moore homology $H_\bullet^{G(\mathbf{d})}(\mathfrak{Z}_\mathbf{d})$ of the corresponding Steinberg-type variety

$$\mathfrak{Z}_\mathbf{d} = \mathfrak{Q}_\mathbf{d} \times_{\operatorname{Rep}(Q,\mathbf{d})} \mathfrak{Q}_\mathbf{d}$$

is an associative algebra with respect to the convolution product, and has a faithful representation on $H_\bullet^{G(\mathbf{d})}(\mathfrak{Q}_\mathbf{d})$, isomorphic to a direct sum of polynomial rings. The convolution algebra $H_\bullet^{G(\mathbf{d})}(\mathfrak{Z}_\mathbf{d})$ is generated by elements which act by polynomial multiplication, "crossings" resembling Demazure operators, and certain idempotents parametrizing the types of complete quiver flags. The algebra $H_\bullet^{G(\mathbf{d})}(\mathfrak{Z}_\mathbf{d})$ is also isomorphic to the extension algebra of the pushforward of the constant sheaf on $\mathfrak{Q}_\mathbf{d}$ along (1.12), and hence comes equipped with a natural grading.

It was shown by Varagnolo and Vasserot [137] that $H_\bullet^{G(\mathbf{d})}(\mathfrak{Z}_\mathbf{d})$ gives a geometric realization of the KLR, or quiver Hecke, algebras $R_\mathbf{d}$ introduced independently by Khovanov and Lauda [88] and Rouquier [114]. These algebras play an important role in the categorification of quantum groups. For example, it was shown in [88, 137] that there is an isomorphism

$$\begin{array}{ccc} \bigoplus_{\mathbf{d}} K_0(R_{\mathbf{d}}\text{-mod}_{gp}) & \cong & U_q^+(\mathfrak{g}_Q) \\ \{\text{indecomposable projectives}\} & \longleftrightarrow & \{\text{Canonical basis}\} \end{array}$$

between the direct sum of the Grothendieck groups of the categories of finitely generated graded projective modules over $R_{\mathbf{d}}$, ranging over all the possible dimension vectors, and Lusztig's integral form of the positive half of the quantum group associated to the underlying graph of the quiver Q. Multiplication and comultiplication on $U_q^+(\mathfrak{g}_Q)$ corresponds to induction and restriction functors for KLR algebras.

Quiver Schur algebras and cohomological Hall algebras. If we replace the space $\mathfrak{Q}_{\mathbf{d}}$ of complete flags of quiver representations with the bigger space of all *partial* flags, we obtain a convolution algebra $\mathcal{Z}_{\mathbf{d}}$, called the *quiver Schur algebra*. These algebras were introduced by Stroppel and Webster [131] and later studied in [99]. Quiver Schur algebras, just like KLR algebras, play an important role in categorification. For example, it was shown in [131] that the quiver Schur algebra associated to the cyclic quiver categorifies the generic nilpotent Hall algebra, and its higher level version categorifies the higher level q-Fock space. It was also shown in [99] that a certain completion of the quiver Schur algebra is isomorphic to a completion of the q-Schur algebra, appearing naturally in the representation theory of p-adic general linear groups.

The first goal of chapter 5 is to study the basic structural properties of quiver Schur algebras. We construct an explicit "Bott-Samelson" basis of $\mathcal{Z}_{\mathbf{d}}$, consisting of pushforwards of fundamental classes of certain vector bundles on diagonal Bott-Samelson varieties, and show that fundamental classes called merges and splits, together with certain invariant polynomials, form generators of $\mathcal{Z}_{\mathbf{d}}$ (Theorem A in §5.1). We also explicitly describe the faithful polynomial representation of the quiver Schur algebra (Theorem 5.4.7) and relate it to Demazure operators.

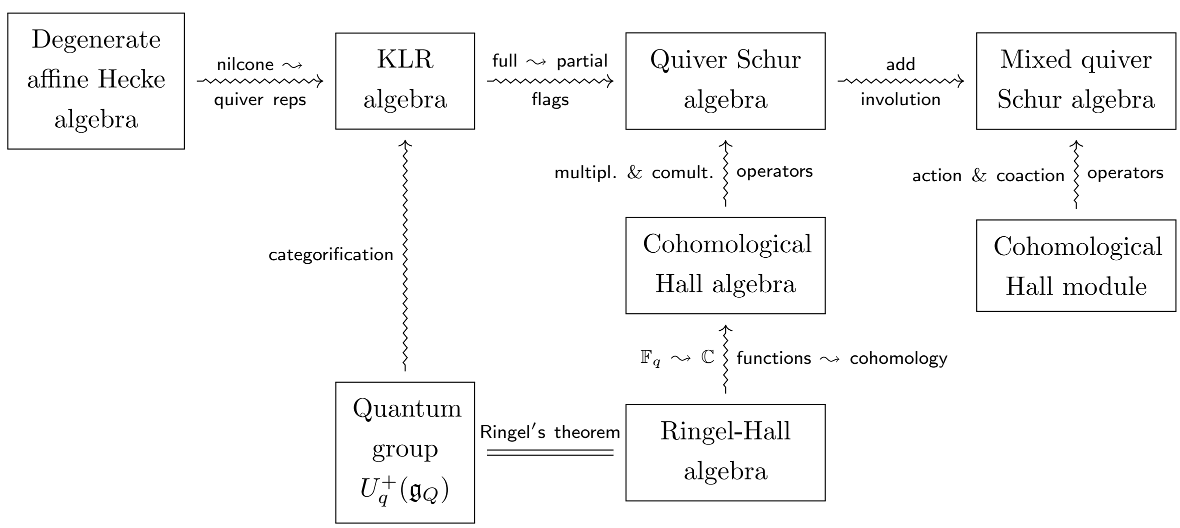
Our second goal is to establish a connection between quiver Schur algebras and cohomological Hall algebras (CoHA's). The latter were introduced by Kontsevich and Soibelman [90] as a categorification of Donaldson-Thomas invariants of three dimensional Calabi-Yau categories. Roughly speaking, the CoHA can be seen as a generalization of the Ringel-Hall algebra of the category of quiver representations over finite fields. Multiplication in the Ringel-Hall algebra is defined as convolution of conjugation-invariant functions via pullback and pushforward along the following diagram

$$\begin{array}{ccccc} & & Fl_2(\mathrm{Rep}_{\mathbb{F}_q}(Q)) & & \\ & \swarrow & & \searrow & \\ \mathrm{Rep}_{\mathbb{F}_q}(Q) \times \mathrm{Rep}_{\mathbb{F}_q}(Q) & & & & \mathrm{Rep}_{\mathbb{F}_q}(Q) \end{array}$$

where $Fl_2(\mathrm{Rep}_{\mathbb{F}_q}(Q))$ denotes the space of flags of length two. If we replace the field $\mathbb{F}_q$ by the field of complex numbers, and invariant functions by equivariant cohomology, we obtain the CoHA. A precise definition can be found in §5.6.1. CoHA's and their generalizations have found numerous applications in representation theory, including a new proof of the Kac positivity conjecture [36], as well as new realizations of the elliptic Hall algebra [124] and Yangians [38, 125, 144].

The Kontsevich-Soibelman CoHA is not a bialgebra, but it carries incompatible algebra and coalgebra structures. Our main result (Theorem B in §5.1) shows that the relations between multiplication and comultiplication in the CoHA are controlled by the quiver Schur algebra. This is quite remarkable, since quiver Schur algebras and CoHA's have very different mathematical origins. As an application, we interpret the description of the CoHA as a shuffle algebra from [90] in terms of Demazure operators (Proposition 5.6.8).

The third goal of chapter 5 is to define a generalization of quiver Schur algebras associated to quivers with a contravariant involution. As we have already mentioned, KLR and quiver Schur algebras can be realized as convolution algebras, or, equivalently, extension algebras of a certain semisimple complex of sheaves on the moduli stack of representations of a quiver. If the quiver admits a contravariant involution θ, this construction can be generalized by replacing the stack of representations of the quiver with the stack of its self-dual representations. We refer to the resulting Ext-algebra as the *mixed quiver Schur algebra*. The mixed quiver Schur algebra has similar structural properties to the ordinary quiver Schur algebra: it has a Bott-Samelson basis (Theorem 5.5.21) and is generated by elementary merges, elementary splits and polynomials (Corollary 5.5.22). Our main result about mixed quiver Schur algebras (Theorem D in §5.1) establishes a connection between them and a certain module over the CoHA, called the cohomological Hall module (CoHM), introduced by Young [146]. As an application, we obtain an explicit description of the faithful polynomial representation of the mixed quiver Schur algebra, and reinterpret the action of the CoHA on the CoHM in terms of Demazure operators of type A-D.



Chapter 2

Rational Cherednik algebras

2.1 Summary of key definitions and facts

In this section we recall the relevent definitions and facts concerning rational Cherednik algebras. For an exhaustive treatment of rational Cherednik algebras, we refer the reader to, e.g., [9, 49]. We work over the field of complex numbers throughout.

2.1.1. Definition of rational Cherednik algebras. We start by recalling the definition of complex reflection groups.

Definition 2.1.1. Let $\mathfrak{h}$ be a finite dimensional complex vector space. An element $s \in GL(\mathfrak{h})$ is called a *complex reflection* if $\ker(s - \mathrm{Id}_{\mathfrak{h}})$ has codimension one in $\mathfrak{h}$. A *complex reflection group* is a finite subgroup of $GL(\mathfrak{h})$ generated by complex reflections.

Complex reflections groups were classified into one infinite family $G(l, p, n)$ and 34 exceptional cases by Shephard and Todd. For more information about complex reflection groups, we refer the reader to, e.g., [24].

Given a complex reflection group $G \subset GL(\mathfrak{h})$, let $\mathcal{S}$ be the set of complex reflections in G and let $\mathbb{C}[\mathcal{S}]^G$ be the set of functions $\mathbf{c}\colon \mathcal{S} \to \mathbb{C}$ invariant under conjugation, i.e.,

$$\mathbf{c}(gsg^{-1}) = \mathbf{c}(s)$$

for all $s \in \mathcal{S}$ and $g \in G$. Let G act diagonally on $\mathfrak{h} \times \mathfrak{h}^*$. For each $s \in \mathcal{S}$, fix non-zero elements $\alpha_s \in \mathrm{Im}(s-1)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathrm{Im}(s-1)|_{\mathfrak{h}}$ satisfying $\alpha_s(\alpha_s^\vee) = 2$.

Definition 2.1.2 ([48])**.** The *rational Cherednik algebra* $\mathtt{H}_{t,\mathbf{c}}(G)$, associated to the complex reflection group G and parameters $t \in \mathbb{C}$, $\mathbf{c} \in \mathbb{C}[\mathcal{S}]$, is the quotient of the cross-product $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[G]$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = t \cdot x(y) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)(y, \alpha_s)(\alpha_s^\vee, x)s,$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$.

2.1.2. Main properties of rational Cherednik algebras. We will now recall a number of fundamental results about the structure of rational Cherednik algebras.

We start with the so-called PBW theorem. Setting

$$\deg x = \deg y = 1, \quad \deg g = 0$$

for each $x \in \mathfrak{h}^*, y \in \mathfrak{h}$ and $g \in G$ defines a filtration on $\mathtt{H}_{t,\mathbf{c}}(G)$. Let $\mathsf{gr}\, \mathtt{H}_{t,\mathbf{c}}(G)$ be the associated graded algebra.

Theorem 2.1.3. *The tautological embedding $\mathfrak{h} \oplus \mathfrak{h}^* \hookrightarrow \mathsf{gr}\, \mathtt{H}_{t,\mathbf{c}}(G)$ extends to a graded algebra isomorphism*

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes \mathbb{C}[G] \xrightarrow{\sim} \mathsf{gr}\, \mathtt{H}_{t,\mathbf{c}}(G) \tag{2.1}$$

called the PBW isomorphism.

Proof. See [48, Theorem 1.3]. □

The behaviour of rational Cherednik algebras depends crucially on the parameter t, which controls the centre of $\mathtt{H}_{t,\mathbf{c}}(G)$. The following theorem collects the most important results about $Z(\mathtt{H}_{t,\mathbf{c}}(G))$.

Theorem 2.1.4. *The following hold.*

a) *We have: $Z(\mathtt{H}_{t,\mathbf{c}}(G)) = \mathbb{C}$ if and only if $t \neq 0$.*
b) *There is an inclusion $\mathbb{C}[\mathfrak{h}]^G \otimes \mathbb{C}[\mathfrak{h}^*]^G \subset Z(\mathtt{H}_{0,\mathbf{c}}(G))$. The algebra $Z(\mathtt{H}_{0,\mathbf{c}}(G))$ is a free $\mathbb{C}[\mathfrak{h}]^G \otimes \mathbb{C}[\mathfrak{h}^*]^G$-module of rank $|G|$.*
c) *The PBW isomorphism restricts to an isomorphism*

$$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^G \xrightarrow{\sim} \mathsf{gr}\, Z(\mathtt{H}_{0,\mathbf{c}}(G)).$$

d) *There is an algebra isomorphism, called the Satake isomorphism,*

$$Z(\mathtt{H}_{0,\mathbf{c}}(G)) \xrightarrow{\sim} e\mathtt{H}_{0,\mathbf{c}}(G)e, \quad z \mapsto z \cdot e, \tag{2.2}$$

where $e = \frac{1}{|G|} \sum_{g \in G} g$ is the trivial (or symmetrizing) idempotent.
e) *If* $\operatorname{Spec} Z(\mathtt{H}_{0,\mathbf{c}}(G))$ *is smooth then the functor*

$$\mathtt{H}_{0,\mathbf{c}}(G)\text{-}\mathsf{mod} \to e\mathtt{H}_{0,\mathbf{c}}(G)e\text{-}\mathsf{mod}, \quad M \mapsto e \cdot M, \tag{2.3}$$

is an equivalence of categories.

Proof. Part a) is [23, Proposition 7.2], part b) is [48, Proposition 4.15], part c) is [48, Theorem 3.3], and part d) is [48, Theorem 3.1]. For part e), see the proof of [48, Proposition 3.8] and the remark following it. □

2.2 Rational Cherednik algebras of type $G(l, 1, n)$

In this thesis we are primarily interested in rational Cherednik algebras associated to complex reflection groups of type $G(l, 1, n)$ at $t = 0$. Below we recall their main properties.

2.2.1. Partitions and multipartitions. We must first introduce some combinatorics. Let k be a non-negative integer. A partition λ of k is an infinite non-increasing sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ of non-negative integers such that $\sum_{i=1}^{\infty} \lambda_i = k$. We write $|\lambda| = k$ and denote the set of all partitions of k by $\mathcal{P}(k)$. Let $\ell(\lambda)$ be the positive integer i such that $\lambda_i \neq 0$ but $\lambda_{i+1} = 0$. We say that $\mu = (\mu_1, \mu_2, \mu_3, \dots)$

is a subpartition of λ if μ is a partition of some positive integer $m \le k$ and $\mu_i \le \lambda_i$ for all $i = 1, 2, \ldots$. A subpartition μ of λ is called a restriction of λ, denoted $\mu \uparrow \lambda$, if $|\mu| = k - 1$. Let $\varnothing = (0, 0, \ldots)$ denote the empty partition.

An l-composition α of k is an l-tuple $(\alpha_0, \ldots, \alpha_{l-1})$ of non-negative integers such that $\sum_{i=0}^{l-1} \alpha_i = k$. An l-multipartition $\underline{\lambda}$ of k is an l-tuple $(\lambda^0, \ldots, \lambda^{l-1})$ such that each λ^i is a partition and $\sum_{i=0}^{l-1} |\lambda^i| = k$. We consider the upper indices modulo l. Let $\mathcal{P}(l,k)$ denote the set of l-multipartitions of k. We say that $\underline{\mu} = (\mu^0, \ldots, \mu^{l-1})$ is a submultipartition of $\underline{\lambda}$ if each μ^i is a subpartition of λ^i. We call a submultipartition $\underline{\mu}$ of $\underline{\lambda}$ a restriction of $\underline{\lambda}$, denoted $\underline{\mu} \uparrow \underline{\lambda}$, if $\sum_{i=0}^{l-1} |\mu^i| = k - 1$.

If λ is a partition we denote its transpose by λ^t. If $\underline{\lambda} = (\lambda^0, \ldots, \lambda^{l-1}) \in \mathcal{P}(l,k)$, we call $\underline{\lambda}^t = ((\lambda^0)^t, \ldots, (\lambda^{l-1})^t)$ the *transpose* multipartition and $\underline{\lambda}^\flat := (\lambda^{l-1}, \lambda^{l-2}, \ldots, \lambda^0)$ the *reverse* multipartition. Finally, we set

$$\mathcal{P} := \bigsqcup_{k \in \mathbb{Z}_{\ge 0}} \mathcal{P}(k), \quad \underline{\mathcal{P}} := \bigsqcup_{k \in \mathbb{Z}_{\ge 0}} \mathcal{P}(l,k).$$

2.2.2. Wreath products. Let us fix once and for all two positive integers n, l. We regard the symmetric group S_n as the group of permutations of the set $\{1, \ldots, n\}$. For $1 \le i < j \le n$ let $s_{i,j}$ denote the transposition swapping numbers i and j. We abbreviate $s_i = s_{i,i+1}$ for $i = 1, \ldots, n-1$. Let $C_l := \mathbb{Z}/l\mathbb{Z} = \langle \epsilon \rangle$ and set

$$\Gamma_n := C_l \wr S_n = (C_l)^n \rtimes S_n,$$

the wreath product of C_l and S_n. It is a complex reflection group of type $G(l,1,n)$. For $1 \le i \le n$ and $1 \le j \le l-1$ let ϵ_i^j denote the element $(1, \ldots, 1, \epsilon^j, 1, \ldots, 1) \in (C_l)^n$ which is non-trivial only in the i-th coordinate. Let

$$e_n := (l^n n!)^{-1} \sum_{g \in \Gamma_n} g$$

be the symmetrizing idempotent and let triv denote the trivial Γ_n-module.

We regard S_{n-1} as the subgroup of S_n generated by the transpositions $s_2, \ldots, s_{n-1}$. We also regard $(C_l)^{n-1}$ as a subgroup of $(C_l)^n$ consisting of elements whose first coordinate is equal to one. This determines an embedding $\Gamma_{n-1} \hookrightarrow \Gamma_n$. Note that $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} = e_{n-1}\mathbb{C}\Gamma_n$ and $|(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}| = nl$.

Isomorphism classes of irreducible Γ_n-modules are naturally parametrized by $\mathcal{P}(l,n)$. We use the parametrization given in [115, §6.1.1]. Let $S(\underline{\lambda})$ denote the irreducible Γ_n-module corresponding to the l-multipartition $\underline{\lambda}$. We will later need the following branching rule [108, Theorem 10].

Proposition 2.2.1. *Let $\underline{\lambda} \in \mathcal{P}(l,n)$. Then*

$$S(\underline{\lambda})|_{\Gamma_{n-1}} := \operatorname{Res}^{\Gamma_n}_{\Gamma_{n-1}} S(\underline{\lambda}) = \bigoplus_{\underline{\mu} \uparrow \underline{\lambda}} S(\underline{\mu}).$$

2.2.3. Rational Cherednik algebras of type $G(l,1,n)$. We recall an explicit definition of the rational Cherednik algebra of type $G(l,1,n)$. Set $\eta := e^{2\pi i/l}$. Let $\mathfrak{h}$ be the n-dimensional representation of Γ_n with basis $y_1, \ldots, y_n$ such that $\epsilon_i \sigma . y_j = \eta^{-\delta_{i,\sigma(j)}} y_{\sigma(j)}$ for any $\sigma \in S_n$. Let $x_1, \ldots, x_n$ be the dual basis of $\mathfrak{h}^*$.

Definition 2.2.2. Let $G = \Gamma_n$. We will use a different parametrization of the rational Cherednik algebra from the standard one introduced in Definition 2.1.1. Since there are l conjugacy classes of reflections in $\mathcal{S}$, we identify the parameter space with $\mathbb{C}^l$ via the rule:

$$\mathbb{C}^l \xrightarrow{\sim} \mathbb{C}[\mathcal{S}]^{\Gamma_n}, \quad \mathbf{h} \mapsto \mathbf{c_h}, \quad \mathbf{c_h}(\epsilon^k) = \sum_{m=0}^{l-1} \eta^{-mk} H_m, \quad \mathbf{c_h}(s_1) = h,$$

where $\mathbf{h} = (h, H_1, \ldots, H_{l-1}) \in \mathbb{Q}^l$ and $H_0 = -(H_1 + \ldots + H_{l-1})$. The rational Cherednik algebra $\mathtt{H}_{t,\mathbf{h}}$ associated to Γ_n is the quotient of the cross-product $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}\Gamma_n$ by the relations

- $[x_i, x_j] = [y_i, y_j] = 0$ for all $1 \le i, j \le n$,
- $[x_i, y_j] = -h \sum_{k=0}^{l-1} \eta^k s_{i,j} \epsilon_i^k \epsilon_j^{-k}$ for all $1 \le i \neq j \le n$,
- $[x_i, y_i] = t + h \sum_{j \neq i} \sum_{k=0}^{l-1} s_{i,j} \epsilon_i^k \epsilon_j^{-k} + \sum_{k=0}^{l-1} (\sum_{m=0}^{l-1} \eta^{-mk} H_m) \epsilon_i^k$ for all $1 \le i \le n$.

We abbreviate $\mathtt{H}_\mathbf{h} := \mathtt{H}_{0,\mathbf{h}}$ and $\mathtt{Z}_\mathbf{h} = Z(\mathtt{H}_\mathbf{h})$.

Example 2.2.3. When $G = S_n$, the relations simplify to:

- $[x_i, x_j] = [y_i, y_j] = 0 \ \ (1 \le i, j \le n)$,
- $[x_i, y_j] = -h s_{i,j} \ \ (1 \le i \neq j \le n)$,
- $[x_i, y_i] = -t + h \sum_{j \neq i} s_{i,j} \ \ (1 \le i \le n)$.

2.2.4. The restricted rational Cherednik algebra. From now on we set $t = 0$. There is a $\mathbb{C}^*$-action on $\mathtt{H}_\mathbf{h}$ defined by the rule

$$z.x_i = zx_i, \quad z.y_i = z^{-1} y_i, \quad z.g = g,$$

where $1 \le i \le n$, $g \in \Gamma_n$ and $z \in \mathbb{C}^*$. This action defines a $\mathbb{Z}$-grading on $\mathtt{H}_\mathbf{h}$ such that $\deg x_i = 1$, $\deg y_i = -1$ and $\deg g = 0$. The action restricts to actions on $e_n \mathtt{H}_\mathbf{h} e_n$ and $\mathtt{Z}_\mathbf{h}$, with respect to which (2.2) is equivariant.

Notation 1. *Given a $\mathbb{Z}$-graded vector space V with finite-dimensional homogeneous components, let* $\mathrm{ch}_q V \in \mathbb{Z}[[q, q^{-1}]]$ *denote its Poincaré series (or, equivalently, its $\mathbb{C}^*$-character).*

Definition 2.2.4. Let $\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n}$ (resp. $\mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n}$) denote the ideal of $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$ (resp. $\mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$) generated by homogeneous elements of positive (resp. negative) degree, in the grading defined by the $\mathbb{C}^*$-action on $\mathtt{H}_\mathbf{h}$. The quotient

$$\overline{\mathtt{H}}_\mathbf{h} := \mathtt{H}_\mathbf{h} / \langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} + \mathbb{C}[\mathfrak{h}^*]_-^{\Gamma_n} \rangle$$

is called the *restricted rational Cherednik algebra*. It is a finite-dimensional algebra.

Let $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n} := \mathbb{C}[\mathfrak{h}] / \mathbb{C}[\mathfrak{h}].\mathbb{C}[\mathfrak{h}]_+^{\Gamma_n}$ be the algebra of coinvariants with respect to the Γ_n-action. It follows from the PBW theorem for rational Cherednik algebras [48, Theorem 1.3] that there is an isomorphism of graded vector spaces $\overline{\mathtt{H}}_\mathbf{h} \cong \mathbb{C}[\mathfrak{h}]^{co\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \otimes \mathbb{C}\Gamma_n$. Moreover, $\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n$ is a subalgebra of $\overline{\mathtt{H}}_\mathbf{h}$.

Definition 2.2.5. Let $\underline{\lambda} \in \mathcal{P}(l, n)$. The irreducible $\mathbb{C}\Gamma_n$-module $S(\underline{\lambda})$ becomes a module over $\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n$ by means of the projection $\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n \to \mathbb{C}\Gamma_n$. The *baby Verma module* associated to λ is the induced module

$$\bar{\Delta}(\underline{\lambda}) := \overline{\mathtt{H}}_\mathbf{h} \otimes_{\mathbb{C}[\mathfrak{h}^*]^{co\Gamma_n} \rtimes \mathbb{C}\Gamma_n} S(\underline{\lambda}).$$

We consider $\bar{\Delta}(\underline{\lambda})$ as a graded $\overline{\mathtt{H}}_\mathbf{h}$-module with $1 \otimes S(\underline{\lambda})$ in degree 0.

Proposition 2.2.6. *Let $\underline{\lambda} \in \mathcal{P}(l,n)$. The baby Verma module $\bar{\Delta}(\underline{\lambda})$ is indecomposable with simple head $L(\underline{\lambda})$. Moreover, $\{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathcal{P}(l,n)\}$ form a complete and irredundant set of representatives of isomorphism classes of graded simple $\overline{\mathtt{H}}_{\mathbf{h}}$-modules, up to a grading shift.*

Proof. See [63, Proposition 4.3]. □

2.2.5. The variety $\mathcal{Y}_{\mathbf{h}}$.

Let
$$\mathcal{Y}_{\mathbf{h}} := \operatorname{Spec} \mathtt{Z}_{\mathbf{h}}.$$

We will always assume that the parameter $\mathbf{h}$ is chosen so that the variety $\mathcal{Y}_{\mathbf{h}}$ is *smooth.* A criterion for smoothness can be found in e.g. [64, Lemma 4.3].

Let $\operatorname{Irrep}(\mathtt{H}_{\mathbf{h}})$ denote the set of isomorphism classes of irreducible representations of $\mathtt{H}_{\mathbf{h}}$. If $[M] \in \operatorname{Irrep}(\mathtt{H}_{\mathbf{h}})$, let $\chi_M : \mathtt{Z}_{\mathbf{h}} \to \mathbb{C}$ denote the character by which $\mathtt{Z}_{\mathbf{h}}$ acts on M. By [48, Theorem 1.7], there is a bijection
$$\operatorname{Irrep}(\mathtt{H}_{\mathbf{h}}) \longleftrightarrow \operatorname{MaxSpec} \mathtt{Z}_{\mathbf{h}}, \quad [M] \mapsto \ker \chi_M. \tag{2.4}$$

We are now going to recall another description of $\mathcal{Y}_{\mathbf{h}}$.

Definition 2.2.7. Let $\operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ be the variety of all algebra homomorphisms $\mathtt{H}_{\mathbf{h}} \to \operatorname{End}_{\mathbb{C}}(\mathbb{C}\Gamma_n)$ whose restriction to $\mathbb{C}\Gamma_n \subset \mathtt{H}_{\mathbf{h}}$ is the $\mathbb{C}\Gamma_n$-action by left multiplication, i.e., the regular representation. This is an affine algebraic variety.

Let $\phi \in \operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$. The one-dimensional vector space $e_n\mathbb{C}\Gamma_n$ is stable under all the endomorphisms in $\phi(e_n\mathtt{H}_{\mathbf{h}}e_n)$. Therefore, $\phi|_{e_n\mathtt{H}_{\mathbf{h}}e_n}$ composed with the Satake isomorphism (see Theorem 2.1.4.d) yields an algebra homomorphism $\chi_\phi : \mathtt{Z}_{\mathbf{h}} \cong e_n\mathtt{H}_{\mathbf{h}}e_n \to \operatorname{End}_{\mathbb{C}}(e_n\mathbb{C}\Gamma_n) \cong \mathbb{C}$. We obtain in this way a morphism of algebraic varieties
$$\pi : \operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}}) \to \mathcal{Y}_{\mathbf{h}}, \quad \phi \mapsto \ker \chi_\phi. \tag{2.5}$$

The $\mathbb{C}^*$-action on $\mathtt{H}_{\mathbf{h}}$ induces $\mathbb{C}^*$-actions on the varieties $\operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ and $\mathcal{Y}_{\mathbf{h}}$, with respect to which π is equivariant.

Let $\operatorname{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ be the group of $\mathbb{C}$-linear Γ_n-equivariant automorphisms of $\mathbb{C}\Gamma_n$. The group $\operatorname{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ acts naturally on $\operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$: if $g \in \operatorname{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n)$ and $\phi \in \operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ then $(g.\phi)(z) = g\phi(z)g^{-1}$, for all $z \in \mathtt{H}_{\mathbf{h}}$. By [48, Theorem 3.7], there exists an irreducible component $\operatorname{Rep}^o_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ of $\operatorname{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ such that (2.5) induces a $\mathbb{C}^*$-equivariant isomorphism of algebraic varieties
$$\operatorname{Rep}^o_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}}) /\!\!/ \operatorname{Aut}_{\Gamma_n}(\mathbb{C}\Gamma_n) \xrightarrow{\sim} \mathcal{Y}_{\mathbf{h}}. \tag{2.6}$$

Next, consider the $(\mathtt{H}_{\mathbf{h}}, e_n\mathtt{H}_{\mathbf{h}}e_n)$-bimodule $\mathtt{H}_{\mathbf{h}}e_n$ together with the $\mathbb{C}^*$-action inherited from $\mathtt{H}_{\mathbf{h}}$. The bimodule $\mathtt{H}_{\mathbf{h}}e_n$ defines a $\mathbb{C}^*$-equivariant coherent sheaf $\widetilde{\mathtt{H}_{\mathbf{h}}e_n}$ on $\operatorname{Spec} e_n\mathtt{H}_{\mathbf{h}}e_n \cong \mathcal{Y}_{\mathbf{h}}$. Since we are assuming that $\mathcal{Y}_{\mathbf{h}}$ is smooth, [48, Theorem 1.7] implies that this sheaf is locally free.

Definition 2.2.8. Let $\mathcal{R}_{\mathbf{h}}$ denote the $\mathbb{C}^*$-equivariant vector bundle whose sheaf of sections is $\widetilde{\mathtt{H}_{\mathbf{h}}e_n}$.

The group Γ_n acts naturally on every fibre of $\mathcal{R}_{\mathbf{h}}$ from the left. Let $\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}} = e_{n-1}\mathcal{R}_{\mathbf{h}}$ be the subbundle of $\mathcal{R}_{\mathbf{h}}$ consisting of Γ_{n-1}-invariants and let $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}}$ denote its fibre at $\chi_{\underline{\lambda}}$.

2.2.6. $\mathbb{C}^*$-fixed points.

Let us recall from [63] the classification of $\mathbb{C}^*$-fixed points in $\mathcal{Y}_{\mathbf{h}}$ in terms of l-multipartitions of n. By [48, Proposition 4.15], the subalgebra $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$ of $\mathtt{H}_{\mathbf{h}}$ is contained in $\mathtt{Z}_{\mathbf{h}}$ and $\mathtt{Z}_{\mathbf{h}}$ is a free $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n}$-module of rank $|\Gamma_n|$. The inclusion $\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \otimes \mathbb{C}[\mathfrak{h}^*]^{\Gamma_n} \hookrightarrow \mathtt{Z}_{\mathbf{h}}$

induces a $\mathbb{C}^*$-equivariant morphism of algebraic varieties

$$\Upsilon \colon \mathcal{Y}_{\mathbf{h}} \to \mathfrak{h}/\Gamma_n \times \mathfrak{h}^*/\Gamma_n.$$

The only $\mathbb{C}^*$-fixed point in $\mathfrak{h}/\Gamma_n \times \mathfrak{h}^*/\Gamma_n$ is 0. Since the group $\mathbb{C}^*$ is connected and the fibre $\Upsilon^{-1}(0)$ is finite, $\mathcal{Y}_{\mathbf{h}}^{\mathbb{C}^*} = \Upsilon^{-1}(0)$. By Theorem 5.6 in [63], there is a bijection between the closed points of $\Upsilon^{-1}(0)$ and isomorphism classes of simple modules over the restricted rational Cherednik algebra $\overline{\mathtt{H}}_{\mathbf{h}}$. Hence there is a bijection

$$\mathcal{P}(l,n) \longleftrightarrow (\operatorname{MaxSpec} \mathtt{Z}_{\mathbf{h}})^{\mathbb{C}^*}, \quad \underline{\lambda} \mapsto \ker \chi_{L(\underline{\lambda})}.$$

We will also write $\chi_{\underline{\lambda}}$ for $\chi_{L(\underline{\lambda})}$.

Chapter 3

Rational Cherednik algebras and Hilbert schemes

3.1 Introduction

In this chapter we consider rational Cherednik algebras $\mathtt{H}_{\mathbf{h}} := \mathtt{H}_{t=0,\mathbf{h}}(\Gamma_n)$ associated to complex reflection groups $\Gamma_n := (\mathbb{Z}/l\mathbb{Z}) \wr S_n$ of type $G(l,1,n)$ at $t=0$.

3.1.1. Rational Cherednik algebras and quiver varieties. As we explained in the general introduction to this thesis, the rational Cherednik algebra $\mathtt{H}_{\mathbf{h}}$ has a large centre, which can be realized as the coordinate ring of a certain Nakajima quiver variety. This fact was used by Gordon [64] to establish a connection between generalized Calogero-Moser spaces and Hilbert schemes of points in the plane. We will now look at this connection in more detail.

Assume that the variety $\mathcal{Y}_{\mathbf{h}} = \operatorname{Spec} Z(\mathtt{H}_{\mathbf{h}})$ is smooth. Etingof and Ginzburg showed in [48] that $\mathcal{Y}_{\mathbf{h}}$ is isomorphic to a cyclic quiver variety $\mathcal{X}_\theta(n\delta)$ (see §3.2.2) generalizing Wilson's construction of the Calogero-Moser space in [142]. Considering $\mathcal{X}_\theta(n\delta)$ as a hyper-Kähler manifold, one can use reflection functors, defined by Nakajima in [103], to construct a hyper-Kähler isometry $\mathcal{X}_\theta(n\delta) \to \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$ between quiver varieties associated to different parameters. Furthermore, rotation of complex structure yields a diffeomorphism between $\mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$ and a certain GIT quotient $\mathcal{M}_{-\mathbf{1}}(\gamma)$ (see §3.2.2). The latter is isomorphic to an irreducible component $\operatorname{Hilb}_K^\nu$ of $\operatorname{Hilb}_K^{\mathbb{Z}/l\mathbb{Z}}$, where Hilb_K denotes the Hilbert scheme of K points in $\mathbb{C}^2$. The following diagram summarizes all the maps involved:

$$\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{X}_\theta(n\delta) \xrightarrow{\mathsf{Refl.Fun.}} \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) \xrightarrow{\mathsf{Rotation}} \mathcal{M}_{-\mathbf{1}}(\gamma) \hookrightarrow \operatorname{Hilb}_K^{\mathbb{Z}/l\mathbb{Z}}. \tag{3.1}$$

Let us explain the parameters. The affine symmetric group $\tilde{S}_l$ acts on dimension vectors and the parameter space associated to the cyclic quiver with l vertices (see §3.3.8, §3.3.9). We apply this action to the dimension vector $n\delta$ and the parameter $-\frac{\mathbf{1}}{\mathbf{2}} := -\frac{1}{2l}(1,\dots,1)$. Fix $w \in \tilde{S}_l$ and set $\theta := w^{-1}\cdot(-\frac{\mathbf{1}}{\mathbf{2}})$ and $\gamma := w * n\delta$. Then $\gamma = n\delta + \gamma_0$, where γ_0 is the l-residue of a uniquely determined l-core partition ν. Set $K := nl + |\nu|$. The relation between the parameters $\mathbf{h}$ and θ is explained in §3.2.3.

Both $\mathcal{Y}_{\mathbf{h}}$ and Hilb_K carry natural $U(1)$-actions with respect to which (3.1) is equivariant. As we saw in §2.2.6, the closed $\mathbb{C}^*$-fixed points in $\mathcal{Y}_{\mathbf{h}}$ are labelled by l-multipartitions of n. On the other hand, the $\mathbb{C}^*$-fixed points in Hilb_K correspond to monomial ideals in $\mathbb{C}[x,y]$ of colength K and are therefore labelled by partitions of K. In particular, the $\mathbb{C}^*$-fixed points in $\operatorname{Hilb}_K^\nu$ are labelled by partitions of K

with l-core ν. Since (3.1) is equivariant, it induces a bijection

$$\mathcal{P}(l,n) \longleftrightarrow \mathcal{P}_\nu(K), \tag{3.2}$$

where $\mathcal{P}_\nu(K)$ denotes the set of partitions of K with l-core ν and $\mathcal{P}(l,n)$ the set of l-multipartitions of n.

3.1.2. Quick summary of the main results and applications. Our main result is an explicit combinatorial description of (3.2). We note that such a description already appeared in [64], but its proof in *loc. cit.* is incorrect (see Remark 3.9.5). Our result has a number of interesting applications. For example, we use it to establish a higher-level version of the q-hook formula. Our result has also recently been used by Bonnafé and Maksimau [18] in their study of fixed-point subvarieties in Calogero-Moser spaces. Moreover, as we explain in §3.1.6, the combinatorial description of (3.2) is a key ingredient in several older results, such as Bezrukavnikov and Finkelberg's [16], as well as Losev's [95], proofs of Haiman's wreath Macdonald positivity conjecture.

3.1.3. Main results. Our first result gives a classification as well as an explicit description of $\mathbb{C}^*$-fixed points in quiver varieties associated to the cyclic quiver. We also consider tautological bundles on these varieties and calculate the characters of their fibres at the $\mathbb{C}^*$-fixed points.

Theorem A. *Let $u \in \tilde{S}_l$, $\xi := u * n\delta = n\delta + \xi_0$ and let ω be the transpose of the l-core corresponding to ξ_0. Set $L := nl + |\omega|$. Let $\alpha \in \mathbb{Q}^l$ be any parameter such that $\mathcal{X}_\alpha(\xi)$ is smooth. Let $\mathcal{V}_\alpha(\xi)$ denote the tautological bundle on $\mathcal{X}_\alpha(\xi)$ (see §3.2.2). Then:*

a) *The $\mathbb{C}^*$-fixed points in $\mathcal{X}_\alpha(\xi)$ are naturally labelled by $\mathcal{P}_\omega(L)$. We construct them explicitly as equivalence classes of quiver representations.*
b) *Let $\mu \in \mathcal{P}_\omega(L)$. Then the $\mathbb{C}^*$-character of the fibre of $\mathcal{V}_\alpha(\xi)$ at μ is given by*

$$\mathrm{ch}_q\, \mathcal{V}_\alpha(\xi)_\mu = \mathrm{Res}_\mu(q) := \sum_{\square \in \mu} q^{c(\square)}.$$

This theorem combines the results of Theorem 3.4.14, Proposition 3.4.15 and Corollary 3.4.16 below. Let us briefly explain our description of the $\mathbb{C}^*$-fixed points. To each partition $\mu \in \mathcal{P}_\omega(L)$ we associate a quadruple of matrices depending on α, ξ and the Frobenius form of μ. Our construction can be regarded as a generalization of Wilson's description of the $\mathbb{C}^*$-fixed points in the (classical) Calogero-Moser space [142, Proposition 6.11]. We also expect that, using an appropriate functor from the category of representations of the infinite linear quiver to that of the cyclic quiver, one can relate our constructions to earlier work on quiver varieties of type A_∞ [58, 118].

Our second result describes the bijection between the $\mathbb{C}^*$-fixed points induced by the Etingof-Ginzburg isomorphism.

Theorem B (Theorem 3.6.18)**.** *The map $\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{X}_\theta(n\delta)$ induces a bijection*

$$\mathcal{P}(l,n) \to \mathcal{P}_\varnothing(nl), \quad \underline{\mathsf{Quot}}(\mu)^\flat \mapsto \mu,$$

where $\underline{\mathsf{Quot}}(\mu)^\flat$ denotes the reverse of the l-quotient of μ (see §2.2.1, §3.3.5) and $\varnothing$ is the empty partition.

The proof of Theorem B occupies sections 3.5 and 3.6. We use the Dunkl-Opdam subalgebra of $\mathtt{H}_{\mathbf{h}}$

to construct a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_{\mathbf{h}} & \xrightarrow{\quad \mathsf{EG} \quad} & \mathcal{X}_\theta(n\delta) \\ & {\scriptstyle \rho_1} \searrow \quad \swarrow {\scriptstyle \rho_2} & \\ & \mathbb{C}^n/S_n & \end{array}$$

where ρ_1 sends a fixed point labelled by $\underline{\lambda}$ to its residue and ρ_2 sends a quiver representation to a certain subset of its eigenvalues. Given a partition μ, we use our description of the corresponding fixed point from Theorem A to obtain an explicit formula for $\rho_2(\mu)$ in terms of the Frobenius form of μ. We then use a combinatorial argument to show that this formula describes the residue of $\underline{\mathsf{Quot}}(\mu)^\flat$.

We next consider reflection functors, which were introduced by Nakajima [103] and also studied by Maffei [96] and Crawley-Boevey and Holland [28,29]. Assuming smoothness, reflection functors are $U(1)$-equivariant hyper-Kähler isometries between quiver varieties associated to different parameters. They satisfy Weyl group relations and have been used by Nakajima to define Weyl group representations on homology groups of quiver varieties.

Let us explain the role reflection functors play in our setting. To each simple reflection $\sigma_i \in \tilde{S}_l$, we associate a reflection functor $\mathfrak{R}_i : \mathcal{X}_\alpha(\xi) \to \mathcal{X}_{\sigma_i \cdot \alpha}(\sigma_i * \xi)$. One can show that $\sigma_i * \xi = n\delta + \sigma_i * \xi_0$, where $\sigma_i * \xi_0$ is the l-residue of a uniquely determined l-core ω'. Then the reflection functor $\mathfrak{R}_i$ induces a bijection between the labelling sets of $\mathbb{C}^*$-fixed points

$$\mathbf{R}_i : \mathcal{P}_\omega(L) \to \mathcal{P}_{(\omega')^t}(L'), \tag{3.3}$$

where $L' := nl + |\omega'|$.

Our third result gives a combinatorial description of this bijection. We use the action of $\tilde{S}_l$ on the set of all partitions defined by Van Leeuwen in [94]. This action involves combinatorial ideas reminiscent of those describing the $\hat{\mathfrak{sl}}_l$-action on the Fock space. More precisely, if μ is a partition then $\sigma_i * \mu$ is the partition obtained by simultaneously removing and adding all the removable (resp. addable) cells of content i mod l from (to) the Young diagram of μ. It is noteworthy that this action also plays a role in the combinatorics describing the Schubert calculus of the affine Grassmannian [92,93].

Theorem C (Theorem 3.8.11)**.** *Let* $\mu \in \mathcal{P}_\omega(L)$. *Then*

$$\mathbf{R}_i(\mu) = (\sigma_i * \mu^t)^t.$$

Combining Theorem B with (iterated applications of) Theorem C allows us to give an explicit combinatorial description of bijection (3.2).

Theorem D (Theorem 3.9.3)**.** *The map* (3.1) *induces the following bijections*

$$\begin{array}{ccccccc} \mathcal{Y}_{\mathbf{h}}^{\mathbb{C}^*} & \longrightarrow & \mathcal{X}_\theta(n\delta)^{\mathbb{C}^*} & \longrightarrow & \mathcal{X}_{-\frac{1}{2}}(\gamma)^{\mathbb{C}^*} & \longrightarrow & (\mathrm{Hilb}^\nu_K)^{\mathbb{C}^*} \\ \mathcal{P}(l,n) & \longrightarrow & \mathcal{P}_\varnothing(nl) & \longrightarrow & \mathcal{P}_{\nu^t}(K) & \longrightarrow & \mathcal{P}_\nu(K) \\ \underline{\mathsf{Quot}}(\mu)^\flat & \longmapsto & \mu & \longmapsto & (w * \mu^t)^t & \longmapsto & w * \mu^t. \end{array}$$

Moreover, $\nu = w * \varnothing$.

Let us rephrase our result slightly. Given $w \in \tilde{S}_l$, we define the *w-twisted l-quotient bijection* to be the map

$$\tau_w : \mathcal{P}(l,n) \to \mathcal{P}_\nu(K), \quad \underline{\mathsf{Quot}}(\mu) \mapsto w * \mu.$$

Corollary E (Corollary 3.9.4)**.** *Bijection* (3.2) *is given by*

$$\underline{\lambda} \mapsto \tau_w(\underline{\lambda}^t).$$

3.1.4. Orderings on category $\mathcal{O}$. One of Gordon's motivations in [64] was to give a geometric interpretation of highest weight structures on category $\mathcal{O}_{\mathbf{h}}$ for rational Cherednik algebras $\mathtt{H}_{t=1,\mathbf{h}}(\Gamma_n)$ at $t = 1$. Consider the combinatorial ordering $\prec_{\mathbf{h}}^{\mathsf{com}}$ on $\mathcal{P}(l,n)$ defined by

$$\underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{com}} \underline{\lambda} \iff \tau_w(\underline{\lambda}^t) \trianglelefteq \tau_w(\underline{\mu}^t),$$

where $\trianglelefteq$ denotes the dominance ordering on partitions. It was shown by Dunkl and Griffeth [42, Theorem 1.2] that $\mathcal{O}_{\mathbf{h}}$ is a highest weight category with respect to this ordering. There is also a geometric ordering $\prec_{\mathbf{h}}^{\mathsf{geo}}$ on $\mathcal{P}(l,n)$, defined by the closure relations between the attracting sets of $\mathbb{C}^*$-fixed points in $\mathcal{M}_{2\theta}(n\delta)$. Using Corollary E and the results of Nakajima from [104] we deduce the following partial geometric interpretation of the combinatorial ordering.

Corollary F (Corollary 3.9.6)**.** *Let* $\underline{\mu}, \underline{\lambda} \in \mathcal{P}(l,n)$*. Then* $\underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{geo}} \underline{\lambda} \ \Rightarrow \ \underline{\mu} \preceq_{\mathbf{h}}^{\mathsf{com}} \underline{\lambda}$.

We remark that the statements of Corollaries E and F first appeared in [64] (see Proposition 7.10 and its proof). However, the proof of Proposition 7.10 in [64] is incorrect - see Remark 3.9.5 for an explanation.

3.1.5. The higher level q-hook formula. Our results have several interesting applications. One of them is a new proof as well as a generalization of the q-hook formula, which we now recall.

Given a partition $\mu \in \mathcal{P}(n)$, let d_μ be the number of standard Young tableaux of shape μ. It is equal to the dimension of the Specht module $S(\mu)$. The *hook length formula* (1.6) states that d_μ is related to the product of the lengths of all hooks in the corresponding Young diagram. Let us abbreviate $h_\mu = \prod_{\square \in \mu} h_\mu(\square)$. The branching rule from Proposition 2.2.1 implies that $d_\mu = \sum_{\lambda \uparrow \mu} d_\lambda$, i.e., d_μ is equal to the sum of dimensions of Specht modules associated to all the Young diagrams obtained from μ by deleting a single cell. Applying (1.6) to each d_λ, we get

$$\sum_{\lambda \uparrow \mu} \frac{(n-1)!}{h_\lambda} = \frac{n!}{h_\mu}.$$

After dividing both sides by $(n-1)!$ and rearranging the formula we obtain the formula:

$$n = \sum_{\lambda \uparrow \mu} \frac{h_\mu}{h_\lambda}. \tag{3.4}$$

This formula admits the following q-analogue:

$$\sum_{\square \in \mu} q^{c(\square)} = [n]_q \sum_{\lambda \uparrow \mu} \frac{f_\lambda(q)}{f_\mu(q)}, \tag{3.5}$$

called the *q-hook formula.* Here $c(\square)$ is the content of $\square$ and $f_\mu(q)$ is the fake degree polynomial associated to μ. The RHS of (3.5) can also be reformulated in terms of Schur functions and hook length polynomials. The q-hook formula has been proven by Kerov [86], Garsia and Haiman [60] and Chen and Stanley [33] using probabilistic, combinatorial and algebraic methods, respectively. We prove the following generalization.

Theorem G (Theorem 3.7.1)**.** *Let $\mu \in \mathcal{P}_{\varnothing}(nl)$. Then:*

$$\sum_{\square \in \mu} q^{c(\square)} = [nl]_t \sum_{\underline{\lambda} \uparrow \underline{\mathsf{Quot}}(\mu)^{\flat}} \frac{f_{\underline{\lambda}}(q)}{f_{\underline{\mathsf{Quot}}(\mu)^{\flat}}(q)}. \tag{3.6}$$

We call (3.6) the *higher-level* q-hook formula. Setting $l = 1$ we recover the classical q-hook formula. Our proof of Theorem G is geometric in nature. Let us briefly explain the main idea behind it. Let e_n denote the symmetrizing idempotent in Γ_n. The right $e_n \mathtt{H}_{\mathbf{h}} e_n$-module $\mathtt{H}_{\mathbf{h}} e_n$ defines a coherent sheaf on $\mathcal{Y}_{\mathbf{h}}$. Since we are assuming that the variety $\mathcal{Y}_{\mathbf{h}}$ is smooth, this sheaf is also locally free. Let $\mathcal{R}_{\mathbf{h}}$ denote the corresponding vector bundle. It was shown in [48] that there exists an isomorphism of vector bundles $\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}} \xrightarrow{\sim} \mathcal{V}_{\theta}(n\delta)$ lifting the Etingof-Ginzburg isomorphism $\mathcal{Y}_{\mathbf{h}} \xrightarrow{\sim} \mathcal{X}_{\theta}(n\delta)$. Let $\mu \in \mathcal{P}_{\varnothing}(nl)$. By Theorem B, the Etingof-Ginzburg map sends the fixed point labelled by $\underline{\mathsf{Quot}}(\mu)^{\flat}$ to the fixed point labelled by μ. We obtain the higher level q-hook formula (3.6) by comparing the $\mathbb{C}^*$-characters of the corresponding fibres $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\mathsf{Quot}}(\mu)^{\flat}}$ and $\mathcal{V}_{\theta}(n\delta)_{\mu}$.

3.1.6. Wreath Macdonald polynomials.

Let us mention a few other applications of our results. The first is related to Haiman's wreath Macdonald positivity conjecture. The original positivity conjecture, proven by Haiman in [69], asserts that Kostka-Macdonald polynomials, which express the change of basis between transformed Macdonald functions and Schur functions, have non-negative coefficients. Haiman [68] later proposed a generalized conjecture, known as the *wreath Macdonald positivity conjecture*, in which the ring of symmetric functions is replaced by the space of virtual characters $\Xi_{q,t}(\Gamma_n)$ of the group Γ_n with coefficients in $\mathbb{Q}(q,t)$. This conjecture was proven by Finkelberg and Bezrukavnikov [16] and Losev [95].

Theorem 3.1.1 ([16, 68, 95])**.** *Fix an l-core ν. Let $K = nl + |\nu|$. There exists a basis $\{H_\mu(q,t)\}$ of $\Xi_{q,t}(\Gamma_n)$ indexed by partitions $\mu \in \mathcal{P}_\nu(K)$, characterized by the following properties:*

a) $H_\mu(q,t) \otimes \sum_i (-q)^i \operatorname{Char}(\wedge^i \mathfrak{h}) \in \mathbb{Q}(q,t)\{\chi^{\underline{\mathsf{Quot}}(\lambda)} \mid \lambda \geq \mu, \lambda \in \mathcal{P}_\nu(K)\}$,

b) $H_\mu(q,t) \otimes \sum_i (-t)^{-i} \operatorname{Char}(\wedge^i \mathfrak{h}) \in \mathbb{Q}(q,t)\{\chi^{\underline{\mathsf{Quot}}(\lambda)} \mid \lambda \leq \mu, \lambda \in \mathcal{P}_\nu(K)\}$,

c) $\langle H_\mu(q,t), 1_{\Gamma_n} \rangle = 1$.

Moreover, the characters $H_\mu(q,t)$ have coefficients in $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]$ and are the graded characters of the fibers of the Procesi bundle on $\operatorname{Hilb}_K^\nu$ at the $\mathbb{C}^*$-fixed points.

We will now explain the role the description of the bijection (3.2) from Corollary E plays in the above-mentioned proofs of the wreath Macdonald positivity conjecture. The key step in Bezrukavnikov and Finkelberg's proof is a characterization of the support of certain Verma modules in positive characteristic [16, Proposition 2.6]. Losev's proof also relies on a calculation of the supports of certain quotients of Procesi bundles [95, Proposition 5.3]. The proofs of these two statements invoke [16, Lemma 3.8]. But the latter implicitly uses Corollary E (see also [16, §2.3]).

3.1.7. Other applications.

We mention two other applications of ours results. Gordon and Martino [65] gave a combinatorial description of the blocks of the restricted rational Cherednik algebra of type $G(l,1,n)$ (also for parameters $\mathbf{h}$ for which the corresponding Calogero-Moser space $\mathcal{Y}_{\mathbf{h}}$ is singular) in terms of J-classes of partitions. Corollary E is an important ingredient in their proof.

More recently, Bonnafé and Maksimau [18] studied the irreducible components of the fixed point subvariety under the action of a finite cyclic group on a smooth Calogero-Moser space. They use Theorem B to give an explicit description of these components for Calogero-Moser spaces of type $G(l,1,n)$.

3.1.8. Structure of the chapter. Let us summarize the contents of the chapter. In sections 2 and 3 we recall some standard material about Nakajima quiver varieties and combinatorics, respectively. In section 4 we explicitly construct the $\mathbb{C}^*$-fixed points in cyclic quiver varieties, and prove Theorem A. Sections 5 and 6 are devoted to proving Theorem B. In section 7 we prove the higher-level version of the q-hook formula (Theorem G). In section 8 we study the bijections between $\mathbb{C}^*$-fixed points induced by reflection functors, and prove Theorem C. In section 9 we make the connection to the Hilbert scheme and prove Theorem D as well as Corollaries E and F.

3.2 Quiver varieties

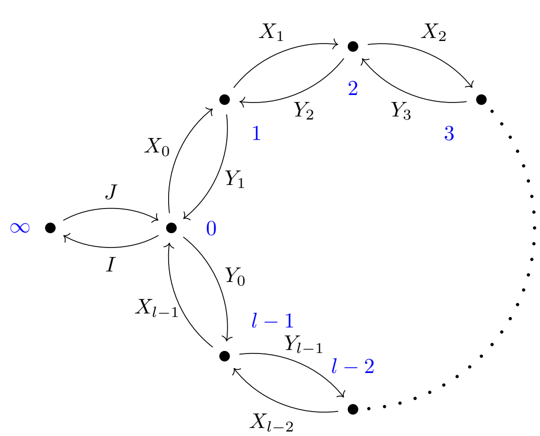
In this section we recall the connection between rational Cherednik algebras and cyclic quiver varieties via the Etingof-Ginzburg isomorphism.

3.2.1. The cyclic quiver. Let $\mathbf{Q}$ be the cyclic quiver with l vertices and a cyclic orientation. We label the vertices as $0, 1, \ldots, l-1$ (considered as elements of $\mathbb{Z}/l\mathbb{Z}$) in such a way that there is a (unique) arrow $i \to j$ if and only if $j = i+1$. Let $\overline{\mathbf{Q}}$ be the double of $\mathbf{Q}$, i.e., the quiver obtained from $\mathbf{Q}$ by adding, for each arrow a in $\mathbf{Q}$, an arrow a^* going in the opposite direction. Moreover, let $\mathbf{Q}_\infty$ be the quiver obtained from $\mathbf{Q}$ by adding an extra vertex, denoted ∞, and an extra arrow $a_\infty : \infty \to 0$. We write $\overline{\mathbf{Q}}_\infty$ for the double of $\mathbf{Q}_\infty$.

Let $\mathbf{d} = (d_0, \ldots, d_{l-1}) \in (\mathbb{Z}_{\geq 0})^l$. We interpret $\mathbf{d}$ as the dimension vector for $\overline{\mathbf{Q}}$ so that the dimension associated to the vertex i is d_i. For each $i = 0, \ldots, l-1$ let $\mathbf{V}_i$ be a complex vector space of dimension d_i. Set $\widehat{\mathbf{V}} := \bigoplus_{i=0}^{l-1} \mathbf{V}_i$. Moreover, let $\mathbf{V}_\infty$ be a one-dimensional complex vector space and set $\mathbf{V} := \mathbf{V}_\infty \oplus \widehat{\mathbf{V}}$. Define

$$\operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d}) := \bigoplus_{i=0}^{l-1} \operatorname{Hom}(\mathbf{V}_i, \mathbf{V}_{i+1}) \oplus \bigoplus_{i=0}^{l-1} \operatorname{Hom}(\mathbf{V}_i, \mathbf{V}_{i-1}) \oplus \operatorname{Hom}(\mathbf{V}_0, \mathbf{V}_\infty) \oplus \operatorname{Hom}(\mathbf{V}_\infty, \mathbf{V}_0).$$

We denote an element of $\operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ as $(\mathbf{X}, \mathbf{Y}, I, J) = (X_0, \ldots, X_{l-1}, Y_0, \ldots, Y_{l-1}, I, J)$ accordingly. There is a natural isomorphism of varieties $\operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d}) \cong T^* \operatorname{Rep}(\mathbf{Q}_\infty, \mathbf{d})$, through which we can equip $\operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ with a symplectic structure.



The algebraic group $G(\mathbf{d}) := \prod_{i=0}^{l-1} \operatorname{GL}(\mathbf{V}_i)$ acts on $\operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ by change of basis. If $\mathbf{g} = (g_0, \ldots, g_{l-1}) \in G(\mathbf{d})$ and $(\mathbf{X}, \mathbf{Y}, I, J) \in \operatorname{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ then

$$\mathbf{g}.(\mathbf{X}, \mathbf{Y}, I, J) = (g_1 X_0 g_0^{-1}, \ldots, g_0 X_{l-1} g_{l-1}^{-1}, g_{l-1} Y_0 g_0^{-1}, \ldots, g_{l-2} Y_{l-1} g_{l-1}^{-1}, I g_0^{-1}, g_0 J).$$

The action of $G(\mathbf{d})$ on $\mathrm{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ is Hamiltonian. The moment map for this action is given by

$$\mu_{\mathbf{d}} : \mathrm{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d}) \to \mathfrak{g}(\mathbf{d})^* \cong \mathfrak{g}(\mathbf{d}) := \mathrm{Lie}\, G(\mathbf{d}), \quad (\mathbf{X}, \mathbf{Y}, I, J) \mapsto [\mathbf{X}, \mathbf{Y}] + JI.$$

3.2.2. Quiver varieties. If $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^l$ and $\theta = (\theta_0, \ldots, \theta_{l-1}) \in \mathbb{Q}^l$, we will also write $\theta = (\theta_0 \,\mathrm{id}_0, \theta_1 \,\mathrm{id}_1, \ldots, \theta_{l-1} \,\mathrm{id}_{l-1}) \in \mathfrak{g}(\mathbf{d})$, where $\mathrm{id}_i = \mathrm{id}_{\mathbf{V}_i}$ $(i = 0, \ldots, l-1)$. Define

$$\mathcal{X}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(\theta) /\!\!/ G(\mathbf{d}) := \mathrm{Spec}\, \mathbb{C}[\mu_{\mathbf{d}}^{-1}(\theta)]^{G(\mathbf{d})}.$$

We will always assume that the parameter θ is chosen in such a way that the variety $\mathcal{X}_\theta(\mathbf{d})$ is smooth. Moreover, define the GIT quotient

$$\mathcal{M}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(0) /\!\!/_\theta G(\mathbf{d}) = \mathrm{Proj} \bigoplus_{i \geq 0} \mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_\theta^i},$$

where $\chi_\theta : G(\mathbf{d}) \to \mathbb{C}^*$ is the character sending $\mathbf{g}$ to $\prod (\det g_i)^{\theta_i}$ and $\mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_\theta^i}$ denotes the space of semi-invariant functions on $\mu_{\mathbf{d}}^{-1}(0)$, i.e., those functions f satisfying $\mathbf{g}.f = \chi_\theta^i(\mathbf{g}) f$. By definition, the space $\mathbb{C}[\mu_{\mathbf{d}}^{-1}(0)]^{\chi_\theta^i}$ is zero unless $i\theta \in \mathbb{Z}^l$.

Example 3.2.1. Let $l = 1$ so that $\mathbf{d} = d$ is just a non-negative integer. Then the variety $\mathcal{M}_{-1}(d)$ is naturally isomorphic to the Hilbert scheme of d points in the plane (see §3.9.1 for more details), and the natural map $\mathcal{M}_{-1}(d) \to \mathcal{M}_0(d) = \mathcal{X}_0(d) = \mathrm{Sym}^d(\mathbb{C}^2)$ can be identified with the Hilbert-Chow morphism. On the other hand, the variety $\mathcal{X}_1(d)$ is isomorphic to the classical Calogero-Moser space.

The varieties $\mathcal{X}_\theta(\mathbf{d})$ and $\mathcal{M}_\theta(\mathbf{d})$ can be endowed with hyper-Kähler structures (see e.g. [64, §3.6]). Moreover, the group $\mathbb{C}^*$ acts on $\mathrm{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$ by the rule $t.(\mathbf{X}, \mathbf{Y}, I, J) = (t^{-1}\mathbf{X}, t\mathbf{Y}, I, J)$ for $t \in \mathbb{C}^*$. This action descends to actions on $\mathcal{X}_\theta(\mathbf{d})$ and $\mathcal{M}_\theta(\mathbf{d})$. By $\mathcal{X}_\theta(\mathbf{d})^{\mathbb{C}^*}$ and $\mathcal{M}_\theta(\mathbf{d})^{\mathbb{C}^*}$ we will always mean the sets of *closed* $\mathbb{C}^*$-fixed points.

Let us recall the definition of the tautological bundle on a quiver variety. Assume that the group $G(\mathbf{d})$ acts freely on the fibre $\mu_{\mathbf{d}}^{-1}(\theta)$ and consider the trivial vector bundle $\widehat{\mathcal{V}}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(\theta) \times \widehat{\mathbf{V}}$ on $\mu_{\mathbf{d}}^{-1}(\theta)$. We regard $\widehat{\mathcal{V}}_\theta(\mathbf{d})$ as a $\mathbb{C}^*$-equivariant vector bundle by letting $\mathbb{C}^*$ act trivially on $\widehat{\mathbf{V}}$. Let $G(\mathbf{d})$ act diagonally on $\widehat{\mathcal{V}}_\theta(\mathbf{d})$. The vector bundle $\widehat{\mathcal{V}}_\theta(\mathbf{d})$ descends to a $\mathbb{C}^*$-equivariant vector bundle

$$\mathcal{V}_\theta(\mathbf{d}) := \mu_{\mathbf{d}}^{-1}(\theta) \times^{G(\mathbf{d})} \widehat{\mathbf{V}} = (\mu_{\mathbf{d}}^{-1}(\theta) \times \widehat{\mathbf{V}}) /\!\!/ G(\mathbf{d})$$

on $\mathcal{X}_\theta(\mathbf{d})$, which is called the *tautological bundle*.

Notation 2. *We will always consider the subscript i in the expressions $d_i, \mathbf{V}_i, g_i, X_i, Y_i, \theta_i$ modulo l (unless $i = \infty$).*

3.2.3. The Etingof-Ginzburg isomorphism. Throughout this subsection we assume $\mathbf{d} = n\delta$, where $\delta := (1, \ldots, 1) \in \mathbb{Z}^l$. Given $\mathbf{h}$ as in Definition 2.2.2, set

$$\theta_{\mathbf{h}} = (\theta_0, \ldots, \theta_{l-1}) = (-h + H_0, H_1, \ldots, H_{l-1}). \tag{3.7}$$

Since we are assuming that the parameter $\mathbf{h}$ is generic, the group $G(n\delta)$ acts freely on the fibre $\mu_{n\delta}^{-1}(\theta_{\mathbf{h}})$ (see [48, Proposition 11.11]). We abbreviate

$$\mathcal{C}_{\mathbf{h}} := \mathcal{X}_{\theta_{\mathbf{h}}}(n\delta), \quad \mathcal{V}_{\mathbf{h}} := \mathcal{V}_{\theta_{\mathbf{h}}}(n\delta).$$

Example 3.2.2. If $l = 1$ and $h = 1$, then

$$\mathcal{C}_{\mathbf{h}} = \{(X, Y, u, v) \in \mathrm{Mat}_{n\times n}(\mathbb{C})^{\oplus 2} \times \mathbb{C}^n \times (\mathbb{C}^n)^* \mid [X, Y] + JI = -\,\mathrm{id}\} /\!\!/ GL_n(\mathbb{C}).$$

This variety is the *classical Calogero-Moser space* from [142].

Consider $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}$ as a module over $\langle \epsilon_1 \rangle = \mathbb{Z}/l\mathbb{Z}$. It decomposes as a direct sum of n-dimensional isotypic components $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}_{\chi_i}$, where χ_i is the character $\chi_i : \epsilon_1 \mapsto \eta^i$. Fix a linear isomorphism

$$(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \to \widehat{\mathbf{V}} \tag{3.8}$$

mapping each $(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}_{\chi_i}$ onto $\mathbf{V}_i$. It induces an isomorphism of endomorphism algebras

$$\varpi : \mathrm{End}_{\mathbb{C}}\left((\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}\right) \to \mathrm{End}_{\mathbb{C}}(\widehat{\mathbf{V}}). \tag{3.9}$$

Definition 3.2.3. Recall $\mathrm{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ from Definition 2.2.7. Each $\phi \in \mathrm{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ defines endomorphisms $\phi(x_1), \phi(y_1) : \mathbb{C}\Gamma_n \to \mathbb{C}\Gamma_n$, where $x_1, y_1 \in \mathtt{H}_{\mathbf{h}}$ are as in Definition 2.2.2. Set

$$\mathbf{X}(\phi) := \varpi\left(\phi(x_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}}\right), \quad \mathbf{Y}(\phi) := \varpi\left(\phi(y_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}}\right).$$

Consider the maps

$$\Psi\colon\ \mathrm{Rep}^o_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}}) \to \mathrm{Rep}(\overline{\mathbf{Q}}, n\delta), \quad \phi \mapsto (\mathbf{X}(\phi), \mathbf{Y}(\phi)), \tag{3.10}$$

$$p\colon\ \mathrm{Rep}(\overline{\mathbf{Q}}_\infty, n\delta) \to \mathrm{Rep}(\overline{\mathbf{Q}}, n\delta), \quad (\mathbf{X}, \mathbf{Y}, I, J) \mapsto (\mathbf{X}, \mathbf{Y}) \tag{3.11}$$

Lemma 3.2.4. *The maps Ψ and p are $\mathbb{C}^*$-equivariant.*

Proof. The equivariance of p is obvious. Let $t \in \mathbb{C}^*, \phi \in \mathrm{Rep}_{\mathbb{C}\Gamma_n}(\mathtt{H}_{\mathbf{h}})$ and $z \in \mathtt{H}_{\mathbf{h}}$. We have $(t.\phi)(z) = \phi(t^{-1}.z)$, so $(t.\phi)(x_1) = t^{-1}\phi(x_1)$ and $(t.\phi)(y_1) = t\phi(y_1)$. Hence

$$\begin{aligned}\Psi(t.\phi) =& \left(\varpi\left(t^{-1}\phi(x_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}}\right), \varpi\left(t\phi(y_1)|_{(\mathbb{C}\Gamma_n)^{\Gamma_{n-1}}}\right)\right)\\ =& (t^{-1}\mathbf{X}(\phi), t\mathbf{Y}(\phi)) = t.(\mathbf{X}(\phi), \mathbf{Y}(\phi)).\end{aligned}$$ □

The proof of [48, Proposition 11.24] carries over directly to yield the following generalization.

Theorem 3.2.5. *Maps* (3.10) *and* (3.11) *induce a $\mathbb{C}^*$-equivariant isomorphism of varieties*

$$\mathsf{EG}\colon \mathcal{Y}_{\mathbf{h}} \xrightarrow{\sim} \mathcal{C}_{\mathbf{h}} \tag{3.12}$$

and vector bundles

$$\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}} \xrightarrow{\sim} \mathcal{V}_{\mathbf{h}}. \tag{3.13}$$

We call (3.12) the *Etingof-Ginzburg isomorphism.*

3.3 Combinatorics

We recall several combinatorial notions which will be used throughout this chapter.

3.3.1. Young diagrams. If μ is a partition of k, set $n(\mu) := \sum_{i\geq 1} i\cdot\mu_{i+1}$. If $\underline{\lambda}$ is an l-multipartition of k, define $r(\underline{\lambda}) = \sum_{i=1}^{l-1} i\cdot|\lambda^i|$. Recall the notations

$$[n]_t = \frac{1-t^n}{1-t} = 1+\ldots+t^{n-1}, \qquad (t)_n = (1-t)(1-t^2)\ldots(1-t^n).$$

Let $\mu = (\mu_1,\ldots,\mu_m,0,\ldots)$ be a partition of k, where $\mu_1,\ldots,\mu_m$ are non-zero. Let $\mathbb{Y}(\mu) := \{(i,j) \mid 1\leq i\leq m,\ 1\leq j\leq \lambda_i\}$ denote the Young diagram of μ. We will always display Young diagrams according to the English convention. We call each pair $(i,j)\in\mathbb{Y}(\mu)$ a cell. We will often use the symbol $\square$ to refer to cells. Sometimes we will also abuse notation and write μ instead of $\mathbb{Y}(\mu)$ where no confusion can arise, e.g., $\square\in\mu$ instead of $\square\in\mathbb{Y}(\mu)$. If $\square=(i,j)\in\mathbb{Y}(\mu)$ is a cell, let $c(\square) := j-i$ be the *content* of $\square$. We call

$$\mathrm{Res}_\mu(t) := \sum_{\square\in\mu} t^{c(\square)}$$

the *residue* of μ. We also call $c(\square) \bmod l$ the *l-content* of $\square$ and $\sum_{\square\in\mu} t^{c(\square) \bmod l}$ the *l-residue* of μ. It is clear that a partition is determined uniquely by its residue.

Now suppose that $\underline{\lambda}$ is an l-multipartition of k. By the Young diagram of $\underline{\lambda}$ we mean the l-tuple $\mathbb{Y}(\underline{\lambda}) := (\mathbb{Y}(\lambda^0),\ldots,\mathbb{Y}(\lambda^{l-1}))$. By a cell $\square\in\mathbb{Y}(\underline{\lambda})$ we mean a cell in any of the Young diagrams $\mathbb{Y}(\lambda^i)$. Let $\mathbf{e}=(e_0,\ldots,e_{l-1})\in\mathbb{Q}^l$. We define the $\mathbf{e}$-residue of $\underline{\lambda}$ to be

$$\mathrm{Res}^{\mathbf{e}}_{\underline{\lambda}}(t) := \sum_{i=0}^{l-1} t^{e_i}\,\mathrm{Res}_{\lambda^i}(t).$$

For sufficiently generic $\mathbf{e}$, an l-multipartition is determined uniquely by its $\mathbf{e}$-residue.

3.3.2. Hook length polynomials. Let μ be a partition and fix a cell $(i,j)\in\mathbb{Y}(\mu)$. By the *hook* associated to the cell (i,j) we mean the set $\{(i,j)\}\cup\{(i',j)\in\mathbb{Y}(\mu)\mid i'>i\}\cup\{(i,j')\in\mathbb{Y}(\mu)\mid j'>j\}$. We call (i,j) the *root* of the hook, $\{(i',j)\in\mathbb{Y}(\mu)\mid i'>i\}$ the *leg* of the hook and $\{(i,j')\in\mathbb{Y}(\mu)\mid j'>j\}$ the *arm* of the hook. The cell in the leg of the hook with the largest first coordinate is called the *foot* of the hook, and the cell in the arm of the hook with the largest second coordinate is called the *hand* of the hook.

By a *hook* in $\mathbb{Y}(\mu)$ we mean a hook associated to some cell $\square\in\mathbb{Y}(\mu)$. If H is a hook, let $\mathsf{arm}(H)$ denote its arm and let $\mathsf{leg}(H)$ denote its leg. If $\square$ is the root of H, let $a_\mu(\square) := |\mathsf{arm}(H)|$ and $l_\mu(\square) := |\mathsf{leg}(H)|$. Set $h_\mu(\square) := 1+a_\mu(\square)+l_\mu(\square)$. The *hook length polynomial* of the partition μ is

$$H_\mu(t) := \prod_{\square\in\mu}(1-t^{h_\mu(\square)}).$$

Hook length polynomials are related to Schur functions by the following equality

$$s_\mu(1,t,t^2,\ldots) = \frac{t^{n(\mu)}}{H_\mu(t)}.$$

3.3.3. Frobenius form of a partition. By a *Frobenius hook* in $\mathbb{Y}(\mu)$ we mean a hook whose root is a cell of content zero. Clearly $\mathbb{Y}(\mu)$ is the disjoint union of all its Frobenius hooks. Suppose that $(1,1),(2,2),\ldots,(k,k)$ are the cells of content zero in $\mathbb{Y}(\mu)$. Let F_i denote the Frobenius hook with root (i,i). We endow the set of Frobenius hooks with the natural ordering $F_1<F_2<\ldots<F_k$. We call F_1 the *innermost* or *first* Frobenius hook and F_k the *outermost* or *last* Frobenius hook. Let $a_i = a_\mu(i,i)$

and $b_i = l_\mu(i,i)$. We call $(a_1, \dots, a_k \mid b_1, \dots, b_k)$ the *Frobenius form* of μ.

3.3.4. Bead diagrams. Let us recall the notion of a bead diagram (see e.g. [76, §2.7]). We call an element (i,j) of $\mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\}$ a *point*. We say that the point (i,j) lies to the left of (i,j') if $j < j'$, and that (i,j) lies above (i',j) if $i' < i$.

A *bead diagram* is a function $f : \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\} \to \{0,1\}$ which takes value 1 for only finitely many points. If $f(i,j) = 1$ we say that the point (i,j) is occupied by a *bead*. If $f(i,j) = 0$ we say that the point (i,j) is *empty*. Suppose that a point (i,j) is empty and that there exists an $i' < i$ such that the point (i',j) is occupied by a bead. Then we call the point (i,j) a *gap*.

We say that a point $(i,j) \in \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\}$ is in the $(-i)$-th row and j-th column (or runner) of the bead diagram. We call the i-th row full (empty) if every point (i,k) for $k = 0, \dots, l-1$ is occupied by a bead (is empty). A row is called redundant if it is a full row and if all the rows above it are full. A graphical interpretation of the notion of a bead diagram can be found in Example 3.3.6. We only display the rows containing at least one bead or gap.

Definition 3.3.1. Let $\mu \in \mathcal{P}$ and $p \geq \ell(\mu)$. Set

$$\beta_i^p = \mu_i + p - i \quad (1 \leq i \leq p).$$

We call $\{\beta_i^p \mid 1 \leq i \leq p\}$ a set of β-numbers for μ. Note that $|\{\beta_i^p \mid 1 \leq i \leq p\}| = p$. From each set of β-numbers one can uniquely recover the corresponding partition μ.

Definition 3.3.2. Given a set of β-numbers $\{\beta_i^p \mid 1 \leq i \leq p\}$ we can naturally associate to it a bead diagram by the rule

$$f(i,j) = 1 \iff -(i+1) \cdot l + j \in \{\beta_i^p \mid 1 \leq i \leq p\}.$$

Let μ be as in Definition 3.3.1. If p is the smallest multiple of l satisfying $p \geq \ell(\mu)$ we denote the resulting bead diagram by $\mathbb{B}(\mu)$. The diagram $\mathbb{B}(\mu)$ has no redundant rows and the number of beads in $\mathbb{B}(\mu)$ is a multiple of l.

Remark 3.3.3. Conversely, if we are given a bead diagram f, the set $\{-(i+1) \cdot l + j \mid f(i,j) = 1\}$ is a set of β-numbers for some partition. The relationship between bead diagrams, sets of β-numbers and partitions can therefore be illustrated as follows

$$\{\text{bead diagrams}\} \longleftrightarrow \{\text{sets of } \beta\text{-numbers}\} \twoheadrightarrow \{\text{partitions}\},$$

where the set of partitions contains partitions of an arbitrary integer.

3.3.5. Cores and quotients. Let $f : \mathbb{Z}_{\leq -1} \times \{0, \dots, l-1\} \to \{0,1\}$ be a bead diagram. Suppose that the point (i,j) with $i < -1$ is occupied by a bead, i.e., $f(i,j) = 1$, and that $f(i+1,j) = 0$. To *slide* or *move* the bead in position (i,j) *upward* means to modify the function f by setting $f'(i,j) = 0, f'(i+1,j) = 1$ and $f' = f$ otherwise.

Definition 3.3.4. Let μ be a partition. Take any bead diagram f corresponding to μ. We obtain a new bead diagram f' by sliding beads upward as long as it is possible. We call the partition corresponding to the bead diagram f' the *l-core* of μ, denoted $\mathsf{Core}(\mu)$. Let $\heartsuit(l)$ denote the set of all l-cores. We set

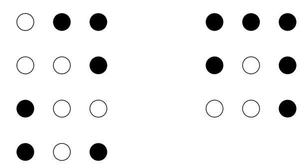
$$\mathcal{P}_\nu(k) := \{\mu \in \mathcal{P}(k) \mid \mathsf{Core}(\mu) = \nu\}.$$

Definition 3.3.5. Consider the bead diagram $\mathbb{B}(\mu)$. Each column of $\mathbb{B}(\mu)$ can itself be considered as a bead diagram for $l = 1$. Let $Q^i(\mu)$ denote the partition corresponding to the i-th column. We call the l-multipartition $\underline{\mathsf{Quot}}(\mu) := (Q^0(\mu), Q^1(\mu), \ldots, Q^{l-1}(\mu))$ the *l-quotient* of μ.

A partition is determined uniquely by its l-core and l-quotient [76, Theorem 2.7.30]. In particular, there is a bijection

$$\mathcal{P}_{\varnothing}(nl) \to \mathcal{P}(l, n), \quad \mu \mapsto \underline{\mathsf{Quot}}(\mu).$$

Example 3.3.6. Consider the partition $\mu = (6, 5, 3, 3, 1, 1)$ and take $l = 3$. The first-column hook-lengths are $11, 9, 6, 5, 2, 1$. They form a set of β-numbers. The bead diagram below on the left illustrates $\mathbb{B}(\mu)$ while the diagram on the right illustrates the effect of sliding all the beads upward.



Let us read off the 3-core of μ from the diagram on the right. We can ignore all the beads before the first empty point, which we label as zero. We carry on counting. The remaining two beads get labels 1 and 4. These form a set of β-numbers, which we can interpret as the first-column hook-lengths corresponding to the partition $(3, 1)$. It follows that $\mathsf{Core}(\mu) = (3, 1)$. To determine the 3-quotient of μ we divide the diagram on the left into three columns and consider each separately. We read off the β-numbers as before - they are $2, 3$ for the first column, 0 for the second column and 1 for the third column. It follows that $\underline{\mathsf{Quot}}(\mu) = ((2, 2), \varnothing, (1))$.

3.3.6. Rim-hooks. The *rim* of $\mathbb{Y}(\mu)$ is the subset of $\mathbb{Y}(\mu)$ consisting of the cells (i, j) such that $(i+1, j+1)$ does not lie in $\mathbb{Y}(\mu)$. Fix a cell $(i, j) \in \mathbb{Y}(\mu)$. Recall that by the hook associated to (i, j) we mean the subset of $\mathbb{Y}(\mu)$ consisting of all the cells (i, k) with $k \geq j$ and all the cells (k, j) with $k \geq i$. We define the *rim-hook* associated to the cell (i, j) to be the intersection of the set $\{(i', j') \mid i' \geq i, j' \geq j\}$ with the rim of $\mathbb{Y}(\mu)$. We call a rim-hook an *l-rim-hook* if it contains l cells. The l-core of μ can also be characterised as the subpartition μ' of μ obtained from μ by a successive removal of l-rim-hooks, in whichever order (see [76, Theorem 2.7.16]).

Lemma 3.3.7 ([76, Lemma 2.7.13])**.** *Let R be an l-rim-hook in μ and set $\mu' := \mu - R$. Then $\underline{\mathsf{Quot}}(\mu') = \underline{\mathsf{Quot}}(\mu) - \Box$ for some $\Box \in \underline{\mathsf{Quot}}(\mu)$.*

3.3.7. The $\tilde{S}_l$-action on partitions. Assume for the rest of this section that $l > 1$. All subscripts should be regarded modulo l. Let $\tilde{S}_l$ denote the affine symmetric group. It has a Coxeter presentation with generators $\sigma_0, \ldots, \sigma_{l-1}$ and relations

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (0 \leq i \leq l-1).$$

Let us recall the $\tilde{S}_l$-action on the set $\mathcal{P}$ of all partitions from [94, §4]. We will later use this action to describe the behaviour of the $\mathbb{C}^*$-fixed points under reflection functors. We need the following definition, reminiscent of the combinatorics of the Fock space.

Definition 3.3.8. Let $k \in \{0, \ldots, l-1\}$. Consider the Young diagram $\mathbb{Y}(\mu)$ as a subset of the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ space. We say that a cell $(i, j) \in \mathbb{Y}(\mu)$ is *removable* if $\mathbb{Y}(\mu) - \{(i, j)\}$ is the Young diagram of a partition. We say that it is *k-removable* if additionally $c(i, j) = j - i = k \mod l$. We call a cell $(i, j) \notin \mathbb{Y}(\mu)$ *addable* if $\mathbb{Y}(\mu) \cup \{(i, j)\}$ is the Young diagram of a partition. We call it *k-addable* if additionally $c(i,j) = j - i = k \mod l$.

We discuss the combinatorics of removability and addability in more detail in §3.8.5. We will, in particular, require Lemma 3.8.10 proven there.

Definition 3.3.9. Suppose that $\mu \in \mathcal{P}$ and $k \in \{0, \dots, l-1\}$. Define $\mathbf{T}_k(\mu)$ to be the partition such that

$$\mathbb{Y}(\mathbf{T}_k(\mu)) = \mathbb{Y}(\mu) \cup \{\square \text{ is } k\text{-addable}\} - \{\square \text{ is } k\text{-removable}\}. \tag{3.14}$$

The group $\tilde{S}_l$ acts on $\mathcal{P}$ by the rule

$$\sigma_i * \mu = \mathbf{T}_i(\mu) \quad (\mu \in \mathcal{P}, i \in \mathbb{Z}/l\mathbb{Z}).$$

This action also plays a role in the combinatorics of the Schubert calculus of the affine Grassmannian, see [92, §8.2] and [93, §11]. By [93, Proposition 22], we have $\tilde{S}_l * \varnothing = \heartsuit(l)$. Let us recall how the $\tilde{S}_l$-action behaves with respect to cores and quotients. Consider the finite symmetric group S_l as the group of permutations of the set $\{0, \dots, l-1\}$. Let $s_i \in S_l$ $(i = 1, \dots, l-1)$ be the transposition swapping $i-1$ and i. Let s_0 be the transposition swapping 0 and $l-1$. Note that our conventions for S_l differ from those for S_n introduced in §2.2.2. The finite symmetric group S_l acts on the set $\underline{\mathcal{P}}$ of all l-multipartitions by the rule

$$w \cdot \underline{\lambda} = (\lambda^{w^{-1}(0)}, \dots, \lambda^{w^{-1}(l-1)}), \quad w \in S_l.$$

Consider the group homomorphism

$$\mathsf{pr} : \tilde{S}_l \twoheadrightarrow S_l, \quad \sigma_i \mapsto s_i \ (i = 0, \dots, l-1).$$

Proposition 3.3.10 ([94, Proposition 4.13])**.** *Let $\mu \in \mathcal{P}$ and $\sigma \in \tilde{S}_l$. Then*

$$\mathsf{Core}(\sigma * \mu) = \sigma * \mathsf{Core}(\mu), \quad \underline{\mathsf{Quot}}(\sigma * \mu) = \mathsf{pr}(\sigma) \cdot \underline{\mathsf{Quot}}(\mu).$$

3.3.8. Partitions and the cyclic quiver. Let $N_i(\lambda)$ be the number of cells of l-content i in $\mathbb{Y}(\lambda)$. Using this notation, the l-residue of λ equals $\sum_{i=0}^{l-1} N_i(\lambda) t^i$. Consider the map

$$\mathfrak{d} : \mathcal{P} \to \mathbb{Z}^l, \quad \lambda \mapsto \mathbf{d}_\lambda := (N_0(\lambda), \dots, N_{l-1}(\lambda)). \tag{3.15}$$

We interpret this map as assigning to every partition a dimension vector for the cyclic quiver with l vertices. Let

$$\mathbb{Z}_\heartsuit = \{\mathbf{d} \in (\mathbb{Z}_{\geq 0})^l \mid \mathbf{d} = \mathbf{d}_\nu \text{ for some } \nu \in \heartsuit(l)\}$$

be the set of all dimension vectors corresponding to l-cores. By [76, Theorem 2.7.41] an l-core is determined uniquely by its l-residue. Hence (3.15) restricts to a bijection

$$\mathfrak{d} : \heartsuit(l) \longleftrightarrow \mathbb{Z}_\heartsuit, \quad \nu \mapsto \mathbf{d}_\nu.$$

There is an $\tilde{S}_l$-action on $\mathbb{Z}^l$ defined as follows. Let $\mathbf{d} \in \mathbb{Z}^l$. Then $\sigma_i * \mathbf{d} = \mathbf{d}'$ with $d'_j = d_j$ $(j \neq i)$ and

$$d'_i = d_{i+1} + d_{i-1} - d_i \quad (i \neq 0), \quad d'_0 = d_1 + d_{l-1} - d_0 + 1 \quad (i = 0).$$

The following proposition follows by an elementary calculation from Lemma 3.8.10.

Proposition 3.3.11. *The following diagram is $\tilde{S}_l$-equivariant*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathfrak{d}} & \mathbb{Z}^l \\ \uparrow & & \uparrow \\ \heartsuit(l) & \xrightarrow[\mathfrak{d}]{\sim} & \mathbb{Z}_{\heartsuit} \end{array}$$

Let $\sigma_i \in \tilde{S}_l$ and $\nu \in \heartsuit(l)$. Then $\sigma_i * (n\delta + \mathbf{d}_\nu) = n\delta + \sigma_i * \mathbf{d}_\nu$ and $\sigma_i * \mathbf{d}_\nu = \mathfrak{d}(\sigma_i * \nu)$. By [76, Theorem 2.7.41] any partition λ of $nl + |\sigma_i * \nu|$ such that $\mathfrak{d}(\lambda) = n\delta + \sigma_i * \mathbf{d}_\nu$ has l-core $\sigma_i * \nu$. Hence

$$\mathcal{P}_{\sigma_i * \nu}(nl + |\sigma_i * \nu|) = \mathfrak{d}^{-1}(n\delta + \sigma_i * \mathbf{d}_\nu).$$

3.3.9. Reflection functors. The group $\tilde{S}_l$ also acts on the parameter space $\mathbb{Q}^l$ for the quiver $\overline{\mathbf{Q}}_\infty$ by the rule $\sigma_i \cdot \theta = \theta'$ with

$$\theta'_i = -\theta_i, \quad \theta'_{i-1} = \theta_{i-1} + \theta_i, \quad \theta'_{i+1} = \theta_{i+1} + \theta_i, \quad \theta'_j = \theta_j \quad (j \notin \{i-1, i, i+1\})$$

Fix $i \in \{0, \ldots, l-1\}$. Let $\theta \in \mathbb{Q}^l$ be such that $\theta_i \neq 0$. Choose $\nu \in \heartsuit(l)$. Let

$$\mathfrak{R}_i : \mathcal{X}_\theta(n\delta + \mathbf{d}_\nu) \to \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \sigma_i * \mathbf{d}_\nu) \tag{3.16}$$

be the reflection functor associated to the simple reflection $\sigma_i \in \tilde{S}_l$. These functors were defined by Nakajima [103, §3] and Crawley-Boevey and Holland [28, §2], [29, §5]. One can endow the varieties $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)$, $\mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \sigma_i * \mathbf{d}_\nu)$ with hyper-Kähler structures with respect to which the reflection functor $\mathfrak{R}_i$ is a $U(1)$-equivariant hyper-Kähler isometry.

3.4 $\mathbb{C}^*$-fixed points in quiver varieties

Assuming smoothness, in this section we explicitly construct the $\mathbb{C}^*$-fixed points in the quiver varieties $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)$ as conjugacy classes of quadruples of certain matrices. Our description generalizes the work of Wilson, who classified the $\mathbb{C}^*$-fixed points in the special case $l = 1$ in [142, Proposition 6.11]. Our construction depends on the Frobenius form of a partition. In §3.4.1 we define the matrices representing the fixed points in the special case when a partition consists of a single Frobenius hook. In §3.4.2 we define more general matrices for arbitrary partitions. In §3.4.3 we interpret our matrices as quiver representations and show that the corresponding orbits are in fact fixed under the $\mathbb{C}^*$-action. We finish by computing the character of the fibre of the tautological bundle at each fixed point.

3.4.1. The matrix $A(m, r)$. Fix $\theta \in \mathbb{Q}^l$. The subscript in θ_i should always be considered modulo l. Suppose that M is a matrix. Let M_{ij} denote the entry of M in the i-th row and j-th column.

Definition 3.4.1. Let $m \geq 1$ and $1 \leq r \leq m$. We let $\Lambda(m)$ denote the $m \times m$ matrix with 1's on the first diagonal and all other entries equal to 0. Let $A(m, r)$ denote the $m \times m$ matrix whose only nonzero entries lie on the (-1)-st diagonal and satisfy

$$A(m,r)_{j+1,j} = \begin{cases} \displaystyle\sum_{i=1}^{j} \theta_{r-i} & \text{if} \quad 1 \le j < r \\ \displaystyle -\sum_{i=0}^{m-j-1} \theta_{-m+r+i} & \text{if} \quad r \le j \le m-1. \end{cases}$$

Lemma 3.4.2. *The matrix $[\Lambda(m), A(m,r)]$ is diagonal with eigenvalues*

$$[\Lambda(m), A(m,r)]_{j,j} = \begin{cases} \theta_{r-j} & \text{if} \quad 1 \le j \ne r \le m \\ \displaystyle -\sum_{i=1}^{r-1} \theta_{r-i} - \sum_{i=0}^{m-r-1} \theta_{-m+r+i} & \text{if} \quad j = r. \end{cases}$$

Proof. Let $\alpha_j := A(m,r)_{j+1,j}$. We have $\Lambda(m)A(m,r) = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_{m-1}, 0)$ and $A(m,r)\Lambda(m) = \mathrm{diag}(0, \alpha_1, \alpha_2, \dots, \alpha_{m-1})$. Hence $[\Lambda(m), A(m,r)] = \mathrm{diag}(\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_{m-1} - \alpha_{m-2}, -\alpha_{m-1})$. $\square$

Example 3.4.3. Let $l = 3, m = 8, r = 5$. Then $A(m,r)$ is the following matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1+\theta_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_2+\theta_1+\theta_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2+2\theta_1+\theta_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2-\theta_1-\theta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta_1-\theta_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta_0 & 0 \end{pmatrix}$$

3.4.2. The matrix $A(\mu)$.

Let $\nu \in \heartsuit(l)$ and $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Let us write it in Frobenius form $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ (see §3.3.3). For each $1 \le i \le k$, let $r_i = b_i + 1$, $m_i = a_i + b_i + 1$ and $\beta_i = \theta_0 + \sum_{i=1}^{r_i-1} \theta_{r_i-i} + \sum_{i=0}^{m-r_i-1} \theta_{-m+r_i+i}$.

Definition 3.4.4. We define $A(\mu)$ to be the matrix with diagonal blocks $A(\mu)^{ii} = A(m_i, r_i)$ and off-diagonal blocks $A(\mu)^{ij}$, where $A(\mu)^{ij}$ is the unique $m_i \times m_j$ matrix with nonzero entries only on the $(r_j - r_i - 1)$-th diagonal satisfying

$$\Lambda(m_i)A(\mu)^{ij} - A(\mu)^{ij}\Lambda(m_j) = -\beta_i E(r_i, r_j), \tag{3.17}$$

where $E(r_i, r_j)$ is the $m_i \times m_j$ matrix with $E(r_i, r_j)_{s,t} = 0$ unless $s = r_i, t = r_j$ and $E(r_i, r_j)_{r_i, r_j} = 1$.

Explicitly, if $i > j$ then the non-zero diagonal of $A(\mu)^{ij}$ has r_i entries equal to β_i followed by $m_i - r_i$ entries equal to zero. If $i < j$ then the non-zero diagonal of $A(\mu)^{ij}$ has $r_j - 1$ entries equal to 0 followed by $n_j - r_j + 1$ entries equal to $-\beta_i$.

Example 3.4.5. Let $l = 3$ and $\mu = (3, 1 \mid 2, 1)$. Then $m_1 = 6, m_2 = 3$ and $r_1 = 3, r_2 = 2$. Set

$h = \theta_2 + \theta_1 + \theta_0$. Then $A(\mu)$ is the matrix

$$\left(\begin{array}{cccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_2+\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_2-\theta_1-\theta_0 & 0 & 0 & 0 & 0 & -2h & 0 \\ 0 & 0 & 0 & -\theta_1-\theta_0 & 0 & 0 & 0 & 0 & -2h \\ 0 & 0 & 0 & 0 & -\theta_0 & 0 & 0 & 0 & 0 \\ \hline h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & \theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_2 & 0 \end{array}\right)$$

Definition 3.4.6. Let $\Lambda(\mu) = \bigoplus_{i=1}^{k} \Lambda(m_i)$. Setting $q_i = \sum_{s=1}^{i-1} m_s + r_i$, let $J(\mu)$ be the $nl \times 1$ matrix with entry β_i in the q_i-th row (for $1 \le i \le k$) and all other entries zero. Furthermore, let $I(\mu)$ be the $1 \times nl$ matrix with entry 1 in the q_i-th column (for $1 \le i \le k$) and all other entries zero. Finally, we set

$$\mathbf{A}(\mu) := (\Lambda(\mu), A(\mu), I(\mu), J(\mu)).$$

3.4.3. The fixed points. Let us fix an l-core ν and a parameter $\theta \in \mathbb{Q}^l$ such that the variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_\tau)$ is smooth, where $\tau = \nu^t$. Let $\mathbf{d}_\tau := (d_0, \dots, d_{l-1})$ and $\mathbf{d} = n\delta + \mathbf{d}_\tau$. Fix a complex vector space $\mathbf{V}_i^\tau$ of dimension $n + d_i$ for each $i = 0, \dots, l-1$. Additionally, let $\mathbf{V}_\infty$ be a complex vector space of dimension one. Set $\widehat{\mathbf{V}}^\tau = \bigoplus_{i=0}^{l-1} \mathbf{V}_i^\tau$ and $\mathbf{V}^\tau = \widehat{\mathbf{V}}^\tau \oplus \mathbf{V}_\infty$.

We are now going to interpret $\mathbf{A}(\mu)$ as a quiver representation. With this goal in mind we choose a suitable ordered basis of the vector space $\mathbf{V}^\tau$. We show that the endomorphisms of $\mathbf{V}^\tau$ defined by $\mathbf{A}(\mu)$ with regard to this basis respect the quiver grading and thus constitute a quiver representation. We next show that this quiver representation lies in the fibre of the moment map at θ. This allows us to conclude that the conjugacy class of $\mathbf{A}(\mu)$ is a point in the quiver variety $\mathcal{X}_\theta(\mathbf{d})$. We finish by showing that this point is fixed under the $\mathbb{C}^*$-action.

Definition 3.4.7. Consider the sequence $\mathsf{Seq} := (1, \dots, m_1, 1, \dots, m_2, \dots, 1, \dots, m_k)$. We call each increasing subsequence of the form $(1, \dots, m_i)$ the p-th *block* of Seq and denote it by Seq_p. Let u_j be the j-th element in Seq. Let $\zeta : \{1, \dots, nl + |\nu|\} \to \{1, \dots, k\}$ be the function given by the rule

$$\zeta(j) = p \iff u_j \in \mathsf{Seq}_p.$$

For each $1 \le j \le nl + |\nu|$ let

$$\psi(j) = (r_{\zeta(j)} - u_j) \bmod l.$$

If $p, p' \in \mathbb{N}$, let $\delta(p, p') = 1$ if $p = p'$ and $\delta(p, p') = 0$ otherwise. For each $0 \le i \le l-1$ and $0 \le j \le nl + |\nu|$, let $\omega_i(j)$ be defined recursively by the formula

$$\omega_i(0) = 0, \quad \omega_i(j) = \omega_i(j-1) + \delta(\psi(j), i).$$

For each $0 \le i \le l-1$, fix a basis $\{v_i^1, \dots, v_i^{n+d_i}\}$ of $\mathbf{V}_i^\tau$. We define a function

$$\mathsf{Bas}\colon \{1, \dots, nl + |\nu|\} \to \{v_i^{e_i} \mid 0 \le i \le l-1,\ 1 \le e_i \le n + d_i\}, \quad j \mapsto v_{\psi(j)}^{\omega_{\psi(j)}(j)}.$$

We also define a function $\mathsf{Cell} : \{1, \dots, nl + |\nu|\} \to \mathbb{Y}(\mu^t)$ associating to a natural number j a cell in the Young diagram of μ. We define $\mathsf{Cell}(j)$ to be the u_j-th cell in the $\zeta(j)$-th Frobenius hook of μ^t, counting from the hand of the hook, moving to the left towards the root of the hook and then down towards the foot.

Lemma 3.4.8. *The functions* Cell *and* Bas *are bijections.*

Proof. The fact that Cell is a bijection follows directly from the definitions. Observe that $\psi(j)$ equals the l-content of $\mathsf{Cell}(j)$. We thus have a commutative diagram

$$\begin{array}{ccc} \{1,\dots,nl+|\nu|\} & \xrightarrow{\mathsf{Bas}} & \{v_i^j \mid 0 \le i \le l-1,\ 1 \le e_i \le n+d_i\} \\ \Big\downarrow{\scriptstyle \mathsf{Cell}} & & \Big\downarrow{\scriptstyle v_i^e \mapsto i} \\ \mathbb{Y}(\mu^t) & \xrightarrow[l\text{-content}]{} & \{0,\dots,l-1\}. \end{array}$$

By [76, Theorem 2.7.41], the l-residue of μ^t equals $\sum_{i=0}^{l-1}(n+d_i)t^i$ because the l-core of μ^t is τ. Hence for each $0 \le i \le l-1$ there are exactly $n+d_i$ elements $j \in \{1,\dots,nl+|\nu|\}$ such that the l-content of $\mathsf{Cell}(j)$ equals i. By the commutativity of our diagram, we conclude that there are exactly $n+d_i$ elements $j \in \{1,\dots,nl+|\nu|\}$ such that $\mathsf{Bas}(j) \in \mathbf{V}_i^\tau$.

Now suppose that $j < j'$ and $\mathsf{Bas}(j), \mathsf{Bas}(j') \in \mathbf{V}_i^\tau$. Then $\psi(j) = \psi(j')$. Since $j < j'$ and the function $\omega_{\psi(j')}(-)$ is non-decreasing we have $\omega_{\psi(j')}(j') = \omega_{\psi(j')}(j'-1)+1 > \omega_{\psi(j')}(j'-1) \ge \omega_{\psi(j)}(j)$. Hence $\mathsf{Bas}(j) \neq \mathsf{Bas}(j')$. We conclude that the function Bas is injective. Since the domain and codomain have the same cardinality, Bas is also bijective. □

Definition 3.4.9. Let $\mathbb{B} := (\mathsf{Bas}(1), \mathsf{Bas}(2), \dots, \mathsf{Bas}(nl+|\tau|))$. By Lemma 3.4.8, $\mathbb{B}$ is an ordered basis of $\widehat{\mathbf{V}}^\tau$. From now on we consider the matrices $\Lambda(\mu)$ and $A(\mu)$ as linear endomorphisms of $\widehat{\mathbf{V}}^\tau$ relative to the ordered basis $\mathbb{B}$. Let us choose a nonzero vector $v_\infty \in \mathbf{V}_\infty$. We consider the matrix $I(\mu)$ as a linear transformation $\widehat{\mathbf{V}}^\tau \to \mathbf{V}_\infty$ relative to the ordered bases $\{v_\infty\}$ and $\mathbb{B}$. We also consider the matrix $J(\mu)$ as a linear transformation $\mathbf{V}_\infty \to \widehat{\mathbf{V}}^\tau$ relative to the ordered bases $\mathbb{B}$ and $\{v_\infty\}$.

Let $\mu \in \mathcal{P}_\nu(nl+|\nu|)$. Suppose that $\mu = (a_1,\dots,a_k \mid b_1,\dots,b_k)$ is the Frobenius form of μ. As before, set $r_i = b_i+1$, $m_i = a_i+b_i+1$ and $q_i = \sum_{j<i} m_j + r_i$.

Lemma 3.4.10. *Suppose that* $1 \le i \le k$. *Then:*

- *If* $0 \le j < a_i$ *then* $A(\mu)(\mathsf{Bas}(q_i+j)) = \sum_{p=1}^{i} c_p \mathsf{Bas}(q_p+j+1)$,
- *if* $0 \le j = a_i$ *then* $A(\mu)(\mathsf{Bas}(q_i+j)) = \sum_{p=1}^{i-1} c_p \mathsf{Bas}(q_p+j+1)$,
- *if* $0 > j \ge -b_i$ *then* $A(\mu)(\mathsf{Bas}(q_i+j)) = \sum_{p=1}^{k} c_p \mathbf{1}_{-j \le b_p+1} \mathsf{Bas}(q_p+j+1)$

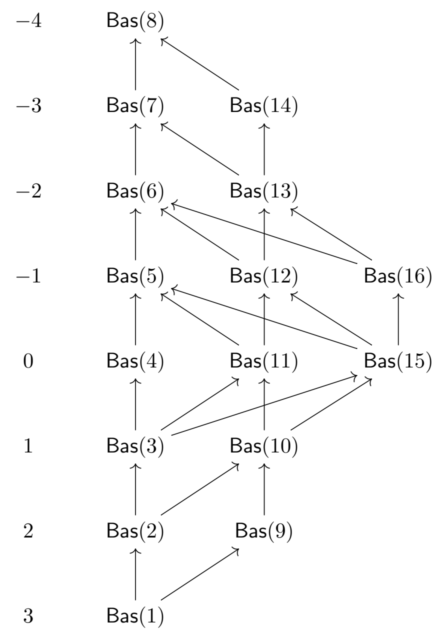
for some coefficients $c_m \in \mathbb{C}$, *where* $\mathbf{1}_{-j\le b_p+1}$ *is the indicator function taking value one if* $-j \le b_p+1$ *and zero otherwise. Moreover, for a generic parameter* θ *the coefficients* c_p *are all non-zero.*

Proof. This is immediate from Definition 3.4.4. □

Lemma 3.4.10 has a very intuitive diagrammatic interpretation. We explain it using the following example.

Example 3.4.11. Consider the partition $\mu = (5,5,4,2)$. Its Frobenius form is $(4,3,1 \mid 3,2,0)$. We have $q_1 = 4, q_2 = 11$ and $q_3 = 15$. The diagram below should be interpreted in the following way: $A(\mu)(\mathsf{Bas}(j))$ is a linear combination of those vectors $\mathsf{Bas}(j')$ for which there is an arrow $\mathsf{Bas}(j) \to$

Bas(j').



We have also introduced a numbering of the rows of the diagram. It is easy to see that $\mathsf{Bas}(j) \in \mathbf{V}_i^\tau$ if and only if $\mathsf{Bas}(j)$ lies in a row whose label is congruent to i mod l.

Lemma 3.4.12. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $\mathbf{A}(\mu) \in \mathrm{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$.*

Proof. We need to check that for each $0 \le i \le l-1$ the following holds:

$$\mathrm{Im}(A(\mu)|_{\mathbf{V}_i^\tau}) \subseteq \mathbf{V}_{i-1}^\tau, \quad \mathrm{Im}\left(\Lambda(\tau)|_{\mathbf{V}_i^\tau}\right) \subseteq \mathbf{V}_{i+1}^\tau, \quad \mathrm{Im}(J(\mu)) \subseteq \mathbf{V}_0^\tau, \quad \bigoplus_{i=1}^{l-1} \mathbf{V}_i^\tau \subseteq \ker(I(\mu)).$$

Let us first show the first statement. We can draw a diagram as in Example 3.4.11. The subspace $\mathbf{V}_i^\tau$ has a basis consisting of vectors $\mathsf{Bas}(j)$ in rows labelled by numbers congruent to i mod l. The diagram shows that $A(\mu)(\mathsf{Bas}(j))$ is a linear combination of basis vectors in the row above $\mathsf{Bas}(j)$. But that row is labelled by a number congruent to $i-1$ mod l. Hence $A(\mu)(\mathsf{Bas}(j)) \in \mathbf{V}_{i-1}^\tau$. The argument for $\Lambda(\mu)$ is analogous.

Let us prove the last claim. Let $j \in \{1, \dots, nl + |\nu|\}$ and suppose that $\mathsf{Bas}(j) \notin \mathbf{V}_0^\tau$. Let $1 \le p \le k$ and set $q_p = \sum_{s=1}^{p-1} m_s + r_p$. Since $\psi(q_p) = r_p - r_p = 0$ we conclude that $p \notin \{q_1, \dots, q_k\}$. But the only non-zero entries of $I(\mu)$ are those in columns numbered q_p, for $1 \le p \le k$. Hence $\mathsf{Bas}(j) \in \ker I(\mu)$. The calculation for $J(\mu)$ is similar. □

Proposition 3.4.13. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $\mathbf{A}(\mu) \in \mu_{\mathbf{d}}^{-1}(\theta)$.*

Proof. By the previous lemma, we know that $\mathbf{A}(\mu) \in \mathrm{Rep}(\overline{\mathbf{Q}}_\infty, \mathbf{d})$. Lemma 3.4.2 together with (3.17) implies that $[\Lambda(\mu), A(\mu)] + J(\mu)I(\mu) = \theta$, so $\mathbf{A}(\mu) \in \mu_{\mathbf{d}}^{-1}(\theta)$. □

Theorem 3.4.14. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then $[\mathbf{A}(\mu)] := G(\mathbf{d}).\mathbf{A}(\mu)$ is a $\mathbb{C}^*$-fixed point in the quiver variety $\mathcal{X}_\theta(\mathbf{d})$.*

Proof. Let $t \in \mathbb{C}^*$. We have $t.\mathbf{A}(\mu) = (t^{-1}\Lambda(\mu), tA(\mu), I(\mu), J(\mu))$. We need to find a matrix N in $G(\mathbf{d})$ such that $Nt.\mathbf{A}(\mu)N^{-1} = \mathbf{A}(\mu)$.

For every $t \in \mathbb{C}^*$, let $Q(t) = \operatorname{diag}(1, t^{-1}, \dots, t^{-nl-|\nu|+1})$. Conjugating an $(nl+|\nu|)\times(nl+|\nu|)$ matrix by $Q(t)$ multiplies the j-th diagonal by t^j. In particular, we have $Q(t)(\bigoplus_{i=1}^k tA(m_i, r_i))Q(t)^{-1} = \bigoplus_{i=1}^k A(m_i, r_i)$ and $Q(t)t^{-1}\Lambda(\mu)Q(t)^{-1} = \Lambda(\mu)$.

Now consider the effect of conjugating $A(\mu)$ by $Q(t)$ on the off-diagonal block $A(\mu)^{ij}$ $(i \neq j)$. This block contains only one nonzero diagonal. Counting within the block, it is the diagonal labelled $r_j - r_i - 1$. Counting inside the entire matrix $A(\mu)$, it is the diagonal labelled $q_j - q_i - 1$. It follows that conjugation by $Q(t)$ multiplies the block $A(\mu)^{ij}$ by $t^{q_j - q_i - 1}$. Hence we have

$$Q(t)\left(\bigoplus_{1\leq i\neq j\leq k} tA(\mu)^{ij}\right)Q(t)^{-1} = \bigoplus_{1\leq i\neq j\leq k} t^{q_j-q_i}A(\mu)^{ij}.$$

Let $P(t) = \bigoplus_{i=1}^k t^{q_i}\operatorname{Id}_{m_i}$. Conjugating $A(\mu)$ by $P(t)$ doesn't change the diagonal blocks but multiplies each off-diagonal block $A(\mu)^{ij}$ by $t^{q_i - q_j}$. We conclude that

$$P(t)Q(t)tA(\mu)Q(t)^{-1}P(t)^{-1} = A(\mu).$$

Since the matrix $\Lambda(\mu)$ contains only diagonal blocks, conjugating by $P(t)$ doesn't have any impact. Hence

$$P(t)Q(t)t^{-1}\Lambda(\mu)Q(t)^{-1}P(t)^{-1} = \Lambda(\mu).$$

The nonzero rows of $J(\mu)$ are precisely rows number $q_1, q_2, \dots, q_k$. But the q_i-th entry of $P(t)$ is t^{q_i} and the q_i-th entry of $Q(t)$ is t^{1-q_i}. Hence $P(t)Q(t)J(\mu) = tJ(\mu)$. Similarly, $I(\mu)q(t)^{-1}P(t)^{-1} = t^{-1}I(\mu)$. Let $D(t) = t^{-1}\operatorname{Id}_{nl+|\nu|}$. Since $D(t)$ is a scalar matrix, conjugating by $D(t)$ doesn't change $A(\mu)$ or $\Lambda(\mu)$. On the other hand, $D(t)P(t)Q(t)J(\mu) = J(\mu)$ and $I(\mu)q(t)^{-1}P(t)^{-1}D(t)^{-1} = I(\mu)$.

The matrices $D(t), Q(t), P(t)$ are diagonal, so they represent linear automorphisms in $G(\mathbf{d})$. Hence $\mathbf{A}(\mu)$ and $t.\mathbf{A}(\mu)$ lie in the same $G(\mathbf{d})$-orbit, which is equivalent to saying that $\mathbf{A}(\mu)$ is a $\mathbb{C}^*$-fixed point in $\mathcal{X}_\theta(\mathbf{d})$. □

3.4.4. Characters of the fibres of $\mathcal{V}$ at the fixed points. Recall the tautological bundle $\mathcal{V}_\theta(\mathbf{d})$ on $\mathcal{X}_\theta(\mathbf{d})$ from §3.2.2. Let us abbreviate $\mathcal{V} := \mathcal{V}_\theta(\mathbf{d})$. Let $\mathcal{V}_\mu$ denote the fibre of $\mathcal{V}$ at the fixed point $[\mathbf{A}(\mu)] := G(\mathbf{d}).\mathbf{A}(\mu)$.

Proposition 3.4.15. *Let $\mu \in \mathcal{P}_\nu(nl + |\nu|)$. Then*

$$\operatorname{ch}_t \mathcal{V}_\mu = \operatorname{Res}_\mu(t) := \sum_{\square\in\mu} t^{c(\square)}.$$

Proof. Consider $[(\mathbf{A}(\mu), v)] := G(\mathbf{d}).(\mathbf{A}(\mu), v) \in \mu_{\mathbf{d}}^{-1}(\theta) \times^{G(\mathbf{d})} \widehat{\mathbf{V}}^\tau = \mathcal{V}$. We have

$$\begin{aligned} t.(\mathbf{A}(\mu), v) = (t.\mathbf{A}(\mu), v) &\sim (D(t)P(t)Q(t)(t.\mathbf{A}(\mu))(D(t)P(t)Q(t))^{-1}, (D(t)P(t)Q(t))^{-1}v) \\ &= (\mathbf{A}(\mu), Q(t)^{-1}P(t)^{-1}D(t)^{-1}v). \end{aligned}$$

The basis vectors $\{\mathsf{Bas}(1), \mathsf{Bas}(2), \dots, \mathsf{Bas}(nl + |\nu|)\}$ are eigenvectors of $(D(t)P(t)Q(t))^{-1}$ with corresponding eigenvalues

$$\{t^{1-r_1}, t^{2-r_1}, \dots, t^{m_1-r_1}; t^{1-r_2}, t^{2-r_2}, \dots, t^{m_2-r_2}; \dots; t^{1-r_k}, t^{2-r_k}, \dots, t^{m_k-r_k}\}.$$

Moreover, these eigenvalues are precisely the contents of the cells in the Young diagram of μ, counting from the foot of the innermost Frobenius hook upward and later to the right, before passing to subsequent Frobenius hooks. Hence $\mathrm{ch}_t\, \mathcal{V}_\mu = \sum_{\Box \in \mu} t^{c(\Box)}$. $\Box$

Recall that we have assumed that the parameter θ is chosen so that the variety $\mathcal{X}_\theta(\mathbf{d})$ is smooth. By [93, Proposition 22] there exists $w \in \tilde{S}_l$ such that $w * \mathbf{d}_\tau = 0$. Let $w = \sigma_{i_1} \cdots \sigma_{i_m}$ be a reduced expression for w in $\tilde{S}_l$. Furthermore, let $w \cdot \theta = (\vartheta_0, \dots, \vartheta_{l-1})$ and $H_1 = \vartheta_1, \dots, H_{l-1} = \vartheta_{l-1}$, $h = -\sum_{i=0}^{l-1} \vartheta_i$, $\mathbf{h} = (h, H_1, \dots, H_{l-1})$. Composing the Etingof-Ginzburg map with reflection functors we obtain a $\mathbb{C}^*$-equivariant isomorphism

$$\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{C}_{\mathbf{h}} = \mathcal{X}_{w\cdot\theta}(n\delta) \xrightarrow{\mathfrak{R}_{i_m} \circ \cdots \circ \mathfrak{R}_{i_1}} \mathcal{X}_\theta(\mathbf{d}).$$

Corollary 3.4.16. *The map*

$$\mathcal{P}_\nu(nl + |\nu|) \to \mathcal{X}_\theta(\mathbf{d})^{\mathbb{C}^*}, \quad \mu \mapsto [\mathbf{A}(\mu)] := G(\mathbf{d}).\mathbf{A}(\mu) \tag{3.18}$$

is a bijection.

Proof. The $\mathbb{C}^*$-fixed points in MaxSpec $\mathsf{Z}_{\mathbf{h}}$ are in bijection with l-multipartitions of n, which are themselves in bijection with partitions of $nl + |\nu|$ with l-core ν. But $\mathcal{Y}_{\mathbf{h}}$ is $\mathbb{C}^*$-equivariantly isomorphic to $\mathcal{X}_\theta(\mathbf{d})$, so $|\mathcal{X}_\theta(\mathbf{d})^{\mathbb{C}^*}| = |(\mathrm{MaxSpec}\, \mathsf{Z}_{\mathbf{h}})^{\mathbb{C}^*}| = |\mathcal{P}(l,n)| = |\mathcal{P}_\nu(nl + |\nu|)|$.

Since a partition is uniquely determined by its residue, $\mu \neq \mu'$ implies $\mathrm{ch}_t\, \mathcal{V}_\mu \neq \mathrm{ch}_t\, \mathcal{V}_{\mu'}$, which in turn implies that $[\mathbf{A}(\mu)] \neq [\mathbf{A}(\mu')]$. It follows that (3.18) is a bijection because it is an injective function between sets of the same cardinality. $\Box$

3.5 Degenerate affine Hecke algebras

In this section we use degenerate affine Hecke algebras and a version of the Chevalley restriction map to associate to each $\mathbb{C}^*$-fixed point in $\mathcal{Y}_{\mathbf{h}}$ and $\mathcal{C}_{\mathbf{h}}$ a distinct point in $\mathbb{C}^n/S_n$ in a manner which is compatible with the Etingof-Ginzburg isomorphism.

3.5.1. Degenerate affine Hecke algebras. Degenerate affine Hecke algebras associated to complex reflection groups of type $G(l,1,n)$ were defined in [109]. Let us recall their definition and basic properties.

Definition 3.5.1. Let $\kappa \in \mathbb{C}$. The degenerate affine Hecke algebra associated to Γ_n is the $\mathbb{C}$-algebra $\mathcal{H}_\kappa$ generated by Γ_n and pairwise commuting elements $z_1, \dots, z_n$ satisfying the following relations:

$$\epsilon_j z_i = z_i \epsilon_j \ (1 \le i, j \le n), \quad s_i z_j = z_j s_i \ (j \neq i, i+1),$$

$$s_i z_{i+1} = z_i s_i + \kappa \sum_{k=0}^{l-1} \epsilon_i^{-k} \epsilon_{i+1}^{k} \ (1 \le i \le n-1).$$

Let $\mathcal{Z}_\kappa$ denote the centre of $\mathcal{H}_\kappa$.

Proposition 3.5.2. *The algebra $\mathcal{H}_\kappa$ has the following properties.*

a) *As a vector space, $\mathcal{H}_\kappa$ is canonically isomorphic to $\mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}\Gamma_n$.*
b) *There is an injective algebra homomorphism $\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow \mathcal{Z}_\kappa$.*
c) *The algebra $\mathcal{H}_\kappa$ has a maximal commutative subalgebra $\mathfrak{C}_\kappa$ which is isomorphic to $\mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}(\mathbb{Z}/l\mathbb{Z})^n$.*

d) *Suppose that* $\mathbf{h} = (h, H_1, \dots, H_{l-1}) \in \mathbb{Q}^l$ *satisfies* $h = \kappa$. *Then there exists an injective algebra homomorphism* $\mathcal{H}_\kappa \hookrightarrow \mathtt{H}_{\mathbf{h}}$ *defined by*

$$g \mapsto g \ (g \in \Gamma_n), \quad z_i \mapsto y_i x_i + \kappa \sum_{1 \le j < i} \sum_{k=0}^{l-1} s_{i,j} \epsilon_i^k \epsilon_j^{-k} + \sum_{k=1}^{l-1} c_k \sum_{m=0}^{l-1} \eta^{-mk} \epsilon_i^m, \tag{3.19}$$

where the c_k*'s are the parameters obtained from* $\mathbf{h}$ *as in [64, §2.7]. This homomorphism restricts to a homomorphism*

$$\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow \mathtt{Z}_{\mathbf{h}}. \tag{3.20}$$

Proof. See Propositions 1.1, 2.1, 2.3 and §3.1 in [40] as well as Proposition 10.1 and Corollary 10.1 in [67]. □

Let us recall the construction of some irreducible $\mathcal{H}_\kappa$-modules.

Definition 3.5.3. Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and $b = (b_1, \dots, b_n) \in (\mathbb{Z}/l\mathbb{Z})^n$. Let $\mathbb{C}_{a,b}$ be the one-dimensional representation of the commutative algebra $\mathfrak{C}_\kappa = \mathbb{C}[z_1, \dots, z_n] \otimes \mathbb{C}(\mathbb{Z}/l\mathbb{Z})^n$ defined by $z_i.v = a_i v$, $\epsilon_i.v = \eta^{b_i} v$ for each $1 \le i \le n$ and $v \in \mathbb{C}_{a,b}$. Define

$$M(a, b) := \mathcal{H}_\kappa \otimes_{\mathfrak{C}_\kappa} \mathbb{C}_{a,b}.$$

Proposition 3.5.4 ([40, Theorem 4.9]). *Let* $a \in \mathbb{C}^n$ *and* $b \in (\mathbb{Z}/l\mathbb{Z})^n$. *If* $a_i - a_j \neq 0, \pm l\kappa$ *for all* $1 \le i \ne j \le n$ *then the* $\mathcal{H}_\kappa$*-module* $M(a, b)$ *is irreducible.*

3.5.2. Restricting $\mathtt{H}_{\mathbf{h}}$-modules to $\mathcal{H}_h$-modules.

Fix $\mathbf{h} \in \mathbb{Q}^l$ such that $\mathcal{Y}_{\mathbf{h}}$ is smooth and set $\kappa = h$. We are going to consider the generic behaviour of simple modules over $\mathtt{H}_{\mathbf{h}}$ under the restriction functor to $\mathcal{H}_h$-modules.

Definition 3.5.5. Set

$$\mathcal{D} := \{a = (a_1, \dots, a_n) \in \mathbb{C}^n \mid a_i - a_j \neq 0, \pm l\kappa \text{ for all } 1 \le i \ne j \le n\}.$$

Observe that $\mathcal{D}$ is a dense open subset of $\mathbb{C}^n$. Proposition 3.5.4 implies that for all $a \in \mathcal{D}$ and $b \in (\mathbb{Z}/l\mathbb{Z})^n$ the module $M(a, b)$ is irreducible. Consider the diagram

$$\mathbb{C}^n \xrightarrow{\phi} \mathbb{C}^n/S_n \xleftarrow{\rho_1} \mathcal{Y}_{\mathbf{h}},$$

where ϕ is the canonical map and ρ_1 is the dominant morphism induced by (3.20). Set

$$\mathcal{U} := \rho_1^{-1}(\phi(\mathcal{D})). \tag{3.21}$$

Lemma 3.5.6. *The subset* $\mathcal{U}$ *is open and dense in* $\mathcal{Y}_{\mathbf{h}}$.

Proof. The set $\mathcal{U}$ is open because ϕ is a quotient map and ρ_1 is continuous. Since the morphism ρ_1 is dominant, $\rho_1(\mathcal{Y}_{\mathbf{h}})$ is dense in $\mathbb{C}^n/S_n$. Therefore, since $\phi(\mathcal{D})$ is open in $\mathbb{C}^n/S_n$, we have $\phi(\mathcal{D}) \cap \rho_1(\mathcal{Y}_{\mathbf{h}}) \neq \varnothing$. Hence $\mathcal{U}$ is nonempty. The fact that the variety $\mathcal{Y}_{\mathbf{h}}$ is irreducible (see e.g. [49, Corollary 3.9]) now implies that $\mathcal{U}$ is dense. □

Let $\hat{e} = \frac{1}{(n-1)!} \sum_{g \in S_{n-1} \subset \Gamma_n} g$ and $\mathbf{0} = (0, \dots, 0) \in (\mathbb{Z}/l\mathbb{Z})^n$. For the rest of this subsection fix an irreducible $\mathtt{H}_{\mathbf{h}}$-module L whose support is contained in $\mathcal{U}$ (i.e. $\chi_L \in \mathcal{U}$). Consider L as an $\mathcal{H}_h$-module using the embedding (3.19).

Lemma 3.5.7. *There exists an injective homomorphism of $\mathcal{H}_h$-modules $M(a,\mathbf{0}) \hookrightarrow L$ for some $a \in \mathcal{D}$.*

Proof. We have a $(\mathbb{Z}/l\mathbb{Z})^n$-module decomposition $L = \bigoplus_{b\in(\mathbb{Z}/l\mathbb{Z})^n} L(b)$, where $L(b)$ is the subspace of L such that $\epsilon_i.w = \eta^{b_i}w$ for all $w \in L(b)$. Since the z_i's commute with the ϵ_j's, each subspace $L(b)$ is preserved under the action of the z_i's. In particular, $z_1,\ldots,z_n$ define commuting linear operators on $L(\mathbf{0})$, so they have some common eigenvector $v \in L(\mathbf{0})$. Let $a_1,\ldots,a_n$ be the respective eigenvalues of the z_i's. Since the support of L is contained in $\mathcal{U}$, we have $a = (a_1,\ldots,a_n) \in \mathcal{D}$.

Let $v_{a,\mathbf{0}} \in \mathbb{C}_{a,\mathbf{0}}$. Then the map $1 \otimes v_{a,\mathbf{0}} \mapsto v$ defines a non-zero $\mathcal{H}_h$-module homomorphism $M(a,\mathbf{0}) \to L$. Since $a = (a_1,\ldots,a_n) \in \mathcal{D}$, the module $M(a,\mathbf{0})$ is simple and so this homomorphism is injective. ☐

Recall that $L^{\Gamma_{n-1}}$ is a module over $\langle\epsilon_1\rangle \cong \mathbb{Z}/l\mathbb{Z}$. Let $L^{\Gamma_{n-1}}_{\chi_0}$ denote the isotypic component corresponding to the trivial character.

Lemma 3.5.8. *We have $\hat{e}M(a,\mathbf{0}) = L^{\Gamma_{n-1}}_{\chi_0}$. Moreover, $\hat{e}M(a,\mathbf{0})$ is stable under the action of z_1 and the eigenvalues of z_1 on $\hat{e}M(a,\mathbf{0})$ are $a_1,\ldots,a_n$.*

Proof. The action of ϵ_1 on $M(a,\mathbf{0})$ is trivial by definition. We have a vector space isomorphism $M(a,\mathbf{0}) \cong \mathbb{C}S_n \otimes \mathbb{C}_{a,\mathbf{0}}$. Therefore $\{\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}} \mid 1 \le j \le n\}$ form a basis of $\hat{e}M(a,\mathbf{0})$ for any nonzero $v_{a,\mathbf{0}} \in \mathbb{C}_{a,\mathbf{0}}$. In particular, $\dim \hat{e}M(a,\mathbf{0}) = n$. Let us show that each of the basis elements we defined is fixed under the action of Γ_{n-1}. We first note that since for each $g \in S_{n-1} \subset \Gamma_n$ we have $g\hat{e} = \hat{e}$, the subgroup S_{n-1} fixes each $\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$. Now consider $\epsilon_i.\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$ with $2 \le i \le n$. We have $\epsilon_i.\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}} = \sum_{g\in S_{n-1}} gs_{1,j}\epsilon_{i(g)} \otimes v_{a,\mathbf{0}}$, where $i(g)$ is an index depending on g. But each $\epsilon_{i(g)}$ acts on $v_{a,\mathbf{0}}$ by the identity, so we conclude that ϵ_i fixes $\hat{e}s_{1,j} \otimes v_{a,\mathbf{0}}$. The stability of $\hat{e}M(a,\mathbf{0})$ under the action of z_1 follows from the fact that z_1 commutes with $\hat{e}$. The calculation of the eigenvalues is similar to the calculation in the proof of [8, Lemma 4.7]. ☐

3.5.3. Connection to the Etingof-Ginzburg isomorphism.

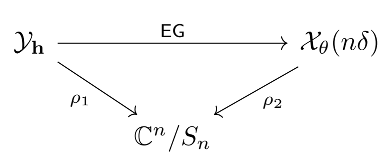
Suppose that L is an irreducible $\mathtt{H}_{\mathbf{h}}$-module whose support is contained in $\mathcal{U}$. Let $M(a,\mathbf{0})$ be as in Lemma 3.5.7. Using Lemma 3.5.8 and (3.8) we can identify $\hat{e}M(a,\mathbf{0}) = L^{\Gamma_{n-1}}_{\chi_0} \cong (\mathbb{C}\Gamma_n)^{\Gamma_{n-1}} \cong \mathbf{V}_0$. Suppose that $\mathsf{EG}(\chi_L) = [(\mathbf{X},\mathbf{Y},I,J)]$. The embedding (3.19) sends

$$z_1 \quad\mapsto\quad x_1y_1 + \sum_{k=1}^{l-1} c_k \sum_{m=0}^{l-1} \eta^{-mk}\epsilon_1^m.$$

Since ϵ_1 acts trivially on $M(a,\mathbf{0})$, the action of z_1 on $\hat{e}M(a,\mathbf{0})$ can be identified with the action of y_1x_1 on $L^{\Gamma_{n-1}}_{\chi_0}$. Using the Etingof-Ginzburg isomorphism, the latter can be identified, up to conjugation, with the matrix Y_1X_0.

Definition 3.5.9. Let $\rho_2 : \mathcal{C}_{\mathbf{h}} \to \mathbb{C}^n/S_n$ be the morphism sending $[(\mathbf{X},\mathbf{Y},I,J)]$ to the multiset of the generalized eigenvalues of the matrix Y_1X_0.

Proposition 3.5.10. *The following diagram commutes.*



Proof. Since EG is an isomorphism, it suffices to show there exists a dense open subset of $\mathcal{Y}_{\mathbf{h}}$ for which the diagram commutes. Consider the dense open subset $\mathcal{U}$ from (3.21). Since $\mathcal{Y}_{\mathbf{h}}$ is smooth, for each

$\chi \in \mathcal{U}$, there exists a unique simple $\mathtt{H}_{\mathbf{h}}$-module L such that $\chi = \chi_L$. Moreover, there is an injective $\mathcal{H}_h$-module homomorphism $M(a, \mathbf{0}) \hookrightarrow L$ for some $a \in \mathcal{D}$, by Lemma 3.5.7. Set $[(\mathbf{X}, \mathbf{Y}, I, J)] := \mathsf{EG}(\chi_L)$. The remarks at the beginning of this subsection imply that the matrix $Y_1 X_0$ describes the action of z_1 on $\hat{e}M(a, \mathbf{0})$. Hence the eigenvalues of $Y_1 X_0$ are the same as the eigenvalues of the operator $z_1|_{\hat{e}M(a,\mathbf{0})}$. By Lemma 3.5.8, these eigenvalues are $a_1, \dots, a_n$. Hence $\rho_2 \circ \mathsf{EG}(\chi_L) = \phi(a) \in \mathbb{C}^n/S_n$.

On the other hand, consider the composition

$$\mathbb{C}[z_1, \dots, z_n]^{S_n} \hookrightarrow \mathtt{Z}_{\mathbf{h}} \xrightarrow{\chi_L} \mathbb{C}. \tag{3.22}$$

By the definition of $M(a, \mathbf{0})$, a symmetric polynomial $f(z_1, \dots, z_n)$ acts on $1 \otimes \mathbb{C}_{a,\mathbf{0}}$ by the scalar $f(a_1, \dots, a_n)$. Since $f(z_1, \dots, z_n)$ is central in $\mathtt{H}_{\mathbf{h}}$, it acts by this scalar on all of L. Therefore, the kernel of (3.22) equals the maximal ideal in $\mathbb{C}[z_1, \dots, z_n]^{S_n}$ consisting of those symmetric polynomials f which satisfy $f(a_1, \dots, a_n) = 0$, which is the vanishing ideal of $\phi(a)$. □

3.5.4. The images of the $\mathbb{C}^*$-fixed points in $\mathbb{C}^n/S_n$. We are now going to identify the images of the $\mathbb{C}^*$-fixed points under ρ_1 and ρ_2. Set $\mathbf{e} = (e_0, \dots, e_{l-1}) \in \mathbb{Q}^l$, where $e_0 = 0$ and $e_i = \sum_{j=1}^{i} H_j$ for $i = 1, \dots, l-1$. Set $\theta := \theta_{\mathbf{h}}$ as in (3.7). For the rest of this section fix $\mu \in \mathcal{P}_\varnothing(nl)$. Let us identify a point $(a_1, \dots, a_n) \in \mathbb{C}^n/S_n$ with the "polynomial" $\sum_{i=1}^n t^{a_i}$.

Lemma 3.5.11 ([98, §5.4])**.** *Let* $\underline{\lambda} \in \mathcal{P}(l, n)$*. Then* $\rho_1(\chi_{\underline{\lambda}}) = \mathrm{Res}^{\mathbf{e}}_{\underline{\lambda}}(t^h)$*.*

Definition 3.5.12. Let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be the Frobenius form of μ. For each $1 \le i \le k$, let $r_i = b_i + 1$ and $m_i = a_i + b_i + 1$. Recall the matrices $\Lambda(m_i)$ and $A(m_i, r_i)$ from Definition 3.4.1. If $\Lambda(m_i)A(m_i, r_i) = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_{m_i-1}, \alpha_{m_i})$, then we define

$$\mathrm{Eig}(\mu, i) = \sum_{\substack{1 \le j \le m_i, \\ j = r_i - 1 \bmod l}} t^{\alpha_j}, \quad \mathrm{Eig}(\mu) = \sum_{i=1}^{k} \mathrm{Eig}(\mu, i).$$

Lemma 3.5.13. *We have* $\rho_2([\mathbf{A}(\mu)]) = \mathrm{Eig}(\mu)$*.*

Proof. The polynomial $\mathrm{Eig}(\mu)$ picks out exactly the eigenvalues of the restricted endomorphism $\Lambda(\mu)A(\mu)|_{\mathbf{V}_1}$ from all the eigenvalues of $\Lambda(\mu)A(\mu)$. But these are the same as the eigenvalues of $A(\mu)\Lambda(\mu)|_{\mathbf{V}_0}$. The fact that $\rho_2([\mathbf{A}(\mu)]) = \mathrm{Eig}(\mu)$ now follows immediately from the definition of ρ_2. □

By Proposition 3.5.10, Lemma 3.5.11, Lemma 3.5.13 and the fact that a multipartition is uniquely determined by its $\mathbf{e}$-residue for generic $\mathbf{e}$, we have

$$\mathrm{Eig}(\mu) = \rho_2([\mathbf{A}(\mu)]) = \rho_1(\chi_{\underline{\lambda}}) = \mathrm{Res}^{\mathbf{e}}_{\underline{\lambda}}(t^h) \tag{3.23}$$

for unique $\underline{\lambda} \in \mathcal{P}(l, n)$.

Definition 3.5.14. Define $\underline{\mathsf{Eig}}(\mu) := \big(\mathsf{Eig}(\mu)^0, \mathsf{Eig}(\mu)^1, \dots, \mathsf{Eig}(\mu)^{l-1}\big) \in \mathcal{P}(l, n)$ by the equation

$$\mathrm{Res}^{\mathbf{e}}_{\underline{\mathsf{Eig}}(\mu)}(t^h) = \mathrm{Eig}(\mu).$$

We thus have a bijection

$$\underline{\mathsf{Eig}} : \mathcal{P}_\varnothing(nl) \to \mathcal{P}(l, n), \quad \mu \mapsto \underline{\mathsf{Eig}}(\mu).$$

Proposition 3.5.15. *Let $\mu \in \mathcal{P}_\varnothing(nl)$. The inverse of the Etingof-Ginzburg isomorphism sends the $\mathbb{C}^*$-fixed point $[\mathbf{A}(\mu)]$ in $\mathcal{C}_\mathbf{h}$ to the $\mathbb{C}^*$-fixed point $\chi_{\underline{\mathsf{Eig}}(\mu)}$ in $\mathcal{Y}_\mathbf{h}$.*

Proof. This follows directly from (3.23). □

3.5.5. Calculation of $\mathrm{Eig}(\mu)$**.** Set $\mathbf{e}' = (e'_0, e'_1, \dots, e'_{l-1})$ and $\mathbf{e}'' = (e''_0, e''_1 \dots, e''_{l-1})$ with $e'_0 = -h$, $e''_{l-1} = 0$ and

$$e'_i = e_i \quad (i = 1, \dots, l-1), \qquad e''_i = h + \sum_{j=1}^{l-i-1} H_j \quad (i = 0, \dots, l-2).$$

In these notations all the lower indices are to be considered mod l.

Lemma 3.5.16. *Let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k) \in \mathcal{P}_\varnothing(nl)$. Then*

$$\mathrm{Eig}(\mu) = \sum_{i=1}^{k} \left(\left(t^{e'_{b_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h} \right) + \left(t^{e''_{a_i}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h} \right) \right). \tag{3.24}$$

Proof. It suffices to show that for each $i = 1, \dots, k$ we have

$$\mathrm{Eig}(\mu, i) = \left(t^{e'_{b_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h} \right) + \left(t^{e''_{a_i}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h} \right). \tag{3.25}$$

We can write

$$\mathrm{Eig}(\mu, i) := \sum_{\substack{1 \le j \le m_i, \\ j = r_i - 1 \bmod l}} t^{\alpha_j} = \sum_{\substack{1 \le j \le r_i - 1, \\ j = r_i - 1 \bmod l}} t^{\alpha_j} + \sum_{\substack{r_i \le j \le m_i, \\ j = r_i - 1 \bmod l}} t^{\alpha_j}. \tag{3.26}$$

Note that $r_i - 1 = b_i = l \cdot (\lceil b_i/l \rceil - 1) + d_i$, where d_i is an integer such that $1 \le d_i \le l$. The $j's$ satisfying $1 \le j \le r_i - 1$ and $j = r_i - 1 \bmod l$ are therefore precisely $b_i, b_i - l, b_i - 2l, \dots, b_i - (\lceil b_i/l \rceil - 1) \cdot l = d_i$. Recall that $\theta_0 + \theta_1 + \ldots + \theta_{l-1} = -h$. Hence $\alpha_{d_i + pl} = \alpha_{d_i} - ph$ for $p = 0, \dots, \lceil b_i/l \rceil - 1$, by Definition 3.4.1. Therefore

$$\sum_{\substack{1 \le j \le r_i - 1, \\ j = r_i - 1 \bmod l}} t^{\alpha_j} = t^{\alpha_{d_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}.$$

Observe that $d_i = b_i \bmod l$ if $1 \le d_i < l$. Hence $\alpha_{d_i} = \sum_{j=1}^{d_i} \theta_{b_i+1-j} = \sum_{j=1}^{d_i} \theta_{d_i+1-j} = \sum_{j=1}^{d_i} \theta_j = e'_{d_i} = e'_{b_i}$. If $d_i = l$ then $\alpha_{d_i} = \alpha_l = \sum_{j=1}^{l} \theta_{b_i+1-j} = \sum_{j=1}^{l} \theta_j = -h = e'_0$. This shows that

$$\sum_{\substack{1 \le j \le r_i - 1, \\ j = r_i - 1 \bmod l}} t^{\alpha_j} = t^{e'_{b_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}.$$

Let us now consider the second sum on the RHS os (3.26). We have $m_i - r_i + 1 = a_i + 1 = l \cdot \lfloor (a_i+1)/l \rfloor + c_i$ with $0 \le c_i < l$. The $j's$ satisfying $r_i \le j \le m_i$ and $j = r_i - 1 \bmod l$ are therefore precisely $b_i + l, b_i + 2l, \dots, b_i + \lfloor (a_i+1)/l \rfloor \cdot l$. Note that $b_i + \lfloor (a_i+1)/l \rfloor \cdot l = m_i - c_i$. Hence $\alpha_{m_i - c_i - pl} = \alpha_{m_i - c_i} + ph$ for $p = 0, \dots, \lfloor (a_i+1)/l \rfloor - 1$. One computes, in a similar fashion as above, that $\alpha_{m_i-c_i} = e''_{a_i}$. This shows that

$$\sum_{\substack{r_i \le j \le m_i,\\ j = r_i - 1 \bmod l}} t^{\alpha_j} = t^{e''_{a_i}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h}. \qquad \square$$

3.6 $\mathbb{C}^*$-fixed points under the Etingof-Ginzburg isomorphism

We will now identify the multipartition $\underline{\mathsf{Eig}}(\mu)$ and thereby establish the correspondence between the $\mathbb{C}^*$-fixed points under the Etingof-Ginzburg isomorphism.

3.6.1. The strategy. Our next goal is to show that $\underline{\mathsf{Eig}}(\mu) = \underline{\mathsf{Quot}}(\mu)^\flat$. We will use the following strategy. Recall Lemma 3.3.7. We first prove that an analogous statement holds for the multipartition $\underline{\mathsf{Eig}}(\mu)$. This will allow us to argue by induction on n. We then prove that $\underline{\mathsf{Eig}}(\mu) = \underline{\mathsf{Quot}}(\mu)^\flat$ for partitions μ with the special property that only a unique l-rim-hook can be removed from μ. We then deduce the result for arbitrary $\mu \in \mathcal{P}_\varnothing(nl)$.

3.6.2. Types and contributions of Frobenius hooks. We need to introduce some notation to break down formula (3.24) into simpler pieces. Throughout this section fix $\mu \in \mathcal{P}_\varnothing(nl)$.

Definition 3.6.1. Let $\mu = (a_1, \ldots, a_k \mid b_1, \ldots, b_k)$ be the Frobenius form of μ. Let $F_1, \ldots, F_k$ be the Frobenius hooks in $\mathbb{Y}(\mu)$ so that (i,i) is the root of F_i. Let

$$\mathsf{type}_\mu(L,i) := b_i \bmod l, \quad \mathsf{type}_\mu(A,i) := -(a_i+1) \bmod l \quad (i = 1, \ldots, k).$$

We call the number $\mathsf{type}_\mu(L,i)$ the *type* of $\mathsf{leg}(F_i)$ and the number $\mathsf{type}_\mu(A,i)$ the *type* of $\mathsf{arm}(F_i)$. Let $\mathbf{e}$, $\mathbf{e}'$ and $\mathbf{e}''$ be as in §3.5.4 and §3.5.5. Define

$$\Xi_\mu(L,i) := t^{e'_{b_i}} \sum_{j=1}^{\lceil b_i/l \rceil} t^{-(j-1)h}, \quad \Xi_\mu(A,i) := t^{e''_{a_i}} \sum_{j=1}^{\lfloor (a_i+1)/l \rfloor} t^{(j-1)h}.$$

We call $\Xi_\mu(L,i)$ the *contribution* of $\mathsf{leg}(F_i)$ and $\Xi_\mu(A,i)$ the *contribution* of $\mathsf{arm}(F_i)$.

By (3.25) we have

$$\mathrm{Eig}(\mu, i) = \Xi_\mu(L,i) + \Xi_\mu(A,i). \tag{3.27}$$

Lemma 3.6.2. *We have*

$$t^{e_j}\,\mathrm{Res}_{\mathsf{Eig}(\mu)^j}(t^h) = \sum_{\substack{1 \le i \le k,\\ \mathsf{type}_\mu(A,i) = j}} \Xi_\mu(A,i) \; + \sum_{\substack{1 \le i \le k,\\ \mathsf{type}_\mu(L,i) = j}} \Xi_\mu(L,i). \tag{3.28}$$

Proof. Each summand t^d on the RHS of (3.24) corresponds (non-canonically) to a cell in the multipartition $\underline{\mathsf{Eig}}(\mu)$ in the sense that it describes that cell's $\mathbf{e}$-shifted content. For generic $\mathbf{e}$ we can write $t^d = t^{e_i} t^c$ for unique $i = 0, \ldots, l-1$ and $c \in \mathbb{Z}$. The summand t^d corresponds to a cell in the partition $\mathsf{Eig}(\mu)^j$ if and only if $i = j$, i.e., $t^d = t^{e_j} t^c$. Since $\mathrm{Eig}(\mu) = \sum_{p=1}^k \mathrm{Eig}(\mu, p)$, formula (3.27) implies that there exists an $1 \le p \le k$ such that t^d is a summand in $\Xi_\mu(L,p)$ or $\Xi_\mu(A,p)$. In the former case $t^d = t^{e_j} t^c$ if and only if $j = b_p \bmod l = \mathsf{type}_\mu(L,p)$. In the latter case $t^d = t^{e_j} t^c$ if and only if $e''_{a_p} = h + e_j$, which is the case if and only if $j = -(a_p+1) \bmod l = \mathsf{type}_\mu(A,p)$. $\square$

3.6.3. Removal of rim-hooks. We will now investigate the effect of removing a rim-hook from μ on the multipartition $\underline{\mathsf{Eig}}(\mu)$. Let $\mathbb{Y}_{+/-/0}(\mu)$ denote the subset of $\mathbb{Y}(\mu)$ consisting of cells of positive/negative/zero content.

Lemma 3.6.3. *Let R be an l-rim-hook in $\mathbb{Y}(\mu)$ and suppose that $R \subset (\mathbb{Y}_0(\mu) \cup \mathbb{Y}_+(\mu))$. Suppose that R intersects r Frobenius hooks, labelled $F_{p+1}, \ldots, F_{p+r}$ so that (i,i) is the root of F_i. Let $\mu' := \mu - R$. Then*

$$\mathsf{type}_{\mu'}(A,j) = \mathsf{type}_{\mu}(A,j+1), \quad \Xi_{\mu'}(A,j) = \Xi_{\mu}(A,j+1) \quad (j = p+1, \ldots, p+r-1),$$

$$\mathsf{type}_{\mu'}(A,p+r) = \mathsf{type}_{\mu}(A,p+1), \quad \Xi_{\mu'}(A,p+r) = \Xi_{\mu}(A,p+1) - M,$$

where M is the (monic) monomial in $\Xi_{\mu}(A,p)$ of highest degree.

Proof. It is clear that R must intersect adjacent Frobenius hooks. Recall that the residue of R is of the form $\mathrm{Res}_R(t) = \sum_{i=i_0}^{i=i_0+l-1} t^i$ with $i_0 \geq 0$. Moreover, we have

$$\mathrm{Res}_{R \cap F_j}(t) = \sum_{i=i_{r+p-j}}^{i_{r+p-j+1}-1} t^i \tag{3.29}$$

for some integers $i_0 < i_1 < \ldots < i_r = i_0 + l$. One can easily see that these integers satisfy

$$i_{r+p-j+1} - 1 = a_j, \tag{3.30}$$

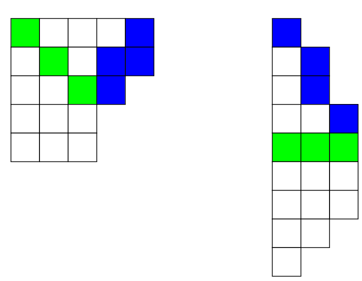
where $a_j = |\mathsf{arm}(F_j)| = \max_{\square \in F_j} c(\square)$. Set $d_j := \max_{\square \in F_j - R} c(\square)$. If $F_j - R = \varnothing$ set $d_j = -1$. From (3.29) and (3.30) we easily deduce that

$$d_j = a_{j+1} \quad (j = p+1, \ldots, p+r-1), \quad d_{p+r} = a_{p+1} - l. \tag{3.31}$$

By definition, the type and contribution of $\mathsf{arm}(F_j)$ resp. $\mathsf{arm}(F_j - R)$ depend only on the numbers a_j and d_j. The lemma now follows immediately from the definitions. $\square$

One can easily formulate a version of Lemma 3.6.3 for $R \subset (\mathbb{Y}_0(\mu) \cup \mathbb{Y}_-(\mu))$. The proof is completely analogous. Lemma 3.6.3 admits the following graphical interpretation.

Example 3.6.4. The figure on the left shows the Young diagram of the partition $(5,5,4,3,3)$. The cells of content zero are marked as green. The blue cells form a 4-rim-hook. The figure on the right shows the same Young diagram rearranged so that cells of the same content occupy the same row.



Using this visual representation we can easily determine the impact of removing the blue rim-hook. The length of the first arm after the removal equals the length of the second arm before the removal. Similarly, the length of the second arm after the removal equals the length of the third arm before the removal. Finally, the length of the third arm after the removal equals the length of the first arm before the removal minus four. This is precisely the content of Lemma 3.6.3.

Proposition 3.6.5. *Let R be a rim-hook in μ and set $\mu' := \mu - R$. Then $\underline{\mathsf{Eig}}(\mu') = \underline{\mathsf{Eig}}(\mu) - \blacksquare$ for some $\blacksquare \in \underline{\mathsf{Eig}}(\mu)$.*

Proof. There are three possibilities: $R \subset \mathbb{Y}_0(\mu) \cup \mathbb{Y}_+(\mu)$, $R \subset \mathbb{Y}_0(\mu) \cup \mathbb{Y}_-(\mu)$ or $R \cap \mathbb{Y}_+(\mu) \neq \varnothing, R \cap \mathbb{Y}_-(\mu) \neq \varnothing$. Consider the first case. Lemma 3.6.3 and Lemma 3.6.2 imply that there exists a $j \in \{0, \dots, l-1\}$ such that $\mathsf{Eig}(\mu')^i = \mathsf{Eig}(\mu)^i$ if $i \neq j$ and $t^{e_j} \operatorname{Res}_{\mathsf{Eig}(\mu')^j}(t^h) = t^{e_j} \operatorname{Res}_{\mathsf{Eig}(\mu)^j}(t^h) - t^{s_j} M$ for some monic monomial $M = t^{qh} \in \mathbb{Z}[t^h]$. Hence $\underline{\mathsf{Eig}}(\mu') = \underline{\mathsf{Eig}}(\mu) - \blacksquare$ for some $\blacksquare \in \underline{\mathsf{Eig}}(\mu)$ with $c(\blacksquare) = q$.

The second case is analogous. Now consider the third case. We claim that $\Xi_\mu(A, i) = 0$ for every Frobenius hook F_i whose arm intersects R nontrivially. Indeed, by definition $\Xi_\mu(A, i) \neq 0$ only if $|\mathsf{arm}(F_i)| + 1 \geq l$. We have $|\mathsf{arm}(F_i)| = \max_{\square \in \mathsf{arm}(F_i)} c(\square) = \max_{\square \in \mathsf{arm}(F_i) \cap R} c(\square)$. Hence $|\mathsf{arm}(F_i)| \leq \max_{\square \in R} c(\square)$. However, since $R \cap \mathbb{Y}_-(\mu) \neq \varnothing$, the rim-hook R must contain a cell of content -1. The fact that $\operatorname{Res}_R(t) = t^q \sum_{p=0}^{l-1} t^p$ for some $q \in \mathbb{Z}$ implies that $\max_{\square \in R} c(\square) \leq l - 2$. Hence $|\mathsf{arm}(F_i)| + 1 \leq l - 1$ and so $\Xi(A, i) = 0$. Therefore the removal of R does not affect the contribution of the arm of any Frobenius hook.

Now set $R' := R \cap (\mathbb{Y}_0(\mu) \cup \mathbb{Y}_-(\mu))$. We have reduced the third case back to the second case, with the modification that R' is now a truncated rim-hook. We can still apply Lemma 3.6.3 with minor adjustments. In particular, equations (3.31) are still true with the exception that the final equation becomes $d_{p+r} = 0$. Let j be the smallest integer such that $\mathsf{leg}(F_j) \cap R \neq \varnothing$. Using the same argument as before, we conclude that $t^{e_j} \operatorname{Res}_{\mathsf{Eig}(\mu')^j}(t^h) = t^{e_j} \operatorname{Res}_{\mathsf{Eig}(\mu)^j}(t^h) - t^{e_j - h}$ and $\mathsf{Eig}(\mu')^i = \mathsf{Eig}(\mu)^i$ if $i \neq j$. □

3.6.4. Partitions with a unique removable rim-hook.

In this section we show that $\underline{\mathsf{Eig}}(\mu) = \underline{\mathsf{Quot}}(\mu)^\flat$ for a certain class of partitions which we call *l-special*.

Definition 3.6.6. We say that a partition μ is *l-special* if the rim of $\mathbb{Y}(\mu)$ contains a unique l-rim-hook R. We call R the *unique removable l-rim-hook* in $\mathbb{Y}(\mu)$. Let $\mathcal{P}^{sp}_\varnothing(k)$ denote the set of partitions of k which are l-special and have a trivial l-core.

Our goal now is to describe partitions of nl which are l-special and have a trivial l-core. Throughout this subsection we assume that $\mu \in \mathcal{P}^{sp}_\varnothing(nl)$. We let R denote the unique removable l-rim-hook in $\mathbb{Y}(\mu)$ and set $\mu' := \mu - R$. Sometimes, for the sake of brevity, we will just write "rim-hook" instead of l-rim-hook.

Lemma 3.6.7. *Let $\mu \in \mathcal{P}^{sp}_\varnothing(nl)$. Then:*

a) *Every column of $\mathbb{B}(\mu)$ contains the same number of beads.*
b) *Sliding distinct beads up results in the removal of distinct l-rim-hooks from $\mathbb{Y}(\mu)$.*
c) *The bead diagram $\mathbb{B}(\mu)$ contains $l-1$ columns with no gaps and one column with a unique string of adjacent gaps.*

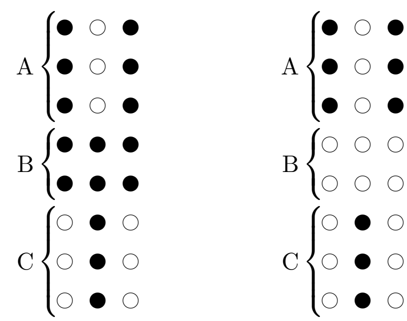
Proof. (a) The empty bead diagram describes the trivial partition. But bead diagrams which describe the trivial partition and have the property that the number of beads in the diagram is divisible by l are unique up to adding or deleting full rows at the top of the diagram. Hence any such diagram consists of consecutive full rows at the top. Since μ has a trivial l-core, the process of sliding beads upward in $\mathbb{B}(\mu)$ must result in a bead diagram of this shape. But this is only possible if every column of $\mathbb{B}(\mu)$ contains the same number of beads.

(b) We can see this by considering the quotients of partitions corresponding to the bead diagrams obtained by moving up distinct beads. If the beads moved are on distinct runners, then a box is removed from distinct partitions in $\underline{\mathsf{Quot}}(\mu)$, so distinct multipartitions arise. If the beads are on the same runner, sliding upward distinct beads implies changing the first-column hook lengths in different ways in the same partition, so different multipartitions arise as well. But a trivial-core partition is uniquely determined by its quotient, so these distinct multipartitions are quotients of distinct partitions of $l(n-1)$.
(c) Since only one rim-hook can be removed from μ, only one bead in our bead diagram can be moved upward. This implies that $l-1$ runners contain no gaps (i.e. they contain a consecutive string of beads counting from the top). The remaining runner must contain a unique gap or a unique string of gaps. □

Lemma 3.6.8. *The bead diagram $\mathbb{B}(\mu)$ can be decomposed into three blocks A, B and C, counting from the top. Each block consists of identical rows. Rows in block A are full except for one bead. Let's say that the gap due to the absent bead is on runner k. Rows in block B are either all full or all empty. Rows in block C are empty except for one bead on runner k. Moreover, the number of rows in block A equals the number of rows in block C.*

Proof. This is an immediate consequence of Lemma 3.6.7. □

Example 3.6.9. Let $l = 3$. According to Lemma 3.6.8 the following bead diagrams correspond to 3-special partitions:



The bead diagram on the left describes the partition $(8, 6, 4, 3^7, 2^2, 1^2)$ while the bead diagram on the right describes the partition $(14, 12, 10, 3, 2^2, 1^2)$.

In the sequel we will only consider the case where all the rows in block B are full. All the following claims can easily be adapted to the case of empty rows.

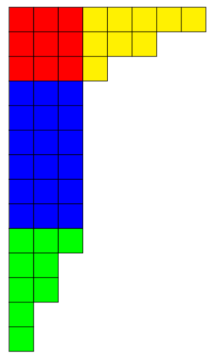
Lemma 3.6.10. *Suppose that block A of $\mathbb{B}(\mu)$ has m rows and block B of $\mathbb{B}(\mu)$ has p rows. Then $\mathbb{Y}(\mu)$ can be decomposed into four blocks $\hat{A}, \hat{B}, \hat{C}, \hat{D}$:*

- *$\hat{A}$ is the Young diagram of the partition corresponding to the bead diagram A.*
- *$\hat{B}$ is a rectangle consisting of m columns and $l \cdot p$ rows.*
- *If $k \neq 0$ block $\hat{C}$ is the Young diagram of the partition corresponding to the bead diagram C. If $k = 0$ block $\hat{C}$ is the the Young diagram of the partition corresponding to the bead diagram obtained from C by inserting an extra row at the top, which is full except for the empty point in column $l-1$.*
- *Block $\hat{D}$ is a square with m rows and columns.*

We recover $\mathbb{Y}(\mu)$ from these blocks by placing $\hat{A}$ at the bottom, stacking $\hat{B}$ on top, then stacking $\hat{D}$ on top and finally placing $\hat{C}$ on the right hand side of $\hat{D}$.

Proof. This follows from Lemma 3.6.8 by a routine calculation - one merely has to recover the first column hook lengths from the positions of the beads. □

Example 3.6.11. Consider the first bead diagram from Example 3.6.9. Below we illustrate the block decomposition of the corresponding Young diagram as in Lemma 3.6.10.



Block $\hat{A}$ Block $\hat{B}$ Block $\hat{C}$ Block $\hat{D}$

We are now ready to investigate the effect of removing the rim-hook R.

Lemma 3.6.12. *The partition μ can be decomposed into m Frobenius hooks. The unique removable rim-hook lies on the outermost Frobenius hook.*

Proof. By Lemma 3.6.10, the 0-th diagonal of the Young diagram of μ is contained in $\hat{D}$ and contains m boxes. Hence there are m Frobenius hooks. It follows easily from Lemma 3.6.8 and Lemma 3.6.10 that the outermost Frobenius hook F_m in μ is the partition of the form $(k+1, 1^{l\cdot p+l-k-1})$. In particular, it contains $l\cdot(p+1)\geq l$ cells. Let us first consider F_m as a partition in its own right and check whether it contains an l-rim-hook. If $p=0$ then F_m is itself a rim-hook. If $p>1$ then the subset $1^{l\cdot p+l-k-1}$ of F_m contains a rim-hook. Hence, in either case, F_m contains a rim-hook. But a rim-hook of the outermost Frobenius hook F_m is also a rim-hook of μ (because the outermost Frobenius hook is part of the rim). □

Proposition 3.6.13. *We have $\underline{\mathsf{Eig}}(\mu')=\underline{\mathsf{Eig}}(\mu)-\blacksquare$, where $\blacksquare$ is the unique removable cell in $\mathsf{Eig}(\mu)^{l-k-1}$ with content $-p$.*

Proof. The outermost Frobenius hook F_m in μ is the partition of the form $(k+1, 1^{l\cdot p+l-k-1})$. By removing the rim-hook R we obtain the partition $(k+1, 1^{l\cdot p-k-1})$ if $p\geq 1$ or the trivial partition if $p=0$. Since the rim-hook R is contained in the outermost Frobenius hook, its removal does not affect the type and contribution of the arms and legs of the other Frobenius hooks.

There are several cases to be considered. Let $a_m=|\mathsf{arm}(F_m)|$ and $b_m=|\mathsf{leg}(F_m)|$. If $k\neq l-1$, then $\Xi_\mu(L,m)=t^{e'_{b_m \bmod l}}\sum_{i=0}^{p}t^{-ih}$ and $\mathsf{type}_\mu(L,m)=b_m=l-k-1$ (while $\Xi_\mu(A,m)=0$). If $k=l-1$ and $p>0$ then $\Xi_\mu(L,m)=\sum_{i=1}^{p}t^{-ih}$ and $\mathsf{type}_\mu(L,m)=0$ (while $\Xi_\mu(A,m)=1$). If $k=l-1, p=0$ then $\Xi_\mu(A,m)=1$ and $\mathsf{type}_\mu(A,m)=0$ (while $\Xi_\mu(L,m)=0$).

Upon removing the rim-hook the polynomials listed above change as follows. We have $\Xi_{\mu'}(L,m)$ $=t^{e'_{b_m \bmod l}}\sum_{i=0}^{p-1}t^{-ih}$ in the first case, $\Xi_{\mu'}(L,m)=\sum_{i=1}^{p-1}t^{-ih}$ in the second case and $\Xi_{\mu'}(A,m)=0$ in third case. The types do not change. We observe that in each case a monomial of degree t^{-ph} (up to a shift) is subtracted, which corresponds to removing a cell of content $-p$ in $\mathsf{Eig}(\mu)^{l-k-1}$. □

We obtain an analogous result for the multipartition $\underline{\mathsf{Quot}}(\mu)$.

Lemma 3.6.14. *The following hold:*

a) $\underline{\mathsf{Quot}}(\mu) = (Q^0(\mu), \dots, Q^{l-1}(\mu))$ *is a multipartition consisting of* $l-1$ *trivial partitions and one non-trivial partition.*
b) *Suppose that the* k*-th column in* $\mathbb{B}(\mu)$ *is the unique column which contains gaps. Then* $Q^k(\mu)$ *is the unique non-trivial partition in* $\underline{\mathsf{Quot}}(\mu)$*. If that column has a string of* m *gaps followed by a string of* $q = m + p$ *beads then the Young diagram of* $Q^k(\mu)$ *is a rectangle consisting of* m *columns and* q *rows.*

Proof. The l-quotient of μ can be deduced directly from the bead diagram $\mathbb{B}(\mu)$. The description of the latter in Lemma 3.6.8 immediately implies the present lemma. □

Recall that $\mu' := \mu - R$, where R is the unique rim-hook which can be removed from μ.

Lemma 3.6.15. *We have* $\underline{\mathsf{Quot}}(\mu') = \underline{\mathsf{Quot}}(\mu) - \square$*, where* $\square$ *is the box in the bottom right corner of the rectangle described in Lemma 3.6.14. That box has content* $-p$*.*

Proof. This is the only cell which can be removed from $\underline{\mathsf{Quot}}(\mu)$ by Lemma 3.6.14, so the claim now follows from Lemma 3.3.7. □

The lemma implies in particular that $\underline{\mathsf{Quot}}(\mu')^\flat = \underline{\mathsf{Quot}}(\mu)^\flat - \square$, where $\square$ is a box of content $-p$ in the $(l-k-1)$-th partition in $\underline{\mathsf{Quot}}(\mu)^\flat$.

3.6.5. Induction.

We will now use induction on n to show that $\underline{\mathsf{Quot}}(\mu)^\flat = \underline{\mathsf{Eig}}(\mu)$. There are two cases to be considered: $\mu \in \mathcal{P}^{sp}_{\varnothing}(nl)$ and $\mu \notin \mathcal{P}^{sp}_{\varnothing}(nl)$.

Proposition 3.6.16. *Suppose that* $\underline{\mathsf{Quot}}(\lambda)^\flat = \underline{\mathsf{Eig}}(\lambda)$ *for any partition* $\lambda \vdash l(n-1)$ *with trivial* l*-core. Let* $\mu \in \mathcal{P}^{sp}_{\varnothing}(nl)$*. Then* $\underline{\mathsf{Quot}}(\mu)^\flat = \underline{\mathsf{Eig}}(\mu)$*.*

Proof. By induction, $\underline{\mathsf{Quot}}(\mu')^\flat = \underline{\mathsf{Eig}}(\mu')$. But by Proposition 3.6.13 and Lemma 3.6.15 both $\underline{\mathsf{Quot}}(\mu)^\flat$ and $\underline{\mathsf{Eig}}(\mu)$ arise from $\underline{\mathsf{Quot}}(\mu')^\flat = \underline{\mathsf{Eig}}(\mu')$ by adding a box of content $-p$ to the $(l-k-1)$-th partition. Hence $\underline{\mathsf{Quot}}(\mu)^\flat = \underline{\mathsf{Eig}}(\mu)$. □

Proposition 3.6.17. *Suppose that* $\underline{\mathsf{Quot}}(\lambda)^\flat = \underline{\mathsf{Eig}}(\lambda)$ *for any partition* $\lambda \vdash l(n-1)$ *with trivial* l*-core. Let* $\mu \notin \mathcal{P}^{sp}_{\varnothing}(nl)$*. Then* $\underline{\mathsf{Quot}}(\mu)^\flat = \underline{\mathsf{Eig}}(\mu)$*.*

Proof. Since $\mu \notin \mathcal{P}^{sp}_{\varnothing}(\mu)$ we can remove two distinct (but possibly overlapping) rim-hooks R' and R'' from μ. Let $\mu' = \mu - R'$ and $\mu'' = \mu - R''$. Then $\mu' \neq \mu''$ and so $\underline{\mathsf{Quot}}(\mu') \neq \underline{\mathsf{Quot}}(\mu'')$ (because the quotient of a partition with trivial core determines that partition uniquely). By Lemma 3.3.7 we have

$$\underline{\mathsf{Quot}}(\mu')^\flat = \underline{\mathsf{Quot}}(\mu)^\flat - \square, \quad \underline{\mathsf{Quot}}(\mu'')^\flat = \underline{\mathsf{Quot}}(\mu)^\flat - \hat{\square}$$

with $\square \neq \hat{\square} \in \underline{\mathsf{Quot}}(\mu)^\flat$. By Proposition 3.6.5 we have

$$\underline{\mathsf{Eig}}(\mu') = \underline{\mathsf{Eig}}(\mu) - \blacksquare, \quad \underline{\mathsf{Eig}}(\mu'') = \underline{\mathsf{Eig}}(\mu) - \hat{\blacksquare}$$

for some $\blacksquare, \hat{\blacksquare} \in \underline{\mathsf{Eig}}(\mu)$. We know that $\underline{\mathsf{Eig}}$ establishes a bijection between l-partitions of $n-1$ and partitions of $l(n-1)$ with a trivial l-core. Hence $\blacksquare \neq \hat{\blacksquare}$. By the inductive hypothesis in our lemma,

$$\underline{\mathsf{Quot}}(\mu')^\flat = \underline{\mathsf{Eig}}(\mu'), \quad \underline{\mathsf{Quot}}(\mu'')^\flat = \underline{\mathsf{Eig}}(\mu'').$$

Hence

$$\underline{\mathsf{Quot}}(\mu)^\flat - \square = \underline{\mathsf{Eig}}(\mu) - \blacksquare, \quad \underline{\mathsf{Quot}}(\mu)^\flat - \hat{\square} = \underline{\mathsf{Eig}}(\mu) - \hat{\blacksquare}$$

and so

$$\underline{\mathsf{Eig}}(\mu) = \underline{\mathsf{Quot}}(\mu)^\flat - \square + \blacksquare = \underline{\mathsf{Quot}}(\mu)^\flat - \hat{\square} + \hat{\blacksquare}.$$

Since $\square \neq \hat{\square}$ and $\blacksquare \neq \hat{\blacksquare}$ we conclude that $\square = \blacksquare$ and $\hat{\square} = \hat{\blacksquare}$. Therefore $\underline{\mathsf{Eig}}(\mu) = \underline{\mathsf{Quot}}(\mu)^\flat$. □

Theorem 3.6.18. *Let $\mu \in \mathcal{P}_\varnothing(nl)$. Then $\underline{\mathsf{Quot}}(\mu)^\flat = \underline{\mathsf{Eig}}(\mu)$. The bijection between the labelling sets of $\mathbb{C}^*$-fixed points induced by the Etingof-Ginzburg isomorphism is given by*

$$\mathcal{P}(l,n) \to \mathcal{P}_\varnothing(nl), \quad \underline{\mathsf{Quot}}(\mu)^\flat \mapsto \mu.$$

Proof. The first claim follows directly from Propositions 3.6.16 and 3.6.17. The second claim follows from Proposition 3.5.15. □

Remark 3.6.19. As a corollary, we also obtain the following explicit formula for the residue of the l-quotient of a partition $\mu = (a_1, \ldots, a_k \mid b_1, \ldots, b_k) \in \mathcal{P}_\varnothing(nl)$. Let us write $\underline{\mathsf{Quot}}(\mu) = (Q^0, \ldots, Q^{l-1})$. Then

$$\mathrm{Res}_{Q^{l-j-1}}(t) = \sum_{\substack{1 \le i \le k, \\ -(a_i+1) = j \bmod l}} t^{p_{a_i}} \sum_{m=1}^{\lfloor (a_i+1)/l \rfloor} t^{(m-1)} + \sum_{\substack{1 \le i \le k, \\ b_i = j \bmod l}} t^{p'_{b_i}} \sum_{m=1}^{\lceil b_i/l \rceil} t^{-(m-1)},$$

where $p_i = 1$ for $i = 0, \ldots, l-2$ and $p_{l-1} = 0$ while $p'_0 = -1$ and $p'_i = 0$ for $i = 1, \ldots, l-1$. Indeed, the RHS of the formula above equals $\mathrm{Res}_{\mathsf{Eig}(\mu)^j}(t)$ by (3.28). But $\mathsf{Eig}(\mu)^j = Q^{l-j-1}$ by Theorem 3.6.18.

3.7 The higher level q-hook formula

We will now use Theorem 3.6.18 to obtain the following "higher level" generalization of the q-hook formula.

Theorem 3.7.1. *Let $\mu \in \mathcal{P}_\varnothing(nl)$. Then*

$$\sum_{\square \in \mu} t^{c(\square)} = [nl]_t \sum_{\underline{\lambda} \uparrow \underline{\mathsf{Quot}}(\mu)^\flat} \frac{f_{\underline{\lambda}}(t)}{f_{\underline{\mathsf{Quot}}(\mu)^\flat}(t)}. \tag{3.32}$$

Our proof is based on comparing the $\mathbb{C}^*$-characters of the vector bundles $\mathcal{V}_{\mathbf{h}}$ and $\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}}$, and uses Proposition 3.4.15, Theorem 3.6.18 as well as the Etingof-Ginzburg isomorphism. The remaining ingredient is a calculation of the $\mathbb{C}^*$-characters of the fibres $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}}$ (see §2.2.5). We carry out this calculation below. We first identify $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}}$ with a graded shift of $e_{n-1}L(\underline{\lambda})$. We next recall the graded multiplicity with which $L(\underline{\lambda})$ occurs in $\bar{\Delta}(\underline{\lambda})$ and calculate the character of $e_{n-1}\bar{\Delta}(\underline{\lambda})$. Finally, we use the equation

$$\mathrm{ch}_t\, e_{n-1} L(\underline{\lambda}) = \frac{\mathrm{ch}_t\, e_{n-1}\bar{\Delta}(\underline{\lambda})}{[\bar{\Delta}(\underline{\lambda}) : L(\underline{\lambda})]_{\mathsf{gr}}}.$$

Note that, setting $l = 1$ in (3.32), we recover the usual q-hook formula, so our result also gives a new geometric proof of this well-known combinatorial identity.

Remark 3.7.2. As explained in the introduction, (3.32) can be regarded as a quantization of the classical hook-length formula (1.6) or its corollary (3.4). These formulas generalize easily to the wreath product case. Since irreducible modules over $S_n \wr \mathbb{Z}_l$ are defined by induction from modules over parabolic subgroups of S_n, we have

$$d_{\underline{\mu}} = \frac{n!}{\prod_{i=0}^{l-1} |\mu_i|!} \prod_{i=0}^{l-1} d_{\mu_i} = \frac{n!}{\prod_{i=0}^{l-1} |\mu_i|!} \prod_{i=0}^{l-1} \frac{|\mu_i|!}{h_{\mu_i}} = n! \prod_{i=0}^{l-1} \frac{1}{h_{\mu_i}}.$$

Moreover,

$$n = \sum_{\substack{\underline{\lambda} \uparrow \underline{\mu}, \\ \lambda^i \neq \mu^i}} \frac{h_{\mu_i}}{h_{\lambda_i}}. \tag{3.33}$$

We can regard (3.32) as a quantization of (3.33). However, there is a crucial difference between the classical and quantized formulas. While the classical wreath product formula is derived from the corresponding formula for the symmetric group, we were not able to derive the higher level q-hook formula from the ordinary q-hook formula purely by algebraic manipulations.

3.7.1. Coinvariant algebras and fake degree polynomials. The algebra $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ is a graded Γ_n-module. It is well known that $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ is isomorphic to the regular representation $\mathbb{C}\Gamma_n$ as an ungraded Γ_n-module.

Let $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}}$ denote the restriction of $\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}$ to a Γ_{n-1}-module. Let $\mathfrak{h}' \subset \mathfrak{h}$ denote the subspace spanned by $y_2, \ldots, y_n$. We choose a splitting $\mathfrak{h} \twoheadrightarrow \mathfrak{h}'$ with kernel spanned by y_1. This splitting induces an inclusion $\mathbb{C}[\mathfrak{h}'] \subset \mathbb{C}[\mathfrak{h}]$.

Lemma 3.7.3. *We have an isomorphism of graded Γ_{n-1}-modules*

$$\mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}} \cong \mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes U,$$

where U is a graded vector space with Poincaré polynomial $\mathrm{ch}_t\, U = [nl]_t$.

Proof. We have a sequence of inclusions of graded Γ_{n-1}-modules

$$\mathbb{C}[\mathfrak{h}]^{\Gamma_n} \hookrightarrow \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} \hookrightarrow \mathbb{C}[\mathfrak{h}]$$

such that each ring is a free graded module over the previous ring. Hence there is an isomorphism of graded Γ_{n-1}-modules

$$\mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \rangle \cong \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_{n-1}} \rangle \otimes \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \rangle.$$

Observe that there is also an isomorphism of graded Γ_{n-1}-modules

$$\mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_{n-1}} \rangle \cong \mathbb{C}[\mathfrak{h}']/\langle \mathbb{C}[\mathfrak{h}']_+^{\Gamma_{n-1}} \rangle = \mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}}.$$

To prove the lemma it now suffices to find the Poincaré polynomial of the graded vector space $\mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \rangle$. We know that $\mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}$ is a polynomial algebra with generators in degrees $l, 2l, \ldots, (n-1)l$ and an additional generator in degree 1. The ring $\mathbb{C}[\mathfrak{h}]^{\Gamma_n}$ is a polynomial algebra with generators in degrees $l, 2l, \ldots, nl$. Hence

$$\mathrm{ch}_t\, \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}} = \frac{1}{1-t} \prod_{i=1}^{n-1} \frac{1}{1-t^{il}}, \quad \mathbb{C}[\mathfrak{h}]^{\Gamma_n} = \prod_{i=1}^{n} \frac{1}{1-t^{il}}.$$

It follows that $\mathrm{ch}_t\, \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}/\langle \mathbb{C}[\mathfrak{h}]_+^{\Gamma_n} \rangle = \frac{\mathrm{ch}_t\, \mathbb{C}[\mathfrak{h}]^{\Gamma_{n-1}}}{\mathrm{ch}_t\, \mathbb{C}[\mathfrak{h}]^{\Gamma_n}} = \frac{1-t^{nl}}{1-t} = [nl]_t$. ∎

Definition 3.7.4. Suppose that we are given an l-multipartition $\underline{\lambda} \in \mathcal{P}(l,n)$ and the corresponding irreducible representation $S(\underline{\lambda})$ of Γ_n. We regard $S(\underline{\lambda})$ as a graded Γ_n-module concentrated in degree zero. The *fake degree polynomial* associated to $\underline{\lambda}$ is defined as

$$f_{\underline{\lambda}}(t) := \sum_{k \in \mathbb{Z}} [\mathbb{C}[\mathfrak{h}]^{co\Gamma_n} : S(\underline{\lambda})^*[k]] t^k.$$

Theorem 3.7.5 ([130, Theorem 5.3]). *Let $\underline{\lambda} \in \mathcal{P}(l,n)$. We have*

$$f_{\underline{\lambda}}(t) = t^{r(\underline{\lambda})}(t^l)_n \prod_{i=0}^{l-1} \frac{t^{l\cdot n(\lambda^i)}}{H_{\lambda^i}(t^l)} = t^{r(\underline{\lambda})}(t^l)_n \prod_{i=0}^{l-1} s_{\lambda^i}(1, t^l, t^{2l}, \ldots).$$

In particular, if λ is a partition of n then $f_\lambda = (t)_n \frac{t^{n(\lambda)}}{H_\lambda(t)} = (t)_n s_\lambda(1, t, t^2, \ldots)$.

3.7.2. Auxiliary calculations. Fix $\underline{\lambda} \in \mathcal{P}(l,n)$. Let $q(\underline{\lambda})$ denote the degree in which the trivial Γ_n-module triv occurs in $L(\underline{\lambda})$.

Lemma 3.7.6. *We have a graded $\mathtt{H}_{\mathbf{h}}$-module isomorphism*

$$\mathcal{R}_{\mathbf{h},\underline{\lambda}} = \mathtt{H}_{\mathbf{h}} e_n \otimes_{e_n \mathtt{H}_{\mathbf{h}} e_n} e_n L_{\underline{\lambda}} \cong L(\underline{\lambda})[-q(\underline{\lambda})]$$

and hence a graded vector space isomorphism

$$(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}} = e_{n-1}\mathcal{R}_{\mathbf{h},\underline{\lambda}} \cong e_{n-1}L(\underline{\lambda})[-q(\underline{\lambda})].$$

Proof. As ungraded $\mathtt{H}_{\mathbf{h}}$-modules, $\mathcal{R}_{\mathbf{h},\underline{\lambda}}$ and $L(\underline{\lambda})$ are clearly isomorphic. Since they are simple, one is a graded shift of the other. The trivial Γ_n-representation triv occurs in $L(\underline{\lambda})$ in degree $q(\underline{\lambda})$. On the other hand, we can identify triv with the subspace $e_n \otimes_{e_n \mathtt{H}_{\mathbf{h}} e_n} e_n L_{\underline{\lambda}}$ of $\mathcal{R}_{\mathbf{h},\underline{\lambda}}$, so triv occurs in $\mathcal{R}_{\mathbf{h},\underline{\lambda}}$ in degree zero. □

Let us calculate the graded multiplicity of $L(\underline{\lambda})$ in $\bar{\Delta}(\underline{\lambda})$.

Lemma 3.7.7. *Let $\underline{\lambda} \in \mathcal{P}(l,n)$. The simple $\overline{\mathtt{H}}_{\mathbf{h}}$-module $L(\underline{\lambda})$ occurs in $\bar{\Delta}(\underline{\lambda})$ with graded multiplicity*

$$\sum_{k \in \mathbb{Z}} [\bar{\Delta}(\underline{\lambda}) : L(\underline{\lambda})[k]] t^k = t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t).$$

Proof. By [10, Lemma 3.3], we have

$$\operatorname{ch}_t L(\underline{\lambda}) = \frac{t^{q(\underline{\lambda})} \operatorname{ch}_t \bar{\Delta}(\underline{\lambda})}{f_{\underline{\lambda}}(t)}.$$

Hence

$$\sum_{k \in \mathbb{Z}} [\bar{\Delta}(\underline{\lambda}) : L(\underline{\lambda})[k]] t^k = \frac{\operatorname{ch}_t \bar{\Delta}(\underline{\lambda})}{\operatorname{ch}_t L(\underline{\lambda})} = t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t).$$ □

Let us calculate the character of $e_{n-1}\bar{\Delta}(\underline{\lambda})$.

Lemma 3.7.8. *We have*

$$\operatorname{ch}_t e_{n-1}\bar{\Delta}(\underline{\lambda}) = [ln]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} f_{\underline{\mu}}(t). \tag{3.34}$$

Proof. By Lemma 3.7.3 and Proposition 2.2.1, we have isomorphisms of graded Γ_{n-1}-modules

$$\begin{aligned}\bar{\Delta}(\underline{\lambda})|_{\Gamma_{n-1}} &\cong \mathbb{C}[\mathfrak{h}]^{co\Gamma_n}|_{\Gamma_{n-1}} \otimes S(\underline{\lambda})|_{\Gamma_{n-1}} \\ &\cong \left(\mathbb{C}[\mathfrak{h}']^{co\Gamma_{n-1}} \otimes U\right) \otimes \bigoplus_{\underline{\mu}\uparrow\underline{\lambda}} S(\underline{\mu}) \cong \bigoplus_{\underline{\mu}\uparrow\underline{\lambda}} \bar{\Delta}(\underline{\mu}) \otimes U,\end{aligned}$$

where U is a graded vector space with character $\mathrm{ch}_t\, U = [ln]_t$. Hence

$$e_{n-1}\bar{\Delta}(\underline{\lambda}) \cong \bigoplus_{\underline{\mu}\uparrow\underline{\lambda}} e_{n-1}\bar{\Delta}(\underline{\mu}) \otimes U \tag{3.35}$$

as graded Γ_{n-1}-modules. For each $\underline{\mu} \uparrow \underline{\lambda}$, we have

$$\mathrm{ch}_t\, e_{n-1}\bar{\Delta}\left(\underline{\mu}\right) = \sum_{k\in\mathbb{Z}} [\bar{\Delta}(\underline{\mu}) : \mathsf{triv}[k]] t^k = f_{\underline{\mu}}(t). \tag{3.36}$$

The first equality above is obvious, for the second see, e.g., the proof of [63, Theorem 5.6]. Combining (3.35) with (3.36) we obtain (3.34). □

3.7.3. The character of the fibre. We can now put our calculations together to obtain the character of $(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}}$.

Theorem 3.7.9. *Let $\underline{\lambda} \in \mathcal{P}(l,n)$. Then*

$$\mathrm{ch}_t(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}} = [ln]_t \sum_{\underline{\mu}\uparrow\underline{\lambda}} \frac{f_{\underline{\mu}}(t)}{f_{\underline{\lambda}}(t)}.$$

Proof. By Lemmas 3.7.6, 3.7.7 and 3.7.8, we have

$$\begin{aligned}\mathrm{ch}_t(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}} &= t^{-q(\underline{\lambda})} \cdot \mathrm{ch}_t\, e_{n-1}L(\underline{\lambda}) = (t^{-q(\underline{\lambda})} \cdot \mathrm{ch}_t\, e_{n-1}\bar{\Delta}(\underline{\lambda}))/(t^{-q(\underline{\lambda})} f_{\underline{\lambda}}(t)) \\ &= \mathrm{ch}_t\, e_{n-1}\bar{\Delta}(\underline{\lambda})/f_{\underline{\lambda}}(t) = [ln]_t \sum_{\underline{\mu}\uparrow\underline{\lambda}} \frac{f_{\underline{\mu}}(t)}{f_{\underline{\lambda}}(t)}.\end{aligned}$$

□

Corollary 3.7.10. *We have*

$$\mathrm{ch}_t(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\lambda}} = \frac{1}{1-t}\sum_{i=0}^{l-1} t^{-i} \sum_{\substack{\underline{\mu}\uparrow\underline{\lambda},\\ \mu^i\neq\lambda^i}} \frac{s_{\mu^i}(1,t^l,t^{2l},\ldots)}{s_{\lambda^i}(1,t^l,t^{2l},\ldots)} = \frac{1}{1-t}\sum_{i=0}^{l-1} t^{-i} \sum_{\substack{\underline{\mu}\uparrow\underline{\lambda},\\ \mu^i\neq\lambda^i}} \frac{t^{l\cdot n(\mu^i)} H_{\lambda^i}(t^l)}{t^{l\cdot n(\lambda^i)} H_{\mu^i}(t^l)}. \tag{3.37}$$

In particular, if $l=1$ then $\mathrm{ch}_t(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\lambda} = \frac{1}{1-t}\sum_{\mu\uparrow\lambda} \frac{s_\mu(1,t,t^2,\ldots)}{s_\lambda(1,t,t^2,\ldots)} = \frac{1}{1-t}\sum_{\mu\uparrow\lambda} \frac{t^{n(\mu)}}{t^{n(\lambda)}}\frac{H_\lambda(t)}{H_\mu(t)}$.

Proof. This follows immediately from Theorems 3.7.5 and 3.7.9. □

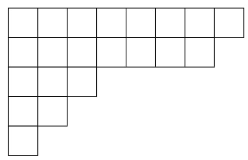
We can now prove Theorem 3.7.1.

Proof of Theorem 3.7.1. Choose any parameter $\mathbf{h} \in \mathbb{Q}^l$ such that $\operatorname{Spec} \mathsf{Z}_{\mathbf{h}}$ is smooth. By Theorem 3.2.5 the Etingof-Ginzburg map induces an isomorphism of vector bundles $\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}} \cong \mathcal{V}_{\mathbf{h}}$. Since the Etingof-Ginzburg map is $\mathbb{C}^*$-equivariant, we have

$$\mathrm{ch}_t(\mathcal{V}_{\mathbf{h}})_\mu = \mathrm{ch}_t(\mathcal{R}_{\mathbf{h}}^{\Gamma_{n-1}})_{\underline{\mathsf{Quot}}(\mu)^\flat}, \tag{3.38}$$

by Theorem 3.6.18. Proposition 3.4.15 yields the formula for the LHS of (3.38) while Theorem 3.7.9 yields the formula for the RHS. □

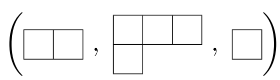
3.7.4. Example. We would like to illustrate the higher level q-hook formula by means of an explicit example. Set $l = 3$ and consider the following partition

$\mu =$ 

of 21. We see directly that

$$\sum_{\square \in \mu} t^{c(\square)} = (t^{-4} + t^{-3} + t^{6} + t^{7}) + 2(t^{-2} + t^{-1} + t + t^{2} + t^{3} + t^{4} + t^{5}) + 3. \tag{3.39}$$

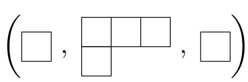
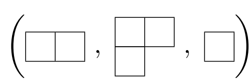
The reverse of the quotient of μ is

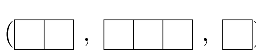
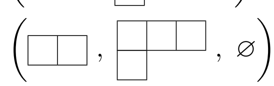
$\underline{\lambda} = \underline{\mathsf{Quot}}(\mu)^{\flat} =$  .

The corresponding hook polynomials are given by

$$H_{\lambda^0}(t^3) = (1 - t^3)(1 - t^6), \quad H_{\lambda^1}(t^3) = (1 - t^{12})(1 - t^6)(1 - t^3)^2, \quad H_{\lambda^2}(t^3) = (1 - t^3).$$

The multipartition $\underline{\mathsf{Quot}}(\mu)^{\flat}$ has four submultipartitions, namely:

$\underline{\nu} =$  , $\underline{\xi} =$  ,

$\underline{\zeta} =$  , $\underline{\gamma} =$  .

We need the following hook polynomials:

$$H_{\nu^0}(t^3) = (1 - t^3), \quad H_{\xi^1}(t^3) = (1 - t^9)(1 - t^3)^2, \quad H_{\zeta^1}(t^3) = (1 - t^3)(1 - t^6)(1 - t^9), \quad H_{\gamma^2}(t^3) = 1.$$

The relevant ratios of the hook polynomials entering the formula (3.37) are:

$$\frac{H_{\lambda^0}(t^3)}{H_{\nu^0}(t^3)} = (1 - t^6), \quad \frac{H_{\lambda^1}(t^3)}{H_{\xi^1}(t^3) + t^{l \cdot n(\lambda^1)} H_{\zeta^1}(t^3)} = \frac{1 - t^{12}}{t^3}, \quad \frac{H_{\lambda^2}(t^3)}{H_{\nu^2}(t^3)} = (1 - t^3).$$

Putting these into (3.37), we get the equality

$$[21]_t \sum_{\underline{\mu} \uparrow \underline{\lambda}} \frac{f_{\underline{\mu}}(t)}{f_{\underline{\lambda}}(t)} = \frac{(1 - t^6) + (1 - t^3)t^{-2} + (1 - t^{12})t^{-4}}{1 - t} = [6]_t + [3]_t t^{-2} + [12]_t t^{-4},$$

which, after an easy calculation, turns out to be equal to (3.39).

3.8 $\mathbb{C}^*$-fixed points under reflection functors

Assume from now on that $l > 1$. In this section we compute the bijections between $\mathbb{C}^*$-fixed points induced by reflection functors (see §3.3.9).

Fix $\nu \in \heartsuit(l)$ and let $\mathbf{d}_\nu = (d_0, \dots, d_{l-1})$ be the corresponding dimension vector (see §3.3.7). Assume that $\theta \in \mathbb{Q}^l$ is chosen so that $\theta_i \neq 0$ and the quiver variety $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)$ is smooth. The reflection functor

$$\mathfrak{R}_i : \mathcal{X}_\theta(n\delta + \mathbf{d}_\nu) \to \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \mathbf{d}_{\sigma_i * \nu}) \tag{3.40}$$

is a $\mathbb{C}^*$-equivariant isomorphism and hence induces a bijection $\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)^{\mathbb{C}^*} \longleftrightarrow \mathcal{X}_{\sigma_i \cdot \theta}(n\delta + \mathbf{d}_{\sigma_i * \nu})^{\mathbb{C}^*}$ between the $\mathbb{C}^*$-fixed points. Composing it with the bijections from Corollary 3.4.16, we obtain a bijection

$$\mathbf{R}_i : \mathcal{P}_{\nu^t}(nl + |\nu^t|) \to \mathcal{P}_{(\sigma_i * \nu)^t}(nl + |(\sigma_i * \nu)^t|). \tag{3.41}$$

Fix $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$. We are going to show that

$$\mathbf{R}_i(\mu) = (\mathbf{T}_i(\mu^t))^t,$$

where $\mathbf{T}_i(\mu^t)$ is the partition obtained from μ^t by adding all i-addable and removing all i-removable cells relative to μ^t, as in Definition 3.3.9.

3.8.1. The strategy. Our goal is to describe the action of reflection functors on the $\mathbb{C}^*$-fixed points combinatorially. In §3.8.2 we endow the vector space $\widehat{\mathbf{V}}^\nu$ with a $\mathbb{Z}$-grading, which we call the "μ-grading". A $\mathbb{C}^*$-fixed point is characterized uniquely by this grading. In §3.8.4 we compute the $\mathbf{R}_i(\mu)$-grading on the vector space $\widehat{\mathbf{V}}^{\sigma_i * \nu}$. In §3.8.5 and §3.8.6 we use this calculation to give a combinatorial description of the partition $\mathbf{R}_i(\mu)$.

3.8.2. The μ-grading. Fix a $\mathbb{Z}/l\mathbb{Z}$-graded complex vector space $\widehat{\mathbf{V}}^\nu := \bigoplus_{i=0}^{l-1} \mathbf{V}_i^\nu$ with $\dim \mathbf{V}_i^\nu = n + d_i$. Set $\mathbf{V}^\nu = \widehat{\mathbf{V}}^\nu \oplus \mathbf{V}_\infty$ with $\dim \mathbf{V}_\infty = 1$. We are now going to introduce a $\mathbb{Z}$-grading on $\widehat{\mathbf{V}}^\nu$ which "lifts" the $\mathbb{Z}/l\mathbb{Z}$-grading.

Definition 3.8.1. We call a $\mathbb{Z}$-grading $\widehat{\mathbf{V}}^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_i$ a *μ-grading* if it satisfies the following condition:

(C) for each $i \in \mathbb{Z}$ we have

$$A(\mu)(\mathbf{W}_i) \subseteq \mathbf{W}_{i-1}, \quad \Lambda(\mu)(\mathbf{W}_i) \subseteq \mathbf{W}_{i+1}, \quad J(\mu)(\mathbf{V}_\infty) \subseteq \mathbf{W}_0, \quad I(\mu)(\mathbf{W}_0) = \mathbf{V}_\infty$$

where $A(\mu), \Lambda(\mu), J(\mu)$ and $I(\mu)$ are as in Definition 3.4.6.

Proposition 3.8.2. *A μ-grading on $\widehat{\mathbf{V}}^\nu$ exists and is unique.*

Proof. We first prove existence. Let $\mu = (a_1, \dots, a_k \mid b_1, \dots, b_k)$ be the Frobenius form of μ. Set $r_i = b_i + 1$, $m_i = a_i + b_i + 1$ and $q_i = \sum_{j<i} m_j + r_i$ for $1 \le i \le k$. Recall the ordered basis $\{\mathsf{Bas}(i) \mid 1 \le i \le nl + |\nu|\}$ of $\widehat{\mathbf{V}}^\nu$ from §3.4.3. We now define $\mathbf{W}_j$ by the rule that for each $1 \le i \le k$:

$$\mathsf{Bas}(q_i - j) \in \mathbf{W}_j \quad (0 \le j \le b_i), \qquad \mathsf{Bas}(q_i + j) \in \mathbf{W}_{-j} \quad (1 \le j \le a_i). \tag{3.42}$$

It follows directly from the construction of the matrices $A(\mu), \Lambda(\mu), J(\mu)$ and $I(\mu)$ that this grading satisfies condition (**C**) in Definition 3.8.1. Hence it is a μ-grading. Finally we observe that the definition of $\Lambda(\mu)$ and Lemma 3.4.10 imply that

$$\mathbf{W}_j = (A(\mu))^j(\mathbf{W}_0) \quad (j < 0), \qquad \mathbf{W}_j = (\Lambda(\mu))^j(\mathbf{W}_0) \quad (j > 0). \tag{3.43}$$

We now prove uniqueness. Let $\widehat{\mathbf{V}}^\nu = \bigoplus_{i\in\mathbb{Z}} \mathbf{W}'_i$ be a μ-grading. Choose $0 \neq v_\infty \in \mathbf{V}_\infty$. Then by the definition of $J(\mu)$ for a generic parameter θ there exist $c_1, \ldots, c_k \in \mathbb{C}^*$ such that $u_1 := c_1\mathsf{Bas}(q_1) + \ldots + c_k\mathsf{Bas}(q_k) = J(\mu)(v_\infty) \in \mathbf{W}'_0$. Lemma 3.4.10 together with the fact that the parameter θ is generic implies that $0 \neq (A(\mu))^{a_1}(\mathsf{Bas}(q_1)) = t(A(\mu))^{a_1}(u_1)$ for some scalar $t \in \mathbb{C}^*$. Since $u_1 \in \mathbf{W}'_0$ and the operator $A(\mu)$ lowers degree by one, we have $0 \neq (A(\mu))^{a_1}(\mathsf{Bas}(q_1)) \in \mathbf{W'}_{-a_1}$. It now follows from the definition of the matrices $A(\mu)$ and $\Lambda(\mu)$ and the genericity of θ that $(\Lambda(\mu))^{a_1} \circ (A(\mu))^{a_1}(u_1) = t'\mathsf{Bas}(q_1)$ for some scalar $t' \in \mathbb{C}^*$. Since the operator $\Lambda(\mu)$ raises degree by one, we have $\mathsf{Bas}(q_1), c_2\mathsf{Bas}(q_2) + \ldots + c_k\mathsf{Bas}(q_k) \in \mathbf{W}'_0$.

We can now apply an analogous argument to the vectors $\mathsf{Bas}(q_2)$ and $u_2 := c_2\mathsf{Bas}(q_2) + \ldots + c_k\mathsf{Bas}(q_k)$. By Lemma 3.4.10 we have $0 \neq (A(\mu))^{a_2}(\mathsf{Bas}(q_2)) = t(A(\mu))^{a_2}(u_2) + t'(A(\mu))^{a_2}(\mathsf{Bas}(q_1))$ for some scalars $t, t' \in \mathbb{C}^*$. Since u_2 and $\mathsf{Bas}(q_1)$ are homogeneous elements of degree zero and $A(\mu)$ lowers degree by one, we get $0 \neq (A(\mu))^{a_2}(\mathsf{Bas}(q_2)) \in \mathbf{W'}_{-a_2}$. Moreover, Lemma 3.4.10 implies that $(A(\mu))^{a_2}(\mathsf{Bas}(q_2))$ is a linear combination of $\mathsf{Bas}(q_1 + a_2)$ and $\mathsf{Bas}(q_2 + a_2)$. But $\mathsf{Bas}(q_1 + a_2) = (A(\mu))^{a_2}(\mathsf{Bas}(q_1))$ up to multiplication by a non-zero scalar, so $\mathsf{Bas}(q_1 + a_2) \in \mathbf{W'}_{-a_2}$. Hence $\mathsf{Bas}(q_2 + a_2) \in \mathbf{W'}_{-a_2}$ and so $\mathsf{Bas}(q_2) = (\Lambda(\mu))^{a_2}(\mathsf{Bas}(q_2 + a_2)) \in \mathbf{W}'_0$. We conclude that $\mathsf{Bas}(q_2), u_3 := c_3\mathsf{Bas}(q_3) + \ldots + c_k\mathsf{Bas}(q_k) \in \mathbf{W}'_0$. Repeating this argument sufficiently many times shows that $\mathsf{Bas}(q_1), \ldots, \mathsf{Bas}(q_k) \in \mathbf{W}'_0$. It follows that $\mathbf{W}'_0 = \mathbf{W}_0$. Condition (**C**) and (3.43) now imply that $\mathbf{W}'_i = \mathbf{W}_i$ for all $i \in \mathbb{Z}$. □

Thanks to Proposition 3.8.2, we can talk about *the* μ-grading on $\widehat{\mathbf{V}}^\nu$. Let us denote it by $\widehat{\mathbf{V}}^\nu = \bigoplus_{i\in\mathbb{Z}} \mathbf{W}^\mu_i$. We write $\deg_\mu v = i$ if $v \in \mathbf{W}^\mu_i$. Moreover, let $P_\mu := \sum_{i\in\mathbb{Z}} \dim \mathbf{W}^\mu_i t^i$ denote the Poincaré polynomial of $\mathbf{V}^\nu$ with respect to the μ-grading. Consider the function

$$\mathcal{X}_\theta(n\delta + \mathbf{d}_\nu)^{\mathbb{C}^*} \to \mathbb{Z}[t, t^{-1}], \quad [\mathbf{A}(\mu)] \mapsto P_\mu. \tag{3.44}$$

Proposition 3.8.3. *We have $P_\mu = \mathrm{Res}_{\mu^t}(t)$. Moreover, the function (3.44) is injective.*

Proof. Fix $j \geq 0$. By (3.42), $\mathsf{Bas}(q_i - j) \in \mathbf{W}^\mu_j$ if and only if $j \leq b_i$, for $i = 1, \ldots, k$. Moreover, $\{\mathsf{Bas}(q_i - j) \mid j \leq b_i\}$ form a basis of $\mathbf{W}^\mu_j$. Hence $\dim \mathbf{W}^\mu_j = \sum_{i=1}^k \mathbf{1}_{j\leq b_i}$, where $\mathbf{1}_{j\leq b_i}$ is the indicator function taking value 1 if $j \leq b_i$ and 0 otherwise. But $\sum_{i=1}^k \mathbf{1}_{j\leq b_i}$ is precisely the number of cells of content $-j$ in μ, which is the same as the number of cells of content j in μ^t. The argument for $j < 0$ is analogous. This proves the first claim. The second claim follows immediately from the fact that partitions are determined uniquely by their residues. □

Suppose that we are given a $\mathbb{C}^*$-fixed point and want to find out which partition it is labelled by. Proposition 3.8.3 implies that to do so we only need to compute the $\mathbb{Z}$-grading on $\widehat{\mathbf{V}}^\nu$ associated to the fixed point and the corresponding Poincaré polynomial.

Lemma 3.8.4. *The following hold:*

a) *The restricted maps*

$$A(\mu) : \mathbf{W}^\mu_i \to \mathbf{W}^\mu_{i-1} \ (i \leq 0), \quad \Lambda(\mu) : \mathbf{W}^\mu_i \to \mathbf{W}^\mu_{i+1}, (i \geq 0)$$

are surjective.

b) *The restricted maps*

$$A(\mu) : \mathbf{W}_i^\mu \to \mathbf{W}_{i-1}^\mu \ (i > 0), \quad \Lambda(\mu) : \mathbf{W}_i^\mu \to \mathbf{W}_{i+1}^\mu \ (i < 0)$$

are injective.

c) *We have* $\mathbf{V}_j^\nu = \bigoplus_{i \in \mathbb{Z}} \mathbf{W}_{j+il}^\mu$ *for each* $j = 0, \ldots, l-1$.

Proof. The first claim is just a restatement of (3.43). The second claim follows directly from Lemma 3.4.10. The third claim follows from (3.43) and the fact that $A(\mu)$ (resp. $\Lambda(\mu)$) is a homogeneous operator of degree -1 (resp. 1) with respect to the $\mathbb{Z}/l\mathbb{Z}$-grading on $\widehat{\mathbf{V}}^\nu$. □

Remark 3.8.5. We may interpret Lemma 3.8.4(c) as saying that the μ-grading on $\widehat{\mathbf{V}}^\nu$ is a *lift* of the $\mathbb{Z}/l\mathbb{Z}$-grading. Furthermore, we may think of the μ-grading as the grading on the representation of the A_∞-quiver corresponding to the fixed point $[\mathbf{A}(\mu)]$. More details about the connection between quiver varieties of type A_∞ and the $\mathbb{C}^*$-fixed points in cyclic quiver varieties can be found in [118, §4].

3.8.3. Explicit definition of reflection functors.

Let us recall the explicit definition of reflection functors from [103, Proposition 3.19] and [29, §5]. Fix a $\mathbb{Z}/l\mathbb{Z}$-graded complex vector space $\widehat{\mathbf{V}}^{\sigma_i * \nu} := \bigoplus_{j=0}^{l-1} \mathbf{V}_j^{\sigma_i * \nu}$ with $\dim \mathbf{V}_j^{\sigma_i * \nu} = n + d_j'$. Set $\mathbf{V}^{\sigma_i * \nu} := \widehat{\mathbf{V}}^{\sigma_i * \nu} \oplus \mathbf{V}_\infty$. To simplify notation, set $\mathbf{d} := \mathbf{d}_\nu$ and $\mathbf{d}' := \sigma_i * \mathbf{d}_\nu$.

Let us first define a map $\mu_{n\delta+\mathbf{d}}^{-1}(\theta) \to \mu_{n\delta+\mathbf{d}'}^{-1}(\sigma \cdot \theta)$ lifting (3.40), which we also denote by $\mathfrak{R}_i$. Let $\rho = (\mathbf{X}, \mathbf{Y}, I, J) \in \mu_{n\delta+\mathbf{d}}^{-1}(\theta)$. The reflected quiver representation

$$\mathfrak{R}_i(\rho) =: (\mathbf{X}', \mathbf{Y}', I', J')$$

is defined as follows. Suppose that $i \neq 0$. We have maps

$$\mathbf{V}_i^\nu \xrightarrow{Y_i - X_i} \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \xrightarrow{X_{i-1} + Y_{i+1}} \mathbf{V}_i^\nu. \tag{3.45}$$

Set $\psi_i := Y_i - X_i$ and $\phi_i := X_{i-1} + Y_{i+1}$ (the indices should be considered mod l). The preprojective relations (i.e. the relations defining the fibre $\mu_{n\delta+\mathbf{d}}^{-1}(\theta)$) ensure that we have a splitting $\mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \operatorname{Im} \psi_i \oplus \ker \phi_i$. The underlying vector space of the quiver representation $\mathfrak{R}_i(\rho)$ is obtained from $\mathbf{V}^\nu$ by replacing $\mathbf{V}_i^\nu$ with $\ker \phi_i$. We have an isomorphism of vector spaces $\mathbf{V}^{\sigma_i * \nu} \cong \ker \phi_i \oplus \bigoplus_{j \neq i} \mathbf{V}_j^\nu \oplus \mathbf{V}_\infty$ preserving the quiver grading. Let us define $\mathfrak{R}_i(\rho)$. We have $X_j' = X_j$ unless $j \in \{i-1, i\}$. We also have $Y_j' = Y_j$ unless $j \in \{i, i+1\}$. Set $I' = I$ and $J' = J$. The maps X_i' and Y_i' are defined as the composite maps

$$X_i' : \ \ker \phi_i \hookrightarrow \ker \phi_i \oplus \operatorname{Im} \psi_i = \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \twoheadrightarrow \mathbf{V}_{i+1}^\nu,$$
$$Y_i' : \ \ker \phi_i \hookrightarrow \ker \phi_i \oplus \operatorname{Im} \psi_i = \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu \twoheadrightarrow \mathbf{V}_{i-1}^\nu.$$

The maps X_{i-1}' and Y_{i+1}' are defined as the composite maps

$$X_{i-1}' : \mathbf{V}_{i-1}^\nu \xrightarrow{\cdot(-\theta_i)} \mathbf{V}_{i-1}^\nu \hookrightarrow \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \ker \phi_i \oplus \operatorname{Im} \psi_i \twoheadrightarrow \ker \phi_i, \tag{3.46}$$

$$Y_{i+1}' : \mathbf{V}_{i+1}^\nu \xrightarrow{\cdot(-\theta_i)} \mathbf{V}_{i+1}^\nu \hookrightarrow \mathbf{V}_{i-1}^\nu \oplus \mathbf{V}_{i+1}^\nu = \ker \phi_i \oplus \operatorname{Im} \psi_i \twoheadrightarrow \ker \phi_i \xrightarrow{\cdot(-1)} \ker \phi_i. \tag{3.47}$$

The minus signs before X_i in (3.45) as well as in the last arrow above come from the fact that our quiver does not have a sink at the vertex i as in [29, §5] - hence the need for these adjustments. If

$i = 0$ one defines $\mathfrak{R}_i(\rho)$ analogously using maps

$$\mathbf{V}_0^\nu \xrightarrow{\psi_0} \mathbf{V}_{l-1}^\nu \oplus \mathbf{V}_1^\nu \oplus \mathbf{V}_\infty \xrightarrow{\phi_0} \mathbf{V}_0^\nu \tag{3.48}$$

with $\psi_0 = Y_0 - X_0 + I$ and $\phi_0 = X_{l-1} + Y_1 + J$.

3.8.4. The reflected grading. Let us apply the definitions from §3.8.3 to $\rho = \mathbf{A}(\mu)$. More specifically, for $i = 0, \dots, l-1$, we set $X_i = \Lambda(\mu)|_{\mathbf{V}_i^\nu}$, $Y_i = A(\mu)|_{\mathbf{V}_i^\nu}$, $I = I(\mu)$ and $J = J(\mu)$. The reflected quiver representation $\mathfrak{R}_i(\mathbf{A}(\mu)) \in \mu_{n\delta+\mathbf{d}'}^{-1}(\sigma \cdot \theta)$ is conjugate under the $G(n\delta + \mathbf{d}')$-action to $\mathbf{A}(\mathbf{R}_i(\mu))$. We now compute the $\mathbf{R}_i(\mu)$-grading on $\widehat{\mathbf{V}}^{\sigma_i * \nu}$.

We have direct sum decompositions

$$\psi_i := \bigoplus_{j\in\mathbb{Z}} \psi_i^j, \quad \phi_i := \bigoplus_{j\in\mathbb{Z}} \phi_i^j, \tag{3.49}$$

with $\psi_i^j = \psi_i|_{\mathbf{W}^\mu_{jl+i}}$ and $\phi_i^j = \phi_i|_{\mathbf{W}^\mu_{jl+i-1}\oplus\mathbf{W}^\mu_{jl+i+1}}$ ($i \neq 0$ or $j \neq 0$) and $\psi_0^0 := \psi_0|_{\mathbf{W}_0^\mu}$. Hence $\mathbf{V}_i^{\sigma_i*\nu} = \ker\phi_i = \bigoplus_{j\in\mathbb{Z}} \ker\phi_i^j$. Moreover, (3.45) and (3.48) decompose as direct sums of maps

$$\mathbf{W}^\mu_{jl+i} \xrightarrow{\psi_i^j} \mathbf{W}^\mu_{jl+i-1} \oplus \mathbf{W}^\mu_{jl+i+1} \xrightarrow{\phi_i^j} \mathbf{W}^\mu_{jl+i} \quad (j \neq 0 \text{ or } i \neq 0).$$
$$\mathbf{W}^\mu_0 \xrightarrow{\psi_0^0} \mathbf{W}^\mu_{-1} \oplus \mathbf{W}^\mu_1 \oplus \mathbf{V}_\infty \xrightarrow{\phi_0^0} \mathbf{W}^\mu_0.$$

These direct sum decompositions together with the preprojective relations imply the following lemma.

Lemma 3.8.6. *Let $i \in \{0, \dots, l-1\}$ and $j \in \mathbb{Z}$. If $i = 0$ and $j = 0$ then $\operatorname{Im}\psi_0^0 \oplus \ker\phi_0^0 = \mathbf{W}^\mu_{-1} \oplus \mathbf{W}^\mu_1 \oplus \mathbf{V}_\infty$. Otherwise $\operatorname{Im}\psi_i^j \oplus \ker\phi_i^j = \mathbf{W}^\mu_{jl+i-1} \oplus \mathbf{W}^\mu_{jl+i+1}$.*

Define

$$\mathbf{U}_j := \mathbf{W}^\mu_j \quad (j \neq i \bmod l), \qquad \mathbf{U}_{lj+i} := \ker\phi_i^j \quad (j \in \mathbb{Z}).$$

Proposition 3.8.7. *The $\mathbb{Z}$-grading $\widehat{\mathbf{V}}^{\sigma_i*\nu} = \bigoplus_{j\in\mathbb{Z}} \mathbf{U}_j$ is the $\mathbf{R}_i(\mu)$-grading on $\widehat{\mathbf{V}}^{\sigma_i*\nu}$.*

Proof. It suffices to show that $\widehat{\mathbf{V}}^{\sigma_i*\nu} = \bigoplus_{j\in\mathbb{Z}} \mathbf{U}_j$ satisfies condition (**C**) in Definition 3.8.1. Suppose $i \neq 0$. Then for each $j \in \mathbb{Z}$ we need to check that

$$A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i+1}) \subseteq \mathbf{U}_{jl+i}, \quad \Lambda(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i-1}) \subseteq \mathbf{U}_{jl+i},$$

$$A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i}) \subseteq \mathbf{U}_{jl+i-1}, \quad \Lambda(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i}) \subseteq \mathbf{U}_{jl+i+1}.$$

If $i = 0$ we additionally need to check that $I(\mathbf{R}_i(\mu))(\mathbf{U}_0) = \mathbf{V}_\infty$ and $J(\mathbf{R}_i(\mu))(\mathbf{V}_\infty) \subseteq \mathbf{U}_0$.

All of the inclusions above follow directly from Lemma 3.8.6 and the definition of reflection functors. For example, let us assume that $i \neq 0$ or $j \neq 0$ and consider the inclusion $A(\mathbf{R}_i(\mu))(\mathbf{U}_{jl+i+1}) \subseteq \mathbf{U}_{jl+i} := \ker\phi_i^j$. The map $A(\mathbf{R}_i(\mu)) : \mathbf{V}_{i+1}^{\sigma_i*\nu} = \mathbf{V}_{i+1}^\nu \to \mathbf{V}_i^{\sigma_i*\nu} = \ker\phi_i$ is given by (3.47). Consider its restriction to the subspace $\mathbf{U}_{jl+i+1} = \mathbf{W}^\mu_{jl+i+1} \subseteq \mathbf{V}_{i+1}^\nu$. By Lemma 3.8.6, we have $\mathbf{W}^\mu_{jl+i+1} \subseteq \mathbf{W}^\mu_{jl+i-1} \oplus \mathbf{W}^\mu_{jl+i+1} = \ker\phi_i^j \oplus \operatorname{Im}\psi_i^j$. Hence $A(\mathbf{R}_i(\mu))$, restricted to $\mathbf{W}^\mu_{jl+i+1}$, is the composition

$$\mathbf{W}^\mu_{jl+i+1} \xrightarrow{\cdot(-\theta_i)} \mathbf{W}^\mu_{jl+i+1} \hookrightarrow \mathbf{W}^\mu_{jl+i-1} \oplus \mathbf{W}^\mu_{jl+i+1} = \ker\phi_i^j \oplus \operatorname{Im}\psi_i^j \twoheadrightarrow \ker\phi_i^j \xrightarrow{\cdot(-1)} \ker\phi_i^j.$$

In particular, $A(\mathbf{R}_i(\mu))(\mathbf{W}^\mu_{jl+i+1}) \subseteq \ker\phi_i^j$, as desired. The other inclusions are proven analogously. □

We have described the $\mathbf{R}_i(\mu)$-grading on $\widehat{\mathbf{V}}^{\sigma_i * \nu}$. We are now going to compute the corresponding Poincaré polynomial $P_{\mathbf{R}_i(\mu)}$.

Lemma 3.8.8. *Let $\widehat{\mathbf{V}}^\nu = \bigoplus_{i\in\mathbb{Z}} \mathbf{W}_i^\mu$ be the μ-grading on $\widehat{\mathbf{V}}^\nu$. Then*

$$\dim\ker\phi_i^j = \dim\mathbf{W}^\mu_{lj+i+1} + \dim\mathbf{W}^\mu_{lj+i-1} - \dim\mathbf{W}^\mu_{lj+i}$$

if $j \neq 0$ or $i \neq 0$. Otherwise

$$\dim\ker\phi_0^0 = \dim\mathbf{W}^\mu_1 + \dim\mathbf{W}^\mu_{-1} - \dim\mathbf{W}^\mu_0 + 1.$$

Proof. Assume that $j \neq 0$ or $i \neq 0$. Recall that $\psi_i^j = A(\mu)|_{\mathbf{W}^\mu_{jl+i}} - \Lambda(\mu)|_{\mathbf{W}^\mu_{jl+i}}$. Lemma 3.8.4 implies that either $A(\mu)|_{\mathbf{W}^\mu_{jl+i}}$ or $\Lambda(\mu)|_{\mathbf{W}^\mu_{jl+i}}$ is injective. Hence ψ_i^j is injective. Therefore we have $\dim\operatorname{Im}\psi_i^j = \dim\mathbf{W}^\mu_{lj+i}$. The equality $\dim\ker\phi_i^j = \dim\mathbf{W}^\mu_{lj+i+1} + \dim\mathbf{W}^\mu_{lj+i-1} - \dim\operatorname{Im}\psi_i^j$ now implies the lemma. The case $i = j = 0$ is similar. □

Write $P_\mu = \sum_{j\in\mathbb{Z}} a_j^\mu t^j$ with $a_j^\mu = \dim\mathbf{W}_j^\mu$ and $P_{\mathbf{R}_i(\mu)} = \sum_{j\in\mathbb{Z}} a_j^{\mathbf{R}_i(\mu)} t^j$ with $a_j^{\mathbf{R}_i(\mu)} = \dim\mathbf{W}_j^{\mathbf{R}_i(\mu)}$. Proposition 3.8.7 and Lemma 3.8.8 directly imply the following.

Corollary 3.8.9. *We have $a_j^\mu = a_j^{\mathbf{R}_i(\mu)}$ for $j \not\equiv i \bmod l$. Moreover $a_{lj+i}^{\mathbf{R}_i(\mu)} = a^\mu_{lj+i+1} + a^\mu_{lj+i-1} - a^\mu_{lj+i}$ if $j \neq 0$ or $i \neq 0$ and $a_0^{\mathbf{R}_0(\mu)} = a_1^\mu + a_{-1}^\mu - a_0^\mu + 1$ otherwise.*

3.8.5. Removable and addable cells.

Throughout this subsection let μ be an arbitrary partition. Recall the partition $\mathbf{T}_k(\mu)$ from Definition 3.3.9 obtained from μ by adding all k-addable cells and removing all k-removable cells. We will now interpret addability and removability in terms of the residue of μ.

Lemma 3.8.10. *Let us write $\operatorname{Res}_\mu(t) = \sum_{j\in\mathbb{Z}} b_j t^j$. Let $k \in \mathbb{Z}$.*

a) *The following are equivalent: (1) a cell of content k is removable; (2) $b_{k-1} = b_k, b_{k+1} = b_k - 1$ $(k > 0)$ or $b_{k+1} = b_k, b_{k-1} = b_k - 1$ $(k < 0)$ or $b_{-1} = b_1, b_0 = b_1 + 1$ $(k = 0)$.*
b) *The following are equivalent: (1) a cell of content k is addable; (2) $b_{k+1} = b_k, b_{k-1} = b_k + 1$ $(k > 0)$ or $b_{k-1} = b_k, b_{k+1} = b_k + 1$ $(k < 0)$ or $b_{-1} = b_0 = b_1$ $(k = 0)$.*
c) *The following are equivalent: (1) no cell of content k is addable or removable; (2) $[b_{k+1} = b_k = b_{k-1}$ or $b_{k+1} = b_k - 1 = b_{k-1} - 2$ $(k > 0)]$, or $[b_{k+1} = b_k = b_{k-1}$ or $b_{k+1} = b_k + 1 = b_{k-1} + 2$ $(k < 0)]$, or $[b_{-1} = b_0 = b_1 + 1$ or $b_1 = b_0 = b_{-1} + 1$ $(k = 0)]$.*

Proof. Let $k > 0$. Suppose that the cell (i, j) is removable from $\mathbb{Y}(\mu)$ and has content k. Then $j - i = k$ and $(1, k+1), (2, k+2), \dots, (i, j)$ are precisely the cells of content k in $\mathbb{Y}(\mu)$. Since $\mathbb{Y}(\mu)$ has a corner at k, the cells of content $k-1$ in $\mathbb{Y}(\mu)$ are precisely $(1, k), (2, k+1), \dots, (i, j-1)$ and the cells of content $k+1$ in $\mathbb{Y}(\mu)$ are precisely $(1, k+2), (2, k+3), \dots, (i-1, j)$. Hence $b_k = i, b_{k-1} = i, b_{k+1} = i - 1$, which yields the desired equalities. Conversely, suppose that $b_{k-1} = b_k, b_{k+1} = b_k - 1$. Then the cell $(b_k, b_k + k)$ is removable. Indeed, $b_{k-1} = b_k$ implies that $(b_k + 1, b_k + k) \notin \mathbb{Y}(\mu)$ and $b_{k+1} = b_k - 1$ implies that $(b_k, b_k + k + 1) \notin \mathbb{Y}(\mu)$. The proofs of the remaining cases are analogous. □

3.8.6. Combinatorial interpretation of reflection functors.

We can now interpret the effect of applying reflection functors to the fixed points combinatorially.

Theorem 3.8.11. *Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$ and $0 \le k \le l - 1$. We have $\mathbf{R}_k(\mu) = (\mathbf{T}_k(\mu^t))^t$.*

Proof. It suffices to show that the residue of $\mathbf{T}_k(\mu^t)$ equals $P_{\mathbf{R}_k(\mu)}$. Let us write

$$\mathrm{Res}_{\mathbf{T}_k(\mu^t)}(t) =: \sum_{i\in\mathbb{Z}} a_i t^i, \quad \mathrm{Res}_{\mu^t}(t) =: \sum_{i\in\mathbb{Z}} b_i t^i, \quad P_{\mathbf{R}_k(\mu)} =: \sum_{i\in\mathbb{Z}} c_i t^i, \quad P_\mu =: \sum_{i\in\mathbb{Z}} d_i t^i.$$

By Lemma 3.8.3, we have $b_i = d_i$. Suppose that $i \neq k \bmod l$. Then $a_i = b_i = d_i = c_i$. The first equality follows from the fact that no cells of content i are added to or removed from μ^t when we transform μ^t into $\mathbf{T}_k(\mu^t)$. The third equality follows from Corollary 3.8.9.

Now suppose that $i = k \bmod l$ and $i > 0$. Then $c_i = d_{i+1} + d_{i-1} - d_i$ by Corollary 3.8.9. Hence $c_i = b_{i+1} + b_{i-1} - b_i$. We now argue that $b_{i+1} + b_{i-1} - b_i = a_i$. There are three possibilities: $a_i = b_i + 1$ and one cell of content i is addable to μ^t, or $a_i = b_i - 1$ and one cell of content i is removable from μ^t, or $a_i = b_i$ and no cell of content i is addable to or removable from μ^t. In the first case we have $b_i = b_{i+1}$ and $b_{i-1} = b_i + 1$. In the second case we have $b_i = b_{i-1}$ and $b_{i+1} = b_i - 1$. In the third case we have $b_{i-1} = b_i = b_{i+1}$ or $b_i = b_{i+1} + 1 = b_{i-1} - 1$. These equalities follow immediately from Lemma 3.8.10. In each of the three cases we see that the equality $b_{i+1} + b_{i-1} - b_i = a_i$ holds. Hence $a_i = c_i$. The proof for $i \leq 0$ is analogous. □

Recall that if $\lambda \in \mathcal{P}$ then

$$\underline{\mathsf{Quot}}(\lambda^t) = (\underline{\mathsf{Quot}}(\lambda)^t)^\flat. \tag{3.50}$$

Corollary 3.8.12. *Let $\mu \in \mathcal{P}_{\nu^t}(nl + |\nu^t|)$ and $i \in \{0, \dots, l-1\}$. Then $\mathbf{R}_i(\mu) = (\sigma_i * \mu^t)^t$. Moreover,*

$$\mathsf{Core}(\mathbf{R}_i(\mu)) = (\sigma_i * \nu)^t = (\mathbf{T}_i(\nu))^t, \quad \underline{\mathsf{Quot}}(\mathbf{R}_i(\mu)) = s_{l-i} \cdot \underline{\mathsf{Quot}}(\mu).$$

Proof. The first claim follows directly from Proposition 3.8.11 and the definition of the $\tilde{S}_l$-action on partitions in §3.3.7. The formula for $\mathsf{Core}(\mathbf{R}_i(\mu))$ follows directly from Proposition 3.3.10. The formula for $\underline{\mathsf{Quot}}(\mathbf{R}_i(\mu))$ follows from Proposition 3.3.10 and (3.50). Indeed,

$$\begin{aligned}\underline{\mathsf{Quot}}(\mathbf{R}_i(\mu)) = \underline{\mathsf{Quot}}((\sigma_i * \mu^t)^t) &= ((\underline{\mathsf{Quot}}(\sigma_i * \mu^t))^t)^\flat \\ &= ((s_i \cdot \underline{\mathsf{Quot}}(\mu^t))^t)^\flat \\ &= (s_i \cdot (\underline{\mathsf{Quot}}(\mu))^\flat)^\flat = s_{l-i} \cdot \underline{\mathsf{Quot}}(\mu).\end{aligned}$$ □

Note that, by Proposition 3.3.10, we also have:

$$\mathsf{Core}((\mathbf{R}_i(\mu))^t) = \sigma_i * \nu = \mathbf{T}_i(\nu), \quad \underline{\mathsf{Quot}}((\mathbf{R}_i(\mu))^t) = \mathrm{pr}(\sigma_i) \cdot \underline{\mathsf{Quot}}(\mu) = s_i \cdot \underline{\mathsf{Quot}}(\mu). \tag{3.51}$$

3.9 Connection to the Hilbert scheme

3.9.1. The Hilbert scheme. Let K be a positive integer. We let Hilb_K denote the *Hilbert scheme* of K points in $\mathbb{C}^2$. The underlying set of the scheme Hilb_K consists of ideals of $\mathbb{C}[z_1, z_2]$ of colength K, i.e., ideals $I \subset \mathbb{C}[z_1, z_2]$ such that $\dim \mathbb{C}[z_1, z_2]/I = K$.

We let $\mathbb{C}^*$ act on $\mathbb{C}[z_1, z_2]$ by the rule $t.z_1 = tz_1, t.z_2 = t^{-1}z_2$. This action induces an action on Hilb_K. The $\mathbb{C}^*$-fixed points in Hilb_K are precisely the monomial ideals in $\mathbb{C}[z_1, z_2]$. Let $\lambda \in \mathcal{P}(K)$. Let I_λ be the $\mathbb{C}$-span of the monomials $\{z_1^i z_2^j \mid (i+1, j+1) \notin \mathbb{Y}(\lambda)\}$. We have a bijection

$$\mathcal{P}(K) \longleftrightarrow \mathrm{Hilb}_K^{\mathbb{C}^*}, \quad \lambda \mapsto I_\lambda.$$

Let $\mathcal{T}_K$ denote the tautological bundle on Hilb_K. Its fibre $(\mathcal{T}_K)_I$ at I is isomorphic to $\mathbb{C}[z_1, z_2]/I$. The following lemma follows immediately from the definitions.

Lemma 3.9.1. *Let $\lambda \in \mathcal{P}(K)$. We have $\mathrm{ch}_t(\mathcal{T}_K)_{I_\lambda} = \mathrm{Res}_{\lambda^t}(t)$.*

There is also a $\mathbb{Z}/l\mathbb{Z}$-action on Hilb_K induced by the $\mathbb{Z}/l\mathbb{Z}$-action on $\mathbb{C}[z_1, z_2]$ given by $\epsilon.z_1 = \eta^{-1}z_1, \epsilon.z_2 = \eta z_2$.

3.9.2. Connection to rational Cherednik algebras. Set $-\mathbf{1} := -\frac{1}{l}(1, \dots, 1) \in \mathbb{Q}^l$ and $-\frac{\mathbf{1}}{\mathbf{2}} := -\frac{1}{2l}(1, \dots, 1) \in \mathbb{Q}^l$. Let $w \in \tilde{S}_l$ and set $\theta := w^{-1} \cdot (-\frac{\mathbf{1}}{\mathbf{2}}) \in \mathbb{Q}^l$ as well as $\gamma := w * n\delta \in \mathbb{Z}^l$. We have $\gamma = n\delta + \gamma_0$, where $\gamma_0 = w * \varnothing$. Let $\nu := \mathfrak{d}^{-1}(\gamma_0)$ be the l-core corresponding to γ_0. By [64, Lemmas 4.3, 7.2], the quiver variety $\mathcal{X}_\theta(n\delta)$ is smooth. Set $\mathbf{h} := (h, H_1, \dots, H_{l-1})$ with $H_j = \theta_j$ $(1 \le j \le l-1)$ and $h = -\theta_0 - \sum_{j=1}^{l-1} H_j$. Let us fix a reduced expression $w = \sigma_{i_1} \cdots \sigma_{i_m}$ for w in $\tilde{S}_l$. Composing reflection functors yields a $U(1)$-equivariant hyper-Kähler isometry

$$\mathfrak{R}_{i_1} \circ \cdots \circ \mathfrak{R}_{i_m} : \mathcal{X}_\theta(n\delta) \xrightarrow{\sim} \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma). \tag{3.52}$$

By [64, §3.7] there exists a $U(1)$-equivariant diffeomorphism

$$\mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) \xrightarrow{\sim} \mathcal{M}_{-\mathbf{1}}(\gamma). \tag{3.53}$$

Set $K = nl + |\nu|$. By [64, Lemma 7.8], there is a $\mathbb{C}^*$-equivariant embedding

$$\mathcal{M}_{-\mathbf{1}}(\gamma) \hookrightarrow \mathrm{Hilb}_K. \tag{3.54}$$

Its image is the component Hilb_K^ν of $\mathrm{Hilb}_K^{\mathbb{Z}/l\mathbb{Z}}$ whose generic points have the form $V(I_\nu) \cup O$, where O is a union of n distinct free $\mathbb{Z}/l\mathbb{Z}$-orbits in $\mathbb{C}^2$. Moreover, $(\mathrm{Hilb}_K^\nu)^{\mathbb{C}^*} = \{I_\lambda \mid \lambda \in \mathcal{P}_\nu(K)\}$. Let

$$\Phi : \mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) \to \mathcal{M}_{-\mathbf{1}}(\gamma) \to \mathrm{Hilb}_K^\nu \tag{3.55}$$

be the composition of (3.53) and (3.54). It induces a bijection between the $\mathbb{C}^*$-fixed points and hence also a bijection between their labelling sets

$$\Psi : \mathcal{P}_{\nu^t}(K) \to \mathcal{P}_\nu(K), \quad \mu \mapsto \lambda,$$

where the partition λ is defined by the equation $I_\lambda = \Phi([\mathbf{A}(\mu)])$.

Lemma 3.9.2. *Let $\mu \in \mathcal{P}_{\nu^t}(K)$. We have $\Psi(\mu) = \mu^t$.*

Proof. Let $\mathcal{V}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) := \mu_\gamma^{-1}(-\frac{\mathbf{1}}{\mathbf{2}}) \times^{G(\gamma)} \widehat{\mathbf{V}}^\nu$ denote the tautological bundle on $\mathcal{X}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$. The diffeomorphism (3.55) lifts to a $U(1)$-equivariant isomorphism of tautological vector bundles

$$\mathcal{V}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma) \xrightarrow{\cong} \mathcal{T}_K^\nu, \tag{3.56}$$

where $\mathcal{T}_K^\nu$ denotes the restriction of $\mathcal{T}_K$ to the subscheme Hilb_K^ν. Proposition 3.4.15 implies that $\mathrm{ch}_t\, \mathcal{V}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)_{[\mathbf{A}(\mu)]} = \mathrm{Res}_\mu(t)$. Hence the $\mathbb{C}^*$-characters of the fibres of $\mathcal{V}_{-\frac{\mathbf{1}}{\mathbf{2}}}(\gamma)$ at any two distinct $\mathbb{C}^*$-fixed points are distinct. By Lemma 3.9.1, we have $\mathrm{ch}_t(\mathcal{T}_K^\nu)_{I_\mu} = \mathrm{Res}_{\mu^t}(t)$. The $U(1)$-equivariance of (3.56) implies that $\Phi([\mathbf{A}(\mu)]) = I_{\mu^t}$ and so $\Psi(\mu) = \mu^t$. □

We are now ready to collect all our results about the $\mathbb{C}^*$-fixed points. We have a sequence of equivariant isomorphisms (of varieties or manifolds):

$$\mathcal{Y}_{\mathbf{h}} \xrightarrow{\mathsf{EG}} \mathcal{X}_\theta(n\delta) \xrightarrow{\mathfrak{R}_{i_1}\circ\cdots\circ\mathfrak{R}_{i_m}} \mathcal{X}_{-\frac{1}{2}}(\gamma) \xrightarrow{\Phi} \mathrm{Hilb}_K^\nu . \tag{3.57}$$

They induce bijections between the labelling sets of $\mathbb{C}^*$-fixed points.

Theorem 3.9.3. *The map* (3.57) *induces the following bijections*

$$\begin{array}{ccccccc} \mathcal{P}(l,n) & \longrightarrow & \mathcal{P}_\varnothing(nl) & \longrightarrow & \mathcal{P}_{\nu^t}(K) & \longrightarrow & \mathcal{P}_\nu(K) \\ \underline{\mathsf{Quot}}(\mu)^\flat & \longmapsto & \mu & \longmapsto & (w*\mu^t)^t & \longmapsto & w*\mu^t. \end{array}$$

Moreover,

$$\nu = w * \varnothing = \mathbf{T}_{i_1}\circ\ldots\circ\mathbf{T}_{i_m}(\varnothing), \quad \underline{\mathsf{Quot}}(w*\mu^t) = \mathsf{pr}(w)\cdot\underline{\mathsf{Quot}}(\mu^t).$$

Proof. The theorem just collects the results of Theorem 3.6.18, Corollary 3.8.12, Lemma 3.9.2 and (3.51). □

Let us rephrase our result slightly. Given $w \in \tilde{S}_l$, we define the *w-twisted l-quotient bijection* to be the map

$$\tau_w : \mathcal{P}(l,n) \to \mathcal{P}_\nu(K), \quad \underline{\mathsf{Quot}}(\mu) \mapsto w*\mu.$$

Corollary 3.9.4. *The bijection $\mathcal{P}(l,n) \to \mathcal{P}_\nu(K)$ induced by* (3.57) *is given by*

$$\underline{\lambda} \mapsto \tau_w(\underline{\lambda}^t). \tag{3.58}$$

Proof. Suppose that $\underline{\lambda} = \underline{\mathsf{Quot}}(\mu)^\flat$. Then Theorem 3.9.3 implies that $\underline{\lambda}$ is sent to $w*\mu^t$. On the other hand, $\underline{\lambda}^t = (\underline{\mathsf{Quot}}(\mu)^\flat)^t = \underline{\mathsf{Quot}}(\mu^t)$ by (3.50). Hence $\tau_w(\underline{\lambda}^t) = w*\mu^t$. □

Remark 3.9.5. The statement of Corollary 3.9.4 appears in the proof of [64, Proposition 7.10]. However, the proof of this statement in [64] is incorrect. The problem lies in an incorrect assumption about the function $c_{\mathbf{h}} : \mathcal{P}(l,n)\times\mathbb{Q}^l \to \mathbb{Q}$, defined by:

$$c_{\mathbf{h}}(\underline{\lambda}) = l\sum_{i=1}^{l-1}|\lambda^i|(H_1+\ldots+H_i) - l\left(\frac{n(n-1)}{2} + \sum_{i=0}^{l-1} n(\lambda^i) - n((\lambda^i)^t)\right)h. \tag{3.59}$$

Given $\mathbf{h}\in\mathbb{Q}^l$, the function $c_{\mathbf{h}}$ induces an ordering on $\mathcal{P}(l,n)$, called the c-order, given by the rule

$$\underline{\mu} <_{\mathbf{h}} \underline{\lambda} \iff c_{\mathbf{h}}(\underline{\mu}) < c_{\mathbf{h}}(\underline{\lambda}).$$

Dependence of this order on $\mathbf{h}$ decomposes the parameter space $\mathbb{Q}^l$ into a finite number of so-called c-chambers. It is stated in [64, §2.5] that the c-order is a total order inside c-chambers. This, however, is not true. Let us consider counterexamples in which the c-order is not total for all values of $\mathbf{h}$. For example, take $l = 1$. Then

$$c_{\mathbf{h}}(\lambda) = -\left(\frac{n(n-1)}{2} + n(\lambda) - n(\lambda^t)\right)h. \tag{3.60}$$

It follows immediately from (3.60) that $c_{\mathbf{h}}(\lambda) = c_{\mathbf{h}}(\mu)$ for all values of $\mathbf{h}$ if λ and μ are two symmetric partitions in $\mathcal{P}(n)$. There are other examples. Take $\mu = (6,3,2,2,2)$. Then $n(\mu) = 3+4+6+8 = 21$. Since $\mu^t = (5,5,2,1,1,1)$ we have $n(\mu^t) = 5+4+3+4+5 = 21$. It follows that μ and μ^t are incomparable in the c-order for all values of $\mathbf{h}$.

Gordon's proof of (3.58) relies on comparing the values of some Morse functions on the quiver variety $\mathcal{M}_{2\theta}(n\delta)$ and the Hilbert scheme at the $\mathbb{C}^*$-fixed points. This approach would work if the Morse functions assigned distinct values to each fixed point. However, this isn't the case because the Morse function on $\mathcal{M}_{2\theta}(n\delta)$, evaluated at the fixed points, is given by $c_{\mathbf{h}}$.

3.9.3. The combinatorial and geometric orderings. In [64, §5.4] Gordon defines a *geometric ordering* $\preceq_w^{\mathsf{geo}}$ on $\mathcal{P}(l,n)$ using the closure relations between the attracting sets of $\mathbb{C}^*$-fixed points in $\mathcal{M}_{2\theta}(n\delta)$ (Gordon uses the notation $\preceq_{\mathbf{h}}$). We also have a *combinatorial ordering* on $\mathcal{P}(l,n)$ given by:

$$\underline{\mu} \preceq_w^{\mathsf{com}} \underline{\lambda} \iff \tau_w(\underline{\lambda}^t) \trianglelefteq \tau_w(\underline{\mu}^t),$$

where $\trianglelefteq$ denotes the dominance ordering on partitions.

Corollary 3.9.6. *Let $w \in \tilde{S}_l$ and $\underline{\mu}, \underline{\lambda} \in \mathcal{P}(l,n)$. Then $\underline{\mu} \preceq_w^{\mathsf{geo}} \underline{\lambda} \;\Rightarrow\; \underline{\mu} \preceq_w^{\mathsf{com}} \underline{\lambda}$.*

Proof. Using the closure relations between the attracting sets of $\mathbb{C}^*$-fixed points in Hilb_K^ν, one can also define a geometric ordering $\preceq^{\mathsf{geo}}$ on $\mathcal{P}_\nu(K)$. By construction, the isomorphism $\mathcal{M}_{2\theta}(n\delta) \xrightarrow{\sim} \mathrm{Hilb}_K^\nu$ intertwines the two geometric orderings. Hence, by Corollary 3.9.4, $\underline{\mu} \preceq_w^{\mathsf{geo}} \underline{\lambda} \iff \tau_w(\underline{\mu}^t) \preceq^{\mathsf{geo}} \tau_w(\underline{\lambda}^t)$. But, by [104, (4.13)], the ordering $\preceq^{\mathsf{geo}}$ is refined by the anti-dominance ordering on $\mathcal{P}_\nu(K)$. $\square$

Chapter 4

The Suzuki functor

In this chapter we will work with rational Cherednik algebras associated to the symmetric group S_m at parameters $t \in \mathbb{C}$ and $h = 1$. We abbreviate $\mathtt{H}_t := \mathtt{H}_{t,1}(S_m)$ and $\mathtt{Z} := Z(\mathtt{H}_0)$.

4.1 Introduction

Arakawa and Suzuki [3] introduced a family of functors from the category $\mathcal{O}$ for $\mathfrak{sl}_n$ to the category of finite-dimensional representations of the degenerate affine Hecke algebra associated to the symmetric group S_m. These functors have been generalized in many different ways, connecting the representation theory of various Lie algebras with the representation theory of various degenerations of affine and double affine Hecke algebras.

Lie algebra	"Hecke" algebra	
$\mathfrak{sl}_n$	degenerate affine Hecke algebra	Arakawa-Suzuki [3]
$\widehat{\mathfrak{sl}}_n$	trigonometric DAHA	Arakawa-Suzuki-Tsuchiya [4]
$\widehat{\mathfrak{gl}}_n$	rational DAHA ($t \neq 0$)	Suzuki [132]
$\widehat{\mathfrak{gl}}_n$	cyclotomic rational DAHA ($t \neq 0$)	Varagnolo-Vasserot [138]

Figure 4.1: Functors relating Lie algebras and "Hecke" algebras in type A

Other generalizations of the Arakawa-Suzuki functor may be found in, e.g., [26,30,44,47,77,78,106]. Here we are concerned with the third functor in the table above, introduced by Suzuki, and later studied by Varagnolo and Vasserot [138], under the assumption that $t \neq 0$, and the level κ is not critical. It is a functor

$$\mathsf{F}_\kappa \colon \mathscr{C}_\kappa \to \mathtt{H}_{\kappa+n}\text{-}\mathsf{mod} \tag{4.1}$$

from the category $\mathscr{C}_\kappa$ of smooth $\widehat{\mathfrak{gl}}_n$-modules of level κ to the category of modules over the rational Cherednik algebra $\mathtt{H}_{\kappa+n}$ associated to S_m and parameters $t = \kappa + n$, $h = 1$. It assigns to each $\widehat{\mathfrak{gl}}_n$-module a certain space of coinvariants:

$$M \mapsto H_0(\mathfrak{gl}_n[z], \mathbb{C}[x_1, \dots, x_m] \otimes (\mathbf{V}^*)^{\otimes m} \otimes M).$$

We study the limit of the functor F_κ as

$$\kappa \to c = -n, \quad t \to 0.$$

As we saw in Theorems 2.1.4 and 3.2.5, $\mathtt{H}_t$ has a non-trivial centre if and only if $t = 0$, and $\mathtt{Z} = Z(\mathtt{H}_0)$ can be identified with the algebra of functions on the classical Calogero-Moser space. An analogous pattern occurs in the representation theory of $\hat{\mathfrak{g}} := \widehat{\mathfrak{gl}}_n$ - the centre of the completed universal enveloping algebra $\widehat{\mathbf{U}}_\kappa$ of $\hat{\mathfrak{g}}$ is trivial unless the level κ is critical. In the latter case, the centre $\mathfrak{Z}$ of $\widehat{\mathbf{U}}_c$ is a completion of a polynomial algebra in infinitely many variables, and, by a theorem of Feigin and Frenkel [50], it can be identified with the algebra of functions on the space of opers on the punctured disc.

The existence of an interesting connection between the two centres $\mathtt{Z}$ and $\mathfrak{Z}$, or, equivalently, between the Calogero-Moser space and opers, is suggested by the close relationship between the Calogero-Moser integrable system and the KP hierarchy. For example, Ben-Zvi and Nevins [14] investigated this relationship from the perspective of noncommutative geometry, identifying the Calogero-Moser space with a certain moduli space of sheaves, called micro-opers, on quantized cotangent bundles. There is also a more direct connection between $\mathtt{Z}$ and $\mathfrak{Z}$ via the Bethe algebra of the Gaudin model associated to $\mathfrak{g}$. By the work of Chervov and Talalaev [34], the Bethe algebra can be obtained as the image of $\mathfrak{Z}$ under the canonical projection from $\widehat{\mathbf{U}}_c$ to $\mathbf{U}(\mathfrak{g}[t^{-1}])$. A surjective homomorphism from the Bethe algebra to the centre of the rational Cherednik algebra was later constructed by Mukhin, Tarasov and Varchenko [101].

Inspired by these intriguing connections, we study the relationship between the two centres from a more algebraic point of view. We consider $\mathtt{Z}$ and $\mathfrak{Z}$ as centres of the respective categories of modules and show that the functor F_c induces (in a sense which will be made precise below) a surjective algebra homomorphism $\Theta\colon \mathfrak{Z} \twoheadrightarrow \mathtt{Z}$. This homomorphism encodes a lot of information about the functor, allowing us to deduce a number of interesting results (see Corollaries A-E). For example, we are able to prove that every simple $\mathtt{H}_0$-module is in the image of F_c, describe the maps between endomorphism rings induced by F_c, and compute the functor on Arakawa and Fiebig's restricted category $\mathcal{O}$. Furthermore, we interpret Θ as an embedding of the Calogero-Moser space into the space of opers on the punctured disc and provide a partial geometric description of this embedding. We expect that there is a connection between our approach and the work of Mukhin, Tarasov and Varchenko, but we do not understand this connection precisely.

4.1.1. Generalization of the Suzuki functor.

Our first theorem, which collects the results of Corollary 4.4.12 and §4.5.2 below, yields a generalization of the functor (4.1) originally defined by Suzuki.

Theorem A. *For all $\kappa \in \mathbb{C}$, there is a colimit preserving functor*

$$\mathsf{F}_\kappa\colon \widehat{\mathbf{U}}_\kappa\text{-mod} \to \mathtt{H}_{\kappa+n}\text{-mod}.$$

When $\kappa \neq c$, the restriction of this functor to $\mathscr{C}_\kappa$ coincides with (4.1).

Our next result describes the images of some important $\widehat{\mathbf{U}}_\kappa$-modules under the functor F_κ. Let us briefly explain the motivation for studying these modules. It comes from the representation theory of the rational Cherednik algebra.

It was proven in [48] that isomorphism classes of simple $\mathtt{H}_0$-modules are in bijection with maximal ideals in $\mathtt{Z} := Z(\mathtt{H}_0)$. Moreover, every simple $\mathtt{H}_0$-module occurs as a quotient of a generalized Verma module $\Delta_0(a, \lambda)$, introduced in [7]. These modules can be defined for any $t \in \mathbb{C}$, and depend on a vector $a \in \mathbb{C}^m$, together with an irreducible representation λ of a parabolic subgroup of S_m. When $a = 0$, they are the usual Verma modules for $\mathtt{H}_t$. The following theorem shows that generalized Verma modules as well as the regular module are in the image of the functor F_κ.

Theorem B (Theorems 4.6.11-4.6.13)**.** *Let $\kappa \in \mathbb{C}$. There exist $\widehat{\mathbf{U}}_\kappa$-modules $\mathbb{H}_\kappa$ and $\mathbb{W}_\kappa(a, \lambda)$ such that*

$$\mathsf{F}_\kappa(\mathbb{H}_\kappa) = \mathtt{H}_{\kappa+n}, \quad \mathsf{F}_\kappa(\mathbb{W}_\kappa(a, \lambda)) = \Delta_{\kappa+n}(a, \lambda).$$

Moreover,

$$\mathsf{F}_\kappa(\mathbb{M}_\kappa(\lambda)) = \Delta_{\kappa+n}(\lambda).$$

Here $\mathbb{M}_\kappa(\lambda)$ denotes the Verma module for $\hat{\mathfrak{g}}$. When $a = 0$, the modules $\mathbb{W}_\kappa(\lambda) := \mathbb{W}_\kappa(0, \lambda)$ coincide with the Weyl modules from [84]. Therefore, we call $\mathbb{W}_\kappa(a, \lambda)$ "generalized Weyl modules".

4.1.2. Suzuki functor and the centres.

From now on assume that $n = m$. One of our main goals is to understand how the centres of the categories $\widehat{\mathbf{U}}_c$-mod and $\mathtt{H}_0$-mod behave under the functor F_c. This is of vital importance because the centres, to a large extent, control morphisms in these categories. For example, it was shown in [56] that the endomorphism rings of Verma and Weyl modules for $\mathbf{U}_c(\hat{\mathfrak{g}})$ are quotients of $\mathfrak{Z}$.

In general, a functor of additive categories does not induce a homomorphism between their centres. We circumvent this problem by introducing the notions of an F-centre of a category and an F-central subcategory. More precisely, we consider the canonical maps

$$\mathfrak{Z} \cong Z(\widehat{\mathbf{U}}_c\text{-mod}) \xrightarrow{\alpha} \mathrm{End}(\mathsf{F}_c) \xleftarrow{\beta} Z(\mathtt{H}_0\text{-mod}) \cong \mathtt{Z}.$$

from the two centres to the endomorphism ring of the functor F_c. Since $\mathtt{H}_0$ lies in the image of F_c, the map β is injective and $\mathtt{Z}$ can be identified with the subring $\mathrm{Im}\,\beta$ of $\mathrm{End}(\mathsf{F}_c)$. We call $Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c) := \alpha^{-1}(\mathtt{Z}) \subset \mathfrak{Z}$ the F_c-*centre* of $\widehat{\mathbf{U}}_c$-mod. Restricting α to $Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c)$ gives a natural algebra homomorphism

$$Z(\mathsf{F}_c) := \alpha|_{Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c)} \colon \; Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c) \longrightarrow \mathtt{Z}$$

making the diagram

$$\begin{array}{ccc} Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c) & \xrightarrow{Z(\mathsf{F}_c)} & \mathtt{Z} \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \mathrm{End}_{\widehat{\mathbf{U}}_c}(M) & \xrightarrow{\mathsf{F}_c} & \mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(M)) \end{array} \tag{4.2}$$

commute for all $\widehat{\mathbf{U}}_c$-modules M. The homomorphism $Z(\mathsf{F}_c)$ contains partial information about all the maps between endomorphism rings induced by the functor F_c.

Our next result gives a partial description of $Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c)$. We consider the subalgebra

$$\mathscr{L}_c := \mathbb{C}[\mathrm{id}[r], {}^c\mathbf{L}_{r+1}]_{r \leq 0} \subset \mathfrak{Z}$$

consisting of certain first- and second-order Segal-Sugawara operators (see §4.3.5 for a precise definition).

Theorem C (Theorem 4.7.5)**.** *The algebra $\mathscr{L}_c$ lies in the F_c-centre of $\widehat{\mathbf{U}}_c$-mod, i.e.,*

$$\mathscr{L}_c \subseteq Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c).$$

We give an explicit description of the associated homomorphism

$$Z(\mathsf{F}_c)|_{\mathscr{L}_c}\colon \mathscr{L}_c \to \mathsf{Z} \tag{4.3}$$

in (4.74)-(4.75).

It is natural to ask whether $Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c)$ coincides with $\mathfrak{Z}$. Unfortunately, this is far from being the case. Our solution to this problem is to relax the condition that the diagram (4.2) should commute for all $\widehat{\mathbf{U}}_c$-modules M. We introduce the notion of a subcategory $\mathcal{A}$ of $\widehat{\mathbf{U}}_c$-mod being F_c-*central* (see Definition 4.7.2 for details), which has the consequence that there exists a unique algebra homomorphism $Z_{\mathcal{A}}(\mathsf{F}_c)\colon \mathfrak{Z} \to \mathsf{Z}$ making the diagram

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{Z_{\mathcal{A}}(\mathsf{F}_c)} & \mathsf{Z} \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \mathrm{End}_{\widehat{\mathbf{U}}_c}(M) & \xrightarrow{\mathsf{F}_c} & \mathrm{End}_{\mathsf{H}_0}(\mathsf{F}_c(M)) \end{array} \tag{4.4}$$

commute for all $M \in \mathcal{A}$. Our next result identifies an important F_c-central subcategory of $\widehat{\mathbf{U}}_c$-mod.

Theorem D (Theorem 4.7.9)**.** *The full subcategory $\mathscr{C}_{\mathbb{H}}$ of $\widehat{\mathbf{U}}_c$-mod projectively generated by $\mathbb{H}_c$ is F_c-central.*

The category $\mathscr{C}_{\mathbb{H}}$ contains all the Verma and generalized Weyl modules which are not annihilated by F_c. The associated homomorphism

$$\Theta = Z_{\mathscr{C}_{\mathbb{H}}}(\mathsf{F}_c)\colon\ \mathfrak{Z} \longrightarrow \mathsf{Z}$$

plays a key role in our study of the functor F_c. The following theorem, whose representation theoretic and geometric consequences are discussed in the next subsection, is the main result of this chapter.

Theorem E (Theorem 4.9.6)**.** *The homomorphism $\Theta\colon \mathfrak{Z} \to \mathsf{Z}$ is surjective.*

Let us briefly comment on the proof of Theorem E. We first show that Θ factors through $\mathfrak{Z}^{\leqslant 2}(\hat{\mathfrak{g}})$ (see §4.10.4 for the definition), and that the homomorphism $\Theta\colon \mathfrak{Z}^{\leqslant 2}(\hat{\mathfrak{g}}) \to \mathsf{Z}$ is filtered with respect to the standard filtration on Z and a certain "height" filtration on $\mathfrak{Z}^{\leqslant 2}(\hat{\mathfrak{g}})$ (see §4.8.2 and §4.9.1) We compute the associated graded homomorphism $\mathsf{gr}\,\Theta$ and use it to deduce the surjectivity of Θ. In our calculations, we rely heavily on the explicit construction of Segal-Sugawara operators due to Chervov and Molev [32].

We also consider the Poisson algebra structures on $\mathfrak{Z}$ and Z given by the Hayashi bracket [71]. The map Θ is not a Poisson homomorphism. However, the following is true.

Theorem F (Theorem 4.10.9)**.** *The restriction of Θ to $\mathscr{L}_c$ is a homomorphism of Poisson algebras.*

The partial compatibility of the Poisson structures on $\mathfrak{Z}$ and Z is a shadow of the fact that the functor F_κ is defined for all levels κ. We remark that the Poisson subalgebra $\mathscr{L}_c \subset \mathfrak{Z}$ can be described quite explicitly. It is isomorphic to a certain subalgebra of $S(\mathsf{Heis} \rtimes \mathsf{Vir})$, the symmetric algebra on the semi-direct product of the Heisenberg and the Virasoro Lie algebras.

4.1.3. Applications. Our main result (Theorem E) has several applications. First of all, we can use it to gain more information about the homomorphisms between endomorphism rings induced by F_c.

Corollary A (Corollary 4.10.1)**.** *The ring homomorphisms*

$$\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(a,\lambda)) \twoheadrightarrow \mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda)), \quad \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{M}_c(\lambda)) \twoheadrightarrow \mathrm{End}_{\mathtt{H}_0}(\Delta_0(\lambda)).$$

induced by F_c *are surjective.*

Secondly, we are able to deduce from Corollary A that every simple $\mathtt{H}_0$-module lies in the image of F_c. This result is, on the one hand, analogous to similar results [132, 138] in the $\kappa \neq c$ case. On the other hand, the situation at the critical level is very different because there are uncountably many non-isomorphic simple $\mathtt{H}_0$-modules. This is reflected by the fact that our proof relies on completely different techniques from those used in [132, 138].

Corollary B (Corollary 4.10.3)**.** *Every simple* $\mathtt{H}_0$*-module is in the image of the functor* F_c*.*

We next connect the functor F_c with the work of Arakawa and Fiebig. In [1, 2], they studied a restricted version of category $\mathcal{O}$, obtained by "killing" the action of the centre $\mathfrak{Z}$. This category contains restricted Verma modules $\overline{\mathbb{M}}_c(\lambda)$ as well as, analogously defined, restricted versions of Weyl modules $\overline{\mathbb{W}}_c(\lambda)$. In our third corollary, we describe the image of these modules under F_c.

Corollary C (Corollaries 4.10.6-4.10.7)**.** *We have*

$$\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) = \mathsf{F}_c(\overline{\mathbb{W}}_c(\lambda)) = \mathsf{F}_c(\mathbb{L}(\lambda)) = L_\lambda,$$

where $\mathbb{L}(\lambda)$ *(resp.* L_λ*) is the unique graded simple quotient of* $\mathbb{M}_c(\lambda)$ *(resp.* $\Delta_0(\lambda)$*).*

Fourthly, we give a partial geometric description of the homomorphism $\Theta\colon \mathfrak{Z} \to \mathtt{Z}$ in terms of opers. By a theorem of Feigin and Frenkel [50], $\mathfrak{Z}$ is canonically isomorphic to the algebra of functions on the space $\mathrm{Op}_{\check{G}}(\mathbb{D}^\times)$ of opers on the punctured disc. Therefore, Θ induces a closed embedding $\Theta^*\colon \operatorname{Spec}\mathtt{Z} \hookrightarrow \mathrm{Op}_{\check{G}}(\mathbb{D}^\times)$. We show that the image of this embedding lies in the space $\mathrm{Op}_{\check{G}}(\mathbb{D})^{\leq 2}$ of opers with singularities of order at most two.

We are also able to obtain some information about the residue and monodromy of the opers in the image of Θ^*. To state our results, we first need to recall some facts about the affine variety $\operatorname{Spec}\mathtt{Z}$ and a canonical map $\pi\colon \operatorname{Spec}\mathtt{Z} \to \mathbb{C}^n/S_n$ (see (4.48)). Bellamy showed in [7,8] that each fibre of π decomposes as a disjoint union of subvarieties $\Omega_{\mathbf{a},\lambda}$, which can be identified with supports of the generalized Verma modules $\Delta_0(a,\lambda)$. Moreover, $\mathtt{Z}$ surjects onto the endomorphism rings $\mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$, and $\operatorname{Spec}\mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda)) \cong \Omega_{\mathbf{a},\lambda}$.

Endomorphism rings of the Weyl modules $\mathbb{W}_c(\lambda)$ also admit a geometric interpretation. Frenkel and Gaitsgory [56] showed that $\mathfrak{Z}$ surjects onto $\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(\lambda))$, and identified the latter with the algebra of functions on the space $\mathrm{Op}^{\lambda}_{\check{G}}(\mathbb{D})$ of opers with residue $\varpi(-\lambda-\rho)$ and trivial monodromy.

Using the results of [51], we show that the image of $\Omega_{\mathbf{a},\lambda}$ under Θ^* is contained in the space $\mathrm{Op}^{\leqslant 2}_{\check{G}}(\mathbb{D})_{\mathbf{a}}$ of opers with singularities of order at most two and 2-residue $\mathbf{a}$. Moreover, we show that the image of Ω_λ is contained in $\mathrm{Op}^{\lambda}_{\check{G}}(\mathbb{D})$.

Corollary D (Corollary 4.10.14)**.** *The following hold.*

a) *The map* $\Theta\colon \mathfrak{Z} \to \mathtt{Z}$ *induces a closed embedding*

$$\Theta^*\colon \operatorname{Spec}\mathtt{Z} \hookrightarrow \mathrm{Op}_{\check{G}}(\mathbb{D})^{\leqslant 2}.$$

b) *We have*

$$\Theta^*(\Omega_{\mathbf{a},\lambda}) \subseteq \mathrm{Op}^{\leqslant 2}_{\check{G}}(\mathbb{D})_{\mathbf{a}}.$$

Hence the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} \mathtt{Z} & \overset{\Theta^*}{\hookrightarrow} & \mathrm{Op}_{\check{G}}(\mathbb{D})^{\leqslant 2} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_2} \\ \mathbb{C}^n/S_n & \xrightarrow{\sim} & \mathfrak{t}^*/S_n \end{array}$$

c) *If* $\mathbf{a} = 0$ *then*

$$\Theta^*(\Omega_\lambda) \subseteq \mathrm{Op}^{\lambda}_{\check{G}}(\mathbb{D}).$$

Finally, we study the behaviour of self-extensions under F_c.

Corollary E. *Suppose that M is a $\widehat{\mathbf{U}}_c$-module with a filtration by Weyl modules. Then F_c induces a linear map*

$$\mathrm{Ext}^1_{\widehat{\mathbf{U}}_c}(M, M) \to \mathrm{Ext}^1_{\mathtt{H}_0}(\mathsf{F}_c(M), \mathsf{F}_c(M)). \tag{4.5}$$

We conjecture (see Conjecture 4.10.17) that (4.5) extends to a surjective homomorphism between extension algebras, and that it admits an interpretation in terms of differential forms on opers and the Calogero-Moser space.

4.1.4. Structure of the chapter. Let us summarize the contents of the chapter. In sections 2-3 we recall the relevant definitions and facts concerning affine Lie algebras and vertex algebras. These sections contain no new results. In Section 4 we recall Suzuki's construction of the functor F_κ and generalize it to the critical level. In section 5 we further generalize the functor F_κ to the category of all $\widehat{\mathbf{U}}_\kappa$-modules, proving Theorem A. Section 6 is devoted to the proof of Theorem B. In Section 7 we study the relationship between the two centres $\mathfrak{Z}$ and $\mathtt{Z}$ via the functor F_c. Section 7 contains the proofs of Theorems C-D. In Section 8 we define graded and filtered analogues of the Suzuki functor, which are later used in Section 9 to set up our "associated graded" argument. All of section 9 is devoted to the proof of Theorem E. In Section 10 we study the applications of Theorem E, proving Corollaries A-E.

4.2 Preliminaries

4.2.1. General conventions. Fix once and for all two positive integers n and m. The parameter n refers to the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ while m refers to the rational Cherednik algebra $\mathtt{H}_{t,c}$ associated to the symmetric group S_m. Once again, we work over the field of complex numbers throughout. If V is a vector space, let $T(V)$ denote the tensor algebra and $S(V)$ the symmetric algebra on V.

For a unital associative algebra A, with unit 1_A, we denote by A-mod the category of left A-modules. Given a left A-module M and a left ideal I in A, let $M^I := \{m \in M \mid I \cdot M = 0\}$ be the set of I-invariants. We will also work with the full subcategory A-fpmod of A-mod consisting of finitely presented modules, i.e., modules M such that there exists a short exact sequence $A^k \to A^l \to M \to 0$ for some $k, l \geq 0$. If B is another algebra, let (A, B)-nmod be the full subcategory of $A \otimes B$-mod consisting of modules M with the property that the action of B normalizes the action of A, i.e., $[A, B] \subseteq A$ in the endomorphism ring of M.

Given a subalgebra $B \subset A$, let $Z_A(B)$ denote the centralizer of B in A. In particular, $Z(A) := Z_A(A)$ is the centre of A. Recall that the centre $Z(\mathcal{C})$ of an additive category $\mathcal{C}$ is the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{C}}$. We can naturally identify $Z(A) \cong Z(A\text{-}\mathsf{mod})$, $z \mapsto \{z_M \mid M \in A\text{-}\mathsf{mod}\}$, where z_M is the endomorphism of M given by the left action of z.

Suppose that A is a commutative algebra and M is an A-module. Let $\operatorname{Ann}_A(M) := \{a \in A \mid a \cdot M = 0\}$ be the annihilator of M in A. The affine variety $\operatorname{supp}_A(M) := \operatorname{Spec} A/\operatorname{Ann}_A(M)$ is called the support of M in $\operatorname{Spec} A$.

4.2.2. Combinatorics. We will now introduce some combinatorics. We remark that in the present chapter we follow slightly different notational conventions from those introduced in §2.2.1. Let $l \geq 1$. We say that $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{Z}_+^l$ is a *composition* of m of *length* l if $\nu_1 + \ldots + \nu_l = m$. Let $\mathcal{C}_l(m)$ denote the set of all such compositions. Set $\nu_{\leq i} = \nu_1 + \ldots + \nu_i$ for each $1 \leq i \leq l$ with $\nu_{\leq 0} = 0$ by convention.

The symmetric group S_m on m letters acts naturally on $\mathfrak{h} = \mathbb{C}^m$ by permuting the coordinates. If $a \in \mathfrak{h}$, let $S_m(a)$ denote its stabilizer in S_m. We abbreviate $s_i := s_{i,i+1}$. Given $\nu \in \mathcal{C}_l(m)$, let $S_\nu := S_{\nu_1} \times \ldots \times S_{\nu_l}$ denote the parabolic subgroup of $\mathfrak{S}_m$ generated by the simple transpositions $s_1, \dots, s_{m-1}$ with the omission of $s_{\nu_{\leq 1}}, s_{\nu_{\leq 2}}, \dots, s_{\nu_{\leq l-1}}$.

A sequence $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ is a *partition* if $\lambda_1 \geq \ldots \geq \lambda_n$. Let $\mathcal{P}_n(m)$ denote the set of all partitions of m of length n. We call $\lambda = (\lambda^1, \dots, \lambda^l) \in \prod_{i=1}^l \mathcal{P}_{n_i}(m_i)$ an *l-multipartition* of m if $\sum_{i=1}^l m_i = m$ and each $m_i \neq 0$. We say that λ has *length* n if $\sum_{i=1}^l n_i = n$, and *length type* μ if $(n_1, \dots, n_l) = \mu \in \mathcal{C}_l(n)$. We say that λ is of *size type* ν if $(m_1, \dots, m_l) = \nu \in \mathcal{C}_l(m)$. Let $\mathcal{P}_\mu(m)$ denote the set of multipartitions of m of length type μ and let $\mathcal{P}_n(\nu)$ denote the set of all multipartitions of length n of size type ν (where we let l vary over all positive integers). Set

$$\mathcal{P}_\mu(\nu) := \mathcal{P}_\mu(m) \cap \mathcal{P}_n(\nu), \quad \mathcal{P}_\mu := \bigsqcup_{m \geq 0} \mathcal{P}_\mu(m), \quad \mathcal{P}(\nu) := \bigcup_{n \geq 0} \mathcal{P}_n(\nu).$$

In the union on the RHS we identify l-multipartitions λ and χ whenever each pair of partitions λ^i and χ^i differ only by the number of parts equal to zero.

If $\lambda \in \mathcal{P}_n(m)$, let $\mathsf{Sp}(\lambda)$ denote the corresponding Specht module. Given $\nu \in \mathcal{C}_l(m)$ and $\lambda \in \mathcal{P}_n(\nu)$, set $\mathsf{Sp}(\lambda) := \mathsf{Sp}(\lambda^1) \otimes \ldots \otimes \mathsf{Sp}(\lambda^l)$. It is a S_ν-module. Let $\mathsf{Sp}_\nu(\lambda) := \mathbb{C}S_m \otimes_{\mathbb{C}S_\nu} \mathsf{Sp}(\lambda)$ be the corresponding S_m-module obtained by induction.

4.2.3. Lie algebras. Given a Lie algebra $\mathfrak{a}$, let $\mathbf{U}(\mathfrak{a})$ denote its universal enveloping algebra, with unit $1_\mathfrak{a} := 1_{\mathbf{U}(\mathfrak{a})}$ and augmentation ideal $\mathbf{U}_+(\mathfrak{a})$. If M is an $\mathfrak{a}$-module and $k \geq 0$, let $H_k(\mathfrak{a}, M)$ denote the k-th homology group of $\mathfrak{a}$ with coefficients in M. In particular, $H_0(\mathfrak{a}, M) = M/\mathbf{U}_+(\mathfrak{a}).M = M/\mathfrak{a}.M$. Given a Lie subalgebra $\mathfrak{c} \subset \mathfrak{a}$ and a $\mathfrak{c}$-module N, let $\operatorname{Ind}_\mathfrak{c}^\mathfrak{a} N := \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{c})} N$ be the induced module. For a surjective Lie algebra homomorphism $\mathfrak{d} \twoheadrightarrow \mathfrak{c}$, let $\operatorname{Inf}_\mathfrak{c}^\mathfrak{d} N$ denote N regarded as a $\mathfrak{d}$-module.

Let $G = GL_n(\mathbb{C})$ be the general linear group and $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ its Lie algebra. Let e_{kl} be the (k,l)-matrix unit and let id denote the identity matrix. We use the standard triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ with respect to the strictly lower triangular, diagonal and strictly upper triangular matrices, and abbreviate $\mathfrak{b}_+ := \mathfrak{t} \oplus \mathfrak{n}_+$. For $1 \leq k \leq n$, let $\epsilon_k \in \mathfrak{t}^*$ be the function defined by $\epsilon_k(e_{ll}) = \delta_{k,l}$.

Given $\mu \in \mathcal{C}_l(n)$, let $\mathfrak{l}_\mu := \prod_{i=1}^l \mathfrak{gl}_{\mu_i} \subseteq \mathfrak{g}$ be the corresponding standard Levi subalgebra. We next recall the connection between multipartitions and weights. A weight $\lambda = \sum_i \lambda_i \epsilon_i \in \mathfrak{t}^*$ is called *μ-dominant* and integral if each $\lambda_i \in \mathbb{Z}$ and $\lambda_i - \lambda_{w(i)} \in \mathbb{Z}_{\geq 0}$ whenever $w(i) > i$, for all $w \in S_\mu$. Let Π_μ^+ denote the set of μ-dominant integral weights with the property that each $\lambda_i \in \mathbb{Z}_{\geq 0}$. If $\mu = (n)$, we abbreviate $\Pi^+_\mu = \Pi^+$. There is a natural bijection

$$\Pi^+_\mu \cong \mathcal{P}_\mu, \quad \lambda \mapsto (\lambda^1, \dots, \lambda^l), \tag{4.6}$$

where $\lambda^i := (\lambda_{\mu_{\leq i-1}+1}, \dots, \lambda_{\mu_{\leq i}})$. From now on we will implicitly identify weights with partitions using this bijection.

4.2.4. Schur-Weyl duality. Given $\lambda \in \Pi^+_\mu$, let $L(\lambda)$ be the corresponding simple $\mathfrak{l}_\mu$-module of highest weight λ. Let $\mathbf{V} \cong L(\epsilon_1)$ be the standard representation of $\mathfrak{g}$, with standard basis $\{e_i \mid 1 \leq i \leq n\}$ and the corresponding dual basis $\{e_i^* \mid 1 \leq i \leq n\}$ of $\mathbf{V}^*$. If $n = m$, set $e^*_{\mathsf{id}} := e^*_1 \otimes \ldots \otimes e^*_n$ and, for $w \in S_n$,

$$e^*_w := e^*_{w^{-1}(1)} \otimes \ldots \otimes e^*_{w^{-1}(n)} \in (\mathbf{V}^*)^{\otimes n}. \tag{4.7}$$

Given $\mu \in \mathcal{C}_l(n)$ and $\nu \in \mathcal{C}_l(m)$, let $\mathbf{V}^*_i$ be the subspace of $\mathbf{V}$ spanned by $e^*_{\mu_{\leq i-1}+1}, \dots, e^*_{\mu_{\leq i}}$ and

$$(\mathbf{V}^*)^{\otimes m}_{(\mu,\nu)} := \bigotimes_{i=1}^{l} (\mathbf{V}^*_i)^{\otimes \nu_i} \subseteq (\mathbf{V}^*)^{\otimes m}.$$

There is an analogue of classical Schur-Weyl duality (see, e.g., [107, Proposition 9.1.2]) for $\mathfrak{l}_\mu$ and $S_m \ltimes \mathbb{Z}_l^m$ - their actions on $\mathbf{V}^{\otimes m}$ centralize each other (see, e.g., [91, Theorem 6.1]). We will need the following application, whose proof can be found in [138, Proposition 3.8(a)].

Proposition 4.2.1. *Let $\lambda \in \mathfrak{t}^*$. Then*

a) $H_0(\mathfrak{l}_\mu, (\mathbf{V}^*)^{\otimes m} \otimes L(\lambda)) = 0$ *unless* $\lambda \in \mathcal{P}_\mu(m)$.
b) *If $\nu \in \mathcal{C}_l(m)$ and $\lambda \in \mathcal{P}_\mu(\nu)$ then*

$$H_0(\mathfrak{l}_\mu, (\mathbf{V}^*)^{\otimes m}_{(\mu,\nu)} \otimes L(\lambda)) \cong \mathsf{Sp}(\lambda), \quad H_0(\mathfrak{l}_\mu, (\mathbf{V}^*)^{\otimes m} \otimes L(\lambda)) \cong \mathsf{Sp}_\nu(\lambda) \tag{4.8}$$

as $\mathbb{C}S_\nu$- resp. $\mathbb{C}S_m$-modules.

In the case $\mu = (n)$, classical Schur-Weyl duality also implies the following.

Corollary 4.2.2. *Let $\lambda \in \mathfrak{t}^*$. Then:*

a) $H_0(\mathfrak{b}_+, (\mathbf{V}^*)^{\otimes m} \otimes \mathbb{C}_\lambda) = 0$ *unless* $\lambda \in \mathcal{P}_n(m)$.
b) *If $\lambda \in \mathcal{P}_n(m)$ then there is a natural $\mathbb{C}S_m$-module isomorphism*

$$H_0(\mathfrak{b}_+, (\mathbf{V}^*)^{\otimes m} \otimes \mathbb{C}_\lambda) \cong \mathsf{Sp}(\lambda). \tag{4.9}$$

Proof. The space $H_0(\mathfrak{b}_+, (\mathbf{V}^*)^{\otimes m} \otimes \mathbb{C}_\lambda)$ can be identified with the space of lowest weight vectors of weight $\lambda^* = (\lambda_n, \dots, \lambda_1)$ in $(\mathbf{V}^*)^{\otimes m}$. By Schur-Weyl duality, $(\mathbf{V}^*)^{\otimes m} = \bigoplus_{\xi \in \mathcal{P}_n(m)} L(\xi) \otimes \mathsf{Sp}(\xi)$. Hence the space of lowest weight vectors of weight λ^* is isomorphic to $\mathsf{Sp}(\lambda)$ if $\lambda \in \mathcal{P}_n(m)$ and is zero otherwise. □

4.2.5. The affine Lie algebra. We recall the definition of the affine Lie algebra associated to $\mathfrak{g}$.

Definition 4.2.3. Let $\kappa \in \mathbb{C}$. The *affine Lie algebra* $\hat{\mathfrak{g}}_\kappa$ is the central extension

$$0 \to \mathbb{C}\mathbf{1} \to \hat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0 \tag{4.10}$$

associated to the cocycle $(X \otimes f, Y \otimes g) \mapsto \langle X, Y\rangle_\kappa \operatorname{Res}_{t=0}(g\partial_t f)$, where

$$\langle -, -\rangle_\kappa := \begin{cases} \kappa \operatorname{Tr}_{\mathfrak{g}} & \text{if } \kappa \neq -n, \\ -\frac{1}{2}\operatorname{Kil}_{\mathfrak{g}} & \text{if } \kappa = -n \end{cases}$$

and $\operatorname{Tr}_{\mathfrak{g}}$ and $\operatorname{Kil}_{\mathfrak{g}}$ are the trace form and the Killing form on $\mathfrak{g}$, respectively. Note that, if we restrict to $\mathfrak{sl}_n$, we have an equality $-n\operatorname{Tr} = -\frac{1}{2}\operatorname{Kil}$. In the $\mathfrak{gl}_n$ case, we use the trace form when $\kappa \neq -n$ to ensure that the centre of the completed universal enveloping algebra (see (4.12)) is trivial. We use the Killing form when $\kappa = -n$ so that shifts of the identity matrix are in the aforementioned centre - this has the advantage of simplifying many formulas.

Explicitly, the Lie bracket in $\hat{\mathfrak{g}}_\kappa$ is given by:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \langle X, Y\rangle_\kappa \operatorname{Res}_{t=0}(g\partial_t f)\mathbf{1}, \quad [X \otimes f, \mathbf{1}] = [\mathbf{1}, \mathbf{1}] = 0$$

for $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}((t))$.

We will also use the central extension $\tilde{\mathfrak{g}}_\kappa$ obtained by replacing $\mathfrak{g}((t))$ with $\mathfrak{g}[t^{\pm 1}]$ in (4.10). Given $X \in \mathfrak{g}$ and $k \in \mathbb{Z}$, set

$$X[k] := X \otimes t^k \in \hat{\mathfrak{g}}_\kappa, \quad \mathfrak{g}[k] := \mathfrak{g} \otimes t^k \subset \hat{\mathfrak{g}}_\kappa.$$

We next introduce notation for the following Lie subalgebras of $\hat{\mathfrak{g}}_\kappa$:

$$\hat{\mathfrak{g}}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}], \quad \hat{\mathfrak{g}}_+ := \mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}, \quad \hat{\mathfrak{g}}_{\geq r} := \mathfrak{g} \otimes t^r\mathbb{C}[[t]], \quad \hat{\mathfrak{g}}_{\leq -r} := \mathfrak{g} \otimes t^{-r}\mathbb{C}[t^{-1}],$$

where $r \geq 0$. Moreover, we abbreviate

$$\hat{\mathfrak{b}}_+ := \hat{\mathfrak{n}}_+ \oplus \mathfrak{t} \oplus \mathbb{C}\mathbf{1}, \quad \hat{\mathfrak{t}}_+ := \mathfrak{t} \oplus \hat{\mathfrak{g}}_{\geq 1} \oplus \mathbb{C}\mathbf{1}.$$

Let $\tilde{\mathfrak{g}}_+, \tilde{\mathfrak{g}}_{\geq r}$, etc., denote the corresponding Lie subalgebras of $\tilde{\mathfrak{g}}_\kappa$.

4.2.6. The completed universal enveloping algebra. We are interested in modules on which $\mathbf{1}$ acts as the identity endomorphism. Therefore we consider the quotient algebra

$$\mathbf{U}_\kappa(\hat{\mathfrak{g}}) := \mathbf{U}(\hat{\mathfrak{g}}_\kappa)/\langle \mathbf{1} - 1_{\hat{\mathfrak{g}}_\kappa}\rangle.$$

Definition 4.2.4. The parameter κ is called the *level*. The value $c := -n$ is called the *critical level*.

We next recall the definition of a certain completion of $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$ (see, e.g., [54, §2.1.2]). There is a topology on $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$ defined by declaring the left ideals $I_r := \mathbf{U}_\kappa(\hat{\mathfrak{g}}).\hat{\mathfrak{g}}_{\geq r}$ $(r \geq 0)$ to be a basis of open neighbourhoods of zero. Let $\widehat{\mathbf{U}}_\kappa$ be the completion of $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$ with respect to this topology. Equivalently, we can write

$$\widehat{\mathbf{U}}_\kappa = \varprojlim \mathbf{U}_\kappa(\hat{\mathfrak{g}})/I_r. \tag{4.11}$$

It is a complete topological algebra with a basis of open neighbourhoods of zero given by the left ideals $\hat{I}_r := \widehat{\mathbf{U}}_\kappa.\hat{\mathfrak{g}}_{\geq r}$. The following proposition illustrates the special nature of the critical level.

Proposition 4.2.5 ([54, Proposition 4.3.9]). *$Z(\widehat{\mathbf{U}}_\kappa) = \mathbb{C}$ if and only if $\kappa \neq c$.*

We abbreviate

$$\mathfrak{z} := Z(\widehat{\mathbf{U}}_c). \tag{4.12}$$

4.2.7. Smooth modules. We will mostly deal with smooth $\widehat{\mathbf{U}}_\kappa$-modules. Let us recall their definition (see, e.g., [54, §1.3.6] or [84, §1.9]).

Definition 4.2.6. A $\widehat{\mathbf{U}}_\kappa$-module M is called *smooth* if $M = \bigcup_{r\geq 0} M^{\hat{I}_r}$. Let $\mathscr{C}_\kappa$ denote the full subcategory of $\widehat{\mathbf{U}}_\kappa$-$\mathsf{mod}$ whose objects are smooth modules. Let $\mathscr{C}_\kappa(r)$ denote the full subcategory of $\mathscr{C}_\kappa$ consisting of all modules M generated by $M^{\hat{I}_r}$.

One can analogously define smooth $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$- and $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$-modules. It is easy to see that the corresponding categories of smooth modules coincide with $\mathscr{C}_\kappa$. The following lemma, whose proof is standard, shows that the concept of smoothness defined above is analogous to that familiar from the representation theory of p-adic groups.

Lemma 4.2.7. *Let M be a $\widehat{\mathbf{U}}_\kappa$-module. The following are equivalent:*

a) *M is smooth,*
b) *M, endowed with the discrete topology, is a topological $\widehat{\mathbf{U}}_\kappa$-module,*
c) $\mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$ *is an open left ideal in $\widehat{\mathbf{U}}_\kappa$ for all $v \in M$.*

Proof. Assume (a). Then for every $v \in M$, $\mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$ contains the open ideal $\hat{I}_r$ for some $r \geq 0$. Hence, by [75, Lemma 2.15], $\mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$ is also an open ideal. Conversely, assume (c). Since $\mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$ is an open neighbourhood of zero, and the ideals $\hat{I}_r$ form a neighbourhood basis of zero, there exists $r \geq 0$ such that $\hat{I}_r \subseteq \mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$.

Now assume (b). The singleton $\{0\}$ is open in the discrete topology on M. For each $v \in M$, the action map $a_v \colon \widehat{\mathbf{U}}_\kappa \times \{v\} \to M$ is continuous, so $a_v^{-1}(\{0\}) = \mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v)$ is open. Conversely, assume (c) and consider the action map $a\colon \widehat{\mathbf{U}}_\kappa \times M \to M$. It suffices to check that $a^{-1}(\{0\})$ is open. But this holds because $a^{-1}(\{0\})$ is a disjoint union of the open sets $\mathrm{Ann}_{\widehat{\mathbf{U}}_\kappa}(v) \times \{v\}$ $(v \in M)$. □

4.3 Recollections on vertex algebras

In this section we recall the definition of the vertex algebra associated to the vacuum module $\mathsf{Vac}_\kappa := \mathbf{U}(\hat{\mathfrak{g}}_\kappa)/\mathbf{U}(\hat{\mathfrak{g}}_\kappa).\hat{\mathfrak{g}}_+$. We also recall the main results about the centre of this vertex algebra and its connection to $\mathfrak{Z}$.

4.3.1. Vertex algebras. Given an algebra R, let $f(z) = \sum_{r\in\mathbb{Z}} f_{(-r-1)} z^r$ and $g(z) = \sum_{r\in\mathbb{Z}} g_{(-r-1)} z^r$ be formal power series in $R[[z,z^{-1}]]$. Their *normally ordered product* $:f(z)g(z):$ is defined to be the formal power series

$$:f(z)g(z): = f_+(z)g(z) + g(z)f_-(z), \quad f_+(z) = \sum_{r\geq 0} f_{(-r-1)} z^r, \quad f_-(z) = \sum_{r<0} f_{(-r-1)} z^r.$$

Given $f_1(z), \dots, f_l(z) \in R[[z,z^{-1}]]$, set

$$:f_1(z)\cdots f_l(z): = :f_1(z)\cdots(:f_{l-2}(z)(:f_{l-1}(z)f_l(z):):):$$

Let W be a vector space. A series $f(z) = \sum_{r\in\mathbb{Z}} f_{(-r-1)} z^r \in (\mathrm{End}_{\mathbb{C}} W)[[z,z^{-1}]]$ is called a *field* on W if for every $v \in W$ there exists an integer $k \geq 0$ such that $f_{(r)}.v = 0$ for all $r \geq k$. Fields are preserved by the normally ordered product.

A *vertex algebra* is a quadruple $(W, |0\rangle, \mathbb{Y}, T)$ consisting of a complex vector space W, a distinguished element $|0\rangle \in W$, called the vacuum vector, a linear map

$$\mathbb{Y}\colon W \to (\operatorname{End}_{\mathbb{C}} W)[[z, z^{-1}]], \quad a \mapsto \mathbb{Y}(a, z) = \sum_{r\in\mathbb{Z}} a_{(-r-1)} z^r$$

sending vectors to fields on W, called the *state-field correspondence*, and a linear map $T\colon W \to W$ called the *translation operator*. These data must satisfy a list of axioms, see, e.g., [13, Definition 1.3.1].

Let us briefly recall the construction of a functor

$$\widetilde{U}\colon \{\mathbb{Z}\text{-graded vertex algebras}\} \to \{\text{complete topological associative algebras}\}.$$

Given a $\mathbb{Z}$-graded vertex algebra W, one considers a completion of the Lie algebra of Fourier coefficients associated to W, and takes its universal enveloping algebra. To obtain $\widetilde{U}(W)$, one again needs to form a completion and take a quotient by certain relations. The precise definition can be found in [13, §4.3.1].

4.3.2. The affine vertex algebra. Let $\kappa \in \mathbb{C}$. The vacuum module Vac_κ can be endowed with the structure of a vertex algebra, as in [13, §2.4]. Let us explicitly recall the state-field correspondence. Let $\rho : \mathbf{U}_\kappa(\hat{\mathfrak{g}}) \to \operatorname{End}_{\mathbb{C}}(\mathsf{Vac}_\kappa)$ be the representation of $\hat{\mathfrak{g}}_\kappa$ on Vac_κ. The state-field correspondence $\mathbb{Y}$ is given by $\mathbb{Y}(|0\rangle, z) = \mathrm{id}$ and

$$X(z) := \mathbb{Y}(X[-1], z) = \sum_{r\in\mathbb{Z}} \rho(X[r]) z^{-r-1}, \tag{4.13}$$

$$\mathbb{Y}(X_1[k_1] \dots X_l[k_l], z) = \frac{1}{(-k_1-1)!} \cdots \frac{1}{(-k_l-1)!} {:}\partial_z^{-k_1-1} X_1(z) \dots \partial_z^{-k_l-1} X_l(z){:} \tag{4.14}$$

for $X, X_1, \dots, X_l \in \mathfrak{g}$ and $k_1, \dots, k_l \le -1$. Given $X \in \mathfrak{g}$ we also define a power series

$$X\langle z\rangle := \mathbb{Y}\langle X[-1], z\rangle := \sum_{r\in\mathbb{Z}} X[r] z^{-r-1}.$$

Applying formula (4.14) with each $X_i(z)$ replaced by $X_i\langle z\rangle$ we can associate a power series $\mathbb{Y}\langle A, z\rangle = \sum_{r\in\mathbb{Z}} A_{\langle -r-1\rangle} z^r \in \widehat{\mathbf{U}}_\kappa[[z, z^{-1}]]$ to an arbitrary element $A \in \mathsf{Vac}_\kappa$.

4.3.3. The Feigin-Frenkel centre. Let $Z(\mathsf{Vac}_\kappa)$ denote the centre of the vertex algebra Vac_κ. It is a commutative vertex algebra, which is also a commutative ring. A precise definition can be found in [54, §3.3.1].

Proposition 4.3.1 ([54, Proposition 3.3.3])**.** *$Z(\mathsf{Vac}_\kappa) = \mathbb{C}|0\rangle$ if and only if $\kappa \neq c$.*

The commutative vertex algebra $\mathfrak{z}(\hat{\mathfrak{g}}) := Z(\mathsf{Vac}_c)$ is known as the *Feigin-Frenkel centre*. Elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ are called *Segal-Sugawara vectors*. We are now going to recall an explicit description of $\mathfrak{z}(\hat{\mathfrak{g}})$ due to Chervov and Molev. Identify $\mathbf{U}(\hat{\mathfrak{g}}_-) \xrightarrow{\sim} \mathsf{Vac}_c,\ X \mapsto X \cdot |0\rangle$ as vector spaces and consider the maps

$$S(\mathfrak{g}) \overset{i}{\hookrightarrow} S(\hat{\mathfrak{g}}_-) \overset{\sigma}{\twoheadleftarrow} \mathbf{U}(\hat{\mathfrak{g}}_-),$$

where $i(X) = X[-1]$ for $X \in \mathfrak{g}$ and σ is the principal symbol map with respect to the PBW filtration.

Definition 4.3.2 ([32, §2.2])**.** One calls $A_1, \dots, A_n \in \mathfrak{z}(\hat{\mathfrak{g}}) \subset \mathbf{U}(\hat{\mathfrak{g}}_-)$ a *complete set of Segal-Sugawara vectors* if there exist algebraically independent generators $B_1, \dots, B_n$ of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ such that $i(B_1) = \sigma(A_1), \dots, i(B_n) = \sigma(A_n)$.

Theorem 4.3.3 ([55, Theorem 9.6]). *If $A_1, \dots, A_n$ are a complete set of Segal-Sugawara vectors then*

$$\mathfrak{z}(\hat{\mathfrak{g}}) = \mathbb{C}[T^k A_r \mid r = 1, \dots, n,\ k \geq 0], \tag{4.15}$$

where T is the translation operator.

Example 4.3.4. Let $\hat{\hat{\mathfrak{g}}}_\kappa$ be the extension $0 \to \hat{\mathfrak{g}}_\kappa \to \hat{\hat{\mathfrak{g}}}_\kappa \to \mathbb{C}\tau \to 0$ defined by the relations $[\tau, X \otimes f] = -X \otimes \partial_t f$ and $[\tau, \mathbf{1}] = [\tau, \tau] = 0$. The subspace $\hat{\hat{\mathfrak{g}}}_- := \hat{\mathfrak{g}}_- \oplus \mathbb{C}\tau$ is a Lie subalgebra of $\hat{\hat{\mathfrak{g}}}_\kappa$. Consider the matrix $E_\tau \in \mathrm{Mat}_{n\times n}(\mathbf{U}(\hat{\hat{\mathfrak{g}}}_-))$ defined as

$$E_\tau := \begin{pmatrix} \tau + e_{11}[-1] & e_{12}[-1] & \cdots & e_{1n}[-1] \\ e_{21}[-1] & \tau + e_{22}[-1] & \cdots & e_{2n}[-1] \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}[-1] & e_{n2}[-1] & \cdots & \tau + e_{nn}[-1] \end{pmatrix}.$$

The traces $\mathrm{Tr}(E_\tau^k)$ are elements of $\mathbf{U}(\hat{\hat{\mathfrak{g}}}_-)$. In light of the canonical vector space isomorphism $\mathbf{U}(\hat{\hat{\mathfrak{g}}}_-) \cong \mathbf{U}(\hat{\mathfrak{g}}_-) \otimes \mathbb{C}[\tau]$, we can regard $\mathrm{Tr}(E_\tau^k)$ as polynomials in τ with coefficients in $\mathbf{U}(\hat{\mathfrak{g}}_-) \cong \mathsf{Vac}_c$. Define $\mathbf{T}_{k;l}$ $(0 \leq l \leq k \leq n)$ to be the coefficients of the polynomial

$$\mathrm{Tr}(E_\tau^k) = \mathbf{T}_{k;0}\tau^k + \mathbf{T}_{k;1}\tau^{k-1} + \ldots + \mathbf{T}_{k;k-1}\tau + \mathbf{T}_{k;k}$$

and set $\mathbf{T}_k := \mathbf{T}_{k;k}$. By [32, Theorem 3.1], the set $\{\mathbf{T}_k \mid 1 \leq k \leq n\}$ is a complete set of Segal-Sugawara vectors in $\mathfrak{z}(\hat{\mathfrak{g}})$.

4.3.4. The centre of the enveloping algebra. If A is a Segal-Sugawara vector, the coefficients $A_{\langle r \rangle}$ of the power series $\mathbb{Y}\langle A, z\rangle$ are called *Segal-Sugawara operators*. Given a complete set of Segal-Sugawara vectors $A_1, \dots, A_n$ such that $\deg A_i = \ \ -i$, let

$$\mathscr{Z} := \mathbb{C}[A_{i,\langle l \rangle}]_{i=1,\dots,n}^{l \in \mathbb{Z}}. \tag{4.16}$$

be the free polynomial algebra generated by the corresponding Segal-Sugawara operators. For $k > 0$, let J_k be the ideal in $\mathscr{Z}$ generated by the $A_{i,\langle l \rangle}$ with $l \geq ik$.

Theorem 4.3.5. *There exist natural algebra isomorphisms*

$$\widetilde{U}(\mathsf{Vac}_c) \cong \widehat{\mathbf{U}}_c, \quad \widetilde{U}(\mathfrak{z}(\hat{\mathfrak{g}})) \cong \mathfrak{Z}. \tag{4.17}$$

Moreover, $\mathfrak{Z} = \varprojlim (\mathscr{Z}/J_k)$.

Proof. For the isomorphisms (4.17), see [54, Lemma 3.2.2, Proposition 4.3.4]. For the second statement, see [54, §4.3.2] or [55, §12.2]. □

4.3.5. Quadratic Segal-Sugawara operators. Let $\kappa \in \mathbb{C}$. An important role is played by the vector

$${}^\kappa\mathbf{L} = \frac{1}{2} \sum_{1 \leq k,l \leq n} e_{kl}[-1]e_{lk}[-1] \in \mathsf{Vac}_\kappa. \tag{4.18}$$

Writing $\mathbb{Y}\langle {}^{\kappa}\mathbf{L}, z\rangle = \sum_{r\in\mathbb{Z}} {}^{\kappa}\mathbf{L}_{\langle r\rangle} z^{-r-1}$, we have the formula

$$ {}^{\kappa}\mathbf{L}_r := {}^{\kappa}\mathbf{L}_{\langle r+1\rangle} = \frac{1}{2}\sum_{1\le k,l\le n}\left(\sum_{i\le -1} e_{kl}[i]e_{lk}[r-i] + \sum_{i\ge 0} e_{lk}[r-i]e_{kl}[i]\right) \in \widehat{\mathbf{U}}_{\kappa}(\hat{\mathfrak{g}}). \tag{4.19}$$

Proposition 4.3.6. *If $\kappa = c$ then ${}^{c}\mathbf{L} \in \mathfrak{z}(\hat{\mathfrak{g}})$ and ${}^{c}\mathbf{L}_r \in \mathfrak{Z}$ for each $r \in \mathbb{Z}$. Moreover,*

$$[{}^{\kappa}\mathbf{L}_{-1}, X\otimes f] = -(\kappa+n)X\otimes \partial_t f$$

for all $X \in \mathfrak{g}$ and $f \in \mathbb{C}((t))$.

Proof. The proposition follows from a direct calculation using operator product expansions. This calculation can be found in, e.g., [54, §3.1.1]. □

4.4 Suzuki functor for all levels

In [132], Suzuki defined a functor $\mathsf{F}_{\kappa}\colon \mathscr{C}_{\kappa} \to \mathtt{H}_{\kappa+n}\text{-}\mathsf{mod}$ for $\kappa \neq c$. In this section we generalize his construction to the $\kappa = c$ case. Throughout this section assume that m, n are any positive integers and $\kappa \in \mathbb{C}$ unless stated otherwise.

We first need to introduce some notation. Define

$$\mathbb{C}[\mathfrak{h}]^{\rtimes} := \mathbb{C}[\mathfrak{h}] \rtimes \mathbb{C}S_m, \quad \mathbb{C}[\mathfrak{h}^*]^{\rtimes} := \mathbb{C}S_m \ltimes \mathbb{C}[\mathfrak{h}^*].$$

Set $\delta := \prod_{1\le i<j\le m}(x_i - x_j)$ and $\delta_z = \prod_{j=1}^{m}(z - x_j)$. Let $\mathfrak{h}_{\mathsf{reg}} \subset \mathfrak{h}$ be the open subset on which S_m acts freely. Define

$$\mathcal{R} := \mathbb{C}[\mathfrak{h}_{\mathsf{reg}}] = \mathbb{C}[x_1,\dots,x_m][\delta^{-1}], \quad \mathcal{R}^{\rtimes} := \mathcal{R}\rtimes \mathbb{C}S_m, \quad \mathcal{R}_z := \mathcal{R}[z][\delta_z^{-1}].$$

4.4.1. Simultaneous affinization. Let $\mathbb{V}^*_{\kappa} := \mathrm{Ind}^{\hat{\mathfrak{g}}_{\kappa}}_{\hat{\mathfrak{g}}_+} \circ \mathrm{Inf}^{\hat{\mathfrak{g}}_+}_{\mathfrak{g}\oplus\mathbb{C}\mathbf{1}} \mathbf{V}^*$, where $\mathbf{1}$ acts on $\mathbf{V}^*$ as the identity endomorphism. We start by recalling (see e.g. [85, §9.9, 9.11]) the construction of a $\mathfrak{g}\otimes\mathcal{R}_z$-action on

$$\mathbb{T}_{\kappa}(M) := \mathcal{R}\otimes(\mathbb{V}^*_{\kappa})^{\otimes m}\otimes M, \tag{4.20}$$

for any module M in $\mathscr{C}_{\kappa}$. For that purpose we first recall the definition of an auxiliary Lie algebra $\mathfrak{G}_R$.

Let R be a commutative unital algebra. We fix formal variables $t_1,\dots,t_m,t_{\infty}$. Set $\mathfrak{g}(i)_R := \mathfrak{g}\otimes R((t_i))$, $\mathfrak{g}(i) := \mathfrak{g}(i)_{\mathbb{C}}$. Consider the R-Lie algebra

$$\mathfrak{G}_R := \bigoplus_{i=1}^{m}\mathfrak{g}(i)_R \oplus \mathfrak{g}(\infty)_R = \mathfrak{g}\otimes(\bigoplus_{i=1}^{m} R((t_i)) \oplus R((t_{\infty}))). \tag{4.21}$$

We denote a pure tensor on the RHS of (4.21) by $X\otimes(f_i)$, where $X \in \mathfrak{g}$ and $f_i \in R((t_i))$ for $i = 1,\dots,m,\infty$. Define $\hat{\mathfrak{G}}_{R,\kappa}$ to be the central extension

$$0 \to R\mathbf{1} \to \hat{\mathfrak{G}}_{R,\kappa} \to \mathfrak{G}_R \to 0 \tag{4.22}$$

associated to the cocycle $(X\otimes(f_i), Y\otimes(g_i)) \mapsto \langle X, Y\rangle_{\kappa}\sum_{i\in\{1,\dots,m,\infty\}}\mathrm{Res}_{t_i=0}(g_i df_i)$. If $R = \mathbb{C}$ we abbreviate $\hat{\mathfrak{G}}_{\kappa} := \hat{\mathfrak{G}}_{\mathbb{C},\kappa}$. Set

$$\mathbf{U}_{\kappa}(\hat{\mathfrak{G}}_R) := \mathbf{U}(\hat{\mathfrak{G}}_{R,\kappa})/\langle \mathbf{1} - 1_{\hat{\mathfrak{G}}_{R,\kappa}}\rangle.$$

A $\mathbf{U}_\kappa(\hat{\mathfrak{G}}_R)$-module M is called *smooth* if for every vector $v \in M$ there exists a positive integer k such that $\mathfrak{g} \otimes (\bigoplus_{i=1}^m t_i^k R((t_i)) \oplus t_\infty^k R((t_\infty))).v = 0$. Suppose that $M_1, \dots, M_m, M_\infty$ are smooth $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$-modules. Then $R \otimes \bigotimes_{i=1}^m M_i \otimes M_\infty$ is a smooth $\mathbf{U}_\kappa(\hat{\mathfrak{G}}_R)$-module with the action of the dense subalgebra $R \otimes \mathbf{U}_\kappa(\hat{\mathfrak{G}})$ given by the formula

$$r \otimes X \otimes (f_i) \mapsto \sum_{i=1,\dots,m,\infty} r \otimes (X \otimes f_i)^{(i)}, \tag{4.23}$$

where $(X \otimes f_i)^{(i)} := \mathrm{id}^{i-1} \otimes (X \otimes f_i) \otimes \mathrm{id}^{m-i}$. Note that if R were an infinite-dimensional algebra and the modules M_i were not smooth, the action of $R \otimes \mathbf{U}_\kappa(\hat{\mathfrak{G}})$ would not necessarily extend to an action of $\mathbf{U}_\kappa(\hat{\mathfrak{G}}_R)$.

4.4.2. Conformal coinvariants. We next recall the connection between the Lie algebras $\mathfrak{G}_R$ and $\mathfrak{g} \otimes \mathcal{R}_z$. Consider $\mathcal{R}_z$ as an $\mathcal{R}$-subalgebra of $\mathcal{R}(z)$. We thus view elements of $\mathcal{R}_z$ as rational functions which may have poles at $x_1, \dots, x_m$ and ∞. Set $z_i := z - x_i$.

Definition 4.4.1. For $1 \le i \le m$, let $\iota_{\mathcal{R},i} \colon \mathcal{R}_z \to \mathcal{R}((z_i))$ (resp. $\iota_{\mathcal{R},\infty} \colon \mathcal{R}_z \to \mathcal{R}((z^{-1}))$) be the $\mathcal{R}$-algebra homomorphism sending a function in $\mathcal{R}_z$ to its Laurent series expansion at x_i (resp. ∞). Let

$$\iota_{\mathcal{R}} : \mathcal{R}_z \hookrightarrow \bigoplus_{i=1}^m \mathcal{R}((t_i)) \oplus \mathcal{R}((t_\infty)) \tag{4.24}$$

be the injective $\mathcal{R}$-algebra homomorphism given by $(\iota_{\mathcal{R},1}, \dots, \iota_{\mathcal{R},m}, \iota_{\mathcal{R},\infty})$ followed by the assignment $z_i \mapsto t_i, z^{-1} \mapsto t_\infty$.

The map (4.24) induces the Lie algebra homomorphism

$$\mathfrak{g} \otimes \mathcal{R}_z \hookrightarrow \mathfrak{G}_{\mathcal{R}}, \quad X \otimes f \mapsto X \otimes \iota_{\mathcal{R}}(f), \tag{4.25}$$

which, by the residue theorem, lifts to an injective Lie algebra homomorphism

$$\mathfrak{g} \otimes \mathcal{R}_z \hookrightarrow \hat{\mathfrak{G}}_{\mathcal{R},\kappa}. \tag{4.26}$$

Let M be a smooth $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$-module. The vector space $\mathbb{T}_\kappa(M)$ is a smooth $\mathbf{U}_\kappa(\hat{\mathfrak{G}}_{\mathcal{R}})$-module (with the action given by (4.23)). We consider it as a $\mathbf{U}(\mathfrak{g} \otimes \mathcal{R}_z)$-module via (4.26). It also carries a natural $\mathcal{R}^\rtimes$-action: $\mathcal{R}$ acts by multiplication and S_m acts by permuting the factors of the tensor product $(\mathbb{V}_\kappa^*)^{\otimes m}$ and the x_i's. The next lemma follows directly from the definitions.

Lemma 4.4.2. *The $\mathcal{R}^\rtimes$-action on $\mathbb{T}_\kappa(M)$ normalizes the $\mathbf{U}(\mathfrak{g} \otimes \mathcal{R}_z)$-action. Therefore we have functors*

$$\mathbb{T}_\kappa \colon \mathscr{C}_\kappa \to (\mathbf{U}(\mathfrak{g} \otimes \mathcal{R}_z), \mathcal{R}^\rtimes)\text{-nmod}, \quad M \mapsto \mathbb{T}_\kappa(M), \tag{4.27}$$

$$\mathbb{F}_\kappa \colon \mathscr{C}_\kappa \to \mathcal{R}^\rtimes\text{-mod}, \quad M \mapsto H_0(\mathfrak{g} \otimes \mathcal{R}_z, \mathbb{T}_\kappa(M)). \tag{4.28}$$

4.4.3. The Knizhnik-Zamolodchikov connection. We are going to extend the $\mathcal{R}^\rtimes$-action on $\mathbb{T}_\kappa(M)$ and $\mathbb{F}_\kappa(M)$ to an action of $\mathtt{H}_{\kappa+n}$.

Definition 4.4.3. Let $\kappa \in \mathbb{C}$. The *deformed Weyl algebra* $\mathcal{D}_\kappa$ is the algebra generated by $x_1, \dots, x_m$ and $q_1, \dots, q_m$ subject to the relations

$$[x_i, x_j] = [q_i, q_j] = 0, \quad [q_i, x_j] = (\kappa + n)\delta_{ij} \quad (1 \le i, j \le m).$$

Note that $\mathcal{D}_c = \mathbb{C}[x_1, \dots, x_m, q_1, \dots, q_m]$ (where $c := -n$). Set

$$\mathcal{D}_\kappa^{\rtimes} := \mathcal{D}_\kappa \rtimes \mathbb{C}S_m, \quad \mathcal{D}_{\kappa,\mathsf{reg}}^{\rtimes} := \mathcal{D}_\kappa^{\rtimes}[\delta^{-1}].$$

Suppose that M is a $\mathbb{C}[\mathfrak{h}]^{\rtimes}$- (resp. $\mathcal{R}^{\rtimes}$-) module. A *good connection* on M is a representation of $\mathcal{D}_\kappa^{\rtimes}$ (resp. $\mathcal{D}_{\kappa,\mathsf{reg}}^{\rtimes}$) on M extending the given $\mathbb{C}[\mathfrak{h}]^{\rtimes}$- (resp. $\mathcal{R}^{\rtimes}$-) module structure.

Lemma 4.4.4. *Let M be a $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-module. If $\rho\colon \mathcal{D}_\kappa^{\rtimes} \to \mathrm{End}_{\mathbb{C}}(M)$ is a good connection on M, then ρ', defined as*

$$\rho'(q_i) := \rho(q_i) + \sum_{j\neq i} \frac{1}{x_i - x_j},$$

is a good connection on the $\mathcal{R}^{\rtimes}$-module $M_{\mathsf{reg}} := \mathcal{R} \otimes_{\mathbb{C}[\mathfrak{h}]} M$.

Proof. The lemma follows by a direct calculation, as in [138, Proposition 1.8]. □

Let M be a smooth $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$-module. Consider the $\mathcal{R}^{\rtimes}$-module $\mathbb{T}_\kappa(M)$ and the operators

$${}^\kappa\mathbb{W}_i := (\kappa + n)\partial_{x_i} + {}^\kappa\mathbf{L}_{-1}^{(i)} \quad (1 \le i \le m)$$

on $\mathbb{T}_\kappa(M)$. The following proposition extends [13, Lemma 13.3.7] to the critical level case.

Proposition 4.4.5. *Let $\kappa \in \mathbb{C}$.*

a) *The assignment*

$${}^\kappa\mathbb{W}\colon \mathcal{D}_{\kappa,\mathsf{reg}}^{\rtimes} \to \mathrm{End}_{\mathbb{C}}(\mathbb{T}_\kappa(M)), \quad q_i \mapsto {}^\kappa\mathbb{W}_i$$

defines a good connection (known as the Knizhnik-Zamolodchikov connection) on $\mathbb{T}_\kappa(M)$.

b) *The operators ${}^\kappa\mathbb{W}_i$ normalize the $\mathfrak{g}\otimes\mathcal{R}_z$-action on $\mathbb{T}_\kappa(M)$, i.e., $[{}^\kappa\mathbb{W}_i, \mathfrak{g}\otimes\mathcal{R}_z] \subset \mathfrak{g}\otimes\mathcal{R}_z$. Hence ${}^\kappa\mathbb{W}$ descends to a good connection on $\mathbb{F}_\kappa(M)$.*

Proof. It suffices to consider the case $\kappa = c$. The operators ${}^c\mathbb{W}_i = {}^c\mathbf{L}_{-1}^{(i)}$ act on different factors $\mathbb{V}_c^*$ of the tensor product $\mathbb{T}_c(M) = \mathcal{R} \otimes (\mathbb{V}_c^*)^{\otimes m} \otimes M$, so they commute. Moreover, the operators x_j act only on the first factor $\mathcal{R}$ and so they commute with the operators ${}^c\mathbb{W}_i$ as well. Hence ${}^c\mathbb{W}$ is a representation of $\mathcal{D}_c$, which clearly extends to a representation of $\mathcal{D}_{c,\mathsf{reg}}^{\rtimes}$. The second statement follows immediately from the fact that ${}^c\mathbf{L}_{-1} \in \mathfrak{Z}$. □

To obtain representations of the rational Cherednik algebra on $\mathbb{T}_\kappa(M)$ and $\mathbb{F}_\kappa(M)$, we are going to compose the connection ${}^\kappa\mathbb{W}'$ with the Dunkl embedding, whose definition we now recall.

Proposition 4.4.6 ([48, Proposition 4.5])**.** *There is an injective algebra homomorphism, called the* Dunkl embedding*,*

$$\mathtt{H}_{\kappa+n} \hookrightarrow \mathcal{D}_{\kappa,\mathsf{reg}}^{\rtimes}, \quad x_i \mapsto x_i,\ w \mapsto w,\ y_i \mapsto D_i := q_i + \sum_{j\neq i} \frac{1}{x_i - x_j}(s_{i,j} - 1), \tag{4.29}$$

with $1 \le i \le m$ and $w \in S_m$.

Proposition 4.4.7. *Composing* (4.29) *with ${}^\kappa\mathbb{W}'$ yields representations of $\mathtt{H}_{\kappa+n}$ on $\mathbb{T}_\kappa(M)$ and $\mathbb{F}_\kappa(M)$. Moreover, the functors* (4.27) *and* (4.28) *extend to functors*

$$\mathbb{T}_\kappa\colon \mathscr{C}_\kappa \to (\mathbf{U}(\mathfrak{g}\otimes\mathcal{R}_z), \mathtt{H}_{\kappa+n})\text{-}\mathsf{nmod}, \quad \mathbb{F}_\kappa\colon \mathscr{C}_\kappa \to \mathtt{H}_{\kappa+n}\text{-}\mathsf{mod}.$$

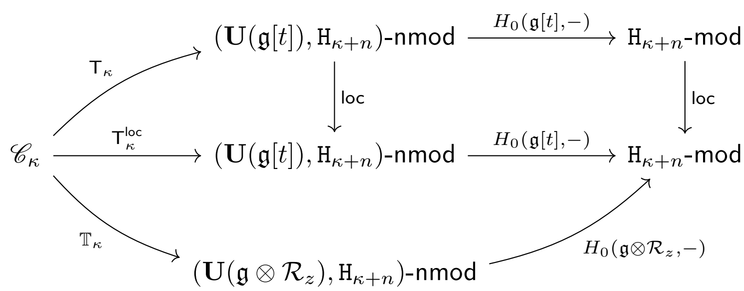
Proof. By Lemma 4.4.4 and Proposition 4.4.5, ${}^{\kappa}\mathbb{V}'$ is a good connection on $\mathbb{T}_\kappa(M)$, which descends to a good connection on $\mathbb{F}_\kappa(M)$. It therefore yields representations of $\mathcal{D}^{\rtimes}_{\kappa,\mathsf{reg}}$ on $\mathbb{T}_\kappa(M)$ and $\mathbb{F}_\kappa(M)$, which become representations of $\mathtt{H}_{\kappa+n}$ via the Dunkl embedding.

Let us check that $\mathbb{T}_\kappa$ and $\mathbb{F}_\kappa$ are functors. Let $f\colon M \to N$ be a morphism in $\mathscr{C}_\kappa$. It induces a map $\mathbb{T}_\kappa(f)\colon \mathbb{T}_\kappa(M) \to \mathbb{T}_\kappa(N)$. Since the $\mathtt{H}_{\kappa+n}$-action doesn't affect the last factor (as in (4.20)) in these tensor products, $\mathbb{T}_\kappa(f)$ commutes with the $\mathtt{H}_{\kappa+n}$-action. The fact that f is a $\hat{\mathfrak{g}}_\kappa$-module homomorphism also implies that $\mathbb{T}_\kappa(f)$ commutes with the $\mathfrak{g}\otimes\mathcal{R}_z$-action on $\mathbb{T}_\kappa(M)$ and $\mathbb{T}_\kappa(N)$. Hence $\mathbb{T}_\kappa(f)$ descends to a $\mathtt{H}_{\kappa+n}$-module homomorphism $\mathbb{F}_\kappa(f)\colon \mathbb{F}_\kappa(M) \to \mathbb{F}_\kappa(N)$. □

4.4.4. The current Lie algebra action. Given a smooth $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$-module M, set

$$\mathsf{T}_\kappa(M) := \mathbb{C}[\mathfrak{h}] \otimes (\mathbf{V}^*)^{\otimes m} \otimes M, \quad \mathsf{T}^{\mathsf{loc}}_\kappa(M) := \mathcal{R} \otimes (\mathbf{V}^*)^{\otimes m} \otimes M.$$

We will show that the functors T_κ and $\mathsf{T}^{\mathsf{loc}}_\kappa$ fit into the following commutative diagram



where loc is the localization functor sending N to $N_{\mathsf{reg}} := \mathcal{R} \otimes_{\mathbb{C}[\mathfrak{h}]} N$. The Suzuki functor is the composition of T_κ with $H_0(\mathfrak{g}[t],-)$. Let us explain this diagram in more detail. The current Lie algebra $\mathfrak{g}[t]$ acts on $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$ by the rule

$$Y[k] \mapsto \sum_{i=1}^{m} x_i^k \otimes Y^{(i)} + 1 \otimes (Y[-k])^{(\infty)} \quad (Y \in \mathfrak{g},\ k \geq 0). \tag{4.30}$$

The $\mathcal{R}^{\rtimes}$-action on $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$ is analogous to that on $\mathbb{T}_\kappa(M)$. It follows directly from the definitions that the $\mathfrak{g}[t]$-action and the $\mathcal{R}^{\rtimes}$-action commute. We next recall how the $\mathcal{R}^{\rtimes}$-action can be extended to an $\mathtt{H}_{\kappa+n}$-action on $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$.

Definition 4.4.8. Let $1 \leq i, j \leq m$ and $p \geq 0$. Consider

$$\Omega^{(i,j)} := \sum_{1\leq k,l\leq n} e^{(i)}_{kl} e^{(j)}_{lk}, \qquad \Omega^{(i,\infty)}_{[p+1]} := \sum_{1\leq k,l\leq n} e^{(i)}_{kl} e_{lk}[p+1]^{(\infty)},$$

$$\mathfrak{L}^{(i)} := -\sum_{1\leq j\neq i\leq m} \frac{\Omega^{(i,j)}}{x_i - x_j} + \sum_{p\geq 0} x_i^p \Omega^{(i,\infty)}_{[p+1]}, \qquad {}^{\kappa}\nabla_i := (\kappa+n)\partial_{x_i} + \mathfrak{L}^{(i)}.$$

as operators on $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$. They are well-defined because M is smooth.

Lemma 4.4.9. *The assignment*

$${}^{\kappa}\nabla\colon \mathcal{D}^{\rtimes}_\kappa \to \mathrm{End}_{\mathbb{C}}(\mathsf{T}^{\mathsf{loc}}_\kappa(M)), \quad q_i \mapsto {}^{\kappa}\nabla_i$$

defines a good connection on $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$.

Proof. One needs to check that ${}^\kappa\nabla$ is a well-defined ring homomorphisms, i.e., show that $[{}^\kappa\nabla_i, {}^\kappa\nabla_j] = 0$ and $[{}^\kappa\nabla_i, x_j] = (\kappa+n)\delta_{ij}$. These commutation relations are calculated in [87, Lemma 3.2-3.3]. □

Proposition 4.4.10. *Composing* (4.29) *with* ${}^\kappa\nabla'$ *yields a representation of* $\mathtt{H}_{\kappa+n}$ *on* $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$*. The element* y_i *acts as the operator*

$$ {}^\kappa\overline{y}_i = (\kappa+n)\partial_{x_i} + \sum_{1\le j\neq i\le m} \frac{\Omega^{(i,j)}}{x_i - x_j}(\underline{s_{i,j}} - 1) + \sum_{p\ge 0} x_i^p \Omega^{(i,\infty)}_{[p+1]}, \tag{4.31}$$

where $\underline{s_{i,j}}$ *acts by permuting the* x_i*'s but not the factors of the tensor product. Moreover,* $\mathsf{T}_\kappa(M)$ *is a subrepresentation.*

Proof. By Lemma 4.4.4 and Lemma 4.4.9, ${}^\kappa\nabla'$ is a good connection, which implies the first statement. For the second part, observe that

$$ {}^\kappa\nabla'(D_i) = {}^\kappa\nabla_i + \sum_{j\neq i}\frac{1}{x_i-x_j}s_{i,j} = (\kappa+n)\partial_{x_i} + \sum_{j\neq i}\frac{1}{x_i-x_j}(s_{i,j} - \Omega^{(i,j)}) + \sum_{p\ge 0} x_i^p\Omega^{(i,\infty)}_{[p+1]}. $$

The equality of operators $s_{i,j} = \Omega^{(i,j)}\underline{s_{i,j}}$ implies (4.31). The third statement follows from the fact that the operators $\frac{-1+\underline{s_{i,j}}}{x_i-x_j}$ and ∂_{x_i} preserve $\mathbb{C}[\mathfrak{h}] \subset \mathcal{R}$. □

4.4.5. Suzuki functor. We next consider the relationship between the functors $\mathsf{T}^{\mathsf{loc}}_\kappa$ and $\mathbb{T}_\kappa$. Both $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$ and $\mathbb{T}_\kappa(M)$ carry representations of $\mathcal{D}^{\rtimes}_\kappa$ given by ${}^\kappa\nabla'$ and ${}^\kappa\mathbb{V}'$, respectively. The following result is well known (see, e.g., [138, Proposition 2.18]).

Proposition 4.4.11. *The connection* ${}^\kappa\nabla'$ *normalizes the* $\mathfrak{g}[t]$*-action on* $\mathsf{T}^{\mathsf{loc}}_\kappa(M)$ *and descends to a good connection on* $H_0(\mathfrak{g}[t], \mathsf{T}^{\mathsf{loc}}_\kappa(M))$*. Moreover, there is a* $\mathcal{D}^{\rtimes}_\kappa$*-module isomorphism*

$$ H_0(\mathfrak{g}\otimes\mathcal{R}_z, \mathbb{T}_\kappa(M)) \cong H_0(\mathfrak{g}[t], \mathsf{T}^{\mathsf{loc}}_\kappa(M)). \tag{4.32}$$

Proof. A detailed proof of the first statement can be found in [87, §3.2], and of the second statement in [138, Proposition 3.6]). □

By Proposition (4.4.10), composing the Dunkl embedding with the connection ${}^\kappa\nabla'$ yields a representation of $\mathtt{H}_{\kappa+n}$ on $\mathsf{T}_\kappa(M)$. Proposition 4.4.11 implies that this representation descends to a representation on $H_0(\mathfrak{g}[t], \mathsf{T}_\kappa(M))$.

Corollary 4.4.12. *For all* $\kappa\in\mathbb{C}$*, we have functors*

$$ \mathsf{T}_\kappa\colon \mathscr{C}_\kappa \to (\mathbf{U}(\mathfrak{g}[t]), \mathtt{H}_{\kappa+n})\text{-nmod}, \quad M\mapsto \mathsf{T}_\kappa(M), \tag{4.33}$$

$$ \mathsf{F}_\kappa\colon \mathscr{C}_\kappa \to \mathtt{H}_{\kappa+n}\text{-mod}, \quad M\mapsto H_0(\mathfrak{g}[t], \mathsf{T}_\kappa(M)). \tag{4.34}$$

Moreover, (4.32) *is an* $\mathtt{H}_{\kappa+n}$*-module isomorphism and the functors* $\mathbb{F}_\kappa$ *and* $\mathsf{loc}\circ\mathsf{F}_\kappa$ *are naturally isomorphic.*

Proof. The first statement follows from Proposition 4.4.10 and Proposition 4.4.11 as explained above. The second statement follows directly from (4.32). □

Definition 4.4.13. Given $\kappa\in\mathbb{C}$, we call

$$ \mathsf{F}_\kappa\colon \mathscr{C}_\kappa \to \mathtt{H}_{\kappa+n}\text{-mod}, \quad M\mapsto H_0(\mathfrak{g}[t], \mathbb{C}[\mathfrak{h}]\otimes(\mathbf{V}^*)^{\otimes m}\otimes M) \tag{4.35}$$

the *Suzuki functor* (of level κ).

The functor (4.35) extends Suzuki's construction from [132] to the critical level case. Indeed, setting $\kappa = c$, we get the functor

$$\mathsf{F}_c \colon \mathscr{C}_c \to \mathtt{H}_0\text{-}\mathsf{mod}$$

relating the affine Lie algebra at the critical level to the rational Cherednik algebra at $t = 0$.

Remark 4.4.14. In [138] Varagnolo and Vasserot constructed functors from $\mathscr{C}_\kappa$ ($\kappa \neq c$) to the category of modules over the rational Cherednik algebra ($t \neq 0$) associated to the wreath product $(\mathbb{Z}/l\mathbb{Z}) \wr S_m$. We expect that our approach to extending the Suzuki functor to the $\kappa = c$, $t = 0$ case can also be applied to their functors.

4.5 Suzuki functor - further generalizations

The Suzuki functor has so far been defined on smooth $\widehat{\mathbf{U}}_\kappa$-modules. We now extend its definition to all $\widehat{\mathbf{U}}_\kappa$-modules using a certain inverse limit construction. Let $\kappa \in \mathbb{C}$ and $t = \kappa + n$ throughout this section.

4.5.1. Pro-smooth modules. We are going to define the category of pro-smooth modules and the pro-smooth completion functor. If $\mathscr{I}$ is an inverse system in some category, we write $\lim \mathscr{I}$ or $\lim_{M_i \in \mathscr{I}} M_i$, where the M_i run over the objects in $\mathscr{I}$, for its inverse limit. We start with the following auxiliary lemma.

Lemma 4.5.1. *Let M be any $\widehat{\mathbf{U}}_\kappa$-module, N a smooth module and $f \colon M \to N$ a $\widehat{\mathbf{U}}_\kappa$-module homomorphism. Then $M/\ker f$ is a smooth module.*

Proof. Let $v \in M$ and let $\bar{v}$ be the image of v in $M/\ker f$. Since N is smooth, there exists $r \geq 0$ such that $\hat{I}_r \cdot f(v) = 0$. Hence $f(\hat{I}_r \cdot v) = 0$, $\hat{I}_r \cdot v \subseteq \ker f$ and so $\hat{I}_r \cdot \bar{v} = 0$. □

Definition 4.5.2. A $\widehat{\mathbf{U}}_\kappa$-module M is called *pro-smooth* if M is the inverse limit of an inverse system of smooth $\widehat{\mathbf{U}}_\kappa$-modules. Let $\widetilde{\mathscr{C}_\kappa}$ denote the full subcategory of $\widehat{\mathbf{U}}_\kappa$-$\mathsf{mod}$ whose objects are pro-smooth modules.

Definition 4.5.3. Let M be a $\widehat{\mathbf{U}}_\kappa$-module. The smooth quotients of M form an inverse system $\mathscr{I}_M$ partially ordered by projections. Let

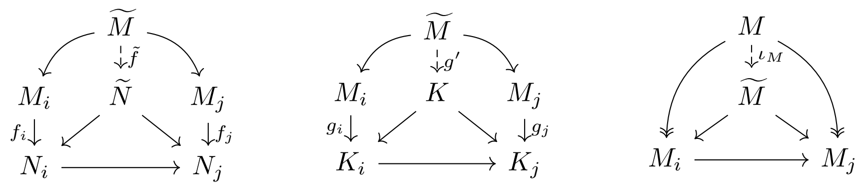
$$\widetilde{M} := \lim \mathscr{I}_M.$$

Proposition 4.5.4. *There exists a "pro-smooth completion" functor*

$$\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod} \to \widetilde{\mathscr{C}_\kappa}, \quad M \mapsto \widetilde{M}, \ f \mapsto \tilde{f} \tag{4.36}$$

left adjoint to the inclusion functor $\widetilde{\mathscr{C}_\kappa} \hookrightarrow \widehat{\mathbf{U}}_\kappa$-$\mathsf{mod}$.

Proof. We first construct $\tilde{f}$ explicitly. Let $f \colon M \to N$ be a homomorphism of $\widehat{\mathbf{U}}_\kappa$-modules. Given a smooth quotient N_i of N, let f_i be the map $M \xrightarrow{f} N \twoheadrightarrow N_i$. By Lemma 4.5.1, $M_i := M/\ker f_i$ is a smooth module. Hence, there is a canonical map $\widetilde{M} \to M_i$ as part of the inverse limit data. Consider the diagram on the LHS below, where N_j is another smooth quotient of N and all the unnamed maps are part of the inverse system or inverse limit data. Since the outer pentagon commutes, the universal property of the inverse limit $\widetilde{N}$ implies that there exists a unique map $\tilde{f}$ making the diagram commute.



Next we construct the adjunction. Let $g\colon M \to K$ be a homomorphism of $\widehat{\mathbf{U}}_\kappa$-modules, and assume that K is the inverse limit of an inverse system of smooth modules. Given such a smooth module K_i, let g_i be the composition of g with the canonical map $K \to K_i$. By Lemma 4.5.1, g factors through the smooth module $M_i := M/\ker g_i$. An analogous argument to the one above shows that there exists a unique map g' making the middle diagram above commute. The universal property of the inverse limit $\widetilde{M}$ also yields a unique map ι_M making the diagram on the RHS above commute.

It is easy to check that the maps

$$\mathrm{Hom}_{\widetilde{\mathscr{C}}_\kappa}(\widetilde{M}, K) \cong \mathrm{Hom}_{\widehat{\mathbf{U}}_\kappa}(M, K), \quad h \mapsto h \circ \iota_M,\ g' \leftarrow\!\shortmid g \tag{4.37}$$

are mutually inverse bijections. This gives the adjunction. □

Proposition 4.5.5. *The restriction of* (4.36) *to* $\mathscr{C}_\kappa$ *or* $\widehat{\mathbf{U}}_\kappa$-fpmod *is naturally isomorphic to the identity functor.*

Proof. If M is smooth then M is the greatest element in the inverse system $\mathscr{I}_M$, so $\widetilde{M} = M$. Next suppose that M is finitely presented with presentation

$$(\widehat{\mathbf{U}}_\kappa)^{\oplus a} \xrightarrow{f} (\widehat{\mathbf{U}}_\kappa)^{\oplus b} \to M \to 0.$$

We first show that $(\widehat{\mathbf{U}}_\kappa)^{\widetilde{}} = \widehat{\mathbf{U}}_\kappa$. The inverse system $\mathscr{I}' := \{\widehat{\mathbf{U}}_\kappa/\hat{I}_r \mid r \geq 0\}$ is a subsystem of $\mathscr{I} := \mathscr{I}_{\widehat{\mathbf{U}}_\kappa}$. Suppose that $N = \widehat{\mathbf{U}}_\kappa/J$ is a smooth quotient and let $\bar{1}$ be the image of 1 in N. Then, by smoothness, $\hat{I}_r.\bar{1} = 0$ for some $r \geq 0$. Hence $\hat{I}_r \subseteq J$ and N is a quotient of $\widehat{\mathbf{U}}_\kappa/\hat{I}_r$. Therefore $\mathscr{I}'$ is a cofinal subsystem of $\mathscr{I}$ and

$$(\widehat{\mathbf{U}}_\kappa)^{\widetilde{}} := \lim \mathscr{I} = \lim \mathscr{I}' = \widehat{\mathbf{U}}_\kappa.$$

The fact that limits commute with finite direct sums implies that the pro-smooth completion functor sends $(\widehat{\mathbf{U}}_\kappa)^{\oplus a}$ to itself. Hence $\iota_{(\widehat{\mathbf{U}}_\kappa)^{\oplus a}} = \mathrm{id}$ and $\tilde{f} = f'$, using the notation from (4.37). The adjunction (4.37), therefore, implies that $\tilde{f} = f$. By Proposition 4.5.4, the pro-smooth completion functor is left adjoint and, hence, right exact. Hence $(\mathrm{coker}\, f)^{\widetilde{}} = \mathrm{coker}\, \tilde{f} = \mathrm{coker}\, f = M$. □

We will also need the following lemma.

Lemma 4.5.6. *Let M and N be $\widehat{\mathbf{U}}_\kappa$-modules. Then*

$$\mathrm{Hom}_{\widetilde{\mathscr{C}}_\kappa}(\widetilde{M}, \widetilde{N}) = \lim_{N_i \in \mathscr{I}_N} \operatorname*{colim}_{M_j \in \mathscr{I}_M} \mathrm{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i).$$

Proof. The equality $\mathrm{Hom}_{\widetilde{\mathscr{C}}_\kappa}(\widetilde{M}, \widetilde{N}) = \lim_{N_i \in \mathscr{I}_N} \mathrm{Hom}_{\widehat{\mathbf{U}}_\kappa}(\widetilde{M}, N_i)$ follows from the general properties of limits. Therefore it suffices to show that, for each $N_i \in \mathscr{I}_N$,

$$\mathrm{Hom}_{\widehat{\mathbf{U}}_\kappa}(\widetilde{M}, N_i) = \operatorname*{colim}_{M_j \in \mathscr{I}_M} \mathrm{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i). \tag{4.38}$$

Let us check that the LHS of (4.38) satisfies the universal property of the colimit. Suppose that we are given a vector space X and linear functions $\chi_{M_j}\colon \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i) \to X$, for each $M_j \in \mathscr{I}_M$, which commute with the natural inclusions between the Hom-spaces. We are now going to define a map $\chi\colon \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(\widetilde{M}, N_i) \to X$. If $f \in \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(\widetilde{M}, N_i)$, then, by Lemma 4.5.1, the module $\overline{M} := \widetilde{M}/\ker f$ is smooth. Let $\bar{f}\colon \overline{M} \to N_i$ be the homomorphism induced by f. We define χ by setting $\chi(f) := \chi_{\overline{M}}(\bar{f})$. One can easily see that χ is the unique map making the diagram below commute (where M_k is another smooth quotient of M and all the unnamed maps are the canonical ones).

$$
\begin{array}{ccccc}
\operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i) & & \longrightarrow & & \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_k, N_i) \\
 & \searrow^{\chi_{M_j}} & & \swarrow^{\chi_{M_k}} & \\
 & & X & & \\
 & & \downarrow \chi & & \\
 & \searrow & \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(\widetilde{M}, N_i) & \swarrow &
\end{array}
$$

□

4.5.2. Extension to all modules. We start by extending the Suzuki functor from Definition (4.4.13) to the category $\widehat{\mathbf{U}}_\kappa$-fpmod. Suppose that M is a finitely presented $\widehat{\mathbf{U}}_\kappa$-module. By Proposition 4.5.5, we have $M = \widetilde{M} := \lim \mathscr{J}_M$. Set

$$\mathsf{F}_\kappa(M) := \lim_{M_i \in \mathscr{I}_M} \mathsf{F}_\kappa(M_i), \tag{4.39}$$

where the limit is taken in the category $\mathtt{H}_t$-mod. If M is smooth then M is the maximal element in the inverse system $\mathscr{I}_M$, so (4.39) is compatible with the previous definition of F_κ.

Proposition 4.5.7. *The functor* (4.34) *extends to a right exact functor*

$$\mathsf{F}_\kappa\colon \widehat{\mathbf{U}}_\kappa\text{-fpmod} \to \mathtt{H}_t\text{-mod}. \tag{4.40}$$

Proof. We need to construct maps between Hom-sets. Suppose that $N = \widetilde{N}$ is another finitely presented $\widehat{\mathbf{U}}_\kappa$-module. Let $N_i \in \mathscr{I}_N$. For all $M_j \in \mathscr{J}_M$, we have maps

$$\phi_j\colon \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i) \xrightarrow{\mathsf{F}_\kappa} \operatorname{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(M_j), \mathsf{F}_\kappa(N_i)) \to \operatorname{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(M), \mathsf{F}_\kappa(N_i))$$

compatible with the transition maps of the direct system $\{\operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i) \mid M_j \in \mathscr{J}_M\}$. The universal property of the colimit and (4.38) yield a canonical map

$$\psi_i\colon \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M, N_i) = \operatorname*{colim}_{M_j \in \mathscr{I}_M} \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M_j, N_i) \to \operatorname{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(M), \mathsf{F}_\kappa(N_i)).$$

The maps ψ_i are compatible with the transition maps of the inverse system $\{\operatorname{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(M), \mathsf{F}_\kappa(N_i)) \mid N_i \in \mathscr{I}_N\}$. Hence the universal property of the limit yields a canonical map

$$\operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M, N) = \lim_{N_i \in \mathscr{I}_N} \operatorname{Hom}_{\widehat{\mathbf{U}}_\kappa}(M, N_i) \to \operatorname{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(M), \mathsf{F}_\kappa(N)). \tag{4.41}$$

Therefore (4.40) is in fact a functor.

We now prove right exactness. Suppose that we have a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{4.42}$$

in $\widehat{\mathbf{U}}_\kappa$-fpmod. By Proposition 4.5.5, these modules are pro-smooth, and there exists a short exact sequence of inverse systems of smooth quotients

$$\{0 \to A_i \to B_i \to C_i \to 0 \mid i \in \mathbb{Z}_{\geq 0}\}$$

whose limit is (4.42). Since we are dealing with inverse systems of smooth quotients, the structure maps are all epimorphisms. Next, note that the functor F_κ is right exact on smooth modules since T_κ is exact and taking coinvariants is right exact. Hence, after applying F_κ, we get a short exact sequence of inverse systems of $\mathtt{H}_t$-modules

$$\{\mathsf{F}_\kappa(A_i) \to \mathsf{F}_\kappa(B_i) \to \mathsf{F}_\kappa(C_i) \to 0 \mid i \in \mathbb{Z}_{\geq 0}\},$$

where the structure maps are still epimorphisms. By [133, Lemma 10.86.1], after taking the inverse limit, we get the sequence

$$\mathsf{F}_\kappa(A) \to \mathsf{F}_\kappa(B) \to \mathsf{F}_\kappa(C) \to 0,$$

proving right-exactness. □

Corollary 4.5.8. *The space $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ is a $(\mathtt{H}_t, \widehat{\mathbf{U}}_\kappa)$-bimodule. There exists a natural isomorphism of functors*

$$\mathsf{F}_\kappa(-) \cong \mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa) \otimes_{\widehat{\mathbf{U}}_\kappa} - \colon \widehat{\mathbf{U}}_\kappa\text{-}\mathsf{fpmod} \to \mathtt{H}_t\text{-}\mathsf{mod}. \tag{4.43}$$

Proof. If we take $M = N = \widehat{\mathbf{U}}_\kappa$ then (4.41) is an algebra homomorphism

$$\widehat{\mathbf{U}}_\kappa^{op} \to \mathrm{End}_{\mathtt{H}_t}(\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa), \mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa))$$

giving the right $\widehat{\mathbf{U}}_\kappa$-module structure.

The second statement is proven in the same way as the Eilenberg-Watts theorem (see, e.g., [112, Theorem 5.45]). Let us briefly summarize the argument. One first uses the fact that F_κ preserves finite direct sums to show that the isomorphism (4.43) holds for the category of finitely generated free $\widehat{\mathbf{U}}_\kappa$-modules. One then concludes that (4.43) holds for arbitrary finitely presented modules by using the right exactness of F_κ together with the five lemma. □

We now introduce the final and most general definition of the Suzuki functor.

Definition 4.5.9. The functor (4.40), in the realization (4.43), extends to the functor

$$\mathsf{F}_\kappa(-) := \mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa) \otimes_{\widehat{\mathbf{U}}_\kappa} - \;:\; \widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod} \to \mathtt{H}_t\text{-}\mathsf{mod}. \tag{4.44}$$

From now on we will refer to (4.44) as *the Suzuki functor.*

Remark 4.5.10. In Corollary 4.5.8 we had to restrict ourselves to the category $\widehat{\mathbf{U}}_\kappa$-fpmod because inverse limits do not commute with infinite coproducts. However, the functor (4.44) preserves all colimits because it is left adjoint to the functor $N \mapsto \mathrm{Hom}_{\mathtt{H}_t}(\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa), N)$.

4.5.3. A generic functor. Considering t as an indeterminate, one obtains flat $\mathbb{C}[t]$-algebras $\widehat{\mathbf{U}}_{\mathbb{C}[t]}$ and $\mathtt{H}_{\mathbb{C}[t]}$ such that

$$\widehat{\mathbf{U}}_{\mathbb{C}[t]}/(t-\xi)\widehat{\mathbf{U}}_{\mathbb{C}[t]} \cong \widehat{\mathbf{U}}_{\xi-n}, \quad \mathtt{H}_{\mathbb{C}[t]}/(t-\xi)\mathtt{H}_{\mathbb{C}[t]} \cong \mathtt{H}_\xi$$

for all $\xi \in \mathbb{C}$. More details on the algebra $\mathtt{H}_{\mathbb{C}[t]}$, often called the *generic* rational Cherednik algebra, can be found in [19, §3]. We have specialization functors

$$\mathsf{spec}_{t=\xi}\colon \ \widehat{\mathbf{U}}_{\mathbb{C}[t]}\text{-}\mathsf{mod} \to \widehat{\mathbf{U}}_{\xi-n}\text{-}\mathsf{mod}, \quad M \mapsto M/(t-\xi)\cdot M,$$
$$\mathsf{spec}_{t=\xi}\colon \ \mathtt{H}_{\mathbb{C}[t]}\text{-}\mathsf{mod} \to \mathtt{H}_{\xi}\text{-}\mathsf{mod}, \quad M \mapsto M/(t-\xi)\cdot M.$$

One can easily verify that our construction of the functor F_κ still makes sense if we treat t as a variable throughout. Therefore, we obtain the *generic Suzuki functor*

$$\mathsf{F}_{\mathbb{C}[t]}\colon \widehat{\mathbf{U}}_{\mathbb{C}[t]}\text{-}\mathsf{mod} \to \mathtt{H}_{\mathbb{C}[t]}\text{-}\mathsf{mod},$$

which commutes with the specialization functors, i.e., $\mathsf{spec}_{t=\xi} \circ \mathsf{F}_{\mathbb{C}[t]} = \mathsf{F}_{\xi-n} \circ \mathsf{spec}_{t=\xi}$.

4.6 Computation of the Suzuki functor

In this section we compute the Suzuki functor on certain induced $\mathbf{U}_\kappa(\hat{\mathfrak{g}})$-modules, showing that the regular module $\mathtt{H}_t$ as well as certain generalized Verma modules (Definition 4.6.6) are in the image of F_κ. Let $\kappa \in \mathbb{C}$ and $t = \kappa + n$ throughout.

4.6.1. Induced modules. We start by recalling the definition of Verma modules for the affine Lie algebra.

Definition 4.6.1. Let $\lambda \in \mathfrak{t}^*$ and let $\mathbb{C}_{\lambda,1}$ be the one-dimensional $\mathfrak{t} \oplus \mathbb{C}\mathbf{1}$-module of weight $(\lambda, 1)$. The corresponding *Verma module* is

$$\mathbb{M}_\kappa(\lambda) := \mathrm{Ind}_{\hat{\mathfrak{b}}_+}^{\hat{\mathfrak{g}}_\kappa} \circ \mathrm{Inf}_{\mathfrak{t}\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{b}}_+} \mathbb{C}_{\lambda,1}.$$

We next define certain induced modules which generalize the Weyl modules from [84, §2.4] (see also [54, §9.6]). Given $l \geq 1$ and $\mu \in \mathcal{C}_l(n)$, define

$$\hat{\mathfrak{l}}_\mu^+ := \mathfrak{l}_\mu \oplus \hat{\mathfrak{g}}_{\geq 1} \oplus \mathbb{C}\mathbf{1} \subseteq \hat{\mathfrak{g}}_+, \quad \bar{\mathfrak{l}}_\mu := \hat{\mathfrak{l}}_\mu^+/\mathfrak{j}_\mu, \tag{4.45}$$

where $\mathfrak{j}_\mu := \mathfrak{n}_-[1] \oplus \mathfrak{n}_+[1] \oplus (\mathfrak{t}[1] \cap [\mathfrak{l}_\mu, \mathfrak{l}_\mu][1]) \oplus \hat{\mathfrak{g}}_{\geq 2}$.

Lemma 4.6.2. *The subspace $\mathfrak{j}_\mu$ is an ideal in the Lie algebra $\hat{\mathfrak{l}}_\mu^+$. Moreover, there is a Lie algebra isomorphism $\bar{\mathfrak{l}}_\mu \cong \mathfrak{l}_\mu \oplus \mathfrak{z}_\mu[1] \oplus \mathbb{C}\mathbf{1}$, where $\mathfrak{z}_\mu$ denotes the centre of $\mathfrak{l}_\mu$.*

Proof. Since $\mathfrak{j}_\mu \subset \hat{\mathfrak{g}}_{\geq 1}$, we have $[\hat{\mathfrak{g}}_{\geq 1}, \mathfrak{j}_\mu] \subseteq \hat{\mathfrak{g}}_{\geq 2} \subset \mathfrak{j}_\mu$. Therefore it suffices to show that $[\mathfrak{l}_\mu, \mathfrak{j}_\mu] \subseteq \mathfrak{j}_\mu$. This follows from the fact that $\mathfrak{j}_\mu = [\mathfrak{l}_\mu, \mathfrak{l}_\mu][1] \oplus \mathfrak{r}[1] \oplus \hat{\mathfrak{g}}_{\geq 2}$, where $\mathfrak{r}$ is the direct sum of the nilradical of the standard parabolic containing $\mathfrak{l}_\mu$ and the nilradical of the opposite parabolic, together with the following three inclusions. Firstly, we have $[\mathfrak{l}_\mu, \hat{\mathfrak{g}}_{\geq 2}] \subseteq \hat{\mathfrak{g}}_{\geq 2} \subset \mathfrak{j}_\mu$. Secondly, $[\mathfrak{l}_\mu, \mathfrak{l}_\mu[1]] \subseteq [\mathfrak{l}_\mu, \mathfrak{l}_\mu][1] \subset \mathfrak{j}_\mu$. Thirdly, $[\mathfrak{l}_\mu, \mathfrak{r}[1]] \subseteq \mathfrak{r}[1] \subset \mathfrak{j}_\mu$. The second statement of the lemma follows immediately. ◻

Let $\mathbf{U}_1(\bar{\mathfrak{l}}_\mu) := \mathbf{U}(\bar{\mathfrak{l}}_\mu)/\langle \mathbf{1} - 1\rangle$. Consider the functor

$$\mathsf{Ind}_{\mu,\kappa} = \mathrm{Ind}_{\hat{\mathfrak{l}}_\mu^+}^{\hat{\mathfrak{g}}_\kappa} \circ \mathrm{Inf}_{\bar{\mathfrak{l}}_\mu}^{\hat{\mathfrak{l}}_\mu^+} \colon \mathbf{U}_1(\bar{\mathfrak{l}}_\mu)\text{-}\mathsf{mod} \to \mathscr{C}_\kappa. \tag{4.46}$$

In the case $\mu = (1^n)$ we abbreviate $\mathsf{Ind}_\kappa := \mathsf{Ind}_{(1^n),\kappa}$. Note that $\hat{\mathfrak{l}}_{(1^n)}^+ = \hat{\mathfrak{t}}_+$. Set $\mathfrak{i} := \mathfrak{j}_{(1^n)} = \mathfrak{n}_-[1] \oplus \mathfrak{n}_+[1] \oplus \hat{\mathfrak{g}}_{\geq 2}$ and $\bar{\mathfrak{t}} := \hat{\mathfrak{t}}_+/\mathfrak{i}$. By Lemma 4.6.2, we have $\bar{\mathfrak{t}} \cong \mathfrak{t} \oplus \mathfrak{t}[1]$.

Definition 4.6.3. Let $\mu \in \mathcal{C}_l(n)$, $\lambda \in \Pi^+_\mu$ and $a \in (\mathfrak{t}[1])^*$ with $S_n(a) = S_\mu$ (with respect to the usual Weyl group action). Extend $L(\lambda)$ to an $\mathbf{U}_1(\bar{\mathfrak{l}}_\mu)$-module $L(a,\lambda)$ by letting $\mathfrak{z}_\mu[1]$ act via the weight a. We define the *Weyl module* of type (a,λ,κ) to be

$$\mathbb{W}_\kappa(a,\lambda) := \mathsf{Ind}_{\mu,\kappa}(L(a,\lambda)).$$

Remark 4.6.4. As a special case, when $a = 0$, we obtain modules $\mathbb{W}_\kappa(\lambda) := \mathbb{W}_\kappa(0,\lambda)$ which coincide with the Weyl modules from [84, §2.4].

Definition 4.6.5. Assume that $n = m$. Let $\mathfrak{I}_\kappa$ be the left ideal in $\mathbf{U}(\hat{\mathfrak{g}}_\kappa)$ generated by $e_{ii} - 1_{\hat{\mathfrak{g}}_\kappa}$ $(1 \le i \le n)$, $\mathbf{1} - 1_{\hat{\mathfrak{g}}_\kappa}$ and $\mathfrak{i} := \mathfrak{n}_-[1] \oplus \mathfrak{n}_+[1] \oplus \hat{\mathfrak{g}}_{\ge 2}$. Define

$$\mathbb{H}_\kappa := \mathbf{U}(\hat{\mathfrak{g}}_\kappa)/\mathfrak{I}_\kappa = \mathsf{Ind}_\kappa(\mathcal{I}),$$

where $\mathcal{I} := \mathrm{Ind}^{\bar{\mathfrak{t}}}_{\mathfrak{t}\oplus\mathbb{C}\mathbf{1}} \mathbb{C}_{(1^n,1)} \cong S(\mathfrak{t}[1])$.

The module $\mathbb{H}_\kappa$ is cyclic, generated by the image $1_{\mathbb{H}}$ of $1_{\hat{\mathfrak{g}}_\kappa} \in \mathbf{U}(\hat{\mathfrak{g}}_\kappa)$. From now on, whenever $n = m$, let us identify

$$\mathcal{I} \cong S(\mathfrak{t}[1]) \to \mathbb{C}[\mathfrak{h}^*], \quad e_{ii}[1] \mapsto -y_i. \tag{4.47}$$

4.6.2. Generalized Verma modules for rational Cherednik algebras. We will now recall the definition of generalized Verma modules for the rational Cherednik algebra $\mathtt{H}_t$ from [7]. Let $l \ge 1$, $\nu \in \mathcal{C}_l(m)$, $\lambda \in \mathcal{P}_m(\nu)$ and $a \in \mathfrak{h}^*$ with $S_m(a) = S_\nu$. Extend the $\mathbb{C}S_\nu$-module $\mathsf{Sp}(\lambda)$ to a $\mathbb{C}S_\nu \ltimes \mathbb{C}[\mathfrak{h}^*]$-module $\mathsf{Sp}(a,\lambda)$ by letting each y_i act on $\mathsf{Sp}(\lambda)$ by the scalar $a_i := a(y_i)$.

Definition 4.6.6 ([7, §1.3])**.** The *generalized Verma module* of type (a,λ) is

$$\Delta_t(a,\lambda) := \mathtt{H}_t \otimes_{\mathbb{C}S_\nu \ltimes \mathbb{C}[\mathfrak{h}^*]} \mathsf{Sp}(a,\lambda).$$

We abbreviate $\Delta_t(\lambda) := \Delta_t(0,\lambda)$.

Remark 4.6.7. When $t \neq 0$, the modules $\Delta_t(\lambda)$ play the role of standard modules in the category $\mathcal{O}(\mathtt{H}_t)$ defined in [61]. Using the results of [12], Bonnafé and Rouquier [19] also defined a highest weight category for $\mathtt{H}_0$ with graded shifts of $\Delta_0(\lambda)$ as the standard modules.

Theorem 4.6.8 ([7, Theorem 2])**.** *The following hold.*

a) *The canonical map* $\mathtt{Z} \to \mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$ *is surjective.*
b) *The ring* $\mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$ *is isomorphic to a polynomial ring in* m *variables.*
c) *The* $\mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$*-module* $\mathbf{e}\Delta(a,\lambda)$ *is free of rank one.*

Theorem 4.6.8 allows us to construct simple $\mathtt{H}_0$-modules as quotients of generalized Verma modules.

Lemma 4.6.9. *Let L be a simple $\mathtt{H}_0$-module. Then there exist $l \ge 1$, $\nu \in \mathcal{C}_l(m)$, $\lambda \in \mathcal{P}_m(\nu)$ and $a \in \mathfrak{h}^*$ with $S_m(a) = S_\nu$ such that $L \cong \Delta_0(a,\lambda)/I \cdot \Delta_0(a,\lambda)$ for some maximal ideal $I \lhd \mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$.*

Proof. The commuting operators $y_1, \ldots, y_m$ have a simultaneous eigenvector $v \in L$. Let $a \in \mathfrak{h}^*$ be the corresponding eigenvalue. Without loss of generality, we may assume that $S_m(a) = S_\nu$ for some $\nu \in \mathcal{C}_l(m)$. The subspace $S_\nu \cdot v \subset L$ is $\mathbb{C}[\mathfrak{h}^*]$-stable and decomposes as a sum of simple S_ν-modules. Suppose that this sum contains a simple module isomorphic to $\mathsf{Sp}(\lambda)$. Then there is a surjective homomorphism $\Delta_0(a,\lambda) \twoheadrightarrow L$. Let K denote its kernel.

We abbreviate $E(a,\lambda) := \mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$. Since, by part a) of Theorem 4.6.8, $\mathtt{Z}$ surjects onto $E(a,\lambda)$, all endomorphisms in $E(a,\lambda)$ preserve $\mathbf{e}K$. Hence $\mathbf{e}K$ is an $E(a,\lambda)$-submodule of $\mathbf{e}\Delta_0(a,\lambda)$.

But, by part c) of Theorem 4.6.8, $\mathbf{e}\Delta_0(a,\lambda)$ is a free $E(a,\lambda)$-module of rank one. Hence $\mathbf{e}K = I \cdot \mathbf{e}\Delta_0(a,\lambda) = \mathbf{e}I \cdot \Delta_0(a,\lambda)$ for some ideal $I \lhd E(a,\lambda)$.

By the definition of K and part d) of Theorem 2.1.4, there is a short exact sequence

$$0 \to \mathbf{e}I \cdot \Delta_0(a,\lambda) \to \mathbf{e}\Delta_0(a,\lambda) \to \mathbf{e}L \to 0.$$

Since, by part e) of Theorem 2.1.4, $\mathbf{e}L \cong \mathbb{C}$, it follows that I is a maximal ideal. The fact that (2.3) is an equivalence implies that the sequence

$$0 \to I \cdot \Delta_0(a,\lambda) \to \Delta_0(a,\lambda) \to L \to 0$$

is exact as well. Hence $K = I \cdot \Delta_0(a,\lambda)$. ☐

By [8, §1.1], the support of the module $\Delta_0(a,\lambda)$ only depends on $\mathbf{a} := \varpi(a)$, where $\varpi\colon \mathfrak{h}^* \to \mathfrak{h}^*/S_m$ is the canonical map. Therefore we can define

$$\Omega_{\mathbf{a},\lambda} := \mathrm{supp}_{\mathtt{Z}}(\Delta_0(a,\lambda)).$$

Let

$$\pi\colon \operatorname{Spec}\mathtt{Z} \to \mathfrak{h}^*/S_m \tag{4.48}$$

be the morphism of affine varieties induced by the inclusion $\mathbb{C}[\mathfrak{h}^*]^{S_m} \hookrightarrow \mathtt{Z}$.

Proposition 4.6.10. *We have*

$$\pi^{-1}(\mathbf{a})_{\mathsf{red}} = \bigsqcup_{\lambda \in \mathcal{P}(\nu)} \Omega_{\mathbf{a},\lambda}$$

with $\Omega_{\mathbf{a},\lambda} \cong \operatorname{Spec}\operatorname{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda)) \cong \mathbb{A}^m$.

Proof. The first statement follows from [8, Proposition 4.9] and the second statement from Theorem 4.6.8.b). ☐

4.6.3. Statement of the results. We state the three main results of this section. The first one implies that the regular module appears in the image of the Suzuki functor.

Theorem 4.6.11. *Let $n = m$. The map*

$$\Upsilon\colon \mathtt{H}_t \xrightarrow{\sim} \mathsf{F}_\kappa(\mathbb{H}_\kappa), \tag{4.49}$$

$$f(x_1,\dots,x_n)wg(y_1,\dots,y_n) \mapsto [f(x_1,\dots,x_n)\otimes e_w^* \otimes g(-e_{11}[1],\dots,-e_{nn}[1])1_{\mathbb{H}}] \tag{4.50}$$

is an isomorphism of $\mathtt{H}_t$-modules.

The next theorem states that the Suzuki functor sends generalized Weyl modules to generalized Verma modules.

Theorem 4.6.12. *Let $n = m$. Take $l \geq 1$, $\mu \in \mathcal{C}_l(n)$, $\lambda \in \mathcal{P}_\mu(\mu)$ and $a \in \mathfrak{h}^* \cong (\mathfrak{t}[1])^*$ with $S_n(a) = S_\mu$. There is an $\mathtt{H}_t$-module isomorphism*

$$\mathsf{F}_\kappa(\mathbb{W}_\kappa(a,\lambda)) \cong \Delta_t(a,\lambda).$$

We remark that the $a = 0$ case of the preceding theorem also follows from [138, Proposition 6.3]. Our third theorem shows that the Suzuki functor sends Verma modules to Verma modules.

Theorem 4.6.13. *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathfrak{t}^*$. Then $\mathsf{F}_\kappa(\mathbb{M}_\kappa(\lambda)) \neq 0$ if and only if $\lambda \in \mathcal{P}_n(m)$. If $\lambda \in \mathcal{P}_n(m)$ then there is an $\mathtt{H}_t$-module isomorphism*

$$\mathsf{F}_\kappa(\mathbb{M}_\kappa(\lambda)) \cong \Delta_t(\lambda).$$

4.6.4. Partial Suzuki functors.

The proof of Theorems 4.6.11-4.6.13 requires some preparation. We start by recalling a few facts about induction.

Lemma 4.6.14. *Let $\mathfrak{d} \subset \mathfrak{a}$ be Lie algebras, M a $\mathfrak{d}$-module and N an $\mathfrak{a}$-module.*

a) *There exists a linear isomorphism $H_0(\mathfrak{a}, \mathrm{Ind}_\mathfrak{d}^\mathfrak{a} M) \cong H_0(\mathfrak{d}, M)$.*
b) *There is an $\mathfrak{a}$-module isomorphism*

$$\mathrm{Ind}_\mathfrak{d}^\mathfrak{a}(N \otimes M) \xrightarrow{\sim} N \otimes \mathrm{Ind}_\mathfrak{d}^\mathfrak{a} M, \quad a \otimes n \otimes m \mapsto \sum a_1 n \otimes a_2 \otimes m,$$

called the tensor identity, *where $\sum a_1 \otimes a_2$ is the coproduct of $a \in \mathbf{U}(\mathfrak{a})$. It restricts to the linear isomorphism*

$$\mathbb{C}1_\mathfrak{a} \otimes (N \otimes M) \xrightarrow{\sim} N \otimes (\mathbb{C}1_\mathfrak{a} \otimes M), \quad 1_\mathfrak{a} \otimes n \otimes m \mapsto n \otimes 1_\mathfrak{a} \otimes m.$$

Proof. The first part of the lemma follows directly from the definitions. For the proof of the second part see, e.g., [89, Proposition 6.5]. □

We next define "partial Suzuki functors". Let $l \geq 1$ and $\mu \in \mathcal{C}_l(n)$. Suppose that $M \in \mathbf{U}_1(\bar{\mathfrak{l}}_\mu)\text{-}\mathsf{mod}$. The diagonal $\mathfrak{g}$-action on

$$\mathbf{T}(M) := (\mathbf{V}^*)^{\otimes m} \otimes M$$

restricts to an action of the Lie subalgebra $\mathfrak{l}_\mu$. The symmetric group acts on $\mathbf{T}(M)$, as usual, by permuting the factors of the tensor product. We extend this action to an action of $\mathbb{C}[\mathfrak{h}^*]^{\rtimes}$ by letting each y_i act as the operator

$$y_i \mapsto \sum_{1 \leq k \leq n} e_{kk}^{(i)} e_{kk}[1]^{(\infty)}. \tag{4.51}$$

Lemma 4.6.15. *The $\mathfrak{l}_\mu$-action and the $\mathbb{C}[\mathfrak{h}^*]^{\rtimes}$-action on $\mathbf{T}(M)$ commute.*

Proof. The fact that the S_m-action commutes with the $\mathfrak{l}_\mu$-action follows from Schur-Weyl duality. Therefore we only need to show that the operators (4.51) commute with the $\mathfrak{l}_\mu$-action. Let $e_{rs} \in \mathfrak{l}_\mu$. We have an equality of operators on $\mathbf{T}(M)$:

$$y_i \sum_{j=1,\ldots,n,\infty} e_{rs}^{(j)} = \sum_{j \neq i, \infty} \sum_{k=1}^{n} e_{kk}^{(i)} e_{rs}^{(j)} e_{kk}[1]^{(\infty)} \tag{4.52}$$

$$+ \sum_{k=1}^{n} e_{kk}^{(i)} e_{rs}^{(i)} e_{kk}[1]^{(\infty)} + \sum_{k=1}^{n} e_{kk}^{(i)} e_{kk}[1]^{(\infty)} e_{rs}^{(\infty)}. \tag{4.53}$$

Consider the first summand in (4.53):

$$\sum_{k=1}^{n} e_{kk}^{(i)} e_{rs}^{(i)} e_{kk}[1]^{(\infty)} = \sum_{k=1}^{n} e_{rs}^{(i)} e_{kk}^{(i)} e_{kk}[1]^{(\infty)} + e_{rs}^{(i)} (e_{rr}[1]^{(\infty)} - e_{ss}[1]^{(\infty)}). \tag{4.54}$$

Since M is an $\bar{\mathfrak{l}}_\mu$-module, $e_{rr}[1] - e_{ss}[1] = 0$ as operators on M and the second summand on the RHS of (4.54) vanishes. Next consider the second summand in (4.53):

$$\sum_{k=1}^{n} e_{kk}^{(i)} e_{kk}[1]^{(\infty)} e_{rs}^{(\infty)} = \sum_{k=1}^{n} e_{kk}^{(i)} e_{rs}^{(\infty)} e_{kk}[1]^{(\infty)} + (e_{rr}^{(i)} - e_{ss}^{(i)}) e_{rs}[1]^{(\infty)}. \tag{4.55}$$

If $r = s$ then the second summand on the RHS of (4.55) vanishes. If $r \neq s$ it vanishes as well since M is an $\bar{\mathfrak{l}}_\mu$-module and $e_{rs}[1]$ acts trivially on M. □

By Lemma 4.6.15, there is an induced $\mathbb{C}[\mathfrak{h}^*]^{\rtimes}$-representation on $H_0(\mathfrak{l}_\mu, \mathbf{T}(M))$ and, therefore, a functor

$$\overline{\mathsf{F}}^{\mu} \colon \mathbf{U}_1(\bar{\mathfrak{l}}_\mu)\text{-}\mathsf{mod} \to \mathbb{C}[\mathfrak{h}^*]^{\rtimes}\text{-}\mathsf{mod}, \quad M \mapsto H_0(\mathfrak{l}_\mu, \mathbf{T}(M)),$$

which we call a *partial Suzuki functor*. For $\mu = (1^n)$ we also write $\overline{\mathsf{F}} := \overline{\mathsf{F}}^{\mu}$. Set

$$\mathtt{Hind}_t \colon \mathbb{C}[\mathfrak{h}^*]^{\rtimes}\text{-}\mathsf{mod} \to \mathtt{H}_t\text{-}\mathsf{mod}, \quad N \mapsto \mathtt{H}_t \otimes_{\mathbb{C}[\mathfrak{h}^*]^{\rtimes}} N.$$

Proposition 4.6.16. *The diagram*

$$\begin{array}{ccc} \mathscr{C}_\kappa & \xrightarrow{\mathsf{F}_\kappa} & \mathtt{H}_t\text{-}\mathsf{mod} \\ {\scriptstyle \mathsf{Ind}_{\mu,\kappa}}\uparrow & & \uparrow{\scriptstyle \mathtt{Hind}_t} \\ U_1(\bar{\mathfrak{l}}_\mu)\text{-}\mathsf{mod} & \xrightarrow{\overline{\mathsf{F}}^{\mu}} & \mathbb{C}[\mathfrak{h}^*]^{\rtimes}\text{-}\mathsf{mod} \end{array}$$

commutes, i.e., there exists a natural isomorphism of functors $\mathsf{F}_\kappa \circ \mathsf{Ind}_{\mu,\kappa} \cong \mathtt{Hind}_t \circ \overline{\mathsf{F}}^{\mu}$*. Explicitly, for each* $M \in \mathbf{U}_1(\bar{\mathfrak{l}}_\mu)\text{-}\mathsf{mod}$*, this isomorphism is given by*

$$\phi \colon \mathtt{Hind}_t(\overline{\mathsf{F}}^{\mu}(M)) = \mathbb{C}[\mathfrak{h}] \otimes H_0(\mathfrak{l}_\mu, \mathbf{T}(M)) \xrightarrow{\sim} \mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(M)) \tag{4.56}$$

$$f(x_1, \ldots, x_m) \otimes [v \otimes u] \mapsto [f(x_1, \ldots, x_m) \otimes v \otimes i(u)], \tag{4.57}$$

where $v \in (\mathbf{V}^*)^{\otimes m}, u \in M$ *and* $i : M \hookrightarrow \mathrm{Ind}_{\hat{\mathfrak{l}}_\mu^+}^{\hat{\mathfrak{g}}_\kappa} M$ *is the natural inclusion.*

Proof. We first show that (4.56) is an isomorphism of $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-modules. Since the first equality in (4.56) follows directly from the PBW theorem (2.1), we only need to prove the second isomorphism. Consider $\mathrm{Ind}_{\hat{\mathfrak{l}}_\mu^+}^{\hat{\mathfrak{g}}_\kappa} M$ as a $\mathfrak{g}[t]$-module using the Lie algebra homomorphism

$$\mathfrak{g}[t] \xrightarrow{\sim} \mathfrak{g}[t^{-1}] \hookrightarrow \hat{\mathfrak{g}}_\kappa, \;\; X[k] \mapsto X[-k]. \tag{4.58}$$

The map (4.58) induces a $\mathfrak{g}[t]$-module isomorphism $\mathrm{Ind}_{\hat{\mathfrak{l}}_\mu^+}^{\hat{\mathfrak{g}}_\kappa} M \xrightarrow{\sim} \mathrm{Ind}_{\mathfrak{l}_\mu}^{\mathfrak{g}[t]} M$. Hence, by Lemma 4.6.14.b), we have a $\mathfrak{g}[t]$-module isomorphism

$$\mathrm{Ind}_{\mathfrak{l}_\mu}^{\mathfrak{g}[t]}(\mathbb{C}[\mathfrak{h}] \otimes (\mathbf{V}^*)^{\otimes m} \otimes M) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}] \otimes (\mathbf{V}^*)^{\otimes m} \otimes (\mathrm{Ind}_{\mathfrak{l}_\mu}^{\mathfrak{g}[t]} M), \tag{4.59}$$

where $\mathfrak{g}[t]$ acts on the LHS as in (4.30), sending

$$1_{\mathfrak{g}[t]} \otimes f(x_1, \ldots, x_m) \otimes v \otimes u \mapsto \; f(x_1, \ldots, x_m) \otimes v \otimes 1_{\mathfrak{g}[t]} \otimes u. \tag{4.60}$$

Next notice that, by Lemma 4.6.14.a), we have linear isomorphisms

$$H_0(\mathfrak{g}[t], \mathrm{Ind}_{\mathfrak{l}_\mu}^{\mathfrak{g}[t]}(\mathbb{C}[\mathfrak{h}] \otimes \mathbf{T}(M))) \xrightarrow{\sim} H_0(\mathfrak{l}_\mu, \mathbb{C}[\mathfrak{h}] \otimes \mathbf{T}(M)) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}] \otimes H_0(\mathfrak{l}_\mu, \mathbf{T}(M)). \tag{4.61}$$

Applying $H_0(\mathfrak{g}[t], -)$ to the inverse of (4.59) and composing with (4.61), we obtain an isomorphism

$$\mathsf{F}_\kappa(\mathrm{Ind}_{\mathfrak{l}_\mu^+}^{\hat{\mathfrak{g}}_\kappa} M) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}] \otimes H_0(\mathfrak{l}_\mu, \mathbf{T}(M)). \tag{4.62}$$

It is clear from (4.60) that (4.62) sends the equivalence class $[f(x_1, \ldots, x_m) \otimes v \otimes i(u)]$ to $f(x_1, \ldots, x_m) \otimes [v \otimes u]$. This implies, in particular, that (4.62) is $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-equivariant.

We next prove that (4.56) is an isomorphism of $\mathtt{H}_t$-modules. Since $\mathtt{H}_t$ is generated as a $\mathbb{C}$-algebra by $\mathbb{C}[\mathfrak{h}]^{\rtimes}$ and $\mathbb{C}[\mathfrak{h}^*]$, it suffices to show that ϕ intertwines the $\mathbb{C}[\mathfrak{h}^*]$-actions. Moreover, since the subspace $W := 1_{\mathbb{C}[\mathfrak{h}]} \otimes H_0(\mathfrak{l}_\mu, \mathbf{T}(M))$ generates $\mathtt{Hind}_t(\overline{\mathsf{F}}^\mu(M))$ as a $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-module, it is enough to check that $\phi|_W$ intertwines the $\mathbb{C}[\mathfrak{h}^*]$-actions.

Consider the subspace $U := 1_{\mathbb{C}[\mathfrak{h}]} \otimes (\mathbf{V}^*)^{\otimes m} \otimes M \subset \mathsf{T}_\kappa(\mathsf{Ind}_{\mu,\kappa}(M))$ and its image $\overline{U}$ in $\mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(M))$. By (4.57), ϕ restricts to a linear isomorphism $\phi|_W : W \xrightarrow{\sim} \overline{U}$. The element $y_i \in \mathtt{H}_t$ acts on $\mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(M))$ as the operator ${}^\kappa\overline{y}_i$ (see (4.31)). The operators ∂_{x_i} and $(1 - \underline{s_{i,j}})$ vanish on the subspace $\overline{U}$. Moreover, $\Omega_{[p+1]}^{(i,\infty)}$ $(p \geq 1)$ and $e_{kl}[1]^{(\infty)}$ $(k \neq l)$ act trivially on all of $\mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(M))$. Therefore

$${}^\kappa\overline{y}_i = \Omega_{[1]}^{(i,\infty)} = \sum_{1 \leq k \leq m} e_{kk}^{(i)} e_{kk}[1]^{(\infty)} \tag{4.63}$$

as operators on $\overline{U}$. On the other hand, the action of y_i on W is given by (4.51). It now follows directly from (4.57) that ϕ is $\mathbb{C}[\mathfrak{h}^*]$-equivariant. □

4.6.5. Proofs of Theorems 4.6.11-4.6.13

We now prove the theorems from §4.6.3.

Proof of Theorem 4.6.11. Combining the left $\mathbb{C}S_n$-module isomorphism

$$\mathbb{C}S_n \xrightarrow{\sim} ((\mathbf{V}^*)^{\otimes n})_{(-1,\ldots,-1)}, \quad w \mapsto e_w^* \tag{4.64}$$

with (4.47) allows us to identify

$$\overline{\Upsilon} \colon \overline{\mathsf{F}}(\mathcal{I}) \cong ((\mathbf{V}^*)^{\otimes n})_{(-1,\ldots,-1)} \otimes \mathcal{I} \cong \mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h}^*] \tag{4.65}$$

as $\mathbb{C}S_n$-modules. We claim that (4.65) also intertwines the $\mathbb{C}[\mathfrak{h}^*]$-actions.

Let us prove the claim. Consider the subspace $U := e_{\mathsf{id}}^* \otimes \mathcal{I} \subset (\mathbf{V}^*)^{\otimes n} \otimes \mathcal{I}$ and its image $\overline{U}$ in $\overline{\mathsf{F}}(\mathcal{I})$. The map $\overline{\Upsilon}$ restricts to a linear isomorphism $\overline{\Upsilon}' \colon \overline{U} \cong \mathbb{C}[\mathfrak{h}^*]$. Since $\mathbb{C}[\mathfrak{h}^*]$ generates $\mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h}^*]$ as an S_n-module, it suffices to show that $\overline{\Upsilon}'$ is $\mathbb{C}[\mathfrak{h}^*]$-equivariant. The action of y_i on $\overline{\mathsf{F}}(\mathcal{I})$ is given by formula (4.51). Observe that $e_{kk}^{(i)}.e_{\mathsf{id}}^* = -\delta_{k,i} e_{\mathsf{id}}^*$ and $e_{kk}[1]$ acts as multiplication by $e_{kk}[1]$ on $\mathcal{I} \cong \mathrm{Sym}(\mathfrak{t}[1])$. Hence y_i acts on $\overline{U}$ as multiplication by $-e_{ii}[1]$. On the other hand, y_i acts on $\mathbb{C}[\mathfrak{h}^*] \subset \mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h}^*]$ as multiplication by y_i. It is clear from (4.47) that $\overline{\Upsilon}'$ intertwines these two actions, which completes the proof of the claim.

We now prove the theorem. By definition, $\mathsf{F}_\kappa(\mathbb{H}_\kappa) = \mathsf{F}_\kappa(\mathsf{Ind}_\kappa(\mathcal{I}))$ and, by Proposition 4.6.16, $\mathsf{F}_\kappa(\mathsf{Ind}_\kappa(\mathcal{I})) \cong \mathtt{Hind}_t(\overline{\mathsf{F}}(\mathcal{I}))$. The claim above implies that $\mathtt{Hind}_t(\overline{\mathsf{F}}(\mathcal{I})) \cong \mathtt{Hind}_t(\mathbb{C}S_n \ltimes \mathbb{C}[\mathfrak{h}^*]) = \mathtt{H}_t$. Formula (4.50) also follows from Proposition 4.6.16. □

Proof of Theorem 4.6.12. Set $P_j(\mu) = \{\mu_{\leq j-1}, \ldots, \mu_{\leq j}\}$ so that $\{1, \ldots, n\} = \bigsqcup_{j=1}^{l} P_j(\mu)$. Write $r \sim s$ if and only if there exists j such that both $r, s \in P_j(\mu)$. By Proposition 4.2.1, there is a natural $\mathbb{C}S_n$-module isomorphism

$$\overline{\Upsilon}_{\mu,a} \colon \overline{\mathsf{F}}^\mu(L(a,\lambda)) \cong \mathbb{C}[\mathfrak{h}^*]^{\rtimes} \otimes_{\mathbb{C}S_\mu \ltimes \mathbb{C}[\mathfrak{h}^*]} \mathsf{Sp}(a,\lambda) =: \mathsf{Sp}_\mu(a,\lambda). \tag{4.66}$$

We claim that (4.66) is an isomorphism of $\mathbb{C}[\mathfrak{h}^*]^{\rtimes}$-modules.

It suffices to show that $\overline{\Upsilon}_{\mu,a}$ is an isomorphism of $\mathbb{C}[\mathfrak{h}^*]$-modules. Consider the subspace $U := (\mathbf{V}^*)^{\otimes n}_{(\mu,\mu)} \otimes L(a,\lambda) \subset (\mathbf{V}^*)^{\otimes n} \otimes L(a,\lambda)$ and its image $\overline{U}$ in $\overline{\mathsf{F}}^{\mu}(L(a,\lambda))$. The map $\overline{\Upsilon}_{\mu,a}$ restricts to a $\mathbb{C}S_\mu$-module isomorphism $\overline{\Upsilon}'_{\mu,a} \colon \overline{U} \cong \mathsf{Sp}(a,\lambda)$. Since $\mathsf{Sp}(a,\lambda)$ generates $\mathsf{Sp}_\mu(a,\lambda)$ as an S_n-module, it suffices to show that $\overline{\Upsilon}'_{\mu,a}$ is $\mathbb{C}[\mathfrak{h}^*]$-equivariant. The action of y_i on $\overline{\mathsf{F}}^{\mu}(L(a,\lambda))$ is given by formula (4.51). Let $v = v_1 \otimes \ldots \otimes v_n \in (\mathbf{V}^*)^{\otimes n}_{(\mu,\mu)}$. Suppose that $i \in P_j(\mu)$. Observe that $e^{(i)}_{kk}.v = 0$ unless $k \sim i$ and $\sum_{k \in P_j(\mu)} e^{(i)}_{kk}.v = -v$. Moreover, the elements $e_{kk}[1]$ ($k \in P_j(\mu)$) act on $L(a,\lambda)$ by the same scalar $-a_i := -a(y_i)$. Hence y_i acts on $\overline{U}$ as multiplication by a_i. This agrees with the definition of the y_i-action on $\mathsf{Sp}(a,\lambda)$, completing the proof of the claim.

We now prove the theorem. By definition, $\mathsf{F}_\kappa(\mathbb{W}_\kappa(a,\lambda)) = \mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(L(a,\lambda)))$ and, by Proposition 4.6.16, $\mathsf{F}_\kappa(\mathsf{Ind}_{\mu,\kappa}(L(a,\lambda))) \cong \mathtt{Hind}_t(\overline{\mathsf{F}}^{\mu}(L(a,\lambda)))$. The claim above implies that $\mathtt{Hind}_t(\overline{\mathsf{F}}^{\mu}(L(a,\lambda))) \cong \mathtt{Hind}_t(\mathsf{Sp}_\mu(a,\lambda)) = \Delta_t(a,\lambda)$. □

Proof of Theorem 4.6.13. In analogy to Proposition 4.6.16, one can show that, for each $\lambda \in \mathfrak{t}^*$, there is a $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-module isomorphism

$$\mathbb{C}[\mathfrak{h}] \otimes H_0(\mathfrak{b}_+, \mathbf{T}(\mathbb{C}_\lambda)) \xrightarrow{\sim} \mathsf{F}_\kappa(\mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\hat{\mathfrak{b}}_+} \mathbb{C}_{\lambda,1}) = \mathsf{F}_\kappa(\mathbb{M}_\kappa(\lambda)) \tag{4.67}$$

$$f(x_1,\ldots,x_m) \otimes [v \otimes u] \mapsto [f(x_1,\ldots,x_m) \otimes v \otimes i(u)], \tag{4.68}$$

where $v \in (\mathbf{V}^*)^{\otimes m}$, $u \in \mathbb{C}_\lambda$ and $i : \mathbb{C}_\lambda \hookrightarrow \mathrm{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\hat{\mathfrak{b}}_+} \mathbb{C}_{\lambda,1}$ is the natural inclusion.

The first statement of the theorem now follows directly from (4.67) and Corollary 4.2.2. So consider the second statement. Let $\lambda \in \mathcal{P}_n(m)$. By Corollary 4.2.2 and (2.1), we can identify $\Delta_t(\lambda) \cong \mathbb{C}[\mathfrak{h}] \otimes H_0(\mathfrak{b}_+, \mathbf{T}(\mathbb{C}_\lambda))$ as $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-modules. Let Υ_λ be the composition of this isomorphism with (4.67). We need to check that Υ_λ intertwines the $\mathbb{C}[\mathfrak{h}^*]$-actions. Observe that, by (4.68), Υ_λ restricts to a linear isomorphism $\mathsf{Sp}(\lambda) \to \overline{U}$, where $\overline{U}$ is the image of $U := 1_{\mathbb{C}[\mathfrak{h}]} \otimes (\mathbf{V}^*)^{\otimes m} \otimes \mathbb{C}_{\lambda,1}$ in $\mathsf{F}_\kappa(\mathbb{M}_\kappa(\lambda))$. Since $\mathsf{Sp}(\lambda)$ generates $\Delta_t(\lambda)$ as a $\mathbb{C}[\mathfrak{h}]^{\rtimes}$-module, it suffices to show that $\Upsilon_\lambda|_{\mathsf{Sp}(\lambda)}$ intertwines the $\mathbb{C}[\mathfrak{h}^*]$-actions. By definition, each y_i acts trivially on $\mathsf{Sp}(\lambda)$. On the other hand, since each $e_{kk}[1]$ acts trivially on $\mathbb{C}_{\lambda,1}$, the operator ${}^{\kappa}\overline{y}_i$ also vanishes on $\overline{U}$. □

4.7 Relationship between the centres

Assume that $n = m$ throughout this section. The fact that the algebras $\widehat{\mathbf{U}}_c$ and $\mathtt{H}_0$ have large centres has many implications for their representation theory. For example, Verma-type modules have large endomorphism and extension rings (see §4.10 for a more detailed discussion). To understand how these behave under the Suzuki functor, we must, therefore, understand the relationship between the centres of the categories $\widehat{\mathbf{U}}_c$-mod and $\mathtt{H}_0$-mod. In general, a functor of additive categories does not induce a homomorphism between their centres. In §4.7.1 below we propose two ways to get around this problem. In §4.7.2 and §4.7.3, we apply them to the Suzuki functor, and construct a map $\mathfrak{Z} \to \mathtt{Z}$ between the two centres.

4.7.1. Centres of categories. Suppose $F \colon \mathcal{A} \to \mathcal{B}$ is an additive functor between additive categories. Recall that the centre $Z(\mathcal{A})$ of $\mathtt{Z}$ is the ring of endomorphisms of the identity functor $\mathrm{id}_{\mathcal{A}}$. An element of $z \in Z(\mathcal{A})$ is thus a collection of endomorphisms $\{z_M \in \mathrm{End}_{\mathcal{A}}(M) \mid M \in \mathcal{A}\}$ such that $f \circ z_M = z_N \circ f$ for all $f \in \mathrm{Hom}_{\mathcal{A}}(M,N)$.

The functor F does not necessarily induce a ring homomorphism $Z(\mathcal{A}) \to Z(\mathcal{B})$. For example, if F is not essentially surjective, then the collection $\{F(z_M) \in \mathrm{End}_{\mathcal{B}}(F(M)) \mid M \in \mathcal{A}\}$ does not contain an endomorphism for every object of $\mathcal{B}$. If F is not full, then the endomorphisms $F(z_M)$ may fail to commute with some of the morphisms in $\mathcal{B}$. Hence $\{F(z_M) \in \mathrm{End}_{\mathcal{B}}(F(M)) \mid M \in \mathcal{A}\}$ is not necessarily an endomorphism of the identity functor $\mathrm{id}_{\mathcal{B}}$. We remark that some sufficient conditions for the existence of a canonical homomorphism $Z(\mathcal{A}) \to Z(\mathcal{B})$ are known - for instance F being a Serre quotient functor (see [111, Lemma 4.3]).

We therefore pursue a different approach to construct a sensible ring homomorphism $Z(\mathcal{A}) \to Z(\mathcal{B})$ encoding information about the functor F. There are canonical ring homomorphisms

$$Z(\mathcal{A}) \xrightarrow{\alpha} \mathrm{End}(F) \xhookleftarrow{\beta} Z(\mathcal{B})$$

with α taking $\{z_M \mid M \in \mathcal{A}\}$ to $\{F(z_M) \mid M \in \mathcal{A}\}$ and β taking $\{z_K \mid K \in \mathcal{B}\}$ to $\{z_{F(M)} \mid M \in \mathcal{A}\}$. We assume that β is injective, and identify $Z(\mathcal{B})$ with a subring of $\mathrm{End}(F)$.

Definition 4.7.1. We call $Z_F(\mathcal{A}) := \alpha^{-1}(Z(\mathcal{B})) \subset Z(\mathcal{A})$ the *F-centre* of $\mathcal{A}$. If $\mathcal{A} = A\text{-}\mathsf{mod}$ is the category of modules over some algebra A, we will also write $Z_F(A) := Z_F(\mathcal{A})$.

Restricting α to $Z_F(\mathcal{A})$ gives a natural algebra homomorphism from the F-centre of $\mathcal{A}$ to the centre of $\mathcal{B}$:

$$Z(F) := \alpha|_{Z_F(\mathcal{A})} \colon \; Z_F(\mathcal{A}) \longrightarrow Z(\mathcal{B}). \tag{4.69}$$

For any object $M \in \mathcal{A}$, the homomorphism $Z(F)$ fits into the following commutative diagram

$$\begin{array}{ccc} Z_F(\mathcal{A}) & \xrightarrow{Z(F)} & Z(\mathcal{B}) \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \mathrm{End}_{\mathcal{A}}(M) & \xrightarrow{F} & \mathrm{End}_{\mathcal{B}}(F(M)) \end{array} \tag{4.70}$$

Therefore, $Z(F)$ contains partial information about all the maps between endomorphism rings induced by the functor F.

In general, $Z_F(\mathcal{A}) \neq Z(\mathcal{A})$. In that case, we would like to extend $Z(F)$ to a homomorphism $Z(\mathcal{A}) \to Z(\mathcal{B})$. Of course, there is a price to pay - such a homomorphism cannot make the diagram (4.70) commute for all objects $M \in \mathcal{A}$. Instead, we impose the condition that the diagram should commute for all M from some subcategory of $\mathcal{A}$.

Given a full additive subcategory $\mathcal{A}'$, let $F' \colon \mathcal{A}' \to \mathcal{B}$ be the restricted functor. Restriction to objects in $\mathcal{A}'$ yields canonical homomorphisms $q \colon Z(\mathcal{A}) \to Z(\mathcal{A}')$ and $\mathrm{End}(F) \to \mathrm{End}(F')$. We assume that the canonical map $\beta' \colon Z(\mathcal{B}) \to \mathrm{End}(F')$ is injective, and identify $Z(\mathcal{B})$ with a subring of $\mathrm{End}(F')$. The following commutative diagram illustrates all the maps we have just defined:

$$\begin{array}{ccccc} Z(\mathcal{A}) & \xrightarrow{\alpha} & \mathrm{End}(F) & \xhookleftarrow{\beta} & Z(\mathcal{B}) \\ {\scriptstyle q}\downarrow & & \downarrow & & \| \\ Z(\mathcal{A}') & \xrightarrow{\alpha'} & \mathrm{End}(F') & \xhookleftarrow{\beta'} & Z(\mathcal{B}) \end{array} \tag{4.71}$$

Definition 4.7.2. We say that a full subcategory $\mathcal{A}'$ of $\mathcal{A}$ is *F-central* if $\mathrm{Im}(\alpha' \circ q) \subseteq Z(\mathcal{B})$.

If $\mathcal{A}'$ is F-central, then there is a natural algebra homomorphism

$$Z_{\mathcal{A}'}(F) := \alpha' \circ q \colon \; Z(\mathcal{A}) \longrightarrow Z(\mathcal{B})$$

extending (4.69), and making the diagram

$$\begin{array}{ccc} Z(\mathcal{A}) & \xrightarrow{Z_{\mathcal{A}'}(F)} & Z(\mathcal{B}) \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \operatorname{End}_{\mathcal{A}}(M) & \xrightarrow{F} & \operatorname{End}_{\mathcal{B}}(F(M)) \end{array}$$

commute for all $M \in \mathcal{A}'$. The homomorphism $Z_{\mathcal{A}'}(F)$ contains partial information about all the maps between endomorphism rings induced by the restricted functor F'.

4.7.2. The F_c-centre. For the rest of this section, we will use the canonical identifications

$$\mathfrak{Z} \cong Z(\widehat{\mathbf{U}}_c\text{-mod}), \quad \mathtt{Z} \cong Z(\mathtt{H}_0\text{-mod}), \quad \widehat{\mathbf{U}}_\kappa^{op} \cong \operatorname{End}_{\widehat{\mathbf{U}}_\kappa}(\widehat{\mathbf{U}}_\kappa).$$

Let us apply the framework developed in §4.7.1 to the functor $\mathsf{F}_c\colon \widehat{\mathbf{U}}_c\text{-mod} \to \mathtt{H}_0\text{-mod}$. We have canonical maps

$$\mathfrak{Z} \xrightarrow{\alpha} \operatorname{End}(\mathsf{F}_c) \xleftarrow{\beta} \mathtt{Z}.$$

By Theorem 4.6.11, the regular module $\mathtt{H}_0$ is in the image of F_c. The fact that $\mathtt{Z}$ acts faithfully on $\mathtt{H}_0$ implies that β is injective.

Our first goal is to give a partial description of the F_c-centre of $\widehat{\mathbf{U}}_c$-mod. For any $\kappa \in \mathbb{C}$, define

$$\mathscr{L}_\kappa := \langle {}^\kappa\mathbf{L}_{r+1}, \mathrm{id}[r] \mid r \le 0\rangle \subset \widehat{\mathbf{U}}_\kappa. \tag{4.72}$$

When $\kappa = c$, it follows from Theorem 4.3.3 and §4.3.4 that the generators on the RHS of (4.72) are algebraically independent. Hence

$$\mathscr{L}_c = \mathbb{C}[{}^c\mathbf{L}_{r+1}, \mathrm{id}[r]]_{r\le 0}. \tag{4.73}$$

We will show that $\mathscr{L}_c$ is a subalgebra of the F_c-centre of $\widehat{\mathbf{U}}_c$-mod. The proof requires some preparations.

Let κ be arbitrary and set $t = \kappa + n$. Let $1_{\hat{\mathfrak{g}}}$ denote the unit in $\widehat{\mathbf{U}}_\kappa$. Consider the image $[1\otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}]$ of $1\otimes e^*_{\mathsf{id}}\otimes 1_{\hat{\mathfrak{g}}} \in \mathsf{T}_\kappa(\widehat{\mathbf{U}}_\kappa)$ in $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$. Let K_t be the $\mathtt{H}_t$-submodule of $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ generated by $[1\otimes e^*_{\mathsf{id}}\otimes 1_{\hat{\mathfrak{g}}}]$.

Lemma 4.7.3. *There is an $\mathtt{H}_t$-module isomorphism $K_t \cong \mathtt{H}_t$.*

Proof. Since F_κ is right exact, it induces an epimorphism

$$\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa) \twoheadrightarrow \mathsf{F}_\kappa(\mathbb{H}_\kappa) \cong \mathtt{H}_t, \quad [1\otimes e^*_{\mathsf{id}}\otimes 1_{\hat{\mathfrak{g}}}] \mapsto [1\otimes e^*_{\mathsf{id}}\otimes 1_{\mathbb{H}}] = 1_{\mathtt{H}},$$

which restricts to an isomorphism $K_t \cong \mathtt{H}_t$. □

Let N_t be the subalgebra of $\operatorname{End}_{\mathtt{H}_t}(\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa))$ consisting of endomorphisms which preserve the submodule K_t. Let $\rho_t\colon N_t \to \operatorname{End}_{\mathtt{H}_t}(K_t) \cong \mathtt{H}_t^{op}$ be the map given by restriction of endomorphisms of $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ to those of K_t.

Lemma 4.7.4. *The following hold.*

a) *The image of $\mathscr{L}_\kappa^{op}$ under $\operatorname{End}_{\widehat{\mathbf{U}}_\kappa}(\widehat{\mathbf{U}}_\kappa) \xrightarrow{\mathsf{F}_\kappa} \operatorname{End}_{\mathtt{H}_t}(\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa))$ is contained in N_t.*

b) *The map $\rho_t \circ \mathsf{F}_\kappa|_{\mathscr{L}_\kappa^{op}}$ is given by:*

$$\mathrm{id}[r] \mapsto \sum_{i=1}^{n} x_i^{-r} \quad (r \leq 0), \tag{4.74}$$

$$^{\kappa}\mathbf{L}_r \mapsto -\sum_{i=1}^{n} x_i^{1-r} y_i + \sum_{i<j} c_{-r}(x_i, x_j) s_{i,j} + \frac{n(1-r)}{2} \sum_{i=1}^{n} x_i^{-r} \quad (r \leq 1), \tag{4.75}$$

where $c_{-r}(x_i, x_j)$ is the complete homogeneous symmetric polynomial of degree $-r$ in x_i and x_j, if $r \leq 0$, and $c_{-1}(x_i, x_j) = 0$.

c) *When $\kappa = c$, the image of $\mathsf{F}_c|_{\mathscr{L}_c}$ lies in the image of $\mathtt{Z}$ in $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\widehat{\mathbf{U}}_c))$.*

Proof. A homomorphism from K_t to $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ is determined by where it sends the generator $[1 \otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}]$. Let z be any of our distinguished generators (see (4.72)) of $\mathscr{L}_\kappa$. The corresponding endomorphism of $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ sends $[1 \otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}]$ to $[1 \otimes e^*_{\mathsf{id}} \otimes z \cdot 1_{\hat{\mathfrak{g}}}]$. We are going to use the $\mathfrak{g}[t]$-action (4.30) to show that $1 \otimes e^*_{\mathsf{id}} \otimes z \cdot 1_{\hat{\mathfrak{g}}}$ is in the same equivalence class in $\mathsf{F}_\kappa(\widehat{\mathbf{U}}_\kappa)$ as an element of the form (4.74) or (4.75). First take $z = \mathrm{id}[r]$ with $r \leq 0$. By (4.30), we have

$$[1 \otimes e^*_{\mathsf{id}} \otimes \mathrm{id}[r] \cdot 1_{\hat{\mathfrak{g}}}] = \sum_{i=1}^{n} [x_i^{-r} \otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}].$$

This yields formula (4.74). Secondly, take $z = {}^{\kappa}\mathbf{L}_r$ with $r \leq 1$. By (4.30), we have the following equalities of operators on $\mathsf{F}_c(\widehat{\mathbf{U}}_c)$ evaluated at $[1 \otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}]$:

$$\sum_{s \geq 1} \sum_{k,l} (e_{kl}[r-s] e_{lk}[s])^{(\infty)} = \sum_{s \geq 1} \sum_{i} x_i^{s-r} \sum_{k,l} e^{(i)}_{kl} e_{lk}[s]^{(\infty)} = -\sum_{i} x_i^{1-r} y_i,$$

$$\sum_{r \leq s \leq 0} \sum_{k,l} (e_{kl}[s] e_{lk}[r-s])^{(\infty)} = \sum_{r \leq s \leq 0} \sum_{i,j} x_i^{-s} x_j^{s-r} \Omega^{(i,j)} = 2 \sum_{i<j} c_{-r}(x_i, x_j) s_{i,j} + n(1-r) \sum_{i=1}^{n} x_i^{-r},$$

yielding formula (4.75). We have thus shown that the endomorphisms in $\mathsf{F}_\kappa(\mathscr{L}_\kappa^{op})$ send the generator $[1 \otimes e^*_{\mathsf{id}} \otimes 1_{\hat{\mathfrak{g}}}]$ of K_t to other elements of K_t. Hence $\mathsf{F}_\kappa(\mathscr{L}_\kappa^{op}) \subseteq N_t$, proving parts a) and b) of the lemma. Part c) can be checked by a direct calculation - it suffices to compute that the elements on the RHS of (4.74) and (4.75) lie in $\mathtt{Z}$. It also follows from Theorem 4.7.9, which has a more conceptual proof. □

Theorem 4.7.5. *We have $\mathscr{L}_c \subseteq Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c)$. Moreover, $Z(\mathsf{F}_c)$ is given by formulae* (4.74) *and* (4.75).

Proof. We need to check that, for any $M \in \widehat{\mathbf{U}}_c$-$\mathsf{mod}$ and $z \in \mathscr{L}_c$, the endomorphism $\mathsf{F}_c(z_M)$ lies in the image of $\mathtt{Z}$ in $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(M))$. By Definition 4.5.9, $\mathsf{F}_c(M) = \mathsf{F}_c(\widehat{\mathbf{U}}_c) \otimes_{\widehat{\mathbf{U}}_c} M$. The corresponding endomorphism $\mathsf{F}_c(z_M)$ of $\mathsf{F}_c(M)$ sends $r \otimes m \mapsto r \otimes z \cdot m = r \cdot z \otimes m$, for $m \in M$ and $r \in \mathsf{F}_\kappa(\widehat{\mathbf{U}}_c)$. Hence $\mathsf{F}_c(z_M) = \mathsf{F}_c(z_{\widehat{\mathbf{U}}_c}) \otimes \mathrm{id}$. But $\mathsf{F}_c(z_{\widehat{\mathbf{U}}_c})$ lies in the image of $\mathtt{Z}$ in $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\widehat{\mathbf{U}}_c))$ by part c) of Lemma 4.7.4. Hence $\mathsf{F}_c(z_M)$ lies in the image of $\mathtt{Z}$ in $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(M))$, proving the first statement. The second statement follows directly from part b) of Lemma 4.7.4. □

4.7.3. An F_c-central subcategory. The following lemma shows that the F_c-centre of $\widehat{\mathbf{U}}_c$-mod is a proper subalgebra of $\mathfrak{Z}$.

Lemma 4.7.6. *We have $Z_{\mathsf{F}_c}(\widehat{\mathbf{U}}_c) \neq \mathfrak{Z}$.*

Proof. Consider the element $\mathrm{id}[1] \in \mathfrak{Z}$. It follows from (4.50) that $-\alpha(\mathrm{id}[1])_{\mathsf{F}_c(\mathbb{H}_c)}$ is the endomorphism of $\mathsf{F}_c(\mathbb{H}_c) \cong \mathtt{H}_0$ given by multiplication with $y_1 + \ldots + y_n$. On the other hand, take, for example, the quotient M of $\mathbf{U}_c(\hat{\mathfrak{g}})$ by the left ideal generated by $\hat{\mathfrak{g}}_{\geq 3}$. One sees easily from (4.31) that $-\alpha(\mathrm{id}[1])_{\mathsf{F}_c(M)}$ does not coincide with the endomorphism of $\mathsf{F}_c(M)$ induced by $y_1 + \ldots + y_n$. □

Our next goal is to find a reasonable F_c-central subcategory of $\widehat{\mathbf{U}}_c$-mod.

Definition 4.7.7. Let $\mathscr{C}_{\mathbb{H}}$ be the full subcategory of $\widehat{\mathbf{U}}_c$-mod containing precisely the quotients of direct sums of $\mathbb{H}_c$. Let $\mathsf{F}_{\mathbb{H}}$ be the restriction of F_c to $\mathscr{C}_{\mathbb{H}}$.

As the lemma below shows, category $\mathscr{C}_{\mathbb{H}}$ contains interesting objects such as Verma and Weyl modules.

Lemma 4.7.8. *The following hold.*

a) *If $\lambda \in \mathcal{P}_n(n)$, then the Verma module $\mathbb{M}_c(\lambda)$ is an object of $\mathscr{C}_{\mathbb{H}}$.*
b) *Let $l \geq 1$, $\mu \in \mathcal{C}_l(n)$, $\lambda \in \mathcal{P}_\mu(\mu)$ and $a \in \mathbb{C}^n$ with $S_n(a) = S_\mu$. Then the Weyl module $\mathbb{W}_c(a, \lambda)$ is an object of $\mathscr{C}_{\mathbb{H}}$.*

Proof. Let us prove b). The definition of $\mathbb{H}_c$ implies that

$$\mathrm{Hom}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c, \mathbb{W}_c(a,\lambda)) \cong \mathbb{W}_c(a,\lambda)^{\mathfrak{i}}_{(1,\ldots,1)},$$

where $\mathfrak{i} = \mathfrak{n}_-[1] \oplus \mathfrak{n}_+[1] \oplus \hat{\mathfrak{g}}_{\geq 2}$. The subspace $L(a,\lambda) \subset \mathbb{W}_c(a,\lambda)$ is annihilated by $\mathfrak{i}$. It is easy to check that, since $\lambda \in \mathcal{P}_\mu(\mu)$, the difference $\lambda - (1,\ldots,1)$ is a sum of positive roots of $\mathfrak{l}_\mu$. Since $(1,\ldots,1)$ is a dominant weight, it follows that $L(a,\lambda)_{(1,\ldots,1)} \neq \{0\}$. Since $L(a,\lambda)$ is simple as an $\mathfrak{l}_\mu$-module, any non-zero vector generates $\mathbb{W}_c(a,\lambda)$ as a $\widehat{\mathbf{U}}_c$-module. It follows that there exists an epimorphism $\mathbb{H}_c \twoheadrightarrow \mathbb{W}_c(a,\lambda)$. Hence $\mathbb{W}_c(a,\lambda) \in \mathscr{C}_{\mathbb{H}}$. The proof of a) is analogous. □

To state the next theorem, we need to introduce some notation:

$$\Phi\colon \mathfrak{Z} \to \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c), \qquad z \mapsto z_{\mathbb{H}_c},$$
$$\Psi\colon \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) \to \mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0) \cong \mathtt{H}_0^{op}, \quad \phi \mapsto \mathsf{F}_c(\phi),$$

$$\Theta := \Psi \circ \Phi.$$

These maps fit into the following commutative diagram.

$$\begin{array}{ccccccc} \mathfrak{Z} & \xrightarrow{q} & Z(\mathscr{C}_{\mathbb{H}}) & \xrightarrow{\alpha'} & \mathrm{End}(\mathsf{F}_{\mathbb{H}}) & \hookleftarrow & \mathtt{Z} \\ \| & & \downarrow & & \downarrow & & \| \\ \mathfrak{Z} & \xrightarrow{\Phi} & \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) & \xrightarrow{\Psi} & \mathtt{H}_0^{op} & \hookleftarrow & \mathtt{Z} \end{array} \tag{4.76}$$

where the vertical arrows send an endomorphism of the identity functor (resp. $\mathsf{F}_{\mathbb{H}}$) to the corresponding endomorphism of $\mathbb{H}_c$ (resp. $\mathtt{H}_0$).

The following theorem is the main result of this section.

Theorem 4.7.9. *The subcategory $\mathscr{C}_{\mathbb{H}}$ is F_c-central and $Z_{\mathscr{C}_{\mathbb{H}}}(\mathsf{F}_c) = \Theta$.*

The proof of Theorem 4.7.9 will be presented in §4.7.4. We note the following corollary, which will be useful later.

Corollary 4.7.10. *Let $M \in \mathscr{C}_{\mathbb{H}}$ and $z \in \mathfrak{Z}$. Then $\Theta(z)_{\mathsf{F}_c(M)} = \mathsf{F}_c(z_M)$. In particular, $\Theta(\mathrm{Ann}_{\mathfrak{Z}}(M)) \subseteq \mathrm{Ann}_{\mathtt{Z}}(\mathsf{F}_c(M))$.*

Proof. By Theorem 4.7.9, we have $\mathsf{F}_c(z_M) = (\alpha' \circ q(z))_M = \Theta(z)_M$. If $z \in \mathrm{Ann}_{\mathfrak{Z}}(M)$, then $z_M = 0$ and so $\Theta(z)_{\mathsf{F}_c(M)} = \mathsf{F}_c(z_M) = 0$. □

4.7.4. Proof of Theorem 4.7.9. The proof of Theorem 4.7.9 requires some preperations. We first prove the following lemma.

Lemma 4.7.11. *The two vertical arrows in* (4.76) *are injective.*

Proof. Let M be an object of $\mathscr{C}_{\mathbb{H}}$. Since M is a quotient of $\mathbb{H}_c^I$ (direct sum over some index set I), there exists an epimorphism $p\colon \mathbb{H}_c^I \twoheadrightarrow M$. Suppose that $z \in Z(\mathscr{C}_{\mathbb{H}})$. Then $z_M \circ p = p \circ z_{\mathbb{H}_c^I}$ and it follows that z_M is uniquely determined by $z_{\mathbb{H}_c^I}$. But $z_{\mathbb{H}_c^I} = \oplus_I z_{\mathbb{H}_c}$, so z_M is in fact uniquely determined by $z_{\mathbb{H}_c}$. This proves the injectivity of the left vertical arrow.

Now suppose that $\phi \in \mathrm{End}(\mathsf{F}_{\mathbb{H}})$. Let ϕ_M be the corresponding endomorphism of $\mathsf{F}_c(M)$. Since F_c is right exact, $\mathsf{F}_c(p)\colon \mathtt{H}_0^I \to \mathsf{F}_c(M)$ is also an epimorphism. Since ϕ is a natural transformation, we have $\mathsf{F}_c(p) \circ \phi_{\mathbb{H}_c^I} = \phi_M \circ \mathsf{F}_c(p)$. It follows that ϕ_M is determined uniquely by $\phi_{\mathbb{H}_c^I}$. But $\phi_{\mathbb{H}_c^I} = \oplus_I \phi_{\mathbb{H}_c}$, so ϕ_M is uniquely determined by $\phi_{\mathbb{H}_c}$. This proves the injectivity of the right vertical arrow. □

Theorem 4.7.9 states that $\mathscr{C}_{\mathbb{H}}$ is F_c-central, i.e., $\mathrm{Im}\,\alpha' \circ q \subseteq \mathtt{Z}$. By Lemma 4.7.11, this is equivalent to showing that $\mathrm{Im}\,\Theta \subseteq \mathtt{Z}$. The rest of this subsection is dedicated to this goal. The main idea is to establish the following two facts: $\mathrm{Im}\,\Theta \subseteq Z_{\mathtt{H}_0^{op}}(\mathbb{C}[\mathfrak{h}^*]^{\rtimes})$ and $Z_{\mathtt{H}_0^{op}}(\mathbb{C}[\mathfrak{h}^*]^{\rtimes}) = \mathtt{Z}$.

We start by recalling some information about the $G((t))$-action on $\widehat{\mathbf{U}}_c$. There is an adjoint action

$$G((t)) \times \mathfrak{g}((t)) \to \mathfrak{g}((t)), \quad (g, X) \mapsto g(X) := gXg^{-1}$$

of $G((t))$ on its Lie algebra $\mathfrak{g}((t))$. It extends to an action on $\hat{\mathfrak{g}}_c$ if we let $G((t))$ act trivially on $\mathbf{1}$. This action induces an action on the universal enveloping algebra $\mathbf{U}_c(\hat{\mathfrak{g}})$ and its completion $\widehat{\mathbf{U}}_c$.

Proposition 4.7.12 ([54, Proposition 4.3.8])**.** *The $G((t))$-action on $\mathfrak{Z} \subseteq \widehat{\mathbf{U}}_c$ is trivial.*

The $G((t))$-action restricts to an S_n-action on $\widehat{\mathbf{U}}_c$, where we identify the symmetric group S_n with the subgroup of permutation matrices in $G \subset G((t))$. The S_n-action preserves the ideal $\mathfrak{I}_c \subset \mathbf{U}(\hat{\mathfrak{g}}_c)$ and, hence, induces an action on the module $\mathbb{H}_c$.

We now define an induced action on $\mathsf{F}_c(\mathbb{H}_c)$. Let S_n act on $(\mathbf{V}^*)^{\otimes n}$ by the rule $e_{i_1}^* \otimes \ldots \otimes e_{i_n}^* \mapsto e_{w(i_{w^{-1}(1)})}^* \otimes \ldots \otimes e_{w(i_{w^{-1}(n)})}^*$. One easily checks that $w \cdot e_\tau^* = e_{w\tau w^{-1}}^*$, where e_τ^* is as in (4.64). Combining the S_n-actions on $\mathbb{H}_c$ and $(\mathbf{V}^*)^{\otimes n}$ defined above with the natural permutation action on $\mathbb{C}[\mathfrak{h}]$ we obtain an action

$$S_n \times \mathsf{T}_c(\mathbb{H}_c) \to \mathsf{T}_c(\mathbb{H}_c), \quad (w, f \otimes u \otimes h) \mapsto w \cdot f \otimes w \cdot u \otimes w \cdot h. \tag{4.77}$$

It is easy to check that if $X[k] \in \mathfrak{g}[t]$ and $w \in S_n$ then $w \circ X[k] = w(X)[k] \circ w$ as operators on $\mathsf{T}_c(\mathbb{H}_c)$. Hence the subspace $\mathfrak{g}[t].\mathsf{T}_c(\mathbb{H}_c)$ is S_n-stable, and (4.77) descends to an action

$$\star\colon S_n \times \mathsf{F}_c(\mathbb{H}_c) \to \mathsf{F}_c(\mathbb{H}_c). \tag{4.78}$$

Note that this action is different from the S_n-action defined in §4.4.4.

There is also a natural conjugation action

$$S_n \times \mathtt{H}_0 \to \mathtt{H}_0, \quad (w, h) \mapsto whw^{-1}. \tag{4.79}$$

In the next lemma we compare the induced actions on endomorphism algebras.

Lemma 4.7.13. *The map Θ is S_n-equivariant.*

Proof. We factor Θ as a product of the maps Φ, $\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) \to \mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\mathbb{H}_c))$ and the isomorphism $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\mathbb{H}_c)) \cong \mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0)$ induced by Υ from (4.49). The first two maps are S_n-equivariant by construction. So we only need to check that Υ intertwines the two actions (4.78) and (4.79). Abbreviating $e_k := e_{kk}$, we have

$$\begin{aligned}\Upsilon(wf(x_1,\dots,x_n)ug(y_1,\dots,y_n)w^{-1}) &= \Upsilon(f(x_{w(1)},\dots,x_{w(n)})wuw^{-1}g(y_{w(1)},\dots,y_{w(n)}))\\ &= [f(x_{w(1)},\dots,x_{w(n)})\otimes e^*_{wuw^{-1}}\otimes g(-e_{w(1)}[1],\dots,-e_{w(n)}[1]).1_{\mathbb{H}}]\end{aligned}$$

and

$$\begin{aligned}w\star\Upsilon(f(x_1,\dots,x_n)ug(y_1,\dots,y_n)) &= w\star[f(x_1,\dots,x_n)\otimes e^*_u\otimes g(-e_1[1],\dots,-e_n[1]).1_{\mathbb{H}}]\\ &= [f(x_{w(1)},\dots,x_{w(n)})\otimes e^*_{wuw^{-1}}\otimes g(-e_{w(1)}[1],\dots,-e_{w(n)}[1]).1_{\mathbb{H}}],\end{aligned}$$

as required. □

Proposition 4.7.14. *We have* $Z_{\mathtt{H}_0}(\mathbb{C}[\mathfrak{h}^*]^{\rtimes}) = \mathtt{Z}$.

Proof. Write $\mathtt{H}_{\mathsf{reg}} := \mathbb{C}[\mathfrak{h}_{\mathsf{reg}}\times\mathfrak{h}^*]\rtimes\mathbb{C}S_n$. We first prove that

$$Z_{\mathtt{H}_{\mathsf{reg}}}(\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}S_n) = Z(\mathtt{H}_{\mathsf{reg}}) = \mathbb{C}[\mathfrak{h}_{\mathsf{reg}}\times\mathfrak{h}^*]^{S_n}.$$

We only need to show that $Z_{\mathtt{H}_{\mathsf{reg}}}(\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}S_n)\subseteq Z(\mathtt{H}_{\mathsf{reg}})$, the other inclusion being obvious. Let $z\in Z_{\mathtt{H}_{\mathsf{reg}}}(\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}S_n)$. We can uniquely write $z=\sum_{w\in S_n}f_w w$ with $f_w\in\mathbb{C}[\mathfrak{h}_{\mathsf{reg}}\times\mathfrak{h}^*]$. Since, by assumption, z commutes with $\mathbb{C}S_n$, for any $u\in S_n$ we have $z=uzu^{-1}=\sum_{w\in S_n}uf_wwu^{-1}=\sum_{w\in S_n}f^u_{u^{-1}wu}w$, where $f^u(a)=f(u^{-1}\cdot a)$. Hence $f_1=f_1^u$ for all $u\in S_n$, i.e., $f_1\in\mathbb{C}[\mathfrak{h}_{\mathsf{reg}}\times\mathfrak{h}^*]^{S_n}$. Next, since z commutes with $\mathbb{C}[\mathfrak{h}^*]$, $0=[z,g]=\sum_{w\in S_n}f_w(g^w-g)w$ for all $g\in\mathbb{C}[\mathfrak{h}^*]$. But S_n acts faithfully on $\mathfrak{h}\subset\mathbb{C}[\mathfrak{h}^*]$, so for each $w\in S_n$ there exists $a\in\mathfrak{h}$ such that $w^{-1}(a)\neq a$. This forces $f_w=0$ for each $w\neq 1$.

Using the Dunkl embedding (see (4.29)), we view $\mathtt{H}_0$ as a subalgebra of $\mathtt{H}_{\mathsf{reg}}$. The following are obvious:

$$Z_{\mathtt{H}_0}(\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}S_n) = Z_{\mathtt{H}_{\mathsf{reg}}}(\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}S_n)\cap\mathtt{H}_0,\quad Z(\mathtt{H}_{\mathsf{reg}})\cap\mathtt{H}_0\subseteq\mathtt{Z}.$$

Since $\mathtt{H}_{\mathsf{reg}}=\mathtt{H}_0[\delta^{-1}]$ and δ^{-1} is central in $\mathtt{H}_{\mathsf{reg}}$, we also have $\mathtt{Z}\subseteq Z(\mathtt{H}_{\mathsf{reg}})\cap\mathtt{H}_0$. □

Remark 4.7.15. Proposition 4.7.14 generalizes to rational Cherednik algebras at $t=0$ associated to any complex reflection group.

Proposition 4.7.16. *We have* $\operatorname{Im}\Theta\subseteq\mathtt{Z}$.

Proof. Lemma 4.7.13 and Proposition 4.7.12 imply that $\operatorname{Im}\Theta\subseteq Z_{\mathtt{H}_0^{op}}(\mathbb{C}S_n)$. Therefore, it suffices to show that $\operatorname{Im}\Theta\subseteq Z_{\mathtt{H}_0^{op}}(\mathbb{C}[\mathfrak{h}^*])$, because then Proposition 4.7.14 implies that $\operatorname{Im}\Theta\subseteq Z_{\mathtt{H}_0^{op}}(\mathbb{C}[\mathfrak{h}^*]^{\rtimes})=\mathtt{Z}$.

By the definition of $\mathbb{H}_c$, there is a natural isomorphism

$$\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c)\xrightarrow{\sim}(\mathbb{H}_c)^{\mathfrak{i}}_{(1,\dots,1)}.\tag{4.80}$$

Observe that $\mathrm{Sym}(\mathfrak{t}[1]).1_{\mathbb{H}}\subset(\mathbb{H}_c)^{\mathfrak{i}}_{(1,\dots,1)}$. Indeed, $\mathrm{Sym}(\mathfrak{t}[1]).1_{\mathbb{H}}$ has $\mathfrak{t}$-weight $(1,\dots,1)$, and since $\mathfrak{i}$ is an ideal in $\hat{\mathfrak{t}}_+$ and $1_{\mathbb{H}}$ is annihilated by $\mathfrak{i}$, so is $\mathrm{Sym}(\mathfrak{t}[1]).1_{\mathbb{H}}$. Hence elements of $\mathrm{Sym}(\mathfrak{t}[1]).1_{\mathbb{H}}$ define endomorphisms of $\mathbb{H}_c$.

By construction, $\operatorname{Im}\Phi\subseteq Z(\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c))$, and so $\operatorname{Im}\Phi$ commutes with the endomorphisms defined by $\mathrm{Sym}(\mathfrak{t}[1]).1_{\mathbb{H}}$. Hence $\operatorname{Im}\Theta=\Psi(\operatorname{Im}\Phi)$ must commute with the image of these endomorphisms under Ψ.

But Theorem 4.6.11 implies that they are mapped to $\mathbb{C}[\mathfrak{h}^*] \subset \mathtt{H}^{op}$. It follows that $\operatorname{Im}\Theta \subseteq Z_{\mathtt{H}_0^{op}}(\mathbb{C}[\mathfrak{h}^*])$, as required. □

We are now ready to complete the proof of Theorem 4.7.9.

Proof of Theorem 4.7.9. By Proposition 4.7.16, $\operatorname{Im}\Theta \subseteq \mathtt{Z}$. Lemma 4.7.11 and the commutativity of diagram (4.76), therefore, imply $\operatorname{Im}(\alpha' \circ q) \subseteq \mathtt{Z}$. The second statement of the theorem also follows directly from the commutativity of the diagram. □

4.8 Filtered and graded versions of the Suzuki functor

Our next goal is to show that $\operatorname{Im}\Theta = \mathtt{Z}$. The proof in §4.9 relies on a filtered version of the Suzuki functor, which we construct in this section. We also introduce a graded version. Assume that $\kappa \in \mathbb{C}$ and m, n are arbitrary unless indicated otherwise.

4.8.1. Background from filtered and graded algebra. We refer the reader to [5] and [129] for basic definitions from filtered and graded algebra. All filtrations we consider are increasing, exhaustive and separated. If M is a graded vector space (or module or algebra) we denote the i-th graded piece by M_i. If M is a filtered vector space (or module or algebra), we denote the i-th filtered piece by $M_{\leq i}$.

Now suppose that A is a filtered algebra and M, N are two filtered A-modules. An A-module homomorphism $f : M \to N$ is called *filtered* of degree i if $f(M_{\leq r}) \subseteq N_{\leq r+i}$ for all $r \in \mathbb{Z}$. We say that f is a *filtered isomorphism* if f is an isomorphism of A-modules and $f(M_{\leq r}) = N_{\leq r}$ for all $r \in \mathbb{Z}$. Let $\operatorname{Hom}_A(M,N)_{\leq i}$ denote the vector space of filtered homomorphisms of degree i and set $\operatorname{Hom}_A^{\mathsf{fil}}(M,N) := \bigcup_{i \in \mathbb{Z}} \operatorname{Hom}_A(M,N)_{\leq i}$. If M is finitely generated as an A-module then $\operatorname{Hom}_A(M,N) = \operatorname{Hom}_A^{\mathsf{fil}}(M,N)$. Observe that $\operatorname{Hom}_A^{\mathsf{fil}}(M,N)$ is a filtered vector space and $\operatorname{Hom}_A^{\mathsf{fil}}(M,M)$ is also a filtered algebra.

We next define two categories whose objects are filtered (left) A-modules. The first category, denoted A-fmod, has Hom-sets of the form $\operatorname{Hom}_A^{\mathsf{fil}}(M,N)$. The second category, denoted A-fmod_0, has Hom-sets of the form $\operatorname{Hom}_A(M,N)_0$. We regard A-fmod as a category enriched in the category $\mathbb{C}$-fmod_0 of filtered vector spaces (where $\mathbb{C}$ is endowed with the trivial filtration).

Analogous definitions make sense in the graded setting. In particular, if A is a $\mathbb{Z}$-graded algebra then we have two categories of graded modules A-gmod and A-gmod_0. We regard A-gmod as a category enriched in the category $\mathbb{C}$-gmod_0 of graded vector spaces.

If A is a filtered algebra, with associated graded $\mathsf{gr}\,A$, let $\sigma\colon A \to \mathsf{gr}\,A$ be the principal symbol map. For $v \in A$, set $\deg v := \deg \sigma(v)$. If $f\colon A \to B$ is a degree zero filtered algebra homomorphism, let $\mathsf{gr}\,f\colon \mathsf{gr}\,A \to \mathsf{gr}\,B$ be the associated graded algebra homomorphism.

4.8.2. Filtrations and gradings. We consider two filtrations and a grading on $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$.

Definition 4.8.1. Suppose that $l \geq 0$, $X_1, \dots, X_l \in \mathfrak{g}$ and $j_1, \dots, j_l \in \mathbb{Z}$. An expression of the form $\mathbf{m} = X_1[j_1] \dots X_l[j_l] \in \mathbf{U}_\kappa(\tilde{\mathfrak{g}})$ is called a *monomial* of *length* l, *height* $j_1 + \ldots + j_l$ and *absolute height* $|j_1| + \ldots + |j_l|$. For $r \in \mathbb{Z}$, define:

(a) $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})_r \quad = \langle$ monomials of height r $\rangle$,

(b) $\mathbf{U}_\kappa^{\mathsf{pbw}}(\tilde{\mathfrak{g}})_{\leq r} = \langle$ monomials of length $\leq r$ $\rangle$,

(c) $\mathbf{U}_\kappa^{\mathsf{abs}}(\tilde{\mathfrak{g}})_{\leq r} = \langle$ monomials of absolute height $\leq r$ $\rangle$, where the brackets denote $\mathbb{C}$-span. Observe that (a) defines a grading while (b) and (c) define filtrations on $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$. Filtration (b) is the usual PBW filtration. We call filtration (c) the *absolute height filtration.* Denote by $\mathbf{U}_\kappa^{\mathsf{pbw}}(\tilde{\mathfrak{g}})$ and $\mathbf{U}_\kappa^{\mathsf{abs}}(\tilde{\mathfrak{g}})$ the corresponding filtered algebras.

Definition 4.8.2. Let $\mathscr{C}_\kappa^{\mathsf{abs}}$ (resp. $\mathscr{C}_\kappa^{\mathsf{abs}}(r)$) be the full subcategory of $\mathbf{U}_\kappa^{\mathsf{abs}}(\tilde{\mathfrak{g}})$-$\mathsf{fmod}$ whose objects are filtered modules with the property that the underlying unfiltered module is an object of $\mathscr{C}_\kappa$ (resp. $\mathscr{C}_\kappa(r)$). Similarly, let $\mathscr{C}_\kappa^{\mathsf{gr}}$ be the full subcategory of $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$-$\mathsf{gmod}$ whose objects are graded modules with the property that the underlying ungraded module is an object of $\mathscr{C}_\kappa$.

Remark 4.8.3. Consider the associated graded algebra $\mathsf{gr}\,\mathbf{U}_\kappa^{\mathsf{abs}}(\tilde{\mathfrak{g}})$. It is easy to see that the relation

$$[\sigma(X\otimes t^r),\sigma(Y\otimes t^l)]=\delta_{|r|+|l|,|r+l|}\sigma([X,Y]\otimes t^{r+l})$$

holds in $\mathsf{gr}\,\mathbf{U}_\kappa^{\mathsf{abs}}(\tilde{\mathfrak{g}})$. Hence

$$\mathsf{gr}\,\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\geq 0})\cong\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\geq 0}),\quad \mathsf{gr}\,\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\leq 0})\cong\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\leq 0}).$$

Moreover, we have $[\mathsf{gr}\,\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\geq 1}),\mathsf{gr}\,\mathbf{U}^{\mathsf{abs}}(\tilde{\mathfrak{g}}_{\leq -1})]=0$.

We next consider a family of filtrations and a grading on rational Cherednik algebras.

Definition 4.8.4. Setting $\deg x_i=-1$, $\deg y_i=1$ and $\deg S_m=0$ defines a grading on $\mathtt{H}_t$. We denote the corresponding graded algebra simply by $\mathtt{H}_t$. For each $k\geq 1$, setting $\deg x_i=1$, $\deg y_i=k$ and $\deg S_m=0$ yields a filtration on $\mathtt{H}_t$, and we denote the corresponding filtered algebra by $\mathtt{H}_t^{(k)}$. When $k=1$, the resulting filtration is known as the PBW filtration, and we abbreviate $\mathtt{H}_t:=\mathtt{H}_t^{(1)}$. We consider $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}[\mathfrak{h}]^{\rtimes}$ and $\mathbb{C}[\mathfrak{h}^*]$ as graded (resp. filtered) subalgebras of $\mathtt{H}_t$.

4.8.3. Filtered lift of the Suzuki functor

Let M be a filtered module in $\mathscr{C}_\kappa^{\mathsf{abs}}$. We equip $(\mathbf{V}^*)^{\otimes m}$ with the trivial filtration and $\mathsf{T}_\kappa(M)$ with the tensor product filtration. Explicitly,

$$\mathsf{T}_\kappa(M)_{\leq r}=\sum_{k+l=r}\mathbb{C}[\mathfrak{h}]_{\leq k}\otimes(\mathbf{V}^*)^{\otimes m}\otimes M_{\leq l}. \tag{4.81}$$

Consider the quotient map

$$\psi\colon \mathsf{T}_\kappa(M)\twoheadrightarrow\mathsf{F}_\kappa(M). \tag{4.82}$$

We endow $\mathsf{F}_\kappa(M)$ with the quotient filtration given by $\mathsf{F}_\kappa(M)_{\leq r}:=\psi(\mathsf{T}_\kappa(M)_{\leq r})$. The following proposition connects the absolute height filtration on $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$ with the filtrations on $\mathtt{H}_{\kappa+n}$.

Proposition 4.8.5. *For each $r\geq 2$, the functor F_κ lifts to a functor*

$$\mathsf{F}_\kappa^{(r)}:\mathscr{C}_\kappa^{\mathsf{abs}}(r)\to\mathtt{H}_{\kappa+n}^{(2r-3)}\text{-}\mathsf{fmod}$$

enriched in $\mathbb{C}$-fmod_0.

Proof. Let $M\in\mathscr{C}_\kappa^{\mathsf{abs}}(r)$. We first show that $\mathsf{F}_\kappa(M)$ is a filtered $\mathtt{H}_{\kappa+n}^{(2r-3)}$-module. The only non-trivial thing to show is that $y_i\mathsf{F}_\kappa(M)_{\leq s}\subseteq\mathsf{F}_\kappa(M)_{\leq s+2r-3}$ for $s\in\mathbb{Z}$ and $1\leq i\leq m$. Recall that the action of y_i is given by (4.31). Clearly each of ∂_{x_i} and $\Omega^{(i,j)}(x_i-x_j)^{-1}(1-\underline{s_{i,j}})$ either vanishes or lowers degree by one. Hence it is enough to show that for each $p\geq 0$, the operator $x_i^p\Omega^{(i,\infty)}_{[p+1]}$ raises degree by at most $2r-3$. Observe that x^p raises degree by p and $e_{kl}^{(i)}$ doesn't change degree. Therefore it is in fact enough to show that each $e_{lk}[p+1]^{(\infty)}$ changes degree by at most $-p+2r-3$.

If $p \leq r-2$ then the fact that M is a filtered module implies that $e_{lk}[p+1]$ raises degree by at most $r-1$. But $r-1 \leq (r-1)+(r-2-p) = -p+2r-3$. So assume $p > r-2$. Let $v \in M$. Because $M \in \mathscr{C}^{\mathsf{abs}}_\kappa(r)$, we can assume without loss of generality that $v = X_1[a_1]\dots X_z[a_z].u$, with u satisfying $\hat{\mathfrak{g}}_{\geq r}.u = 0$, $X_1,\dots,X_z \in \mathfrak{g}$ and $a_1 \leq \dots \leq a_z < r$. We argue by induction on z (i.e. by induction on the PBW filtration). If $z = 1$ then

$$e_{lk}[p+1].v = X_1[a_1]e_{lk}[p+1].u + [e_{lk}, X_1][p+1+a_1].u = [e_{lk}, X_1][p+1+a_1].u. \tag{4.83}$$

Note that $[e_{lk}, X_1][p+1+a_1].u = 0$ unless $a_1 \leq -p+r-2$. Let us now calculate the difference in degree between v and (4.83). First assume $a_1 \leq -p-1$. Then $|p+1+a_1| - |a_1| = -(p+1+a_1)+a_1 = -p-1$. But $-p-1 \leq -p+2r-3$ since $r \geq 2$. Next assume $-p \leq a_1 \leq -p+r-2 < 0$. Then $|p+1+a_1|-|a_1| = p+2a_1+1 \leq -p+2r-3$. Hence $e_{lk}[p+1]^{(\infty)}$ changes degree by at most $-p+2r-3$, as required.

Now let $z > 1$. We have

$$e_{lk}[p+1].v = X_1[a_1]e_{lk}[p+1].v' + [e_{lk}, X_1][p+1+a_1].v',$$

where $v' = X_2[a_2]\dots X_z[a_z].u$. By induction we know that $e_{lk}[p+1]$ changes the degree of v' by at most $-p+2r-3$. Hence $\deg X_1[a_1]e_{lk}[p+1].v' \leq \deg v' + a_1 - p + 2r - 3 \leq \deg v - p + 2r - 3$. Moreover, since M is a filtered module, $[e_{lk}, X_1][p+1+a_1]$ changes the degree of v' by at most $|p+1+a_1|$. By triangle inequality, $|p+1+a_1|-|a_1| \leq |p+1| = p+1$, so $\deg[e_{lk}, X_1][p+1+a_1].v' \leq \deg v - p - 1 < \deg v - p + 2r - 3$. It follows that $e_{lk}[p+1]^{(\infty)}$ changes degree by at most $-p+2r-3$, as required.

We now show that $\mathsf{F}^{(r)}_\kappa$ is an enriched functor. Suppose that M and N are two filtered modules in $\mathscr{C}^{\mathsf{abs}}_\kappa(r)$. Let $h : M \to N$ be a filtered homomorphism of degree i. We need to show that $\mathsf{F}_\kappa(h)$ is also a filtered homomorphism of degree i. So let $v \in \mathsf{F}_\kappa(M)_{\leq s}$. Recall the projection (4.82). Since $\mathsf{F}_\kappa(M)$ is endowed with the quotient filtration, we can choose $\tilde{v} \in \mathsf{T}_\kappa(M)_{\leq s}$ with $\psi(\tilde{v}) = v$. We can assume without loss of generality that $\tilde{v} = f(x_1,\dots,x_m) \otimes u \otimes z$ with $u \in (\mathbf{V}^*)^{\otimes m}$, $z \in M$ and f some polynomial. Since h is filtered of degree i, we have $\mathsf{T}_\kappa(h)(\tilde{v}) = f(x_1,\dots,x_m) \otimes u \otimes h(z) \in \mathsf{T}_\kappa(N)_{\leq s+i}$. However, $\psi' \circ \mathsf{T}_\kappa(h)(\tilde{v}) = \mathsf{F}_\kappa(h)(v)$, where ψ' is the projection $\psi' : \mathsf{T}_\kappa(N) \twoheadrightarrow \mathsf{F}_\kappa(N)$. It follows that $\mathsf{F}_\kappa(h)(v) \in \mathsf{F}_\kappa(N)_{\leq s+i}$, as required. □

In the following proposition assume that $\kappa = c$, $m = n$ and consider the module $\mathbb{H}_c = \mathbf{U}^{\mathsf{abs}}_c(\tilde{\mathfrak{g}})/(\mathfrak{I}_c \cap \mathbf{U}^{\mathsf{abs}}_c(\tilde{\mathfrak{g}}))$ as a filtered $\mathbf{U}^{\mathsf{abs}}_c(\tilde{\mathfrak{g}})$-module endowed with the quotient filtration.

Proposition 4.8.6. *The isomorphism* $\Upsilon \colon \mathtt{H}_0 \xrightarrow{\sim} \mathsf{F}_c(\mathbb{H}_c)$ *from* (4.49) *lifts to an isomorphism in the category* $\mathtt{H}_0\text{-}\mathsf{fmod}_0$. *Moreover, the map* $\Psi \colon \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) \to \mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0)$ *is a filtered algebra homomorphism.*

Proof. Since it is difficult to work with quotient filtrations, we first show that $\mathsf{F}_c(\mathbb{H}_c)$ is isomorphic to another module with a more explicit filtration. Consider the $\mathtt{H}_0$-module $\mathsf{T}_c(\mathbb{H}_c)$. One easily checks that the subspace $M = \mathbb{C}[\mathfrak{h}] \otimes ((\mathbf{V}^*)^{\otimes n})_{(-1,\dots,-1)} \otimes \mathcal{I}$ is a $\mathtt{H}_0$-submodule of $\mathsf{T}_c(\mathbb{H}_c)$. Moreover, it follows from Theorem 4.6.11 that $\mathsf{T}_c(\mathbb{H}_c) = M \oplus \mathfrak{g}[t] \cdot \mathsf{T}_c(\mathbb{H}_c)$ and $\mathsf{F}_c(\mathbb{H}_c) \cong M$. The latter isomorphism is filtered if we endow $\mathsf{F}_c(\mathbb{H}_c)$ with the quotient filtration and M with the subspace filtration. It follows from (4.50) that composing Υ with $\mathsf{F}_c(\mathbb{H}_c) \cong M$ yields an $\mathtt{H}_0$-module isomorphism $\mathtt{H}_0 \cong M$ given by

$$f(x_1,\dots,x_n)wg(y_1,\dots,y_n) \mapsto f(x_1,\dots,x_n) \otimes e^*_w \otimes g(-e_{11}[1],\dots,-e_{nn}[1])1_{\mathbb{H}}.$$

This formula together with the definition of the filtration on $\mathtt{H}_0$ and (4.81) imply that the isomorphism

$\mathtt{H}_0 \cong M$ is in fact filtered. This proves the first part of the proposition.

The filtered isomorphism Υ^{-1} induces a filtered isomorphism of endomorphism rings $\mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\mathbb{H}_c)) \cong \mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0)$. But Ψ is a composition of the latter with the homomorphism $\mathsf{F}_c\colon\ \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) \to \mathrm{End}_{\mathtt{H}_0}(\mathsf{F}_c(\mathbb{H}_c))$, which is filtered by Proposition 4.8.5. □

4.8.4. Graded lift of the Suzuki functor. Suppose that M is a graded module in $\mathscr{C}_\kappa^{\mathsf{gr}}$. Consider $(\mathbf{V}^*)^{\otimes m}$ as a graded vector space concentrated in degree zero. Endow $\mathsf{T}_\kappa(M)$ with the tensor product grading in analogy to (4.81). It follows immediately from (4.30) that $\mathsf{F}_\kappa(M)$ is a quotient of $\mathsf{T}_\kappa(M)$ by a graded subspace. Hence the grading on $\mathsf{T}_\kappa(M)$ descends to a grading on $\mathsf{F}_\kappa(M)$.

Proposition 4.8.7. *The functor* F_κ *lifts to a functor*

$$\mathsf{F}_\kappa^{\mathsf{gr}}\colon \mathscr{C}_\kappa^{\mathsf{gr}} \to \mathtt{H}_{\kappa+n}\text{-}\mathsf{gmod}$$

enriched in $\mathbb{C}\text{-}\mathsf{gmod}_0$.

Proof. Let $M \in \mathscr{C}_\kappa^{\mathsf{gr}}$. We first prove that $\mathsf{F}_\kappa(M)$ is a graded $\mathtt{H}_{\kappa+n}$-module. It suffices to show that $y_i\mathsf{F}_\kappa(M)_s \subseteq \mathsf{F}_\kappa(M)_s$ for $s \in \mathbb{Z}$ and $1 \leq i \leq m$. Recall that the action of y_i is given by (4.31). Clearly ∂_{x_i} and $\Omega^{(i,j)}(x_i - x_j)^{-1}(1 - \underline{s_{i,j}})$ either vanish or raise degree by one. Since M is a graded $\mathbf{U}_\kappa(\tilde{\mathfrak{g}})$-module, the same holds for $x_i^p\Omega_{[p+1]}^{(i,\infty)}$ for each $p \geq 0$, as required. The proof of the fact that $\mathsf{F}_\kappa^{\mathsf{gr}}$ is an enriched functor is analogous to the proof of Proposition 4.8.5. □

4.9 Surjectivity of Θ

In this section we show that $\mathrm{Im}\,\Theta = \mathtt{Z}$. Assume that $n = m$ and $\kappa = c$ throughout.

4.9.1. The associated graded map. Consider the following commutative diagram in the category of vector spaces.

$$\begin{array}{ccccccc}
\mathbb{H}_c^{\mathfrak{I}_c} & \hookrightarrow & \mathbb{H}_c & \xrightarrow{v \mapsto 1_{\mathbb{C}[\mathfrak{h}]} \otimes e_{\mathrm{id}}^* \otimes v} & \mathsf{T}_c(\mathbb{H}_c) & \twoheadrightarrow & \mathsf{F}_c(\mathbb{H}_c) \\
\uparrow \wr & & & & & & \wr\downarrow \Upsilon^{-1} \\
\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c) & & & \xrightarrow{\Psi} & & & \mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0) \cong \mathtt{H}_0 \\
\uparrow & & & & & & \uparrow \\
\mathrm{Im}\,\Phi & & & \xrightarrow{\Psi} & & & \mathtt{Z}
\end{array} \tag{4.84}$$

Note that the fact that $\Psi(\mathrm{Im}\,\Phi) \subseteq \mathtt{Z}$ follows from Proposition 4.7.16. We endow each of the vector spaces above with a filtration:

- $\mathbb{H}_c = \mathbf{U}_c^{\mathsf{abs}}(\tilde{\mathfrak{g}})/(\mathfrak{I}_c \cap \mathbf{U}_c^{\mathsf{abs}}(\tilde{\mathfrak{g}}))$ carries the quotient filtration and $\mathbb{H}_c^{\mathfrak{I}_c} \subset \mathbb{H}_c$ has the subspace filtration,
- $\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c)$ carries the filtration induced by the one on $\mathbb{H}_c$ and $\mathrm{Im}\,\Theta \subset \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{H}_c)$ has the subspace filtration,
- $\mathsf{T}_c(\mathbb{H}_c)$ has the filtration from (4.81) and $\mathsf{F}_c(\mathbb{H}_c)$ has the corresponding quotient filtration,
- $\mathtt{H}_0$ has the PBW filtration, $\mathrm{End}_{\mathtt{H}_0}(\mathtt{H}_0)$ carries the induced filtration and $\mathtt{Z} \subset \mathtt{H}_0$ the subspace filtration.

Lemma 4.9.1. *Each map in the diagram* (4.84) *is filtered.*

Proof. Every map is filtered by definition except for Ψ and Υ^{-1}. The fact that the latter two are filtered follows from Proposition 4.8.6. □

We will show that $\operatorname{Im}\Theta = \mathtt{Z}$ by computing the associated graded algebra homomorphism

$$\mathsf{gr}\,\Psi\colon\ \mathsf{gr}\,\operatorname{Im}\Phi \to \mathsf{gr}\,\mathtt{Z}. \tag{4.85}$$

We split the task of computing (4.85) into two parts. We first compute the principal symbols of the images of Segal-Sugawara operators in $\mathbb{H}_c$. We then compute the images of these principal symbols under the associated graded of the map $\mathbb{H}_c \to \mathtt{H}_0$ arising from the upper right corner of the diagram (4.84).

4.9.2. Calculation of principal symbols. The ideal $(\mathbf{U}(\hat{\mathfrak{g}}_-)(\mathfrak{n}_+\otimes t\mathbb{C}[t^{-1}]))\cap\mathbf{U}(\hat{\mathfrak{g}}_-)^{\mathrm{ad}\,\mathfrak{t}}$ in $\mathbf{U}(\hat{\mathfrak{g}}_-)^{\mathrm{ad}\,\mathfrak{t}}$ is two-sided (see e.g. [100]). Hence the corresponding projection

$$\mathsf{AHC}\colon\ \mathbf{U}(\hat{\mathfrak{g}}_-)^{\mathrm{ad}\,\mathfrak{t}} \twoheadrightarrow \mathbf{U}(\mathfrak{t}\otimes t\mathbb{C}[t^{-1}])$$

is an algebra homomorphism, often called the *affine Harish-Chandra homomorphism.* Note that AHC is, moreover, a filtered homomorphism with respect to the PBW filtrations.

Lemma 4.9.2. *Let $1 \le k \le n$. The Segal-Sugawara vector $\mathbf{T}_k$ from Example 4.3.4 can be written as*

$$\mathbf{T}_k = \mathbf{P}_k + Q_k + Q_k',$$

where $\mathbf{P}_k := (e_{11}[-1])^k + \ldots + (e_{nn}[-1])^k$,

$$Q_k \in (\mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\le k-1})^{\mathrm{ad}\,\mathfrak{t}}, \quad Q_k' \in (\mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\le k})^{\mathrm{ad}\,\mathfrak{t}}$$

and $Q_k' \in \ker\mathsf{AHC}$.

Proof. Consider the algebra $\mathbf{U}(\hat{\hat{\mathfrak{g}}}_-)$ from Example 4.3.4 equipped with a modified PBW filtration in which τ has degree zero. One easily sees that the principal symbol of $\mathrm{Tr}(E_\tau^k)$ equals $\mathrm{Tr}(E^{(-1)})^k$, where $E^{(-1)} := (e_{ij}[-1])_{i,j=1}^n$ is a matrix with coefficients in $S(\hat{\mathfrak{g}}_-)$. But $\mathsf{gr}\,\mathsf{AHC}(\mathrm{Tr}(E^{(-1)})^k) = \mathbf{P}_k$. □

Definition 4.9.3. Suppose that $A \in \mathbf{U}(\hat{\mathfrak{g}}_-)$. We write $A_l := A_{\langle -l-1\rangle}$ so that $\mathbb{Y}\langle A, z\rangle = \sum_{l\in\mathbb{Z}} A_l z^l$ (note that the same notation was used with a different meaning in (4.19)). In particular, for $1 \le k \le n$, we write $\mathbf{T}_{k,l} := \mathbf{T}_{k,\langle -l-1\rangle}$ (not to be confused with $\mathbf{T}_{k;l}$ from Example 4.3.4). We also write

$$\widehat{A}_l := \widehat{\Phi}(A_l), \quad \overline{A}_l := \sigma^{\mathsf{abs}}(\widehat{A}_l),$$

where $\sigma^{\mathsf{abs}}\colon \mathbb{H}_c \to \mathsf{gr}\,\mathbb{H}_c$ is the principal symbol map with respect to the absolute height filtration and $\widehat{\Phi} : \widehat{\mathbf{U}}_c \twoheadrightarrow \widehat{\mathbf{U}}_c/\widehat{\mathbf{U}}_c.\mathfrak{I}_c = \mathbb{H}_c$ is the canonical map. If $v \in \mathbb{H}_c$, set $\deg v := \deg \sigma^{\mathsf{abs}}(v)$.

The proof of the following key proposition is rather technical and has been relegated to the appendix.

Proposition 4.9.4. *Let $1 \le k \le n$. Then:*

$$\widehat{\mathbf{T}}_{k,l} = 0 \quad (l < -2k), \qquad \widehat{\mathbf{T}}_{k,-2k} = \widehat{\mathbf{P}}_{k,-2k} = \sum_{i=1}^n (e_{ii}[1])^k.1_{\mathbb{H}},$$

$$\overline{\mathbf{T}}_{k,-2k+2+b} = \overline{\mathbf{P}}_{k,-2k+2+b} = k\sum_{i=1}^{n} e_{ii}[-b-1](e_{ii}[1])^{k-1}.1_{\mathbb{H}} \;+\; (\mathbb{H}_c)_{\leq k+b-1} \quad (b \geq 0).$$

Proof. See Appendix A. □

4.9.3. The main result. Recall from Theorem 2.1.4 that $\mathsf{gr}\,\mathsf{Z} = \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{S_n}$. The latter is known as the ring of diagonal invariants or multisymmetric polynomials. Given $a, b \in \mathbb{Z}_{\geq 0}$, the multisymmetric power-sum polynomial of degree (a,b) is defined as $\mathsf{p}_{a,b} := x_1^a y_1^b + \ldots + x_n^a y_n^b$. We call $a+b$ the total degree of $\mathsf{p}_{a,b}$.

Proposition 4.9.5. *The polynomials $\mathsf{p}_{a,b}$ with $a+b \leq n$ generate $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{S_n}$.*

Proof. See, e.g., [116, Corollary 8.4]. □

We are ready to prove our main result: the surjectivity of Θ. We also partially describe the kernel of Θ, compute Θ on Segal-Sugawara operators corresponding to $\mathbf{T}_1$ and $\mathbf{T}_2$, and compute the principal symbols of the images of "higher-order" Segal-Sugawara operators under Θ.

Theorem 4.9.6. *The map $\Theta\colon \mathfrak{Z} \to \mathsf{Z}$ is surjective with*

(i) $\Theta(\mathbf{T}_{k,l}) = 0 \quad (l < -2k)$,

(ii) $\Theta(\mathbf{T}_{1,l}) = \mathsf{p}_{l+1,0} \quad (l \geq 0)$,

(iii) $\Theta(\mathbf{T}_{2,l}) = -2\mathsf{p}_{l+3,1} + \sum_{i<j} 2c_{l+2}(x_i,x_j)s_{i,j} + ((n+1)l+3n+1)\sum_{i=1}^n x_i^{l+2} \quad (l \geq -2)$,

(iv) $\Theta(\mathbf{T}_{k,-2k}) = (-1)^k \mathsf{p}_{0,k}$,

(v) $\sigma(\Theta(\mathbf{T}_{k,-2k+2+b})) = (-1)^{k-1}k\mathsf{p}_{b+1,k-1} \quad (b \geq 0)$,

where $1 \leq k \leq n$, $c_r(x_i,x_j)$ is the complete homogeneous symmetric polynomial of degree r in x_i and x_j, and $\sigma\colon \mathsf{Z} \to \mathsf{gr}\,\mathsf{Z}$ is the principal symbol map.

Proof. Part (i) follows directly from Proposition 4.9.4, while (ii)-(iii) follow from Lemma 4.7.4 and the fact that $\mathbf{T}_2 = 2\cdot{}^c\mathbf{L} + \mathrm{id}[-2]$. Proposition 4.9.4 together with (4.50) implies that Υ^{-1} sends

$$[1_{\mathbb{C}[\mathfrak{h}]} \otimes e^*_{\mathsf{id}} \otimes \widehat{\mathbf{T}}_{k,-2k}] = [1_{\mathbb{C}[\mathfrak{h}]} \otimes e^*_{\mathsf{id}} \otimes \sum_{i=1}^n (e_{ii}[1])^k.1_{\mathbb{H}}] \;\mapsto\; (-1)^k \mathsf{p}_{0,k},$$

which proves (iv). Moreover, Proposition 4.9.4 together with (4.30) and (4.50) implies that $\mathsf{gr}\,\Upsilon^{-1}$ sends

$$[1_{\mathbb{C}[\mathfrak{h}]} \otimes e^*_{\mathsf{id}} \otimes \overline{\mathbf{T}}_{k,-2k+2+b}] = k\sum_{i=1}^n x_i^{b+1} \otimes e^*_{\mathsf{id}} \otimes (e_{ii}[1])^{k-1}.1_{\mathbb{H}} \;\mapsto\; (-1)^{k-1}k\mathsf{p}_{b+1,k-1},$$

which proves (v) because $\mathsf{gr}\,\Psi(\overline{\mathbf{T}}_{k,r}) = \sigma(\Theta(\mathbf{T}_{k,r}))$ for $r \geq -2k+2$.

It follows from (iv) and (v) that the multisymmetric power-sum polynomials of total degree $\leq n$ all lie in the image of $\mathsf{gr}\,\Psi$. But, by Proposition 4.9.5, these polynomials generate $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{S_n} = \mathsf{gr}\,\mathsf{Z}$. Hence the map $\mathsf{gr}\,\Psi\colon \mathsf{gr}\,\mathrm{Im}\,\Phi \to \mathsf{gr}\,\mathsf{Z}$ is surjective. By [129, Lemma 1(e)], the map $\Psi\colon \mathrm{Im}\,\Theta \to \mathsf{Z}$ is surjective as well because the filtration on Z is exhaustive and discrete. The surjectivity of $\Theta = \Psi\circ\Phi$ follows. □

4.10 Applications and connections to other topics

We present several applications of Theorem 4.9.6. Assume that $n = m$ throughout.

4.10.1. Endomorphism rings and simple modules. We prove that the homomorphisms between endomorphism rings of Weyl and Verma modules induced by the Suzuki functor are surjective and use this fact to show that every simple $\mathtt{H}_0$-module is in the image of F_c.

Corollary 4.10.1. *The functor F_c induces surjective ring homomorphisms:*

$$\mathsf{F}_c\colon \operatorname{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(a,\lambda)) \to \operatorname{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda)), \tag{4.86}$$

for $l \geq 1$, $\nu \in \mathcal{C}_l(n)$, $\lambda \in \mathcal{P}_n(\nu)$ and $a \in \mathfrak{h}^*$ with $S_n(a) = S_\nu$; and

$$\mathsf{F}_c\colon \operatorname{End}_{\widehat{\mathbf{U}}_c}(\mathbb{M}_c(\lambda)) \to \operatorname{End}_{\mathtt{H}_0}(\Delta_0(\lambda)), \tag{4.87}$$

for $\lambda \in \mathcal{P}(n)$. Moreover, the homomorphisms (4.87) *are graded.*

Proof. The existence of the ring homomorphisms (4.86) and (4.87) follows from the fact that $\mathsf{F}_c(\mathbb{W}_c(a,\lambda)) \cong \Delta_0(a,\lambda)$ (Theorem 4.6.12) and $\mathsf{F}_c(\mathbb{M}_c(\lambda)) \cong \Delta_0(\lambda)$ (Theorem 4.6.13). Let us prove their surjectivity. Corollary 4.7.10 implies that we have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z} & \xrightarrow{\quad\Theta\quad} & \mathtt{Z} \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ \operatorname{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(a,\lambda)) & \xrightarrow{\mathsf{F}_c} & \operatorname{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda)) \end{array}$$

By Theorem 4.9.6, Θ is surjective, and, by Theorem 4.6.8.b), the right vertical map is surjective as well. Hence the lower horizontal map must be surjective, too. The proof in the case of the Verma modules $\mathbb{M}_c(\lambda)$ is analogous. The fact that (4.87) is a graded homomorphism follows from Proposition 4.8.7. □

We need the following lemma.

Lemma 4.10.2. *Let M be a $\widehat{\mathbf{U}}_c$-module and $A \subseteq \operatorname{End}_{\widehat{\mathbf{U}}_c}(M)$ be a vector subspace. Then*

$$\mathsf{F}_c(M/AM) = \mathsf{F}_c(M)/\mathsf{F}_c(A)\mathsf{F}_c(M).$$

Proof. Let $\mathcal{B}$ be a basis of A. By definition, $M/AM = M/\sum_{f\in\mathcal{B}} \operatorname{Im} f$. Consider the exact sequence

$$\bigoplus_{f\in\mathcal{B}} M \xrightarrow{\oplus_{f\in\mathcal{B}} f} M \to M/\sum_{f\in\mathcal{B}} \operatorname{Im} f \to 0.$$

By Remark 4.5.10, the functor F_c preserves colimits. In particular, it preserves (possibly infinite) direct sums and cokernels. Hence

$$\begin{aligned} \mathsf{F}_c(M/\textstyle\sum_{f\in\mathcal{B}} \operatorname{Im} f) &= \mathsf{F}_c(\operatorname{coker}(\oplus_{f\in\mathcal{B}} f)) \\ &= \operatorname{coker}(\oplus_{f\in\mathcal{B}} \mathsf{F}_c(f)) = \mathsf{F}_c(M)/\textstyle\sum_{f\in\mathcal{B}} \operatorname{Im} \mathsf{F}_c(f). \end{aligned}$$

But $\sum_{f\in\mathcal{B}} \operatorname{Im} \mathsf{F}_c(f) = \mathsf{F}_c(A)\mathsf{F}_c(M)$. □

Corollary 4.10.3. *Every simple* $\mathtt{H}_0$*-module is in the image of the functor* F_c*.*

Proof. Let L be a simple $\mathtt{H}_0$-module. By Lemma 4.6.9, there exists a generalized Verma module $\Delta_0(a,\lambda)$ such that $L \cong \Delta_0(a,\lambda)/I \cdot \Delta_0(a,\lambda)$ for some ideal $I \subset \mathrm{End}_{\mathtt{H}_0}(\Delta_0(a,\lambda))$. Let $J := \mathsf{F}_c^{-1}(I) \subset \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(a,\lambda))$. Corollary 4.10.1 implies that $\mathsf{F}_c(J) = I$. Hence, by Lemma 4.10.2,

$$\mathsf{F}_c(\mathbb{W}_c(a,\lambda)/J \cdot \mathbb{W}_c(a,\lambda)) = \Delta_0(a,\lambda)/I \cdot \Delta_0(a,\lambda) \cong L. \qquad \square$$

Remark 4.10.4. When $\kappa \neq c$, it has been shown (see [132, Theorem 4.3] and [138, Theorem A.5.1]) that, under some mild assumptions, every simple $\mathtt{H}_{\kappa+n}$-module in category $\mathcal{O}(\mathtt{H}_{\kappa+n})$ is in the image of F_κ. It is noteworthy that the proofs in [132] and [138] employ very different techniques from those used by us in the $\kappa = c$ case.

4.10.2. Restricted Verma and Weyl modules. We are going to compute the Suzuki functor on restricted Verma and Weyl modules as well as the simple modules in category $\mathcal{O}$.

Consider the algebra $\mathscr{Z}$ from (4.16) equipped with the natural $\mathbb{Z}$-grading induced from $\mathbf{U}_c(\hat{\mathfrak{g}})$. In [2, §3.2], Arakawa and Fiebig consider the *restriction functor*

$$\mathscr{C}_c \to \mathscr{C}_c, \quad M \mapsto \overline{M} := M/\sum_{0\neq i\in\mathbb{Z}} \mathscr{Z}_i \cdot M. \tag{4.88}$$

This functor is right exact because it is left adjoint to the invariants functor $M \mapsto \underline{M} := \{m \in M \mid z\cdot m = 0 \text{ for all } z \in \mathscr{Z}_i, i \neq 0\}$. Given $\mu \in \mathfrak{t}^*$, in [2, §3.5], Arakawa and Fiebig define the corresponding *restricted Verma module* as $\overline{\mathbb{M}}_c(\mu)$. By [2, Lemma 3.5],

$$\overline{\mathbb{M}}_c(\mu) = \mathbb{M}_c(\mu)/\mathscr{Z}_- \cdot \mathbb{M}_c(\mu),$$

where $\mathscr{Z}_- = \bigoplus_{i<0} \mathscr{Z}_i$.

Consider $\mathbb{M}_c(\mu)$ as a graded $\hat{\mathfrak{g}}_c$-module with the subspace $\mathbb{C}_{\lambda,1} \subset \mathrm{Ind}_{\hat{\mathfrak{b}}_+}^{\hat{\mathfrak{g}}_\kappa} \mathbb{C}_{\lambda,1}$ lying in degree zero, or, equivalently, as a module over the Kac-Moody algebra $\hat{\mathfrak{g}}_c \rtimes \mathbb{C}{}^c\mathbf{L}_0$, with the Segal-Sugawara operator ${}^c\mathbf{L}_0$ (see (4.19)) acting by zero on $\mathbb{C}_{\lambda,1}$. It is known (see, e.g., [79, Proposition 9.2.c)] that $\mathbb{M}_c(\mu)$ has a unique graded simple quotient $\mathbb{L}(\mu)$.

Lemma 4.10.5. *If* $\mu \notin \mathcal{P}(n) \subset \mathfrak{t}^*$ *then* $\mathsf{F}_c(\mathbb{L}(\mu)) = 0$*.*

Proof. By Theorem 4.6.13, the module $\mathbb{M}_c(\mu)$ is killed by F_c. Since F_c is right exact, its quotient $\mathbb{L}(\mu)$ is killed as well. $\square$

We also consider $\Delta_0(\lambda)$, for $\lambda \in \mathcal{P}(n)$, as a graded $\mathtt{H}_0$-module. It follows from [63, Proposition 4.3] that $\Delta_0(\lambda)$ has a unique graded simple quotient L_λ (not to be confused with $L(\lambda)$ from §4.2.4).

Corollary 4.10.6. *Let* $\lambda \in \mathcal{P}(n)$*. Then* $\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) \cong \mathsf{F}_c(\mathbb{L}(\lambda)) \cong L_\lambda$*.*

Proof. Consider the short exact sequence

$$0 \to K \to \overline{\mathbb{M}}_c(\lambda) \to \mathbb{L}(\lambda) \to 0.$$

By [1, Lemma 4.2(5)], K has a (possibly infinite) filtration with each subquotient isomorphic to a graded shift of a simple module of the form $\mathbb{L}(w\cdot\lambda)$, where $e \neq w \in S_n$ and $w\cdot\lambda = w(\lambda+\rho)-\rho$. In particular, none of the weights $w\cdot\lambda$ are dominant. Therefore, there exists a surjective homomorphism from a direct sum of graded shifts of Verma modules of the form $\mathbb{M}_c(w\cdot\lambda)$, one for each subquotient in the filtration, to K. By Theorem 4.6.13, this direct sum of Verma modules is killed by F_c. Right exactness implies that $\mathsf{F}_c(K) = 0$. Hence, by another application of right exactness, $\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) \cong \mathsf{F}_c(\mathbb{L}(\lambda))$.

We next prove that $\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) \cong L_\lambda$. Abbreviate $\mathbb{E}_\lambda := \operatorname{Im}\mathscr{Z} \subset \operatorname{End}_{\widehat{\mathbf{U}}_c}(\mathbb{M}_c(\lambda))$ and $E_\lambda := \operatorname{End}_{\mathtt{H}_0}(\Delta_0(\lambda))$. These rings are $\mathbb{Z}_{\leq 0}$-graded. Let $\mathbb{E}_\lambda^- \lhd \mathbb{E}_\lambda$ and $E_\lambda^- \lhd E_\lambda$ denote their maximal graded ideals. It follows from the proof of Corollary 4.10.1 that the restriction of (4.87) to $\mathbb{E}_\lambda$ is surjective (in fact, by [54, Theorem 9.5.3], $\mathbb{E}_\lambda = \operatorname{End}_{\widehat{\mathbf{U}}_c}(\mathbb{M}_c(\lambda))$, but we do not need to use this fact). Since (4.87) is a graded homomorphism, it follows that $\mathsf{F}_c(\mathbb{E}_\lambda^-) = E_\lambda^-$. Therefore, Lemma 4.10.2 implies that

$$\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) = \mathsf{F}_c(\mathbb{M}_c(\lambda)/\mathbb{E}_\lambda^- \cdot \mathbb{M}_c(\lambda)) = \Delta_0(\lambda)/E_\lambda^- \cdot \Delta_0(\lambda).$$

Arguing as in the proof of Lemma 4.6.9, one concludes that $\Delta_0(\lambda)/E_\lambda \cdot \Delta_0(\lambda) = L_\lambda$. □

Given $\lambda \in \mathcal{P}(n)$, we define the corresponding *restricted Weyl module* to be $\overline{\mathbb{W}}_c(\lambda)$. Since $\mathscr{Z}_+ = \bigoplus_{i>0} \mathscr{Z}_i$ annihilates $\mathbb{W}_c(\lambda)$, we have

$$\overline{\mathbb{W}}_c(\lambda) = \mathbb{W}_c(\lambda)/\mathscr{Z}_- \cdot \mathbb{W}_c(\lambda).$$

Corollary 4.10.7. *Let $\lambda \in \mathcal{P}(n)$. Then $\mathsf{F}_c(\overline{\mathbb{W}}_c(\lambda)) \cong L_\lambda$.*

Proof. Let $M(\lambda)$ denote the Verma module over $\mathfrak{g}$ with highest weight λ. The canonical surjection $M(\lambda) \twoheadrightarrow L(\lambda)$ induces a surjection $\mathbb{M}_c(\lambda) = \operatorname{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}_\kappa} L(\lambda) \twoheadrightarrow \operatorname{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}_\kappa} M(\lambda) = \mathbb{W}_c(\lambda)$. Let K denote its kernel. The functor F_c sends the exact sequence $0 \to K \to \mathbb{M}_c(\lambda) \to \mathbb{W}_c(\lambda) \to 0$ to the exact sequence $\mathsf{F}_c(K) \to \Delta_0(\lambda) \xrightarrow{f} \Delta_0(\lambda) \to 0$. But $\Delta_0(\lambda)$ is a cyclic $\mathtt{H}_0$-module, so f must be an isomorphism. It follows that $\mathsf{F}_c(K) = 0$. Moreover, $\mathsf{F}_c(\overline{K}) = 0$ because $\overline{K}$ is a quotient of K.

Since the restriction functor (4.88) is right exact, we also have an exact sequence $\overline{K} \to \overline{\mathbb{M}}_c(\lambda) \to \overline{\mathbb{W}}_c(\lambda) \to 0$. The functor F_c sends it to the exact sequence $0 = \mathsf{F}_c(\overline{K}) \to L_\lambda \to \mathsf{F}_c(\overline{\mathbb{W}}_c(\lambda)) \to 0$ because $\mathsf{F}_c(\overline{\mathbb{M}}_c(\lambda)) \cong L_\lambda$, by Corollary 4.10.6. It follows that $\mathsf{F}_c(\overline{\mathbb{W}}_c(\lambda)) \cong L_\lambda$. □

4.10.3. Poisson brackets. Suppose that A is an algebraic deformation of an associative algebra A_0, i.e., A is a free $\mathbb{C}[\hbar]$-algebra such that $A/\hbar A = A_0$. Then there is a canonical Poisson bracket on $Z(A_0)$, called the Hayashi bracket, given by

$$\{a, b\} := \frac{1}{\hbar}[\tilde{a}, \tilde{b}] \mod \hbar,$$

where $\tilde{a}, \tilde{b}$ are arbitrary lifts of a and b, respectively. This Poisson bracket was introduced by Hayashi in [71]. Applying this construction to $\widehat{\mathbf{U}}_c$ and $\mathtt{H}_t$, we get Poisson brackets on $\mathfrak{Z}$ and $\mathtt{Z}$.

Lemma 4.10.8. *The vector space spanned by* 1, $\mathrm{id}[r]$ *and* ${}^c\mathbf{L}_r$ *is, under the Poisson bracket, a Lie subalgebra of* $\mathfrak{Z}$ *isomorphic to the semidirect product of the Heisenberg algebra with the Virasoro algebra. Moreover, the subspace spanned by* $\mathrm{id}[r]$ *and* ${}^c\mathbf{L}_{r+1}$ $(r \leq 0)$ *is a Lie subalgebra.*

Proof. This follows from, e.g., [54, (3.1.3)]. □

By Lemma 4.10.8, the algebra $\mathscr{L}_c$ from (4.72) is a Poisson subalgebra of $\mathfrak{Z}$. Since the generators ${}^\kappa\mathbf{L}_{r+1}$, $\mathrm{id}[r]$ $(r \leq 0)$ of $\mathscr{L}_\kappa$ are defined for any κ, they have canonical lifts to $\widehat{\mathbf{U}}_{\mathbb{C}[t]}$. Let $\mathscr{L}_{\mathbb{C}[t]}$ be the $\mathbb{C}[t]$-subalgebra of $\widehat{\mathbf{U}}_{\mathbb{C}[t]}$ generated by them. The map $\rho_t \circ \mathsf{F}_\kappa|_{\mathscr{L}_\kappa^{op}}$ from Lemma 4.7.4 also lifts to a map $\rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]}|_{\mathscr{L}_{\mathbb{C}[t]}^{op}} : \mathscr{L}_{\mathbb{C}[t]}^{op} \to \mathtt{H}_{\mathbb{C}[t]}^{op}$.

Theorem 4.10.9. *The map* $\Theta \colon \mathscr{L}_c \to \mathtt{Z}$ *is a homomorphism of Poisson algebras.*

Proof. It follows from Lemma 4.7.4.c), Theorem 4.7.5 and Theorem 4.7.9 that we can identify $\Theta|_{\mathscr{L}_c}$ with $\rho_0 \circ \mathsf{F}_c|_{\mathscr{L}_c}$. Since Θ is an algebra homomorphism, it suffices to check that Θ preserves the Poisson bracket on multiplicative generators of $\mathscr{L}_c$. Let a_c, b_c be any two of the generators ${}^c\mathbf{L}_{r+1}$, $\mathrm{id}[r]$ $(r \leq 0)$ and let a and b be their canonical lifts to $\mathscr{L}^{op}_{\mathbb{C}[t]}$. Let us interpret a_c and b_c as endomorphisms of $\widehat{\mathbf{U}}_c$. Then

$$\begin{aligned}
\Theta(\{a_c, b_c\}) &= \rho_0 \circ \mathsf{F}_c(\{a_c, b_c\}) \\
&= - \ \rho_0 \circ \mathsf{F}_c \left(\mathsf{spec}_{t=0} \left(\frac{1}{t}[a, b] \right) \right) \\
&= \ - \mathsf{spec}_{t=0} \left(\frac{1}{t} [\rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]}(a), \rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]}(b)] \right) \\
&= \{\rho_0 \circ \mathsf{F}_c(a_c), \rho_0 \circ \mathsf{F}_c(b_c)\} = \{\Theta(a_c), \Theta(b_c)\}
\end{aligned}$$

The second equality follows from the definition of the Poisson bracket. The third equality follows from the easily verifiable fact that $\mathsf{spec}_{t=0} \circ \rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]} = \rho_0 \circ \mathsf{F}_c \circ \mathsf{spec}_{t=0}$. The fourth equality follows from part b) of Lemma 4.7.4, which implies that $\rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]}(a)$ and $\rho_{\mathbb{C}[t]} \circ \mathsf{F}_{\mathbb{C}[t]}(b)$ are, respectively, lifts of $\rho_0 \circ \mathsf{F}_c(a)$ and $\rho_0 \circ \mathsf{F}_c(b)$ to $\mathtt{H}^{op}_{\mathbb{C}[t]}$. The minus signs in the second and third lines arise because we work with lifts in the opposite algebras. □

Remark 4.10.10. It would be interesting to know whether there exists a bigger subalgebra $\mathscr{L}_c \subset A \subset \mathfrak{Z}$ such that $\Theta|_A$ is a homomorphism of Poisson algebras.

Remark 4.10.11. The image of the "grading element" ${}^c\mathbf{L}_0$ under Θ is the so-called Euler element **eu** in $\mathtt{Z}$. Moreover, since ${}^c\mathbf{L}_1, -2{}^c\mathbf{L}_0, -{}^c\mathbf{L}_{-1}$ form an $\mathfrak{sl}_2$-triple under the Poisson bracket, we obtain an $\mathfrak{sl}_2$-action on $\mathtt{Z}$. This action is not integrable, in contrast to the well-studied ([11, 15]) action of the $\mathfrak{sl}_2$-triple $\sum_i x_i^2$, **eu**, $\sum_i y_i^2$. For example, the subspace of $\mathtt{Z}$ spanned by $\sum_i x_i^r$ $(r \geq 0)$ is isomorphic to the contragredient Verma module of weight zero while the subspace spanned by $\Theta({}^c\mathbf{L}_r)$ $(r \leq 1)$ is isomorphic to the contragredient Verma module of weight two. It would be interesting to know in more detail how $\mathtt{Z}$ decomposes under our $\mathfrak{sl}_2$-action.

4.10.4. A description of Θ in terms of opers We are going to show that Θ induces an embedding of the Calogero-Moser space into the space of opers on the punctured disc and describe some of its properties. Let us first introduce some notation. Set $\mathbb{D} := \operatorname{Spec} \mathbb{C}[[t]]$ and $\mathbb{D}^\times := \operatorname{Spec} \mathbb{C}((t))$. Let $B \subset G$ be the standard Borel subgroup and $N := [B, B]$.

The notion of a G-oper on $\mathbb{D}^\times$ was introduced by Drinfeld and Sokolov in [41]. It was later generalized by Beilinson and Drinfeld in [6] for arbitrary smooth curves. Roughly speaking, a G-oper is a triple consisting of a principal G-bundle, a connection as well as a reduction of the structure group to B, satisfying a certain transversality condition.

We will work with an explicit description of G-opers on $\mathbb{D}^\times$ from [41, §3] in terms of certain operators (see also [54, §4.2.2]), which we now recall. Let $\mathrm{Loc}'_G(\mathbb{D}^\times)$ be the space of operators of the form

$$\nabla = \partial_t + u(t), \quad u(t) \in \mathfrak{g}((t)).$$

There is an action of $G((t))$ on $\mathrm{Loc}_G(\mathbb{D}^\times)$ by the rule $g \cdot (\partial_t + A(t)) = \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g$. Elements of the orbit space $\mathrm{Loc}_G(\mathbb{D}^\times) = \mathrm{Loc}'_G(\mathbb{D}^\times)/G((t))$ are called *G-local systems* on $\mathbb{D}^\times$. Let $\mathrm{Op}_G(\mathbb{D}^\times)$ be the space of $N((t))$-equivalence classes of operators of the form

$$\nabla = \partial_t + p_{-1} + v(t), \quad v(t) \in \mathfrak{b}((t)),$$

where $p_{-1} = e_{2,1} + \ldots + e_{n,n-1} \in \mathfrak{g}$. Elements of $\mathrm{Op}_G(\mathbb{D}^\times)$ are called *G-opers* on $\mathbb{D}^\times$. There is a natural map $\mathrm{Op}_G(\mathbb{D}^\times) \to \mathrm{Loc}_G(\mathbb{D}^\times)$ sending an $N((t))$-equivalence class to a $G((t))$-equivalence class. An oper has *trivial monodromy* if it is in the $G((t))$-orbit of the local system ∂_t. Let $\mathrm{Op}_G(\mathbb{D}^\times)^0$ denote the subspace of opers with trivial monodromy.

A G-oper on $\mathbb{D}$ with *singularity of order* at most r (see [6, §3.8.8]), where $r \geq 1$, is an $N[[t]]$-equivalence class of operators of the form

$$\nabla = \partial_t + t^{-r}(p_{-1} + v(t)), \quad v(t) \in \mathfrak{b}[[t]]. \tag{4.89}$$

Let $\mathrm{Op}_G^{\leqslant r}(\mathbb{D})$ be the space of all such G-opers. By [6, Proposition 3.8.9], the natural map $\mathrm{Op}_G^{\leqslant r}(\mathbb{D}) \to \mathrm{Op}_G(\mathbb{D}^\times)$ sending an $N[[t]]$-equivalence class of operators to their $N((t))$-equivalence class is injective. The space $\mathrm{Op}_G^{\leqslant r}(\mathbb{D})$ can be endowed with the structure of a scheme and $\mathrm{Op}_G(\mathbb{D}^\times)$ with the structure of an ind-scheme (see, e.g., [6, §3.1.11]).

For an operator (4.89), its r-th residue ($r \geq 1$) is defined in [51, §4.3] as $\mathrm{Res}_r(\nabla) := p_{-1} + v(0)$. Under conjugation by an element $A(t) \in N[[t]]$, $\mathrm{Res}_r(\nabla)$ is conjugated by $A(0)$. Hence the projection of $\mathrm{Res}_r(\nabla)$ onto $\mathfrak{g}/G \cong \mathfrak{t}/S_n$ (identified via the Chevalley isomorphism) is well defined, and we have a map

$$\mathrm{Res}_r\colon\ \mathrm{Op}_G^{\leqslant r}(\mathbb{D}) \to \mathfrak{t}/S_n.$$

For each $z \in \mathfrak{t}/S_n$, let $\mathrm{Op}_G^{\leqslant r}(\mathbb{D})_z := \mathrm{Res}_r^{-1}(z)$.

Let $\check{G}$ denote the Langlands dual of G. Let $\mathrm{Op}_{\check{G}}(\mathbb{D}^\times)$ be the space obtained by replacing all the algebraic groups and Lie algebras by their Langlands duals in the definitions above. Noting that $\check{\mathfrak{t}} = \mathfrak{t}^*$, let

$$\varpi\colon \mathfrak{t}^* \to \mathfrak{t}^*/S_n = \check{\mathfrak{t}}/S_n, \quad \vartheta\colon \mathfrak{g}^* \to \mathfrak{g}^*/G \cong \mathfrak{t}^*/S_n = \check{\mathfrak{t}}/S_n$$

be the canonical projections. For $\lambda \in \Pi^+$, we abbreviate

$$\mathrm{Op}_{\check{G}}^{\lambda}(\mathbb{D}) := \mathrm{Op}_{\check{G}}^{\leqslant 1}(\mathbb{D})^0_{\varpi(-\lambda-\rho)}.$$

We are next going to recall the connection between opers and the algebra $\mathfrak{Z}$. Consider $\mathfrak{Z}$ as a graded algebra, with the grading induced by the grading on $\widehat{\mathbf{U}}_c$, and, moreover, as a filtered algebra, with the filtration induced by the PBW filtration on $\widehat{\mathbf{U}}_c$. Let $\mathfrak{Z}^{\leqslant r}(\hat{\mathfrak{g}})$ be the quotient of $\mathfrak{Z}$ by the ideal topologically generated by elements of graded degree i and PBW degree j, satisfying $-i < j(1-r)$.

Theorem 4.10.12. *The following hold.*

a) *There is a canonical algebra isomorphism*

$$\mathfrak{Z} \cong \mathbb{C}[\mathrm{Op}_{\check{G}}(\mathbb{D}^\times)]. \tag{4.90}$$

b) *The isomorphism* (4.90) *induces, for each* $r \geq 0$, *isomorphisms*

$$\mathfrak{Z}^{\leqslant r}(\hat{\mathfrak{g}}) \cong \mathbb{C}[\mathrm{Op}_{\check{G}}^{\leqslant r}(\mathbb{D})]$$

c) *For each* $\lambda \in \Pi^+$, *the canonical map* $\mathfrak{Z} \to \mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(\lambda))$ *is surjective. Moreover,*

$$\mathrm{End}_{\widehat{\mathbf{U}}_c}(\mathbb{W}_c(\lambda)) \cong \mathbb{C}[\mathrm{Op}_{\check{G}}^{\lambda}(\mathbb{D})].$$

Proof. Part a) is [54, Theorem 4.3.6], part b) is [6, Proposition 3.8.6] and part c) is [54, Theorem 9.6.1]. □

For $\chi \in \mathfrak{g}^* \cong \mathfrak{g}[r-1]^*$, let $\mathbb{I}_{r,\chi} := \operatorname{Ind}_{\hat{\mathfrak{g}}_{\geq r-1}\oplus\mathbb{C}\mathbf{1}}^{\hat{\mathfrak{g}}_c} \mathbb{C}_\chi$, with $\hat{\mathfrak{g}}_{\geq r-1}$ acting on $\mathbb{C}_\chi$ via $\hat{\mathfrak{g}}_{\geq r-1} \twoheadrightarrow \mathfrak{g}[r-1] \xrightarrow{\chi} \mathbb{C}$ and $\mathbf{1}$ acting as the identity. Set $\mathbb{U}_r := \mathbb{I}_{r+1,0}$.

Theorem 4.10.13 ([51, Theorem 5.6.(1)-(2)]). *We have*

$$\operatorname{supp}_{\mathfrak{Z}} \mathbb{U}_r \subseteq \operatorname{Op}_{\check{G}}^{\leqslant r}(\mathbb{D}), \quad \operatorname{supp}_{\mathfrak{Z}} \mathbb{I}_{r,\chi} \subseteq \operatorname{Op}_{\check{G}}^{\leqslant r}(\mathbb{D})_{\vartheta(\chi)}.$$

Let us identify $\mathfrak{t}^* \cong \mathfrak{h}^*$ via (4.47) and $\mathfrak{t} \cong \mathfrak{t}[1]$, $z \mapsto z[1]$. Recall the map π and the varieties $\Omega_{\mathbf{a},\lambda}$ from §4.6.2. The following corollary gives a partial description of Θ in terms of opers.

Corollary 4.10.14. *The following hold.*

a) *The map* $\Theta\colon \mathfrak{Z} \to \mathtt{Z}$ *induces a closed embedding*

$$\Theta^*\colon \operatorname{Spec}\mathtt{Z} \hookrightarrow \operatorname{Op}_{\check{G}}(\mathbb{D})^{\leqslant 2}.$$

b) *Let* $l \geq 1$, $\nu \in \mathcal{C}_l(n)$, $\lambda \in \mathcal{P}_n(\nu)$, $a \in \mathfrak{h}^*$ *with* $S_n(a) = S_\nu$ *and* $\mathbf{a} = \varpi(a)$. *We have*

$$\Theta^*(\Omega_{\mathbf{a},\lambda}) \subseteq \operatorname{Op}_{\check{G}}^{\leqslant 2}(\mathbb{D})_{\mathbf{a}}.$$

Hence the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec}\mathtt{Z} & \xhookrightarrow{\Theta^*} & \operatorname{Op}_{\check{G}}(\mathbb{D})^{\leqslant 2} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\operatorname{Res}_2} \\ \mathfrak{h}^*/S_n & \xrightarrow{\sim} & \mathfrak{t}^*/S_n \end{array} \tag{4.91}$$

c) *If* $\mathbf{a} = 0$ *then*

$$\Theta^*(\Omega_\lambda) \subseteq \operatorname{Op}_{\check{G}}^{\lambda}(\mathbb{D}). \tag{4.92}$$

Proof. By Theorem (4.9.6), Θ is surjective, so it induces a closed embedding $\Theta^*\colon \operatorname{Spec}\mathtt{Z} \hookrightarrow \operatorname{Op}_{\check{G}}(\mathbb{D}^\times)$. Corollary 4.7.10 implies that

$$\Theta^*(\operatorname{Spec}\mathtt{Z}) = \Theta^*(\operatorname{supp}_{\mathtt{Z}}(\mathtt{H}_0)) \subseteq \operatorname{supp}_{\mathfrak{Z}} \mathbb{H}_c.$$

Since $\mathbb{H}_c$ is a quotient of $\mathbb{U}_2$, it follows from Theorem 4.10.13 that

$$\operatorname{supp}_{\mathfrak{Z}} \mathbb{H}_c \subseteq \operatorname{supp}_{\mathfrak{Z}} \mathbb{U}_2 \subseteq \operatorname{Op}_{\check{G}}^{\leqslant 2}(\mathbb{D}).$$

This proves part a). Let us prove part b). Corollary 4.7.10 implies that

$$\Theta^*(\operatorname{supp}_{\mathtt{Z}}(\Delta_0(a,\lambda)) \subseteq \operatorname{supp}_{\mathfrak{Z}} \mathbb{W}_c(a,\lambda). \tag{4.93}$$

If we take $\chi \in \mathfrak{g}[1]^*$ with $\chi|_{\mathfrak{n}_-[1]\oplus\mathfrak{n}_+[1]} = 0$ and $\chi|_{\mathfrak{t}[1]} = a$ then $\mathbb{W}_c(a,\lambda)$ is a quotient of $\mathbb{I}_{2,\chi}$. Hence Theorem 4.10.13 implies that

$$\operatorname{supp}_{\mathfrak{Z}} \mathbb{W}_c(a,\lambda) \subseteq \operatorname{supp}_{\mathfrak{Z}} \mathbb{I}_{2,\chi} \subseteq \operatorname{Op}_{\check{G}}^{\leqslant 2}(\mathbb{D})_{\mathbf{a}}.$$

The commutativity of the diagram (4.91) now follows directly from Proposition 4.6.10. Let us next prove part c). As a special case of (4.93), we have $\Theta^*(\operatorname{supp}_{\mathtt{Z}}(\Delta_0(\lambda)) \subseteq \operatorname{supp}_{\mathfrak{Z}} \mathbb{W}_c(\lambda)$. Theorem

4.10.12.c) implies that

$$\operatorname{supp}_{\mathfrak{Z}} \mathbb{W}_c(\lambda) = \mathrm{Op}^{\lambda}_{\check{G}}(\mathbb{D}),$$

completing the proof. □

4.10.5. Extensions and differential forms. Let $\kappa \in \mathbb{C}$. We are going to show that the first derived functor of F_κ vanishes on modules which admit a filtration by Weyl modules. We also formulate a conjecture that F_c induces a map between certain extension algebras.

We say that a $\widehat{\mathbf{U}}_\kappa$-module has a Δ-*filtration* if it has a finite filtration with each subquotient isomorphic to $\mathbb{W}_\kappa(\lambda)$ for some $\lambda \in \mathcal{P}(n)$. Let $\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$ be the full subcategory of $\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}$ consisting of modules with a Δ-filtration.

Proposition 4.10.15. *We have $L^1\mathsf{F}_\kappa(M) = 0$ for all $M \in \widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$. Hence F_κ is exact on $\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$.*

Proof. Consider the augmentation map $\varepsilon\colon \mathbf{U}(\mathfrak{g}) \to \mathbb{C}$. Tensoring with $\mathbb{C}$ over $\mathbf{U}(\mathfrak{sl}_n)$ we obtain a map $\varepsilon'\colon \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{sl}_n)} \mathbb{C} \to \mathbb{C}$. Let $K := \ker \varepsilon'$. By [72, Proposition VI.16.1], adapted to the Lie algebra homology setting, we have a long exact sequence

$$H_1(\mathfrak{sl}_n, N) \xrightarrow{\mathsf{cores}} H_1(\mathfrak{g}, N) \to N \otimes_{\mathbf{U}(\mathfrak{g})} K \to H_0(\mathfrak{sl}_n, N) \xrightarrow{\mathsf{cores}} H_0(\mathfrak{g}, N) \to 0 \tag{4.94}$$

for any $U(\mathfrak{g})$-module N, where cores is the corestriction map. If N is finite-dimensional then, by Whitehead's first lemma (see e.g. [72, Proposition VII.6.1]), $H_1(\mathfrak{sl}_n, N) = 0$. If, moreover, the corestriction map $H_0(\mathfrak{sl}_n, N) \to H_0(\mathfrak{g}, N)$ is an isomorphism, the long exact sequence (4.94) forces $H_1(\mathfrak{g}, N) = 0$.

Now let $\lambda \in \mathcal{P}(n)$ and take $N = (\mathbf{V}^*)^{\otimes n} \otimes L(\lambda)$. We claim that the corestriction map is an isomorphism. We need to show that $\mathfrak{sl}_n \cdot N = \mathfrak{g} \cdot N$, which is equivalent to showing that any trivial $\mathfrak{sl}_n$-submodule of N is also trivial as a $\mathfrak{g}$-module. If $\mu = \sum_i a_i \epsilon_i$ is a weight of $(\mathbf{V}^*)^{\otimes n}$ then $\phi(\mu) := \sum_i a_i = -n$. Similarly, if μ is a weight of $L(\lambda)$, then $\phi(\mu) = n$. Hence, for any weight μ of N, we must have $\phi(\mu) = 0$. But a non-trivial $\mathfrak{g}$-module which is trivial when restricted to $\mathfrak{sl}_n$ must have weights of the form $\chi = a \sum_i \epsilon_i$ for $0 \neq a \in \mathbb{Z}$, which implies that $\phi(\chi) \neq 0$. This proves the claim. It follows that

$$H_1(\mathfrak{g}, (\mathbf{V}^*)^{\otimes n} \otimes L(\lambda)) = 0. \tag{4.95}$$

Since homology commutes with induction, using the tensor identity and arguing as in the proof of Proposition 4.6.16, one shows that

$$L^i\mathsf{F}_\kappa(\mathbb{W}_\kappa(\lambda)) = H_i(\mathfrak{g}[t], \mathsf{T}_\kappa(\mathbb{W}_\kappa(\lambda))) = \mathbb{C}[\mathfrak{h}] \otimes H_i(\mathfrak{g}, (\mathbf{V}^*)^{\otimes n} \otimes L(\lambda)).$$

Together with (4.95), this implies that $L^1\mathsf{F}_\kappa(\mathbb{W}_\kappa(\lambda)) = 0$. One shows that $L^1\mathsf{F}_\kappa(M) = 0$ for all $M \in \widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$ by induction on the length of the Δ-filtration. □

Corollary 4.10.16. *The functor F_κ induces a linear map*

$$\mathrm{Ext}^1_{\widehat{\mathbf{U}}_\kappa}(M, M) \to \mathrm{Ext}^1_{\mathbb{H}_{\kappa+n}}(\mathsf{F}_\kappa(M), \mathsf{F}_\kappa(M))$$

for all M in $\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$.

Proof. This follows from Proposition 4.10.15 because the category $\widehat{\mathbf{U}}_\kappa\text{-}\mathsf{mod}_\Delta$ is closed under one-step extensions. □

Corollary 4.10.16 admits, at least conjecturally, a geometric interpretation when $\kappa = c$. Frenkel and Teleman consider in [57] the category of $(\widehat{\mathbf{U}}_c, G[[t]])$-bimodules. They conjecture, for $\mu \in \Pi^+$ (and prove for $\mu = 0$), that $\mathrm{Ext}^\bullet_{\widehat{\mathbf{U}}_c, G[[t]]}(\mathbb{W}_c(\mu), \mathbb{W}_c(\mu))$ is isomorphic to the algebra of differential forms on $\mathrm{Op}^\mu_{\check{G}}(\mathbb{D})$. Note that if this conjecture holds, the algebra of self-extensions is generated by Ext^1. An analogous result for rational Cherednik algebras is proven in [8, Corollary 4.2], stating that $\mathrm{Ext}^\bullet_{\mathtt{H}_0}(\Delta_0(\lambda), \Delta_0(\lambda))$ is isomorphic to the algebra of differential forms on Ω_λ, for $\lambda \in \mathcal{P}(n)$.

Conjecture 4.10.17. *Let $\lambda \in \mathcal{P}(n)$. The functor F_c induces a surjective algebra homomorphism*

$$\mathrm{Ext}^\bullet_{\widehat{\mathbf{U}}_c, G[[t]]}(\mathbb{W}_c(\lambda), \mathbb{W}_c(\lambda)) \to \mathrm{Ext}^\bullet_{\mathtt{H}_0}(\Delta_0(\lambda), \Delta_0(\lambda)),$$

which is given by the restriction of differential forms via the inclusion (4.92).

4.11 Conclusion and open problems

As noted in Remark 4.4.14, Varagnolo and Vasserot [138] constructed a generalization of the Suzuki functor, relating a certain (non-critical) parabolic version of category $\mathcal{O}$ for the affine Lie algebra to category $\mathcal{O}$ for the rational Cherednik algebra $\mathtt{H}_{1,\mathbf{h}}(\Gamma_n)$ of type $G(l,1,n)$ (for $t \neq 0$). We expect that this functor can also be extended to the $\kappa = c$, $t = 0$ case. Depending on the choice of parameters, $\mathrm{Spec}\, Z(\mathtt{H}_{0,\mathbf{h}}(\Gamma_n))$ is not necessarily smooth. In that case, $\mathtt{H}_{0,\mathbf{h}}(\Gamma_n)$ is not Morita equivalent to its centre. In particular, baby Verma modules $\bar{\Delta}(\underline{\lambda})$ for $\mathtt{H}_{0,\mathbf{h}}(\Gamma_n)$ may have non-isomorphic simple modules $L(\underline{\mu})$ as composition factors. The (graded) multiplicities $[\bar{\Delta}(\underline{\lambda}) : L(\underline{\mu})]$ are in general still unknown. It would be interesting to investigate whether this problem can be solved using the Suzuki functor. Unfortunately, there are two major obstacles this approach faces. Firstly, parabolic category $\mathcal{O}$ at the critical level is presently not very well understood. Secondly, there is no reason to expect that the restriction of the Suzuki functor to this category is an exact functor. Indeed, even when the level is not critical, the functor from [138] fails to be exact (see [138, Remark 7.11.a]).

Chapter 5

Quiver Schur algebras and cohomological Hall algebras

5.1 Introduction

Our goal in this chapter is to establish a connection between two algebras, which, historically, appeared in very different mathematical contexts and were introduced with rather different motivations in mind, namely: quiver Schur algebras and cohomological Hall algebras.

Quiver Schur algebras are a generalization of Khovanov and Lauda's [88] and Rouquier's [114] quiver Hecke algebras, nowadays also known as KLR algebras. The latter can be described algebraically by generators and relations, or in terms of a certain diagrammatic calculus. However, the passage from KLR algebras to quiver Schur algebras is easiest to understand from a geometric point of view. Varagnolo and Vasserot [137] (and later Kang, Kashiwara and Park [81], in a somewhat more general setting) constructed KLR algebras as extension algebras of a certain semisimple complex of constructible sheaves on the moduli stack of representations of a quiver. These extension algebras can also be described as convolution algebras in the equivariant Borel-Moore homology of a certain variety of triples, reminiscent of the classical Steinberg variety. The triples consist of a pair of full flags together with a compatible quiver representation. By incorporating partial flags into this construction, Stroppel and Webster [131] arrived at the definition of a quiver Schur algebra. Later, these algebras were studied from a more algebraic point of view in [99].

One of the main motivations for introducing KLR algebras was to construct a categorification of quantum groups and their canonical bases. For results in this direction, we refer the reader to, e.g., [80, 88, 113, 137]. Quiver Schur algebras also play an important role in this context. For example, quiver Schur algebras associated to the cyclic quiver provide a categorification of the generic nilpotent Hall algebra [131, Proposition 2.12], and their higher level versions categorify a higher level q-Fock space [131, Theorem C].

The second protagonist of our story, the cohomological Hall algebra (CoHA), was introduced by Kontsevich and Soibelman [90] as a categorification of Donaldson-Thomas invariants of three dimensional Calabi-Yau categories. One of the primary original motivations for studying the CoHA was to provide a rigorous mathematical definition of the algebra of BPS states from string theory. CoHAs and their generalizations have found numerous applications in representation theory, including a new proof of the Kac positivity conjecture [36], as well as new realizations of the elliptic Hall algebra [124] and Yangians [38, 125, 144].

We restrict ourselves to the relatively simple case of CoHAs associated to quivers with the trivial potential. For more information about this special case, including explicit examples, we refer the reader to, e.g., [43, 53, 110]. One of our main results, described in more detail below, says that the relations between algebra and coalgebra structures on the CoHA can be understood in terms of actions of quiver Schur algebras. It would be interesting to know whether one can associate KLR-type algebras to more general categories than those of quiver representations, and whether the connection between quiver Schur algebras and the CoHA described by us could be extended to such categories.

We also remark that similar connections arise in other settings. For example, Nakajima's original proposal [102, §7] for the mathematical definition of Coulomb branches in terms of the vanishing cycle associated to the Chern-Simons functional was inspired by Donaldson-Thomas theory. On the other hand, the ultimate definition of Coulomb branches from [20] involves a convolution algebra which can be viewed as an infinite dimensional example of Sauter's generalized quiver Hecke algebras from [119] (see [20, Remark 3.9.4]).

5.1.1. Main results. We will now describe our results in more detail. Given a quiver Q and a dimension vector $\mathbf{c}$, we consider the space $\mathfrak{Q}_{\mathbf{c}}$ of flagged representations of Q with dimension vector $\mathbf{c}$, together with the forgetful map onto the space $\mathfrak{R}_{\mathbf{c}}$ of unflagged representations. In contrast to KLR algebras, we allow arbitrary partial flags instead of full flags only. The quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ is the equivariant Borel-Moore homology of the corresponding Steinberg-type variety

$$\mathcal{Z}_{\mathbf{c}} = H_*^{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\mathbf{c}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}_{\mathbf{c}}),$$

equipped with the convolution product as in [35]. We remark that our construction differs slightly from the construction of Stroppel and Webster [131] - they impose the additional condition on the space $\mathfrak{Q}_{\mathbf{c}}$ that each flagged quiver representation is nilpotent and its associated graded must be semisimple. To distinguish the two constructions, we refer to their convolution algebra $\mathcal{Z}_{\mathbf{c}}^{SW}$ as the Stroppel-Webster quiver Schur algebra, and reserve the simpler name "quiver Schur algebra" for $\mathcal{Z}_{\mathbf{c}}$.

Our first result deals with the basic structural properties of quiver Schur algebras. It is well known that KLR algebras are generated by certain distinguished elements, called idempotents, polynomials and crossings, and that they admit a PBW-type basis. We prove an analogous result for quiver Schur algebras, with crossings replaced by fundamental classes called (elementary) merges and splits (see Definition 5.3.4).

Theorem A (Theorem 5.3.25, Corollary 5.3.27)**.** *The following hold:*

a) *The quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ has a "Bott-Samelson" basis consisting of pushforwards of fundamental classes of certain vector bundles on diagonal Bott-Samelson varieties.*
b) *Elementary merges, elementary splits and polynomials generate $\mathcal{Z}_{\mathbf{c}}$ as an algebra.*

The quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ has a natural faithful representation $\mathcal{Q}_{\mathbf{c}}$, called the "polynomial representation", on the direct sum of rings of partial invariants. We give an explicit description of this representation (Theorem 5.4.7) and interpret it in terms of Demazure operators (Proposition 5.4.9). In the special cases of the A_1 quiver (i.e., the quiver with one vertex and no arrows) and the Jordan quiver, we give a complete list of defining relations for the associated reduced quiver Schur algebra (see Theorems 5.4.14 and 5.4.17, as well as [127]), which is defined as the subalgebra of $\mathcal{Z}_{\mathbf{c}}$ generated by merges and splits, without the polynomials. The reduced quiver Schur algebra of the A_1 quiver turns out to be related to the green web category from [27, 135] (see Corollary 5.4.16), which arises naturally in the context of skew Howe duality.

Our next result establishes a connection between quiver Schur algebras and the CoHA associated to the same quiver Q. We first need to introduce some notation. Let $\mathcal{Z} = \bigoplus_{\mathbf{c}} \mathcal{Z}_{\mathbf{c}}$ be the direct sum of all the quiver Schur algebras associated to Q (summing over all dimension vectors $\mathbf{c}$) and let $\mathcal{Q} = \bigoplus_{\mathbf{c}} \mathcal{Q}_{\mathbf{c}}$ be the direct sum of their polynomial representations. We call $\mathcal{Z}$ the *total* quiver Schur algebra. Let us now briefly recall a few facts about the CoHA. It is defined as the direct sum of equivariant cohomology groups

$$\mathcal{H} = \bigoplus_{\mathbf{c}} H^{\bullet}_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{R}_{\mathbf{c}}),$$

equipped with multiplication via a certain pullback-pushforward construction. The CoHA can also be endowed with a coalgebra structure. However, the natural coproduct on the CoHA (see [90, §2.9]) is not compatible with the multiplication in the sense that $\mathcal{H}$ is not a bialgebra. This problem can be remedied at the cost of passing to a localization of $\mathcal{H}$ and working with a localized version of the natural coproduct (see [37]). We do not pursue this approach here. Instead, we are interested in gaining a better understanding of the relations between the natural coalgebra and algebra structures on $\mathcal{H}$. The following theorem shows that these relations are controlled by the total quiver Schur algebra $\mathcal{Z}$.

Theorem B (Theorem 5.6.6)**.** *The faithful polynomial representation $\mathcal{Q}$ of the total quiver Schur algebra $\mathcal{Z}$ can be naturally identified with the tensor algebra $T(\mathcal{H}_+)$ on the augmentation ideal $\mathcal{H}_+$ of the CoHA. This identification induces an injective algebra homomorphism*

$$\mathcal{Z} \hookrightarrow \operatorname{End}(T(\mathcal{H}_+))$$

sending elementary merges in $\mathcal{Z}$ to CoHA multiplication operators and elementary splits in $\mathcal{Z}$ to CoHA comultiplication operators.

The CoHA admits a description as a shuffle algebra [90, Theorem 2] in the sense of Feigin and Odesskii [52]. We interpret this description in terms of Demazure operators (Proposition 5.6.8), connecting it to our description of the polynomial representation $\mathcal{Q}$ of the quiver Schur algebra $\mathcal{Z}$. We expect that the relationship between shuffle algebras and Demazure operators carries over to more general settings. For example, we expect that multiplication in the formal version of the CoHA, defined by Yang and Zhao [143] for any equivariant oriented Borel-Moore homology theory, can be rephrased in terms of the formal Demazure operators from [73].

5.1.2. Geometric realization of the modified quiver Schur algebra.

One of the exciting features of KLR algebras (associated to finite and affine type A quivers) is that, after passing to suitable completions or cyclotomic quotients, they are isomorphic to affine Hecke algebras [25, 114], and endow the latter with interesting gradings. This isomorphism, known in the literature as the Brundan-Kleshchev-Rouquier isomorphism, was later generalized to Schur algebras in [97, 99, 131] (see also [141]).

The main result of [99] says that the convolution algebra $\mathcal{Z}^{SW}_{\mathbf{c}}$ from [131] is, after completion, isomorphic to the affine q-Schur algebra appearing naturally in the representation theory of p-adic general linear groups. The proof of this result relies on the fact that both of these algebras are isomorphic to a certain intermediate algebra $\mathcal{Z}^{MS}_{\mathbf{c}}$, called the modified quiver Schur algebra, which is defined in purely algebraic terms. We show that the modified quiver Schur algebra also admits a geometric realization as a convolution algebra.

Theorem C (Theorem 5.4.10)**.** *There is a natural algebra isomorphism $\mathcal{Z}_{\mathbf{c}} \cong \mathcal{Z}^{MS}_{\mathbf{c}}$ between our quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ and the modified quiver Schur algebra $\mathcal{Z}^{MS}_{\mathbf{c}}$.*

As an application, we deduce that our quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ is also isomorphic to the Stroppel-Webster quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{SW}$ (Theorem 5.4.12).

5.1.3. Mixed quiver Schur algebras. As we have already mentioned, KLR and quiver Schur algebras can be realized as convolution algebras, or, equivalently, extension algebras of a certain semisimple complex of sheaves on the moduli stack of representations of a quiver. If the quiver admits a contravariant involution θ, this construction can be generalized by replacing the stack of representations of the quiver with the stack of its self-dual representations.

This idea was pursued by Varagnolo and Vasserot in [136]. They obtained generalized KLR algebras which are Morita equivalent to affine Hecke algebras of type B, and provide a categorification of highest weight modules over $B_\theta(\mathfrak{g}_Q)$, the algebra introduced by Enomoto and Kashiwara [45,46] in the context of symmetric crystals. The type D case is considered in [83, 128].

Sauter [119–121] took the idea of generalizing KLR algebras further, and replaced the stack of self-dual representations of a quiver with the stack of generalized quiver representations in the sense of Derksen and Weyman [39]. In this generalization, the gauge group acting on the space of quiver representations is no longer a classical group, but an arbitrary reductive group.

We define a generalization of quiver Schur algebras which is close in spirit to the above-mentioned generalizations of KLR algebras. Given a quiver Q with a contravariant involution θ and an extra datum, called a duality structure (see Definition 5.5.1), we consider the stack of a certain type of self-dual representations of Q, introduced by Zubkov [147] under the name of supermixed quiver representations. We refer to the resulting Ext-algebra as the *mixed quiver Schur algebra* and denote it by ${}^\theta\mathcal{Z}_{\mathbf{c}}$. The mixed quiver Schur algebra has similar structural properties to the ordinary quiver Schur algebra: it has a Bott-Samelson basis (Theorem 5.5.21) and is generated by elementary merges, elementary splits and polynomials (Corollary 5.5.22).

The idea of replacing ordinary quiver representations by self-dual representations has also been exploited in the representation theory of Hall algebras (in the finite field setting) [145] and cohomological Hall algebras [146] by Young. In the finite field case, Young defined a "Hall module" over the Hall algebra of Q, and showed that it carries a natural action of the aforementioned Enomoto-Kashiwara algebra $B_\theta(\mathfrak{g}_Q)$. In the cohomological case, he introduced a "cohomological Hall module" ${}^\theta\mathcal{M}$ over the cohomological Hall algebra $\mathcal{H}$ associated to the same quiver Q without the involution θ. The module ${}^\theta\mathcal{M}$ is defined as the direct sum of equivariant cohomology groups

$$ {}^\theta\mathcal{M} = \bigoplus_{\mathbf{c}} H^\bullet_{{}^\theta\mathsf{G}_{\mathbf{c}}}({}^\theta\mathfrak{R}_{\mathbf{c}}) $$

of the spaces ${}^\theta\mathfrak{R}_{\mathbf{c}}$ of self-dual quiver representations, equipped with an $\mathcal{H}$-module structure via certain geometric correspondences. The module ${}^\theta\mathcal{M}$ also carries a natural $\mathcal{H}$-comodule structure, but it fails to be a Hopf module. Our next result shows that the relations between multiplication and comultiplication in the CoHA, as well as its action and coaction on the cohomological Hall module, are controlled by the total mixed quiver Schur algebra ${}^\theta\mathcal{Z} = \bigoplus_{\mathbf{c}} {}^\theta\mathcal{Z}_{\mathbf{c}}$.

Theorem D (Theorem 5.6.12)**.** *There is an injective algebra homomorphism*

$$ {}^\theta\mathcal{Z} \hookrightarrow \mathrm{End}(T(\mathcal{H}_+) \otimes {}^\theta\mathcal{M}) $$

sending elementary merges in ${}^\theta\mathcal{Z}$ to CoHA multiplication and action operators and elementary splits in ${}^\theta\mathcal{Z}$ to CoHA comultiplication and coaction operators.

As an application of Theorem D, we obtain an explicit description of the faithful polynomial representation of a mixed quiver Schur algebra (Theorem 5.6.15). Moreover, we reinterpret the description of the CoHM as a shuffle module [146, Theorem 3.3] in terms of Demazure operators of types A-D (Corollary 5.6.19).

Mixed quiver Schur algebras are also related to the Hall modules defined in the finite field setting. The direct sum ${}^{\theta}\mathcal{Z}$-pmod of the categories of finitely generated graded projective modules over all the mixed quiver Schur algebras carries a natural action of the monoidal category $\mathcal{Z}$-pmod, and its Grothendieck group $K_0({}^{\theta}\mathcal{Z})$ is a module as well as a comodule over $K_0(\mathcal{Z})$ (Proposition 5.5.25). We expect that, via the standard technique of sending the class of a semisimple perverse sheaf to the function given by the super-trace of the Frobenius on its stalks (see, e.g., [122]), $K_0(\mathcal{Z})$ can be identified with a subalgebra of the Hall algebra of Q. For example, in the special case of a Dynkin or cyclic quiver, $K_0(\mathcal{Z})^{op}$ is naturally isomorphic to the generic nilpotent Hall algebra (Proposition 5.5.23). We also expect that $K_0({}^{\theta}\mathcal{Z})$ can be identified with a subspace of the Hall module associated to the category of self-dual representations of Q, and that $K_0({}^{\theta}\mathcal{Z})$ is a semisimple module over the Enomoto-Kashiwara algebra $B_\theta(\mathfrak{g}_Q)$.

5.1.4. Structure of the chapter. Let us summarize the contents of the chapter. In section 2, we recall some basic material about combinatorial and geometric objects associated to quivers. In section 3, we define the quiver Schur algebra, and prove Theorem A about its basis and generators. In section 4, we describe the faithful polynomial representation, prove Theorem C, and give explicit relations for the quiver Schur algebra for the A_1 and Jordan quivers. In section 5, we define mixed quiver Schur algebras. In section 6, we establish a connection between quiver Schur algebras and CoHA's, and prove Theorems B and D.

5.2 Preliminaries

In this section we introduce notation and basic definitions which will be used throughout the chapter. We begin by setting up the notation for quivers and associated combinatorial objects such as dimension vectors and their compositions. We then recall the definitions of some geometric objects associated to quivers, such as quiver flag varieties and the corresponding Steinberg-type varieties. We finish by recalling a few facts about equivariant cohomology and convolution algebras.

5.2.1. Quivers and associated combinatorics. Let us fix for the rest of this section a quiver Q with a finite set of vertices Q_0 and a finite set of arrows Q_1. In particular, we allow multiple edges and edge loops.

If $a \in Q_1$ is an arrow, let $s(a)$ be its source and $t(a)$ its target. Let a_{ij} denote the number of arrows from vertex i to j. Let $\Gamma := \mathbb{Z}_{\geq 0}Q_0$ denote the free commutative monoid of dimension vectors for Q and let $\Gamma_+ := \Gamma\backslash\{0\}$. If $\mathbf{c} = \sum_{i\in Q_0} \mathbf{c}(i)\cdot i \in \Gamma$, write $|\mathbf{c}| = \sum_{i\in Q_0}\mathbf{c}(i) \in \mathbb{Z}$. Given a Q_0-graded vector space V, let $\dim_{Q_0} V \in \Gamma$ denote its Q_0-graded dimension.

Let n be a positive integer. We say that $\beta = (\beta_1, \ldots, \beta_{\ell_\beta}) \in (\mathbb{Z}_{\geq 1})^{\ell_\beta}$ is a *composition* of n if $\sum_j \beta_j = n$. Let $\mathrm{Com}(n)$ denote the set of compositions of n. Given $\beta \in \mathrm{Com}(n)$, let $\mathring{\beta}_j = \beta_1 + \ldots + \beta_j$ for $1 \leq j \leq \ell_\beta$, with $\mathring{\beta}_0 = 0$.

Definition 5.2.1. Let $\mathbf{c} \in \Gamma_+$. We say that $\underline{\mathbf{d}} = (\mathbf{d}_1, \ldots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}) \in \Gamma_+^{\ell_{\underline{\mathbf{d}}}}$ is a *vector composition* of $\mathbf{c}$, denoted $\underline{\mathbf{d}} \vartriangleright \mathbf{c}$, if $\langle\underline{\mathbf{d}}\rangle := \sum_{j=1}^{\ell_{\underline{\mathbf{d}}}} \mathbf{d}_j = \mathbf{c}$. We call $\ell_{\underline{\mathbf{d}}}$ the length of $\underline{\mathbf{d}}$. Let $\mathbf{Com}_{\mathbf{c}}$ denote the set of vector compositions of $\mathbf{c}$ and let $\mathbf{Com}_{\mathbf{c}}^n$ denote the subset of vector compositions of length n. The symmetric group Sym_n acts naturally on $\mathbf{Com}^n_{\mathbf{c}}$ from the right by permutations. For each $i \in Q_0$, we have a map

$$\mathbf{Com}_{\mathbf{c}} \to \mathrm{Com}(\mathbf{c}(i)), \quad \underline{\mathbf{d}} \mapsto \underline{\mathbf{d}}(i) := (\mathbf{d}_1(i), \dots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}(i)).$$

Given two vector compositions $\underline{\mathbf{d}} \rhd \mathbf{a}$ and $\underline{\mathbf{e}} \rhd \mathbf{b}$, let $\underline{\mathbf{d}} \cup \underline{\mathbf{e}} = (\mathbf{d}_1, \dots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}, \mathbf{e}_1, \dots, \mathbf{e}_{\ell_{\underline{\mathbf{e}}}}) \rhd \mathbf{a} + \mathbf{b}$ be their concatenation.

Definition 5.2.2. Suppose that $\beta \in \mathrm{Com}(\ell_{\underline{\mathbf{d}}})$. Define

$$\vee^j_\beta(\underline{\mathbf{d}}) := (\mathbf{d}_{\mathring{\beta}_{j-1}+1}, \dots, \mathbf{d}_{\mathring{\beta}_j}), \quad \wedge_\beta(\underline{\mathbf{d}}) := (\langle \vee^1_\beta(\underline{\mathbf{d}}) \rangle, \dots, \langle \vee^{\ell_\beta}_\beta(\underline{\mathbf{d}}) \rangle).$$

In particular, if $\beta = (1^{k-1}, 2, 1^{\ell_{\underline{\mathbf{d}}}-k-1})$ for some $1 \le k \le \ell_{\underline{\mathbf{d}}} - 1$, then we abbreviate $\wedge_k(\underline{\mathbf{d}}) := \wedge_\beta(\underline{\mathbf{d}})$. We define a partial order on $\mathbf{Com}_{\mathbf{c}}$ by setting

$$\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}} \iff \underline{\mathbf{e}} = \wedge_\beta(\underline{\mathbf{d}})$$

for some $\beta \in \mathrm{Com}(\ell_{\underline{\mathbf{d}}})$. If $\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}}$, we call $\underline{\mathbf{d}}$ a *refinement* of $\underline{\mathbf{e}}$, and $\underline{\mathbf{e}}$ a *coarsening* of $\underline{\mathbf{d}}$.

Example 5.2.3. Consider the A_3 quiver

$$\underset{i_1}{\bullet} \longrightarrow \underset{i_2}{\bullet} \longrightarrow \underset{i_3}{\bullet}$$

Let $\mathbf{c} = 4i_1 + 3i_2 + 3i_3$ and $\underline{\mathbf{d}} = (i_1 + i_3, 2i_1 + i_2, 2i_3, i_1 + i_2, i_2) \rhd \mathbf{c}$ so that $\ell_{\underline{\mathbf{d}}} = 5$. We have $\wedge_{(5)}(\underline{\mathbf{d}}) = \mathbf{c}$ and $\wedge_{(1,1,1,1,1)}(\underline{\mathbf{d}}) = \underline{\mathbf{d}}$. Moreover,

$$\begin{aligned} \wedge_{(2,3)}(\underline{\mathbf{d}}) &= (3i_1 + i_2 + i_3, i_1 + 2i_2 + 2i_3) \\ \wedge_{(2,1,2)}(\underline{\mathbf{d}}) &= (3i_1 + i_2 + i_3, 2i_3, i_1 + 2i_2) \\ \wedge_1(\underline{\mathbf{d}}) = \wedge_{(2,1,1,1)}(\underline{\mathbf{d}}) &= (3i_1 + i_2 + i_3, 2i_1 + i_2, 2i_3, i_1 + i_2, i_2), \\ \wedge_3(\underline{\mathbf{d}}) = \wedge_{(1,1,2,1)}(\underline{\mathbf{d}}) &= (i_1 + i_3, 2i_1 + i_2, i_1 + i_2 + 2i_3, i_2). \end{aligned}$$

Next we assign some products of symmetric groups to the combinatorics developed above. Given a positive integer n and $\alpha \in \mathrm{Com}(n)$, let $\mathsf{Sym}_\alpha = \prod_{j=1}^{\ell_\alpha} \mathsf{Sym}_{\alpha_j}$. Furthermore, set

$$\mathsf{W}_{\mathbf{c}} := \prod_{i \in Q_0} \mathsf{Sym}_{\mathbf{c}(i)}, \quad \mathsf{W}_{\underline{\mathbf{d}}} := \prod_{i \in Q_0} \mathsf{Sym}_{\underline{\mathbf{d}}(i)} \subseteq \mathsf{Sym}_{\mathbf{c}}.$$

We consider the groups $\mathsf{W}_{\mathbf{c}}$ and $\mathsf{W}_{\underline{\mathbf{d}}}$ as Coxeter groups in the usual way. In particular, they are endowed with a length function ℓ and a Bruhat order. Let $s_j(i)$ $(i \in Q_0, 1 \le j \le \mathbf{c}(i) - 1)$ be the standard generators of $\mathsf{W}_{\mathbf{c}}$. Given $\underline{\mathbf{e}}, \underline{\mathbf{f}} \succ \underline{\mathbf{d}} \rhd \mathbf{c}$, let

$${}_{\underline{\mathbf{e}}}\overset{\underline{\mathbf{d}}}{\mathsf{D}}_{\underline{\mathbf{f}}} := [\mathsf{W}_{\underline{\mathbf{e}}} \backslash \mathsf{W}_{\underline{\mathbf{d}}} / \mathsf{W}_{\underline{\mathbf{f}}}]^{\mathsf{min}}$$

denote the set of minimal length double coset representatives. If $\underline{\mathbf{d}} = (\mathbf{c})$, we write ${}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{f}}} := {}_{\underline{\mathbf{e}}}\overset{\underline{\mathbf{d}}}{\mathsf{D}}_{\underline{\mathbf{f}}}$. When $\mathsf{W}_{\underline{\mathbf{e}}}$ is trivial, we abbreviate $\mathsf{D}^{\underline{\mathbf{d}}}_{\underline{\mathbf{f}}} := {}_{\underline{\mathbf{e}}}\overset{\underline{\mathbf{d}}}{\mathsf{D}}_{\underline{\mathbf{f}}}$.

5.2.2. Quiver representations and flag varieties.

Let $\mathbf{c} \in \Gamma_+$. Fix a Q_0-graded $\mathbb{C}$-vector space $\mathbf{V}_{\mathbf{c}} = \bigoplus_{i \in Q_0} \mathbf{V}_{\mathbf{c}}(i)$ with $\dim \mathbf{V}_{\mathbf{c}}(i) = \mathbf{c}(i)$. Let us fix a basis $\{v_k(i) \mid 1 \le k \le \mathbf{c}(i)\}$ of $\mathbf{V}_{\mathbf{c}}(i)$ for each

$i \in Q_0$. Let

$$\mathfrak{R}_{\mathbf{c}} := \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_{\mathbf{c}}(s(a)), \mathbf{V}_{\mathbf{c}}(t(a))), \quad \mathsf{G}_{\mathbf{c}} := \prod_{i \in Q_0} \mathsf{GL}(\mathbf{V}_{\mathbf{c}}(i)).$$

The group $\mathsf{G}_{\mathbf{c}}$ acts naturally on $\mathfrak{R}_{\mathbf{c}}$ by conjugation. Let $\mathsf{T}_{\mathbf{c}}$ be the standard maximal torus in $\mathsf{G}_{\mathbf{c}}$, with fundamental weights $\omega_j(i)$ (for $i \in Q_0$, $1 \leq j \leq \mathbf{c}(i)$), and let $\mathsf{B}_{\mathbf{c}}$ be the standard Borel subgroup. Let $R^+_{\mathbf{c}} \subset R_{\mathbf{c}}$ be the associated (positive) root system. We identify the associated Weyl group with $\mathsf{W}_{\mathbf{c}}$. Given $w \in \mathsf{W}_{\mathbf{c}}$, let $R_{\mathbf{c}}(w) = \{\alpha \in R^+_{\mathbf{c}} \mid w(\alpha) \in -R^+_{\mathbf{c}}\}$. If $\alpha \in R_{\mathbf{c}}$, let U_α be the corresponding unipotent subgroup of $\mathsf{G}_{\mathbf{c}}$, and set $\mathsf{U}_w = \prod_{\alpha \in R_{\mathbf{c}}(w)} \mathsf{U}_\alpha$.

We call a sequence $V_\bullet$ of Q_0-graded subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{\ell_{\underline{\mathbf{d}}}} = \mathbf{V}_{\mathbf{c}}$$

a *flag of type* $\underline{\mathbf{d}} \in \mathbf{Com}_{\mathbf{c}}$ if $\dim_{Q_0} V_j/V_{j-1} = \mathbf{d}_j$. We refer to the flag $\mathbf{V}_{\underline{\mathbf{d}}} = (\mathbf{V}^j_{\underline{\mathbf{d}}})^{\ell_{\underline{\mathbf{d}}}}_{j=0}$, where $\mathbf{V}^j_{\underline{\mathbf{d}}} := \langle v_k(i) \mid 1 \leq k \leq \mathbf{d}_1(i) + \ldots + \mathbf{d}_j(i), i \in Q_0 \rangle$, as the *standard flag* of type $\underline{\mathbf{d}}$. Let

$$\mathsf{P}_{\underline{\mathbf{d}}} := \mathrm{Stab}_{\mathsf{G}_{\mathbf{c}}}(\mathbf{V}_{\underline{\mathbf{d}}}), \quad \mathsf{L}_{\underline{\mathbf{d}}} := \textstyle\prod_{j=1}^{\ell_{\underline{\mathbf{d}}}} \mathsf{G}_{\mathbf{d}_j}.$$

be the parabolic and Levi subgroups, respectively, associated to $\underline{\mathbf{d}}$, and let $R^+_{\underline{\mathbf{d}}} \subseteq R^+_{\mathbf{c}}$ be the corresponding subset of positive roots. Let $\mathfrak{F}_{\underline{\mathbf{d}}} \cong \mathsf{G}_{\mathbf{c}}/\mathsf{P}_{\underline{\mathbf{d}}}$ be the projective variety parametrizing flags of type $\underline{\mathbf{d}}$.

Definition 5.2.4. Let $\rho = (\rho_a) \in \mathfrak{R}_{\mathbf{c}}$. We say that a flag $V_\bullet$ is:

- *ρ-stable* if $\rho_a(V_j(s(a))) \subseteq V_j(t(a))$,
- *strictly ρ-stable* if $\rho_a(V_j(s(a))) \subseteq V_{j-1}(t(a))$,

for all $a \in Q_1$ and $1 \leq j \leq \ell_{\underline{\mathbf{d}}}$.

Given $\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}}$ and $V_\bullet$, a flag of type $\underline{\mathbf{d}}$, let $V_\bullet|_{\underline{\mathbf{e}}}$ denote its coarsening to a flag of type $\underline{\mathbf{e}}$. Let

$$\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} := \{(V_\bullet, \rho) \in \mathfrak{F}_{\underline{\mathbf{d}}} \times \mathfrak{R}_{\mathbf{c}} \mid V_\bullet|_{\underline{\mathbf{e}}} \text{ is } \rho\text{-stable}\}, \quad \mathfrak{Q}_{\underline{\mathbf{d}}} := \mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}. \tag{5.1}$$

be the space of flags of type $\underline{\mathbf{d}}$ together with suitably compatible quiver representations. There is a canonical $\mathsf{G}_{\mathbf{c}}$-equivariant isomorphism

$$\mathsf{G}_{\mathbf{c}} \times^{\mathsf{P}_{\underline{\mathbf{d}}}} \mathfrak{R}_{\underline{\mathbf{e}}} \cong \mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}, \quad (g, \rho) \mapsto (g \cdot \mathbf{V}_{\underline{\mathbf{d}}}, g \cdot \rho), \tag{5.2}$$

where $\mathfrak{R}_{\underline{\mathbf{d}}} := \{\rho \in \mathfrak{R}_{\mathbf{c}} \mid \mathbf{V}_{\underline{\mathbf{d}}} \text{ is } \rho\text{-stable}\}$. Let

$$\mathfrak{F}_{\underline{\mathbf{d}}} \xleftarrow{\tau_{\underline{\mathbf{d}}}} \mathfrak{Q}_{\underline{\mathbf{d}}} \xrightarrow{\pi_{\underline{\mathbf{d}}}} \mathfrak{R}_{\mathbf{c}}$$

be the canonical projections. The first one, $\tau_{\underline{\mathbf{d}}}$, is a vector bundle while the second one, $\pi_{\underline{\mathbf{d}}}$, is a proper map. We abbreviate

$$\mathfrak{F}_{\mathbf{c}} := \bigsqcup_{\underline{\mathbf{d}} \rhd \mathbf{c}} \mathfrak{F}_{\underline{\mathbf{d}}}, \quad \mathfrak{Q}_{\mathbf{c}} := \bigsqcup_{\underline{\mathbf{d}} \rhd \mathbf{c}} \mathfrak{Q}_{\underline{\mathbf{d}}}, \quad \pi_{\mathbf{c}} := \sqcup \pi_{\underline{\mathbf{d}}} \colon \mathfrak{Q}_{\mathbf{c}} \to \mathfrak{R}_{\mathbf{c}}.$$

Definition 5.2.5. Let $\mathfrak{Q}^s_{\underline{\mathbf{d}}}$ and $\mathfrak{R}^s_{\underline{\mathbf{d}}}$ be the varieties obtained by replacing "ρ-stable" with "strictly ρ-stable" in the definitions of $\mathfrak{Q}_{\underline{\mathbf{d}}}$ and $\mathfrak{R}_{\underline{\mathbf{d}}}$, respectively. Set $\mathfrak{Q}^s_{\mathbf{c}} = \bigsqcup_{\underline{\mathbf{d}} \rhd \mathbf{c}} \mathfrak{Q}^s_{\underline{\mathbf{d}}}$.

Remark 5.2.6. We make a few remarks about the relationship between $\mathfrak{Q}_{\underline{\mathbf{d}}}$ and $\mathfrak{Q}^s_{\underline{\mathbf{d}}}$.

(i) The variety $\mathfrak{Q}_{\underline{\mathbf{d}}}$ is isomorphic to the variety of representations ϕ of Q with dimension vector $\mathbf{c}$ together with a filtration by subrepresentations $\phi_1 \subset \ldots \subset \phi_{\ell_{\underline{\mathbf{d}}}}$ such that the dimension vectors of the subquotients form the vector composition $\underline{\mathbf{d}}$. If we impose the additional condition that the subquotients in a filtration are nilpotent and semisimple, we obtain $\mathfrak{Q}^s_{\underline{\mathbf{d}}}$.

(ii) If the quiver Q has no edge loops and each dimension vector $\mathbf{d}_1, \ldots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}$ in the vector composition $\underline{\mathbf{d}}$ is supported only at one vertex, then $\mathfrak{Q}_{\underline{\mathbf{d}}} = \mathfrak{Q}^s_{\underline{\mathbf{d}}}$.

(iii) The variety $\mathfrak{Q}_{\underline{\mathbf{d}}}$ is a generalization of the universal quiver Grassmannian defined in [126].

5.2.3. The Steinberg variety. Given $\underline{\mathbf{d}}, \underline{\mathbf{e}} \rhd \mathbf{c}$, set

$$\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := \mathfrak{Q}_{\underline{\mathbf{e}}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}_{\underline{\mathbf{d}}}, \quad \mathfrak{Z}_{\mathbf{c}} := \mathfrak{Q}_{\mathbf{c}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}_{\mathbf{c}} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{e}} \rhd \mathbf{c}} \mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}},$$

where the fibred product is taken with respect to $\pi_{\mathbf{c}}$. We call $\mathfrak{Z}_{\mathbf{c}}$ the *quiver Steinberg variety.* Let $\mathfrak{Z}^s_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := \mathfrak{Q}^s_{\underline{\mathbf{e}}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}^s_{\underline{\mathbf{d}}}$ and $\mathfrak{Z}^s_{\mathbf{c}} := \mathfrak{Q}^s_{\mathbf{c}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}^s_{\mathbf{c}}$ be the corresponding strictly stable versions.

Example 5.2.7. Let Q be the Jordan quiver, $\mathbf{c} = n$ and $\underline{\mathbf{d}} = (1^n)$. Then $\mathfrak{R}^s_{\mathbf{c}} = \mathcal{N}$ is the nilpotent cone, $\mathfrak{Q}^s_{\underline{\mathbf{d}}} = \widetilde{\mathcal{N}} = T^*(GL_n/B)$ is the cotangent bundle to the flag variety, $\pi_{\underline{\mathbf{d}}} \colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is the Springer resolution and $\mathfrak{Z}^s_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ is the usual Steinberg variety. On the other hand, $\mathfrak{R}_{\mathbf{c}} = \mathfrak{g} = \mathfrak{gl}_n$, $\mathfrak{Q}_{\underline{\mathbf{d}}} = \widetilde{\mathfrak{g}} = GL_n \times^B \mathfrak{b}$ and $\pi_{\underline{\mathbf{d}}} \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer resolution.

We next define a relative position stratification on $\mathcal{Z}_{\mathbf{c}}$. Consider the projection

$$\tau_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \colon \mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \to \mathfrak{F}_{\underline{\mathbf{e}}} \times \mathfrak{F}_{\underline{\mathbf{d}}}$$

remembering the flags and forgetting the quiver representation. Given $w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$, let $\mathcal{O}^{\Delta}_w := \mathsf{G}_{\mathbf{c}} \cdot (e\mathsf{P}_{\underline{\mathbf{e}}}, w\mathsf{P}_{\underline{\mathbf{d}}}) \subset \mathfrak{F}_{\underline{\mathbf{e}}} \times \mathfrak{F}_{\underline{\mathbf{d}}}$ be the diagonal $\mathsf{G}_{\mathbf{c}}$-orbit corresponding to w, and set

$$\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := \tau^{-1}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}(\mathcal{O}^{\Delta}_w), \quad \mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := \bigsqcup_{{}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}} \ni u \leqslant w} \mathfrak{Z}^u_{\underline{\mathbf{e}},\underline{\mathbf{d}}}, \quad \mathfrak{Z}^{<w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} = \mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \backslash \mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}, \tag{5.3}$$

where $u \leqslant w$ stands for the Bruhat order.

Lemma 5.2.8. *For each $w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$, the subvariety $\mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ is closed in $\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$, and the inclusion $\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \hookrightarrow \mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ is an open immersion.*

Proof. By the usual Bruhat decomposition, we have $\overline{\mathcal{O}^{\Delta}_w} = \bigsqcup_{{}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}} \ni u \leqslant w} \mathcal{O}^{\Delta}_u$. Hence $\mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} = \tau^{-1}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}(\overline{\mathcal{O}^{\Delta}_w})$ is closed in $\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$. Since $\mathcal{O}^{\Delta}_w$ is open in $\overline{\mathcal{O}^{\Delta}_w}$, the preimage $\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} = \tau^{-1}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}(\mathcal{O}^{\Delta}_w)$ is also open in $\mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$. □

5.2.4. Cohomology. Below we will always deal with complex algebraic varieties which are also smooth manifolds or admit closed embeddings into smooth manifolds. Let X be a complex algebraic variety with an action of a complex linear algebraic group G. We denote by EG the universal bundle and by BG the classifying space associated to G. The quotient $X_G := EG \times^G X = (EG \times X)/G$ by the diagonal G-action is called the homotopy quotient of X by G. Let $H^\bullet_G(X) := H^\bullet(X_G)$ denote the G-equivariant cohomology ring and $H^G_\bullet(X) := H_\bullet(X_G)$ the G-equivariant Borel-Moore homology of X, with coefficients in $\mathbb{C}$. If $Y \subseteq X$ is a closed G-stable subvariety, let $[Y] \in H^G_\bullet(X)$ denote its G-equivariant fundamental class. Given a G-equivariant complex vector bundle V on X, let $\mathtt{eu}_G(V) = \mathtt{eu}(EG \times^G V) \in H^\bullet_G(X)$ denote its top G-equivariant Chern class, i.e., the equivariant Euler class of the underlying real vector bundle. More information about equivariant homology and cohomology may be found in, e.g., [21, 22].

We will now introduce notation for various equivariant cohomology groups. Define

$$\mathcal{P}_{\mathbf{c}} := H^\bullet(B\mathsf{T}_{\mathbf{c}}) = \bigotimes_{i\in Q_0} \mathbb{C}[x_1(i),\ldots,x_{\mathbf{c}(i)}(i)],$$

where $x_j(i) := \mathsf{eu}(\mathfrak{V}_j(i))$ is the first Chern class of the line bundle

$$\mathfrak{V}_j(i) := E\mathsf{T}_{\mathbf{c}} \times^{\omega_j(i)} \mathbb{C}. \tag{5.4}$$

For each $\underline{\mathbf{d}} \rhd \mathbf{c}$, set

$$\Lambda_{\underline{\mathbf{d}}} := \mathcal{P}_{\mathbf{c}}^{\mathsf{W}_{\underline{\mathbf{d}}}}, \quad \Lambda_{\mathbf{c}} := \bigoplus_{\underline{\mathbf{d}}\rhd\mathbf{c}} \Lambda_{\underline{\mathbf{d}}}.$$

The canonical map $B\mathsf{T}_{\mathbf{c}} \twoheadrightarrow B\mathsf{P}_{\underline{\mathbf{d}}}$ induces an injective algebra homomorphism

$$H^\bullet(B\mathsf{P}_{\underline{\mathbf{d}}}) \hookrightarrow H^\bullet(B\mathsf{T}_{\mathbf{c}})$$

whose image is $\Lambda_{\underline{\mathbf{d}}}$. Given any $\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}}$, we use the homotopy equivalence $\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} \simeq \mathfrak{F}_{\underline{\mathbf{d}}}$ and the fact that $(\mathfrak{F}_{\underline{\mathbf{d}}})_{\mathsf{G}_{\mathbf{c}}} = B\mathsf{P}_{\underline{\mathbf{d}}}$ to identify

$$H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}) \cong H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}}) = \Lambda_{\underline{\mathbf{d}}}.$$

We next introduce notation for various equivariant Borel-Moore homology groups. Set

$$\mathcal{Q}_{\underline{\mathbf{d}}} := H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Q}_{\underline{\mathbf{d}}}), \quad \mathcal{Q}_{\mathbf{c}} := H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Q}_{\mathbf{c}}), \qquad \mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}), \quad \mathcal{Z}_{\mathbf{c}} := H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}_{\mathbf{c}}). \tag{5.5}$$

Since the varieties $\mathfrak{Q}_{\underline{\mathbf{d}}}$ and $\mathfrak{Q}_{\mathbf{c}}$ are smooth, Poincaré duality yields isomorphisms

$$\mathcal{Q}_{\underline{\mathbf{d}}} \cong H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\underline{\mathbf{d}}}) \cong \Lambda_{\underline{\mathbf{d}}}, \quad \mathcal{Q}_{\mathbf{c}} \cong H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\mathbf{c}}) \cong \Lambda_{\mathbf{c}}. \tag{5.6}$$

Moreover, set

$$\mathcal{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}), \quad \mathcal{Z}^e_{\mathbf{c}} := \bigoplus_{\underline{\mathbf{d}}\rhd\mathbf{c}} \mathcal{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}}. \tag{5.7}$$

5.2.5. Convolution. We recall the definition of the convolution product from [35]. Let G be a complex Lie group, X_1, X_2, X_3 be smooth complex G-manifolds, and let $Z_{12} \subset X_1 \times X_2$ and $Z_{23} \subset X_2 \times X_3$ be closed G-stable subsets. Let $p_{ij} \colon X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the projection onto the i-th and j-th factors. Assume that the restriction of p_{13} to $Z_{12} \times_{X_2} Z_{23}$ is proper. Set $Z_{13} = Z_{12} \circ Z_{23} = p_{13}(Z_{12} \times_{X_2} Z_{23})$. Given $c_{12} \in H^G_\bullet(Z_{12})$ and $c_{23} \in H^G_\bullet(Z_{23})$, their *convolution* is defined as

$$c_{12} \star c_{23} := (p_{13})_*((p_{12}^* c_{12}) \cap (p_{23}^* c_{23})) \in H^G_\bullet(Z_{13}),$$

where $\cap$ denotes the intersection pairing. We will often need to compute the convolution of fundamental classes in the following special case.

Lemma 5.2.9. *Assume that $Z_{12} \subset X_1 \times X_2$ and $Z_{23} \subset X_2 \times X_3$ are complex submanifolds. Further, suppose that either of the canonical projections $Z_{12} \to X_2 \leftarrow Z_{23}$ is a submersion, and that the map $p_{13} \colon Z_{12} \times_{X_2} Z_{23} \to Z_{13}$ is an isomorphism. Then $[Z_{12}] \star [Z_{23}] = [Z_{13}]$.*

Proof. The submersion assumption implies that the intersection of $p_{12}^{-1}(Z_{12})$ and $p_{23}^{-1}(Z_{23})$ is transverse (see [35, Remark 2.7.27.(ii)]). Hence, by [35, Proposition 2.6.47], we have $p_{12}^*[Z_{12}] \cap p_{23}^*[Z_{23}] = [Z_{12} \times_{X_2} Z_{23}]$. Since p_{13}, restricted to $Z_{12} \times_{X_2} Z_{23}$, is an isomorphism onto Z_{13}, we get $(p_{13})_*[Z_{12} \times_{X_2} Z_{23}] = [Z_{13}]$. □

Let X be a smooth complex G-manifold, let Y be a possibly singular complex G-variety and let $\pi\colon X \to Y$ be a G-equivariant proper map. Set $X_1 = X_2 = X_3 = X$, $Z = Z_{12} = Z_{23} = X \times_Y X$. Convolution yields a product $H_\bullet^G(Z) \times H_\bullet^G(Z) \to H_\bullet^G(Z)$, which, by [35, Corollary 2.7.41], makes $H_\bullet^G(Z)$ into a unital associative $H^\bullet(BG)$-algebra. The unit is given by $[X_\Delta]$, the G-equivariant fundamental class of X diagonally embedded into Z. Next, let $X_1 = X_2 = X$ and $X_3 = \{pt\}$. Then convolution yields an action $H_\bullet^G(Z) \times H_\bullet^G(X) \to H_\bullet^G(X)$, which makes $H_\bullet^G(X)$ into a left $H_\bullet^G(Z)$-module.

5.3 Quiver Schur algebras

In this section we define the quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ and construct a "Bott-Samelson basis" for $\mathcal{Z}_{\mathbf{c}}$. We deduce that $\mathcal{Z}_{\mathbf{c}}$ is generated by certain special elements called merges, splits and polynomials.

5.3.1. The quiver Schur algebra. Fix $\mathbf{c} \in \Gamma$. We apply the framework of §5.2.5 to the vector bundle $X = \mathfrak{Q}_{\mathbf{c}}$ on the quiver flag variety $\mathfrak{F}_{\mathbf{c}}$, the space of quiver representations $Y = \mathfrak{R}_{\mathbf{c}}$ and the projection $\pi = \pi_{\mathbf{c}}$. Then $Z = \mathfrak{Z}_{\mathbf{c}}$ is the quiver Steinberg variety, and we obtain a convolution algebra structure on its Borel-Moore homology $\mathcal{Z}_{\mathbf{c}} = H_\bullet^{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Z}_{\mathbf{c}})$ and a $\mathcal{Z}_{\mathbf{c}}$-module structure on $\mathcal{Q}_{\mathbf{c}} = H_\bullet^{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\mathbf{c}})$. By (5.6), $\mathcal{Q}_{\mathbf{c}}$ can be identified with the direct sum $\Lambda_{\mathbf{c}}$ of rings of invariant polynomials.

Definition 5.3.1. We call $\mathcal{Z}_{\mathbf{c}}$ the *quiver Schur algebra* associated to $(Q, \mathbf{c})$, and $\mathcal{Q}_{\mathbf{c}}$ its *polynomial representation.*

Remark 5.3.2. Our quiver Schur algebra can be seen as a modification of the quiver Schur algebra introduced by Stroppel and Webster in [131, §2.2]. There are two differences between our construction and theirs. Firstly, Stroppel and Webster only consider cyclic quivers with at least two vertices, while we work with arbitrary finite quivers. Secondly, we use the quiver Steinberg variety $\mathfrak{Z}_{\mathbf{c}}$ while they use its strictly stable version $\mathfrak{Z}_{\mathbf{c}}^s$. We will refer to the algebra from [131] as the "*Stroppel-Webster quiver Schur algebra*" and denote it by $\mathcal{Z}_{\mathbf{c}}^{SW}$.

The following standard result follows from the general theory of convolution algebras (see, e.g., [35, Proposition 8.6.35]).

Proposition 5.3.3. *There are canonical isomorphisms*

$$\mathcal{Z}_{\mathbf{c}} \cong \operatorname{Ext}^\bullet_{\mathsf{G}_{\mathbf{c}}}((\pi_{\mathbf{c}})_*\mathbb{C}_{\mathfrak{Q}_{\mathbf{c}}}, (\pi_{\mathbf{c}})_*\mathbb{C}_{\mathfrak{Q}_{\mathbf{c}}}), \quad \mathcal{Q}_{\mathbf{c}} \cong \operatorname{Ext}^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathbb{C}_{\mathfrak{R}_{\mathbf{c}}}, (\pi_{\mathbf{c}})_*\mathbb{C}_{\mathfrak{Q}_{\mathbf{c}}}) \tag{5.8}$$

intertwining the convolution product with the Yoneda product, and the convolution action with the Yoneda action, respectively.

5.3.2. Merges, splits and polynomials. We will now introduce notation and a diagrammatic calculus for certain special fundamental classes in $\mathcal{Z}_{\mathbf{c}}$. We begin by observing that $\mathcal{Z}_{\mathbf{c}}^e \subset \mathcal{Z}_{\mathbf{c}}$ is a subalgebra and that there is an algebra isomorphism

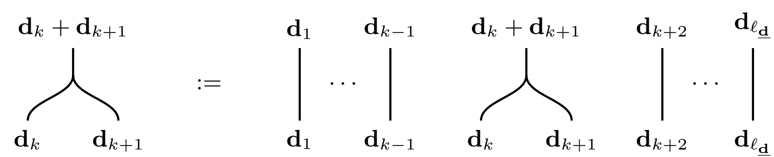
$$\mathcal{Z}_{\mathbf{c}}^e \cong H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\mathbf{c}}) \cong \Lambda_{\mathbf{c}}. \tag{5.9}$$

For this reason, we refer to $\mathcal{Z}^e_{\mathbf{c}}$ as “the polynomials” in $\mathcal{Z}_{\mathbf{c}}$. Next, observe that the fundamental classes $\mathsf{e}_{\underline{\mathbf{d}}} := [\mathfrak{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}}]$ form a complete set of mutually orthogonal idempotents in $\mathcal{Z}_{\mathbf{c}}$. The definition below introduces two other kinds of fundamental classes, which, following [131], we call “merges” and “splits”.

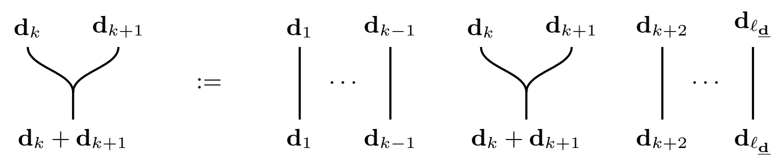
Definition 5.3.4. Given $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$, we call

- $\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} := [\mathfrak{Z}^e_{\underline{\mathbf{e}},\underline{\mathbf{d}}}] \in \mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ a *merge*,
- $\curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} := [\mathfrak{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{e}}}] \in \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}$ a *split*.

We say that a merge or split is *elementary* if $\underline{\mathbf{e}} = \wedge_k(\underline{\mathbf{d}})$ (see Definition 5.2.2) for some $1 \leq k \leq \ell_{\underline{\mathbf{d}}}-1$. We will depict elementary merges and splits diagrammatically in the following way. To the elementary merge $\curlywedge^{\wedge^k(\underline{\mathbf{d}})}_{\underline{\mathbf{d}}}$ we associate the diagram

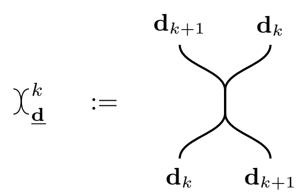


and to the elementary split $\curlyvee^{\underline{\mathbf{d}}}_{\wedge^k(\underline{\mathbf{d}})}$ the diagram



The diagram on the LHS should be understood as shorthand notation for the full diagram on the RHS. Multiplication of elementary merges and splits is depicted through a vertical composition of diagrams. We always read diagrams from the bottom to the top.

We call



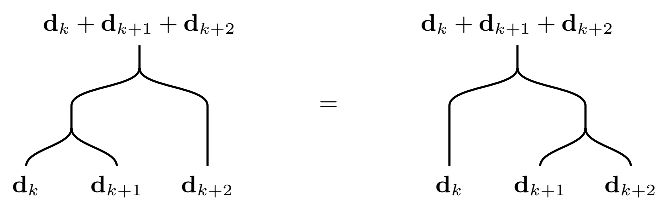
a *crossing*.

Proposition 5.3.5. *We list several basic relations which hold in* $\mathcal{Z}_{\mathbf{c}}$.

a) *Let* $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \succ \underline{\mathbf{f}} \rhd \mathbf{c}$. *Merges and splits satisfy the following* transitivity *relations:*

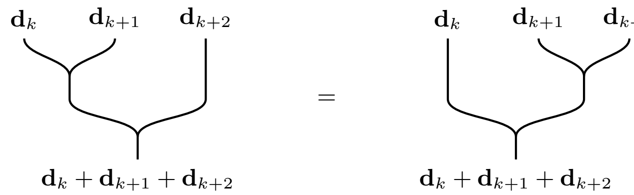
$$\curlywedge^{\underline{\mathbf{f}}}_{\underline{\mathbf{e}}} \star \curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \curlywedge^{\underline{\mathbf{f}}}_{\underline{\mathbf{d}}}, \qquad \curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} \star \curlyvee^{\underline{\mathbf{e}}}_{\underline{\mathbf{f}}} = \curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{f}}}.$$

b) *Let* $\underline{\mathbf{d}} \rhd \mathbf{c}$ *and* $1 \leq k \leq \ell_{\underline{\mathbf{d}}} - 2$. *Elementary merges satisfy the following* associativity *relation:*



(R1)

Elementary splits satisfy the following coassociativity *relation:*



(R2)

Proof. Part a) follows via an easy calculation from Lemma 5.2.9. Part b) follows immediately from part a). □

5.3.3. Relation to KLR algebras. We would like to connect the quiver Schur algebra to the well known quiver Hecke (or KLR) algebra (associated to the same quiver Q), defined diagrammatically by Khovanov and Lauda [88], and algebraically by Rouquier [114]. Let us recall its geometric construction and some generalizations. Define

$$\mathfrak{Q}_{\mathbf{c}}^{\mathsf{KLR}} := \bigsqcup_{\underline{\mathbf{d}}\in \mathrm{Com}_{\mathbf{c}}^{|\mathbf{c}|}} \mathfrak{Q}_{\underline{\mathbf{d}}}, \quad \mathfrak{Z}_{\mathbf{c}}^{\mathsf{KLR}} := \mathfrak{Q}_{\mathbf{c}}^{\mathsf{KLR}} \times_{\mathfrak{R}_{\mathbf{c}}} \mathfrak{Q}_{\mathbf{c}}^{\mathsf{KLR}}, \quad \mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}} := H_{\bullet}^{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Z}_{\mathbf{c}}^{\mathsf{KLR}}). \tag{5.10}$$

Note that the set $\mathrm{Com}_{\mathbf{c}}^{|\mathbf{c}|}$ contains precisely the vector compositions of $\mathbf{c}$ of maximal length, i.e., those parametrizing the types of complete quiver flags. Let $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR},s}$ be the strictly stable version of $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$, obtained by replacing each $\mathfrak{Q}_{\underline{\mathbf{d}}}$ with $\mathfrak{Q}_{\underline{\mathbf{d}}}^{s}$ in (5.10).

In the case when the quiver Q has no edge loops, it was proven by Varagnolo and Vasserot [137] that the convolution algebra $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$ gives a geometric realization of the KLR algebra associated to $(Q, \mathbf{c})$. Note that in this case Remark 5.2.6(ii) implies that $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR},s}$ coincides with $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$.

The case when Q may contain edge loops was studied by Kang, Kashiwara and Park [81]. They showed that the convolution algebra $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR},s}$ gives a geometric realization of the generalized KLR algebra from [82] associated to a symmetrizable Borcherds-Cartan datum.

Let Q be an arbitrary quiver. The following proposition describes $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$ as a subalgebra of $\mathcal{Z}_{\mathbf{c}}$.

Proposition 5.3.6. *The convolution algebra $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$ is a subalgebra of $\mathcal{Z}_{\mathbf{c}}$. It is generated by the polynomials $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e}$ and the crossings $\asymp_{\underline{\mathbf{d}}}^{k}$ (for $\underline{\mathbf{d}} \in \mathrm{Com}_{\mathbf{c}}^{|\mathbf{c}|}$ and $1 \leq k \leq |\mathbf{c}| - 1$).*

Proof. The natural inclusion $\mathfrak{Z}_{\mathbf{c}}^{\mathsf{KLR}} \hookrightarrow \mathfrak{Z}_{\mathbf{c}}$ is an inclusion of connected components, and hence induces an inclusion of the corresponding convolution algebras. This proves the first claim. Given $\underline{\mathbf{d}} \in \mathrm{Com}_{\mathbf{c}}^{|\mathbf{c}|}$, let us identify $\mathsf{W}_{\mathbf{c}}$ with the subgroup of $\mathsf{Sym}_{|\mathbf{c}|}$ preserving $\underline{\mathbf{d}}$. It follows easily from Lemma 5.2.9 that $\asymp_{\underline{\mathbf{d}}}^{k} = [\mathfrak{Z}_{s_k(\underline{\mathbf{d}}),\underline{\mathbf{d}}}^{e}]$ if $s_k(\underline{\mathbf{d}}) \neq \underline{\mathbf{d}}$ and $\asymp_{\underline{\mathbf{d}}}^{k} = [\overline{\mathfrak{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{s_k}}]$ if $s_k(\underline{\mathbf{d}}) = \underline{\mathbf{d}}$. But these fundamental classes, together with the polynomials $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e} \cong \Lambda_{\underline{\mathbf{d}}}$, generate $\mathcal{Z}_{\mathbf{c}}^{\mathsf{KLR}}$ by (a straightforward generalization of) [137, Theorem 3.6]. □

5.3.4. The combinatorics of refinements. Our next goal is to construct a basis for the quiver Schur algebra which is natural both from algebraic and geometric points of view. Algebraically, the basis elements are certain products of merges, splits and polynomials. Geometrically, they will be realized as pushforwards of vector bundles on diagonal Bott-Samelson varieties. In §5.3.4 we develop the combinatorial tools needed to define the basis. We state the basis theorem in §5.3.5, and prove it in §5.3.6.

Definition 5.3.7. Given $i \in Q_0$, let $\mathbf{N}_{\mathbf{c}}(i) := \{(1, i), \ldots, (\mathbf{c}(i), i)\}$, and set $\mathbf{N}_{\mathbf{c}} := \bigsqcup_{i\in Q_0} \mathbf{N}_{\mathbf{c}}(i)$. By a *partitioning* of $\mathbf{c}$ of length $n = \ell_\lambda$ we mean a function $\lambda \colon \mathbf{N}_{\mathbf{c}} \to \mathbb{Z}_{\geq 1}$ such that $\mathrm{Im}\, \lambda = [1, n] :=$

$\{1,\ldots,n\}$. Let λ_i be the restriction of λ to $\mathbf{N_c}(i)$. Let $\mathbf{Par}^n_{\mathbf{c}}$ denote the set of all partitionings of $\mathbf{c}$ of length n and let $\mathbf{Par_c} := \bigsqcup_{1\le n\le |\mathbf{c}|}\mathbf{Par}^n_{\mathbf{c}}$. The following lemma follows directly from the definitions.

Lemma 5.3.8. *There is a bijection between* $\mathbf{Par_c}$ *and the set of coordinate flags in* $\mathbf{V_c}$*, sending* λ *to the flag* $V_\bullet$ *with* $V_r = \langle v_k(i) \mid (k,i)\in\lambda^{-1}([1,r])\rangle$.

Partitionings are related to vector compositions of $\mathbf{c}$ through functions

$$\mathbf{Par_c} \underset{P}{\overset{C}{\rightleftarrows}} \mathbf{Com_c}$$

defined in the following way. If $\lambda\in\mathbf{Par_c}$, then $C(\lambda) = (\mathbf{d}_1,\ldots,\mathbf{d}_{|\lambda|})$ is given by $\mathbf{d}_k(i) = |\lambda_i^{-1}(k)|$. If $\underline{\mathbf{e}}\in\mathbf{Com_c}$, then $P(\underline{\mathbf{e}}) = \lambda$ is given by $\lambda_i^{-1}(k) = \{(\mathring{\mathbf{e}}_{k-1}(i)+1,i),\ldots,(\mathring{\mathbf{e}}_k(i),i)\}$.

Example 5.3.9. Consider $Q,\mathbf{c}$ and $\underline{\mathbf{d}}\in\mathbf{Com}^5_{\mathbf{c}}$ from Example 5.2.3. Let $\lambda = P(\underline{\mathbf{d}})$. Then

$$\lambda^{-1}(1) = \{(1,i_1),(1,i_3)\},\quad \lambda^{-1}(2) = \{(2,i_1),(3,i_1),(1,i_2)\},\quad \lambda^{-1}(3) = \{(2,i_3),(3,i_3)\},$$

$$\lambda^{-1}(4) = \{(4,i_1),(2,i_2)\},\quad \lambda^{-1}(5) = \{(3,i_2)\}.$$

Next, let $\mu\in\mathbf{Par}^4_{\mathbf{c}}$ be given by

$$\begin{aligned} \mu^{-1}(1) &= \{(2,i_2),(3,i_2),(2,i_3)\}, & \mu^{-1}(2) &= \{(4,i_1)\},\\ \mu^{-1}(3) &= \{(2,i_1),(1,i_2),(1,i_3),(3,i_3)\}, & \mu^{-1}(4) &= \{(1,i_1),(3,i_1)\}. \end{aligned}$$

Then $C(\mu) = (2i_2+i_3, i_1, i_1+i_2+2i_3, 2i_1)$.

We let $\mathsf{W_c}$ act on $\mathbf{N_c}$ from the left and Sym_n act on $[1,n]$ from the right. This means that Sym_n acts by permuting places rather than numbers. We get induced actions on $\mathrm{Fun}(\mathbf{N_c},[1,n])$, which preserve $\mathbf{Par}^n_{\mathbf{c}}$, viewed as the subset of $\mathrm{Fun}(\mathbf{N_c},[1,n])$ consisting of surjective functions. Note that the resulting Sym_n-action on $\mathbf{Par}^n_{\mathbf{c}}$ is free. The following lemma follows directly from the definitions.

Lemma 5.3.10. *The functions C and P have the following properties:*

a) *The function C is surjective and* $C\circ P = \mathrm{id}$.
b) *We have* $\ell_{C(\lambda)} = \ell_\lambda$ *and* $\ell_{P(\underline{\mathbf{d}})} = \ell_{\underline{\mathbf{d}}}$*, for all* $\lambda\in\mathbf{Par_c}$ *and* $\underline{\mathbf{d}}\in\mathbf{Com_c}$.
c) *The fibres of C are precisely the* $\mathsf{W_c}$*-orbits in* $\mathbf{Par_c}$*. Hence C induces a set isomorphism*

$$\mathbf{Par_c}/\mathsf{W_c}\cong\mathbf{Com_c}.$$

d) *We have* $\mathrm{Stab}_{\mathsf{W_c}}(P(\underline{\mathbf{d}})) = \mathsf{W}_{\underline{\mathbf{d}}}$ *for any* $\underline{\mathbf{d}}\in\mathbf{Com_c}$.
e) *The map* $C|_{\mathbf{Par}^n_{\mathbf{c}}}$ *is* Sym_n*-equivariant, for each* $1\le n\le|\mathbf{c}|$.
f) *We have* $s_j\cdot P(\underline{\mathbf{d}}) = \widetilde{s_j}\cdot P(s_j\cdot\underline{\mathbf{d}})$ *with* $\widetilde{s_j}$ *being the longest element in* $\mathsf{D}^{\wedge_j(\underline{\mathbf{d}})}_{s_j\cdot\underline{\mathbf{d}}}$*, for any* $\underline{\mathbf{d}}\in\mathbf{Com_c}$ *and* $1\le j\le\ell_{\underline{\mathbf{d}}}-1$ *(note that* $s_j\in\mathsf{Sym}_{\ell_{\underline{\mathbf{d}}}}$ *while* $\widetilde{s_j}\in\mathsf{W_c}$*).*

Next, we define a binary operation $\Cap$ on $\mathbf{Par_c}$.

Definition 5.3.11. Given $\lambda,\mu\in\mathbf{Par_c}$, let $R_{k,l} = \lambda^{-1}(k)\cap\mu^{-1}(l)$ and $S_{\lambda,\mu} = \{R_{k,l}\neq\varnothing \mid 1\le k\le |\lambda|,\ 1\le l\le|\mu|\}$. We define a total order on the set $S_{\lambda,\mu}$ by declaring that $R_{k,l} < R_{r,s}$ if and only if $r>k$ or $r=k$ and $l<s$. We then define the *ordered intersection* $\lambda\Cap\mu = \nu$ of partitionings λ and μ by setting

$$\nu^{-1}(m) = \begin{cases} m\text{-th element of } S_{\lambda,\mu} \text{ in the total order defined above,} & \text{if } 1\le m\le|S_{\lambda,\mu}|,\\ \varnothing, & \text{if } m>|S_{\lambda,\mu}|. \end{cases}$$

One can immediately see that ν is in fact a partitioning of $\mathbf{c}$. The operation Ω is not symmetric. However, the following holds.

Lemma 5.3.12. *Let $\lambda, \mu \in \mathbf{Par_c}$. Then:*

a) *$\lambda \Omega \mu$ and $\mu \Omega \lambda$ are of the same length and lie in the same orbit of $\mathsf{Sym}_{\ell_{\lambda \Omega \mu}}$.*
b) *$C(\lambda \Omega \mu) \succcurlyeq C(\lambda)$ (but in general $C(\lambda \Omega \mu) \not\succcurlyeq C(\mu)$).*
c) *$\mathrm{Stab}_{\mathsf{W_c}}(\lambda \Omega \mu) = \mathrm{Stab}_{\mathsf{W_c}}(\lambda) \cap \mathrm{Stab}_{\mathsf{W_c}}(\mu)$.*
d) *$w \cdot (\lambda \Omega \mu) = (w \cdot \lambda) \Omega (w \cdot \mu)$ for all $w \in \mathsf{W_c}$.*

Proof. Part a) follows immediately from the fact that the sets $S_{\lambda,\mu}$ and $S_{\mu,\lambda}$ are the same if we forget their orderings. Part b) is obvious. For part c), observe that if $\nu \in \mathbf{Par_c}$ then $\mathrm{Stab}_{\mathsf{W_c}}(\nu) = \bigcap_{1\le r\le \ell_\nu} \mathrm{Stab}_{\mathsf{W_c}}(\nu^{-1}(r))$, where $\mathrm{Stab}_{\mathsf{W_c}}(\nu^{-1}(r))$ is the subgroup of $\mathsf{W_c}$ fixing $\nu^{-1}(r)$ setwise. If $x \in \mathsf{W_c}$ stabilizes all the subsets $\lambda^{-1}(k)$ and $\mu^{-1}(l)$ then x also stabilizes all their intersections $R_{k,l}$. Hence it stabilizes $\lambda \Omega \mu$. Conversely, note that $S_{\lambda,\mu}$ is a partition of the set $\mathbf{N_c}$, which refines the partitions $S_{\lambda,\lambda}$ and $S_{\mu,\mu}$. Hence, if x stabilizes all the sets $R_{k,l}$ in $S_{\lambda,\mu}$, then it must also stabilize all the preimages $\lambda^{-1}(k)$ and $\mu^{-1}(l)$. Part d) is clear. □

Example 5.3.13. Suppose that Q_0 is a singleton. We can then identify Γ with $\mathbb{Z}_{\ge 0}$. Let $\mathbf{c} = 8$ and take $\lambda = [1,2,3][4,5][6,7,8]$ and $\mu = [1,2,4,5][6,7][3,8]$. This notation means that, e.g., $\mu^{-1}(1) = \{1,2,4,5\}$ and $\mu^{-1}(2) = \{6,7\}$ and $\mu^{-1}(3) = \{3,8\}$. Then

$$\lambda \Omega \mu = [1,2][3][4,5][6,7][8], \quad \mu \Omega \lambda = [1,2][4,5][6,7][3][8].$$

Again suppose that Q_0 is a singleton, and take $\mathbf{c} = 10$, $\lambda = [3,5,6][1,4][2,8][7,9,10]$ and $\mu = [1,5,9,10][2,3,4,6,7,8]$. Then

$$\lambda \Omega \mu = [5][3,6][1][4][2,8][9,10][7], \quad \mu \Omega \lambda = [5][1][9,10][3,6][4][2,8][7].$$

We see that $\lambda \Omega \mu$ and $\mu \Omega \lambda$ are of the same length and differ only by a permutation.

Example 5.3.14. Suppose that Q is the A_3 quiver from Example 5.2.3 and $\mathbf{c} = 5i_1 + 4i_2 + 3i_3$. Let λ be given by $\lambda^{-1}(1) = \{(1,i_1),(2,i_1),(3,i_1),(1,i_3),(2,i_3)\}$, $\lambda^{-1}(2) = \{(1,i_2),(2,i_2),(3,i_3)\}$ and $\lambda^{-1}(3) = \{(4,i_1),(5,i_1),(3,i_2),(4,i_2)\}$. We assign the colour black to vertex i_1, blue to i_2 and red to i_3 and rewrite λ in the following more readable notation: $\lambda = [1,2,3,1,2][1,2,3][4,5,3,4]$. Let μ be a second partitioning given by $\mu = [1,4,1,3,4,3][2,3,5,2,1,2]$. Then

$$\lambda \Omega \mu = [1][2,3,1,2][1,3][2][4,3,4][5], \quad \mu \Omega \lambda = [1][1,3][4,3,4][2,3,1,2][2][5].$$

Definition 5.3.15. We call a triple $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$, consisting of $\underline{\mathbf{e}}, \underline{\mathbf{d}} \in \mathbf{Com_c}$ and $w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$, an *orbit datum*. This name is motivated by the fact that orbit data naturally label the $\mathsf{G_c}$-orbits in $\mathfrak{F}_\mathbf{c} \times \mathfrak{F}_\mathbf{c}$. Abbreviating $\lambda = P(\underline{\mathbf{e}})$ and $\mu = w \cdot P(\underline{\mathbf{d}})$, we also define

$$\widehat{\underline{\mathbf{e}}} := C(\lambda \Omega \mu), \quad \widehat{\underline{\mathbf{d}}} := C(\mu \Omega \lambda).$$

By Lemma 5.3.12.b), $\widehat{\underline{\mathbf{e}}}$ is a refinement of $\underline{\mathbf{e}}$ and $\widehat{\underline{\mathbf{d}}}$ is a refinement of $\underline{\mathbf{d}}$. By Lemma 5.3.12.a), the partitionings $\mu \Omega \lambda$ and $\lambda \Omega \mu$ are of the same length n and lie in the same orbit of Sym_n. Since the Sym_n-action on $\mathbf{Par}^n_\mathbf{c}$ is free, there exists a unique permutation $u \in \mathsf{Sym}_n$ which sends $\lambda \Omega \mu$ to $\mu \Omega \lambda$. We call the triple $(\widehat{\underline{\mathbf{e}}}, \widehat{\underline{\mathbf{d}}}, u)$ the *refinement datum* corresponding to $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$.

Example 5.3.16. Let Q_0 be a singleton, $\mathbf{c} = 8$, $\underline{\mathbf{e}} = (3, 2, 3)$, $\underline{\mathbf{d}} = (4, 2, 2)$ and $w = s_3 s_4 s_5 s_6$. Then $\lambda = P(\underline{\mathbf{e}}) = [1, 2, 3][4, 5][6, 7, 8]$ and $\mu = w \cdot P(\underline{\mathbf{d}}) = [1, 2, 4, 5][6, 7][3, 8]$, i.e., λ and μ are as in the first case considered in Example 5.3.13. Hence $\widehat{\underline{\mathbf{e}}} = (2, 1, 2, 2, 1)$, $\widehat{\underline{\mathbf{d}}} = (2, 2, 2, 1, 1)$ and $u = s_3 s_2 \in \mathsf{Sym}_5$.

Example 5.3.17. Let Q be the A_3 quiver, $\mathbf{c} = 5i_1 + 4i_2 + 3i_3$, $\underline{\mathbf{e}} = (3i_1 + 2i_3, 2i_2 + i_3, 2i_1, 2i_2)$, $\underline{\mathbf{d}} = (2i_1, 3i_2 + i_3, 3i_1 + i_2 + 2i_3)$ and $w = (s_3 s_2)(s_2 s_3)(s_2 s_1) \in \mathsf{W}_{\mathbf{c}} = \mathsf{Sym}_5 \times \mathsf{Sym}_4 \times \mathsf{Sym}_3$. Then $\lambda = P(\underline{\mathbf{e}}) = [1, 2, 3, 1, 2][1, 2, 3][4, 5, 3, 4]$ and $\mu = w \cdot P(\underline{\mathbf{d}}) = [1, 4, 2, 3, 4, 3][2, 3, 5, 1, 1, 2]$, i.e., λ and μ are as in Example 5.3.14. Hence $\widehat{\underline{\mathbf{e}}} = (i_1, 2i_1 + 2i_3, i_2 + i_3, i_2, i_1 + 2i_2, i_1)$, $\widehat{\underline{\mathbf{d}}} = (i_1, i_2 + i_3, i_1 + 2i_2, 2i_1 + 2i_3, i_2, i_1)$ and $u = s_3 s_2 s_4 \in \mathsf{Sym}_6$. We rewrite the vector compositions $\widehat{\underline{\mathbf{e}}}$ and $\widehat{\underline{\mathbf{d}}}$ in the following more readable notation

$$\widehat{\underline{\mathbf{e}}} = (1, 2 + 2, 1 + 1, 1, 1 + 2, 1), \quad \widehat{\underline{\mathbf{d}}} = (1, 1 + 1, 1 + 2, 2 + 2, 1, 1).$$

Again, we see that $\widehat{\underline{\mathbf{e}}}$ and $\widehat{\underline{\mathbf{d}}}$ are of the same length and differ only by a permutation.

Lemma 5.3.18. *Let $\lambda = P(\underline{\mathbf{e}})$ and $\mu = w \cdot P(\underline{\mathbf{d}})$ as in Definition 5.3.15. Then $P(\widehat{\underline{\mathbf{e}}}) = \lambda \Omega \mu$.*

Proof. A partitioning ν is in the image of P if and only if for all $1 \leq r \leq \ell_\nu - 1$, $i \in Q_0$ and $(x, i) \in \nu_i^{-1}(r)$, $(y, i) \in \nu^{-1}(r + 1)$, we have $x < y$. Since λ is in the image of P, and w is a shortest coset representative, $\lambda \Omega \mu$ satisfies this condition. Hence $\lambda \Omega \mu = P(\underline{\mathbf{f}})$ for some $\underline{\mathbf{f}} \in \mathbf{Com_c}$. But then $P(\widehat{\underline{\mathbf{e}}}) = P \circ C(\lambda \Omega \mu) = P \circ C \circ P(\underline{\mathbf{f}}) = P(\underline{\mathbf{f}}) = \lambda \Omega \mu$. □

Definition 5.3.19. Given a refinement datum $(\widehat{\underline{\mathbf{e}}}, \widehat{\underline{\mathbf{d}}}, u)$, let us choose a reduced expression $u = s_{j_k} \cdot \ldots \cdot s_{j_1}$, where $k = \ell(u)$. Set $u_l = s_{j_l} \cdot \ldots \cdot s_{j_1}$ and let $\underline{\mathbf{e}}^{2l} = u_l(\widehat{\underline{\mathbf{e}}})$, $\underline{\mathbf{e}}^{2l-1} = \wedge_{j_l}(\underline{\mathbf{e}}^{2l-2})$, for $1 \leq l \leq k$, with $\underline{\mathbf{e}}^0 = \widehat{\underline{\mathbf{e}}}$. Observe that $\underline{\mathbf{e}}^{2k} = \widehat{\underline{\mathbf{d}}}$. We call $(\underline{\mathbf{e}}^0, \ldots, \underline{\mathbf{e}}^{2k})$ a *crossing datum* associated to $(\widehat{\underline{\mathbf{e}}}, \widehat{\underline{\mathbf{d}}}, u)$.

The following diagram illustrates the relationships between the different vector compositions in a crossing datum. Vector compositions in the same row (possibly except for $\underline{\mathbf{e}}$ and $\underline{\mathbf{d}}$) are of the same length.

$$\begin{array}{ccccccccccccc} \underline{\mathbf{e}} & & & & \underline{\mathbf{e}}^1 & & & & \underline{\mathbf{e}}^3 & \cdots & \underline{\mathbf{e}}^{2k-1} & & & & \underline{\mathbf{d}} \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & & & \searrow & & \swarrow & \\ & & \underline{\mathbf{e}}^0 & & & & \underline{\mathbf{e}}^2 & & & & & & \underline{\mathbf{e}}^{2k} & & \end{array}$$

Example 5.3.20. Consider the first case from Example 5.3.16. Given the reduced expression $u = s_3 s_2 \in \mathsf{Sym}_5$, we have $\underline{\mathbf{e}}^1 = (2, 3, 2, 1)$, $\underline{\mathbf{e}}^2 = (2, 2, 1, 2, 1)$, $\underline{\mathbf{e}}^3 = (2, 2, 3, 1)$. Next, consider Example 5.3.17. Given the reduced expression $u = s_3 s_2 s_4 \in \mathsf{Sym}_6$, we have $\underline{\mathbf{e}}^1 = (1, 2 + 2, 1 + 1, 1 + 3, 1)$, $\underline{\mathbf{e}}^2 = (1, 2 + 2, 1 + 1, 1 + 2, 1, 1)$, $\underline{\mathbf{e}}^3 = (1, 2 + 1 + 3, 1 + 2, 1, 1)$, $\underline{\mathbf{e}}^4 = (1, 1 + 1, 2 + 2, 1 + 2, 1, 1)$ and $\underline{\mathbf{e}}^5 = (1, 1 + 1, 3 + 2 + 2, 1, 1)$.

We will now explain the connection between $w \in \mathsf{W}_{\mathbf{c}}$ and $u = s_{j_k} \cdot \ldots \cdot s_{j_1} \in \mathsf{Sym}_n$. Let w_l denote the longest element in $\mathsf{D}_{\underline{\mathbf{e}}^{2l}}^{\underline{\mathbf{e}}^{2l-1}}$ and let $\widetilde{u} = w_1 \cdot \ldots \cdot w_k$.

Proposition 5.3.21. *The following hold:*

a) $\ell(\widetilde{u}) = \ell(w_1) + \ldots + \ell(w_k)$ *and* $w_1 \cdot \ldots \cdot w_l \in \mathsf{D}_{\underline{\mathbf{e}}^{2l}}^{\mathbf{c}}$, *for all* $1 \leq l \leq k$.

b) $\widetilde{u} = w$.

c) $\mathsf{W}_{\underline{\mathbf{e}}^0} = \mathsf{W}_{\underline{\mathbf{e}}} \cap (w \mathsf{W}_{\underline{\mathbf{d}}} w^{-1})$.

Proof. We start by proving the first statement in part a). To simplify notation, we assume (without loss of generality) that Q_0 is a singleton. We divide the interval $[1, \mathbf{c}]$ into blocks (i.e. subintervals) of size $\widehat{\mathbf{e}}_1, \ldots, \widehat{\mathbf{e}}_n$. Let $B_1, \ldots, B_n$ be the blocks. The permutation u acts by permuting these blocks. Let $\mathrm{Inv} = \{(i, j) \in [1, \mathbf{c}]^2 \mid i < j,\ \widetilde{u}(i) > \widetilde{u}(j)\}$ and $\mathrm{Inv}_B = \{(i, j) \in [1, n]^2 \mid B_i < B_j,\ u(B_i) > u(B_j)\}$. The length of $\widetilde{u}$ equals the number of inversions, i.e., $\ell(\widetilde{u}) = |\mathrm{Inv}|$. Since $\widetilde{u}$ permutes blocks but does not change the order inside blocks, we have $|\mathrm{Inv}| = \sum_{(i,j) \in \mathrm{Inv}_B} \widehat{\mathbf{e}}_i \cdot \widehat{\mathbf{e}}_j$. We can identify each inversion

(i,j) in Inv_B with some simple transposition s_{j_l} in the reduced expression $u = s_{j_k} \cdot \ldots \cdot s_{j_1}$. But then $\widehat{\mathbf{e}}_i \cdot \widehat{\mathbf{e}}_j = \ell(w_l)$, proving the statement. For the second statement in part a), note that $w_1 \cdot \ldots \cdot w_l$ also permutes blocks but does not change the order inside blocks. But any $x \in \mathsf{W}_{\underline{\mathbf{e}}^{2l}}$ permutes numbers within blocks, increasing the number of inversions. Hence $\ell(w_1 \cdot \ldots \cdot w_l \cdot x) > \ell(w_1 \cdot \ldots \cdot w_l)$, which implies that $w_1 \cdot \ldots \cdot w_l \in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{e}}^{2l}}$.

Let us prove part b). We have the following chain of equalities

$$w \cdot P(\widehat{\underline{\mathbf{d}}}) = w \cdot P(\underline{\mathbf{d}}) \Cap P(\underline{\mathbf{e}}) = u \cdot P(\widehat{\underline{\mathbf{e}}}) = \widetilde{u} \cdot P(u \cdot \widehat{\underline{\mathbf{e}}}) = \widetilde{u} \cdot P(\widehat{\underline{\mathbf{d}}}). \tag{5.11}$$

For the first equality, note that $\widehat{\underline{\mathbf{d}}} = C(w \cdot P(\underline{\mathbf{d}}) \Cap P(\underline{\mathbf{e}})) = C(P(\underline{\mathbf{d}}) \Cap w^{-1} \cdot P(\underline{\mathbf{e}}))$. Hence, switching the roles of $\underline{\mathbf{e}}$ and $\underline{\mathbf{d}}$ in Lemma 5.3.18, we get $P(\widehat{\underline{\mathbf{d}}}) = P(\underline{\mathbf{d}}) \Cap w^{-1} \cdot P(\underline{\mathbf{e}})$. After acting on both sides by w we get the first equality. The second equality follows directly from the definition of u, while the third equality follows from a repeated application of Lemma 5.3.10.f). The final equality holds since $u \cdot \widehat{\underline{\mathbf{e}}} = u \cdot C(P(\underline{\mathbf{e}}) \Cap w \cdot P(\underline{\mathbf{d}})) = C(u(\cdot P(\underline{\mathbf{e}}) \Cap w \cdot P(\underline{\mathbf{d}}))) = C(w \cdot P(\underline{\mathbf{d}}) \Cap P(\underline{\mathbf{e}})) = \widehat{\underline{\mathbf{d}}}$. The claim that $\widetilde{u} = w$ now follows from (5.11), the fact that $\mathrm{Stab}_{\mathsf{W}_{\mathbf{c}}}(P(\widehat{\underline{\mathbf{d}}})) = \mathsf{W}_{\widehat{\underline{\mathbf{d}}}}$ and that both w and $\widetilde{u}$ are in $\mathsf{D}^{\mathbf{c}}_{\widehat{\underline{\mathbf{d}}}}$.

Part c) follows from the following chain of equalities

$$\mathsf{W}_{\underline{\mathbf{e}}^0} = \mathrm{Stab}_{\mathsf{W}_{\mathbf{c}}}(P \circ C(\lambda \Cap \mu)) = \mathrm{Stab}_{\mathsf{W}_{\mathbf{c}}}(\lambda \Cap \mu) = \mathsf{W}_{\underline{\mathbf{e}}} \cap (w \mathsf{W}_{\underline{\mathbf{d}}} w^{-1}),$$

where $\lambda = P(\underline{\mathbf{e}})$ and $\mu = w \cdot P(\underline{\mathbf{d}})$. The first equality follows from the fact that, by definition, $\underline{\mathbf{e}}^0 = \widehat{\underline{\mathbf{e}}} = C(\lambda \Cap \mu)$, and Lemma 5.3.10.d). The second equality follows from Lemma 5.3.18 while the third equality follows from Lemma 5.3.12.c) and Lemma 5.3.10.d). ☐

We have so far treated the ordered intersection operation $\Cap$ on partitionings as a combinatorial device. However, as indicated in Lemma 5.3.8, partitionings also have a geometric meaning since they correspond to coordinate flags in $\mathbf{V}_{\mathbf{c}}$. Based on this observation, we will extend the operation $\Cap$ to all flags in $\mathfrak{F}_{\mathbf{c}}$.

Definition 5.3.22. Let $F = (F_i)_{i=0}^{\ell_{\mathbf{e}}} \in \mathfrak{F}_{\underline{\mathbf{e}}}$ and $F' = (F_j)_{j=0}^{\ell_{\mathbf{d}}} \in \mathfrak{F}_{\underline{\mathbf{d}}}$ be two flags. Let $R_{i,j} = F_i \cap F'_j$ and $S_{F,F'} = \{R_{i,j} \mid 0 \leq i \leq \ell_{\underline{\mathbf{e}}},\ 0 \leq j \leq \ell_{\underline{\mathbf{d}}}\}$. We put a total order on the set $S_{F,F'}$ by declaring that $R_{i,j} < R_{r,s}$ if and only if $r > i$ or $r = i$ and $j < s$. We then define the *ordered intersection* $F \Cap F'$ of flags F and F' by setting $(F \Cap F')_m$ to be the m-th element of $S_{F,F'}$ with respect to the total order defined above, and deleting all the repeated occurrences of subspaces.

It is clear that $F \Cap F'$ is a flag in $\mathfrak{F}_{\mathbf{c}}$. Moreover, if $(F, F') \in \mathcal{O}_w^{\Delta}$ then $F \Cap F' \in \mathfrak{F}_{\widehat{\underline{\mathbf{e}}}}$.

Lemma 5.3.23. *Let $\rho \in \mathfrak{R}_{\mathbf{c}}$. If $F \in \mathfrak{F}_{\underline{\mathbf{e}}}$ and $F' \in \mathfrak{F}_{\underline{\mathbf{d}}}$ are ρ-stable, then so are $F \Cap F'$ and $F' \Cap F$.*

Proof. Since F and F' are ρ-stable, each intersection $R_{i,j} = F_i \cap F'_j$ is preserved by ρ, which implies that $F \Cap F'$ and $F' \Cap F$ are also ρ-stable. ☐

5.3.5. Basis and generators. In this section we state a basis theorem for the quiver Schur algebra and use it to find a convenient generating set for our algebra.

Definition 5.3.24. Let $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$ be an orbit datum with the corresponding refinement datum $(\widehat{\underline{\mathbf{e}}}, \widehat{\underline{\mathbf{d}}}, u)$. Let us fix a reduced decomposition $u = s_{j_k} \cdot \ldots \cdot s_{j_1}$, which determines the corresponding crossing datum $(\underline{\mathbf{e}}^0, \ldots, \underline{\mathbf{e}}^{2k})$. Define

$$\underline{\mathbf{d}} \underset{w}{\Longrightarrow} \underline{\mathbf{e}} \quad := \quad \curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0} \star \chi^{j_1}_{\underline{\mathbf{e}}^2} \star \ldots \star \chi^{j_k}_{\underline{\mathbf{e}}^{2k}} \star \curlyvee^{\underline{\mathbf{e}}^{2k}}_{\underline{\mathbf{d}}} \in \mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}.$$

Given $c \in \mathcal{Z}^{e}_{\underline{\mathbf{e}}^{2k},\underline{\mathbf{e}}^{2k}} \cong \Lambda_{\underline{\mathbf{e}}^{2k}}$, also define

$$\underline{\mathbf{d}} \overset{c}{\underset{w}{\Longrightarrow}} \underline{\mathbf{e}} \quad := \quad \curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^{0}} \star \curlyvee^{j_1}_{\underline{\mathbf{e}}^{2}} \star \ldots \star \curlyvee^{j_k}_{\underline{\mathbf{e}}^{2k}} \star c \star \Upsilon^{\underline{\mathbf{e}}^{2k}}_{\underline{\mathbf{d}}} \in \mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}. \tag{5.12}$$

We now state the basis theorem for the quiver Schur algebra.

Theorem 5.3.25. *If we let $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$ range over all orbit data and c range over a basis of $\Lambda_{\underline{\mathbf{e}}^{2k}}$, then the elements $\underline{\mathbf{d}} \overset{c}{\underset{w}{\Longrightarrow}} \underline{\mathbf{e}}$ form a basis of $\mathcal{Z}_{\mathbf{c}}$.*

Remark 5.3.26. We make a few remarks regarding the theorem.

(i) The basis in Theorem 5.3.25 depends on the choice of a crossing datum, i.e., the choice of a reduced expression for u. One would also obtain other bases by letting c range over a basis of $\mathcal{Z}^{e}_{\underline{\mathbf{e}}^{2l},\underline{\mathbf{e}}^{2l}}$, for any $1 \leq l \leq k$, and redefining the elements (5.12) appropriately.

(ii) The basis in Theorem 5.3.25 is natural from a geometric point of view. As we will see in §5.3.6, it is related to certain generalizations of Bott-Samelson varieties. For this reason, we call it a *Bott-Samelson basis* of the quiver Schur algebra. Our basis also admits a natural interpretation in terms of cohomological Hall algebras (see Theorem 5.6.6).

(iii) Theorem 5.3.25 is analogous to the basis theorem [131, Theorem 3.13] for the Stroppel-Webster quiver Schur algebra. However, the combinatorics of residue sequences developed in [131] is not sufficient to correctly characterize the refined vector compositions used in the definition of the basis. Our combinatorics of partitionings (§5.3.4) fixes this problem.

Using Theorem 5.3.25, we can find a generating set for the quiver Schur algebra.

Corollary 5.3.27. *Elementary merges, elementary splits and the polynomials $\mathcal{Z}^{e}_{\mathbf{c}}$ generate $\mathcal{Z}_{\mathbf{c}}$ as an algebra.*

Proof. This follows directly from Theorem 5.3.25 and Proposition 5.3.5. □

5.3.6. Proof of the basis theorem. We begin by proving four technical lemmas. Set $\mathcal{O}_w := \mathsf{P}_{\underline{\mathbf{e}}} w \mathsf{P}_{\underline{\mathbf{d}}}/\mathsf{P}_{\underline{\mathbf{d}}}$. Consider the following parabolic analogue of the Bott-Samelson variety:

$$\mathfrak{BS}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} := \mathsf{P}_{\underline{\mathbf{e}}} \times_{\mathsf{P}_{\underline{\mathbf{e}}^0}} \mathsf{P}_{\underline{\mathbf{e}}^1} \times_{\mathsf{P}_{\underline{\mathbf{e}}^2}} \ldots \times_{\mathsf{P}_{\underline{\mathbf{e}}^{2k-2}}} \mathsf{P}_{\underline{\mathbf{e}}^{2k-1}} \times_{\mathsf{P}_{\underline{\mathbf{e}}^{2k}}} \mathsf{P}_{\underline{\mathbf{d}}}/\mathsf{P}_{\underline{\mathbf{d}}}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathsf{P}_{\underline{\mathbf{e}}} \times \mathsf{P}_{\underline{\mathbf{e}}^1} \times \mathsf{P}_{\underline{\mathbf{e}}^3} \times \ldots \times \mathsf{P}_{\underline{\mathbf{e}}^{2k-1}} \times \mathsf{P}_{\underline{\mathbf{d}}} & \xrightarrow{\ m\ } & \mathsf{G}_{\mathbf{c}} \\ {\scriptstyle can_1}\downarrow & & \downarrow{\scriptstyle can_2} \\ \mathfrak{BS}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} & \xrightarrow{\quad \phi \quad} & \mathfrak{F}_{\underline{\mathbf{d}}} \end{array} \tag{5.13}$$

where m is the multiplication map and ϕ is the induced map.

Lemma 5.3.28. *The map ϕ is proper, its image equals $\overline{\mathcal{O}_w}$, and it restricts to an isomorphism over $\mathcal{O}_w$.*

Proof. The proof is similar to the proof for the usual Bott-Samelson resolution, where the parabolics $\mathsf{P}_{\underline{\mathbf{e}}}$ and $\mathsf{P}_{\underline{\mathbf{d}}}$ are replaced with a Borel subgroup. Since we could not find an explicit reference, and since the proof relies on the combinatorics from §5.3.4, we sketch it below.

It is clear that can_1 and can_2 are proper. The multiplication map $(\mathsf{G}_{\mathbf{c}})^{k+2} \to \mathsf{G}_{\mathbf{c}}$ is proper and hence its restriction to the closed submanifold $\mathsf{P}_{\underline{\mathbf{e}}} \times \mathsf{P}_{\underline{\mathbf{e}}^1} \times \ldots \times \mathsf{P}_{\underline{\mathbf{e}}^{2k-1}} \times \mathsf{P}_{\underline{\mathbf{d}}}$ is proper as well. Since can_1

is a locally trivial fibration, and properness is a local property, it follows that ϕ is proper, as required. This proves the first statement in the lemma.

By the Bruhat decomposition, we have

$$\mathsf{P}_{\underline{\mathbf{e}}^{2l-1}} = \bigsqcup_{x\in \mathsf{D}^{\underline{\mathbf{e}}^{2l-1}}_{\underline{\mathbf{e}}^{2l}}} \mathsf{B}_{\mathbf{c}} x \mathsf{P}_{\underline{\mathbf{e}}^{2l}}.$$

Proposition 5.3.21.a) implies that

$$\mathsf{P}_{\underline{\mathbf{e}}^{2l-1}} \cdot \mathsf{P}_{\underline{\mathbf{e}}^{2l+1}} = \bigcup_{x\in \mathsf{D}^{\underline{\mathbf{e}}^{2l-1}}_{\underline{\mathbf{e}}^{2l}}} \mathsf{B}_{\mathbf{c}} x \mathsf{P}_{\underline{\mathbf{e}}^{2l+1}} = \bigsqcup_{x\le w_l w_{l+1}\in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{e}}^{2l+2}}} \mathsf{B}_{\mathbf{c}} x \mathsf{P}_{\underline{\mathbf{e}}^{2l+2}}.$$

By induction and Proposition 5.3.21.b), we get

$$\mathsf{P}_{\underline{\mathbf{e}}} \cdot \mathsf{P}_{\underline{\mathbf{e}}^1} \cdot \ldots \cdot \mathsf{P}_{\underline{\mathbf{e}}^{2k-1}} \cdot \mathsf{P}_{\underline{\mathbf{d}}} = \bigsqcup_{x\le w\in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}} \mathsf{P}_{\underline{\mathbf{e}}} x \mathsf{P}_{\underline{\mathbf{d}}}. \tag{5.14}$$

This implies that the image of $can_2 \circ m$ is $\overline{\mathcal{O}_w}$. Since can_1 is surjective, this is also the image of ϕ. This proves the second statement of the lemma.

Next, we claim that

$$\mathsf{P}_{\underline{\mathbf{e}}} w \mathsf{P}_{\underline{\mathbf{d}}} = \bigcup_{y\in \mathsf{W}_{\underline{\mathbf{e}}}} \mathsf{B}_{\mathbf{c}} y w \mathsf{P}_{\underline{\mathbf{d}}} = \bigsqcup_{y\in \mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0}} \mathsf{B}_{\mathbf{c}} y w \mathsf{P}_{\underline{\mathbf{d}}}. \tag{5.15}$$

Let us first show that the union on the RHS of (5.15) is disjoint. Suppose that $y_1 w = y_2 w x$ for some $y_1, y_2 \in \mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0}$ and $x \in \mathsf{W}_{\underline{\mathbf{d}}}$. Then $y_2^{-1} y_1 = w x w^{-1}$, which, by Proposition 5.3.21.c), implies that $y_2^{-1} y_1 \in \mathsf{W}_{\underline{\mathbf{e}}} \cap w \mathsf{W}_{\underline{\mathbf{d}}} w^{-1} = \mathsf{W}_{\underline{\mathbf{e}}^0}$. Hence $y_1 = y_2$, so the union is indeed disjoint. The first equality in (5.15) follows from the Bruhat decomposition of $\mathsf{P}_{\underline{\mathbf{e}}}$ and the fact that $w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$. Next, if $y \in \mathsf{W}_{\underline{\mathbf{e}}}$, we can write it as $y = ab$ with $a \in \mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0}$ and $b \in \mathsf{W}_{\underline{\mathbf{e}}^0}$. But then $yw\mathsf{P}_{\underline{\mathbf{d}}} = aw\mathsf{P}_{\underline{\mathbf{d}}}$, which yields the second equality in (5.15).

Given $y \in \mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0}$, let $\widetilde{U}_y := \mathsf{U}_{y^{-1}} y \times \mathsf{U}_{w_1^{-1}} w_1 \times \ldots \times \mathsf{U}_{w_k^{-1}} w_k$. Set $\widetilde{U} = \bigsqcup_{y\in \mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0}} \widetilde{U}_y$ and $U = can_1(\widetilde{U})$. It is easy to check that m maps $\widetilde{U}_y$ isomorphically onto $\mathsf{U}_{(yw)^{-1}} yw$. It is also well known that the map

$$\mathsf{U}_{(yw)^{-1}} \times \mathsf{P}_{\underline{\mathbf{d}}} \to \mathsf{B}_{\mathbf{c}} y w \mathsf{P}_{\underline{\mathbf{d}}}, \quad (v,p) \mapsto v(yw)p,$$

is an isomorphism. Hence $can_2 \circ m$ maps $\widetilde{U}$ isomorphically onto $\mathcal{O}_w$. By the commutativity of the diagram (5.13), $\phi|_U$ is also an isomorphism onto $\mathcal{O}_w$. An easy argument again based on the Bruhat decomposition also shows that $U = \phi^{-1}(\mathcal{O}_w)$. This completes the proof of the lemma. □

Let $\mathcal{O}_w^\Delta := \mathsf{G}_{\mathbf{c}} \cdot (e\mathsf{P}_{\underline{\mathbf{e}}}, w\mathsf{P}_{\underline{\mathbf{d}}})$ be the diagonal $\mathsf{G}_{\mathbf{c}}$-orbit corresponding to w and define

$$\mathfrak{B}\mathfrak{S}^\Delta_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} := \mathfrak{F}_{\underline{\mathbf{e}}^0} \times_{\mathfrak{F}_{\underline{\mathbf{e}}^1}} \mathfrak{F}_{\underline{\mathbf{e}}^2} \times_{\mathfrak{F}_{\underline{\mathbf{e}}^3}} \ldots \times_{\mathfrak{F}_{\underline{\mathbf{e}}^{2k-1}}} \mathfrak{F}_{\underline{\mathbf{e}}^{2k}}.$$

Let $p\colon \mathfrak{B}\mathfrak{S}^\Delta_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} \to \mathfrak{F}_{\underline{\mathbf{e}}^0} \to \mathfrak{F}_{\underline{\mathbf{e}}}$ and $q\colon \mathfrak{B}\mathfrak{S}^\Delta_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} \to \mathfrak{F}_{\underline{\mathbf{e}}^{2k}} \to \mathfrak{F}_{\underline{\mathbf{d}}}$ be the canonical maps. Set $\psi := p \times q$.

Consider the following diagram of $\mathsf{G}_{\mathbf{c}}$-equivariant maps.

$$\begin{array}{ccc} \mathfrak{BS}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} & \xrightarrow{\psi} & \mathfrak{F}_{\underline{\mathbf{e}}} \times \mathfrak{F}_{\underline{\mathbf{d}}} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle can} & \\ & \mathfrak{F}_{\underline{\mathbf{e}}} & \end{array}$$

Lemma 5.3.29. *The map ψ is proper, its image equals $\overline{\mathcal{O}^{\Delta}_w}$, and it restricts to an isomorphism over $\mathcal{O}^{\Delta}_w$.*

Proof. It is easy to check that p is a locally trivial fibration with fibre $\mathfrak{BS}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$. Let $F_e = p^{-1}(e\mathsf{P}_{\underline{\mathbf{e}}}/\mathsf{P}_{\underline{\mathbf{e}}})$. Lemma 5.3.28 implies that $\psi|_{F_e}$ is proper, $\psi(F_e) = \{(e\mathsf{P}_{\underline{\mathbf{e}}}/\mathsf{P}_{\underline{\mathbf{e}}}, x) \mid x \in \overline{\mathcal{O}_w}\}$ and that the restriction of ψ to $F_e \cap \psi^{-1}(\mathcal{O}^{\Delta}_w)$ is an isomorphism onto $\{(e\mathsf{P}_{\underline{\mathbf{e}}}/\mathsf{P}_{\underline{\mathbf{e}}}, x) \mid x \in \mathcal{O}_w\}$. The lemma now follows from the fact that ψ is $\mathsf{G}_{\mathbf{c}}$-equivariant. $\square$

Next consider the following iterated fibre product

$$\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} := \mathfrak{Q}_{\underline{\mathbf{e}}^0} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^1}} \mathfrak{Q}_{\underline{\mathbf{e}}^2} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^3}} \dots \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^{2k-1}}} \mathfrak{Q}_{\underline{\mathbf{e}}^{2k}}.$$

The closed points of $\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$ correspond to sequences of flags $F^0, F^2, \dots, F^{2k}$ (satisfying appropriate conditions) together with a quiver representation ρ with respect to which each flag is stable. This implies that ψ lifts to a map $\widetilde{\psi}\colon \widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} \to \mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ (sending $(F^0, \dots, F^{2k}, \rho) \mapsto (F^0|_{\underline{\mathbf{e}}}, F^{2k}|_{\underline{\mathbf{d}}}, \rho)$), i.e., there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} & \xrightarrow{\widetilde{\psi}} & \mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \\ {\scriptstyle \widetilde{\pi}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\underline{\mathbf{e}},\underline{\mathbf{d}}}} \\ \mathfrak{BS}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} & \xrightarrow{\psi} & \mathfrak{F}_{\underline{\mathbf{e}}} \times \mathfrak{F}_{\underline{\mathbf{d}}} \end{array} \tag{5.16}$$

where $\widetilde{\pi}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ is the map forgetting the quiver representation.

Lemma 5.3.30. *The map $\widetilde{\psi}$ is proper, its image is contained in $\mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$, and it restricts to an isomorphism over $\mathfrak{Z}^{w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$.*

Proof. It is easy to see that $\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$ is a closed subset inside the fibre product of $\mathfrak{BS}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$ and $\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ over $\mathfrak{F}_{\underline{\mathbf{e}}} \times \mathfrak{F}_{\underline{\mathbf{d}}}$. So $\widetilde{\psi}$ is the restriction of a base change of a proper map to a closed subset, and is therefore proper. By Lemma 5.3.29, $\operatorname{Im}\psi = \overline{\mathcal{O}^{\Delta}_w}$, so $\operatorname{Im}\widetilde{\psi} \subseteq \pi^{-1}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}(\overline{\mathcal{O}^{\Delta}_w}) = \mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$.

It remains to prove the third statement. Let $(F, F', \rho) \in \mathfrak{Z}^{w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$. By Lemma 5.3.29, we know that there exists a unique sequence of flags $(F^0, F^2, \dots, F^{2k}) \in \mathfrak{BS}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$ such that $F = F^0|_{\underline{\mathbf{e}}}$ and $F' = F^{2k}|_{\underline{\mathbf{d}}}$. Since $F^0 = F \between F'$ and $F^{2k} = F' \between F$, Lemma 5.3.23 implies that F^0 and F^{2k} are ρ-stable. Let $1 \leq l \leq k-1$. Since F^{2l} is in relative position $w_1 \cdot \ldots \cdot w_l$ to F^0, F^{2k} is in relative position $w_{l+1} \cdot \ldots \cdot w_k$ to F^{2l} and, by Proposition 5.3.21, $\ell(w) = \ell(w_1 \cdot \ldots \cdot w_l) + \ell(w_{l+1} \cdot \ldots \cdot w_k)$, F^{2l} is also ρ-stable. Hence $(F^0, F^2, \dots, F^{2k}, \rho) \in \widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}$ and so $\widetilde{\psi}$ is surjective. This clearly implies that $\widetilde{\psi}$ is an isomorphism. $\square$

We have the following Cartesian diagram.

$$\begin{array}{ccc} \mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} & \xhookrightarrow{\iota_1} & \mathfrak{Z}^{\leqslant w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \\ \uparrow \wr & & \uparrow \widetilde{\psi} \\ \widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}) & \xhookrightarrow{\iota_2} & \widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} \end{array}$$

Let $q_0 \colon \mathfrak{Q}_{\underline{\mathbf{e}}^0} \times \mathfrak{Q}_{\underline{\mathbf{e}}^2} \times \ldots \times \mathfrak{Q}_{\underline{\mathbf{e}}^{2k}} \to \mathfrak{Q}_{\underline{\mathbf{e}}^{2k}}$ be the canonical map.

Lemma 5.3.31. *Let $B(\Lambda_{\underline{\mathbf{d}}^0})$ be a basis of $\Lambda_{\underline{\mathbf{d}}^0} \cong H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Q}_{\underline{\mathbf{d}}^0})$ and $c \in B(\Lambda_{\underline{\mathbf{d}}^0})$. Then:*

a) $\widetilde{\psi}_*(q_0^* c \cap [\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}]) \;=\; \underline{\mathbf{d}} \underset{w}{\overset{c}{\Longrightarrow}} \underline{\mathbf{e}}$.

b) $\{\iota_1^*(\underline{\mathbf{d}} \underset{w}{\overset{c}{\Longrightarrow}} \underline{\mathbf{e}}) \mid c \in B(\Lambda_{\underline{\mathbf{d}}^0})\}$ *is a basis of* $\mathcal{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$.

Proof. The first part of the lemma follows directly from Lemma 5.2.9 and the fact that

$$\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w} \cong \mathfrak{Z}^e_{\underline{\mathbf{e}},\underline{\mathbf{e}}^0} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^0}} \mathfrak{Z}^e_{\underline{\mathbf{e}}^0,\underline{\mathbf{e}}^1} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^1}} \mathfrak{Z}^e_{\underline{\mathbf{e}}^1,\underline{\mathbf{e}}^2} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^2}} \cdots \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^{2k-1}}} \mathfrak{Z}^e_{\underline{\mathbf{e}}^{2k-1},\underline{\mathbf{e}}^{2k}} \times_{\mathfrak{Q}_{\underline{\mathbf{e}}^{2k}}} \mathfrak{Z}^e_{\underline{\mathbf{e}}^{2k},\underline{\mathbf{d}}}.$$

For the second part, we have

$$\iota_1^*(\underline{\mathbf{d}} \underset{w}{\overset{c}{\Longrightarrow}} \underline{\mathbf{e}}) = \iota_1^*\widetilde{\psi}_*(q_0^* c \cap [\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}]) = \widetilde{\psi}_*\iota_2^*(q_0^* c \cap [\widetilde{\mathfrak{BS}}^{\Delta}_{\underline{\mathbf{e}},\underline{\mathbf{d}},w}]) = \widetilde{\psi}_*(q_0^* c \cap [\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})]),$$

where the second equality follows by proper base change. It now suffices to check that $q_0^* c \cap [\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})]$ form a basis of $H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}))$. Since the variety $\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})$ is an affine bundle over $\mathcal{O}^{\Delta}_w \cong \mathsf{G}_{\mathbf{c}}/(w^{-1}\mathsf{P}_{\underline{\mathbf{e}}} w \cap \mathsf{P}_{\underline{\mathbf{d}}})$, which itself is an affine bundle over $\mathfrak{F}_{\underline{\mathbf{d}}^0}$, the restriction of q_0 to $\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})$ is a homotopy equivalence. Hence the map $\Lambda_{\underline{\mathbf{d}}^0} \to H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}))$ sending $c \mapsto q_0^* c \cap [\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})]$ is an isomorphism, and sends the basis $B(\Lambda_{\underline{\mathbf{d}}^0})$ to a basis of $H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\widetilde{\psi}^{-1}(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}))$, as required. ☐

We are now ready to prove Theorem 5.3.25.

Proof of Theorem 5.3.25. Refine the Bruhat order on ${}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$ to a linear order, which we denote by $\trianglelefteq$. Given $w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$, let $\mathfrak{Z}^{\trianglelefteq w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} := \bigsqcup_{{}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}} \ni u \trianglelefteq w} \mathfrak{Z}^u_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ and $\mathfrak{Z}^{\triangleleft w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} = \mathfrak{Z}^{\trianglelefteq w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \backslash \mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$. Consider the inclusions

$$\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \overset{i}{\hookrightarrow} \mathfrak{Z}^{\trianglelefteq w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} \overset{j}{\hookleftarrow} \mathfrak{Z}^{\triangleleft w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}. \tag{5.17}$$

It follows from Lemma 5.2.8 that i is an open embedding and j is a closed embedding. Since the odd cohomology of $\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ and $\mathfrak{Z}^{\triangleleft w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ vanishes, the long exact sequence in $\mathsf{G}_{\mathbf{c}}$-equivariant Borel-Moore homology associated to (5.17) becomes a short exact sequence of $H^\bullet_{\mathsf{G}_{\mathbf{c}}}(pt)$-modules

$$0 \to H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}^{\triangleleft w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}) \to H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}^{\trianglelefteq w}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}) \to H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}) \to 0.$$

Since, by equivariant formality, the $H^\bullet_{\mathsf{G}_{\mathbf{c}}}(pt)$-module $H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}})$ is free, the short exact sequence splits. Arguing by induction on the refined Bruhat order, we conclude that

$$\mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}} = \bigoplus_{w \in {}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}} \mathcal{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}. \tag{5.18}$$

By Lemma 5.3.31, the elements $\underline{\mathbf{d}} \underset{w}{\overset{c}{\Longrightarrow}} \underline{\mathbf{e}} \in \mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$ ($c \in B(\Lambda_{\underline{\mathbf{d}}^0})$) pull back to a basis of $\mathcal{Z}^w_{\underline{\mathbf{e}},\underline{\mathbf{d}}}$. Therefore, (5.18) implies that, if we let $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$ range over all orbit data, the elements $\underline{\mathbf{d}} \underset{w}{\overset{c}{\Longrightarrow}} \underline{\mathbf{e}}$ indeed form a basis of $\mathcal{Z}_{\mathbf{c}}$. □

5.4 The polynomial representation

In this section we compute the polynomial representation of the quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$, and use it to show that $\mathcal{Z}_{\mathbf{c}}$ gives a geometric realization of the "modified quiver Schur algebra" from [99], thereby connecting $\mathcal{Z}_{\mathbf{c}}$ to the affine q-Schur algebra. We also give a complete list of relations for the quiver Schur algebra associated to the A_1 and Jordan quivers.

5.4.1. $\mathsf{T}_{\mathbf{c}}$-equivariant cohomology and localization. We first recall some facts about $\mathsf{T}_{\mathbf{c}}$-equivariant cohomology of flag varieties. By [134, Theorem 3], there is a ring isomorphism

$$\Phi_{\underline{\mathbf{d}}}\colon H^\bullet(B\mathsf{T}_{\mathbf{c}}) \otimes_{H^\bullet(B\mathsf{G}_{\mathbf{c}})} H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}}) \cong H^\bullet_{\mathsf{T}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}}), \quad a \otimes b \mapsto (\gamma_1^* a)(\gamma_2^* b),$$

where γ_1 is the projection of $\mathfrak{F}_{\underline{\mathbf{d}}}$ onto a point and γ_2 is the canonical map $(\mathfrak{F}_{\underline{\mathbf{d}}})_{\mathsf{T}_{\mathbf{c}}} \twoheadrightarrow (\mathfrak{F}_{\underline{\mathbf{d}}})_{\mathsf{G}_{\mathbf{c}}}$. Let us write a polynomial f in variables $x_j(i)$ as $f(\vec{x})$. We abbreviate $x_j(i) := \Phi_{\underline{\mathbf{d}}}(x_j(i) \otimes 1)$ and $f(\vec{y}) := \Phi_{\underline{\mathbf{d}}}(1 \otimes f(\vec{x}))$ (we substitute variables $y_j(i)$ for $x_j(i)$).

Definition 5.4.1. For $\underline{\mathbf{d}} \succ \underline{\mathbf{e}}$, we define the following polynomials in $\Lambda_{\underline{\mathbf{d}}}$:

$$\mathsf{S}_{\underline{\mathbf{d}}} := \prod_{i \in Q_0} \prod_{r=1}^{\ell_{\underline{\mathbf{d}}}-1} \prod_{k=\mathring{\mathbf{d}}_{r-1}(i)+1}^{\mathring{\mathbf{d}}_r(i)} \prod_{l=\mathring{\mathbf{d}}_r(i)+1}^{\mathbf{c}(i)} (x_l(i) - x_k(i)), \qquad \mathsf{S}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} := \frac{\mathsf{S}_{\underline{\mathbf{d}}}}{\mathsf{S}_{\underline{\mathbf{e}}}}. \tag{5.19}$$

Note that $\mathsf{S}_{\underline{\mathbf{d}}}$ is indeed $\mathsf{W}_{\underline{\mathbf{d}}}$-invariant and $\mathsf{S}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}$ is a polynomial. Explicit examples of these polynomials for specific quivers and dimension vectors can be found in [99, §8].

It is well known that the fixed points $\mathfrak{F}_{\underline{\mathbf{d}}}^{\mathsf{T}_{\mathbf{c}}}$ are parametrized by $\mathsf{D}_{\underline{\mathbf{d}}}^{\mathbf{c}}$. Given $w \in \mathsf{D}_{\underline{\mathbf{d}}}^{\mathbf{c}}$, let $i_w\colon \{w\} \hookrightarrow \mathfrak{F}_{\underline{\mathbf{d}}}$ be the inclusion of the corresponding fixed point, and let $\zeta_w = [w] \in H_\bullet^{\mathsf{T}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}})$ denote the $\mathsf{T}_{\mathbf{c}}$-equivariant fundamental class of this fixed point.

The theorem below summarizes the key facts about the $\mathsf{T}_{\mathbf{c}}$-equivariant cohomology of flag varieties.

Theorem 5.4.2. *The following hold:*

a) *The $\mathsf{T}_{\mathbf{c}}$-equivariant cohomology of $\mathfrak{F}_{\underline{\mathbf{d}}}$ is equal to the quotient*

$$H^\bullet_{\mathsf{T}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}}) = (\mathcal{P}_{\mathbf{c}} \otimes \Phi_{\underline{\mathbf{d}}}(\Lambda_{\underline{\mathbf{d}}}))/I,$$

where I is the ideal generated by $p(\vec{y}) - p(\vec{x})$ as p ranges over all $\mathsf{W}_{\mathbf{c}}$-invariant polynomials of positive degree.

b) *The pullback $i_w^*\colon H^*_{\mathsf{T}_{\mathbf{c}}}(\mathfrak{F}_{\underline{\mathbf{d}}}) \to H^*(B\mathsf{T}_{\mathbf{c}})$ is given by*

$$i_w^*(x_j(i)) = x_j(i), \quad i_w^*(f(\vec{y})) = f(\vec{x}).$$

c) *The $\mathsf{T}_{\mathbf{c}}$-equivariant Euler class of the normal bundle to the fixed point w is given by*

$$\mathtt{eu}_{\mathsf{T}_{\mathbf{c}}}(T_{\{w\}}\mathfrak{F}_{\underline{\mathbf{d}}}) = w \cdot \mathsf{S}_{\underline{\mathbf{d}}}.$$

Proof. See, e.g., [134, Theorem 11]. □

Next, we recall the localization theorem for equivariant cohomology (see, e.g., [22]). Let $\mathcal{K}_{\mathbf{c}}$ denote the fraction field of $\mathcal{P}_{\mathbf{c}} = H^\bullet(B\mathsf{T}_{\mathbf{c}})$.

Theorem 5.4.3. *Let X be a smooth quasi-projective $\mathsf{T}_{\mathbf{c}}$-variety and let Y be the set of connected components of the fixed point set $X^{\mathsf{T}_{\mathbf{c}}}$. Suppose that Y is finite. Then the maps*

$$\mathcal{K}_{\mathbf{c}} \otimes_{\mathcal{P}_{\mathbf{c}}} H^\bullet_{\mathsf{T}_{\mathbf{c}}}(X) \xrightarrow{i^*} \mathcal{K}_{\mathbf{c}} \otimes_{\mathcal{P}_{\mathbf{c}}} H^\bullet_{\mathsf{T}_{\mathbf{c}}}(X^{\mathsf{T}_{\mathbf{c}}}) \xrightarrow{i_*} \mathcal{K}_{\mathbf{c}} \otimes_{\mathcal{P}_{\mathbf{c}}} H^\bullet_{\mathsf{T}_{\mathbf{c}}}(X)$$

are isomorphisms and

$$u = \sum_{y \in Y} \frac{(i_y)_* i_y^*(u)}{\mathtt{eu}_{\mathsf{T}_{\mathbf{c}}}(T_y X)}$$

for all $u \in H^\bullet_{\mathsf{T}_{\mathbf{c}}}(X)$. Here $i\colon X^{\mathsf{T}_{\mathbf{c}}} \hookrightarrow X$ and $i_y\colon y \hookrightarrow X$ are the natural inclusions.

5.4.2. The polynomial representation.

In this subsection we describe the polynomial representation $\mathcal{Q}_{\mathbf{c}}$ of $\mathcal{Z}_{\mathbf{c}}$. The following result is standard.

Proposition 5.4.4. *The $\mathcal{Z}_{\mathbf{c}}$-module $\mathcal{Q}_{\mathbf{c}}$ is faithful.*

Proof. The proof is standard - similar proofs may be found in, e.g., [138, Lemma 1.8(a)] and [136, Proposition 3.1]. Therefore we restrict ourselves to summarizing the main idea of the proof. Consider the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{K}_{\mathbf{c}} \otimes_{\mathcal{P}_{\mathbf{c}}} H^{\mathsf{T}_{\mathbf{c}}}_\bullet(\mathfrak{Z}_{\mathbf{c}}) & \xrightarrow{c} & \mathcal{K}_{\mathbf{c}} \otimes_{\mathcal{P}_{\mathbf{c}}} \mathrm{End}_{\mathcal{P}_{\mathbf{c}}}(H^{\mathsf{T}_{\mathbf{c}}}_\bullet(\mathfrak{Q}_{\mathbf{c}})) \\
{\scriptstyle b}\uparrow & & \uparrow \\
H^{\mathsf{T}_{\mathbf{c}}}_\bullet(\mathfrak{Z}_{\mathbf{c}}) & \longrightarrow & \mathrm{End}_{\mathcal{P}_{\mathbf{c}}}(H^{\mathsf{T}_{\mathbf{c}}}_\bullet(\mathfrak{Q}_{\mathbf{c}})) \\
{\scriptstyle a}\uparrow & & \uparrow \\
H^{\mathsf{G}_{\mathbf{c}}}_\bullet(\mathfrak{Z}_{\mathbf{c}}) & \longrightarrow & \mathrm{End}_{H^{\mathsf{G}_{\mathbf{c}}}_\bullet(pt)}(H^\bullet_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\mathbf{c}}))
\end{array}$$

It is well known (see, e.g., [22, Proposition 1]) that a is injective. Since the $\mathsf{T}_{\mathbf{c}}$-variety $\mathfrak{Z}_{\mathbf{c}}$ is equivariantly formal, b is injective as well, and the injectivity of c follows from a direct calculation of the convolution product on the torus fixed points. Since the diagram is commutative, we conclude that the lower horizontal map must be injective, as required. □

We will now calculate the action of the generators of the quiver Schur algebra on its polynomial representation. As preperation, we first compute the Euler classes of certain normal bundles.

Definition 5.4.5. For $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$, we define the following polynomials in $\Lambda_{\underline{\mathbf{d}}}$:

$$\mathtt{E}_{\underline{\mathbf{d}}} := \prod_{i,j \in Q_0} \prod_{r=1}^{\ell_{\underline{\mathbf{d}}}-1} \prod_{k=\mathring{\mathbf{d}}_{r-1}(i)+1}^{\mathring{\mathbf{d}}_r(i)} \prod_{l=\mathring{\mathbf{d}}_r(j)+1}^{\mathbf{c}(j)} (x_l(j) - x_k(i))^{a_{ij}}, \qquad \mathtt{E}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} := \frac{\mathtt{E}_{\underline{\mathbf{d}}}}{\mathtt{E}_{\underline{\mathbf{e}}}}, \tag{5.20}$$

where a_{ij} is the number of arrows from vertex i to j.

It is easy to see that $\mathtt{E}_{\underline{\mathbf{d}}}$ is $\mathsf{W}_{\underline{\mathbf{d}}}$-invariant and $\mathtt{E}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ is a polynomial. Explicit examples of these polynomials can be found in [99, §8].

Lemma 5.4.6. *We have $\mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{d}}}} \mathfrak{Q}_{\underline{\mathbf{d}},(\mathbf{c})}) = \mathtt{E}_{\underline{\mathbf{d}}}$.*

Proof. We identify $T_{\mathfrak{Q}_{\underline{\mathbf{d}}}} \mathfrak{Q}_{\underline{\mathbf{d}},(\mathbf{c})} \cong \mathsf{G}_{\mathbf{c}} \times^{\mathsf{P}_{\underline{\mathbf{d}}}} (\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{d}}})$ and $(\mathsf{G}_{\mathbf{c}} \times^{\mathsf{P}_{\underline{\mathbf{d}}}} (\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{e}}}))_{\mathsf{G}_{\mathbf{c}}} = (\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{d}}}}$. The pullback of the latter vector bundle on $B\mathsf{P}_{\underline{\mathbf{d}}}$ along the canonical map $B\mathsf{T}_{\mathbf{c}} \twoheadrightarrow B\mathsf{P}_{\underline{\mathbf{d}}}$ equals $(\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{T}_{\mathbf{c}}}$. Observe that

$$(\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{T}_{\mathbf{c}}} \cong \bigoplus_{i,j\in Q_0} \bigoplus_{r=1}^{\ell_{\underline{\mathbf{d}}}-1} \bigoplus_{k=\mathring{\mathbf{d}}_{r-1}(i)+1}^{\mathring{\mathbf{d}}_r(i)} \bigoplus_{l=\mathring{\mathbf{d}}_r(j)+1}^{\mathbf{c}(j)} (\mathfrak{V}_l(j)\otimes\mathfrak{V}_k(i)^*)^{\oplus a_{ij}},$$

where $\mathfrak{V}_l(j)$ is the line bundle from (5.4). By definition, the $\mathsf{T}_{\mathbf{c}}$-equivariant Euler class of the bundle on the RHS equals $\mathtt{E}_{\underline{\mathbf{d}}}$. Since Euler classes commute with pullbacks, it follows that

$$\mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},(\mathbf{c})}) = \mathtt{eu}_{\mathsf{T}_{\mathbf{c}}}((\mathfrak{R}_{\mathbf{c}}/\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{T}_{\mathbf{c}}}) = \mathtt{E}_{\underline{\mathbf{d}}},$$

as desired. □

We also need the following "shuffle operator"

$$\pitchfork_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} := \sum_{w\in\mathsf{D}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}} w \in \mathsf{W}_{\underline{\mathbf{e}}}.$$

Theorem 5.4.7. *Let $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$.*

a) *The action of $\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}$ on $\mathcal{Q}_{\mathbf{c}} \cong \Lambda_{\mathbf{c}}$ is given by*

$$\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}\colon \Lambda_{\underline{\mathbf{d}}} \to \Lambda_{\underline{\mathbf{e}}}, \quad f \mapsto \pitchfork_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}\left(\frac{\mathtt{E}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}}{\mathtt{S}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}} f\right), \qquad \curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}|_{\Lambda_{\underline{\mathbf{b}}}} = 0 \quad \text{if} \quad \underline{\mathbf{b}} \neq \underline{\mathbf{d}}.$$

b) *The action of $\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}$ is given by the inclusion*

$$\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}\colon \Lambda_{\underline{\mathbf{e}}} \hookrightarrow \Lambda_{\underline{\mathbf{d}}}, \quad f \mapsto f, \qquad \curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}|_{\Lambda_{\underline{\mathbf{b}}}} = 0 \quad \text{if} \quad \underline{\mathbf{b}} \neq \underline{\mathbf{e}}.$$

c) *The action of $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e}$ on $\Lambda_{\underline{\mathbf{b}}}$ is trivial unless $\underline{\mathbf{b}} = \underline{\mathbf{d}}$. In the latter case, if we identify $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e} \cong \Lambda_{\underline{\mathbf{d}}}$ as in (5.9), then $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e}$ acts on $\Lambda_{\underline{\mathbf{d}}}$ by usual multiplication.*

Proof. It is obvious that $\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}|_{\Lambda_{\underline{\mathbf{b}}}} = 0$ unless $\underline{\mathbf{b}} = \underline{\mathbf{d}}$. In the latter case, we observe that $\mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}^{e} \times_{\mathfrak{Q}_{\underline{\mathbf{d}}}} \mathfrak{Q}_{\underline{\mathbf{d}}} = \mathfrak{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}^{e} \cong \mathfrak{Q}_{\underline{\mathbf{d}}}$ and that, under this identification, $p_{12}^*(\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}) \cap p_{23}^* f = f$ (with p_{ij} as in §5.2.5). Hence $\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} \star f$ equals the pushforward of f along the canonical map $p_{13}\colon \mathfrak{Q}_{\underline{\mathbf{d}}} \twoheadrightarrow \mathfrak{Q}_{\underline{\mathbf{e}}}$, which factors as follows

$$p_{13}\colon \mathfrak{Q}_{\underline{\mathbf{d}}} \overset{\iota}{\hookrightarrow} \mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} \overset{q}{\twoheadrightarrow} \mathfrak{Q}_{\underline{\mathbf{e}}}. \tag{5.21}$$

We first compute ι_*. By [35, Corollary 2.6.44], we have $\iota_* f = \mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}})f$. The short exact sequence $0 \to T_{\mathfrak{Q}_{\underline{\mathbf{e}}}}\mathfrak{Q}_{\underline{\mathbf{e}},(\mathbf{c})} \to T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},(\mathbf{c})} \to T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} \to 0$ implies that

$$\mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}) = \mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{d}}}}\mathfrak{Q}_{\underline{\mathbf{d}},(\mathbf{c})})/\mathtt{eu}_{\mathsf{G}_{\mathbf{c}}}(T_{\mathfrak{Q}_{\underline{\mathbf{e}}}}\mathfrak{Q}_{\underline{\mathbf{e}},(\mathbf{c})}).$$

It now follows from Lemma 5.4.6 that

$$\iota_* f = \mathtt{E}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} f. \tag{5.22}$$

We will next compute $q_* h$, where $h := \mathtt{E}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} f$. Since $\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} = \mathfrak{F}_{\underline{\mathbf{d}}} \times_{\mathfrak{F}_{\underline{\mathbf{e}}}} \mathfrak{Q}_{\underline{\mathbf{e}}}$, calculating the pushforward q_* reduces to calculating the pushforward along $\bar{q}\colon \mathfrak{F}_{\underline{\mathbf{d}}} \to \mathfrak{F}_{\underline{\mathbf{e}}}$. By Theorems 5.4.2 and 5.4.3, we have

$$h(\vec{y}) = \sum_{w\in\mathsf{D}_{\underline{\mathbf{d}}}^{\mathbf{c}}} \frac{(i_w)_* i_w^*(h(\vec{y}))}{\mathtt{eu}_{\mathsf{T}_{\mathbf{c}}}(T_w\mathfrak{F}_{\underline{\mathbf{d}}})} = \sum_{w\in\mathsf{D}_{\underline{\mathbf{d}}}^{\mathbf{c}}} \frac{w\cdot h(\vec{x})}{w\cdot\mathtt{S}_{\underline{\mathbf{d}}}}\zeta_w = \sum_{u\in\mathsf{D}_{\underline{\mathbf{e}}}^{\mathbf{c}}} \frac{1}{u\cdot\mathtt{S}_{\underline{\mathbf{e}}}} u \sum_{v\in\mathsf{D}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}} \frac{v\cdot h(\vec{x})}{v\cdot\mathtt{S}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}}\zeta_{uv}$$

since $v \cdot \mathsf{S}_{\underline{\mathbf{e}}} = \mathsf{S}_{\underline{\mathbf{e}}}$. Let $g(\vec{y}) := \bar{q}_*(h(\vec{y}))$. Since $\bar{q}_*\zeta_{uv} = \zeta_u$, we have

$$g(\vec{y}) = \sum_{u \in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{e}}}} \frac{1}{u \cdot \mathsf{S}_{\underline{\mathbf{e}}}} u \sum_{v \in \mathsf{D}^{\mathbf{e}}_{\underline{\mathbf{d}}}} \frac{v \cdot h(\vec{x})}{v \cdot \mathsf{S}^{\mathbf{e}}_{\underline{\mathbf{d}}}} \zeta_u.$$

On the other hand, Theorem 5.4.2 implies that

$$g(\vec{y}) = \sum_{u \in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{e}}}} \frac{(i_u)_* i_u^*(g(\vec{y}))}{\mathtt{eu}_{\mathsf{T}_{\mathbf{c}}}(T_u \mathfrak{F}_{\underline{\mathbf{e}}})} = \sum_{u \in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{e}}}} \frac{u \cdot g(\vec{x})}{u \cdot \mathsf{S}_{\underline{\mathbf{e}}}} \zeta_u.$$

Hence

$$g(\vec{x}) = \sum_{v \in \mathsf{D}^{\mathbf{e}}_{\underline{\mathbf{d}}}} v \cdot \frac{h(\vec{x})}{\mathsf{S}^{\mathbf{e}}_{\underline{\mathbf{d}}}}. \tag{5.23}$$

Combining (5.22) with (5.23) yields the first part of the theorem.

An argument analogous to the one at the beginning of the proof shows that $\curlyvee^{\mathbf{d}}_{\underline{\mathbf{e}}}|_{\Lambda_{\underline{\mathbf{b}}}} = 0$ unless $\underline{\mathbf{b}} = \underline{\mathbf{e}}$, and that convolving $\curlyvee^{\mathbf{d}}_{\underline{\mathbf{e}}}$ with a function $f \in \Lambda_{\underline{\mathbf{e}}}$ is the same as taking the pullback with respect to (5.21). A calculation using the localization theorem, similar to the one above, shows that q^* is given by the inclusion of the invariants $\Lambda_{\underline{\mathbf{e}}} \hookrightarrow \Lambda_{\underline{\mathbf{d}}}$, while the second pullback ι^* is just an isomorphism. This yields the second part of the theorem. The third part is standard - see, e.g., [35, Example 2.7.10(i)]. □

We will now relate the action of the merges to Demazure operators.

Definition 5.4.8. Given $s_j(i) \in \mathsf{W}_{\mathbf{c}}$, let

$$\Delta_j(i) = \frac{1 - s_j(i)}{x_j(i) - x_{j+1}(i)}$$

be the corresponding Demazure operator. Given $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$, let $w^{\mathbf{e}}_{\underline{\mathbf{d}}}$ be the longest element in $\mathsf{D}^{\mathbf{e}}_{\underline{\mathbf{d}}}$. Choose a reduced expression $w^{\mathbf{e}}_{\underline{\mathbf{d}}} = s_{j_1}(i_1) \cdot \ldots \cdot s_{j_k}(i_k)$ and define

$$\Delta^{\mathbf{e}}_{\underline{\mathbf{d}}} = \Delta_{j_1}(i_1) \circ \ldots \circ \Delta_{j_k}(i_k).$$

It is well known that $\Delta^{\mathbf{e}}_{\underline{\mathbf{d}}}$ does not depend on the choice of reduced expression for $w^{\mathbf{e}}_{\underline{\mathbf{d}}}$. Let $r^{\mathbf{e}}_{\underline{\mathbf{d}}} = |R^+_{\underline{\mathbf{e}}} - R^+_{\underline{\mathbf{d}}}|$ and $r_{\underline{\mathbf{d}}} = |R^+_{\mathbf{c}} - R^+_{\underline{\mathbf{d}}}|$.

Proposition 5.4.9. *We have an equality of operators*

$$\pitchfork^{\mathbf{e}}_{\underline{\mathbf{d}}} (\mathsf{S}^{\mathbf{e}}_{\underline{\mathbf{d}}})^{-1} = (-1)^{r^{\mathbf{e}}_{\underline{\mathbf{d}}}} \Delta^{\mathbf{e}}_{\underline{\mathbf{d}}}.$$

Proof. See [99, Proposition 8.13]. □

5.4.3. Application: geometric realization of the modified quiver Schur algebra. We now deduce some consequences from Theorem 5.4.7 in the special case when Q is the cyclic quiver with at least two vertices or the infinite (in both directions) linear quiver A_∞, connecting our quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ to exisiting constructions.

Miemietz and Stroppel introduced in [99, Definition 8.4] a *modified quiver Schur algebra*. Let us denote it by $\mathcal{Z}^{MS}_{\mathbf{c}}$ (in [99] the notation $\mathbf{C_i}$ is used). It is defined, purely algebraically, as the subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbf{c}})$ generated by certain linear operators, called idempotents, polynomials, splits and merges. These operators are defined by explicit formulas. We will refer to them as "algebraic", in order to distinguish them from the fundamental classes in Definition 5.3.4.

We must first deal with a minor technical issue. The algebraic merges are defined using "reversed Euler classes", denoted in [99] by $\mathtt{E}_{\mathbf{u}_J}$, and "symmetrisers", denoted by $\mathtt{S}_{\mathbf{u}_J}$ (see [99, (8.1-2)]). Both of them are given by certain product formulas. We define sign-corrected algebraic merges to be the operators obtained by multiplying $\mathtt{E}_{\mathbf{u}_J}$ and $\mathtt{S}_{\mathbf{u}_J}$ by -1 if number of factors in the corresponding product is odd.

The main result of [99] says that the geometrically defined Stroppel-Webster quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{SW}$ is, after completion, isomorphic to the affine q-Schur algebra [66], which naturally appears in the representation theory of p-adic general linear groups. The proof of this result relies on the fact that both of these algebras are isomorphic to the modified quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{MS}$. The following theorem shows that $\mathcal{Z}_{\mathbf{c}}^{MS}$ also admits a geometric realization as a convolution algebra, and that this realization is afforded by our quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$.

Theorem 5.4.10. *There is an algebra isomorphism $\mathcal{Z}_{\mathbf{c}} \cong \mathcal{Z}_{\mathbf{c}}^{MS}$. Explicitly, this isomorphism sends $\curlyvee_{\underline{\mathbf{e}}}^{\mathbf{d}}$, $\curlywedge_{\underline{\mathbf{d}}}^{\mathbf{e}}$, $\mathsf{e}_{\underline{\mathbf{d}}}$ and a polynomial in $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e}$ (where $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$) to the corresponding algebraic split, sign-corrected merge, idempotent and polynomial, respectively.*

Proof. Corollary 5.3.27 says that $\curlyvee_{\underline{\mathbf{e}}}^{\mathbf{d}}$, $\curlywedge_{\underline{\mathbf{d}}}^{\mathbf{e}}$ and $\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^{e}$ generate $\mathcal{Z}_{\mathbf{c}}$. In fact, the corollary makes the stronger statement that it is enough to take polynomials together with *elementary* splits and merges to get a generating set. However, since the definition of $\mathcal{Z}_{\mathbf{c}}^{MS}$ involves arbitrary algebraic splits and merges, we only need the weaker form of Corollary 5.3.27 here. By Proposition 5.4.4, the polynomial representation $\mathcal{Q}_{\mathbf{c}} \cong \Lambda_{\mathbf{c}}$ of $\mathcal{Z}_{\mathbf{c}}$ is faithful. Hence $\mathcal{Z}_{\mathbf{c}}$ is isomorphic to the subalgebra of $\mathrm{End}_{\mathbb{C}}(\Lambda_{\mathbf{c}})$ generated by the linear operators representing splits, merges and polynomials. To complete the proof, one only has to compare the description of these operators from Theorem 5.4.7 with the definition of their algebraic counterparts in [99, Definition 8.4]. □

Corollary 5.3.27 and Theorem 5.4.10 directly imply the following statement about the generators of the modified quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{MS}$, which is not obvious from its algebraic definition.

Theorem 5.4.11. *The modified quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{MS}$ is generated by algebraic polynomials and* elementary *algebraic splits and merges.*

Note that combining Theorems 5.3.25 and 5.4.10 also gives us a basis of $\mathcal{Z}_{\mathbf{c}}^{MS}$.

Moreover, we can relate $\mathcal{Z}_{\mathbf{c}}$ to the Stroppel-Webster quiver Schur algebra.

Theorem 5.4.12. *Let Q be an arbitrary quiver. Then our quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}$ is isomorphic to the Stroppel-Webster quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{SW} = H_{\bullet}^{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Z}_{\mathbf{c}}^{s})$.*

Proof. The definition of the modified quiver Schur algebra $\mathcal{Z}_{\mathbf{c}}^{MS}$, together with Theorem 5.4.10, generalize straightforwardly to arbitrary quivers. We also observe that the proofs of [99, Propositions 9.4, 9.6] do not depend on the choice of cyclic quiver, and hence generalize to arbitrary quivers, yielding the desired isomorphism. □

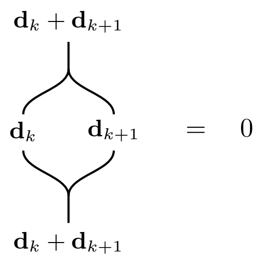
5.4.4. Examples: the A_1 and Jordan quivers. In this subsection we discuss the examples of the A_1 quiver (i.e. one vertex with no arrows) and the Jordan quiver. It is well known (see, e.g., [88,114,121]) that the corresponding KLR algebras are isomorphic to the affine Nil-Hecke algebra and the degenerate affine Hecke algebra, respectively. While it is quite hard to give a presentation by generators and relations for the entire quiver Schur algebra, even for the A_1 and the Jordan quiver, we are able to give a complete list of relations for the following subalgebra.

Definition 5.4.13. Let $\mathcal{Z}_{\mathbf{c}}'$ be the subalgebra of $\mathcal{Z}_{\mathbf{c}}$ generated by all merges and splits. We call it the *reduced quiver Schur algebra.*

We first consider the case where Q is the A_1 quiver. Let $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$. Note that since the quiver has only one vertex, $\mathbf{c}$ is just a positive integer and $\underline{\mathbf{d}}$ and $\underline{\mathbf{e}}$ are compositions of this integer. Since the quiver has no arrows, $\mathtt{E}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = 1$ and $\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \pitchfork^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}(\mathtt{S}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}})^{-1}$. Therefore, by Lemma 5.4.9, we have $\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \Delta^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$, i.e., merges coincide with Demazure operators. Let us look at the special case when $\underline{\mathbf{d}} = (m, n)$ and $\underline{\mathbf{e}} = (\mathbf{c}) = (m + n)$. Then

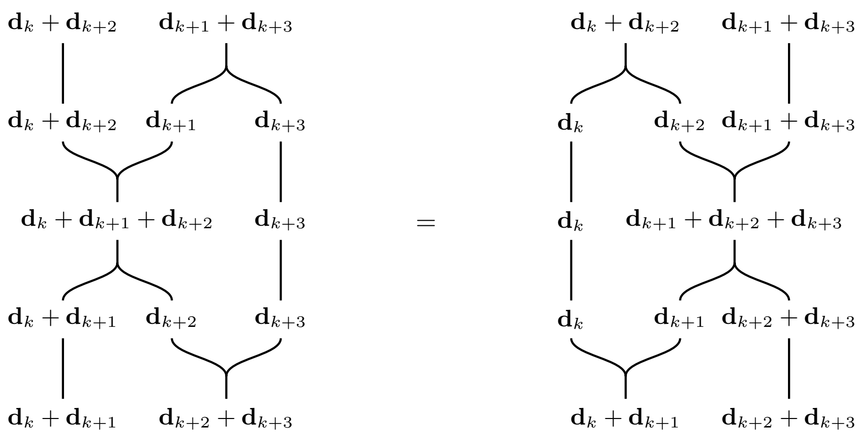
$$\mathtt{S}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \mathtt{S}_{\underline{\mathbf{d}}} = \prod_{k=1}^{m} \prod_{l=m+1}^{m+n} (x_l - x_k), \qquad \curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \sum_{w \in \mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{d}}}} \prod_{k=1}^{m} \prod_{l=m+1}^{m+n} w \cdot (x_l - x_k)^{-1}. \tag{5.24}$$

We will now give a complete list of defining relations in the reduced quiver Schur algebra. We call



(R3)

the *hole removal* relation, and



(R4)

the *ladder* relation.

Theorem 5.4.14. *The reduced quiver Schur algebra $\mathcal{Z}'_{\mathbf{c}}$ associated to the A_1 quiver is generated by elementary merges and splits, subject to the relations* (R1), (R2), (R3) *and* (R4).

A detailed proof of Theorem 5.4.14 can be found in [127]. Below we will sketch the main ideas of the proof. We first need to recall some material about the green web category ∞-$\mathbf{Web}_g$ from [135].

Definition 5.4.15. We define a certain full subcategory $\mathscr{C}_{\mathbf{c}}$ of the green web category ∞-$\mathbf{Web}_g$.

a) One first defines the free web category $\mathscr{C}^f_{\mathbf{c}}$. Its objects are compositions of $\mathbf{c}$ and its morphism spaces $\mathrm{Hom}_{\mathscr{C}^f_{\mathbf{c}}}(\underline{\mathbf{d}}, \underline{\mathbf{e}})$ are generated by elementary merge and split diagrams (which we denote by $\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ and $\curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$, respectively) via vertical composition.
b) We define a filtration on the morphism spaces by setting $\deg \curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = \deg \curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} = \mathbf{d}_k + \mathbf{d}_{k+1}$ if $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$ and $\underline{\mathbf{e}} = \wedge_k(\underline{\mathbf{d}})$ for some $1 \leq k \leq \ell_{\underline{\mathbf{d}}}$.
c) The category $\mathscr{C}_{\mathbf{c}}$ is the quotient of $\mathscr{C}^f_{\mathbf{c}}$ obtained by imposing certain relations on morphisms, called the associativity, coassociativity, digon removal and square switch relations [135, (2-6)-(2-8)].

We remark that ∞-$\mathbf{Web}_g$ is defined in [135] as a $\mathbb{C}(q)$-linear category. For our purposes, however, it is enough to work with the $\mathbb{C}$-linear category obtained by setting $q = 1$.

Consider the filtered algebra

$$\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) = \bigoplus_{\underline{\mathbf{d}},\underline{\mathbf{e}}\in\mathbf{Com}_{\mathbf{c}}} \mathrm{Hom}_{\mathscr{C}_{\mathbf{c}}^f}(\underline{\mathbf{d}}, \underline{\mathbf{e}}).$$

Let $I_{\mathscr{C}_{\mathbf{c}}}$ be the kernel of the canonical map $\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) \twoheadrightarrow \mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$. We endow $I_{\mathscr{C}_{\mathbf{c}}}$ with the subspace filtration and $\mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$ with the quotient filtration. Then the sequence $0 \to I_{\mathscr{C}_{\mathbf{c}}} \to \mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) \to \mathrm{Mor}(\mathscr{C}_{\mathbf{c}}) \to 0$ is strict exact and so, after taking the associated graded, we obtain the short exact sequence

$$0 \to \mathsf{gr}\, I_{\mathscr{C}_{\mathbf{c}}} \to \mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) \to \mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}) \to 0. \tag{5.25}$$

Note that the rule in Definition 5.4.15.b) also defines a grading on $\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f)$, so $\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f)$ is isomorphic as an algebra to its associated graded $\mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f)$.

Proof of Theorem 5.4.14. Firstly, we need to check that the relations (R1)-(R4) hold in $\mathcal{Z}'_{\mathbf{c}}$. By Proposition 5.3.5, the relations (R1) and (R2) hold in any quiver Schur algebra (associated to any quiver). Relations (R3) and (R4) follow easily from the properties of Demazure operators.

Secondly, we need to check that the relations (R1)-(R4) generate all the relations in $\mathcal{Z}'_{\mathbf{c}}$. Let $\widetilde{\mathcal{Z}}'_{\mathbf{c}}$ be the quotient of the free algebra ${}^f\widetilde{\mathcal{Z}}'_{\mathbf{c}}$, generated by elementary merges and splits, by the ideal $I_{\mathbf{c}}$ generated by the relations (R1)-(R4) so that we have a short exact sequence

$$0 \to I_{\mathbf{c}} \to {}^f\widetilde{\mathcal{Z}}_{\mathbf{c}} \to \widetilde{\mathcal{Z}}'_{\mathbf{c}} \to 0. \tag{5.26}$$

It follows from the definitions that $\mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) \cong \mathrm{Mor}(\mathscr{C}_{\mathbf{c}}^f) \cong {}^f\widetilde{\mathcal{Z}}'_{\mathbf{c}}$. One also easily sees that after taking the associated graded the relations (2-6)-(2-8) from [135] become the relations (R1)-(R4). Hence $I_{\mathbf{c}} \cong \mathsf{gr}\, I_{\mathscr{C}_{\mathbf{c}}}$. Comparing (5.25) with (5.26) now implies that $\widetilde{\mathcal{Z}}'_{\mathbf{c}} \cong \mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$.

Next, [135, Theorem 3.20] implies that there is a vector space isomorphism

$$\mathrm{Hom}_{\mathscr{C}_{\mathbf{c}}}(\underline{\mathbf{d}}, \underline{\mathbf{e}}) \cong \mathrm{Hom}_{\mathfrak{gl}_\infty}(\bigwedge^{\underline{\mathbf{d}}} \mathbb{C}^\infty, \bigwedge^{\underline{\mathbf{e}}} \mathbb{C}^\infty),$$

where $\bigwedge^{\underline{\mathbf{d}}} \mathbb{C}^\infty = \bigwedge^{\mathbf{d}_1} \mathbb{C}^\infty \otimes \ldots \otimes \bigwedge^{\mathbf{d}_{\ell_{\underline{\mathbf{d}}}}} \mathbb{C}^\infty$. Hence

$$\dim \widetilde{\mathcal{Z}}'_{\underline{\mathbf{d}},\underline{\mathbf{e}}} = \dim \mathrm{Hom}_{\mathfrak{gl}_\infty}(\bigwedge^{\underline{\mathbf{d}}} \mathbb{C}^\infty, \bigwedge^{\underline{\mathbf{e}}} \mathbb{C}^\infty) = |{}_{\underline{\mathbf{d}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{e}}}| = \dim \mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{e}}},$$

where the second equality can be deduced from Schur-Weyl duality and the last equality follows from Theorem 5.3.25. We conclude that the natural map $\widetilde{\mathcal{Z}}'_{\mathbf{c}} \twoheadrightarrow \mathcal{Z}_{\mathbf{c}}$ is an isomorphism. □

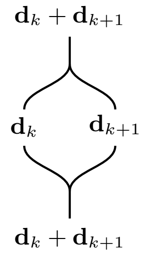
Let us record the following corollary of the proof of Theorem 5.4.14.

Corollary 5.4.16. *If Q is the A_1 quiver, then there is an algebra isomorphism $\mathcal{Z}'_{\mathbf{c}} \cong \mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$.*

Next suppose that Q is the Jordan quiver. We can interpret merges as symmetrization operators between rings of invariants. Indeed, $\mathtt{S}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} = \mathtt{E}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}$ and so

$$\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} = \pitchfork_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}. \tag{5.27}$$

We will now describe the relations in the reduced quiver Schur algebra. We use the following modification of (R3) (with $\underline{\mathbf{e}} = \wedge^k(\underline{\mathbf{d}})$):



$$= \left|\mathsf{D}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}\right| \cdot 1. \tag{R3'}$$

Theorem 5.4.17. *The following hold:*

a) *The reduced quiver Schur algebra algebra $\mathcal{Z}'_{\mathbf{c}}$ associated to the Jordan quiver is generated by elementary merges and splits, subject to the relations* (R1), (R2), (R3') *and* (R4).
b) *The algebra $\mathcal{Z}'_{\mathbf{c}}$ is isomorphic to the convolution algebra $\bigoplus_{(\mathsf{P},\mathsf{P}')} \mathbb{C}[\mathsf{P}\backslash\mathsf{G}_{\mathbf{c}}/\mathsf{P}']$ of complex valued functions on double cosets, where (P,P') runs over all pairs of standard parabolic subgroups of $\mathsf{G}_{\mathbf{c}}$.*

Proof. The fact that the relations (R3') and (R4) hold in $\mathcal{Z}'_{\mathbf{c}}$ follows easily from the properties of symmetrization operators. One can define a filtration on $\widetilde{\mathcal{Z}}'_{\mathbf{c}}$ analogous to the filtration on $\mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$. It is clear that $\mathsf{gr}\,\widetilde{\mathcal{Z}}'_{\mathbf{c}} \cong \mathsf{gr}\,\mathrm{Mor}(\mathscr{C}_{\mathbf{c}})$. Hence one can use the same argument as in the proof of Theorem 5.4.14 to show that (R1), (R2), (R3') and (R4) generate all the relations. This proves the first statement of the theorem. The second statement follows from the description of $\mathcal{Z}'_{\mathbf{c}}$ as the algebra of symmetrization operators in (5.27). □

5.5 Mixed quiver Schur algebras

In this section we define and study a generalization of quiver Schur algebras, depending on a quiver together with a contravariant involution and a duality structure. We call these new algebras *mixed quiver Schur algebras.* From a geometric point of view, our generalization arises by replacing the stack of representations of a quiver with the stack of its supermixed representations in the sense of Zubkov [147].

5.5.1. Involutions and duality structures. We begin by recalling the notion of a contravariant involution and a duality structure. These ideas, in the context of quiver representations, were first studied in [39, 147]. We use the formulation from [146].

Definition 5.5.1. A (contravariant) *involution* of a quiver Q is a pair of involutions $\theta\colon Q_0 \to Q_0$ and $\theta\colon Q_1 \to Q_1$ such that:

a) $s(\theta(a)) = \theta(t(a))$ and $t(\theta(a)) = \theta(s(a))$ for all $a \in Q_1$,
b) if $t(a) = \theta(s(a))$ then $a = \theta(a)$.

A *duality structure* on (Q,θ) is a pair of functions $\sigma\colon Q_0 \to \{\pm 1\}$ and $\varsigma\colon Q_1 \to \{\pm 1\}$ such that $\sigma(\theta(i)) = \sigma(i)$ for all $i \in Q_0$ and $\varsigma(a)\cdot\varsigma(\theta(a)) = \sigma(s(a))\cdot\sigma(t(a))$ for all $a \in Q_1$.

Example 5.5.2. Let $n \geq 1$ and suppose that Q is the A_n quiver

$$\underset{i_1}{\bullet} \xrightarrow{a_1} \underset{i_2}{\bullet} \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} \underset{i_n}{\bullet}$$

There is a unique involution θ on Q. We have $\theta(i_k) = i_{n-k+1}$ for $1 \leq k \leq n$ and $\theta(a_l) = a_{n-l}$ for $1 \leq l \leq n-1$. If n is even then $Q_0^\theta = \varnothing$ and $Q_1^\theta = \{a_{n/2}\}$. If n is odd then $Q_1^\theta = \varnothing$ and $Q_0^\theta = \{i_{(n+1)/2}\}$. There are two inequivalent duality structures: $\sigma = 1$ and $\varsigma = -1$ or $\sigma = -1$ and $\varsigma = -1$.

Example 5.5.3. Suppose that Q is the quiver with one vertex and $m \geq 0$ loops. There is a unique involution on Q. It fixes the vertex and fixes all the loops as well. A duality structure is given by a choice of sign σ and a choice of sign $\varsigma(a)$ for each arrow a. Hence there are 2^{m+1} possible duality structures.

For the rest of this section let us fix a quiver Q together with an involution θ and a duality structure (σ, ς). We will now introduce some combinatorics necessary to describe isotropic flag varieties. Let us fix partitions

$$Q_0 = Q_0^- \sqcup Q_0^\theta \sqcup Q_0^+, \quad Q_1 = Q_1^- \sqcup Q_1^\theta \sqcup Q_1^+$$

such that $\theta(Q_0^+) = Q_0^-$ and $\theta(Q_1^+) = Q_1^-$. The involution θ induces an involution $\theta \colon \Gamma \to \Gamma$ on the monoid of dimension vectors. Let Γ^θ be the submonoid of θ-fixed points. We consider Γ^θ as a Γ-module via the monoid homomorphism

$$\mathrm{D} \colon \Gamma \to \Gamma^\theta, \quad \mathbf{c} \mapsto \mathbf{c} + \theta(\mathbf{c}).$$

Definition 5.5.4. Let $\mathbf{c} \in \Gamma^\theta$. We call a sequence $\underline{\mathbf{d}} = (\mathbf{d}_1, \ldots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}, \mathbf{d}_\infty) \in \Gamma_+^{\ell_{\underline{\mathbf{d}}}} \times \Gamma^\theta$ (where $\ell_{\underline{\mathbf{d}}}$ may equal zero) an *isotropic vector composition* of $\mathbf{c}$, denoted $\underline{\mathbf{d}} \vDash \mathbf{c}$, if $\langle \underline{\mathbf{d}} \rangle_\theta := \mathbf{d}_\infty + \sum_j \mathrm{D}(\mathbf{d}_j) = \mathbf{c}$. We call $\ell_{\underline{\mathbf{d}}}$ the *length* of $\underline{\mathbf{d}}$. Let ${}^\theta\mathbf{Com}_\mathbf{c}$ denote the set of all isotropic vector compositions of $\mathbf{c}$, and let ${}^\theta\mathbf{Com}_\mathbf{c}^m$ denote the subset of compositions of length m. Consider $\mathbb{Z}_2 \wr \mathsf{Sym}_m$ as the group of signed permutations of the set $\{\pm 1, \ldots, \pm m\}$ with s_m changing the sign of m. We endow ${}^\theta\mathbf{Com}_\mathbf{c}^m$ with a right $\mathbb{Z}_2 \wr \mathsf{Sym}_m$-action so that Sym_m acts by permuting the first m dimension vectors and s_m acts by changing $\mathbf{d}_{\ell_{\underline{\mathbf{d}}}}$ to $\theta(\mathbf{d}_{\ell_{\underline{\mathbf{d}}}})$. Set

$$\mathrm{D}(\underline{\mathbf{d}}) := (\mathbf{d}_1, \ldots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}, \mathbf{d}_\infty, \theta(\mathbf{d}_{\ell_{\underline{\mathbf{d}}}}), \ldots, \theta(\mathbf{d}_1)), \quad \underline{\mathbf{d}}^f = (\mathbf{d}_1, \ldots, \mathbf{d}_{\ell_{\underline{\mathbf{d}}}}).$$

Given $\beta \in \mathrm{Com}(\ell_{\underline{\mathbf{d}}} + 1)$, let

$$\wedge_\beta^\theta(\underline{\mathbf{d}}) := (\langle \vee_\beta^1(\underline{\mathbf{d}}) \rangle, \ldots, \langle \vee_\beta^{\ell_\beta - 1}(\underline{\mathbf{d}}) \rangle, \langle \vee_\beta^{\ell_\beta}(\underline{\mathbf{d}}) \rangle_\theta).$$

In particular, if $\beta = (1^{k-1}, 2, 1^{\ell_{\underline{\mathbf{d}}} - k})$ for some $1 \leq k \leq \ell_{\underline{\mathbf{d}}}$, then we abbreviate $\wedge_k^\theta(\underline{\mathbf{d}}) := \wedge_\beta^\theta(\underline{\mathbf{d}})$.

Example 5.5.5. Consider the A_3 quiver together with its unique involution. Let $\mathbf{c} = 4i_1 + 3i_2 + 4i_3 \in \Gamma^\theta$ and $\underline{\mathbf{d}} = (i_1 + i_2, i_3, 2i_1 + i_2 + 2i_3) \vDash \mathbf{c}$. Then $\wedge_1^\theta = (i_1 + i_2 + i_3, 2i_1 + i_2 + 2i_3)$ and $\wedge_2^\theta = (i_1 + i_2, 3i_1 + i_2 + 3i_3)$.

In analogy to Definition 5.2.2, we define a partial order on ${}^\theta\mathbf{Com}_\mathbf{c}$ by setting

$$\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}} \iff \underline{\mathbf{e}} = \wedge_\beta^\theta(\underline{\mathbf{d}})$$

for some $\beta \in \mathrm{Com}(\ell_{\underline{\mathbf{d}}} + 1)$. If $\mathbf{e}_\infty = \mathbf{d}_\infty$, then we write $\underline{\mathbf{d}} \succcurlyeq_f \underline{\mathbf{e}}$. If $\underline{\mathbf{d}} = \underline{\mathbf{d}}' \cup \underline{\mathbf{d}}''$ and $\langle \underline{\mathbf{d}}'' \rangle_\theta = \mathbf{e}_\infty$, we write $\underline{\mathbf{d}} \succcurlyeq_\infty \underline{\mathbf{e}}$.

5.5.2. Isotropic flag varieties.

In this subsection we introduce the notation for isotropic flag varieties, isotropic Steinberg varieties and related objects.

Definition 5.5.6. Let $\mathbf{c} \in \Gamma^\theta$. If $i \in Q_0^\theta$ and $\sigma(i) = -1$, we assume that $\mathbf{c}(i)$ is even. Fix a Q_0-graded $\mathbb{C}$-vector space $\mathbf{V}_\mathbf{c} = \bigoplus_{i \in Q_0} \mathbf{V}_\mathbf{c}(i)$ with $\dim \mathbf{V}_\mathbf{c}(i) = \mathbf{c}(i)$ and a nondegenerate bilinear form

$\langle\cdot,\cdot\rangle\colon \mathbf{V_c}\times\mathbf{V_c}\to\mathbb{C}$ such that:

a) $\mathbf{V_c}(i)$ and $\mathbf{V_c}(j)$ are orthogonal unless $i=\theta(j)$,

b) the restriction of $\langle\cdot,\cdot\rangle$ to $\mathbf{V_c}(i)+\mathbf{V_c}(\theta(i))$ satisfies $\langle u,v\rangle=\sigma(i)\langle v,u\rangle$,

for $i,j\in Q_0$ and $u,v\in\mathbf{V_c}(i)+\mathbf{V_c}(\theta(i))$. Set

$$^{\theta}\mathfrak{R}_{\mathbf{c}}:=\{\rho\in\mathfrak{R}_{\mathbf{c}}\mid\langle\rho_a(u),v\rangle=\varsigma(a)\langle u,\rho_{\theta(a)}(v)\rangle\ \forall a\in Q_1,u\in\mathbf{V_c}(s(a)),v\in\mathbf{V_c}(t(a))\}.$$

There is a vector space isomorphism

$$^{\theta}\mathfrak{R}_{\mathbf{c}}\cong\bigoplus_{a\in Q_1^+}\mathrm{Hom}_{\mathbb{C}}(\mathbf{V_c}(s(a)),\mathbf{V_c}(t(a)))\oplus\bigoplus_{a\in Q_1^\theta}\mathrm{Bil}^{\sigma(s(a))\cdot\varsigma(a)}(\mathbf{V_c}(s(a))),$$

where $\mathrm{Bil}^{\epsilon}(\mathbf{V_c}(s(a)))$ is the vector space of symmetric ($\epsilon=1$) or skew-symmetric ($\epsilon=-1$) bilinear forms on $\mathbf{V_c}(s(a))$.

Definition 5.5.7. Let $^{\theta}\mathsf{G}'_{\mathbf{c}}$ be the subgroup of $\mathsf{G}_{\mathbf{c}}$ which preserves the bilinear form $\langle\cdot,\cdot\rangle$. We have

$$^{\theta}\mathsf{G}'_{\mathbf{c}}\cong\prod_{i\in Q_0^+}\mathsf{GL}(\mathbf{V_c}(i))\times\prod_{\substack{i\in Q_0^\theta,\\ \sigma(i)=1}}\mathsf{O}(\mathbf{V_c}(i))\times\prod_{\substack{i\in Q_0^\theta,\\ \sigma(i)=-1}}\mathsf{Sp}(\mathbf{V_c}(i)). \tag{5.28}$$

The group $^{\theta}\mathsf{G}'_{\mathbf{c}}$ acts naturally on $^{\theta}\mathfrak{R}_{\mathbf{c}}$ by conjugation. Let $^{\theta}\mathsf{G}_{\mathbf{c}}\subseteq{}^{\theta}\mathsf{G}'_{\mathbf{c}}$ be the subgroup obtained from (5.28) by replacing $\mathsf{O}(\mathbf{V_c}(i))$ with $\mathsf{SO}(\mathbf{V_c}(i))$ whenever $\mathbf{c}(i)$ is odd.

Example 5.5.8. Let Q be the Jordan quiver and $\mathbf{c}=2n$. Let $\sigma=1$ so that $^{\theta}\mathsf{G}_{\mathbf{c}}=\mathsf{O}_{2n}$. If $\varsigma=-1$ then $^{\theta}\mathfrak{R}_{\mathbf{c}}=\mathfrak{so}_{2n}$, while if $\varsigma=1$ then $^{\theta}\mathfrak{R}_{\mathbf{c}}=\mathrm{Sym}^2\,\mathbb{C}^{2n}$ as O_{2n}-modules. Next, let $\sigma=-1$ so that $^{\theta}\mathsf{G}_{\mathbf{c}}=\mathsf{Sp}_{2n}$. If $\varsigma=-1$ then $^{\theta}\mathfrak{R}_{\mathbf{c}}=\mathfrak{sp}_{2n}$, while if $\varsigma=1$ then $^{\theta}\mathfrak{R}_{\mathbf{c}}=\bigwedge^2\mathbb{C}^{2n}$ as Sp_{2n}-modules.

Definition 5.5.9. Let $^{\theta}\mathsf{T}_{\mathbf{c}}\subset{}^{\theta}\mathsf{B}_{\mathbf{c}}\subset{}^{\theta}\mathsf{G}_{\mathbf{c}}$ be the standard maximal torus (with fundamental weights $\omega_j(i)$) and Borel subgroup in $^{\theta}\mathsf{G}_{\mathbf{c}}$. Let $^{\theta}\mathsf{W}_{\mathbf{c}}=N_{^{\theta}\mathsf{G}_{\mathbf{c}}}(^{\theta}\mathsf{T}_{\mathbf{c}})/^{\theta}\mathsf{T}_{\mathbf{c}}$ be the corresponding Weyl group. There is an isomorphism

$$^{\theta}\mathsf{W}_{\mathbf{c}}\cong\prod_{i\in Q_0^+}\mathsf{Sym}_{\mathbf{c}(i)}\times\prod_{i\in Q_0^\theta}\mathbb{Z}_2\wr\mathsf{Sym}_{\lfloor\mathbf{c}(i)/2\rfloor}.$$

Given $\underline{\mathbf{d}}\in{}^{\theta}\mathbf{Com}_{\mathbf{c}}$, let $^{\theta}\mathsf{W}_{\underline{\mathbf{d}}}=\mathsf{W}_{\underline{\mathbf{d}}^f}\times{}^{\theta}\mathsf{W}_{\mathbf{d}_\infty}\subset{}^{\theta}\mathsf{W}_{\mathbf{c}}$. If $\underline{\mathbf{e}},\underline{\mathbf{d}}\in{}^{\theta}\mathbf{Com}_{\mathbf{c}}$, let $^{\theta}_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$ denote the set of the shortest representatives in $^{\theta}\mathsf{W}_{\mathbf{c}}$ of the double cosets $^{\theta}\mathsf{W}_{\underline{\mathbf{e}}}\backslash{}^{\theta}\mathsf{W}_{\mathbf{c}}/{}^{\theta}\mathsf{W}_{\underline{\mathbf{d}}}$.

Definition 5.5.10. Given $\underline{\mathbf{d}}\in{}^{\theta}\mathbf{Com}_{\mathbf{c}}$, we call a sequence $V_\bullet$ of Q_0-graded isotropic subspaces

$$\{0\}=V_0\subset V_1\subset V_2\subset\ldots\subset V_{\ell_{\underline{\mathbf{d}}}}\subset\mathbf{V_c}$$

an *isotropic flag* of type $\underline{\mathbf{d}}$ if $\dim_{Q_0}V_j/V_{j-1}=\mathbf{d}_j$ and $\dim_{Q_0}V_{\ell_{\underline{\mathbf{d}}}}^{\perp}/V_{\ell_{\underline{\mathbf{d}}}}=\mathbf{d}_\infty$. Any isotropic flag $V_\bullet$ can be extended to a flag $\mathrm{D}(V_\bullet)\in\mathfrak{F}_{\mathrm{D}(\underline{\mathbf{d}})}$ of length $2\ell_{\underline{\mathbf{d}}}+1$ by setting $V_{2\ell_{\underline{\mathbf{d}}}-k+1}=V_k^{\perp}$ for $k=0,\ldots,\ell_{\underline{\mathbf{d}}}$ (if $\mathbf{d}_\infty=0$ then $V_{\ell_{\underline{\mathbf{d}}}+1}=V_{\ell_{\underline{\mathbf{d}}}}$ is Lagrangian). Let $^{\theta}\mathbf{V}_{\underline{\mathbf{d}}}$ denote the standard isotropic flag of type $\underline{\mathbf{d}}$ (consisting of coordinate subspaces with respect to some fixed basis). Define

$$^{\theta}\mathfrak{R}_{\underline{\mathbf{d}}}:=\{\rho\in{}^{\theta}\mathfrak{R}_{\mathbf{c}}\mid\mathrm{D}(^{\theta}\mathbf{V}_{\underline{\mathbf{d}}})\text{ is }\rho\text{-stable}\},\quad {}^{\theta}\mathsf{P}_{\underline{\mathbf{d}}}:=\mathrm{Stab}_{^{\theta}\mathsf{G}_{\mathbf{c}}}(^{\theta}\mathbf{V}_{\underline{\mathbf{d}}}),\quad {}^{\theta}\mathsf{L}_{\underline{\mathbf{d}}}:=\textstyle\prod_{j=1}^{l_{\underline{\mathbf{d}}}}\mathsf{G}_{\mathbf{d}_j}\times{}^{\theta}\mathsf{G}_{\mathbf{d}_\infty}.$$

Let $^{\theta}\mathfrak{F}_{\underline{\mathbf{d}}}\cong{}^{\theta}\mathsf{G}_{\mathbf{c}}/^{\theta}\mathsf{P}_{\underline{\mathbf{d}}}$ be the projective variety parametrizing isotropic flags of type $\underline{\mathbf{d}}$. Given $\underline{\mathbf{d}}\succcurlyeq\underline{\mathbf{e}}\vDash\mathbf{c}$, define

$$^{\theta}\mathfrak{Q}_{\underline{\mathbf{d}}}:=\{(V_\bullet,\rho)\in{}^{\theta}\mathfrak{F}_{\underline{\mathbf{d}}}\times{}^{\theta}\mathfrak{R}_{\mathbf{c}}\mid\mathrm{D}(V_\bullet)\text{ is }\rho\text{-stable}\}.$$

Let

$$ {}^\theta\mathfrak{F}_{\underline{\mathbf{d}}} \xleftarrow{{}^\theta\tau_{\underline{\mathbf{d}}}} {}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}} \xrightarrow{{}^\theta\pi_{\underline{\mathbf{d}}}} {}^\theta\mathfrak{R}_{\mathbf{c}} $$

be the canonical projections. Note that ${}^\theta\tau_{\underline{\mathbf{d}}}$ is a vector bundle while ${}^\theta\pi_{\underline{\mathbf{d}}}$ is proper. We abbreviate

$$ {}^\theta\mathfrak{F}_{\mathbf{c}} := \bigsqcup_{\underline{\mathbf{d}}\vDash\mathbf{c}} {}^\theta\mathfrak{F}_{\underline{\mathbf{d}}}, \quad {}^\theta\mathfrak{Q}_{\mathbf{c}} := \bigsqcup_{\underline{\mathbf{d}}\vDash\mathbf{c}} {}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}}, \quad {}^\theta\pi_{\mathbf{c}} := \sqcup {}^\theta\pi_{\underline{\mathbf{d}}} \colon {}^\theta\mathfrak{Q}_{\mathbf{c}} \to {}^\theta\mathfrak{R}_{\mathbf{c}}. $$

Definition 5.5.11. Given $\underline{\mathbf{d}}, \underline{\mathbf{e}} \vDash \mathbf{c}$, set

$$ {}^\theta\mathfrak{Z}_{\underline{\mathbf{d}},\underline{\mathbf{e}}} := {}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}} \times_{{}^\theta\mathfrak{R}_{\mathbf{c}}} {}^\theta\mathfrak{Q}_{\underline{\mathbf{e}}}, \quad {}^\theta\mathfrak{Z}_{\mathbf{c}} := {}^\theta\mathfrak{Q}_{\mathbf{c}} \times_{{}^\theta\mathfrak{R}_{\mathbf{c}}} {}^\theta\mathfrak{Q}_{\mathbf{c}} = \bigsqcup_{\underline{\mathbf{d}},\underline{\mathbf{e}}\vDash\mathbf{c}} {}^\theta\mathfrak{Z}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}, $$

where the fibred product is taken with respect to ${}^\theta\pi_{\mathbf{c}}$. We call ${}^\theta\mathfrak{Z}_{\mathbf{c}}$ the *isotropic quiver Steinberg variety*. We define the ${}^\theta\mathsf{G}_{\mathbf{c}}$-equivariant Borel-Moore homology groups ${}^\theta\mathcal{Q}_{\underline{\mathbf{d}}}, {}^\theta\mathcal{Q}_{\mathbf{c}}, {}^\theta\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{e}}}, {}^\theta\mathcal{Z}_{\mathbf{c}}$ in analogy to (5.5), and ${}^\theta\mathfrak{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}}, {}^\theta\mathcal{Z}^e_{\mathbf{c}} := \bigoplus_{\underline{\mathbf{d}}\vDash\mathbf{c}} {}^\theta\mathcal{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}}$ in analogy to (5.3) and (5.7), respectively.

Furthermore, define

$$ {}^\theta\mathcal{P}_{\mathbf{c}} := H^\bullet(B\,{}^\theta\mathsf{T}_{\mathbf{c}}) = \bigotimes_{i\in Q_0^+} \mathbb{C}[x_1(i),\dots,x_{\mathbf{c}(i)}(i)] \otimes \bigotimes_{i\in Q_0^\theta} \mathbb{C}[x_1(i),\dots,x_{\lfloor\frac{\mathbf{c}(i)}{2}\rfloor}(i)], $$

where $x_j(i) := \mathsf{c}_1({}^\theta\mathfrak{V}_j(i))$ is the first Chern class of the line bundle ${}^\theta\mathfrak{V}_j(i) := E\,{}^\theta\mathsf{T}_{\mathbf{c}} \times^{\omega_j(i)} \mathbb{C}$. For each $\underline{\mathbf{d}} \vDash \mathbf{c}$, set

$$ {}^\theta\Lambda_{\underline{\mathbf{d}}} := {}^\theta\mathcal{P}_{\mathbf{c}}^{{}^\theta\mathsf{W}_{\underline{\mathbf{d}}}}, \quad {}^\theta\Lambda_{\mathbf{c}} := \bigoplus_{\underline{\mathbf{d}}\vDash\mathbf{c}} {}^\theta\Lambda_{\underline{\mathbf{d}}}. $$

As in (5.6) and (5.9), we can identify ${}^\theta\mathcal{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}} \cong {}^\theta\mathcal{Q}_{\underline{\mathbf{d}}} \cong {}^\theta\Lambda_{\underline{\mathbf{d}}}$ and ${}^\theta\mathcal{Z}^e_{\mathbf{c}} \cong {}^\theta\mathcal{Q}_{\mathbf{c}} \cong {}^\theta\Lambda_{\mathbf{c}}$.

5.5.3. Quiver Schur algebras for quivers with an involution. We apply the framework of §5.2.5 to the vector bundle $X = {}^\theta\mathfrak{Q}_{\mathbf{c}}$ on the isotropic quiver flag variety ${}^\theta\mathfrak{F}_{\mathbf{c}}$, the space of self-dual quiver representations $Y = {}^\theta\mathfrak{R}_{\mathbf{c}}$ and the projection $\pi = {}^\theta\pi_{\mathbf{c}}$. Then $Z = {}^\theta\mathfrak{Z}_{\mathbf{c}}$ is the isotropic quiver Steinberg variety, and we obtain a convolution algebra structure on its Borel-Moore homology ${}^\theta\mathcal{Z}_{\mathbf{c}} = H^{{}^\theta\mathsf{G}_{\mathbf{c}}}_\bullet({}^\theta\mathfrak{Z}_{\mathbf{c}})$ and a ${}^\theta\mathcal{Z}_{\mathbf{c}}$-module structure on ${}^\theta\mathcal{Q}_{\mathbf{c}} = H^{{}^\theta\mathsf{G}_{\mathbf{c}}}_\bullet({}^\theta\mathfrak{Q}_{\mathbf{c}})$.

Definition 5.5.12. We call ${}^\theta\mathcal{Z}_{\mathbf{c}}$ the *mixed quiver Schur algebra* associated to $(Q,\theta,\sigma,\varsigma,\mathbf{c})$, and ${}^\theta\mathcal{Q}_{\mathbf{c}}$ its *polynomial representation.*

Remark 5.5.13. We would like to remark on the connection between our mixed quiver Schur algebra ${}^\theta\mathcal{Z}_{\mathbf{c}}$ and existing constructions.

(i) In the case when Q is a loopless quiver and θ is an involution with no fixed vertices, the KLR analogue of ${}^\theta\mathcal{Z}_{\mathbf{c}}$, associated to complete (rather than partial) isotropic flags, was defined and studied by Varagnolo and Vasserot in [136].

(ii) Our algebra ${}^\theta\mathcal{Z}_{\mathbf{c}}$ is also related to the parabolic Steinberg algebras defined by Sauter [119–121]. On the one hand, Sauter's construction is somewhat more general since she also works with non-classical gauge groups. On the other hand, Sauter's construction is different from ours since she only allows parabolic flags of a certain fixed type, while we consider all the possible types at once. In effect, special cases of Sauter's parabolic Steinberg algebras appear as subalgebras in ${}^\theta\mathcal{Z}_{\mathbf{c}}$.

The following result carries over, with analogous proof, from the no-involution case.

Proposition 5.5.14. *The ${}^\theta\mathcal{Z}_{\mathbf{c}}$-module ${}^\theta\mathcal{Q}_{\mathbf{c}}$ is faithful. There are canonical isomorphisms*

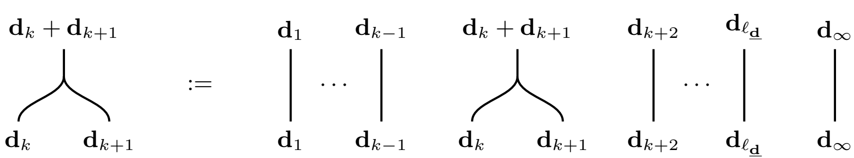
$$ {}^\theta\mathcal{Z}_{\mathbf{c}} \cong \operatorname{Ext}^\bullet_{{}^\theta\mathsf{G}_{\mathbf{c}}}\big(({}^\theta\pi_{\mathbf{c}})_*\mathbb{C}_{{}^\theta\mathfrak{Q}_{\mathbf{c}}}, ({}^\theta\pi_{\mathbf{c}})_*\mathbb{C}_{{}^\theta\mathfrak{Q}_{\mathbf{c}}}\big), \quad {}^\theta\mathcal{Q}_{\mathbf{c}} \cong \operatorname{Ext}^\bullet_{{}^\theta\mathsf{G}_{\mathbf{c}}}\big(\mathbb{C}_{{}^\theta\mathfrak{R}_{\mathbf{c}}}, ({}^\theta\pi_{\mathbf{c}})_*\mathbb{C}_{{}^\theta\mathfrak{Q}_{\mathbf{c}}}\big) \tag{5.29}$$

intertwining the convolution product with the Yoneda product, and the convolution action with the Yoneda action.

Definition 5.5.15. We have the following analogues of merges, splits, idempotents and crossings from Definition 5.3.4 in ${}^\theta\mathcal{Z}_{\mathbf{c}}$:

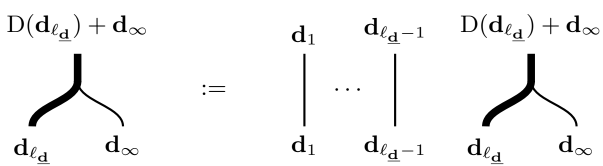
$$ {}^\theta\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} := [{}^\theta\mathfrak{Z}^e_{\underline{\mathbf{e}},\underline{\mathbf{d}}}], \quad {}^\theta\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}} := [{}^\theta\mathfrak{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{e}}}], \quad {}^\theta\mathsf{e}_{\underline{\mathbf{d}}} := [{}^\theta\mathfrak{Z}^e_{\underline{\mathbf{d}},\underline{\mathbf{d}}}], \quad {}^\theta\chi^k_{\underline{\mathbf{d}}} := {}^\theta\curlyvee^{s_k(\underline{\mathbf{d}})}_{\wedge^\theta_k(\underline{\mathbf{d}})} \star {}^\theta\curlywedge^{\wedge^\theta_k(\underline{\mathbf{d}})}_{\underline{\mathbf{d}}}$$

for $\underline{\mathbf{d}} \succcurlyeq \underline{\mathbf{e}} \trianglerighteq \mathbf{c}$ and $1 \le k \le \ell_{\underline{\mathbf{d}}}$. We say that a merge or split is *elementary* if $\underline{\mathbf{e}} = \wedge^\theta_k(\underline{\mathbf{d}})$. If $1 \le k \le \ell_{\underline{\mathbf{d}}} - 1$, we depict elementary merges and splits diagrammatically in analogy to the elementary merges and splits in Definition 5.3.4. More precisely, to the elementary merge ${}^\theta\curlywedge^{\wedge^k(\underline{\mathbf{d}})}_{\underline{\mathbf{d}}}$ we associate the diagram



and to the elementary split ${}^\theta\curlyvee^{\underline{\mathbf{d}}}_{\wedge^k(\underline{\mathbf{d}})}$ the vertically reflected diagram.

If $k = \ell_{\underline{\mathbf{d}}}$, we associate to the elementary merge ${}^\theta\curlywedge^{\wedge^k(\underline{\mathbf{d}})}_{\underline{\mathbf{d}}}$ the new diagram



and to the elementary split ${}^\theta\curlyvee^{\underline{\mathbf{d}}}_{\wedge^k(\underline{\mathbf{d}})}$ the vertically reflected diagram.

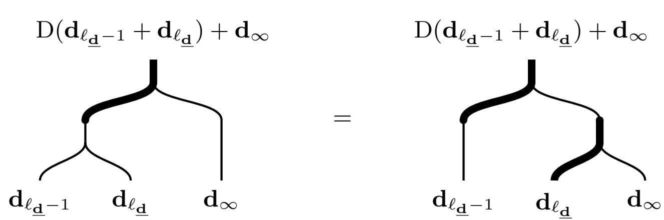
We have the following analogue of Proposition 5.3.5, which also follows directly from Lemma 5.2.9.

Proposition 5.5.16. *We list several basic relations which hold in ${}^\theta\mathcal{Z}_{\mathbf{c}}$.*

a) *Let $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \succ \underline{\mathbf{f}} \trianglerighteq \mathbf{c}$. Merges and splits satisfy the following* transitivity *relations:*

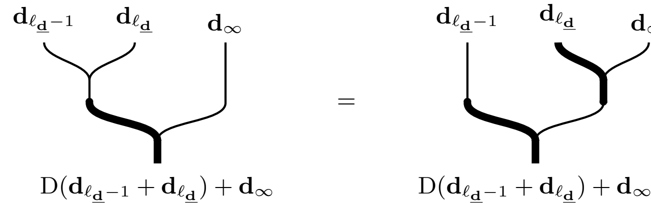
$$ {}^\theta\curlywedge^{\underline{\mathbf{f}}}_{\underline{\mathbf{e}}} \star {}^\theta\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} = {}^\theta\curlywedge^{\underline{\mathbf{f}}}_{\underline{\mathbf{d}}}, \quad {}^\theta\curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} \star {}^\theta\curlyvee^{\underline{\mathbf{e}}}_{\underline{\mathbf{f}}} = {}^\theta\curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{f}}}.$$

b) *Let $\underline{\mathbf{d}} \trianglerighteq \mathbf{c}$. Elementary merges satisfy the relations* (R1) *and the following new relation*



(${}^\theta$R1)

Elementary splits satisfy the relations (R2) *and the following new relation*



($^\theta$R2)

5.5.4. Basis and generators. We want to construct a basis for ${}^\theta\mathcal{Z}_{\mathbf{c}}$ analogous to the Bott-Samelson basis of $\mathcal{Z}_{\mathbf{c}}$ from Theorem 5.3.25. We begin by adapting the combinatorics of refinements (see §5.3.4) to the present setting.

Let $\theta\colon \mathbf{N_c} \to \mathbf{N_c}$ be the involution defined by

$$(k,i) \mapsto \begin{cases} (k,\theta(i)) & \text{if } i \in Q_0^+ \text{ and } 1 \leq k \leq \mathbf{c}(i), \\ (\mathbf{c}(i)-k+1,i) & \text{if } i \in Q_0^\theta \text{ and } 1 \leq k \leq \mathbf{c}(i). \end{cases}$$

If $\lambda \in \mathbf{Par}_{\mathbf{c}}^n$, we say that λ is an *isotropic partitioning* of $\mathbf{c}$ of length ${}^\theta\ell_\lambda = \lfloor n/2 \rfloor$ if $\lambda^{-1}(k) = \theta(\lambda^{-1}(n-k+1))$ for $0 \leq k \leq n$. Let ${}^\theta\mathbf{Par_c} \subset \mathbf{Par_c}$ denote the set of isotropic partitionings of $\mathbf{c}$, and let ${}^\theta\mathbf{Par}_{\mathbf{c}}^m$ denote the subset of those isotropic partitionings which have length m.

Let ${}^\theta C$ and ${}^\theta P$ be the unique functions making the following diagram commute

$$\begin{array}{ccc} \mathbf{Par_c} & \underset{P}{\overset{C}{\rightleftarrows}} & \mathbf{Com_c} \\ \uparrow & & \uparrow \mathrm{D} \\ {}^\theta\mathbf{Par_c} & \underset{{}^\theta P}{\overset{{}^\theta C}{\rightleftarrows}} & {}^\theta\mathbf{Com_c} \end{array}$$

The set ${}^\theta\mathbf{Par}_{\mathbf{c}}^m$ is endowed with natural $\mathbb{Z}_2 \wr \mathsf{Sym}_m$- and ${}^\theta\mathsf{W}_{\mathbf{c}}$-actions. It is easy to check that Lemma 5.3.10 still holds if we replace $\mathbf{Par_c}$, $\mathbf{Com_c}$, C, P, Sym_n, $\mathsf{W}_{\mathbf{c}}$ and $\mathsf{W}_{\underline{\mathbf{d}}}$ by their isotropic analogues. The following lemma follows directly from the definitions.

Lemma 5.5.17. *If* $\lambda, \mu \in {}^\theta\mathbf{Par_c}$ *then* $\lambda \Omega \mu \in {}^\theta\mathbf{Par_c}$.

Definition 5.5.18. We call a triple $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$, consisting of $\underline{\mathbf{e}}, \underline{\mathbf{d}} \in {}^\theta\mathbf{Com_c}$ and $w \in {}^\theta_{\underline{\mathbf{e}}}\overset{\mathbf{c}}{\mathsf{D}}_{\underline{\mathbf{d}}}$, an *isotropic orbit datum*. This name is motivated by the fact that orbit data naturally label the ${}^\theta\mathsf{G}_{\mathbf{c}}$-orbits in ${}^\theta\mathfrak{F}_{\mathbf{c}} \times {}^\theta\mathfrak{F}_{\mathbf{c}}$. We define the corresponding *refinement datum* $(\widehat{\underline{\mathbf{e}}}, \widehat{\underline{\mathbf{d}}}, u)$ in analogy to Definition 5.3.15. More precisely, if we abbreviate $\lambda = {}^\theta P(\underline{\mathbf{e}})$ and $\mu = w \cdot {}^\theta P(\underline{\mathbf{d}})$, then

$$\widehat{\underline{\mathbf{e}}} := {}^\theta C(\lambda \Omega \mu), \quad \widehat{\underline{\mathbf{d}}} := {}^\theta C(\mu \Omega \lambda)$$

and $u \in \mathbb{Z}_2 \wr \mathsf{Sym}_{{}^\theta\ell_{\lambda\Omega\mu}}$ is the unique permutation sending $\lambda \Omega \mu$ to $\mu \Omega \lambda$. We also choose a reduced expression $u = s_{j_k} \cdot \ldots \cdot s_{j_1}$ and define the associated *crossing datum* $(\underline{\mathbf{e}}^0, \ldots, \underline{\mathbf{e}}^{2k})$ in the same way as in Definition 5.3.19.

Example 5.5.19. Let Q by a quiver such that Q_0 is a singleton. Then there is a unique involution on Q. Let $\mathbf{c} = 14$ and choose any duality structure on Q. We identify $7+k = -k$ for $1 \leq k \leq 7$ so that $\mathbf{N_c} = \{\pm 1, \ldots, \pm 7\}$. Let $s_7 \in {}^\theta\mathsf{W}_{\mathbf{c}} = \mathbb{Z}_2 \wr \mathsf{Sym}_7$ be the element swapping 7 and -7. Suppose that

$\underline{\mathbf{e}} = (3,2,4)$, $\underline{\mathbf{d}} = (4,2,2)$ and $w = s_5s_6s_7s_6s_5s_3s_4s_5s_6$. Then

$$\lambda = [1,2,3][4,5][\pm 6,\pm 7][\text{-}5,\text{-}4][\text{-}3,\text{-}2,\text{-}1], \qquad \mu = [1,2,4,\text{-}5][6,7][\pm 3][\text{-}7,\text{-}6][5,\text{-}4,\text{-}2,\text{-}1],$$
$$\lambda \mathbin{\Omega} \mu = [1,2][3][4][5][6,7][\text{-}7,\text{-}6][\text{-}5][\text{-}4][\text{-}3][\text{-}2,\text{-}1], \quad \mu \mathbin{\Omega} \lambda = [1,2][4][\text{-}5][6,7][3][\text{-}7,\text{-}6][\text{-}3][5][\text{-}4][\text{-}2,\text{-}1].$$

Hence

$$\widehat{\underline{\mathbf{e}}} = (2,1,1,1,2,0), \quad \widehat{\underline{\mathbf{d}}} = (2,1,1,2,1,0), \quad u = s_4s_3s_2s_4s_5s_4 \in \mathbb{Z}_2 \wr \mathsf{Sym}_5.$$

Definition 5.5.20. Given an isotropic orbit datum $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$, the corresponding refinement and crossing data, and $c \in {}^\theta\Lambda_{\underline{\mathbf{e}}^{2k}}$, we define elements $\underline{\mathbf{d}} \overset{c}{\underset{w}{\Longrightarrow}} \underline{\mathbf{e}}$ in analogy to Definition 5.3.24, i.e.,

$$\underline{\mathbf{d}} \overset{c}{\underset{w}{\Longrightarrow}} \underline{\mathbf{e}} \quad := \quad {}^\theta\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{e}}^0} \star {}^\theta\between^{j_1}_{\underline{\mathbf{e}}^2} \star \ldots \star {}^\theta\between^{j_k}_{\underline{\mathbf{e}}^{2k}} \star c \star {}^\theta\curlyvee^{\underline{\mathbf{e}}^{2k}}_{\underline{\mathbf{d}}} \in {}^\theta\mathcal{Z}_{\underline{\mathbf{e}},\underline{\mathbf{d}}}.$$

We have the following analogue of Theorem 5.3.25.

Theorem 5.5.21. *If we let $(\underline{\mathbf{e}}, \underline{\mathbf{d}}, w)$ range over all isotropic orbit data and c range over a basis of ${}^\theta\Lambda_{\underline{\mathbf{e}}^{2k}}$, then the elements $\underline{\mathbf{d}} \overset{c}{\underset{w}{\Longrightarrow}} \underline{\mathbf{e}}$ form a basis of ${}^\theta\mathcal{Z}_{\mathbf{c}}$.*

Proof. The proof of Theorem 5.3.25 uses only three ingredients: the Bruhat decomposition, Proposition 5.3.21 and Lemma 5.3.23. The Bruhat decomposition of course generalizes to reductive groups of type B, C and D. Lemma 5.3.23 also generalizes straightforwardly. To generalize Proposition 5.3.21, one only needs to modify its proof by replacing inversions associated to the symmetric group by inversions associated to the Weyl group of type B (see, e.g., [17, Proposition 8.1.1]). □

Theorem 5.5.21 and Proposition 5.5.16 directly imply the following analogue of Corollary 5.3.27.

Corollary 5.5.22. *Elementary merges, elementary splits and the polynomials ${}^\theta\mathcal{Z}^e_{\mathbf{c}}$ generate ${}^\theta\mathcal{Z}_{\mathbf{c}}$ as an algebra.*

5.5.5. Monoidal structure and categorification. We now consider the relationship between the categories of modules over ${}^\theta\mathcal{Z}_{\mathbf{c}}$ and $\mathcal{Z}_{\mathbf{c}}$. In this subsection we view ${}^\theta\mathcal{Z}_{\mathbf{c}}$ and $\mathcal{Z}_{\mathbf{c}}$ as graded algebras, with the gradings imported from the gradings on the corresponding Ext-algebras via the isomorphisms (5.8) and (5.29). We begin by recalling the monoidal structure on the direct sum $\mathcal{Z}$-pmod of the categories of finitely generated graded projective modules over quiver Schur algebras $\mathcal{Z}_{\mathbf{c}}$ for all dimension vectors $\mathbf{c} \in \Gamma$. We then show that the monoidal category $\mathcal{Z}$-pmod acts on the corresponding category ${}^\theta\mathcal{Z}$-pmod of modules over the algebras ${}^\theta\mathcal{Z}_{\mathbf{c}}$. Passing to Grothendieck groups, we obtain a $K_0(\mathcal{Z})$-module and -comodule structure on $K_0({}^\theta\mathcal{Z})$, which we relate to the Hall module of the category of self-dual representations of the quiver Q introduced by Young in [145].

One can easily show (as in, e.g., [131, §2.4] or [88, §2.6]) that there are canonical (non-unital) injective graded ring homomorphisms

$$i_{\mathbf{c},\mathbf{c}'} \colon \mathcal{Z}_{\mathbf{c}} \otimes \mathcal{Z}_{\mathbf{c}'} \hookrightarrow \mathcal{Z}_{\mathbf{c}+\mathbf{c}'}, \tag{5.30}$$

for all $\mathbf{c}, \mathbf{c}' \in \Gamma$, induced by inclusions of the corresponding polynomial representations

$$\mathcal{Q}_{\mathbf{c}} \otimes \mathcal{Q}_{\mathbf{c}'} \hookrightarrow \mathcal{Q}_{\mathbf{c}+\mathbf{c}'}. \tag{5.31}$$

Diagrammatically, these inclusions are depicted by a horizontal composition of diagrams. They define an associative algebra structure on the direct sums $\mathcal{Z} = \bigoplus_{\mathbf{c}\in\Gamma} \mathcal{Z}_{\mathbf{c}}$ and $\mathcal{Q} = \bigoplus_{\mathbf{c}\in\Gamma} \mathcal{Q}_{\mathbf{c}}$, which is referred to as the *horizontal multiplication*. The inclusions (5.30) also give rise to induction and restriction functors

$$\mathrm{Ind}_{\mathbf{c},\mathbf{c}'}\colon \mathcal{Z}_{\mathbf{c}} \otimes \mathcal{Z}_{\mathbf{c}'}\text{-}\operatorname{mod} \to \mathcal{Z}_{\mathbf{c}+\mathbf{c}'}\text{-}\operatorname{mod}, \quad \mathrm{Res}_{\mathbf{c},\mathbf{c}'}\colon \mathcal{Z}_{\mathbf{c}+\mathbf{c}'}\text{-}\operatorname{mod} \to \mathcal{Z}_{\mathbf{c}} \otimes \mathcal{Z}_{\mathbf{c}'}\text{-}\operatorname{mod}, \tag{5.32}$$

where by, e.g., $\mathcal{Z}_{\mathbf{c}}$- mod, we mean the category of finitely generated graded left $\mathcal{Z}_{\mathbf{c}}$-modules. These functors restrict to subcategories of projective modules. Setting

$$M \otimes N = \mathcal{Z}_{\mathbf{c}+\mathbf{c}'} \otimes_{\mathcal{Z}_{\mathbf{c}} \otimes \mathcal{Z}_{\mathbf{c}'}} M \boxtimes N$$

for $M \in \mathcal{Z}_{\mathbf{c}}$- pmod and $N \in \mathcal{Z}_{\mathbf{c}'}$- pmod, and $\mathbf{1} = \mathcal{Z}_0$ defines a monoidal structure on the direct sum of categories

$$\mathcal{Z}\text{-pmod} = \bigoplus_{\mathbf{c}\in\Gamma} \mathcal{Z}_{\mathbf{c}}\text{-pmod}.$$

Let $K_0(\mathcal{Z}) = K_0(\mathcal{Z}\text{-pmod})$ be its Grothendieck group, considered as a $\mathbb{Z}[q^{\pm 1}]$-module. The functors (5.32) induce maps

$$K_0(\mathcal{Z}) \otimes K_0(\mathcal{Z}) \to K_0(\mathcal{Z}), \quad K_0(\mathcal{Z}) \to K_0(\mathcal{Z}) \otimes K_0(\mathcal{Z}),$$

which turn $K_0(\mathcal{Z})$ into a Γ-graded $\mathbb{Z}[q^{\pm 1}]$-bialgebra. For special choices of the quiver Q, the bialgebra $K_0(\mathcal{Z})$ can be identified with (the opposite of) the generic nilpotent Hall algebra associated to the category of representations of Q over finite fields. For more information about this algebra we refer the reader to, e.g., [123].

Proposition 5.5.23. *Let Q be one of the following quivers: a Dynkin quiver, the A_∞ quiver, the Jordan quiver or a cyclic quiver. Then $K_0(\mathcal{Z})^{op}$ is canonically isomorphic to the integral form of the generic nilpotent Hall algebra of the quiver Q.*

Proof. By Theorem 5.4.12, there is an isomorphism of algebras $\mathcal{Z}_{\mathbf{c}} \cong \mathcal{Z}^{SW}_{\mathbf{c}}$ for each $\mathbf{c} \in \Gamma$. The explicit description of this isomorphism from [99, Proposition 9.4, 9.6] implies that there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccc}
\mathcal{Z}^{SW}_{\mathbf{c}'} \otimes \mathcal{Z}^{SW}_{\mathbf{c}} & \hookrightarrow & \mathcal{Z}^{SW}_{\mathbf{c}+\mathbf{c}'} \\
\wr\uparrow{\scriptstyle flip} & & \uparrow \\
\mathcal{Z}^{SW}_{\mathbf{c}} \otimes \mathcal{Z}^{SW}_{\mathbf{c}'} & & \wr \\
\wr\uparrow & & \\
\mathcal{Z}_{\mathbf{c}} \otimes \mathcal{Z}_{\mathbf{c}'} & \hookrightarrow & \mathcal{Z}_{\mathbf{c}+\mathbf{c}'}
\end{array}$$

Passing to Grothendieck groups, we see that $K_0(\mathcal{Z})^{op} \cong K_0(\mathcal{Z}^{SW})$ as algebras. The proposition now follows from [131, Proposition 5.12]. □

We now bring the mixed quiver Schur algebras ${}^\theta\mathcal{Z}_{\mathbf{c}}$ into the picture.

Lemma 5.5.24. *If $\mathbf{a} \in \Gamma$, $\mathbf{b} \in \Gamma^\theta$ satisfy $\mathrm{D}(\mathbf{a}) + \mathbf{b} = \mathbf{c}$, then there is an injective (non-unital) ring homomorphism*

$$i_{\mathbf{a},\mathbf{b}}\colon \mathcal{Z}_{\mathbf{a}} \otimes {}^\theta\mathcal{Z}_{\mathbf{b}} \hookrightarrow {}^\theta\mathcal{Z}_{\mathbf{c}}, \qquad \curlywedge^{\mathbf{e}}_{\underline{\mathbf{d}}} \otimes {}^\theta\curlywedge^{\mathbf{e}'}_{\underline{\mathbf{d}}'} \mapsto {}^\theta\curlywedge^{\mathbf{e}''}_{\underline{\mathbf{d}}''}, \quad \curlyvee^{\mathbf{d}}_{\underline{\mathbf{e}}} \otimes {}^\theta\curlywedge^{\mathbf{e}'}_{\underline{\mathbf{d}}'} \mapsto {}^\theta\curlyvee^{\mathbf{d}''}_{\underline{\mathbf{e}}''}, \tag{5.33}$$

sending a polynomial $f \otimes g \in \Lambda_{\underline{\mathbf{d}}} \otimes {}^\theta\Lambda_{\underline{\mathbf{d}}'}$ to $f \cdot g \in {}^\theta\Lambda_{\underline{\mathbf{d}}''}$, where $\underline{\mathbf{d}}'' = \underline{\mathbf{d}} \cup \underline{\mathbf{d}}'$ and $\underline{\mathbf{d}}'' \succ \underline{\mathbf{e}}'' \trianglerighteq \mathbf{c}$.

Proof. Let ${}^\theta\mathfrak{Z}_{\mathbf{a},\mathbf{b}} := \bigsqcup {}^\theta\mathfrak{Z}_{\underline{\mathbf{d}}'',\underline{\mathbf{e}}''}$, where the disjoint union ranges over all $\underline{\mathbf{d}}'', \underline{\mathbf{e}}'' \trianglerighteq \mathbf{c}$ which can be expressed as a concatenation $\underline{\mathbf{d}} \cup \underline{\mathbf{d}}'$, for some $\underline{\mathbf{d}} \triangleright \mathbf{a}$ and $\underline{\mathbf{d}}' \trianglerighteq \mathbf{b}$ (and analogously for $\underline{\mathbf{e}}''$). Clearly ${}^\theta\mathcal{Z}_{\mathbf{a},\mathbf{b}} := H_\bullet^{{}^\theta\mathsf{G}_\mathbf{c}}({}^\theta\mathfrak{Z}_{\mathbf{a},\mathbf{b}})$ is a convolution subalgebra of ${}^\theta\mathcal{Z}_\mathbf{c}$. The forgetful maps ${}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}''} \to \mathfrak{Q}_{\underline{\mathbf{d}}}$ (remembering only the first $\ell_{\underline{\mathbf{d}}}$ steps in an isotropic flag) and ${}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}''} \to {}^\theta\mathfrak{Q}_{\underline{\mathbf{d}}'}$ (remembering only the last $\ell_{\underline{\mathbf{d}}'} + 1$ steps) induce a map ${}^\theta\mathfrak{Z}_{\mathbf{a},\mathbf{b}} \to \mathfrak{Z}_\mathbf{a} \times {}^\theta\mathfrak{Z}_\mathbf{b}$. The pullback $\mathcal{Z}_\mathbf{a} \otimes {}^\theta\mathcal{Z}_\mathbf{b} \to {}^\theta\mathcal{Z}_\mathbf{c}$ with respect to the latter is injective, and it is easy to check that it is compatible with the convolution product and that, explicitly, it is given by (5.33). □

As before, the inclusions (5.33) are depicted diagrammatically via a horizontal composition of diagrams. They give rise to functors

$$\mathrm{Ind}_{\mathbf{a},\mathbf{b}} \colon \mathcal{Z}_\mathbf{a} \otimes {}^\theta\mathcal{Z}_\mathbf{b}\text{-}\,\mathrm{pmod} \to {}^\theta\mathcal{Z}_{\mathrm{D}(\mathbf{a})+\mathbf{b}}\text{-}\,\mathrm{pmod}, \quad \mathrm{Res}_{\mathbf{a},\mathbf{b}} \colon {}^\theta\mathcal{Z}_{\mathrm{D}(\mathbf{a})+\mathbf{b}}\text{-}\,\mathrm{pmod} \to \mathcal{Z}_\mathbf{a} \otimes {}^\theta\mathcal{Z}_\mathbf{b}\text{-pmod}. \tag{5.34}$$

Let ${}^\theta\mathcal{Z}$-pmod be the direct sum of categories

$${}^\theta\mathcal{Z}\text{-pmod} = \bigoplus_{\mathbf{c} \in \Gamma^\theta} {}^\theta\mathcal{Z}_\mathbf{c}\text{-pmod}$$

and let $K_0({}^\theta\mathcal{Z}) = K_0({}^\theta\mathcal{Z}\text{-pmod})$ be its Grothendieck group. The following proposition, whose proof is standard, summarizes the relation between the categories $\mathcal{Z}$-pmod and ${}^\theta\mathcal{Z}$-pmod.

Proposition 5.5.25. *The following hold.*

a) *The monoidal category* $\mathcal{Z}$- pmod *acts (see, e.g., [70]) on* ${}^\theta\mathcal{Z}$- pmod *via*

$$M * N = {}^\theta\mathcal{Z}_{\mathrm{D}(\mathbf{a})+\mathbf{b}} \otimes_{\mathcal{Z}_\mathbf{a} \otimes {}^\theta\mathcal{Z}_\mathbf{b}} M \boxtimes N,$$

for $M \in \mathcal{Z}_\mathbf{a}$- pmod *and* $N \in {}^\theta\mathcal{Z}_\mathbf{b}$- pmod.

b) *The functors* (5.34) *induce maps*

$$K_0(\mathcal{Z}) \otimes K_0({}^\theta\mathcal{Z}) \to K_0({}^\theta\mathcal{Z}), \quad K_0({}^\theta\mathcal{Z}) \to K_0(\mathcal{Z}) \otimes K_0({}^\theta\mathcal{Z}),$$

which turn $K_0({}^\theta\mathcal{Z})$ *into a* Γ^θ*-graded* $K_0(\mathcal{Z})$*-module and -comodule.*

Remark 5.5.26. In [145], Young defined a Hall module associated to the category of self-dual representations of a quiver with an involution. The Hall module is a module as well as a comodule over the Hall algebra associated to the same quiver. We expect that, for a general quiver Q with an involution θ, $K_0(\mathcal{Z})$ is isomorphic to a subalgebra of the Hall algebra of Q and $K_0({}^\theta\mathcal{Z})$ is isomorphic to a subspace of the Hall module of (Q,θ) stable under the action and coaction of $K_0(\mathcal{Z})$. Since $K_0(\mathcal{Z})$ contains the composition subalgebra associated to Q, [145, Theorem 3.5] implies that $K_0({}^\theta\mathcal{Z})$ is also a module over $B_\theta(\mathfrak{g}_Q)$, the algebra introduced by Enomoto and Kashiwara [45, 46] in the context of symmetric crystals. The KLR analogue of $K_0({}^\theta\mathcal{Z})$ was studied by Varagnolo and Vasserot [136], who showed that it is isomorphic to a certain highest weight module over $B_\theta(\mathfrak{g}_Q)$.

5.6 Connection to cohomological Hall algebras

In this section we relate quiver Schur algebras to the cohomological Hall algebra (CoHA) of a quiver Q (without potential) introduced by Kontsevich and Soibelman [90]. More specifically, we interpret merges and splits as iterated multiplication and comultiplication in the CoHA. This gives an action of quiver Schur algebras on the tensor algebra of the CoHA, which we identify with the direct sum of the polynomial representations of all the quiver Schur algebras associated to Q. In the case of a quiver endowed with an involution and a duality structure, we relate mixed quiver Schur algebras to the cohomological Hall module (CoHM) introduced by Young [146], realizing merges and splits as action and coaction operators. An algebraic manifestation of these connections is a new interpretation of the shuffle description of the CoHA and the CoHM in terms of Demazure operators.

5.6.1. The cohomological Hall algebra. We start by recalling the definition of the CoHA from [90, §2.2]. Let Q be a finite quiver. Given $\mathbf{c} \in \Gamma$ and $\underline{\mathbf{d}} \rhd \mathbf{c}$, set

$$\mathcal{H}_{\mathbf{c}} := H^{\bullet}_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{R}_{\mathbf{c}}), \quad \mathcal{H}_{\underline{\mathbf{d}}} := \bigotimes_{j=1}^{\ell_{\mathbf{d}}} \mathcal{H}_{\mathbf{d}_j}, \quad \mathcal{H} := \bigoplus_{\mathbf{c}\in\Gamma} \mathcal{H}_{\mathbf{c}}.$$

The Künneth map and the homotopy equivalences $\mathfrak{R}_{\underline{\mathbf{d}}} \twoheadrightarrow \prod_j \mathfrak{R}_{\mathbf{d}_j}$ and $\mathsf{P}_{\underline{\mathbf{d}}} \twoheadrightarrow \mathsf{L}_{\underline{\mathbf{d}}}$ yield canonical isomorphisms

$$\mathcal{H}_{\underline{\mathbf{d}}} \cong H^{\bullet}_{\mathsf{L}_{\underline{\mathbf{d}}}}(\textstyle\prod_j^{\ell_{\mathbf{d}}} \mathfrak{R}_{\mathbf{d}_j}) \cong H^{\bullet}_{\mathsf{P}_{\underline{\mathbf{d}}}}(\mathfrak{R}_{\underline{\mathbf{d}}}). \tag{5.35}$$

Definition 5.6.1. Given $\underline{\mathbf{d}} \succ \underline{\mathbf{e}}$, we have a closed embedding $(\mathfrak{R}_{\underline{\mathbf{d}}})_{\mathsf{P}_{\underline{\mathbf{d}}}} \overset{i}{\hookrightarrow} (\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{d}}}}$ and a fibration $(\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{d}}}} \overset{p}{\twoheadrightarrow} (\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{e}}}}$ with smooth and compact fibre. Using the identification (5.35), we get operators

$$\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} \colon \mathcal{H}_{\underline{\mathbf{d}}} \xrightarrow{p_* \circ i_*} \mathcal{H}_{\underline{\mathbf{e}}}, \quad \mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} \colon \mathcal{H}_{\underline{\mathbf{e}}} \xrightarrow{i^* \circ p^*} \mathcal{H}_{\underline{\mathbf{d}}}. \tag{5.36}$$

We abbreviate $\mathsf{m}^{\mathbf{c}}_{\underline{\mathbf{d}}} = \mathsf{m}^{(\mathbf{c})}_{\underline{\mathbf{d}}}$, etc. Let $\mathsf{m} \colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and $\mathsf{com} \colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be the operators defined by the condition that $\mathsf{m}|_{\mathcal{H}_{\underline{\mathbf{d}}}} = \mathsf{m}^{\mathbf{c}}_{\underline{\mathbf{d}}}$, and that the projection of $\mathsf{com}|_{\mathcal{H}_{\mathbf{c}}}$ onto $\mathcal{H}_{\underline{\mathbf{d}}}$ equals $\mathsf{com}^{\underline{\mathbf{d}}}_{\mathbf{c}}$, for all dimension vectors $\mathbf{c} \in \Gamma$ and vector compositions $\underline{\mathbf{d}} \in \mathbf{Com}^2_{\mathbf{c}}$ of length two.

Definition 5.6.2. The *cohomological Hall algebra* associated to the quiver Q is the Γ-graded vector space $\mathcal{H}$ together with multiplication given by m. By [90, Theorem 1], $(\mathcal{H}, \mathsf{m})$ is indeed an associative algebra. The operation com also makes $\mathcal{H}$ into a coassociative coalgebra. However, the multiplication and comultiplication are in general not compatible, i.e., $(\mathcal{H}, \mathsf{m}, \mathsf{com})$ is not a bialgebra.

In light of Definition 5.6.2, the operators (5.36) can be viewed as multifactor versions of multiplication and comultiplication in $\mathcal{H}$.

Definition 5.6.3. Let $\mathbb{T}(\mathcal{H}) := T(\mathcal{H}_+)$ be the tensor algebra of $\mathcal{H}_+ := \bigoplus_{\mathbf{c}\in\Gamma_+} \mathcal{H}_{\mathbf{c}}$. We regard it as a Γ-graded vector space in the following way:

$$\mathbb{T}(\mathcal{H}) = \bigoplus_{\mathbf{c}\in\Gamma} \mathbb{T}_{\mathbf{c}}(\mathcal{H}), \quad \mathbb{T}_{\mathbf{c}}(\mathcal{H}) := \bigoplus_{\underline{\mathbf{d}}\rhd\mathbf{c}} \mathcal{H}_{\underline{\mathbf{d}}}. \tag{5.37}$$

We consider $\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ and $\mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$ as operators on $\mathbb{T}_{\mathbf{c}}(\mathcal{H})$. Given $\gamma \in \mathcal{H}_{\underline{\mathbf{d}}}$, let $\cup_\gamma = \gamma \cup \text{-} : \mathcal{H}_{\underline{\mathbf{d}}} \to \mathcal{H}_{\underline{\mathbf{d}}}$ be the operator given by taking the cup product with γ.

5.6.2. The CoHA and quiver Schur algebras. We will now explain the connection between the cohomological Hall algebra $\mathcal{H}$ and quiver Schur algebras associated to the same quiver Q.

Lemma 5.6.4. *For each $\mathbf{c} \in \Gamma$, there is a vector space isomorphism*

$$\mathbb{T}_{\mathbf{c}}(\mathcal{H}) \xrightarrow{\sim} \mathcal{Q}_{\mathbf{c}}. \tag{5.38}$$

Proof. It is easy to see that the Borel constructions $(\mathfrak{R}_{\underline{\mathbf{d}}})_{\mathsf{P}_{\underline{\mathbf{d}}}}$ and $(\mathfrak{Q}_{\underline{\mathbf{d}}})_{\mathsf{G}_{\mathbf{c}}}$ are naturally isomorphic. Composing (5.35) with the induced isomorphism of equivariant cohomology groups $H^{\bullet}_{\mathsf{P}_{\underline{\mathbf{d}}}}(\mathfrak{R}_{\underline{\mathbf{d}}}) \cong H^{\bullet}_{\mathsf{G}_{\mathbf{c}}}(\mathfrak{Q}_{\underline{\mathbf{d}}})$ yields an isomorphism $\mathcal{H}_{\underline{\mathbf{d}}} \xrightarrow{\sim} \mathcal{Q}_{\underline{\mathbf{d}}}$. The lemma follows by summing over all $\underline{\mathbf{d}} \rhd \mathbf{c}$. □

Remark 5.6.5. If we sum over all dimension vectors $\mathbf{c} \in \Gamma$, the identification (5.38) gives rise to an isomorphism between the entire tensor algebra $\mathbb{T}(\mathcal{H})$ and the direct sum $\mathcal{Q} = \bigoplus_{\mathbf{c}\in\Gamma} \mathcal{Q}_{\mathbf{c}}$ of the polynomial representations of all the quiver Schur algebras associated to the quiver Q. Under this isomorphism, multiplication in the tensor algebra corresponds to the horizontal multiplication on $\mathcal{Q}$ defined by the inclusions (5.31).

Since, by Proposition 5.4.4, the $\mathcal{Z}_{\mathbf{c}}$-module $\mathcal{Q}_{\mathbf{c}}$ is faithful, (5.38) induces an injective algebra homomorphism

$$\mathcal{Z}_{\mathbf{c}} \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{T}_{\mathbf{c}}(\mathcal{H})). \tag{5.39}$$

The following theorem gives an explicit description of this homomorphism.

Theorem 5.6.6. *The algebra homomorphism* (5.39) *is given by*

$$\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}} \mapsto \mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}, \quad \curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}} \mapsto \mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}, \quad \gamma \mapsto \cup_\gamma,$$

where $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \rhd \mathbf{c}$ *and* $\gamma \in \mathcal{Q}_{\underline{\mathbf{d}}} \cong \mathcal{H}_{\underline{\mathbf{d}}}$.

Proof. By Corollary 5.3.27, $\mathcal{Z}_{\mathbf{c}}$ is generated by merges, splits and polynomials. Therefore, it suffices to describe the image of these elements. We have a commutative diagram

$$\begin{array}{ccccc} (\mathfrak{R}_{\underline{\mathbf{d}}})_{\mathsf{P}_{\underline{\mathbf{d}}}} & \xrightarrow{i} & (\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{d}}}} & \xrightarrow{p} & (\mathfrak{R}_{\underline{\mathbf{e}}})_{\mathsf{P}_{\underline{\mathbf{e}}}} \\ \wr\| & & \wr\| & & \wr\| \\ (\mathfrak{Q}_{\underline{\mathbf{d}}})_{\mathsf{G}_{\mathbf{c}}} & \xrightarrow{\iota} & (\mathfrak{Q}_{\underline{\mathbf{d}},\underline{\mathbf{e}}})_{\mathsf{G}_{\mathbf{c}}} & \xrightarrow{q} & (\mathfrak{Q}_{\underline{\mathbf{e}}})_{\mathsf{G}_{\mathbf{c}}} \end{array} \tag{5.40}$$

where ι and q are as in (5.21). As explained in the proof of Theorem 5.4.7, the action of $\curlywedge^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ is given by the pushforward along the two lower horizontal maps in (5.40). But this is the same as the pushforward along the two upper horizontal maps, which is, by definition, $\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$. Similarly, the action of $\curlyvee^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$ is given by the pullback along the two lower horizontal maps in (5.40), and this is the same as the pullback along the two upper horizontal maps, which is, by definition, $\mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$. The third statement is clear. □

Remark 5.6.7. We make several remarks about Theorem 5.6.6.

(i) In light of Theorem 5.6.6, the associativity of the merges (R1) and the coassociativity of the splits (R2) relations in the quiver Schur algebra express the fact that $\mathcal{H}$ is an associative algebra and a coassociative coalgebra, respectively.

(ii) When Q is the A_1 quiver, $\mathcal{H}$ is isomorphic to the exterior algebra in infinitely many variables (see [90, §2.5]). This fact explains the connection between quiver Schur algebras associated to the A_1 quiver and web categories, discussed in §5.4.4.

Next, we interpret multiplication in the cohomological Hall algebra in terms of Demazure operators.

Proposition 5.6.8. *Let* $\mathbf{a}, \mathbf{b} \in \Gamma$, $\mathbf{c} = \mathbf{a} + \mathbf{b}$ *and* $\underline{\mathbf{d}} = (\mathbf{a}, \mathbf{b})$. *Given* $f \in \mathcal{H}_{\mathbf{a}}$ *and* $g \in \mathcal{H}_{\mathbf{b}}$, *the multiplication of* f *and* g *is given by*

$$\mathsf{m}(f, g) = (-1)^{r_{\underline{\mathbf{d}}}} \cdot \Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f \cdot g \cdot \mathtt{E}_{\underline{\mathbf{d}}}),$$

where $\cdot$ *stands for polynomial multiplication (i.e., the cup product).*

Proof. The proposition follows directly from Proposition 5.4.9 and the shuffle formula for multiplication in $\mathcal{H}$ from [90, Theorem 2]. □

Remark 5.6.9. Yang and Zhao defined in [143] a formal version of the CoHA associated to any equivariant oriented Borel-Moore homology theory and described multiplication in the formal CoHA in terms of a shuffle formula depending on a formal group law. We expect this formula can be rephrased in terms of the formal Demazure operators from [73].

5.6.3. Cohomological Hall modules.

We recall the definition of the cohomological Hall module from [146, §3.1]. Suppose that Q admits an involution θ and a duality structure (σ, ς). Given $\mathbf{c} \in \Gamma^\theta$ and $\underline{\mathbf{d}} \trianglerighteq \mathbf{c}$, let

$$ {}^\theta\mathcal{M}_{\mathbf{c}} := H^\bullet_{{}^\theta\mathsf{G}_{\mathbf{c}}}({}^\theta\mathfrak{R}_{\mathbf{c}}), \quad {}^\theta\mathcal{M}_{\underline{\mathbf{d}}} := \bigotimes_{j=1}^{\ell_{\underline{\mathbf{d}}}} \mathcal{H}_{\mathbf{d}_j} \otimes {}^\theta\mathcal{M}_{\mathbf{d}_\infty}, \quad {}^\theta\mathcal{M} := \bigoplus_{\mathbf{c}\in\Gamma^\theta} {}^\theta\mathcal{M}_{\mathbf{c}}. $$

In analogy to (5.35), we have canonical isomorphisms

$$ {}^\theta\mathcal{M}_{\underline{\mathbf{d}}} \cong H^\bullet_{{}^\theta\mathsf{L}_{\underline{\mathbf{d}}}}\big(\textstyle\prod_{j=1}^{\ell_{\underline{\mathbf{d}}}} \mathfrak{R}_{\mathbf{d}_j} \times {}^\theta\mathfrak{R}_{\mathbf{d}_\infty}\big) \cong H^\bullet_{{}^\theta\mathsf{P}_{\underline{\mathbf{d}}}}({}^\theta\mathfrak{R}_{\underline{\mathbf{d}}}). \tag{5.41} $$

Definition 5.6.10. Given $\underline{\mathbf{d}} \succ \underline{\mathbf{e}}$, we have a closed embedding ${}^\theta i\colon ({}^\theta\mathfrak{R}_{\underline{\mathbf{d}}})_{{}^\theta\mathsf{P}_{\underline{\mathbf{d}}}} \hookrightarrow ({}^\theta\mathfrak{R}_{\underline{\mathbf{e}}})_{{}^\theta\mathsf{P}_{\underline{\mathbf{d}}}}$ and a fibration ${}^\theta p\colon ({}^\theta\mathfrak{R}_{\underline{\mathbf{e}}})_{{}^\theta\mathsf{P}_{\underline{\mathbf{d}}}} \twoheadrightarrow ({}^\theta\mathfrak{R}_{\underline{\mathbf{e}}})_{{}^\theta\mathsf{P}_{\underline{\mathbf{e}}}}$. Using the identification (5.41), we get operators

$$ {}^\theta\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}\colon {}^\theta\mathcal{M}_{\underline{\mathbf{d}}} \xrightarrow{{}^\theta p_* \circ {}^\theta i_*} {}^\theta\mathcal{M}_{\underline{\mathbf{e}}}, \quad {}^\theta\mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}\colon {}^\theta\mathcal{M}_{\underline{\mathbf{e}}} \xrightarrow{{}^\theta i^* \circ {}^\theta p^*} {}^\theta\mathcal{M}_{\underline{\mathbf{d}}}. \tag{5.42} $$

Let $\mathsf{act}\colon \mathcal{H} \otimes {}^\theta\mathcal{M} \to {}^\theta\mathcal{M}$ and $\mathsf{coact}\colon {}^\theta\mathcal{M} \to \mathcal{H} \otimes {}^\theta\mathcal{M}$ be the operators defined by the condition that $\mathsf{act}|_{{}^\theta\mathcal{M}_{\underline{\mathbf{d}}}} = \mathsf{m}^{\mathbf{c}}_{\underline{\mathbf{d}}}$, and that the projection of $\mathsf{coact}|_{{}^\theta\mathcal{M}_{\mathbf{c}}}$ onto ${}^\theta\mathcal{M}_{\underline{\mathbf{d}}}$ equals $\mathsf{com}^{\underline{\mathbf{d}}}_{\mathbf{c}}$, for all $\mathbf{c} \in \Gamma^\theta$ and $\underline{\mathbf{d}} \in {}^\theta\mathbf{Com}^1_{\mathbf{c}}$.

Definition 5.6.11. The *cohomological Hall module* associated to $(Q, \theta, \sigma, \varsigma)$ is the Γ^θ-graded vector space ${}^\theta\mathcal{M}$ together with the $\mathcal{H}$-action given by act. By [146, Theorem 3.1], $({}^\theta\mathcal{M}, \mathsf{act})$ is indeed an $\mathcal{H}$-module. The operation coact also makes ${}^\theta\mathcal{M}$ into an $\mathcal{H}$-comodule. However, the action and the coaction are in general not compatible, i.e., $({}^\theta\mathcal{M}, \mathsf{act}, \mathsf{coact})$ is not a Hopf module.

Let us interpret the operators (5.42) in the two special cases when $\underline{\mathbf{d}} \succ_f \underline{\mathbf{e}}$ or $\underline{\mathbf{d}} \succ_\infty \underline{\mathbf{e}}$. If $\underline{\mathbf{d}} \succ_f \underline{\mathbf{e}}$ then ${}^\theta\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ and ${}^\theta\mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$ are multifactor multiplication and comultiplication operators, respectively. On the other hand, if $\underline{\mathbf{d}} \succ_\infty \underline{\mathbf{e}}$ then ${}^\theta\mathsf{m}^{\underline{\mathbf{e}}}_{\underline{\mathbf{d}}}$ and ${}^\theta\mathsf{com}^{\underline{\mathbf{d}}}_{\underline{\mathbf{e}}}$ can be interpreted as iterated action and coaction operators, respectively.

5.6.4. The CoHM and mixed quiver Schur algebras.

Let $\mathbb{T}({}^\theta\mathcal{M}) := \mathbb{T}(\mathcal{H}) \otimes {}^\theta\mathcal{M}$. We regard it as a Γ^θ-graded vector space as follows:

$$ \mathbb{T}({}^\theta\mathcal{M}) = \bigoplus_{\mathbf{c}\in\Gamma^\theta} \mathbb{T}_{\mathbf{c}}({}^\theta\mathcal{M}), \quad \mathbb{T}_{\mathbf{c}}({}^\theta\mathcal{M}) := \bigoplus_{\underline{\mathbf{d}}\trianglerighteq\mathbf{c}} {}^\theta\mathcal{M}_{\underline{\mathbf{d}}}. $$

In analogy to Lemma 5.6.4, one easily shows that there is a vector space isomorphism

$$ \mathbb{T}_{\mathbf{c}}({}^\theta\mathcal{M}) \xrightarrow{\sim} {}^\theta\mathcal{Q}_{\mathbf{c}}. \tag{5.43} $$

Since the $\mathcal{Z}_{\mathbf{c}}$-module $\mathcal{Q}_{\mathbf{c}}$ is faithful, (5.43) induces an injective algebra homomorphism

$$ {}^\theta\mathcal{Z}_{\mathbf{c}} \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{T}_{\mathbf{c}}({}^\theta\mathcal{M})). \tag{5.44} $$

Theorem 5.6.6 carries over, with analogous proof (using Corollary 5.5.22), to our current setting, yielding an explicit description of this homomorphism.

Theorem 5.6.12. *The algebra homomorphism* (5.44) *is given by*

$$^{\theta}\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} \mapsto {}^{\theta}\mathsf{m}_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}, \quad {}^{\theta}\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}} \mapsto {}^{\theta}\mathsf{com}_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}, \quad \gamma \mapsto \cup_{\gamma},$$

where $\underline{\mathbf{d}} \trianglerighteq \mathbf{c}$ *and* $\gamma \in {}^{\theta}\mathcal{Q}_{\underline{\mathbf{d}}} \cong {}^{\theta}\mathcal{M}_{\underline{\mathbf{d}}}$.

Remark 5.6.13. Summing over all $\mathbf{c} \in \Gamma^{\theta}$, (5.43) gives an identification of $\mathbb{T}(\mathcal{H}) \otimes {}^{\theta}\mathcal{M}$ with the direct sum of the polynomial representations of all the mixed quiver Schur algebras associated to $(Q, \theta, \sigma, \varsigma)$. Moreover, the relations ($^{\theta}$R1)-($^{\theta}$R2) express the fact that $^{\theta}\mathcal{M}$ is an $\mathcal{H}$-module and -comodule, respectively.

5.6.5. The polynomial representation. We will now use Theorem 5.6.12 to deduce an explicit description of the polynomial representation $^{\theta}\mathcal{Q}_{\mathbf{c}}$ of $^{\theta}\mathcal{Z}_{\mathbf{c}}$ from the corresponding description of the cohomological Hall module $^{\theta}\mathcal{M}$ as a shuffle module in [146].

Definition 5.6.14. Let $\underline{\mathbf{d}} = (\mathbf{a}, \mathbf{b}) \trianglerighteq \mathbf{c}$. We first define an analogue of the classes $\mathtt{S}_{\underline{\mathbf{d}}}$ from (5.19).

If $i \in Q_0^+$, then let

$$^{\theta}\mathtt{S}_{\underline{\mathbf{d}}}(i) = \prod_{k=1}^{\mathbf{a}(i)} \prod_{l=\mathbf{a}(i)+1}^{\mathbf{a}(i)+\mathbf{b}(i)} \prod_{m=\mathbf{a}(i)+\mathbf{b}(i)+1}^{\mathbf{c}(i)} (x_l(i) - x_k(i))(x_m(i) - x_l(i))(x_m(i) - x_k(i)).$$

If $i \in Q_0^{\theta}$ then

$$^{\theta}\mathtt{S}_{\underline{\mathbf{d}}}(i) = g_i(x_1(i), \ldots, x_{\mathbf{a}(i)}(i)) \prod_{1 \le k < l \le \mathbf{a}(i)} (-x_k(i) - x_l(i)) \prod_{k=1}^{\mathbf{a}(i)} \prod_{l=\mathbf{a}(i)+1}^{\lfloor \mathbf{c}(i)/2 \rfloor} (x_k(i)^2 - x_l(i)^2),$$

where

$$g_i(x_1(i), \ldots, x_{\mathbf{a}(i)}(i)) = \begin{cases} (-1)^{\mathbf{a}(i)} \prod_{k=1}^{\mathbf{a}(i)} x_k(i) & \text{if } \sigma(i) = 1 \text{ and } \mathbf{c}(i) \text{ is odd,} \\ (-2)^{\mathbf{a}(i)} \prod_{k=1}^{\mathbf{a}(i)} x_k(i) & \text{if } \sigma(i) = -1, \\ 1 & \text{if } \sigma(i) = 1 \text{ and } \mathbf{c}(i) \text{ is even.} \end{cases}$$

Next, we define an analogue of the classes $\mathtt{E}_{\underline{\mathbf{d}}}$ from (5.20). To simplify exposition, let us write $x_k(\theta(i)) = -x_{\mathbf{a}(i)+\mathbf{b}(i)+k}(i)$ if $i \in Q_0^+$ and $x_k(\theta(i)) = x_k(i)$ if $i \in Q_0^{\theta}$.

If $i \xrightarrow{a} j \in Q_1^+$, then let

$$^{\theta}\mathtt{E}_{\underline{\mathbf{d}}}(a) := {}^{\theta}\mathtt{E}_{\underline{\mathbf{d}}}(a, i) {}^{\theta}\mathtt{E}_j(a, j) \prod_{m=1}^{\mathbf{a}(\theta(j))} \prod_{k=1}^{\mathbf{a}(i)} (-x_m(\theta(j)) - x_k(i)),$$

where

$$^{\theta}\mathtt{E}_{\underline{\mathbf{d}}}(a, i) = \begin{cases} \prod_{l=\mathbf{a}(i)+1}^{\mathbf{a}(i)+\mathbf{b}(i)} \prod_{m=1}^{\mathbf{a}(\theta(j))} (-x_m(\theta(j)) - x_l(i)) & \text{if } i \notin Q_0^{\theta}, \\ \prod_{l=\mathbf{a}(i)+1}^{\lfloor \mathbf{c}(i)/2 \rfloor} \prod_{m=1}^{\mathbf{a}(\theta(j))} (x_m(\theta(j))^2 - x_l(i)^2)(-x_m(j))^{\epsilon(i)} & \text{if } i \in Q_0^{\theta}, \end{cases}$$

$$
{}^\theta\mathtt{E}_{\underline{\mathbf{d}}}(a,j) = \begin{cases} \displaystyle\prod_{k=1}^{\mathbf{a}(i)} \prod_{l=\mathbf{a}(j)+1}^{\mathbf{a}(j)+\mathbf{b}(j)} (x_l(j) - x_k(i)) & \text{if } j \notin Q_0^\theta, \\ \displaystyle\prod_{k=1}^{\mathbf{a}(i)} \prod_{l=\mathbf{a}(j)+1}^{\lfloor \mathbf{c}(j)/2 \rfloor} (x_k(i)^2 - x_l(j)^2)(-x_k(i))^{\epsilon(j)} & \text{if } j \in Q_0^\theta, \end{cases}
$$

and $\epsilon(i) = 1$ if $\mathbf{c}(i)$ is odd, and $\epsilon(i) = 0$ if $\mathbf{c}(i)$ is even.

If $\theta(i) \xrightarrow{a} i \in Q_1^\theta$, then let

$$
{}^\theta\mathtt{E}_{\underline{\mathbf{d}}}(a) := {}^\theta\widetilde{\mathtt{E}}_{\underline{\mathbf{d}}}(a) \prod_{1 \le k \le_{\sigma(i)\varsigma(a)} l \le \mathbf{a}(\theta(i))} (-x_k(\theta(i)) - x_l(\theta(i))),
$$

where $\le_1 = \le$ and $\le_{-1} = <$, and

$$
{}^\theta\widetilde{\mathtt{E}}_{\underline{\mathbf{d}}}(a) = \begin{cases} \displaystyle\prod_{l=\mathbf{a}(i)+1}^{\mathbf{a}(i)+\mathbf{b}(i)} \prod_{m=1}^{\mathbf{a}(\theta(i))} (x_l(i) - x_m(\theta(i))) & \text{if } i \notin Q_0^\theta, \\ \displaystyle\prod_{m=1}^{\mathbf{a}(i)} \prod_{l=\mathbf{a}(i)+1}^{\lfloor \mathbf{c}(i)/2 \rfloor} (x_m(i)^2 - x_l(j)^2)(x_l(i))^{\epsilon(i)} & \text{if } i \in Q_0^\theta. \end{cases}
$$

Finally, define

$$
{}^\theta\mathtt{S}_{\underline{\mathbf{d}}} := \prod_{i \in Q_0^+ \sqcup Q_0^\theta} {}^\theta\mathtt{S}_{\underline{\mathbf{d}}}(i), \quad {}^\theta\mathtt{E}_{\underline{\mathbf{d}}} := \prod_{a \in Q_1^+ \sqcup Q_1^\theta} {}^\theta\mathtt{E}_{\underline{\mathbf{d}}}(a), \quad {}^\theta\pitchfork_{\underline{\mathbf{d}}}^{\mathbf{c}} := \sum_{w \in {}^\theta\mathsf{D}_{\underline{\mathbf{d}}}^{\mathbf{c}}} w \in {}^\theta\mathsf{W}_{\mathbf{c}}.
$$

Theorem 5.6.15. *Let $\underline{\mathbf{d}} \succ \underline{\mathbf{e}} \vDash \mathbf{c}$. The action of the generators of ${}^\theta\mathcal{Z}_{\mathbf{c}}$ on ${}^\theta\mathcal{Q}_{\mathbf{c}} \cong {}^\theta\Lambda_{\mathbf{c}}$ admits the following description.*

a) *The action of ${}^\theta\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}$ is given by the inclusion*

$$
{}^\theta\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}} \colon {}^\theta\Lambda_{\underline{\mathbf{e}}} \hookrightarrow {}^\theta\Lambda_{\underline{\mathbf{d}}}, \quad f \mapsto f, \qquad {}^\theta\curlyvee_{\underline{\mathbf{e}}}^{\underline{\mathbf{d}}}|_{{}^\theta\Lambda_{\underline{\mathbf{f}}}} = 0 \quad \text{if} \quad \underline{\mathbf{f}} \neq \underline{\mathbf{e}}.
$$

b) *The action of ${}^\theta\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^e$ on ${}^\theta\Lambda_{\underline{\mathbf{f}}}$ is trivial unless $\underline{\mathbf{f}} = \underline{\mathbf{d}}$. In the latter case, if we identify ${}^\theta\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^e \cong {}^\theta\Lambda_{\underline{\mathbf{d}}}$, then ${}^\theta\mathcal{Z}_{\underline{\mathbf{d}},\underline{\mathbf{d}}}^e$ acts on ${}^\theta\Lambda_{\underline{\mathbf{d}}}$ by usual multiplication.*

c) *The action of $\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}$ on ${}^\theta\Lambda_{\underline{\mathbf{f}}}$ is trivial unless $\underline{\mathbf{f}} = \underline{\mathbf{d}}$. In the latter case, if $\underline{\mathbf{d}} \succ_f \underline{\mathbf{e}}$, then*

$$
{}^\theta\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}}|_{{}^\theta\Lambda_{\underline{\mathbf{d}}}} = \curlywedge_{\underline{\mathbf{d}}^f}^{\underline{\mathbf{e}}^f}|_{\Lambda_{\underline{\mathbf{d}}^f}} \otimes 1|_{{}^\theta\Lambda_{\mathbf{d}_\infty}}.
$$

If $\underline{\mathbf{d}} = (\mathbf{a}, \mathbf{b})$ and $\underline{\mathbf{e}} = (\mathbf{c})$ then

$$
{}^\theta\curlywedge_{\underline{\mathbf{d}}}^{\underline{\mathbf{e}}} \colon {}^\theta\Lambda_{\underline{\mathbf{d}}} \to {}^\theta\Lambda_{(\mathbf{c})}, \quad f \mapsto {}^\theta\pitchfork_{\underline{\mathbf{d}}}^{\mathbf{c}} \left(\frac{{}^\theta\mathtt{E}_{\underline{\mathbf{d}}}}{{}^\theta\mathtt{S}_{\underline{\mathbf{d}}}} f \right).
$$

Proof. Parts a) and b) are proven in the same was as in Theorem 5.4.7. Part c) follows directly from Theorem 5.6.12 and [146, Theorem 3.3]. ☐

We would like to illuminate the formulas from Definition 5.6.14 by relating them to Demazure operators, generalizing Proposition 5.4.9.

Definition 5.6.16. Let $\underline{\mathbf{d}} \vDash \mathbf{c}$. Let ${}^\theta R_{\mathbf{c}}^+$ and ${}^\theta R_{\underline{\mathbf{d}}}^+$ denote the set of positive roots corresponding to $({}^\theta\mathsf{B}_{\mathbf{c}}, {}^\theta\mathsf{G}_{\mathbf{c}})$ and $({}^\theta\mathsf{B}_{\mathbf{c}}, {}^\theta\mathsf{P}_{\underline{\mathbf{d}}})$, respectively. We abbreviate ${}^\theta r_{\underline{\mathbf{d}}} = |{}^\theta R_{\mathbf{c}}^+ - {}^\theta R_{\underline{\mathbf{d}}}^+|$. Define $\blacktriangle_{\underline{\mathbf{d}}} = \prod_{\alpha \in {}^\theta R_{\underline{\mathbf{d}}}^+} \alpha$ and $\blacktriangle_{\mathbf{c}} = \blacktriangle_{(\mathbf{c})}$. Given $w \in {}^\theta\mathsf{W}_{\mathbf{c}}$, let ${}^\theta\Delta_w$ be the corresponding Demazure operator. Let $w_{\underline{\mathbf{d}}}$ and $w_{\underline{\mathbf{d}}}^{\mathbf{c}}$ be the longest elements in ${}^\theta\mathsf{W}_{\underline{\mathbf{d}}}$ and ${}^\theta\mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{d}}}$, respectively. We set $w_{\mathbf{c}} = w_{(\mathbf{c})}$, ${}^\theta\Delta_{\underline{\mathbf{d}}} = {}^\theta\Delta_{w_{\underline{\mathbf{d}}}}$, ${}^\theta\Delta_{\mathbf{c}} = {}^\theta\Delta_{w_{\mathbf{c}}}$ and ${}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}} = {}^\theta\Delta_{w^{\mathbf{c}}_{\underline{\mathbf{d}}}}$.

Lemma 5.6.17. *Let* $\underline{\mathbf{d}} \vartriangleright \mathbf{c}$.

a) *The Demazure operator* ${}^\theta\Delta_{\underline{\mathbf{d}}}$ *is given by the following explicit formula*

$$ {}^\theta\Delta_{\underline{\mathbf{d}}} = \sum_{w\in {}^\theta\mathsf{W}_{\underline{\mathbf{d}}}} w(\blacktriangle^{-1}_{\underline{\mathbf{d}}}). \tag{5.45}$$

b) *There exists some polynomial* $h \in {}^\theta\mathcal{P}_{\mathbf{c}}$ *such that* ${}^\theta\Delta_{\underline{\mathbf{d}}}(h) = 1$.

c) *If* $h \in {}^\theta\mathcal{P}_{\mathbf{c}}$ *and* $f \in {}^\theta\Lambda_{\underline{\mathbf{d}}}$, *then* ${}^\theta\Delta_{\underline{\mathbf{d}}}(fh) = f \cdot {}^\theta\Delta_{\underline{\mathbf{d}}}(h)$.

d) *If* $\underline{\mathbf{d}} = (\mathbf{a}, \mathbf{b})$, *then* $\blacktriangle_{\mathbf{c}} = (-1)^{{}^\theta r_{\underline{\mathbf{d}}}} \cdot \blacktriangle_{\underline{\mathbf{d}}} \cdot {}^\theta\mathsf{S}_{\underline{\mathbf{d}}}$ *and* ${}^\theta\mathsf{S}_{\underline{\mathbf{d}}} \in {}^\theta\Lambda_{\underline{\mathbf{d}}}$.

Proof. Part a) is proven as in [59, Lemma 12]. The proof of part b) is analogous to the proof of [99, Lemma 8.12] and requires only the following modification: one needs to replace the equality $\Delta_{\mathbf{c}}(x_1^{n-1}x_2^{n-2}\cdot\ldots\cdot x_{n-1} = 1$ by $-{}^\theta\Delta_{\mathbf{c}}(x_1x_2^3\cdot\ldots\cdot x_n^{2n-1}) = (-2)^n$. The latter can be easily proven by induction. Part c) is a standard property of Demazure operators. Part d) follows directly from the observation that ${}^\theta\mathsf{S}_{\underline{\mathbf{d}}} = \prod_{\alpha\in {}^\theta R^+_{\mathbf{c}} - {}^\theta R^+_{\underline{\mathbf{d}}}} -\alpha$. □

Proposition 5.6.18. *There is an equality of operators on* ${}^\theta\Lambda_{\mathbf{c}}$:

$$ {}^\theta \pitchfork^{\mathbf{c}}_{\underline{\mathbf{d}}} ({}^\theta\mathsf{S}_{\underline{\mathbf{d}}})^{-1} = (-1)^{{}^\theta r_{\underline{\mathbf{d}}}} \cdot {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}. $$

Proof. Let $f \in {}^\theta\Lambda_{\underline{\mathbf{d}}}$. We claim that

$$ {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f) = {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f\cdot 1) = {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f\cdot {}^\theta\Delta_{\underline{\mathbf{d}}}(h)) = {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}({}^\theta\Delta_{\underline{\mathbf{d}}}(fh)) = {}^\theta\Delta_{\mathbf{c}}(fh) $$

for some $h \in {}^\theta\mathcal{P}_{\mathbf{c}}$. Indeed, the second equality follows from part b) of Lemma 5.6.17, the third equality from part c) and the last equality from the fact that $w_{\mathbf{c}} = w^{\mathbf{c}}_{\underline{\mathbf{d}}} w_{\underline{\mathbf{d}}}$ and $\ell(w_{\mathbf{c}}) = \ell(w^{\mathbf{c}}_{\underline{\mathbf{d}}}) + \ell(w_{\underline{\mathbf{d}}})$. Next, (5.45) implies that

$$ {}^\theta\Delta_{\mathbf{c}}(fh) = \sum_{u\in {}^\theta\mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{d}}}} \sum_{v\in {}^\theta\mathsf{W}_{\underline{\mathbf{d}}}} uv(fh\cdot \blacktriangle^{-1}_{\mathbf{c}}). $$

Since f and ${}^\theta\mathsf{S}_{\underline{\mathbf{d}}}$ are ${}^\theta\mathsf{W}_{\underline{\mathbf{d}}}$-invariant, part d) of Lemma 5.6.17 implies that

$$ {}^\theta\Delta_{\mathbf{c}}(fh) = (-1)^{{}^\theta r_{\underline{\mathbf{d}}}} \sum_{u\in {}^\theta\mathsf{D}^{\mathbf{c}}_{\underline{\mathbf{d}}}} u(f\cdot ({}^\theta\mathsf{S}_{\underline{\mathbf{d}}})^{-1}) \cdot u \sum_{v\in {}^\theta\mathsf{W}_{\underline{\mathbf{d}}}} v(h\cdot \blacktriangle^{-1}_{\underline{\mathbf{d}}}). $$

By (5.45) and the choice of the polynomial h, we have

$$ \sum_{v\in {}^\theta\mathsf{W}_{\underline{\mathbf{d}}}} v(h\cdot \blacktriangle^{-1}_{\underline{\mathbf{d}}}) = {}^\theta\Delta_{\underline{\mathbf{d}}}(h) = 1, $$

Hence

$$ {}^\theta\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f) = {}^\theta\Delta_{\mathbf{c}}(fh) = (-1)^{{}^\theta r_{\underline{\mathbf{d}}}} \cdot {}^\theta \pitchfork^{\mathbf{c}}_{\underline{\mathbf{d}}} (f\cdot ({}^\theta\mathsf{S}_{\underline{\mathbf{d}}})^{-1}). $$ □

Proposition 5.6.18 yields a new interpretation of the action of the cohomological Hall algebra $\mathcal{H}$ on the cohomological Hall module ${}^\theta\mathcal{M}$ in terms of Demazure operators.

Corollary 5.6.19. *Let* $\mathbf{a} \in \Gamma$, $\mathbf{b} \in \Gamma^\theta$, $\mathbf{c} = \mathrm{D}(\mathbf{a}) + \mathbf{b}$ *and* $\underline{\mathbf{d}} = (\mathbf{a}, \mathbf{b})$. *Given* $f \in \mathcal{H}_{\mathbf{a}}$ *and* $g \in {}^\theta\mathcal{M}_{\mathbf{b}}$, *the action of f on g is given by*

$$\mathsf{act}(f,g) = (-1)^{{}^{\theta}r_{\underline{\mathbf{d}}}} \cdot {}^{\theta}\Delta^{\mathbf{c}}_{\underline{\mathbf{d}}}(f \cdot g \cdot {}^{\theta}\mathtt{E}_{\underline{\mathbf{d}}}),$$

where · stands for polynomial multiplication (i.e., the cup product).

Proof. The corollary follows directly from [146, Theorem 3.3] and Proposition 5.6.18. □

5.7 Conclusion and open problems

There are several new research directions and open problems arising from our work. Firstly, it would be interesting to obtain a more comprehensive description of the structure of the quiver Schur algebras $\mathcal{Z}_{\mathbf{c}}$ and ${}^{\theta}\mathcal{Z}_{\mathbf{c}}$. In particular, we would like to gain a better understanding of the relations in these algebras, with view to giving a complete description by generators and relations. Such a description already exists for KLR algebras. In the case of KLR algebras associated to a quiver with an involution, a description by generators and relations can be deduced from the work of Sauter [119, Proposition 2]. However, as far as quiver Schur algebras are concerned, we have so far only been able to solve the same problem only in the special cases of the A_1 and Jordan quivers (see Theorem 5.4.14 and 5.4.17). We expect that, for general quivers, this is a very hard problem.

Secondly, we would like to develop the theory of the mixed quiver Schur algebras ${}^{\theta}\mathcal{Z}_{\mathbf{c}}$. In particular, it would be interesting to understand their relation to Hecke and Schur algebras associated to p-adic classical groups other than $GL_n(\mathbb{Q}_p)$.

Thirdly, it is natural to ask whether our results concerning the connection between quiver Schur algebras and cohomological Hall algebras can be generalized to other settings. At present, the theory of cohomological Hall algebras appears somewhat richer than the theory of KLR and quiver Schur algebras. For example, there exist cohomological Hall algebras associated to quivers with potential [90], preprojective algebras [143], Hilbert schemes [124] and Higgs sheaves on a curve [117]. It would be interesting to investigate whether one can associate KLR-type algebras to the aforementioned objects, and whether one could fruitfully use such algebras to categorify existing Hopf algebras or to construct new ones.

Appendix A

Proof of Proposition 4.9.4

We work in the following setup. Let $1 \leq a \leq k$, $j_1, \ldots, j_a \geq 1$ and $j_1 + \ldots + j_a = k$. Consider an element $C = X_1[-j_1] \ldots X_a[-j_a] \in \mathbf{U}(\hat{\mathfrak{g}}_-)$, where $X_i \in \{e_{rs} \mid 1 \leq r, s \leq n\}$.

Lemma A.0.1. *The following estimates hold:*

- $\widehat{C}_l = 0$ *if* $l < -(k+a)$,
- $\deg \widehat{C}_{-(k+a)} \leq a$,
- $\deg \widehat{C}_{-(k+a)+1} \leq a-1$,
- $\deg \widehat{C}_{-(k+a)+2+p} \leq a+p \quad (p \geq 0)$.

Moreover, if $C \in \ker \mathsf{AHC} \subset \mathbf{U}(\hat{\mathfrak{g}}_-)^{\mathrm{ad}\, \mathfrak{t}}$ *then* $\widehat{C}_{-(k+a)} = 0$ *and* $\deg \widehat{C}_{-(k+a)+2+p} \leq a+p-2$.

Proof. We proceed by induction on a.

1. The base case. Let us first tackle the base case $a = 1$. Then $C = X_1[-k]$ and, by definition,

$$\mathbb{Y}\langle C, z\rangle = \frac{1}{(k-1)!}\partial_z^{k-1}\mathbb{Y}\langle X_1[-1], z\rangle = \sum_{i\in\mathbb{Z}} \frac{(i+1)\cdots(i+k-1)}{(k-1)!} X_1[-i-k]z^i.$$

Hence

$$C_i = \frac{(i+1)\cdots(i+k-1)}{(k-1)!} X_1[-i-k]. \tag{A.1}$$

In particular,

$$C_i = 0 \quad \text{if} \quad i = -1, \ldots, -k+1. \tag{A.2}$$

We now consider the four cases in the lemma. First suppose that $i < -(k+1)$. Since $-i-k > 1$ and $X_1[b].1_{\mathbb{H}} = 0$ for $b > 1$, formula (A.1) implies that $\widehat{C}_i = C_i.1_{\mathbb{H}} = 0$.

In the second and third cases we have $C_{-(k+1)} = (-1)^{k-1}kX_1[1]$ and $C_{-k} = (-1)^{k-1}X_1$. Hence $\deg \widehat{C}_{-(k+1)} \leq 1$ and $\deg \widehat{C}_{-k} \leq 0$. Finally suppose that $i = -k+p+1$ with $p \geq 0$. Formula (A.1) implies that C_i is a multiple of $X_1[-p-1]$ and so $\deg \widehat{C}_i \leq p+1$.

2. The inductive case - notation. Assume $a \geq 2$. Let us set $k' = j_2 + \ldots + j_a$ and $a' = a-1$. Set $A = X_1[-j_1]$ and $B = X_2[-j_2] \ldots X_a[-j_a]$. By definition of the normally ordered product we have

$$C_l = \sum_{\substack{r+s=l,\\ r\geq 0}} A_r B_s + \sum_{\substack{r+s=l,\\ s<0}} B_r A_s. \tag{A.3}$$

Set $C_l^+ = \sum_{\substack{r+s=l \\ r \geq 0}} A_r B_s$ and $C_l^- = \sum_{\substack{r+s=l \\ s<0}} B_r A_s$ so that $C_l = C_l^+ + C_l^-$. Also set $\widehat{C}_l^+ := \widehat{\Phi}(C_l^+)$ and $\widehat{C}_l^- := \widehat{\Phi}(C_l^-)$.

3. The inductive case - C_l^+. First suppose that $l < -(k+a)+2$. Consider any monomial $A_r B_s$ in C_l^+. Since $r \geq 0$, we have $s = l - r < -(k+a)+2 \leq -(k'+a')$. Therefore, by induction, $\widehat{B}_s = 0$. Hence $\widehat{C}_l^+ = 0$. This takes care of the first three cases.

Now assume that $l = -(k+a)+2+p$ with $p \geq 0$. Since $r \geq 0$, we can write $r = -(j_1+1)+2+p'$ with $p' = r + j_1 - 1 \geq 0$. Then, by the base case, we know that $\deg \widehat{A}_r \leq r + j_1$. We now estimate the degree of $\widehat{B}_s$. We have $s = l - r = -(k'+a')+2+p-(r+j_1+1)$. There are four situations to consider. Firstly, suppose that $p \geq r + j_1 + 1$. Then, by induction (the fourth case), we conclude that $\deg \widehat{B}_s \leq a'+p-(r+j_1+1)$. Hence $\deg \widehat{\Phi}(A_r B_s) \leq \deg \widehat{A}_r + \deg \widehat{B}_s \leq (r+j_1)+(a'+p-(r+j_1+1)) = a'+p-1 = a+p-2$. Secondly, suppose that $p = r+j_1$. Then $s = -(k'+a')+1$ and so, by induction (the third case), we have $\deg \widehat{B}_s \leq a'-1$. Hence $\deg \widehat{\Phi}(A_r B_s) \leq \deg \widehat{A}_r + \deg \widehat{B}_s \leq (r+j_1)+a'-1 = a+p-2$. Thirdly, suppose that $p = r + j_1 - 1$. Then $s = -(k'+a')$ and so, by induction (the second case), we have $\deg \widehat{B}_s \leq a'$. Hence $\deg \widehat{\Phi}(A_r B_s) \leq \deg \widehat{A}_r + \deg \widehat{B}_s \leq (r+j_1)+a' = a+p$. Finally, if $p < r+j_1-1$ then $s < -(k'+a')$. Hence $\widehat{B}_s = 0$ and $\widehat{\Phi}(A_r B_s) = 0$. Overall we conclude that $\deg \widehat{C}_l \leq a+p$.

4. Auxiliary induction. Let us call an expression of the form $MZ[b]$, with $M \in \mathbf{U}_c(\tilde{\mathfrak{g}}), Z \in \mathfrak{g}$ and $b > 1$, a *good word*. A good word is thus an element of the left ideal in $\mathbf{U}_c(\tilde{\mathfrak{g}})$ generated by $\mathfrak{g} \otimes t^2\mathbb{C}[t]$. We will now prove the following claim:

(C) Suppose that $l < -(k+a)$. Then C_l vanishes or can be written as a (possibly infinite) sum of good words.

We proceed by induction on a. The base case $a = 1$ follows immediately from formula (A.1). So assume that $a \geq 2$. We first consider C_l^+. Take any monomial $A_r B_s$ in C_l^+. Since $r \geq 0$, we have $s = l - r < -(k+a)+2 \leq -(k'+a')$. Therefore, by induction, B_s can be written as a sum of good words. The same obviously applies to $A_r B_s$ and, consequently, to C_l^+.

We now consider C_l^-. Take any monomial $B_r A_s$ in C_l^-. If $-j_1 + 1 \leq s < 0$ then $A_s = 0$ by (A.2). If $s \leq -j_1 - 2$ then, by (A.1), A_s is a scalar multiple of $X_1[b]$ with $b \geq 2$. Hence in both of these cases $B_r A_s$ can be written as a sum of good words. There remain two cases to consider. First suppose that $s = -j_1$. Then A_s is a multiple of X_1 by (A.1). We also have $r = l - s < -(k+a+s) = -(k'+a')-1$. Hence, by induction, B_r can be written as a sum of good words. Take any such good word $MZ[b]$. Then $MZ[b]X_1 = MX_1Z[b] + M[Z, X_1][b]$. Hence $B_r A_s$ can also be written as a sum of good words. Secondly, suppose that $s = -j_1 - 1$. Then A_s is a multiple of $X_1[1]$. We also have $r = l - s < -(k+a+s) = -(k'+a')$. Hence, by induction, B_r can be written as a sum of good words. Take any such good word $MZ[b]$. Then $MZ[b]X_1[1] = MX_1[1]Z[b] + M[Z, X_1][b+1]$. Hence $B_r A_s$ can also be written as a sum of good words. This proves **(C)**.

5. The inductive case - C_l^-. First suppose that $l < -(k+a)$. Then, by **(C)**, C_l^- vanishes or can be written as a sum of good words. But every good word annihilates $1_{\mathbb{H}}$, so $\widehat{C}_l^- = 0$.

We now consider the remaining three cases. Regard C_l^- as a sum of monomials $B_r A_s$ as in (A.3). Given that $s < 0$, we have $\widehat{A}_s \neq 0$ only if $s = -j_1$ or $s = -j_1 - 1$, by (A.1) and (A.2). So suppose that $s = -j_1$. Then $A_s = (-1)^{j_1-1}X_1$ and so $\deg \widehat{A}_s = 0$. Firstly, assume that $l = -(k+a)$. Then $r = l - s = -(k'+a') - 1$. Hence, by **(C)**, B_r can be written as a sum of good words. Take any such good word $MZ[b]$. Then $MZ[b]X_1 = MX_1Z[b] + M[Z, X_1][b]$ and so $\widehat{\Phi}(B_r A_s) = 0$.

Secondly, assume that $l = -(k+a)+1$. Then $r = l - s = -(k'+a')$. Hence, by induction, $\deg \widehat{B}_r \le a' = a-1$ and so we can conclude that $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r \le a-1$. Thirdly, assume that $l = -(k+a)+2$. Then $r = l-s = -(k'+a')+1$. Hence, by induction, $\deg \widehat{B}_r \le a'-1 = a-2$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r \le a-2 < a$. Fourthly, assume that $l = -(k+a)+2+p$ with $p > 0$. Then $r = l-s = -(k'+a')+2+(p-1)$. Hence, by induction, $\deg \widehat{B}_r \le a'+p-1 = a+p-2$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r \le a+p-2 < a+p$.

Now suppose that $s = -j_1 - 1$. Then $A_s = (-1)^{j_1-1} j_1 X_1[1]$ and so $\deg \widehat{A}_s \le 1$. Firstly, assume that $l = -(k+a)$. Then $r = l-s = -(k'+a')$. Hence, by induction, $\deg \widehat{B}_r \le a' = a-1$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r + \deg \widehat{A}_s \le a$. Secondly, assume that $l = -(k+a)+1$. Then $r = l-s = -(k'+a')+1$. Hence, by induction, $\deg \widehat{B}_r \le a'-1 = a-2$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r + \deg \widehat{A}_s \le a-1$. Thirdly, assume that $l = -(k+a)+2+p$ with $p \ge 0$. Then $r = l-s = -(k'+a')+2+p$. Hence, by induction, $\deg \widehat{B}_r \le a'+p = a+p-1$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r + \deg \widehat{A}_s \le a+p$. This proves that $\widehat{C}_l^-$ satisfies the required constraints and completes the proof of the first part of the lemma.

6. Another auxiliary induction. We claim that

(C') If $X_i \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$ for some $1 \le i \le a$ then $\widehat{C}_{-(k+a)} = 0$.

If $a = 1$ then $\widehat{C}_{-(k+1)} = (-1)^{k-1} k X_1[1].1_{\mathbb{H}} = 0$ since $X_1 \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$. So suppose $a > 1$. Then, by part 3 of the proof, $\widehat{C}^+_{-(k+a)} = 0$. Let us show that $\widehat{C}^-_{-(k+a)}$ vanishes as well. Part 5 of the proof implies that it suffices to consider the monomial B_rA_s in $\widehat{C}^-_{-(k+a)}$ with $s = -j_1 - 1$. Since $A_s = (-1)^{j_1-1} j_1 X_1[1]$, we have $\widehat{\Phi}(B_rA_s) = 0$ if $X_1 \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$. Otherwise, $X_1 \in \mathfrak{t}$ and $X_i \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$ for some $2 \le i \le a$. By induction, $\widehat{B}_r = 0$. Hence $B_r \in \mathfrak{I}_c$ and B_r can be written as a (finite) sum $\sum_p Z_pY_p$ with $Z_p \in \mathbf{U}_c(\tilde{\mathfrak{g}})$ and $Y_p \in \mathfrak{i}$ or $Y_p = e_{qq} - 1$ for some $1 \le q \le n$. In the first case, we use the fact that, by Lemma 4.6.2, $\mathfrak{i}$ is an ideal in $\hat{\mathfrak{t}}_+$. Since $A_s \in \mathfrak{t}[1]$, we get $[Y_p, A_s] \in \mathfrak{i}$. In the second case, $[Y_p, A_s] = 0$. It follows that $[B_r, A_s] \in \mathfrak{I}_c$.

7. Second part of the lemma. We now prove the second statament of the lemma. First observe that in many parts of the proof so far we have established the stronger inequalities in the second statement of the lemma without even using the assumption that $C \in \ker \mathsf{AHC}$. Let us consider all the remaining cases. The first such case appears in part 3 of the proof: $l = -(k+a)+2+p$ with $p = r + j_1 - 1$. In that case $s = -(k'+a')$. Since $C \in \ker \mathsf{AHC}$, B_s satisfies the hypothesis of **(C')**, from which we conclude that $\widehat{B}_s = 0$ and so $\widehat{\Phi}(A_rB_s) = 0$.

The second case appears in part 5 of the proof: $s = -j_1 - 1$ and $l = -(k+a)$. It follows directly from **(C')** that $\widehat{\Phi}(B_rA_s) = 0$. The third case also appears in part 5 of the proof: $s = -j_1 - 1$ and $l = -(k+a)+2+p$ with $p \ge 0$. In that case $A_s = (-1)^{j_1-1} j_1 X_1[1]$. There are two possibilities. Either $X_1 \in \mathfrak{n}_+ \oplus \mathfrak{n}_-$ or $B_r \in \ker \mathsf{AHC}$. In the first case $\widehat{\Phi}(B_rA_s) = 0$ and in the second case, by induction, $\deg \widehat{B}_r \le a'+p-2 = a+p-3$ and so $\deg \widehat{\Phi}(B_rA_s) \le \deg \widehat{B}_r + \deg \widehat{A}_s \le a+p-2$. This was the last case to consider. We have therefore completed the proof of the lemma. □

Lemma A.0.1 directly implies the following.

Corollary A.0.2. *Suppose that either (i)* $C \in \mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\le k-1}$ *or (ii)* $C \in (\mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\le k})^{\mathrm{ad}\,\mathfrak{t}}$ *and* $C \in \ker \mathsf{AHC}$. *Then:*

$$\widehat{C}_l = 0 \quad (l \le -2k), \qquad \deg \widehat{C}_{-2k+2+p} \le k-2+p \quad (p \ge 0).$$

Lemma A.0.3. *We have:*

- $\widehat{\mathbf{P}}_{k,l} = 0$ *if* $l < -2k$,
- $\widehat{\mathbf{P}}_{k,-2k} = \sum_{i=1}^{n}(e_{ii}[1])^k.1_{\mathbb{H}}$,
- $\widehat{\mathbf{P}}_{k,-2k+1} = k\sum_{i=1}^{n}(e_{ii}[1])^{k-1}.1_{\mathbb{H}}$,
- *if* $b \geq 0$ *then:*

$$\overline{\mathbf{P}}_{k,-2k+2+b} = k\sum_{i=1}^{n} e_{ii}[-b-1](e_{ii}[1])^{k-1}.1_{\mathbb{H}} \;+\; (\mathbb{H}_c)_{\leq k+b-1}.$$

Proof. The first case follows directly from Lemma A.0.1. So consider the remaining three cases. Fix $1 \leq i \leq n$. Let $A = e_{ii}[-1]$, $B = (e_{ii}[-1])^{k-1}$ and $C = AB$. By Lemma A.0.1, we have $B_s.1_{\mathbb{H}} = 0$ for $s < -2k+2$ and $A_s.1_{\mathbb{H}} = 0$ for $s < -2$. Hence (A.3) implies that

$$\widehat{C}_{-2k} = B_{-2k+2}A_{-2}.1_{\mathbb{H}}, \quad \widehat{C}_{-2k+1} = B_{-2k+2}A_{-1}.1_{\mathbb{H}} + B_{-2k+3}A_{-2}.1_{\mathbb{H}}.$$

By induction we know that $B_{-2k+2} = (e_{ii}[1])^{k-1}$ and $B_{-2k+3} = (k-1)(e_{ii}[1])^{k-2}$ modulo $\widehat{\mathbf{U}}_c.\hat{\mathfrak{g}}_{\geq 2}$. Hence $\widehat{C}_{-2k} = A_{-2}B_{-2k+2}.1_{\mathbb{H}} = (e_{ii}[1])^k.1_{\mathbb{H}}$ and

$$\begin{aligned}\widehat{C}_{-2k+1} &= B_{-2k+2}A_{-1}.1_{\mathbb{H}} + B_{-2k+3}A_{-2}.1_{\mathbb{H}}\\ &= B_{-2k+2}.1_{\mathbb{H}} + A_{-2}B_{-2k+3}.1_{\mathbb{H}} = k(e_{ii}[1])^{k-1}.1_{\mathbb{H}}.\end{aligned}$$

This proves the second and third cases. Finally consider the fourth case. We have

$$\widehat{C}_{-2k+2+b} = \sum_{0\leq s\leq b} A_sB_{-2k+2+b-s}.1_{\mathbb{H}} + B_{-2k+3+b}A_{-1}.1_{\mathbb{H}} + B_{-2k+4+b}A_{-2}.1_{\mathbb{H}}.$$

Lemma A.0.1 implies that $A_bB_{-2k+2}.1_{\mathbb{H}} + B_{-2k+4+b}A_{-2}.1_{\mathbb{H}}$ is the leading term of $\widehat{C}_{-2k+2+b}$. By induction we know that $\sigma^{\mathsf{abs}}(\widehat{B}_{-2k+4+b}) = (k-1)e_{ii}[-b-1](e_{ii}[1])^{k-2}.1_{\mathbb{H}}$ and $\widehat{B}_{-2k+2} = (e_{ii}[1])^{k-1}.1_{\mathbb{H}}$. Hence $\sigma^{\mathsf{abs}}(\widehat{C}_{-2k+2+b}) = ke_{ii}[-b-1](e_{ii}[1])^{k-1}.1_{\mathbb{H}}$. Summing over $i = 1, \dots, n$ yields the lemma. □

We can now prove Proposition 4.9.4.

Proof of Proposition 4.9.4. By Lemma 4.9.2, we can write

$$\widehat{\mathbf{T}}_{k,l} = \widehat{Q}_{k,l} + \widehat{Q}'_{k,l} + \widehat{\mathbf{P}}_{k,l},$$

where $Q_k \in (\mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\leq k-1})^{\mathrm{ad}\,\mathfrak{t}}$, $Q'_k \in (\mathbf{U}(\tilde{\mathfrak{g}}_-)_{-k} \cap \mathbf{U}^{\mathsf{pbw}}(\tilde{\mathfrak{g}}_-)_{\leq k})^{\mathrm{ad}\,\mathfrak{t}}$ and $\mathsf{AHC}(Q'_k) = 0$. Hence Corollary A.0.2 implies that $\widehat{Q}_{k,l} = \widehat{Q}'_{k,l} = 0$ for $l \leq -2k$ and

$$\deg \widehat{Q}_{k,-2k+2+p} = \deg \widehat{Q}'_{k,-2k+2+p} \leq k+p-2$$

for $p \geq 0$. On the other hand, we know from Lemma A.0.3 that $\widehat{\mathbf{P}}_{k,l} = 0$ for $l < -2k$, $\deg \widehat{\mathbf{P}}_{k,-2k} = k$ and $\deg \widehat{\mathbf{P}}_{k,-2k+2+p} = k+p$ for $p \geq 0$. It follows that $\widehat{\mathbf{T}}_{k,l} = 0$ if $l < -2k$, $\widehat{\mathbf{T}}_{k,-2k} = \widehat{\mathbf{P}}_{k,-2k}$ and that $\widehat{\mathbf{P}}_{k,l}$ is the leading term of $\widehat{\mathbf{T}}_{k,l}$ if $l \geq -2k+2$, as required. □

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