SISSA

Scuola Internazionale Superiore di Studi Avanzati - Trieste

Area of Mathematics
Ph.D. in Geometry

Thesis

On the E-polynomial of a family of parabolic Sp_{2n}-character varieties

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Thesis submitted in partial fulfillment of the requirements for the degree of Philosophiæ Doctor

Academic Year 2016-2017

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Il presente lavoro costituisce la tesi presentata da Vincenzo Cambò, sotto la direzione del Professor Fernando Rodriguez-Villegas, al fine di ottenere l'attestato di ricerca post-universitaria *Philosophiæ Doctor* presso la Scuola Internazionale Superiore di Studi Avanzati di Trieste (SISSA), Curriculum di Geometria. Ai sensi dell'articolo 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica.*

Trieste, Anno Accademico 2016-2017.

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September 19, 2017

To the odd q

Abstract

In this thesis, we find the E-polynomials of a family of parabolic Sp_{2n}-character varieties $\mathcal{M}_n^\xi$ of Riemann surfaces by constructing a stratification, proving that each stratum has polynomial count, applying a result of Katz regarding the counting functions, and finally adding up the resulting E-polynomials of the strata. To count the number of $\mathbb{F}_q$-points of the strata, we invoke a formula due to Frobenius. Our calculation make use of a formula for the evaluation of characters on semisimple elements coming from Deligne-Lusztig theory, applied to the character theory of $\mathrm{Sp}(2n, \mathbb{F}_q)$, and Möbius inversion on the poset of set-partitions. We compute the Euler characteristic of the $\mathcal{M}_n^\xi$ with these polynomials, and show they are connected.

Ringraziamenti

Innanzitutto, ringrazio il professor Fernando Rodriguez-Villegas per la pazienza e la competenza con cui mi ha seguito in questo mio lavoro di tesi e per avermi introdotto nello studio delle varietà dei caratteri. Ringrazio i miei compagni di dottorato Lorenzo, Jacopo, Giorgio, Luca e Giordano per la loro amicizia, che spero possa durare ancora a lungo, e per aver condiviso con me gioie e dolori del lavoro di ricerca. In ultimo, ringrazio gli amici di sempre, la mia famiglia e la mia ragazza, senza i quali tutto questo non sarebbe stato possibile.

Contents

List of Tables

Chapter 1

Introduction

Let Σ_g be a compact Riemann surface of genus $g \geq 0$ and let G be a complex reductive group. The G-character variety of Σ_g is defined as the moduli space of representations of $\pi_1(\Sigma_g)$ into G. Using the standard presentation of $\pi_1(\Sigma_g)$, we have the following description of this moduli space as an affine GIT quotient:

$$\mathcal{M}_B(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] = \mathrm{Id}_G \right\} //G$$

where $[A : B] := ABA^{-1}B^{-1}$ and G acts by simultaneous conjugation. For complex linear groups $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})$, the representations of $\pi_1(\Sigma_g)$ into G can be understood as G-local systems $E \to \Sigma_g$, hence defining a flat bundle E whose degree is zero.

For $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})$, a natural generalization consists of allowing bundles E of non-zero degree d. In this case, one considers the space of the irreducible G-local systems on Σ_g with prescribed cyclic holonomy around one puncture, which correspond to representations $\rho : \pi_1(\Sigma_g \setminus \{p_0\}) \to G$, where $p_0 \in \Sigma_g$ is a fixed point, and $\rho(\gamma) = e^{\frac{2\pi i d}{n}} \mathrm{Id}_G$, with γ a loop around p_0, giving rise to the moduli space

$$\mathcal{M}_B^d(G) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] = e^{\frac{2\pi i d}{n}} \mathrm{Id}_G \right\} //G.$$

The space $\mathcal{M}_B^d(G)$ is known in the literature as the *Betti moduli space*. These varieties have a very rich structure and they have been the object of study in a broad range of area.

In his seminal work [Hi87], after studying the dimensional reduction of the Yang-Mills equations from four to two dimensions, Hitchin introduced a family of completely integrable Hamiltonian systems. These equations are known as *Hitchin's self-duality equations* on a rank n and degree d bundle on the Riemann surface Σ_g.

The moduli space of solutions comes equipped with a hyperkähler manifold structure on its smooth locus. This hyperkähler structure has two distinguished complex structures, up to equivalence. One is analytically isomorphic to $\mathcal{M}_{Dol}^d(G)$, a moduli space of G-Higgs bundles, and the other is $\mathcal{M}_{DR}^d(G)$, the space of algebraic flat bundles on Σ_g of degree d and rank n, whose algebraic connections on $\Sigma_g \setminus \{p_0\}$ have a logarithmic pole at p_0 with residue $e^{\frac{2\pi i d}{n}}\mathrm{Id}$. By Riemann-Hilbert correspondence ([De70], [Si95]), the space $\mathcal{M}_{DR}^d(G)$ is analytically (but not algebraically) isomorphic to $\mathcal{M}_B^d(G)$ and the theory of harmonic bundles ([Co88], [Si92]) gives an homeomorphism $\mathcal{M}_B^d(G) \cong \mathcal{M}_{Dol}^d(G)$.

When $\gcd(d, n) = 1$, these moduli spaces are smooth and their cohomology has been computed in several particular cases, but mostly from the point of view of the Dolbeaut moduli space $\mathcal{M}_{Dol}^d(G)$.

Hitchin and Gothen computed the Poincaré polynomial for $G = \mathrm{SL}(2, \mathbb{C})$

and $G = \mathrm{SL}(3, \mathbb{C})$ respectively in [Hi87] and [Go94] and their techniques have been improved to compute the compactly supported Hodge polynomials ([GHS14]). Recently, Schiffmann, Mozgovoy and Mellit computed the Betti numbers of $\mathcal{M}^d_{Dol}(G)$ for $G = \mathrm{GL}(n, \mathbb{C})$ respectively in [Sch14], [MS14] and [Mel17].

Hausel and Thaddeus ([HT03]) gave a new perspective for the topological study of these varieties giving the first non-trivial example of the Strominger-Yau-Zaslow Mirror Symmetry ([SYZ96]) using the so called Hitchin system ([Hi87]) for the Dolbeaut space. They conjectured also (and checked for $G = \mathrm{SL}(2, \mathbb{C}), \mathrm{SL}(3, \mathbb{C})$ using the results by Hitchin and Gothen) that a version of the topological mirror symmetry holds, i.e., some Hodge numbers $h^{p,q}$ of $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_{Dol}(G^L)$, for G and Langlands dual G^L, agree. Very recently, Groechenig, Wyss and Ziegler proved the topological mirror symmetry for $G = \mathrm{SL}(n, \mathbb{C})$ in [GWZ17].

Contrarily to $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_{Dol}(G)$ cases, the cohomology of $\mathcal{M}^d_B(G)$ does not have a *pure Hodge structure*. This fact motivates the study of E-*polynomials* of the G-character varieties. The E-polynomial of a variety X is

$$E(X; x, y) := H_c(X; x, y, -1)$$

where

$$H_c(X; x, y, t) := \sum h_c^{p,q;j}(X) x^p y^q t^j$$

the $h_c^{p,q;j}$ being the mixed Hodge numbers with compact support of X ([De71], [De74]). When the E-polynomial only depends on xy, we write $E(X; q)$, meaning

$$E(X; q) = H_c(X; \sqrt{q}, \sqrt{q}, -1).$$

Hausel and Rodriguez-Villegas started computing the E-polynomials of G-character varieties for $G = \mathrm{GL}(n, \mathbb{C}), \mathrm{PGL}(n, \mathbb{C})$ using arithmetic methods inspired on Weil conjectures. In [HRV08] they obtained the E-polynomials of $\mathcal{M}_B^d(\mathrm{GL}(n, \mathbb{C}))$. Following this work, in [Mer15] Mereb computed the E-polynomials of $\mathcal{M}_B^d(\mathrm{SL}(n, \mathbb{C}))$. He proved that these polynomials are palindromic and monic.

Another direction of interest is the moduli space of parabolic bundles. If $p_1, \ldots, p_s$ are s marked points in a Riemann surface Σ_g of genus g, and $\mathcal{C}_i \subseteq G$ semisimple conjugacy classes for $i = 1, \ldots, s$, the corresponding Betti moduli space of parabolic representations (or *parabolic G-character variety*) is

$$\mathcal{M}^{\mathcal{C}_1,\ldots,\mathcal{C}_s}(G) := \Big\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \mid \prod_{i=1}^{g} [A_i : B_i] \prod_{j=1}^{s} C_j = \mathrm{Id}_G,\, C_j \in \mathcal{C}_j,\, j = 1, \ldots, s \Big\} // G.$$

In [Si90], Simpson proved that this space is analytically isomorphic to the moduli space of flat logarithmic G-connections and homeomorphic to a moduli space of Higgs bundles with parabolic structures at $p_1, \ldots, p_s$.

Hausel, Letellier and Rodriguez-Villegas ([HLRV11]) found formulae for the E-polynomials of the parabolic character varieties for $G = \mathrm{GL}(n, \mathbb{C})$ and generic semisimple $\mathcal{C}_1, \ldots, \mathcal{C}_s$.

In this thesis, we consider certain parabolic character varieties for the group $G = \mathrm{Sp}(2n, \mathbb{C})$. For a semisimple element ξ belonging to a conjugacy class

$\mathcal{C} \subseteq \mathrm{Sp}(2n, \mathbb{C})$, we define

$$\begin{aligned}\mathcal{M}_n^{\xi} :&= \left\{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{Sp}(2n, \mathbb{C})^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] = \xi \right\} //C(\xi) \\ &= \left\{(A_1, B_1, \ldots, A_g, B_g) \in \mathrm{Sp}(2n, \mathbb{C})^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] \in \mathcal{C} \right\} //\mathrm{Sp}(2n, \mathbb{C})\end{aligned}$$

where $C(\xi)$ is the centralizer of ξ in $\mathrm{Sp}(2n, \mathbb{C})$.

We assume that ξ satisfies the genericity condition 3.1.1 below; in particular, ξ is a regular semisimple element, hence $C(\xi) = T \cong (\mathbb{C}^{\times})^n$, the maximally split torus in $\mathrm{Sp}(2n, \mathbb{C})$. It turns out that $\mathcal{M}_n^{\xi}$ is a geometric quotient and all the stabilisers are finite subgroups of $\boldsymbol{\mu}_{\mathbf{2}}^n$, the group of diagonal symplectic involutions.

Our goal is to compute the E-polynomials of $\mathcal{M}_n^{\xi}$ for any genus g and dimension n. This is accomplished by arithmetic methods, following the work of Hausel and Rodriguez -Villegas in [HRV08] and Mereb in [Mer15]. Our methods depends on the additive property of the E-polynomial, which allows us to compute this polynomial using stratifications (see 4.1.6).

The strategy to compute the E-polynomials of $\mathcal{M}_n^{\xi}$ is to construct a stratification of $\mathcal{M}_n^{\xi}$ and proving that each stratum $\widetilde{\mathcal{M}}_{n,H}^{\xi}$ has polynomial count, with H varying on the set of the subgroups of $\boldsymbol{\mu}_{\mathbf{2}}^n$ (see Def 3.1.14); i.e., there is a polynomial $E_{n,H}(q) \in \mathbb{Z}[q]$ such that $\left|\widetilde{\mathcal{M}}_{n,H}^{\xi}(\mathbb{F}_q)\right| = E_{n,H}(q)$ for sufficiently many prime powers q in the sense described in Section 4.2. According to Katz's theorem 4.2.3, the E-polynomial of $\widetilde{\mathcal{M}}_{n,H}^{\xi}$ agrees with the counting polynomial

$E_{n,H}$. By Theorem 3.2.5 below, one is reduced to count

$$E_n(q) := \frac{1}{(q-1)^n} \left| \left\{ (A_1, B_1, \ldots, A_g, B_g) \in \mathrm{Sp}(2n, \mathbb{F}_q)^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] = \xi \right\} \right| \tag{1.0.1}$$

with $\mathbb{F}_q$ a finite field containing the eigenvalues of ξ. The number of solutions of an equation like 1.0.1 is given by a Frobenius-type formula involving certain values of the irreducible characters χ of $\mathrm{Sp}(2n, \mathbb{F}_q)$ (see 2.2.3). Thanks to the formula 4.3.37 below for the evaluation of irreducible characters of a finite group of Lie type on a regular semisimple element, the Frobenius formula and Katz's theorem, we are able to compute $E_n(q)$, hence the $E_{n,H}(q)$'s. Adding them up, we eventually obtain the following

Theorem. *The E-polynomial of $\mathcal{M}_n^{\xi}$ satisfies*

$$E\left(\mathcal{M}_n^{\xi}; q\right) = E_n(q) = \frac{1}{(q-1)^n} \sum_{\tau} (H_\tau(q))^{2g-1} C_\tau .$$

Here, $H_\tau(q)$ are polynomials with integer coefficients (see 2.2.64), C_τ are integer constants and the sum is over a well described set (see 2.2.83). It is remarkable that the E-polynomial of $\mathcal{M}_n^{\xi}$ does not depend on the choice of the generic element ξ. A direct consequence of our calculation and of the fact that $\mathcal{M}_n^{\xi}$ is equidimensional is the following

Corollary. *The E-polynomial of $\mathcal{M}_n^{\xi}$ is palindromic and monic. In particular, the parabolic character variety $\mathcal{M}_n^{\xi}$ is connected.*

Our formula also implies

Corollary. *The Euler characteristic $\chi\left(\mathcal{M}_n^{\xi}\right)$ of $\mathcal{M}_n^{\xi}$ vanishes for $g > 1$. For*

$g = 1$, *we have*

$$\sum_{n\geq 0} \frac{\chi\left(\mathcal{M}_n^{\xi}\right)}{2^n n!} T^n = \prod_{k\geq 1} \frac{1}{\left(1-T^k\right)^3}.$$

See Theorem 4.3.19 and Corollaries 4.4.1, and 4.4.2 for details.

For $n = 1$, the formula looks like:

$$\begin{aligned} E\left(\mathcal{M}_1^{\xi}; q\right) &= \left(q^3 - q\right)^{2g-2}\left(q^2 + q\right) + \left(q^2 - 1\right)^{2g-2}(q+1) \\ &\quad + \left(2^{2g} - 2\right)\left(q^2 - q\right)^{2g-2} q. \end{aligned}$$

This result recovers the ones obtained by Logares, Muñoz and Newstead in [LMN13] for small genus g and by Martinez and Muñoz in [MM15] for all possible g because $\mathrm{Sp}(2,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})$.

The present dissertation is organized as follows: In Chapter 2, we go over the basics of combinatorics and representation theory that are going to be needed. In Chapter 3, we study the geometry of the parabolic character varieties to be studied. In Chapter 4, we perform the computation of the E-polynomial of $\mathcal{M}_n^{\xi}$ and prove the corollaries concerning the topological properties of $\mathcal{M}_n^{\xi}$ encoded in $E\left(\mathcal{M}_n^{\xi}; q\right)$.

Chapter 2

Preliminaries

We are going to need some definitions and results from Combinatorics and Representation Theory of finite groups of Lie type. We follow [GP00] for basic definitions and notation on partitions and symbols and [St12] for Möbius Inversion. For the basics on linear algebraic groups we refer to [Bo91], [Ca85] and [Hu75].

2.1 Combinatorics

2.1.1 Partitions and Symbols

Notation. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, we have $\lambda_1 \geq \cdots \geq \lambda_l > 0$ and write $l(\lambda)$ for its length l and $|\lambda|$ for its size $\sum_{i=1}^{l} \lambda_i$. If λ is a partition of a natural number n, we write $\lambda \vdash n$ and we denote by $\mathcal{P}_n$ the set of all partitions of n. For the author, the set of natural numbers $\mathbb{N}$ does not include 0. To do this, we use the symbol $\mathbb{N}_0$.

We usually regard $\lambda_i = 0$ whenever $i > l(\lambda)$ for the ease of notation in formulae.

Definition 2.1.1. Let $i \in \mathbb{N}$, $\lambda \in \mathcal{P}_n$. The *multiplicity of* i *in* λ, which denote by $m_i(\lambda)$, is defined as

$$m_i(\lambda) := |\{j \mid \lambda_j = i\}|\,. \tag{2.1.1}$$

Let $X = \{\beta_1, \dots, \beta_s\}$ be a finite subset in $\mathbb{N}_0$ with $\beta_1 > \cdots > \beta_s$. We call X a *β-set*. The *rank* of X is defined by $\mathrm{rk}(X) := 0$ if $X = \emptyset$ and, otherwise, by

$$\mathrm{rk}(X) := \sum_{i=1}^{s} \beta_i - \binom{s}{2}. \tag{2.1.2}$$

Definition 2.1.2. Let t be a positive integer. If $X = \{\beta_1, \dots, \beta_s\}$ is a β-set, the *t-shift of* X is the β-set X^{+t} defined by

$$X^{+t} := \{\beta_1 + t, \dots, \beta_s + t, t-1, \dots, 1, 0\}\,. \tag{2.1.3}$$

We call the *shift operation* the procedure that allows us to obtain X^{+t} from X for any integer $t \geq 0$.

Remark 2.1.3. According to the Definition 2.1.2, it is easy to see that $\mathrm{rk}(X) = \mathrm{rk}\big(X^{+t}\big)$.

Remark 2.1.4. Let $n \in \mathbb{N}_0$. The t-shift operation generates an equivalence relation in the infinite collection of β-sets of rank n. In particular, any β-set of rank n can be represented in a unique way by a finite set X such that $0 \notin X$.

Let Φ_n be the set of β-sets of rank n modulo the equivalence relation generated by the shift operation, $X \in \Phi_n$ with entries $\beta_1 > \cdots > \beta_s \gneq 0$. Then we associate with X a partition

$$\lambda_X := (\beta_1 + 1 - s, \beta_2 + 2 - s, \ldots, \beta_s)\,. \tag{2.1.4}$$

Notice that

$$|\lambda_X| = \sum_{i=1}^{s} (\beta_i - i + s) = \mathrm{rk}(X) = n \tag{2.1.5}$$

so the assignment

$$X \mapsto \lambda_X \tag{2.1.6}$$

defines a bijection between Φ_n and $\mathcal{P}_n$. On the other hand, we denote by X_λ the β-set in Φ_n associated to the partition $\lambda \vdash n$.

Definition 2.1.5. Let $d \in \mathbb{N}_0$. A *symbol* of defect d is an unordered pair of β-sets $\Lambda = (X, Y)$ such that $||X| - |Y|| = d$. The *rank* of Λ is defined as

$$\mathrm{rk}\,(\Lambda) := \mathrm{rk}\,(X) + \mathrm{rk}\,(Y) + \left\lfloor \frac{d}{2} \right\rfloor. \tag{2.1.7}$$

A symbol of the form (X, X) is said to be *special*.

The shift operation on β-sets induces a shift operation on symbols by

$$\Lambda = (X, Y) \mapsto \Lambda^{+t} := \left(X^{+t}, Y^{+t}\right)$$

with $t \in \mathbb{N}_0$.

Remark 2.1.6. By Remark 2.1.3, we have that the rank and the defect of a symbol are invariant under the shift operation. Thus, the shift operation generates an equivalence relation in the set of symbols of fixed rank n and defect d. In particular, again from Remark 2.1.3, any equivalence class of a symbol has a representative $\Lambda = (X, Y)$ such that $0 \notin X \cap Y$ and $|X| = |Y| + d$. Such a symbol is said to be *reduced*.

We denote by $\Phi_{n,d}$ the set of symbols of rank n and defect d modulo shift. Let $\Lambda \in \Phi_{n,1}$ represented by a reduced symbol (X, Y). Then we associate with Λ the ordered pair of partitions (λ_X, λ_Y), where λ_X and λ_Y are as in 2.1.4. Notice that, according to 2.1.5,

$$|\lambda_X| + |\lambda_Y| = \mathrm{rk}(X) + \mathrm{rk}(Y) = \mathrm{rk}(\Lambda).$$

One can reverse this assignment (for further information, see [Lu77]), so we have a bijection between $\Phi_{n,1}$ and the set of ordered pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = n$ given by

$$\Lambda = (X, Y) \mapsto (\lambda_X, \lambda_Y) \tag{2.1.8}$$

Now, let $\tilde{\Phi}_{n,0}$ be the set of symbol classes of rank n and defect 0, in which each special symbol class is repeated twice. The assignment 2.1.8 gives a bijective correspondence between $\tilde{\Phi}_{n,0}$ and the set of unordered pairs of partitions $\{\lambda, \mu\}$ such that $|\lambda| + |\mu| = n$, with pairs of the form $\{\lambda, \lambda\}$ repeated twice when n is even (again see [Lu77]). We denote such special pairs by $\{\lambda, +\}$ and $\{\lambda, -\}$ for $\lambda \vdash \frac{n}{2}$ and the corresponding special symbols by $(X, +)$ and $(X, -)$ for some β-set X.

2.1.2 Möbius Inversion formula

Let $(P, \leq)$ be a finite poset, i.e., a partially ordered set.

Definition 2.1.7. To every function $f : P \to \mathbb{C}$ we assign

$$\widehat{f} : P \to \mathbb{C}$$

$$\widehat{f}(a) := \sum_{a \leq b} f(b)$$

the *accumulated sum of* f with respect to P.

Remark 2.1.8. We see that $f \mapsto \widehat{f}$ is a linear operator in $\mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{|P|}, \mathbb{C}^{|P|}\right)$ whose matrix M_Σ is given by

$$(M_\Sigma)_{a,b} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

the adjacency matrix of (the digraph induced by) P.

Since P is a poset, this matrix will be upper triangular with respect to some total ordering refining that of P, and will have only 1's in its main diagonal. Therefore, its inverse $\mu \in \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{|P|}, \mathbb{C}^{|P|}\right)$ has only 1's in the diagonal and integral entries, for writing M_Σ as $\mathrm{Id} + A$, with A strictly upper triangular (hence nilpotent), the alternating sum

$$\mathrm{Id} - A + A^2 - A^3 \ldots$$

for $(\mathrm{Id} + A)^{-1}$ results finite.

Definition 2.1.9. The function $\mu\,(a, b)$ given by the entries $(\mu)_{a,b}$ of the matrix μ is called the *Möbius function* for P.

Remark 2.1.10. By definition of μ, we have the *Möbius Inversion formula*:

$$f(a) := \sum_{a \leq b} \mu(a, b) \widehat{f}(b). \tag{2.1.9}$$

for any function $f : P \to \mathbb{C}$.

Example 2.1.11. (*Divisors*). Let P be the set of positive divisors of a fixed n with the ordering given by divisibility, more precisely $a \leq b$ if and only if $a|b$. The accumulated sums for f are $\widehat{f}(d) := \sum_{d|m} f(m)$ and the inversion formula for this case is the well known

$$f(d) := \sum_{d|m} \widehat{f}(m) \mu\left(\frac{m}{d}\right) \tag{2.1.10}$$

where

$$\mu(m) := \begin{cases} (-1)^{\#\{\text{primes dividing}\, m\}} & \text{if}\, m\, \text{is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mu(a, b) = \mu\left(\frac{b}{a}\right)$ if a divides b and 0 if not.

Example 2.1.12. (*Finite vector subspaces*). Let n be a natural number, $q = p^m$ a prime power. Consider P the poset of the vector subspaces of $\mathbb{F}_q^n$, the n-dimensional vector space over the finite field $\mathbb{F}_q$, with the ordering given by the inclusion. If $U, V \subseteq \mathbb{F}_q^n$, the Möbius function is

$$\mu\left(U, V\right) = \begin{cases} (-1)\, q^{\binom{k}{2}} & \text{if}\, U \subseteq V\, \text{and}\, \dim(V) - \dim(U) = k \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.11}$$

Example 2.1.13. (*Set-partitions*). Let us take P as the collection Π_c of the set-partitions of the set $[c] := \{1, \ldots, c\}$ for a fixed c and the order being given by refinement, which we denote by $\preceq$. For instance, in Π_4 we have

$$1|2|3|4 \preceq 1|2|34 \preceq 12|34 \preceq 1234.$$

Obviously, we can speak of the length of an element π in Π_c as in the case of partitions, and denote it by $l(\pi)$ similarly. We now show an application of the Möbius Inversion formula that we will use in the following.

If $x \in \mathbb{N}$ and $h : [c] \to [x]$, define the *kernel* of h, and denote it by $Ker(h)$, to be the partition of $[c]$ induced by the equivalence relation

$$a \equiv b \Longleftrightarrow h(a) = h(b).$$

Fix $\pi \in \Pi_c$ and let

$$\begin{aligned} \Sigma'_\pi &:= \{h : [c] \to [x] \mid Ker(h) = \pi\}, \\ \Sigma_\pi &:= \{h : [c] \to [x] \mid Ker(h) = \sigma \,\text{for some}\, \sigma \succeq \pi\}. \end{aligned} \tag{2.1.12}$$

Clearly, $|\Sigma'_\pi| = (x)_{(l(\pi))} := x\,(x-1)\cdots(x - l\,(\pi) + 1)$ and $|\Sigma_\pi| = x^{l(\pi)}$. On the other hand, if $f(\pi) := |\Sigma'_\pi|$ then $\widehat{f}(\pi) = |\Sigma_\pi|$, so by the Möbius Inversion formula 2.1.9, we have

$$f(\pi) = \sum_{\pi \preceq \sigma} \mu(\pi, \sigma) \widehat{f}(\sigma)$$

that is

$$(x)_{l(\pi)} = \sum_{\pi \preceq \sigma} \mu(\pi, \sigma) x^{l(\sigma)}. \tag{2.1.13}$$

Since 2.2.6 holds for any $x \in \mathbb{N}$ and since both sides of 2.2.6 are polynomials, it is a polynomial identity. In particular, if we specialize it at $x = -1$, we obtain

$$(-1)^{l(\pi)}\,(l(\pi))! = \sum_{\pi \preceq \sigma} (-1)^{l(\sigma)}\,\mu(\pi,\sigma). \tag{2.1.14}$$

2.2 Representation Theory

In this section, we list and prove some facts on Representation Theory we will need. For proof and details on the basics on Representation Theory of finite groups, we refer the reader to [FH91] and [Ser77].

A representation of a finite group G is a finite dimensional $\mathbb{C}$-vector space V together with a group homomorphism

$$\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V).$$

Its character $\chi = \chi_\rho$ is the class function obtained from composition with the trace

$$\chi_\rho(g) := \mathrm{tr}(\rho(g)).$$

Since the character χ_ρ of a representation ρ determines it uniquely up to isomorphism, a representation is usually identified with its character.

A representation is said to be irreducible if it does not have nontrivial invariant subspaces. In such case, its character is also called irreducible.

Similarly, a representation of an algebra A over an algebraically closed field $\mathbb{K}$ is a finite dimensional $\mathbb{K}$-vector space V together with an algebra homomorphism

$$\psi : A \to \mathrm{End}_{\mathbb{K}}(V).$$

The character of a representation of an algebra is defined as in the case of representations of finite groups.

Notation. For a finite group G, we note $\mathrm{Irr}(G)$ the set of its irreducible characters. Whenever it is clear from the context, it will also represents the set of (isomorphism classes of) irreducible representations of G. We will use the same notation for the set of irreducible characters of an algebra. Moreover, we write $\widehat{G}$ to denote $\mathrm{Hom}(G, \mathbb{C}^\times)$

In an irreducible representation (V, ρ), Schur's Lemma says that all the $\mathbb{C}$-linear endomorphisms in $\mathrm{Hom}_\rho(V, V)$, i.e., the set of endomorphisms commuting with ρ, are the scalar multiplication $\zeta\mathrm{Id} : x \mapsto \zeta x$ with $\zeta \in \mathbb{C}$.

Remark 2.2.1. By Schur's Lemma, we have that for G abelian, $\mathrm{Irr}(G) = \widehat{G}$.

There is a natural inner product $\langle \cdot, \cdot \rangle_G$ for characters given by

$$\langle \chi, \chi' \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

Recall the Orthogonality relations: for $\chi, \chi' \in \mathrm{Irr}(G)$ then

$$\langle \chi, \chi' \rangle_G = \begin{cases} 1 & \text{if } \chi = \chi' \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

For $g, h \in G$

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\chi(h) = \begin{cases} |C_G(g)| & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases} \tag{2.2.2}$$

where $C_G(g)$ is the centralizer of g in G.

Notation. For a general group G and elements $x, g \in G$, sometimes we write x^g for gxg^{-1}. We may use the same notation for the conjugate of a group.

Let H be a subgroup of a finite group G, χ a character of G, θ a character of H. Define the restriction character $\mathrm{Res}_H^G(\chi)$ as the character of H satisfying

$$\mathrm{Res}_H^G(\chi)(h) = \chi(h)$$

for any $h \in H$ and the induced character $\mathrm{Ind}_H^G(\theta)$ as the character of G satisfying

$$\mathrm{Ind}_H^G(\theta)(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ g^s \in H}} \theta(g^s)$$

for any $g \in G$. One can give analogous definitions for characters of representations of an algebra. The following Frobenius reciprocity law holds:

$$\left\langle \chi, \mathrm{Ind}_H^G(\theta) \right\rangle_G = \left\langle \mathrm{Res}_H^G(\chi), \theta \right\rangle_H . \tag{2.2.3}$$

Let G_1 and G_2 be two finite groups, $\chi_1 \in \mathrm{Irr}(G_1)$ and $\chi_2 \in \mathrm{Irr}(G_2)$. Then we can define the character $\chi_1 \otimes \chi_2 \in \mathrm{Irr}(G_1 \times G_2)$ by

$$(\chi_1 \otimes \chi_2)\left((g_1, g_2)\right) := \chi_1(g_1)\chi_2(g_2) \tag{2.2.4}$$

for any $g_1 \in G_1$ and $g_2 \in G_2$. It turns out that every irreducible character of the direct product $G_1 \times G_2$ arises uniquely in this way, so $\mathrm{Irr}(G_1 \times G_2) = \mathrm{Irr}(G_1) \times \mathrm{Irr}(G_2)$.

2.2.1 Counting solutions of equations in finite groups

Let G be a finite group and A some abelian group like $\mathbb{C}$ or $\mathbb{C}^{n \times n}$.

Definition 2.2.2. For a function $f : G \to A$, we note

$$\int_G f(x)dx := \frac{1}{|G|}\sum_{x\in G} f(x)$$

the integral of f with respect to the Haar measure of G.

Our goal now is to prove the following formula due to Frobenius (see [FQ93], [FS906] or [Med78]). It will be fundamental for our next computations.

Proposition 2.2.3. (Frobenius formula). *Given $z \in G$, the number of $2g$-tuples $(x_1, y_1, \dots, x_g, y_g)$ satisfying $\prod_{i=1}^{g} [x_i : y_i]z = 1$ is:*

$$\left|\left\{\prod_{i=1}^{g} [x_i : y_i]z = 1\right\}\right| = \sum_{\chi\in \mathrm{Irr}(G)} \chi(z)\left(\frac{|G|}{\chi(1)}\right)^{2g-1}. \tag{2.2.5}$$

Proof. ([Mer15, Proposition 2.2.1]). Given $\rho : G \to \mathrm{Aut}\,(V)$ an irreducible representation of G and χ its character, let us consider for a fixed $y \in G$ the average

$$\int_G \rho\big(xyx^{-1}\big)dx. \tag{2.2.6}$$

Since it commutes with the G-action, it must be, by Schur's Lemma, a scalar map $\zeta\mathrm{Id}$. By taking the traces we get:

$$\begin{aligned}
\int_G \rho\big(xyx^{-1}\big)dx &= \zeta\mathrm{Id}\\
\int_G \mathrm{tr}\big(\rho\big(xyx^{-1}\big)\big)dx &= \zeta\mathrm{tr}(\mathrm{Id})\\
\int_G \chi\big(xyx^{-1}\big)dx &= \zeta\chi(1)\\
\int_G \chi(y)dx &= \zeta\chi(1)\\
\frac{\chi(y)}{\chi(1)} &= \zeta
\end{aligned} \tag{2.2.7}$$

thus, the average 2.2.6 becomes

$$\int_G \rho\big(xyx^{-1}\big)dx = \frac{\chi(y)}{\chi(1)}\mathrm{Id}. \tag{2.2.8}$$

Multiplying by $\rho\big(y^{-1}\big)$ from the right we get:

$$\int_G \rho\big(xyx^{-1}y^{-1}\big)dx = \frac{\chi(y)}{\chi(1)}\rho(y)^{-1}$$

summing over all $y \in G$ and dividing by $|G|$ again we end up with

$$\int\int_{G^2} \rho([x:y])dydx = \int_G \frac{\chi(y)}{\chi(1)}\rho(1)^{-1}dy. \tag{2.2.9}$$

Since the left hand side of this equation is invariant under G-conjugation, it is also a scalar transformation. taking traces again we conclude

$$\int\int_{G^2} \mathrm{tr}(\rho([x:y]))dydx = \left(\int_G \frac{\chi(y)\chi\big(y^{-1}\big)}{\chi(1)^2}dx\right)\mathrm{Id} = \frac{1}{\chi(1)^2}\mathrm{Id}. \tag{2.2.10}$$

Raising 2.2.10 to the g-th power and multiplying by $\rho(z)$ from the right, we will have

$$\begin{aligned} \left(\int\int_{G^2} \rho\big(xyx^{-1}y^{-1}\big)dydx\right)^g \rho(z) &= \frac{1}{\chi(1)^{2g}}\rho(z) \\ \underbrace{\int_G \cdots \int_G}_{2g\,\text{times}} \rho([x_1:y_1]\cdots[x_g:y_g]\,z)dx_1dy_1\ldots dx_gdy_g &= \frac{1}{\chi(1)^{2g}}\rho(z). \end{aligned} \tag{2.2.11}$$

Taking trace at both sides yet one more time we have

$$\int_G \cdots \int_G \chi([x_1:y_1]\cdots[x_g:y_g]\,z)dx_1dy_1\ldots dx_gdy_g = \frac{\chi(z)}{\chi(1)^{2g}}. \tag{2.2.12}$$

Multiplying this by $\chi(1)$ and summing over all $\chi \in \mathrm{Irr}(G)$, we see that the sum

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(z)}{\chi(1)^{2g-1}} \tag{2.2.13}$$

is equal to

$$\int_G \cdots \int_G \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi([x_1 : y_1] \cdots [x_g : y_g]\, z) dx_1 dy_1 \dots dx_g dy_g \tag{2.2.14}$$

and thanks to the orthogonality relation 2.2.2, only those terms with

$$\prod_{i=1}^{g} [x_i : y_i] z = 1$$

survive in 2.2.14, and we get

$$\frac{1}{|G|^{2g-1}} \left| \left\{ \prod_{i=1}^{g} [x_i : y_i] z = 1 \right\} \right| = \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(z)}{\chi(1)^{2g-1}} \tag{2.2.15}$$

from which the proposition follows immediately. □

As a particular case of Proposition 2.2.3, for $z = 1$ we recover the following well known formula (see [FQ93] for instance).

Corollary 2.2.4. *Let Σ_g be a compact Riemann surface of genus g, $\pi_1(\Sigma_g)$ its fundamental group and G a finite group. Then*

$$\frac{1}{|G|} |\mathrm{Hom}(\pi_1(\Sigma_g), G)| = \sum_{\chi \in \mathrm{Irr}(G)} \left(\frac{|G|}{\chi(1)} \right)^{2g-2}.$$

2.2.2 Algebraic groups

In the next sections, we list some basic definitions and facts on the theory of algebraic groups. In order to do this, we follow [Ca85, Chapter 1]. For more details, we refer the reader to [Bo91] and [Hu75].

Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) \neq 2$.

Definition 2.2.5. An (affine) *algebraic group* over $\mathbb{K}$ is a set G which is an affine algebraic variety over $\mathbb{K}$ and also a group, such that the maps

$$m : G \times G \to G$$
$$(x, y) \mapsto xy$$

and

$$i : G \to G$$
$$x \mapsto x^{-1}$$

are morphisms of varieties. If G and G' are two algebraic groups, a map $\alpha : G \to G'$ is called a *homomorphism* if α is both a morphism of varieties and a homomorphism of groups. If α is bijective and α^{-1} is a morphism of varieties, then α is an *isomorphism* of algebraic groups.

Remark 2.2.6. If H is a closed subgroup of an algebraic group G, then it is an algebraic group. If G_1 and G_2 are algebraic groups over $\mathbb{K}$, then the direct product $G_1 \times G_2$ will also be an algebraic group.

Example 2.2.7. The additive group $\mathbb{G}_a := (\mathbb{K}, +)$ and the multiplicative group $\mathbb{G}_m := (\mathbb{K}^\times, \cdot)$ are algebraic groups over $\mathbb{K}$. These are the only algebraic groups of dimension 1.

Example 2.2.8. Let n be a natural number and let $\mathfrak{gl}(n, \mathbb{K})$ be the set of $n \times n$

matrices with coefficients in $\mathbb{K}$. The general linear group of nonsingular matrices

$$\mathrm{GL}(n,\mathbb{K}) = \{A \in \mathfrak{gl}(n,\mathbb{K}) \mid \det(A) \neq 0\}$$

is an algebraic group over $\mathbb{K}$.

Example 2.2.9. For $n \in \mathbb{N}$, any closed subgroup of $\mathrm{GL}(n,\mathbb{K})$ is an affine algebraic group. In particular, the special linear group

$$\mathrm{SL}(n,\mathbb{K}) = \{A \in \mathrm{GL}(2n,\mathbb{K}) \mid \det(A) = 1\}$$

and the group of symplectic matrices

$$\mathrm{Sp}(2n,\mathbb{K}) = \left\{A \in \mathrm{GL}(2n,\mathbb{K}) \mid A^t J A = J\right\} \tag{2.2.16}$$

are algebraic groups, where

$$J = \left(\begin{smallmatrix} 0 & C \\ -C & 0 \end{smallmatrix}\right) \in \mathfrak{gl}(2n,\mathbb{K})$$

and

$$C = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in \mathfrak{gl}(n,\mathbb{K})$$

relatively to a basis $\{e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}\}$.

Remark 2.2.10. The converse of the statement in the previous example 2.2.9 holds. Namely, every affine algebraic group over $\mathbb{K}$ is isomorphic to a closed subgroup of $\mathrm{GL}(n,\mathbb{K})$ for some $n \in \mathbb{N}$.

Let G be an algebraic group over $\mathbb{K}$ and $j : G \hookrightarrow \mathrm{GL}(n,\mathbb{K})$ an isomorphism onto a closed subgroup of $\mathrm{GL}(n,\mathbb{K})$, for some $n \in \mathbb{N}$, provided by Remark

2.2.10.

Definition 2.2.11. An element $x \in G$ is said to be *semisimple* if $j(x)$ is a diagonalizable matrix. x is said to be *unipotent* if $j(x)$ is a matrix whose eigenvalues are all equal to 1.

Remark 2.2.12. The definitions of semisimple and unipotent element of an algebraic group turn out to be independent from the choice of the closed embedding into some general linear group $\mathrm{GL}(n, \mathbb{K})$.

Remark 2.2.13. The property of being unipotent or semisimple for an element of an algebraic group G is preserved by homomorphisms of algebraic groups. In particular, the only semisimple and unipotent element of an algebraic group is the identity of G.

Definition 2.2.14. An algebraic group G is called unipotent if all its elements are unipotent. An algebraic group G is a *torus* if it is isomorphic to a group of the form $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Example 2.2.15. The group $\mathbb{G}_a$ is unipotent. In fact, we have the following closed embedding:

$$j : \mathbb{G}_a \hookrightarrow \mathrm{GL}(2, \mathbb{K})$$

$$c \mapsto j(c) := \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right).$$

Conversely, all elements of $\mathbb{G}_m$ are semisimple, so by Remark 2.2.13, any torus is made by semisimple elements.

Definition 2.2.16. The *unipotent radical* $R_u(G)$ of an algebraic group G is the unique maximal element of the set of closed connected unipotent normal subgroups of G. G is called *reductive* if $R_u(G) = \{1\}$.

Example 2.2.17. $\mathrm{GL}(n, \mathbb{K})$, $\mathrm{SL}(n, \mathbb{K})$ and $\mathrm{Sp}(2n, \mathbb{K})$ are all reductive algebraic groups over $\mathbb{K}$.

We now deal with connected algebraic groups. Recall that a group G is solvable if it has a series of normal subgroups

$$1 = G^{(0)} \subset G^{(1)} \cdots \subset G^{(d-1)} \subset G^{(d)} = G$$

where $G^{(i+1)} := \left[G^{(i)} : G^{(i)}\right]$, the commutator of G^i, for each i.

Definition 2.2.18. A *Borel subgroup* B of an algebraic group G over $\mathbb{K}$ is a maximal closed connected solvable subgroup of G. A *maximal torus* T is a closed subgroup of G that is a torus not properly contained in larger tori.

Remark 2.2.19. All Borel subgroups of an algebraic group G are conjugate. This is true for maximal tori too.

Remark 2.2.20. If T is a maximal torus in G, B a Borel subgroup of G containing T, we have the following decomposition of B as a semidirect product:

$$B = T \ltimes U \tag{2.2.17}$$

where $U = R_u(B)$.

Example 2.2.21. For $G = \mathrm{GL}(n, \mathbb{K})$, we may take $B = T_n(\mathbb{K})$, the subgroup of upper-triangular matrices in G, as a Borel subgroup of G and $T = D_n(\mathbb{K})$, the subgroup of diagonal matrices in G, as a maximal torus of G. Moreover, according to 2.2.17, we have

$$T_n(\mathbb{K}) = D_n(\mathbb{K}) \ltimes U_n(\mathbb{K})$$

where $U_n(\mathbb{K})$ is the subgroup of upper-unitriangular matrices.

Example 2.2.22. For $G = \mathrm{Sp}(2n, \mathbb{K}) \subset \mathrm{GL}(2n, \mathbb{K})$, using the notation of the example 2.2.21, we may take $B = T_{2n}(\mathbb{K}) \cap G$ as a Borel subgroup of G and $T = D_{2n}(\mathbb{K}) \cap G$ as a maximal torus of G, so in particular

$$T = \left\{ \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}\right) \mid \lambda_1, \ldots, \lambda_n \in \mathbb{K}^\times \right\} \tag{2.2.18}$$

hence $\dim(T) = n$.

2.2.3 Roots, coroots and the Weyl group

From now on, we only deal with a connected reductive algebraic group G.

Definition 2.2.23. Let T be a maximal torus of G. The *Weyl group* is defined as $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G.

Remark 2.2.24. The Weyl group W is a finite group and, by Remark 2.2.19, it is uniquely determined up to isomorphism, i.e., different choices of maximal tori produce isomorphic Weyl groups.

We now give a more detailed description of the structure of the Weyl group W.

Let T be a maximal torus of G and let $X := \mathrm{Hom}(T, \mathbb{G}_m)$ be the set of algebraic group homomorphism from T to $\mathbb{G}_m$. X can be made into a group under the operation:

$$+ : X \times X \to X$$

defined by

$$(\chi_1 + \chi_2)(t) := \chi_1(t)\chi_2(t)$$

for any $\chi_1, \chi_2 \in X$, $t \in T$.

Definition 2.2.25. X equipped with the operation $+$ defined above is called the *character group* of T.

Remark 2.2.26. Suppose first that $\dim(T) = 1$. Then T is isomorphic to $\mathbb{G}_m$ and we are considering the group $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$. It is easy to see that the only algebraic homomorphisms from $\mathbb{G}_m$ to itself are the maps

$$\begin{aligned} \mathbb{G}_m &\to \mathbb{G}_m \\ \lambda &\mapsto \lambda^n \end{aligned}$$

where $n \in \mathbb{Z}$. Thus $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$. In general, we shall have $T \cong \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{r \text{ times}}$. Then we have

$$X = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathrm{Hom}\left(\underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{r \text{ times}}, \mathbb{G}_m\right) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \text{ times}}.$$

Thus X is a free abelian group of rank r.

Let $Y := \mathrm{Hom}(\mathbb{G}_m, T)$ be the set of algebraic homomorphism from $\mathbb{G}_m$ into T. As in the case of characters group X, Y can be made into a group under the operation

$$+ : Y \times Y \to Y$$

given by

$$(\gamma_1 + \gamma_2)(\lambda) := \gamma_1(\lambda)\gamma_2(\lambda)$$

for any $\gamma_1, \gamma_2 \in Y$, $\lambda \in \mathbb{K}^\times$.

Definition 2.2.27. Y equipped with the operation $+$ defined above is called the *cocharacters group* (or one-parameter subgroups) of T.

Remark 2.2.28. Analogously to the case of the characters group X, if $\dim(T) = r$, then we have

$$Y \cong \mathrm{Hom}\left(\mathbb{G}_m, \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{r\text{ times}}\right) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r\text{ times}}.$$

Thus Y is also a free abelian group of rank r.

We now define a map from $X \times Y$ into $\mathbb{Z}$ taking (χ, γ) to an integer $\langle \chi, \gamma \rangle$. This integer is defined as follows. Since $\chi \in X$ and $\gamma \in Y$, $\chi \circ \gamma$ lies in $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$, hence $(\chi \circ \gamma)(\lambda) = \lambda^n$ for some $n \in \mathbb{Z}$ and for all $\lambda \in \mathbb{G}_m$. We define $\langle \chi, \gamma \rangle = n$. The map

$$\begin{aligned} X \times Y &\to \mathbb{Z} \\ (\chi, \gamma) &\mapsto \langle \chi, \gamma \rangle \end{aligned}$$

is non-degenerate and gives rise to a duality between X and Y. It gives isomorphisms between X and $\mathrm{Hom}(Y, \mathbb{Z})$ and between Y and $\mathrm{Hom}(X, \mathbb{Z})$.

The Weyl group can be made to act on both X and Y as follows. If $w \in W$ and $\chi \in X$, for any $t \in T$ we define ${}^w\chi \in X$ by

$${}^w\chi(t) := \chi(t^w).$$

Then $\chi \to {}^w\chi$ is an automorphism of X and we have ${}^{w'}({}^w\chi) = {}^{w'w}\chi$. If $\gamma \in Y$, for any $\lambda \in \mathbb{G}_m$ we define γ^w by

$$\gamma^w(\lambda) := \gamma(\lambda)^w.$$

Then $\gamma \to \gamma^w$ is an automorphism of Y and we have $(\gamma^{w'})^w = \gamma^{w'w}$. The W-actions on X and Y are related by the formula

$$\langle \chi, \gamma^w \rangle = \langle {}^w\chi, \gamma \rangle$$

for $\chi \in X, \gamma \in Y$ and $w \in W$.

Let us consider a Borel subgroup B of G containing T. Then $B = T \ltimes U$ according to 2.2.17, where $U = R_u(B)$. It is true that G has a unique Borel subgroup B^- containing T such that $B \cap B^- = T$. B and B^- are called opposite Borel subgroups. Again by 2.2.17, we have that $B^- = T \ltimes U^-$, where $U^- = R_u(B^-)$. U and U^- are connected unipotent groups normalized by T satisfying $U \cap U^- = \{1\}$ and they are maximal unipotent subgroups of G.

We consider the minimal proper subgroups of U and U^- which are normalized by T. They are all connected unipotent groups of dimension 1 as affine varieties, so the statements in examples 2.2.7 and 2.2.15 tell us that all these groups are isomorphic to the additive group $\mathbb{G}_a$. T acts on each of them by conjugation, giving a homomorphism $\alpha : T \to \mathrm{Aut}(\mathbb{G}_a)$ from T to the group of algebraic automorphisms of $\mathbb{G}_a$. However, the only algebraic automorphisms of $\mathbb{G}_a$ are the maps $\lambda \mapsto \mu\lambda$ for some $\mu \in \mathbb{K}^\times$. Thus $\mathrm{Aut}(\mathbb{G}_a)$ is isomorphic to $\mathbb{G}_m$. Hence each of our 1-dimensional unipotent groups determines an element of $\mathrm{Hom}(T, \mathbb{G}_m) = X$.

Definition 2.2.29. The elements of X arising in the way described above form the finite set Φ of the *roots* of G.

Remark 2.2.30. The roots of G are all nonzero elements of X. Distinct 1-dimensional unipotent subgroups give rise to distinct roots and the set of roots

Φ is independent of the choice of the Borel subgroup B containing T.

Definition 2.2.31. For each root $\alpha \in \Phi$, the 1-dimensional unipotent subgroup X_α giving rise to it is called a *root subgroup* of G.

Remark 2.2.32. The roots arising from root subgroup in U^- are the negatives of the roots arising from root subgroups in U. We also have $G = \langle T, X_\alpha \mid \alpha \in \Phi \rangle$.

Let α, $-\alpha$ be a pair of opposite roots. Then we consider the subgroup $\langle X_\alpha, X_{-\alpha} \rangle$ of G generated by the root subgroups X_α and $X_{-\alpha}$. This subgroup is a 3-dimensional simple group, i.e., a group that has no proper closed connected normal subgroups, isomorphic to either $\mathrm{SL}(2, \mathbb{K})$ or to $\mathrm{PGL}(2, \mathbb{K}) := \mathrm{GL}(2, \mathbb{K}) / \{\pm I_2\}$. In fact, there is a homomorphism $\phi_\alpha : \mathrm{SL}(2, \mathbb{K}) \twoheadrightarrow \langle X_\alpha, X_{-\alpha} \rangle$ such that

$$\phi_\alpha(\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}) = X_\alpha,$$

$$\phi_\alpha(\{\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\}) = X_{-\alpha}.$$

Definition 2.2.33. Let $\alpha \in \Phi$. The 1-dimensional torus

$$T_\alpha := \phi_\alpha\big(\big\{\big(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\big)\big\}\big)$$

is called a *root torus*. The cocharacter α^v of T given by

$$\begin{aligned} \alpha^v : \mathbb{G}_m &\to T_\alpha \subseteq T \\ \lambda &\mapsto \phi_\alpha\big(\big(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\big)\big) \end{aligned}$$

is called the *coroot* corresponding to the root α.

Remark 2.2.34. A root α and its corresponding coroot α^v are related by the condition $\langle \alpha, \alpha^v \rangle = 2$. Moreover, the coroots form a finite subset of Y that we denote by Φ^v.

Remark 2.2.35. If $\alpha \in \Phi$, one can prove that the root torus T_α coincides with $T_{-\alpha}$.

We now consider in more detail the actions of the Weyl group on X and Y. W acts faithfully on both these lattices. Each element of W permutes the set of roots Φ in X and the set of coroots Φ^v in Y. For each root α consider the element $\phi_\alpha\big(\big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big)\big) \in \langle X_\alpha, X_{-\alpha}\rangle$. This element lies in $N_G(T)$, so we can define the following assignment

$$\begin{aligned} &\Phi \to W = N_G(T)/T \\ &\alpha \mapsto w_\alpha := \big[\phi_\alpha\big(\big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big)\big)\big]_T . \end{aligned} \tag{2.2.19}$$

The element w_α acts on X by

$$w_\alpha(\chi) := \chi - \langle \chi, \alpha^v \rangle \, \alpha \tag{2.2.20}$$

for $\chi \in X$, and on Y by

$$w_\alpha(\gamma) := \gamma - \langle \alpha, \gamma \rangle \, \alpha^v \tag{2.2.21}$$

for $\gamma \in Y$. We have $w_\alpha = w_{-\alpha}$ and $w_\alpha^2 = 1$. Moreover, the elements w_α for all $\alpha \in \Phi$ generate W.

Definition 2.2.36. For a connected reductive group G, the quadruple (X, Φ, Y, Φ^v) is called a *root datum*. This means that X and Y are free abelian groups of the same finite rank with a nondegenerate map $X \times Y \to \mathbb{Z}$ which puts them into duality. Φ and Φ^v are finite subsets of X and Y respectively, and there is a bijection $\alpha \mapsto \alpha^v$ between them satisfying $\langle \alpha, \alpha^v \rangle = 2$. Finally, for each $\alpha \in \Phi$ we have maps $w_\alpha : X \to X$ and $w_\alpha : Y \to Y$ defined as in 2.2.20 and

2.2.21 satisfying $w_\alpha(\Phi) = \Phi$, $w_\alpha(\Phi^v) = \Phi^v$.

Remark 2.2.37. The basic classification theorem for connected reductive groups asserts that, given any root datum, there is a unique connected reductive group G over $\mathbb{K}$ which gives rise to the root datum in the manner described above.

Definition 2.2.38. Let Φ^+ be the set of roots arising from root subgroup of U and Φ^- be those arising from subgroups of U^-. Roots in Φ^+, Φ^- are called *positive* and *negative* respectively. Let Δ be the set of positive roots which cannot be expressed as a sum of two positive roots. Δ is called the set of *simple roots.*

Remark 2.2.39. The set Δ of simple roots is linearly independent. If $|\Delta| = n$ and $\Delta = \{\alpha_1, \dots, \alpha_n\}$, then each root in Φ^+ has the form $\sum_{i=1}^{n} n_i\alpha_i$ where $n_i \in \mathbb{Z}_{\geq 0}$ and each root in Φ^- has the form $\sum_{i=1}^{n} n_i\alpha_i$ where $n_i \in \mathbb{Z}_{\leq 0}$.

Remark 2.2.40. If $\Delta = \{\alpha_1, \dots, \alpha_n\}$, the Weyl group W is generated by the elements $s_1, \dots, s_n$ corresponding to the simple roots.

Remark 2.2.41. We have $W(\Delta) = \Phi$, so that each root is the image of some simple root under an element of the Weyl group W. Also, there is a unique element $w_0 \in W$ such that $w_0(\Phi^+) = \Phi^-$.

We conclude illustrating the situation described in this section in the following two concrete examples.

Example 2.2.42. Let $G = \mathrm{GL}(n, \mathbb{K})$. Let us take B and T as in example 2.2.21. Then U is the subgroup of upper-unitriangular matrices. We also have for B^- the subgroup of lower-triangular matrices and for U^- the subgroup of lower-unitriangular matrices. For $i = 1, \dots, n$, define $\epsilon_i : T \to \mathbb{G}_m$ as the character of T given by

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_i.$$

Then the roots of G are given by $\alpha_{ij} := \epsilon_i - \epsilon_j$ for $i, j = 1, \ldots, n$ with $i \neq j$. For each root α_{ij}, the corresponding root subgroup $X_{\alpha_{ij}}$ and root torus $T_{\alpha_{ij}}$ are given by

$$X_{\alpha_{ij}} = \{I_n + \lambda E_{ij} \mid \lambda \in \mathbb{K}\},$$

$$T_{\alpha_{ij}} = \left\{ \operatorname{diag}\left(1, \ldots, 1, \overbrace{\lambda}^{i}, 1, \ldots, 1, \overbrace{\lambda^{-1}}^{j}, 1, \ldots, 1\right) \mid \lambda \in \mathbb{K}^\times \right\}$$

where E_{ij} is the elementary matrix with 1 in the (i, j) position and zeroes elsewhere, while the corresponding coroot $\alpha_{ij}^v : \mathbb{G}_m \to T$ is given by

$$\lambda \mapsto \operatorname{diag}\left(1, \ldots, 1, \overbrace{\lambda}^{i}, 1, \ldots, 1, \overbrace{\lambda^{-1}}^{j}, 1, \ldots, 1\right)$$

We observe that the root α_{ij} maps this matrix into λ^2, so that $\left\langle \alpha_{ij}, \alpha_{ij}^v \right\rangle = 2$. The Weyl group W is isomorphic to the symmetric group S_n. For each root α_{ij}, the corresponding element $w_{\alpha_{ij}} \in W$ is the permutation which transposes i and j and fixes the remaining symbols. The set of positive roots is $\Phi^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}$. On the other hand, the set of negative roots is $\Phi^- = \{\alpha_{ij} \mid 1 \leq j < i \leq n\}$. The set Δ of simple roots is given by $\Delta = \{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n,n-1}\}$.

Example 2.2.43. (See [DM91, Chapter 15]). Let $G = \operatorname{Sp}(2n, \mathbb{K})$. Let us take B and T as in example 2.2.22. Then U is the subgroup of upper-unitriangular symplectic matrices. We also have for B^- the subgroup of lower-triangular symplectic matrices and for U^- the subgroup of lower-unitriangular symplectic matrices. Similarly to what we have done in the previous example 2.2.42, for $i = 1, \ldots, n$ define $\epsilon_i : T \to \mathbb{G}_m$ as the character of T given by

$$\operatorname{diag}\left(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}\right) \mapsto \lambda_i.$$

Then the set of roots of G is given by

$$\begin{aligned} \Phi = &\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq n\} \cup \\ &\{\pm 2\epsilon_i \mid 1 \leq i \leq n\}. \end{aligned} \tag{2.2.22}$$

In particular

$$\begin{aligned} \Phi^+ = &\{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \\ &\{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \\ &\{2\epsilon_i \mid 1 \leq i \leq n\}, \\ \Phi^- = &\{-\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq n\} \cup \\ &\{-\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \cup \\ &\{-2\epsilon_i \mid 1 \leq i \leq n\}. \end{aligned}$$

The corresponding root subgroups are the following

$$\begin{aligned} X_{\epsilon_i - \epsilon_j} &= \{1 + t\left(E_{i,j} - E_{-j,-i}\right) \mid t \in \mathbb{K}\}, \\ X_{-\epsilon_i + \epsilon_j} &= \{1 + t\left(E_{-i,-j} - E_{j,i}\right) \mid t \in \mathbb{K}\}; \end{aligned} \tag{2.2.23}$$

$$\begin{aligned} X_{\epsilon_i + \epsilon_j} &= \{1 + t\left(E_{i,-j} + E_{j,-i}\right) \mid t \in \mathbb{K}\}, \\ X_{-\epsilon_i - \epsilon_j} &= \{1 + t\left(-E_{-i,j} - E_{-j,i}\right) \mid t \in \mathbb{K}\}; \end{aligned} \tag{2.2.24}$$

$$\begin{aligned} X_{2\epsilon_i} &= \{1 + tE_{i,-i} \mid t \in \mathbb{K}\}\,, \\ X_{-2\epsilon_i} &= \{1 + tE_{-i,i} \mid t \in \mathbb{K}\}\,; \end{aligned} \tag{2.2.25}$$

while the corresponding root tori are given by

$$\begin{aligned} T_{\epsilon_i-\epsilon_j} = \Bigg\{ \mathrm{diag}\Bigg(& 1,\dots,1,\overbrace{\lambda}^{i},\dots,\overbrace{\lambda^{-1}}^{j},1,\dots \\ & \dots,1,\overbrace{\lambda}^{-j},\dots,\overbrace{\lambda^{-1}}^{-i},1,\dots,1\Bigg) \mid \lambda \in \mathbb{K}^\times \Bigg\}; \end{aligned} \tag{2.2.26}$$

$$\begin{aligned} T_{\epsilon_i+\epsilon_j} = \Bigg\{ \mathrm{diag}\Bigg(& 1,\dots,1,\overbrace{\lambda}^{i},\dots,\overbrace{\lambda}^{j},1,\dots \\ & \dots,1,\overbrace{\lambda^{-1}}^{-j},\dots,\overbrace{\lambda^{-1}}^{-i},1,\dots,1\Bigg) \mid \lambda \in \mathbb{K}^\times \Bigg\}; \end{aligned} \tag{2.2.27}$$

$$T_{2\epsilon_i} = \Bigg\{ \mathrm{diag}\Bigg(1,\dots,1,\overbrace{\lambda}^{i},\dots,\overbrace{\lambda^{-1}}^{-i},1,\dots,1\Bigg) \mid \lambda \in \mathbb{K}^\times \Bigg\}. \tag{2.2.28}$$

For any root $\alpha \in \Phi$, the group $\langle X_\alpha, X_{-\alpha}\rangle$ is isomorphic to $\mathrm{SL}(2,\mathbb{K})$.

The Weyl group W is isomorphic to the semidirect product $S_n \ltimes \boldsymbol{\mu}_{\mathbf{2}}^n$, where $\boldsymbol{\mu}_{\mathbf{2}} := \{\pm 1\}$. For each root of the form $\epsilon_i - \epsilon_j$, the corresponding element $w_{\epsilon_i-\epsilon_j} \in W$ is the permutation of the set $\{1,\dots,n,-n,\dots,-1\}$ which transposes i and j and also $-i$ and $-j$ and fixes the remaining symbols. For each root of the form $\epsilon_i + \epsilon_j$, $w_{\epsilon_i+\epsilon_j}$ is the permutation of the set $\{1,\dots,n,-n,\dots,-1\}$ which transposes i and $-j$ and also $-i$ and j and fixes the remaining symbols. Finally, for each root of the form $2\epsilon_i$, $w_{2\epsilon_i}$ is the permutation of the set $\{1,\dots,n,-n,\dots,-1\}$ which transposes i and $-i$ and fixes the remaining symbols. The set Δ of simple roots is given by $\Delta = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$.

2.2.4 Coxeter groups and generic degrees

Now we make a digression on Representation theory of Coxeter groups. We refer the reader to [GP00].

Definition 2.2.44. A finite group W is a *Coxeter group* if it is presented as an abstract group by the following system of generators and relations:

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \text{ if } i \neq j \rangle . \tag{2.2.29}$$

If $S = \{s_1, \ldots, s_n\}$ is the set of generators of W as a Coxeter group, the pair (W, S) is said to be a *Coxeter system*. The symmetric matrix M defined as $(M)_{ij} := m_{ij}$ is called the *Coxeter matrix* of W.

Remark 2.2.45. The direct product $W_1 \times W_2$ of two Coxeter groups W_1 and W_2 is a Coxeter group.

Definition 2.2.46. If (W, S) is a Coxeter system with $S = \{s_1, \ldots, s_n\}$, define the *sign character* of W as the linear character ε of W such that $\varepsilon(s_i) = -1$ for any $i = 1, \ldots, n$.

Definition 2.2.47. Let W be a Coxeter group, $S = \{s_1, \ldots, s_n\}$ its set of generators and M its Coxeter matrix. W is said to be of type A_{n-1}, and we denote it by $W_{A_{n-1}}$, if, for any $1 \leq i \leq j \leq n$, M satisfies

$$(M)_{ij} = \begin{cases} 3 & \text{if } j - i = 1 \\ 2 & \text{otherwise.} \end{cases}$$

W is said to be of type B_n and we denote it by W_{B_n}, if, for any $1 \leq i \leq j \leq n$,

M satisfies

$$(M)_{ij} = \begin{cases} 3 & \text{if } j-i=1 \text{ and } i \neq n-1 \\ 4 & \text{if } i=n-1,\ j=n \\ 2 & \text{otherwise.} \end{cases}$$

W is said to be of type D_n, and we denote it by W_{D_n}, if, for any $1 \leq i \leq j \leq n$, M satisfies

$$(M)_{ij} = \begin{cases} 3 & \text{if } (j-i=1 \text{ and } i \leq n-2) \text{ or } (i=n-2, j=n) \\ 2 & \text{otherwise.} \end{cases}$$

Notation. If W_1 and W_2 are Coxeter groups of type X_1 and X_2 respectively, then we say that $W_1 \times W_2$ is of type $X_1 \times X_2$.

Example 2.2.48. If G is a connected reductive algebraic group, then its Weyl group W is a Coxeter group. The generators of W are the elements $s_1, \dots, s_n$ corresponding to the simple roots $\alpha_1, \dots . \alpha_n$.

Remark 2.2.49. From Examples 2.2.42 and 2.2.43, it is easy to see by Definition 2.2.47 and Remark 2.2.40 that the Weyl groups of $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{Sp}(2n, \mathbb{K})$ are Coxeter groups of type A_{n-1} and B_n respectively. Thus $W_{A_{n-1}} \cong S_n$ and $W_{B_n} \cong S_n \ltimes \boldsymbol{\mu}_{\mathbf{2}}^n$.

Remark 2.2.50. The Coxeter group W_{D_n} is a subgroup of $W_{B_n} \cong S_n \ltimes \boldsymbol{\mu}_{\mathbf{2}}^n$ of index two. Namely,

$$W_{D_n} = \left\{ (\sigma, (x_1, \dots, x_n)) \in \ S_n \ltimes \boldsymbol{\mu}_{\mathbf{2}}^n \mid \prod_{i=1}^{n} x_i = 1 \right\}. \tag{2.2.30}$$

Definition 2.2.51. The *Poincaré polynomials* of $W_{A_{n-1}}$, W_{B_n} and W_{D_n} are polynomials with integral coefficients defined as follows:

$$\begin{aligned} P_{A_{n-1}}(u) &:= \prod_{i=1}^{n} \left(\frac{u^i - 1}{u - 1} \right), \\ P_{B_n}(u) &:= P_{A_{n-1}}(u) \prod_{i=1}^{n} (1 + u^i), \\ P_{D_n}(u) &:= P_{A_{n-1}}(u) \prod_{i=1}^{n-1} (1 + u^i). \end{aligned} \tag{2.2.31}$$

Remark 2.2.52. Actually, there is a definition of Poincaré polynomial for general Coxeter groups (see for instance [GP00, Section 8.1.8] or also [Ki69]), but we only need this notion in the case of Coxeter groups of type A, B and D. Nevertheless, we will use the following general property for Poincaré polynomials: if W is a Coxeter group and U is a subgroup of W that is Coxeter, then the Poincaré polynomial P_U of U divides the Poincaré polynomial P_W of W.

Notation. Sometimes we denote the Poincaré polynomial of a Coxeter group W by P_W as in the previous Remark 2.2.52.

Remark 2.2.53. The Poincaré polynomial is multiplicative with respect to the direct product, that is, if $W = W_1 \times W_2$ is of type $X_1 \times X_2$ then $P_{X_1 \times X_2}(u) = P_{X_1}(u)P_{X_2}(u)$

Remark 2.2.54. It is easy to see that

1. $P_{A_{n-1}}(1) = n! = \left|W_{A_{n-1}}\right|$.
2. $P_{B_n}(1) = 2^n n! = |W_{B_n}|$.
3. $P_{D_n}(1) = 2^{n-1} n! = |W_{D_n}|$.

The following bijective correspondences are well known (see for instance [Lu77, 2.2, 2.3, 2.5]):

1. $\mathrm{Irr}\big(W_{A_{n-1}}\big) \overset{1:1}{\longleftrightarrow} \mathcal{P}_n$.

2. $\mathrm{Irr}(W_{B_n}) \overset{1:1}{\longleftrightarrow} \bigcup_{k+l=n} \mathcal{P}_k \times \mathcal{P}_l$

3. $\mathrm{Irr}(W_{D_n}) \overset{1:1}{\longleftrightarrow} \{\{\lambda,\mu\} \mid |\lambda|+|\mu|=n\} \cup \big\{\{\lambda,\pm\} \mid \lambda \vdash \frac{n}{2}\big\}$.

Thus using the assignments 2.1.6 and 2.1.8 and referring to the notations used in 2.1.1, we have:

$$\mathrm{Irr}\big(W_{A_{n-1}}\big) \overset{1:1}{\longleftrightarrow} \Phi_n, \tag{2.2.32}$$

$$\mathrm{Irr}(W_{B_n}) \overset{1:1}{\longleftrightarrow} \Phi_{n,1}, \tag{2.2.33}$$

$$\mathrm{Irr}(W_{D_n}) \overset{1:1}{\longleftrightarrow} \tilde{\Phi}_{n,0}. \tag{2.2.34}$$

We conclude this section by defining generic degrees for irreducible characters of Coxeter groups of type A_{n-1}, B_n and D_n. In order to do this, it will be convenient to introduce the following notation.

Let u be an indeterminate. For any integer $m \geq 1$, we set

$$[m](u) := \frac{u^m - 1}{u-1}.$$

Moreover, we set $[m](u)! := [1](u)[2](u)\cdots[m](u)$ and $[0](u) := 1$. Finally, let

$$\Delta(Z,u) := \prod_{\substack{k,l \in Z \\ k \gneq l}} \left(u^k - u^l\right)$$

for any β-set Z.

Definition 2.2.55. (Theorem 10.5.2, 10.5.3 in [GP00]). Let $X \in \Phi_n$ such that $|X| = b$. The *generic degree* of the corresponding irreducible character of

$W_{A_{n-1}}$ is given by

$$d_X(u) := \frac{(u-1)^n\,[n](u)!\Delta(X,u)}{u^{b(b-1)(b-2)/2}\left(\prod_{k\in X}(u-1)^k\,[k](u)!\right)}. \tag{2.2.35}$$

Let $\Lambda = (X,Y) \in \Phi_{n,1}$ such that $|X| = b+1$, $|Y| = b$. The *generic degree* of the corresponding irreducible character of W_{B_n} is given by

$$d_\Lambda(u) := \frac{u^{b(b+1)/2}\,(u-1)^n\,P_{B_n}(u)\Delta(X,u)\Delta(Y,u)\prod_{(k,l)\in X\times Y}\left(u^k+u^l\right)}{2^b u^{b(2b-1)(b+2)/3}\left(\prod_{k\in X}(u-1)^k\,P_{B_k}(u)\right)\left(\prod_{l\in Y}(u-1)^l\,P_{B_l}(u)\right)}. \tag{2.2.36}$$

Let $\Lambda' = (X,Y) \in \tilde{\Phi}_{n,0}$ such that $|X| = |Y| = b$. The *generic degree* of the corresponding irreducible character of W_{D_n} is given by

$$d_{\Lambda'}(u) := \frac{(u-1)^n\,P_{D_n}(u)\Delta(X,u)\Delta(Y,u)\prod_{(k,l)\in X\times Y}\left(u^k+u^l\right)}{2^c u^{b(b-1)(4b-5)/6}\left(\prod_{k\in X}\prod_{h=1}^{k}(u^{2h}-1)\right)\left(\prod_{l\in Y}\prod_{h=1}^{l}(u^{2h}-1)\right)} \tag{2.2.37}$$

where $c = b$ if $\Lambda' = (X, \pm)$ is a special symbol for some β-set X and $c = b-1$ otherwise.

Notation. If χ is an irreducible character of a Coxeter group W of type A_{n-1}, B_n or D_n for some $n \in \mathbb{N}$, sometimes we denote its generic degree by d_χ.

Remark 2.2.56. Let W a Coxeter group of type A_{n-1}, B_n or D_n for some $n \in \mathbb{N}$. If $\chi \in \mathrm{Irr}(W)$, it is true that $d_\chi(1) = \chi(1)$ ([CIK72, Theorem 5.7]).

Remark 2.2.57. Let χ be an irreducible character of a Coxeter group W of type A, B or D, d_χ its generic degree. Then $d_\chi = \frac{1}{c_\chi} f_\chi$, where $f_\chi \in \mathbb{Z}[u]$ is a monic polynomial and $c_\chi \in \mathbb{N}$, both depending on χ ([GP00, Corollary 9.3.6]).

Remark 2.2.58. As in the case of Poincaré polynomials, the generic degree can be defined for a general Coxeter group in such a way that it is multiplicative with respect to the direct product, that is, if $W = W_1 \times W_2$ and $\chi = \chi_1 \otimes \chi_2 \in \mathrm{Irr}(W)$ with $\chi_1 \in \mathrm{Irr}(W_1)$ and $\chi_2 \in \mathrm{Irr}(W_2)$, then $d_\chi = d_{\chi_1} \cdot d_{\chi_2}$.

We conclude this section by stating the following

Proposition 2.2.59. *Let W be a Coxeter group, P_W its Poincaré polynomial, $\chi \in \mathrm{Irr}(W)$ and d_χ the corresponding generic degree. Then*

$$\frac{P_W\left(u^{-1}\right)}{d_\chi(u^{-1})} = \frac{P_W(u)}{d_{\epsilon\chi}(u)} \tag{2.2.38}$$

where ϵ is the sign character of W as defined in Definition 2.2.46.

Proof. The statement is a direct consequence of [Ca85, Lemma 11.3.2]. □

2.2.5 Principal series representations of $\mathrm{Sp}(2n, \mathbb{F}_q)$

We now develop part of the theory of principal series representations for the finite symplectic group that we will need in the following. In particular, we show the results contained in [DL76] and [HK80] adapted to the case of $\mathrm{Sp}(2n, \mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite field of odd prime characteristic p, $q = p^m$ for some $m \in \mathbb{N}$.

Throughout this section, let $G = \mathrm{Sp}\left(2n, \overline{\mathbb{F}_q}\right)$, B, U and T again as in the example 2.2.22. In this setting, we also have the standard Frobenius map

$$\begin{aligned} F : G &\to G \\ (a_{ij}) &\mapsto \left(a_{ij}^q\right). \end{aligned}$$

This is a bijective homomorphism of algebraic groups and B, T and U are F-stable subgroups of G. If H is any F-stable algebraic group, let us denote by H^F the finite group of points in H fixed by F. Then

$$\begin{aligned} G^F &= \mathrm{Sp}(2n, \mathbb{F}_q), & B^F &= B_{2n}(\mathbb{F}_q) \cap \mathrm{Sp}(2n, \mathbb{F}_q), \\ T^F &= D_{2n}(\mathbb{F}_q) \cap \mathrm{Sp}(2n, \mathbb{F}_q), & U^F &= U_{2n}(\mathbb{F}_q) \cap \mathrm{Sp}(2n, \mathbb{F}_q). \end{aligned}$$

Consequently, $B^F = T^F \ltimes U^F$ with

$$T^F = \left\{ \mathrm{diag}\big(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}\big) \mid \lambda_1, \ldots, \lambda_n \in \mathbb{F}_q^\times \right\} \tag{2.2.39}$$

and the normalizer $N_G(T)$ is F-stable. Fix a group homomorphism

$$f : \mathbb{F}_q^\times \to \mathbb{C}^\times \tag{2.2.40}$$

such that if $\mathbb{F}_q^\times = \langle \gamma \rangle$, then $f(\gamma)$ generates a cyclic subgroup in $\mathbb{C}^\times$ of order $q-1$. If $\theta \in \widehat{T^F}$ and

$$t = \mathrm{diag}\big(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}\big)$$

is an element of T^F, then θ is a map of the following form

$$\begin{aligned} \theta : T^F &\to \mathbb{C}^\times \\ t &\mapsto \theta(t) := \prod_{i=1}^{n} f(\lambda_i)^{m_i}. \end{aligned} \tag{2.2.41}$$

with $m_i \in \mathbb{Z}_{q-1}$ for each $i = 1, \ldots, n$, so

$$\left(\widehat{T^F}, \cdot\right) \cong \left(\mathbb{Z}_{q-1}^n, +\right).$$

The Frobenius map F induces the identity map on the Weyl group $W = W_{B_n}$

of G.

Notation. From now on, we denote in the same way an element in $\mathbb{F}_q^\times$ and its image via the homomorphism f defined in 2.2.40.

Definition 2.2.60. Let $\theta \in \widehat{T^F}$. The corresponding *Deligne-Lusztig character* is given by

$$R_T^G(\theta) := \mathrm{Ind}_{B^F}^{G^F}\left(\widetilde{\theta}\right) \tag{2.2.42}$$

where $\widetilde{\theta} := \theta \circ p_1 \in \widehat{B^F}$ and $p_1 : B^F = T^F \ltimes U^F \twoheadrightarrow T^F$ is the natural projection onto T^F. The irreducible components of $R_T^G(\theta)$ are called the *principal series of* θ.

Notation. In the following, sometimes we may use the symbol $R_T^G(\theta)$ also to denote both the associated representation and the set of principal series of θ. So if χ is in the principal series of θ, we write $\chi \in R_T^G(\theta)$.

Now recall the following formula concerning the inner product between Deligne-Lusztig characters.

Proposition 2.2.61. ([DL76, Theorem 6.8]).*Let $\theta, \theta' \in \widehat{T^F}$. Then*

$$\left\langle R_T^G(\theta), R_T^G(\theta')\right\rangle_{G^F} = \frac{\left|\left\{n \in N_G(T)^F \mid \theta^n = \theta'\right\}\right|}{|T^F|} \tag{2.2.43}$$

where $\theta^n(t) := \theta(t^n)$ for every $t \in T^F$. Moreover, there are only two possibilities:

1. $\left\langle R_T^G(\theta), R_T^G(\theta')\right\rangle_{G^F} = 0$*;*

2. $R_T^G(\theta) = R_T^G(\theta')$*.*

Remark 2.2.62. The finite normalizer $N_G(T)^F$ acts on $\widehat{T^F}$ by conjugation, namely we have

$$\begin{aligned} \varphi : N_G(T)^F \times \widehat{T^F} &\to \widehat{T^F} \\ (n, \theta) &\mapsto \theta^n. \end{aligned} \tag{2.2.44}$$

As T^F acts trivially, this action induces an action $\overline{\varphi}$ of the Weyl group $W = N_G(T)^F / T^F$ on $\widehat{T^F}$.

Remark 2.2.63. Let $\theta, \theta' \in \widehat{T^F}$. Since $R_T^G(\theta)$ and $R_T^G(\theta')$ are characters of representations of G^F, we deduce from 2.2.43 that either the principal series of θ and θ' coincide or they are disjoint. So we have that $R_T^G(\theta) \cap R_T^G(\theta') \neq \emptyset$ if and only if there exists $w \in W$ such that $\theta^w = \theta'$.

Let us look more closely at the action $\overline{\varphi}$ induced by φ as in 2.2.44. Because of 2.2.41, we can describe the action of $W_n = S_n \ltimes \boldsymbol{\mu_2^n}$ on $\widehat{T^F} = \mathbb{Z}_{q-1}^n$ in the following way:

$$\begin{aligned} \overline{\varphi} : (S_n \ltimes \boldsymbol{\mu_2^n}) \times \mathbb{Z}_{q-1}^n &\to \mathbb{Z}_{q-1}^n \\ ((\sigma, (\varepsilon_1, \dots, \varepsilon_n)), (k_1, \dots, k_n)) &\mapsto \left(\varepsilon_1 k_{\sigma(1)}, \dots, \varepsilon_n k_{\sigma(n)}\right). \end{aligned} \tag{2.2.45}$$

By this description of the action of $\overline{\varphi}$, we deduce the following

Proposition 2.2.64. *Every W-orbit in $\widehat{T^F}$ is uniquely represented by a character of the form*

$$\theta \sim \left(\overbrace{k_1, \dots, k_1}^{\lambda_1}, \overbrace{k_2, \dots, k_2}^{\lambda_2}, \dots, \overbrace{k_l, \dots, k_l}^{\lambda_l}, \overbrace{0, \dots, 0}^{\alpha_1}, \overbrace{\frac{q-1}{2}, \dots, \frac{q-1}{2}}^{\alpha_\epsilon} \right) \tag{2.2.46}$$

where

1. $\lambda = (\lambda_1 \geq \dots \geq \lambda_l) \vdash c$ *for some natural number* $c \leq n$;

2. $c + \alpha_1 + \alpha_\epsilon = n$;

3. $k_i \in Q := \left\{1, \ldots, \frac{q-3}{2}\right\}$;

4. *for all* $i, j = 1, \ldots, l$, $k_i \neq k_j$ *and* $k_i < k_j$ *if* $\lambda_i = \lambda_j$.

Remark 2.2.65. Let $\theta \in \widehat{T^F}$ of the form 2.2.46 and S_θ the stabilizer of θ with respect to the action of W_n on $\widehat{T^F}$. Then it is easy to see that $S_\theta = S_{\lambda,\alpha_1,\alpha_\epsilon} := \left(\prod_{i=1}^{l} W_{A_{\lambda_i - 1}}\right) \times W_{B_{\alpha_1}} \times W_{B_{\alpha_\epsilon}}$, thus S_θ is a Coxeter group of type $A_{\lambda_1 - 1} \times \cdots \times A_{\lambda_l - 1} \times B_{\alpha_1} \times B_{\alpha_\epsilon}$.

Our next goal is to collect the principal series representations of G^F in families such that:

1. The number of families only depends on n.

2. Members of the same family have the same degree.

In order to do this, we need to recall some definitions and results that can be found in [HK80]. We refer to the notations used in the section 2.2.3.

Let $\theta \in \widehat{T^F}$ of the form 2.2.46 and $\alpha \in \Phi$.

Definition 2.2.66. Let $c_\alpha(\theta)$ be a natural number equal to 1 if $\theta|_{T_\alpha^F}$ is the trivial character of T_α^F and 0 otherwise. The *q-parameter* $q_\alpha(\theta)$ is defined as $q^{c_\alpha(\theta)}$.

Remark 2.2.67. Actually, we are giving an *ad hoc* definition of q-parameters for the sake of simplicity. For a more general definition of these parameters, we refer to [HK80, Lemma 2.6]. In [HK80, Section 4] it is also proved that this general definition is equivalent to our one.

Now define the set

$$\Gamma := \{\beta \in \Phi \mid q_\beta(\theta) \neq 1\} \tag{2.2.47}$$

and let W_{S_θ} be the group generated by the reflections corresponding to roots in Γ. Thus, Γ is the root system of the reflection group W_{S_θ} (for definitions and further information on root systems, see [Hu75, Appendix]). Let

$$D := \left\{w \in S_\theta \mid w(\alpha) \in \Phi^+,\, \forall \alpha \in \Gamma^+\right\}. \tag{2.2.48}$$

where Γ^+ is the set of the positive roots in Γ.

Proposition 2.2.68. [HK80, Lemma 2.9]. *D is a subgroup of S_θ, which normalizes W_{S_θ}, such that $S_\theta = D \ltimes W_{S_\theta}$.*

Proposition 2.2.69. *The group W_{S_θ} is equal to $\left(\prod_{i=i}^{l} W_{A_{\lambda_i - 1}}\right) \times W_{B_{\alpha_1}} \times W_{D_{\alpha_\epsilon}}$. In particular,*

$$S_\theta = \begin{cases} W_{S_\theta} & \text{if } \alpha_\epsilon = 0 \\ \boldsymbol{\mu_2} \ltimes W_{S_\theta} & \text{if } \alpha_\epsilon \neq 0. \end{cases} \tag{2.2.49}$$

Proof. From the definition of W_{S_θ}, it is sufficient to compute the set Γ defined as in 2.2.47. By Definition 2.2.66 of the parameters $q_\alpha(\theta)$ and the structure of the root system and of the root tori of G as described in 2.2.22, 2.2.26, 2.2.27 and 2.2.28, it is easy to see that $\Gamma = A \cup B \cup C$, where

$$\begin{aligned} A &:= \left\{\pm(\epsilon_i - \epsilon_{i+1}) \mid i \neq \lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_{i=1}^{l} \lambda_i, \sum_{i=1}^{l} \lambda_i + \mu_1\right\}, \\ B &:= \{\pm 2\epsilon_i \mid m+1 \leq i \leq m + \alpha_1\}, \\ C &:= \{\pm(\epsilon_i + \epsilon_{i+1}) \mid m+1 \leq i \leq n-1,\, i \neq m + \alpha_1\}. \end{aligned} \tag{2.2.50}$$

Thus the claim easily follows from the fact that W_{S_θ} is generated by the reflections corresponding to the roots in Γ. □

Remark 2.2.70. From the proof of the previous Proposition 2.2.69, we see that the set Γ only depends on the triple $(\lambda, \alpha_1, \alpha_\epsilon)$ associated to a character $\theta \in \widehat{T^F}$ of the form 2.2.46.

Let Σ be the set of simple roots of Γ consisting of roots that are positive in Φ.

Definition 2.2.71. The *generic algebra* $\mathcal{A}(u)$ is the algebra over $\mathbb{C}[u]$ with basis $\{a_w \mid w \in S_\theta\}$ such that, if $w \in S$, $d \in D$ and s is the reflection corresponding to the root $\beta \in \Sigma$, the following relations hold:

1. $a_d a_w = a_{dw}$, $a_w a_d = a_{wd}$.

2. $a_w a_s = a_{ws}$ if $w(\beta) \in \Gamma^+$.

3. $a_w a_s = u_\beta(\theta) a_{ws} + (u_\beta(\theta) - 1)\, a_w$ if $w(\beta) \in \Gamma^-$.

Let $K := \overline{\mathbb{C}(u)}$ and write $\mathcal{A}(u)^K = \mathcal{A}(u) \otimes_{\mathbb{C}[u]} K$. An algebra homomorphism $f : \mathbb{C}[u] \to \mathbb{C}$ makes $\mathbb{C}$ into a $(\mathbb{C}, \mathbb{C}[u])$-bimodule via $(a, p) \cdot c := acf(p)$.

Definition 2.2.72. If $f(u) = b$, the *specialization* $\mathcal{A}(b) := \mathcal{A}(u) \otimes_f \mathbb{C}$ is an algebra over $\mathbb{C}$ with basis $\{a_w \otimes 1 \mid w \in S_\theta\}$ whose members satisfy relations 1, 2 and 3 after replacing u with b.

Before stating the next theorem, recall the definitions of separable algebra and numerical invariants.

An algebra $\mathcal{A}$ over K is said to be *semisimple* if it is isomorphic as an algebra to a finite direct product of matrix algebras over K. If $\mathcal{A} \cong \prod_{i=1}^{n} \mathfrak{gl}(d_i, K)$

is semisimple, the numbers $d_1, \ldots, d_n, n$ are uniquely determined and are called the *numerical invariants* of $\mathcal{A}$. An algebra $\mathcal{A}$ over K is said to be *separable* if $\mathcal{A} \otimes_K L$ is semisimple for every field extension L of K.

Theorem 2.2.73. (Tits, [CIK72, Theorem 1.11]). *$\mathcal{A}(u)^K$ is a separable K-algebra and for each $b \in \mathbb{C}$ such that $\mathcal{A}(b)$ is separable, the algebras $\mathcal{A}(u)^K$ and $\mathcal{A}(b)$ have the same numerical invariants.*

Corollary 2.2.74. $\mathrm{End}_{G^F}\left(R_T^G(\theta)\right) \cong \mathbb{C}S_\theta$ *as $\mathbb{C}$-algebras.*

Proof. By [CIK72, Theorem 2.17, 2.18], we have that

$$\mathcal{A}(q) \cong \mathrm{End}_{G^F}\left(R_T^G(\theta)\right),$$
$$\mathcal{A}(1) \cong \mathbb{C}S_\theta$$

and since both $\mathrm{End}_{G^F}\left(R_T^G(\theta)\right)$ and $\mathbb{C}S_\theta$ are semisimple, they have the same numerical invariants by Theorem 2.2.73 and so are isomorphic. □

From Corollary 2.2.74, we deduce the following fundamental result.

Corollary 2.2.75. *There exists a bijecitve correspondence between the set $R_T^G(\theta)$ of principal series of θ and the set $\mathrm{Irr}(S_\theta)$ of the irreducible characters of S_θ.*

The correspondence established in Corollary 2.2.75 can be stated more precisely by the following

Proposition 2.2.76. [HK80, Lemma 3.4]). *Let χ be an irreducible character of $\mathcal{A}(u)^K$. Then for all $w \in S_\theta$, $\chi(a_w)$ is in the integral closure of $\mathbb{C}[u]$ in K. Let $f : \mathbb{C}[u] \to \mathbb{C}$ be a homomorphism such that $f(u) = b$ and $\mathcal{A}(b)$ is separable, and let f^* be an extension of f to the integral closure of $\mathbb{C}[u]$. Then the linear* *map* $\chi_f : \mathcal{A}(b) \to \mathbb{C}$ *defined by* $\chi_f(a_w \otimes 1) := f^*(\chi(a_w))$*, for all* $w \in S_\theta$*, is an irreducible character of* $\mathcal{A}(b)$*. For a fixed extension* f^* *of* f*, the map* $\chi \longmapsto \chi_f$ *is a bijection between the irreducible characters of* $\mathcal{A}(u)^K$ *and those of* $\mathcal{A}(b)$*.*

Remark 2.2.77. If $\theta, \theta' \in \widehat{T^F}$ are of the form 2.2.46 with the same associated triple $(\lambda, \alpha_1, \alpha_\epsilon)$, then by Proposition 2.2.69 and Definition 2.2.71, we obtain the same generic algebra $\mathcal{A}(u)$ starting from θ or θ'. Thus, by Proposition 2.2.76, we have the same correspondence between the irreducible characters of $\mathcal{A}(u)^K$ and those of $S_\theta = S_{\theta'} = S_{\lambda,\alpha_1,\alpha_\epsilon}$.

From [McG82, Theorem A], we deduce this important result on the multiplicities of the principal series representations.

Proposition 2.2.78. *If* $\chi \in R_T^G(\theta)$ *corresponding to* $\beta \in \mathrm{Irr}(S_\theta)$*, then*

$$\left\langle \chi, R_T^G(\theta) \right\rangle_{G^F} = \beta(1). \tag{2.2.51}$$

Let $\overline{\beta} \in \mathrm{Irr}\Big(\mathcal{A}(u)^K\Big)$, β the corresponding character of S_θ given by Proposition 2.2.76. Define

$$D_\beta(u) := \frac{\overline{\beta}(1) P_W(u)}{\sum\limits_{w \in S_\theta} u_w(\theta)^{-1} \overline{\beta}(a_{w^{-1}}) \overline{\beta}(a_w)} \tag{2.2.52}$$

where $u_w(\theta) := \prod\limits_{\substack{\alpha \in \Gamma^+ \\ w(\alpha) \in \Gamma^-}} u_\alpha(\theta)$ and P_W is the Poincaré polynomial of W. Then we have

Proposition 2.2.79. [HK80, pag. 567, (3.5)]. *If* $\chi \in R_T^G(\theta)$ *is the principal series corresponding to* $\beta \in \mathrm{Irr}(S_\theta)$*, then* $D_\beta(q) = \chi(1)$*.*

Let $\mathcal{B}(u)$ be the subalgebra of $\mathcal{A}(u)$ generated by $\{a_w \mid w \in W_{S_\theta}\}$. Then $\mathcal{B}(u)$ is the generic algebra corresponding to the Coxeter group W_{S_θ} and by

Proposition 2.2.76, there is a bijection between $\mathrm{Irr}\Big(\mathcal{B}(u)^K\Big)$ and $\mathrm{Irr}(W_{S_\theta})$. The group D, defined in 2.2.48, acts as a group of automorphisms of $\mathcal{B}(u)^K$ via $a_w \mapsto a_{dwd^{-1}}$ for $d \in D$, $w \in W_{S_\theta}$. Thus, for each $d \in D$, if $\overline{\varphi} \in \mathrm{Irr}\Big(\mathcal{B}(u)^K\Big)$, the character $\overline{\varphi}^d$ of $\mathcal{B}(u)^K$ determined by $\overline{\varphi}^d(a_w) := \overline{\varphi}(a_{dwd^{-1}})$ is irreducible too.

Proposition 2.2.80. ([HK80, Theorem 3.13]). *Let $\overline{\beta} \in \mathrm{Irr}\Big(\mathcal{A}(u)^K\Big)$, β the corresponding character in S_θ, $\overline{\varphi}$ an irreducible component of $\mathrm{Res}^{\mathcal{A}(u)^K}_{\mathcal{B}(u)^K}(\overline{\beta})$, φ the corresponding character of W_{S_θ}. If $C := \{d \in D \mid \varphi^d = \varphi\}$ and d_φ is the generic degree of φ, then*

$$D_\beta(u) = \frac{P_W(u)}{P_{W_{S_\theta}}(u)\,|C|} d_\varphi(u). \tag{2.2.53}$$

Remark 2.2.81. By Proposition 2.2.69, $W_{S_\theta} = \left(\prod_{i=i}^{l} W_{A_{\lambda_i - 1}}\right) \times W_{B_{\alpha_1}} \times W_{D_{\alpha_\epsilon}}$, so if $\overline{\beta}$, $\overline{\varphi}$, β and φ are as in Proposition 2.2.80, then $\varphi = \left(\bigotimes_{i=1}^{l} \varphi_i\right) \otimes \varphi_{\alpha_1} \otimes \varphi_{\alpha_\epsilon}$, with $\varphi_i \in \mathrm{Irr}\Big(W_{A_{\lambda_i - 1}}\Big)$ for $i = 1, \ldots, l$, $\varphi_{\alpha_1} \in \mathrm{Irr}\big(W_{B_{\alpha_1}}\big)$ and $\varphi_{\alpha_\epsilon} \in \mathrm{Irr}\big(W_{D_{\alpha_\epsilon}}\big)$. Remember that $W = W_{B_n}$, we can rewrite 2.2.53 as follows:

$$D_\beta(u) = \frac{P_{B_n}(u)}{|C|} \left(\prod_{i=1}^{l} \frac{d_{\varphi_i}(u)}{P_{A_{\lambda_i - 1}}(u)}\right) \frac{d_{\varphi_{\alpha_1}}(u)}{P_{B_{\alpha_1}}(u)} \frac{d_{\varphi_{\alpha_\epsilon}}(u)}{P_{D_{\alpha_\epsilon}}(u)} \tag{2.2.54}$$

where the d_{φ_i}'s, $d_{\varphi_{\alpha_1}}$ and $d_{\varphi_{\alpha_\epsilon}}$ are generic degrees as defined in 2.2.35, 2.2.36 and 2.2.37 respectively.

Remark 2.2.82. One can show that, if $\overline{\beta}$, $\overline{\varphi}$, β and φ are as in Proposition 2.2.80, then $\mathrm{Res}^{\mathcal{A}(u)^K}_{\mathcal{B}(u)^K}(\overline{\beta}) = \frac{1}{|C|} \sum_{d \in D} \overline{\varphi}^d$ ([HK80, proof of Theorem 3.13]). Thus, by Proposition 2.2.76, we have that

$$\operatorname{Res}^{S_\theta}_{W_{S_\theta}}(\beta) = \frac{1}{|C|}\sum_{d\in D}\varphi^d. \tag{2.2.55}$$

In particular,

$$\beta(1) = \frac{|D|}{|C|}\varphi(1). \tag{2.2.56}$$

Therefore using Proposition 2.2.80 together with Remark 2.2.54 and 2.2.56, we obtain from 2.2.54

$$D_\beta(1) = [W : S_\theta]\,\beta(1). \tag{2.2.57}$$

Let $\chi \in R_T^G(\theta)$ with θ of the form 2.2.46.

Definition 2.2.83. If β is the irreducible character of S_θ corresponding to χ, define the 4-tuple

$$\tau := (\lambda, \alpha_1, \alpha_\epsilon, \beta) \tag{2.2.58}$$

as the *type* of χ. If $\tau = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$ and ε is the sign character of $S_\theta = S_{\lambda,\alpha_1,\alpha_\epsilon}$ as defined in 2.2.46, define the *type dual to* τ as

$$\tau' := (\lambda, \alpha_1, \alpha_\epsilon, \varepsilon\beta) \tag{2.2.59}$$

Notation. We write $\tau(\chi)$ for the type of the principal series χ and χ_τ to denote a principal series of a fixed type τ.

Proposition 2.2.84. *If $\chi \in R_T^G(\theta)$ with θ of the form 2.2.46, then $\chi(1)$ only depends on $\tau(\chi) = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$.*

Proof. By Proposition 2.2.79 and 2.2.80, it is sufficient to prove that W_{S_θ} and the corresponding character $\overline{\beta} \in \mathcal{A}(u)^K$ only depend on τ. But the generating set Γ of W_{S_θ} only depend on the triple $(\lambda, \alpha_1, \alpha_\epsilon)$ for the proof of Proposition 2.2.69, hence W_{S_θ} too. Moreover, because of this fact, it follows from Remark 2.2.77 that the character $\overline{\beta}$ is invariant with respect to the type τ, so we are done. □

Remark 2.2.85. Let χ_τ be a principal series of $\theta \in \widehat{T^F}$ such that $\tau = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$ with $\beta \in S_{\lambda,\alpha_1,\alpha_\epsilon} = S_\theta$.

If $\alpha_\epsilon = 0$, then $S_\theta = W_{S_\theta}$ by 2.2.49. Thus, D is the trivial group and $\beta = \varphi$ where φ is as in Remark 2.2.81. Thus, specializing 2.2.54 at $u = q$, we have that

$$\chi_\tau(1) = P_{B_n}(q) \left(\prod_{i=1}^{l} \frac{d_{\varphi_i}(q)}{P_{A_{\lambda_i - 1}}(q)} \right) \frac{d_{\varphi_{\alpha_1}}(q)}{P_{B_{\alpha_1}}(q)} \frac{d_{\varphi_{\alpha_\epsilon}}(q)}{P_{D_{\alpha_\epsilon}}(q)} \tag{2.2.60}$$

If $\alpha_\epsilon \neq 0$, then again by 2.2.49, $S_\theta = \boldsymbol{\mu_2} \ltimes W_{S_\theta}$, so $D = \boldsymbol{\mu_2}$ and we have to distinguish two different cases.

If $\mathrm{Res}^{S_\theta}_{W_{S_\theta}}(\beta) = \varphi$, then $|C| = 2$ and we obtain that

$$\chi_\tau(1) = \frac{P_{B_n}(q)}{2} \left(\prod_{i=1}^{l} \frac{d_{\varphi_i}(q)}{P_{A_{\lambda_i - 1}}(q)} \right) \frac{d_{\varphi_{\alpha_1}}(q)}{P_{B_{\alpha_1}}(q)} \frac{d_{\varphi_{\alpha_\epsilon}}(q)}{P_{D_{\alpha_\epsilon}}(q)} \tag{2.2.61}$$

otherwise, we have again the formula 2.2.60 for the degree of χ_τ.

Remark 2.2.86. Since Poincaré polynomials of Coxeter groups are always monic, as one can see from formulas 2.2.31, by 2.2.60, 2.2.61 and Remark 2.2.57 we have that $\chi_\tau(1) = \frac{1}{c_{\chi_\tau}} f_{\chi_\tau}$ with $f_{\chi_\tau} \in \mathbb{Z}[q]$ monic and $c_{\chi_\tau} \in \mathbb{N}$.

Remark 2.2.87. For odd $q \geq 3$, since $G^F = \mathrm{Sp}(2n, \mathbb{F}_q)$ is a perfect group, unless $n = 1$ and $q = 3$, by previous Remark 2.2.86 and by looking at the table 4.1 below, we have that $\chi_\tau(1)$ is independent from q if and only if $\chi_\tau = 1_{\mathrm{Sp}(2n,\mathbb{F}_q)}$.

Finally, we have constructed the decomposition of the set $\mathcal{R}$ of principal series representations satisfying conditions 1 and 2 in pag. 44. Namely, we have

$$\mathcal{R} = \coprod_{\tau} \mathcal{R}_{\tau} \tag{2.2.62}$$

where

$$\mathcal{R}_{\tau} := \{\chi \in \mathcal{R} \mid \tau(\chi) = \tau\}. \tag{2.2.63}$$

Remark 2.2.88. Let $\theta_1, \theta_2 \in \widehat{T^F}$ of the same form 2.2.46. If χ_1 and χ_2 are irreducible constituents of $R_T^G(\theta_1)$ and $R_T^G(\theta_2)$ respectively such that $\tau(\chi_1) = \tau(\chi_2) = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$, then by Proposition 2.2.78 we have that $\langle \chi_1, R_T^G(\theta_1) \rangle_{G^F} = \langle \chi_2, R_T^G(\theta_2) \rangle_{G^F} = \beta(1)$.

Now let us give the following

Definition 2.2.89. For any possible type τ, define

$$H_{\tau}(q) := \frac{|\mathrm{Sp}(2n, \mathbb{F}_q)|}{\chi_{\tau}(1)} \tag{2.2.64}$$

Remark 2.2.90. Since $|\mathrm{Sp}(2n, \mathbb{F}_q)|$ is a monic integral polynomial in q, by Remark 2.2.86, we get that $H_{\tau}(q) \in \mathbb{Z}[q]$.

Remark 2.2.91. If $\tau = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$, combining Proposition 2.2.79 and 2.2.57 we have that $\gcd(\chi_{\tau}(1), q-1) = 1$, considering $\chi_{\tau}(1)$ as an element of $\mathbb{Z}[q]$. Thus, since

$$|\mathrm{Sp}(2n, \mathbb{F}_q)| = q^{n^2} (q-1)^n P_{B_n}(q) \tag{2.2.65}$$

we obtain that $\frac{1}{(q-1)^n} H_{\tau}(q) \in \mathbb{Z}[q]$.

Proposition 2.2.92. *For any possible type τ, we have*

$$H_{\tau}(q^{-1}) = \frac{(-1)^n}{q^{n(2n+1)}} H_{\tau'}(q). \tag{2.2.66}$$

Proof. Let $\theta \in \widehat{T^F}$ of the form 2.2.46, $\tau = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$ with $\beta \in S_\theta$. Then the dual type τ' of τ is given by $(\lambda, \alpha_1, \alpha_\epsilon, \epsilon\beta)$ where ϵ is the sign character of S_θ. If $\varphi \in \mathrm{Irr}(W_{S_\theta})$ as in Proposition 2.2.80, then the corresponding character of W_{S_θ} to $\epsilon\beta$ is $\epsilon\varphi$ (abusing of notation, we denote the sign characters of S_θ and W_{S_θ} by the same symbol ϵ). Using formula 2.2.53 and 2.2.65, we have

$$H_\tau(q) = \frac{q^{n^2} (q-1)^n P_{W_{S_\theta}}(q) \, |C|}{d_\varphi(q)}.$$

Thus we obtain that

$$\begin{aligned} H_\tau(q^{-1}) &= \frac{q^{-n^2} \left(q^{-1}-1\right)^n P_{W_{S_\theta}}\left(q^{-1}\right) |C|}{d_\varphi(q^{-1})} \\ &\overset{2.2.38}{=} \frac{(-1)^n}{q^{n(2n+1)}} \frac{q^{n^2} (q-1)^n P_{W_{S_\theta}}(q) \, |C|}{d_{\epsilon\varphi}(q)} \\ &= \frac{(-1)^n}{q^{n(2n+1)}} H_{\tau'}(q) \end{aligned}$$

and so we are done. □

Chapter 3

Geometry of $\mathcal{M}_n^\xi$

Throughout this chapter, we consider a presentation of the symplectic group different from 2.2.16 to make computations below simpler. So, if $\mathbb{K}$ is an algebraically closed field, n a positive integer, then

$$G = \mathrm{Sp}(2n, \mathbb{K}) := \left\{ A \in \mathfrak{gl}(2n, \mathbb{K}) \mid A^t J A = A \right\}$$

with $J := \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. In particular, the maximal torus T of the diagonal symplectic matrices is given by

$$T = \left\{ \mathrm{diag}\left(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}\right) \mid \lambda_i \in \mathbb{K}^\star,\ i = 1, \ldots, n \right\}$$

3.1 Parabolic Sp_{2n}-character varieties

3.1.1 Basic definitions and facts

Let $g > 0$, $n > 0$ be integers. Let $\mathbb{K}$ be an algebraically closed field with $\mathrm{char}\,(\mathbb{K}) \neq 2$, possessing a primitive m-th root of unity φ, for which there exist natural numbers $m_1, \ldots, m_n$ such that $\varphi^{m_1}, \ldots, \varphi^{m_n}$ satisfy the following non-equalities, for every disjoint sets of indices J and L, not simultaneously empty, and every index i in $[n]$:

$$\begin{gathered} \prod_{j\in J}\varphi^{m_j} \neq \prod_{l\in L}\varphi^{m_l}, \\ \varphi^{2m_i} \neq 1. \end{gathered} \tag{3.1.1}$$

Remark 3.1.1. Specializing 3.1.1 for $J = \{j\}$ and $L = \{l\}$ or $J = \{j, l\}$ and $L = \emptyset$, we have that $\varphi^{m_j} \neq \varphi^{m_l}$ and $\varphi^{m_j} \neq \varphi^{-m_l}$ respectively, so $\varphi^{m_1}, \ldots, \varphi^{m_n}, \varphi^{-m_1}, \ldots, \varphi^{-m_n}$ have to be all different. In particular, $m > n$.

Remark 3.1.2. It is evident from the definition of the conditions 3.1.1 that the elements of any subset of $\{\varphi^{m_1}, \ldots, \varphi^{m_n}\}$ satisfy 3.1.1 too.

Example 3.1.3. Let φ be a primitive $(2^n + 1)$-th root of unity, then the elements $\varphi, \varphi^2, \ldots, \varphi^{2^{n-1}}$ satisfy 3.1.1. In fact, let J, L two disjoint sets in $[n]$ not simultaneously empty. It is evident that the second inequality in 3.1.1 is trivially satisfied. Define k as the index such that $2^{k-1} = \max\limits_{s\in J\cap L}\left\{2^{s-1}\right\}$ and suppose that $k \in J$. Then

$$\sum_{j\in J} 2^{j-1} \geq 2^{k-1} > \sum_{l\in L} 2^{l-1}. \tag{3.1.2}$$

Moreover,

$$\sum_{j\in J} 2^{j-1} \leq \sum_{j=1}^{n} 2^{j-1} = 2^n - 1 < 2^n + 1 \tag{3.1.3}$$

thus the second inequality in 3.1.1 must be satisfied, otherwise, because of 3.1.3, $\sum\limits_{j\in J} 2^{j-1}$ has to be equal to $\sum\limits_{l\in L} 2^{l-1}$, contradicting 3.1.2.

Definition 3.1.4. A diagonal symplectic matrix

$$\xi = \operatorname{diag}\left(\varphi^{m_1}, \ldots, \varphi^{m_n}, \varphi^{-m_1}, \ldots, \varphi^{-m_n}\right) \tag{3.1.4}$$

such that $\varphi^{m_1}, \ldots, \varphi^{m_n}$ satisfy conditions 3.1.1 is said to be a *generic element*.

Remark 3.1.5. By Definition 3.1.4 and Remark 3.1.1, it is easy to see that in particular, ξ is a regular diagonal matrix of finite order.

Consider the following algebraic variety over $\mathbb{K}$:

$$\mathcal{U}_n^\xi := \left\{(A_1, B_1, \ldots, A_g, B_g) \in \operatorname{Sp}(2n, \mathbb{K})^{2g} \mid \prod_{i=1}^{g} [A_i : B_i] = \xi\right\} = \mu^{-1}(\xi) \tag{3.1.5}$$

where $\mu : \operatorname{Sp}(2n, \mathbb{K})^{2g} \to \operatorname{Sp}(2n, \mathbb{K})$ is given by

$$\mu\left(A_1, B_1, \ldots, A_g, B_g\right) := \prod_{i=1}^{g} [A_i : B_i] \tag{3.1.6}$$

and ξ is a generic element according to Definition 3.1.4. If $n = 0$, we will assume that $\mathcal{U}_n^\xi = \{\star\}$. By Remark 3.1.5, the centralizer of ξ in $\operatorname{Sp}(2n, \mathbb{K})$ is the maximal torus T and acts by pointwise conjugation on $\mathcal{U}_n^\xi$:

$$\begin{aligned} \sigma : T \times \mathcal{U}_n^\xi &\to \mathcal{U}_n^\xi \\ (Z, (A_1, B_1, \ldots, A_g, B_g)) &\mapsto \left(A_1^Z, B_1^Z, \ldots, A_g^Z, B_g^Z\right). \end{aligned} \tag{3.1.7}$$

As the center of $Z(G) = \{\pm I_{2n}\}$ of G acts trivially, this action induces an action

$$\bar{\sigma} : T/Z(G) \times \mathcal{U}_n^\xi \to \mathcal{U}_n^\xi.$$

Notation. Let $X \in \mathcal{U}_n^\xi$, $Z \in T$. We write X^Z instead of $\sigma(Z, X)$ or $\overline{\sigma}(Z, X)$.

Moreover, we denote by T_X the stabilizer $\mathrm{Stab}_\sigma(X)$.

Consider the subgroup of the involution matrices in T. This is given by the following set

$$\{\mathrm{diag}(\epsilon_1, \dots, \epsilon_n, \epsilon_1, \dots, \epsilon_n) \mid \epsilon_i \in \{\pm 1\}, \, i = 1, \dots, n\}.$$

Since it is isomorphic to $\boldsymbol{\mu}_2^n$, we denote it in the same way. In the following, it will be clear from the context which group we refer to by this symbol.

Proposition 3.1.6. *Let* $X = (A_1, B_1, \dots, A_g, B_g)$ *be an element of* $\mathcal{U}_n^\xi$. *Then* $T_X \subseteq \boldsymbol{\mu}_2^n$.

Proof. Let Z be an element of T_X. Then

$$\begin{aligned} ZA_i &= A_i Z, \\ ZB_i &= B_i Z \end{aligned} \tag{3.1.8}$$

for every $i = 1, \dots, g$. If $Z \notin \boldsymbol{\mu}_2^n$, then Z has an eigenvalue α different from ± 1. Permute the eigenvalues of Z in order to collect them in groups such that all the elements in the same group are equal. By a further permutation, we can assume that the first group of eigenvalues of Z is made by the α's. This is equivalent to the action of a permutation matrix π by conjugation on Z. It follows from 3.1.8 that

$$\begin{aligned} Z^\pi A_i^\pi &= A_i^\pi Z^\pi, \\ Z^\pi B_i^\pi &= B_i^\pi Z^\pi \end{aligned} \tag{3.1.9}$$

for all $i = 1, \dots, g$, and

$$\prod_{i=1}^{g} [A_i^\pi : B_i^\pi] = \prod_{i=1}^{g} [A_i : B_i]^\pi = \xi^\pi. \tag{3.1.10}$$

Now, by 3.1.9, we have that

$$A_i^\pi = \operatorname{diag}\Big(A_i^1, \dots, A_i^k\Big)$$
$$B_i^\pi = \operatorname{diag}\Big(B_i^1, \dots, B_i^k\Big)$$

for every $i = 1, \dots, g$, where k is the number of different eigenvalues of Z and the A_i^h's and B_i^h are square matrices whose sizes are equal to the multiplicity of the h-th eigenvalue of Z, $h = 1, \dots, k$. From 3.1.10, writing $\xi^\pi = \operatorname{diag}\big(D^1, \dots, D^k\big)$, it follows that

$$\operatorname{diag}\left(\prod_{i=1}^{g} \left[A_i^1 : B_i^1\right], \dots, \prod_{i=1}^{g} \left[A_i^k : B_i^k\right]\right) = \operatorname{diag}\Big(D^1, \dots, D^k\Big).$$

As the determinant of a commutator is 1, the determinant of D^h's has to be 1 for every $h = 1, \dots, k$. In particular, $\det\big(D^1\big) = 1$. But since $\alpha \neq \alpha^{-1}$, there is a φ^{m_j}, for some j, that is an eigenvalue of D^1, but not φ^{-m_j}, so $\det\big(D^1\big)$ cannot be equal to 1, because $\varphi^{m_1}, \dots, \varphi^{m_n}$ satisfy the inequalities 3.1.1, and this is a contradiction. □

Definition 3.1.7. Fix a subgroup H of $\boldsymbol{\mu}_2^n$ containing $Z(G)$. Define the following subsets of $\mathcal{U}_n^\xi$:

$$\widetilde{\mathcal{U}}_{n,H}^\xi := \Big\{X \in \mathcal{U}_n^\xi \mid H = T_X\Big\} \tag{3.1.11}$$

$$\mathcal{U}_{n,H}^\xi := \Big\{X \in \mathcal{U}_n^\xi \mid H \subseteq T_X\Big\}. \tag{3.1.12}$$

Remark 3.1.8. It is evident from the previous Definition 3.1.7 that $\widetilde{\mathcal{U}}_{n,H}^\xi$ is an open subset of the closed affine variety $\mathcal{U}_{n,H}^\xi$. In particular,

$$\left\{\widetilde{\mathcal{U}}_{n,H}^\xi\right\}_{Z(G)\leq H\leq \boldsymbol{\mu}_{\mathbf{2}}^n} \tag{3.1.13}$$

is a stratification of $\mathcal{U}_n^\xi$. Moreover, $\mathcal{U}_{n,Z(G)}^\xi = \mathcal{U}_n^\xi$, so $\widetilde{\mathcal{U}}_{n,Z(G)}^\xi$ is an open subset of $\mathcal{U}_n^\xi$

Proposition 3.1.9. *$\widetilde{\mathcal{U}}_{n,H}^\xi$ and $\mathcal{U}_{n,H}^\xi$ are stable under the action σ of T.*

Proof. Let $X = (A_1, B_1, \dots, A_g, B_g)$ be an element of $\mathcal{U}_{n,H}^\xi$ and $Z \in H$. Then

$$A_i Z = Z A_i,$$
$$B_i Z = Z B_i$$

for all $i = 1, \dots, g$. It follows that, if $\omega \in T$, $X \in \mathcal{U}_{n,H}^\xi$ if and only if $X^\omega \in \mathcal{U}_{n,H^\omega}^\xi$. But since T is abelian, $H^\omega = H$ so $\mathcal{U}_{n,H}^\xi$ is T-stable. The proof for $\widetilde{\mathcal{U}}_{n,H}^\xi$ is completely analogous. □

Remark 3.1.10. Let Z be an element of $\boldsymbol{\mu}_{\mathbf{2}}^n$. Define

$$\mathcal{U}_Z := \left\{X \in \mathcal{U}_n^\xi \mid X^Z = X\right\}.$$

Then, if $Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$, we have that

$$\widetilde{\mathcal{U}}_{n,H}^\xi = \mathcal{U}_{n,H}^\xi \cap \left(\bigcap_{Z\in \boldsymbol{\mu}_{\mathbf{2}}^n \setminus H} \mathcal{U}_Z^\complement\right) \tag{3.1.14}$$

where $\mathcal{U}_Z^\complement := \mathcal{U}_n^\xi \setminus \mathcal{U}_Z$. If $X = (A_1, B_1, \dots, A_g, B_g) \in \mathcal{U}_n^\xi$, for $h, k = 1, \dots, 2n$

and $i = 1, \dots, g$, define the following subsets

$$\begin{aligned} \mathcal{A}_{i,Z} &:= \left\{ X \in \mathcal{U}_n^\xi \mid A_i^Z \neq A_i \right\} \\ \mathcal{B}_{i,Z} &:= \left\{ X \in \mathcal{U}_n^\xi \mid B_i^Z \neq B_i \right\} \end{aligned} \tag{3.1.15}$$

$$\begin{aligned} \mathcal{A}_{i,Z}^{h,k} &:= \left\{ X \in \mathcal{U}_n^\xi \mid \left(A_i^Z\right)_{h,k} \neq (A_i)_{h,k} \right\} \\ \mathcal{B}_{i,Z}^{h,k} &:= \left\{ X \in \mathcal{U}_n^\xi \mid \left(B_i^Z\right)_{h,k} \neq (B_i)_{h,k} \right\}. \end{aligned} \tag{3.1.16}$$

Then

$$\begin{aligned} \mathcal{A}_{i,Z} &= \bigcup_{h,k=1}^{2n} \mathcal{A}_{i,Z}^{h,k} \\ \mathcal{B}_{i,Z} &= \bigcup_{h,k=1}^{2n} \mathcal{B}_{i,Z}^{h,k} \\ \mathcal{U}_Z^\complement &= \bigcup_{i=1}^{g} \left(\mathcal{A}_{i,Z} \cup \mathcal{B}_{i,Z}\right) \end{aligned} \tag{3.1.17}$$

Thus if $m = |\boldsymbol{\mu}_\mathbf{2}^n \setminus H|$, plugging 3.1.17 in 3.1.14, we obtain that

$$\widetilde{\mathcal{U}}_{n,H}^\xi = \bigcup_{\substack{h_1,\dots,h_m; \\ k_1,\dots,k_m=1 \\ i_1,\dots,i_m=1 \\ s=0}}^{\substack{m \\ g \\ 2n}} \mathcal{U}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m} \tag{3.1.18}$$

where

$$\mathcal{U}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m} := \left(\bigcap_{\substack{h \in \{h_1,\dots,h_s\} \\ k \in \{k_1,\dots,k_s\} \\ i \in \{i_1,\dots,i_s\} \\ Z \in \boldsymbol{\mu}_\mathbf{2}^n \setminus H}} \mathcal{A}_{i,Z}^{h,k} \right) \cap \left(\bigcap_{\substack{h \in \{h_{s+1},\dots,h_m\} \\ k \in \{k_{s+1},\dots,k_m\} \\ i \in \{i_{s+1},\dots,i_m\} \\ Z \in \boldsymbol{\mu}_\mathbf{2}^n \setminus H}} \mathcal{B}_{i,Z}^{h,k} \right). \tag{3.1.19}$$

So we have constructed the finite open covering

$$\left\{\mathcal{U}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m}\right\} \tag{3.1.20}$$

of $\widetilde{\mathcal{U}}_{n,H}^\xi$. It is easy to check that this open covering is made by affine T-stable subsets.

Definition 3.1.11. A *parabolic* $\mathrm{Sp}(2n,\mathbb{K})$*-character variety* of a closed Riemann surface of genus g is the categorical quotient

$$\mathcal{M}_n^\xi := \mathcal{U}_n^\xi//T = \mathrm{Spec}\left(\mathbb{K}\left[\mathcal{U}_n^\xi\right]^T\right). \tag{3.1.21}$$

More generally, define the categorical quotient

$$\mathcal{M}_{n,H}^\xi := \mathcal{U}_{n,H}^\xi//T = \mathrm{Spec}\left(\mathbb{K}\left[\mathcal{U}_{n,H}^\xi\right]^T\right). \tag{3.1.22}$$

Remark 3.1.12. Since H acts trivially on $\mathcal{U}_{n,H}^\xi$, we can define $\mathcal{M}_{n,H}^\xi$ as the categorical quotient $\mathcal{U}_{n,H}^\xi//\left(T/H\right)$.

Proposition 3.1.13. *$\mathcal{M}_{n,H}^\xi$ are geometric quotients for every $Z(G) \leq H \leq \boldsymbol{\mu}_\mathbf{2}^n$.*

Proof. Since $\mathcal{M}_{n,H}^\xi$ is a categorical quotient of an affine variety by the action of an affine reductive algebraic group, it is a good quotient (for a definition of a good quotient see [Ho12, Definition 2.36]), so by [Ho12, Corollary 2.39 ii)], it is sufficient to prove that all the orbits are closed. By 3.1.6, for every $X \in \mathcal{U}_{n,H}^\xi$, $\dim(T_X) = 0$. It follows, denoting the orbit of X by TX, that $\dim(TX) = \dim(T)$ from the Orbit-Stabiliser Theorem.

Now, suppose that there exists a non closed orbit. Then, by [Hu75, Proposition in 8.3], its boundary is not empty and it is a union of orbits of strictly smaller dimension. But this contradicts the fact that all the orbits have the same dimension. □

By Proposition 3.1.9, together with the properties of geometric quotients, we can give the following

Definition 3.1.14. For every $Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$, define the geometric quotient

$$\widetilde{\mathcal{M}}_{n,H}^\xi := \widetilde{\mathcal{U}}_{n,H}^\xi / T.$$

Remark 3.1.15. Since $\mathcal{M}_n$ is a geometric quotient because of Proposition 3.1.13, it has the quotient topology, hence, by Proposition 3.1.9 and Remark 3.1.8, $\mathcal{M}_{n,H}^\xi$ is a closed affine variety and $\widetilde{\mathcal{M}}_{n,H}^\xi$ is an open subset of it. In particular,

$$\left\{ \widetilde{\mathcal{M}}_{n,H}^\xi \right\}_{Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n} \tag{3.1.23}$$

is a stratification of $\mathcal{M}_n^g$ and $\widetilde{\mathcal{M}}_{n,Z(G)}^\xi$ is an open subset of the character variety $\mathcal{M}_n^\xi$.

Remark 3.1.16. As in Remark 3.1.12, we can realize $\widetilde{\mathcal{M}}_{n,H}^\xi$ as the geometric quotient of $\widetilde{\mathcal{U}}_{n,H}^\xi$ by the *free* action of the affine algebraic group T/H.

Remark 3.1.17. Thanks to Remark 3.1.10, we get a finite open affine cover of $\widetilde{\mathcal{M}}_{n,H}^\xi$ given by

$$\left\{ \mathcal{M}_{i_1,\ldots,i_m;s}^{h_1,\ldots,h_m;k_1,\ldots,k_m} \right\} \tag{3.1.24}$$

where

$$\mathcal{M}_{i_1,\ldots,i_m;s}^{h_1,\ldots,h_m;k_1,\ldots,k_m} := \mathcal{U}_{i_1,\ldots,i_m;s}^{h_1,\ldots,h_m;k_1,\ldots,k_m} / T \tag{3.1.25}$$

and $\mathcal{U}_{i_1,\ldots,i_m;s}^{h_1,\ldots,h_m;k_1,\ldots,k_m}$ is defined as in 3.1.19.

3.1.2 Regularity and dimension

Now we prove the following

Proposition 3.1.18. *The variety $\mathcal{U}_n^\xi$ is non singular and equidimensional. The dimension of each connected component of $\mathcal{U}_n^\xi$ is given by*

$$\dim\left(\mathcal{U}_n^\xi\right) = (2g-1)\, n\, (2n+1)\,. \tag{3.1.26}$$

Proof. We follow the strategy of [HRV08, Theorem 2.2.5], with slight variations. Assume that $g > 0$. It is enough to show that at a solution $s = (A_1, B_1, \ldots, A_g, B_g) \in \mathrm{Sp}(2n, \mathbb{K})^{2g}$ of the equation

$$[A_1 : B_1] \cdots [A_g : B_g] = \xi \tag{3.1.27}$$

the derivative of μ on the tangent spaces

$$d\mu_s : T_s(\mathrm{Sp}(2n, \mathbb{K}))^{2g} \to T_\xi(\mathrm{Sp}(2n, \mathbb{K}))$$

is surjective. So take $(X_1, Y_1, \ldots, X_g, Y_g) \in T_s(\mathrm{Sp}(2n, \mathbb{K}))^{2g}$. Then differentiate μ to get:

$$d\mu_s(X_1,Y_1,\dots,X_g,Y_g) =$$

$$\begin{aligned}
&\sum_{i=1}^{g}[A_1:B_1]\cdots[A_{i-1}:B_{i-1}]X_iB_iA_i^{-1}B_i^{-1}[A_{i+1}:B_{i+1}]\cdots[A_g:B_g]\\
&+\sum_{i=1}^{g}[A_1:B_1]\cdots[A_{i-1}:B_{i-1}]A_iY_iA_i^{-1}B_i^{-1}[A_{i+1}:B_{i+1}]\cdots[A_g:B_g]\\
&-\sum_{i=1}^{g}[A_1:B_1]\cdots[A_{i-1}:B_{i-1}]A_iB_iA_i^{-1}X_iA_i^{-1}B_i^{-1}[A_{i+1}:B_{i+1}]\cdots[A_g:B_g]\\
&-\sum_{i=1}^{g}[A_1:B_1]\cdots[A_{i-1}:B_{i-1}]A_iB_iA_i^{-1}B_i^{-1}Y_iB_i^{-1}[A_{i+1}:B_{i+1}]\cdots[A_g:B_g]
\end{aligned} \tag{3.1.28}$$

and using 3.1.27, for each of the four terms, we get:

$$d\mu_s(X_1,Y_1,\dots,X_g,Y_g) = \sum_{i=1}^{g} f_i(X_i)+g_i(Y_i), \tag{3.1.29}$$

where we define linear maps

$$f_i : T_{A_i}(\mathrm{Sp}(2n,\mathbb{K})) \to \mathfrak{gl}(2n,\mathbb{K})$$

$$g_i : T_{B_i}(\mathrm{Sp}(2n,\mathbb{K})) \to \mathfrak{gl}(2n,\mathbb{K})$$

by $f_i(X) :=$

$$\prod_{j=1}^{i-1}[A_j:B_j]\big(XA_i^{-1}-A_iB_iA_i^{-1}XB_i^{-1}A_i^{-1}\big)\prod_{j=1}^{i-1}[B_{i-j}:A_{i-j}]\xi$$

and $g_i(Y) :=$

$$\prod_{j=1}^{i-1}[A_j:B_j]\big(A_iYB_i^{-1}A_i^{-1}-A_iB_iA_i^{-1}B_i^{-1}YA_iB_i^{-1}A_i^{-1}\big)\prod_{j=1}^{i-1}[B_{i-j}:A_{i-j}]\xi.$$

We claim that f_i and g_i take values in $T_\xi(\mathrm{Sp}(2n,\mathbb{K}))$. We will prove it only for

f_i, the proof for g_i being completely analogous.

What we have to prove is that $B = f_i(X)\xi^{-1}$ is a Hamiltonian matrix for every $X \in T_{A_i}(\mathrm{Sp}(2n,\mathbb{K}))$, i.e., $B^tJ = -JB$. Call

$$U = \prod_{j=1}^{i-1}[A_j : B_j],$$
$$V = XA_i^{-1} - A_iB_iA_i^{-1}XB_i^{-1}A_i^{-1}.$$

Notice that, since A_j and B_j are symplectic for every $j = 1,\dots,i$ and $X \in T_{A_i}(\mathrm{Sp}(2n,\mathbb{K}))$, U and U^{-1} are symplectic. Moreover, the facts that $A_i^{-1}X$ is hamiltonian and A_iB_i is symplectic imply that $A_iB_iA_i^{-1}XB_i^{-1}A_i^{-1}$ is hamiltonian and since XA_i^{-1} is hamiltonian, V and V^t are hamiltonian too. Then

$$\begin{aligned} B^tJ &= \left(U^{-1}\right)^t V^tU^tJ = \left(U^{-1}\right)^t V^tJU^{-1} \\ &= -\left(U^{-1}\right)^t JVU^{-1} = -JUVU^{-1} = -JB \end{aligned}$$

that is our claim.

Assume that $Z' \in T_{\xi}(\mathrm{Sp}(2n,\mathbb{K}))$ such that

$$\mathrm{Tr}\big(JZ'J^{-1}d\mu_s(X_1,Y_1,\dots,X_g,Y_g)\big) = 0. \tag{3.1.30}$$

By 3.1.29, this is equivalent to

$$\mathrm{Tr}\big(JZ'J^{-1}f_i(X_i)\big) = \mathrm{Tr}\big(JZ'J^{-1}g_i(Y_i)\big) = 0$$

for all i and $X_i \in T_{A_i}(\mathrm{Sp}(2n,\mathbb{K}))$, $Y_i \in T_{B_i}(\mathrm{Sp}(2n,\mathbb{K}))$. We show by induction on i that this implies that, if $Z' = \xi Z$, with Z hamiltonian, $C := JZJ^{-1}$ commutes with A_i and B_i. Notice that C is hamiltonian. Assume we have already proved this for $j \lneq i$ and calculate

$$\begin{aligned} 0 &= \mathrm{Tr}\big(JZ'J^{-1}f_i(X_i)\big) \\ &= \mathrm{Tr}\big(C\left(X_iA_i^{-1} - A_iB_iA_i^{-1}X_iB_i^{-1}A_i^{-1}\right)\big) \\ &= \mathrm{Tr}\big(\left(A_i^{-1}CA_i - B_iA_i^{-1}CA_iB_i\right)A_i^{-1}X_i\big) \end{aligned}$$

for all $X_i \in T_{A_i}(\mathrm{Sp}(2n,\mathbb{K}))$. Since $A_i^{-1}CA_i - B_iA_i^{-1}CA_iB_i$ and $A_i^{-1}X_i$ are hamiltonian, and $\mathrm{Tr}(\cdot,\cdot)$ is a non degenerate symmetric bilinear form over $\mathfrak{sp}(2n,\mathbb{K})$, when $\mathrm{char}(\mathbb{K}) \neq 2$, C commutes with $A_iB_iA_i^{-1}$. Similarly we have

$$\begin{aligned} 0 &= \mathrm{Tr}\big(JZ'J^{-1}g_i(Y_i)\big) \\ &= \mathrm{Tr}\big(\left(B_i^{-1}A_i^{-1}CA_iB_i - A_i^{-1}B_i^{-1}A_i^{-1}CA_iB_iA_i^{-1}\right)B_i^{-1}Y_i\big) \end{aligned}$$

which implies that C commutes with $A_iB_iA_iB_i^{-1}A_i^{-1}$. Thus C commutes with A_i and B_i, hence with $\xi = \prod_{i=1}^{g} [A_i : B_i]$. It follows that

$$C = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n).$$

Arguing as in Proposition 3.1.6, we can prove by contradiction that $C = 0$. Thus there is no non-zero Z' such that 3.1.30 holds for all X_i and Y_i. Since $\varphi(A,B) := \mathrm{Tr}\big(JAJ^{-1}B\big)$ is symmetric non degenerate bilinear form over $T_\xi(\mathrm{Sp}(2n,\mathbb{K}))$ when $\mathrm{char}(\mathbb{K}) \neq 2$, this implies that $d\mu$ is surjective at any solution s of 3.1.27. Thus $\mathcal{U}_n^\xi$ is non singular and equidimensional. Finally, we see that the dimension of (each connected component of) $\mathcal{U}_n^\xi$ is

$$\dim\Big(\mathrm{Sp}(2n,\mathbb{K})^{2g}\Big) - \dim(\mathrm{Sp}(2n,\mathbb{K})) = (2g-1)\,n\,(2n+1)$$

proving the second claim. □

Corollary 3.1.19. *The dimension of (each connected component of) $\mathcal{M}_n^\xi$ is equal to $d_n := (2g-1)\,n\,(2n+1) - n$.*

Proof. By Proposition 3.1.6 and 3.1.13, we have that

$$\dim\left(\mathcal{M}_n^\xi\right) = \dim\left(\mathcal{U}_n^\xi\right) - \dim(T)$$

so the claim easily follows from 3.1.26. □

3.2 Geometry of $\mathcal{U}_{n,H}^\xi$

The goal of this section is to describe the geometry of the variety $\mathcal{U}_{n,H}^\xi$ defined in 3.1.12 for any $Z(G) \le H \le \boldsymbol{\mu}_2^n$.

3.2.1 The Lemmas

Notation. If $Z = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_1, \ldots, \varepsilon_n) \in \boldsymbol{\mu}_2^n$, then we denote Z by $\mathrm{diag}^2(\varepsilon_1, \ldots, \varepsilon_n)$. If $\Phi \in S_n$, we call Φ the corresponding symplectic permutation matrix too. If $A \in \mathrm{Sp}(2n, \mathbb{K})$, we write $\Phi(A)$ instead of A^Φ.

Let H be a subgroup of $\boldsymbol{\mu}_2^n/Z(G)$ of rank k, $\overrightarrow{\mathcal{B}}$ a basis of H. Then $\overrightarrow{\mathcal{B}} = \{Z_1, \ldots, Z_k\}$, where $Z_1, \ldots, Z_k$ are independent matrices, defined up to a sign, such that $-I_{2n} \notin \mathrm{span}\left\{\overline{Z_1}, \ldots, \overline{Z_k}\right\} \le \boldsymbol{\mu}_2^n$, whatever the choice of representatives $\overline{Z_1}, \ldots, \overline{Z_k}$ of $Z_1, \ldots, Z_k$ is. It can be easily shown that there exists a permutation $\Phi \in S_n$ such that, for all $h \in [k]$,

$$\Phi(Z_h) = \operatorname{diag}^2\left(\overbrace{1,\dots,1}^{a_1^h}, \overbrace{-1,\dots,-1}^{a_2^h}, \dots, \overbrace{1,\dots,1}^{a_{2^h-1}^h}, \overbrace{-1,\dots,-1}^{a_{2^h}^h}\right) \tag{3.2.1}$$

where $a_i^{h-1} = a_{2i-1}^h + a_{2i}^h$ for any $i \in [2^{h-1}]$.

This permutation gives rise to the following family of unordered partitions of n:

$$\left\{\left(a_1^h, \dots, a_{2^h}^h\right)\right\}_{h\in[k]}.$$

We will prove that these partitions are uniquely determined by the subgroup H. In order to do this, we have to check the following facts about the set $\{(a_1^h, \dots, a_{2^h}^h)\}_{h\in[k]}$:

1. It does not depend on the choice of the permutation Φ.
2. It does not depend on the choice of the representatives of the elements of the basis $\overrightarrow{\mathcal{B}}$.
3. It does not depend on the choice of the basis $\overrightarrow{\mathcal{B}}$, once the representatives of its elements are fixed.

Lemma 3.2.1. *Let $H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$, $\overrightarrow{\mathcal{B}} = \{Z_1, \dots, Z_k\}$ a basis of H such that the elements of $\overrightarrow{\mathcal{B}}$ are of the form 3.2.1. If $\lambda \in S_n$ such that $\lambda(Z_h) = Z_h$ for all $h \in [k]$, then $\lambda \in \prod_{i=1}^{2^k} S_{a_i^k}$.*

Proof. By induction on $k = \operatorname{rk}(H)$.

$k = 1$: In this case, $\lambda(Z_1) = Z_1$, and since Z_1 is of the form 3.2.1, $\lambda \in S_{a_1^1} \times S_{a_2^1}$.

$k \mapsto k+1$: Let $\lambda \in S_n$ such that $\lambda(Z_h) = Z_h$ for any $h = 1, \dots, k+1$. In particular $\lambda(Z_h) = Z_h$ for any $h \in [k]$. By the inductive hypothesis,

$$\lambda = (\lambda_1, \dots, \lambda_{2^k}) \in \prod_{i=1}^{2^k} S_{a_i^k}.$$

Now, $Z_{k+1} = \operatorname{diag}^2(W_1, \dots, W_{2^k})$, where

$$W_i = \operatorname{diag}\left(\overbrace{1, \dots, 1}^{a_{2i-1}^{k+1}}, \overbrace{-1, \dots, -1}^{a_{2i}^{k+1}}\right)$$

for $i \in [2^k]$ and $\lambda(Z_{k+1}) = Z_{k+1}$. This implies that $\lambda_i(W_i) = W_i$ hence, by the case $k = 1$, $\lambda_i \in S_{a_{2i-1}^{k+1}} \times S_{a_{2i}^{k+1}}$, and this proves the lemma. □

Notation. If $\lambda \in S_n$ and $\overrightarrow{\mathcal{B}} = \{Z_1, \dots, Z_k\}$ an ordered basis of a subspace H of $\boldsymbol{\mu}_\mathbf{2}^n$, then $\lambda\left(\overrightarrow{\mathcal{B}}\right) := \{\lambda(Z_1), \dots, \lambda(Z_k)\}$.

Lemma 3.2.2. *Let $H \leq \boldsymbol{\mu}_\mathbf{2}^n$, $\overrightarrow{\mathcal{B}} = \{Z_1, \dots, Z_k\}$ an ordered basis of H and λ, $\mu \in S_n$ such that $\lambda\left(\overrightarrow{\mathcal{B}}\right)$ and $\mu\left(\overrightarrow{\mathcal{B}}\right)$ are of the form 3.2.1. Then $\lambda\left(\overrightarrow{\mathcal{B}}\right) = \mu\left(\overrightarrow{\mathcal{B}}\right)$.*

Proof. By induction on k.

$k = 1$: The assertion is true because a_1^1 and a_2^1 are equal to the number of eigenvalues equal to 1 and -1 in Z_1 respectively.

$k \mapsto k+1$: If $\overrightarrow{\mathcal{B}} = \{Z_1, \dots, Z_{k+1}\}$ and $\lambda\left(\overrightarrow{\mathcal{B}}\right)$ and $\mu\left(\overrightarrow{\mathcal{B}}\right)$ are of the form 3.2.1, then, by the inductive hypothesis, $\lambda(Z_i) = \mu(Z_i)$ for any $i \in [k]$. In other words, if

$$\left\{\left(a_1^h, \dots, a_{2^h}^h\right)\right\}_{h\in[k+1]},$$
$$\left\{\left(b_1^h, \dots, b_{2^h}^h\right)\right\}_{h\in[k+1]}$$

are the partitions of n determined by λ and μ respectively, then $a_i^h = b_i^h$ for $h \in [k]$.

Now, $\left(\lambda\mu^{-1}\right)(\mu(Z_i)) = \lambda(Z_i) = \mu(Z_i)$ for any $i \in [k]$. By Lemma 3.2.1, $\lambda\mu^{-1} = (\alpha_1, \ldots, \alpha_{2^k}) \in \prod_{i=1}^{2^k} S_{a_i^k}$. Write $\lambda(Z_{k+1}) = \mathrm{diag}^2(W_1, \ldots, W_{2^k})$, $\mu(Z_{k+1}) = \mathrm{diag}^2(V_1, \ldots, V_{2^k})$, where

$$W_i = \mathrm{diag}\left(\overbrace{1, \ldots, 1}^{a_{2i-1}^{k+1}}, \overbrace{-1, \ldots, -1}^{a_{2i}^{k+1}}\right),$$

$$V_i = \mathrm{diag}\left(\overbrace{1, \ldots, 1}^{b_{2i-1}^{k+1}}, \overbrace{-1, \ldots, -1}^{b_{2i}^{k+1}}\right)$$

for $i \in [2^k]$. We have that

$$\begin{aligned}\lambda(Z_{k+1}) &= \left(\lambda\mu^{-1}\right)(\mu(Z_{k+1})) \\ &= \mathrm{diag}^2(\alpha_1(V_1), \ldots, \alpha_{2^k}(V_{2^k})) = \mathrm{diag}^2(W_1, \ldots, W_{2^k}).\end{aligned}$$

It follows that the $\alpha_i(V_i)$'s have the form 3.2.1, and by the case $k = 1$, $\alpha_i(V_i) = V_i$ for $i \in [2^k]$. So $V_i = W_i$ for any $i \in [2^k]$ and this concludes the proof. □

Lemma 3.2.3. *Let $H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$ such that $-I_{2n} \notin H$, $\overrightarrow{\mathcal{B}} = \{Z_1, \ldots, Z_k\}$ an ordered basis of H, $\overrightarrow{\mathcal{B}'} = \{Z_1, \ldots, Z_{j-1}, -Z_j, Z_{j+1}, \ldots, Z_k\}$ and*

$$\begin{gathered}\left\{\left(a_1^h, \ldots, a_{2^h}^h\right)\right\}_{h\in[k]}, \\ \left\{\left(b_1^h, \ldots, b_{2^h}^h\right)\right\}_{h\in[k]}\end{gathered} \tag{3.2.2}$$

the partitions of n associated to $\overrightarrow{\mathcal{B}}$ and $\overrightarrow{\mathcal{B}'}$ respectively. Then for any $h \in [k]$, there exists a permutation $\lambda_h \in S_{2^h}$ such that $b_i^h = a_{\lambda_h(i)}^h$ for all $i \in [2^h]$.

Proof. First of all, notice that partitions 3.2.2 are uniquely determined by Lemma 3.2.2. Let $\mu \in S_n$ such that $\mu\left(\overrightarrow{\mathcal{B}}\right)$ is of the form 3.2.1. Then the matrices of $\mu\left(\overrightarrow{\mathcal{B}'}\right)$ assume the following form:

$$\mu(Z_h) = \operatorname{diag}^2\left(\overbrace{1,\dots,1}^{a_1^h}, \overbrace{-1,\dots,-1}^{a_2^h}, \dots, \overbrace{1,\dots,1}^{a_{2^h-1}^h}, \overbrace{-1,\dots,-1}^{a_{2^h}^h}\right)$$

for $h \neq j$ and

$$\mu(Z_j) = \operatorname{diag}^2\left(\overbrace{-1,\dots,-1}^{a_1^j}, \overbrace{1,\dots,1}^{a_2^j}, \dots, -\overbrace{1,\dots,-1}^{a_{2^j-1}^j}, \overbrace{1,\dots,1}^{a_{2^j}^j}\right).$$

Let $\lambda \in S_n$ such that, for any $l \in [2^j]$, if $\left(\sum_{i=1}^{l-1} a_i^j + 1\right) \leq s \leq \sum_{i=1}^{l} a_i^j$, then

$$\lambda(s) = \begin{cases} s + a_{l+1}^j & \text{if } l \text{ is odd} \\ s - a_{l-1}^j & \text{if } l \text{ is even} \end{cases}$$

(λ is the permutation that exchanges pairwise the blocks of size a_{2i-1}^j, a_{2i}^j, $i \in [2^{j-1}]$). It is easy to check that $\lambda\mu\left(\overrightarrow{\mathcal{B}'}\right)$ is of the form 3.2.1 and that the desired permutations λ_h are the following:

$$\lambda_h = \begin{cases} \mathrm{id}_{[n]} & \text{if } 1 \leq h \leq j-1 \\ \prod_{l=0}^{2\left(2^j-1\right)} \left(\prod_{i=1}^{2^{h-j}} \left(i + l2^{h-j}, i + (l+1)\, 2^{h-j}\right) \right) & \text{if } j \leq h \leq k. \end{cases}$$

□

Lemma 3.2.4. *Let H and $\overrightarrow{\mathcal{B}}$ like in Lemma 3.2.3, $\overrightarrow{\mathcal{B}'}$ another ordered basis of H and consider the partitions of n as in 3.2.2 associated to $\overrightarrow{\mathcal{B}}$ and $\overrightarrow{\mathcal{B}'}$. Then for any $h \in [k]$, there exists a permutation $\lambda_h \in S_{2^h}$ such that $b_i^h = a_{\lambda_h(i)}^h$ for all $i \in [2^h]$.*

Proof. First of all, let us give an explicit description of the action of an element in $\mathrm{GL}(k, \mathbb{Z}_2)$ on an ordered basis of H: if $A = (a_{ij})_{i,j\in[k]} \in \mathrm{GL}(k, \mathbb{Z}_2)$ and $\overrightarrow{\mathcal{B}} = \{Z_1, \ldots, Z_k\}$, then

$$A\left(\overrightarrow{\mathcal{B}}\right) := \left\{\prod_{i=1}^{k} Z_i^{a_{ij}}\right\}_{j\in[k]}.$$

It is known that $\mathrm{GL}\,(k, \mathbb{Z}_2)$ is generated as a group by the matrices of the form $I_k + E_{i,i+1}$ for $i = 1, \ldots, k-1$, where $E_{i,i+1}$ is the $(i, i+1)$-th elementary matrix. Then, it is sufficient to prove the lemma only for the change of basis given by these matrices. For simplicity, we will give the proof only in the case where the change of basis is given by $I_k + E_{1,2}$, being the proof in the other cases completely analogous. Therefore, the basis involved are

$$\begin{aligned}\overrightarrow{\mathcal{B}} &= \{Z_1, \ldots, Z_k\},\\ \overrightarrow{\mathcal{B}'} &= \{Z_1, Z_1Z_2, Z_3, \ldots, Z_k\}.\end{aligned}$$

Let $\mu \in S_n$ such that $\mu\left(\overrightarrow{\mathcal{B}}\right)$ is of the form 3.2.1. Then

$$\mu(Z_1Z_2) = \mathrm{diag}^2\left(\overbrace{1, \ldots, 1}^{a_1^2}, \overbrace{-1, \ldots, -1}^{a_2^2}, \overbrace{-1, \ldots, -1}^{a_3^2}, \overbrace{1, \ldots, 1}^{a_4^2}\right).$$

Let $\lambda \in S_n$ such that

$$\lambda(s) = \begin{cases} s & \text{if } 1 \le s \le a_1^2 + a_2^2 \\ s + a_4^2 & \text{if } a_1^2 + a_2^2 + 1 \le s \le a_1^2 + a_2^2 + a_3^2 \\ s - a_3^2 & \text{otherwise} \end{cases}$$

(λ is the permutation exchanging the blocks of size a_3^2, a_4^2). Then $\lambda\mu\left(\overrightarrow{\mathcal{B}'}\right)$ is of the form 3.2.1 and it turns out that $a_1^1 = b_1^1$, $a_2^1 = b_2^1$ and that, for $h = 2, \dots, k$,

$$b_i^h = \begin{cases} a_i^h & \text{if } 1 \le i \le 2^{h-1} \\ a_{i+2^{h-2}}^h & \text{if } 2^{h-1} + 1 \le i \le 2^{h-1} + 2^{h-2} \\ a_{i-2^{h-2}}^h & \text{if } 2^{h-1} + 2^{h-2} + 1 \le i \le 2^h. \end{cases}$$

□

Summaring up, Lemma 3.2.1 together with Lemma 3.2.2 prove 1, Lemma 3.2.3 proves 2 and Lemma 3.2.4 proves 3.

3.2.2 The Theorem

Finally, we are able to prove the following

Theorem 3.2.5. *Let H be a subgroup of $\boldsymbol{\mu}_{\mathbf{2}}^n$ containing $Z(G)$. There exists a unique set-partition $\{\Pi_i\}_{i \in [2^k]}$ of $\Pi = \{\varphi^{m_1}, \dots, \varphi^{m_n}\}$ such that*

$$\mathcal{U}_{n,H}^\xi \cong \prod_{i=1}^{2^k} \mathcal{U}_{a_i^k}^{\Pi_i(\xi)} \tag{3.2.3}$$

where

$$\Pi_i(\xi) := \operatorname{diag}\Big((\xi_h)_{h\in\Pi_i}\,,(\xi_h^{-1})_{h\in\Pi_i}\Big)$$

and $a_i^k = |\Pi_i|$ for any $i \in [2^k]$.

Remark 3.2.6. For any $i \in [2^k]$, $\Pi_i(\xi)$ is a matrix uniquely determined by Π_i up the action of a permutation $\varphi \in S_{a_i^k}$. So $\mathcal{U}_{a_i^k}^{\Pi_i(\xi)}$ is uniquely determined up to isomorphism induced by such a φ.

Proof of Theorem 3.2.5. Notice that for $i \in [2^k]$, the eigenvalues of $\Pi_i(\xi)$ satisfy 3.1.1 because of Remark 3.1.2, so $\mathcal{U}_{a_i^k}^{\Pi_i(\xi)}$ is well defined. Let $X = (A_1, B_1, \ldots, A_g, B_g) \in \mathcal{U}_n^{\xi}$, $\overrightarrow{\mathcal{B}} = \{Z_1, \ldots, Z_k\}$ an ordered basis of $H/Z(G)$. Then $X \in \mathcal{U}_{n,H}^{\xi}$ if and only if, for $i \in [g]$ and $j \in [k]$,

$$\begin{aligned} A_iZ_j &= Z_jA_i, \\ B_iZ_j &= Z_jB_i. \end{aligned} \tag{3.2.4}$$

Applying a permutation Φ such that $\Phi\Big(\overrightarrow{\mathcal{B}}\Big)$ is of the form 3.2.1 to equations 3.2.4, we have, for $i \in [g]$ and $j \in [k]$,

$$\begin{aligned} \Phi(A_i)\Phi(Z_j) &= \Phi(Z_j)\Phi(A_i), \\ \Phi(B_i)\Phi(Z_j) &= \Phi(Z_j)\Phi(B_i) \end{aligned}$$

For any $i \in [g]$, it can be easily shown that

$$\Phi(A_i) = \begin{pmatrix} A_i^1 & A_i^2 \\ A_i^3 & A_i^4 \end{pmatrix}, \ \Phi(B_i) = \begin{pmatrix} B_i^1 & B_i^2 \\ B_i^3 & B_i^4 \end{pmatrix}$$

where

$$A_i^s = \operatorname{diag}\Big(C_{i,1}^s, \dots, C_{i,2^k}^s\Big),$$
$$B_i^j = \operatorname{diag}\Big(D_{i,1}^s, \dots, D_{i,2^k}^s\Big)$$

and $C_{i,h}^s$, $D_{i,h}^s$ are square matrices of size a_h^k for any $s = 1, \dots, 4$, $h \in [2^k]$. Since the $\Phi(A_i)$'s and the $\Phi(B_i)$'s are symplectic matrices, we have for $i \in [g]$, $h \in [2^k]$

$$\begin{aligned}
&\left(C_{i,h}^1\right)^T C_{i,h}^4 - \left(C_{i,h}^3\right)^T C_{i,h}^2 = I_{a_h^k},\\
&\left(C_{i,h}^3\right)^T C_{i,h}^1 = \left(C_{i,h}^1\right)^T C_{i,h}^3,\\
&\left(D_{i,h}^1\right)^T D_{i,h}^4 - \left(D_{i,h}^3\right)^T D_{i,h}^2 = I_{a_h^k},\\
&\left(D_{i,h}^3\right)^T D_{i,h}^1 = \left(D_{i,h}^1\right)^T D_{i,h}^3.
\end{aligned} \tag{3.2.5}$$

Moreover, Φ determines a partition of $\{\Pi_i\}_{i\in[2^k]}$ of Π, with $a_i^k = |\Pi_i|$, and

$$\Phi(\xi) = \operatorname{diag}\Big((\xi_h)_{h\in\Pi_1}, \dots, (\xi_h)_{h\in\Pi_{2^k}}, \left(\xi_h^{-1}\right)_{h\in\Pi_1}, \dots, \left(\xi_h^{-1}\right)_{h\in\Pi_{2^k}}\Big).$$

Now, let $\lambda \in S_{2n}$ such that

$$(\lambda\Phi)(Z_j) = \operatorname{diag}\left(\overbrace{1, \dots, 1}^{2a_1^j}, \overbrace{-1, \dots, -1}^{2a_2^j}, \dots, \overbrace{1, \dots, 1}^{2a_{2^j-1}^j}, \overbrace{-1, \dots, -1}^{2a_{2^j}^j}\right)$$

for all $j \in [k]$, and λ does not move any element in any of the blocks of size a_h^j. Then

$$(\lambda\Phi)(A_i) = \operatorname{diag}\Big(C_i^1, \dots, C_i^{2^k}\Big),$$
$$(\lambda\Phi)(B_i) = \operatorname{diag}\Big(D_i^1, \dots, D_i^{2^k}\Big)$$

where

$$C_i^h = \begin{pmatrix} C_{i,h}^1 & C_{i,h}^2 \\ C_{i,h}^3 & C_{i,h}^4 \end{pmatrix}, \ D_i^h = \begin{pmatrix} D_{i,h}^1 & D_{i,h}^2 \\ D_{i,h}^3 & D_{i,h}^4 \end{pmatrix}$$

for $h \in [2^k]$, $i \in [g]$, so the C_i^h's and the D_i^h's are symplectic by 3.2.5, and

$$(\lambda\Phi)(\xi) = \mathrm{diag}(\Pi_1(\xi), \dots, \Pi_{2^k}(\xi)).$$

It follows that there is an isomorphism between $\mathcal{U}_{n,H}^{\xi}$ and $\prod_{i=1}^{2^k} \mathcal{U}_{a_i^k}^{\Pi_i(\xi)}$ given by

$$\begin{aligned} f : \mathcal{U}_{n,H}^{\xi} &\xrightarrow{\cong} \prod_{i=1}^{2^k} \mathcal{U}_{a_i^k}^{\Pi_i(\xi)} \\ (A_1, B_1, \dots, A_g, B_g) &\longmapsto \left(\left(C_i^1, D_i^1\right)_{i=1,\dots g}, \dots, \left(C_i^{2^k}, D_i^{2^k}\right)_{i=1,\dots,g} \right) \end{aligned} \tag{3.2.6}$$

induced by the permutation $\lambda\Phi$.

If we choose a different Φ' such that $\Phi'\left(\overrightarrow{\mathcal{B}}\right)$ is of the form 3.2.1, then by Lemma 3.2.1, $\Phi^{-1}\Phi' = (\alpha_1, \dots, \alpha_{2^k}) \in \prod_{i=1}^{2^k} S_{a_i^k}$. Therefore, Φ' induces the same partition $\{\Pi_i\}_{i \in [2^k]}$ of Π and by 3.2.6 we are done. If $\overrightarrow{\mathcal{B}'}$ is a different basis of $H/Z(G)$, by the proofs of Lemma 3.2.3 and 3.2.4, we have that if $\Phi'\left(\overrightarrow{\mathcal{B}'}\right)$ is of the form 3.2.1, $\Phi' \in S_n$, then $\Phi' = \mu\Phi$, where $\mu \in S_n$ permutes the blocks of size a_i^h. It follows that Φ' induces the same partition $\{\Pi_i\}_{i \in [2^k]}$ of Π and we are done again. □

Chapter 4

E-polynomial of $\mathcal{M}_n^\xi/\mathbb{C}$

4.1 Mixed Hodge structures

Motivated by the (then still unproven) Weil Conjectures and Grothendieck's "yoga of weights", which drew cohomological conclusions about complex varieties from the truth of those conjectures, Deligne in [De71] and [De74] proved the existence of *mixed Hodge structures* on the cohomology of a complex algebraic variety.

Proposition 4.1.1. ([De71], [De74]). *Let X be a complex algebraic variety. For each j there is an increasing weight filtration*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2j} = H^j(X, \mathbb{Q}) \tag{4.1.1}$$

and a decreasing Hodge filtration

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0 \tag{4.1.2}$$

such that the filtration induced by F on the complexification of the graded pieces $Gr_l^W := W_l/W_{l-1}$ of the weight filtration endows every graded piece with a pure Hodge structure of weight l, or equivalently, for every $0 \le p \le l$, we have

$$Gr_l^{W^{\mathbb{C}}} = F^p Gr_l^{W^{\mathbb{C}}} \oplus \overline{F^{l-p+1} Gr_l^{W^{\mathbb{C}}}}. \tag{4.1.3}$$

Remark 4.1.2. This mixed Hodge structure of X respects most operations in cohomology, like maps $f^* : H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$ induced by a morphism of varieties $f : X \to Y$, maps induced by field automorphisms $\sigma \in \mathrm{Aut}\,(\mathbb{C}/\mathbb{Q})$, the Künneth isomorphism

$$H^*(X \times Y,\mathbb{Q}) \cong H^*(X,\mathbb{Q}) \otimes H^*(Y,\mathbb{Q}), \tag{4.1.4}$$

cup products, etc.

Using Deligne's construction [De74, 8.3.8] of mixed Hodge structure on relative cohomology, one can define ([DK86]) a well-behaved mixed Hodge structure on compactly supported cohomology $H_c^*(X,\mathbb{Q})$, compatible with Poincarè duality for smooth connected X (see also [PS08]).

Definition 4.1.3. Define the *compactly supported mixed Hodge numbers* by

$$h_c^{p,q;j}(X) := \dim_{\mathbb{C}}\Big(Gr_p^F Gr_{p+q}^{W^{\mathbb{C}}} H_c^j(X,\mathbb{C})\Big). \tag{4.1.5}$$

Form the *compactly supported mixed Hodge polynomial*:

$$H_c(X;x,y,t) := \sum_{p,q,j} h_c^{p,q;j}(X) x^p y^q t^j \tag{4.1.6}$$

and the E-*polynomial* of X:

$$E(X; x, y) := H_c(X; x, y, -1). \tag{4.1.7}$$

Remark 4.1.4. The E-polynomial of an algebraic variety X is an algebraic invariant, as well as mixed Hodge structures and their compactly supported counterpart.

Remark 4.1.5. By definition, we can deduce the following properties of the E-polynomial $E(X; x, y)$ of an algebraic variety X:

- $E(X; 1, 1) = \chi(X)$, the Euler characteristic of X.
- The total degree of $E(X; x, y)$ is twice the dimension of X as a complex algebraic variety.
- The coefficient of $x^{\dim(X)}y^{\dim(X)}$ in $E(X; x, y)$ is the number of the highest dimensional connected components of the variety X.

Remark 4.1.6. If $\{Z_i\}_{i=1,\ldots,n}$ is a stratification of an algebraic variety X, i.e., a finite partition of X into the locally closed subsets Z_i, then

$$E(X; x, y) = \sum_{i=1}^{n} E(Z_i; x, y) \tag{4.1.8}$$

that is, the E-polynomial is additive with respect to stratifications.

4.2 Spreading out and Katz's theorem

Sometimes, the E-polynomial could be calculated using arithmetic algebraic geometry. The setup is the following.

Definition 4.2.1. Let X be a complex algebraic variety, R a finitely generated $\mathbb{Z}$-algebra, $\phi : R \hookrightarrow \mathbb{C}$ a fixed embedding. We say that a separated scheme $\mathfrak{X}/R$ is a *spreading out* of X if its extension of scalars $\mathfrak{X}_{\phi}$ is isomorphic to X.

Definition 4.2.2. Suppose that a complex algebraic variety X has a spreading out $\mathfrak{X}$ such that for every ring homomorphism $\psi : R \to \mathbb{F}_q$, the number of points of $\mathfrak{X}_{\psi}(\mathbb{F}_q)$ is given by $P_X(q)$ for some fixed $P_X(t) \in \mathbb{Z}[t]$. We say that X is a *polynomial count variety* and that P_X is the counting polynomial.

Then we have the following fundamental result:

Theorem 4.2.3. ([HRV08, Katz (2.18)]). *Let X be a variety over $\mathbb{C}$. Assume X is polynomial count with counting polynomial $P_X(t) \in \mathbb{Z}[t]$, then the E-polynomial of X is given by $E(X; x, y) = P_X(xy)$.*

In this case, and more generally, when the E-polynomial only depends on xy, we write

$$E(X; q) := E(X; \sqrt{q}, \sqrt{q}).$$

Remark 4.2.4. By 4.1.5, for a variety X whose E-polynomial is given by $P_X(q)$, the Euler characteristic of X is equal to $P_X(1)$, while the leading coefficient of $P_X(q)$ is the number of highest dimensional connected components of X.

Remark 4.2.5. Informally, Katz's theorem says that if we can count the number of solutions of the equations defining our variety over $\mathbb{F}_q$, and this number turns out to be some universal polynomial in q, then this polynomial determines the E-polynomial of the variety.

Example 4.2.6. Let $X = \mathbb{C}^{\times}$. A spreading out of X over $\mathbb{Z}$ is given by $\mathfrak{X} = \mathbb{Z}^{\times}$. Clearly, there is only one possible ring homomorphism $\psi : \mathbb{Z} \to \mathbb{F}_q$, and trivially

$|\mathfrak{X}_{\psi}(\mathbb{F}_q)| = \left|\mathbb{F}_q^{\times}\right| = q-1$. Thus $\mathbb{C}^{\times}$ is a polynomial count variety and by Katz theorem 4.2.3, $E(\mathbb{C}^{\times}; q) = q-1$.

Actually, generally we do not need to perform the computation of the rational points of an algebraic variety over all possible finite fields in order to apply Theorem 4.2.3. In fact, one can restrict the computations via a suitable choice of the finitely generated $\mathbb{Z}$-algebra which a spreading out of the variety X is defined over. We list some examples to explain this situation in more detail.

Example 4.2.7. ([HRV08, Example 2.1.10]). Fix a non-zero integer $m \in \mathbb{Z}$ and let us consider the complex algebraic variety $X \subset \mathbb{C}^2$ defined by the following equation:

$$xy = m. \tag{4.2.1}$$

A possible spreading out of X is the scheme $\mathfrak{X}$ over $\mathbb{Z}$ determined by the equation 4.2.1. The extension of scalar $\mathfrak{X}_{\psi}$ determined by a ring homomorphism $\psi : \mathbb{Z} \to \mathbb{F}_q$ is given by the same equation 4.2.1 now viewed over $\mathbb{F}_q$. It is easy to count solutions to 4.2.1. Let p be the characteristic of $\mathbb{F}_q$ (so that q is a power of p). Then

$$|\mathfrak{X}_{\psi}(\mathbb{F}_q)| = \begin{cases} 2q-1 & \text{if } p \mid m \\ q-1 & \text{otherwise.} \end{cases}$$

Therefore X does not admit an universal polynomial in q counting the number of its $\mathbb{F}_q$-points with this choice of a spreading out. But if we consider the spreading out $\mathfrak{X}$ over $\mathbb{Z}\left[\frac{1}{m}\right]$, then we eliminate the primes dividing m and find that in all cases $|\mathfrak{X}_{\psi}(\mathbb{F}_q)| = q-1$, hence X has polynomial count and by

Theorem 4.2.3, $E(X;q) = q - 1$ This is consistent with the fact that $X \cong \mathbb{C}^\times$ using Remark 4.1.4.

Example 4.2.8. ([Mer15, Example 2.4]). Let us take the affine curve

$$C = \left\{(x,y) \in \mathbb{C}^2 \mid 2x^2 + 3y^2 = 5\right\}.$$

We want a scheme $\mathcal{X}$ defined by the same equation over a finitely generated $\mathbb{Z}$-algebra. It is easy to check that $|C(\mathbb{F}_p)| = p - \left(\frac{-6}{p}\right)$ for $p > 5$ prime, where $\left(\frac{-6}{p}\right)$ is the Legendre symbol, and that $|C(\mathbb{F}_2)| = 2$, $|C(\mathbb{F}_3)| = 6$ and $|C(\mathbb{F}_5)| = 9$. To have a polynomial count for C, we need to exclude some primes. To get rid of 2, 3 and 5, we consider the scheme $\mathcal{X}$ over $\mathbb{Z}\left[\frac{1}{30}\right]$, ending up with a quasi-polynomial, since the term $\left(\frac{-6}{p}\right)$ is periodic. To satisfy the hypotheses of Theorem 4.2.3, we still have to exclude all primes p such that -6 is a quadratic non-residue modulo p. This can be accomplished by adding $\sqrt{-6}$ to the base ring. The scheme $\mathcal{X}/\mathbb{Z}\left[\frac{1}{30}, \sqrt{-6}\right]$ is a spreading out for C with polynomial count $P_C(p) = p - 1$. By Katz's result, the E-polynomial of C is $E(C;x,y) = xy - 1$. Again, this agrees with the fact that $C \cong \mathbb{C}^\times$.

4.3 Computation of the E-polynomial of $\mathcal{M}_n^\xi/\mathbb{C}$

In this section, we compute the E-polynomial of $\mathcal{M}_n^\xi/\mathbb{C}$, where ξ is a generic element of the form 3.1.4 with φ a primitive m-th root of unity. By using the stratification 3.1.23 of $\mathcal{M}_n^\xi/\mathbb{C}$ and Remark 4.1.6, we have that

Proposition 4.3.1. *The E-polynomial of $\mathcal{M}_n^\xi/\mathbb{C}$ satisfies*

$$E\Big(\mathcal{M}_n^\xi/\mathbb{C}; x, y\Big) = \sum_{Z(G) \leq H \leq \boldsymbol{\mu}_2^n} E\Big(\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}; x, y\Big). \tag{4.3.1}$$

So we reduce to compute the E-polynomial of the stratum $\widetilde{\mathcal{M}}^{\xi}_{n,H}$ for any $Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$. In order to do this, we prove that each stratum has polynomial counting function over finite fields possessing a primitive $2m$-th root of unity. Then by Theorem 4.2.3, this function will be the E-polynomial of the stratum. But firstly, we need to find a suitable spreading out schemes of the strata.

Notation. Throughout this chapter, we denote by ζ a primitive $2m$-th root of unity.

4.3.1 Spreading out of $\widetilde{\mathcal{M}}^{\xi}_{n,H}/\mathbb{C}$

Define the following finitely generated $\mathbb{Z}$-algebra:

$$R := \mathbb{Z}\left[\zeta, \frac{1}{2m}\right] \tag{4.3.2}$$

Then we have

Proposition 4.3.2. *For all $Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$, the variety $\widetilde{\mathcal{M}}^{\xi}_{n,H}/\mathbb{C}$ admits a spreading out scheme over R.*

Proof. From the definition 3.1.12 of $\widetilde{\mathcal{U}}^{\xi}_{n,H}$ it is clear that it can be viewed as a subscheme $\mathfrak{X}_H$ of $\mathrm{Sp}(2n, R)^{2g}$ and we can do the same thing for the open T-stable affine piece $\mathcal{U}^{h_1,\ldots,h_m;k_1,\ldots,k_m}_{i_1,\ldots,i_m;s}$ defined as in 3.1.19, calling $\mathfrak{X}^{h_1,\ldots,h_m;k_1,\ldots,k_m}_{i_1,\ldots,i_m;s}$ the corresponding subscheme over R for any possibile values of the indices. Let $\rho : R \to \mathbb{C}$ be an embedding, then $\mathfrak{X}_H$ and $\mathfrak{X}^{h_1,\ldots,h_m;k_1,\ldots,k_m}_{i_1,\ldots,i_m;s}$ are spreading out of $\widetilde{\mathcal{U}}^{\xi}_{n,H}/\mathbb{C}$ and $\mathcal{U}^{h_1,\ldots,h_m;k_1,\ldots,k_m}_{i_1,\ldots,i_m;s}/\mathbb{C}$ respectively.

On the other hand, the group scheme $T(R)$, defined as the centralizer of ξ in $\mathrm{Sp}(2n, R)$, acts on $\mathfrak{X}_H$ and $\mathfrak{X}^{h_1,\ldots,h_m;k_1,\ldots,k_m}_{i_1,\ldots,i_m;s}$ by conjugation, so using Seshadri's extension of geometric invariant theory quotients for schemes (see

[Se77]), remembering Proposition 3.1.13, we can take the geometric quotient

$$\mathfrak{Y}_H := \mathfrak{X}_H/T(R) \tag{4.3.3}$$

and we can define the affine schemes

$$\mathfrak{Y}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m} := \mathrm{Spec}\left(R\left[\mathfrak{X}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m}\right]^{T(R)}\right)$$

over R for any possible choice of the indices. Then $\left\{\mathfrak{Y}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m}\right\}$ is an open cover of affine subschemes of $\mathfrak{Y}_H$. Because $\rho : R \to \mathbb{C}$ is a flat morphism, [Se77, Lemma 2] implies that $\mathfrak{Y}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m}$ is a spreading out scheme of the complex variety $\mathcal{M}_{i_1,\dots,i_m;s}^{h_1,\dots,h_m;k_1,\dots,k_m}$ as defined in 3.1.24 for any possible choice of the indices, so $\mathfrak{Y}_H$ is a spreading out of $\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}$ because of the local nature of fibered product for schemes. □

4.3.2 The number of $\mathbb{F}_q$-points of $\widetilde{\mathcal{M}}_{n,H}^\xi$

Let q be a power of a prime $p \geq 3$. From now on, let us assume that $\mathbb{F}_q$ contains a primitive $2m$-th root of unity ζ such that ζ^2 satisfies conditions 3.1.1. In particular

$$q \underset{2m}{\equiv} 1. \tag{4.3.4}$$

For any $Z(G) \leq H \leq \boldsymbol{\mu}_{\mathbf{2}}^n$, define

$$\widetilde{N}_{n,H}^\xi(q) := \left|\widetilde{\mathcal{U}}_{n,H}^\xi(\mathbb{F}_q)\right| \tag{4.3.5}$$

$$N_{n,H}^\xi(q) := \left|\mathcal{U}_{n,H}^\xi(\mathbb{F}_q)\right|. \tag{4.3.6}$$

These quantities are the number of rational points of $\widetilde{\mathcal{U}}_{n,H}^\xi/\overline{\mathbb{F}_q}$ and $\mathcal{U}_{n,H}^\xi/\overline{\mathbb{F}_q}$

respectively. When $H = Z(G)$, we simply write $\widetilde{N}_n^\xi(q)$ and $N_n^\xi(q)$. From 3.1.11 and 3.1.12, it is easy to see that

$$N_{n,H}^\xi(q) = \sum_{H \le S \le \boldsymbol{\mu}_\mathbf{2}^n} \widetilde{N}_{n,S}^\xi(q) \tag{4.3.7}$$

hence by applying the Möbius inversion formula 2.1.9 in the poset of subgroups of $\boldsymbol{\mu}_\mathbf{2}^n$, we have

$$\widetilde{N}_{n,H}^\xi(q) = \sum_{H \le S \le \boldsymbol{\mu}_\mathbf{2}^n} \mu(H,S) N_{n,S}^\xi(q). \tag{4.3.8}$$

Here, the Möbius function μ is that one in 2.1.11 once replaced q with 2, so for $H \le S \le \boldsymbol{\mu}_\mathbf{2}^n$, we have

$$\mu(H,S) = (-1)^{\mathrm{rk}(S)-\mathrm{rk}(H)}\, 2^{\binom{\mathrm{rk}(S)-\mathrm{rk}(H)}{2}}. \tag{4.3.9}$$

By Theorem 3.2.5, if $Z(G) \le S \le \boldsymbol{\mu}_\mathbf{2}^n$, there exists a unique partition $\{\Pi_i\}_{i \in [2^{\mathrm{rk}(S)}]}$ of $\Pi = \{\varphi^{m_1}, \ldots, \varphi^{m_n}\}$ such that

$$N_{n,S}^\xi(q) = \prod_{i=1}^{2^{\mathrm{rk}(S)}} N_{a_i^{\mathrm{rk}(S)}}^{\Pi_i(\xi)}(q) \tag{4.3.10}$$

where $a_i^{\mathrm{rk}(S)} = |\Pi_i|$ and $\Pi_i(\xi)$ is a generic element in the sense of Definition 3.1.4 for any $i \in 2^{\mathrm{rk}(S)}$. Plugging 4.3.9 and 4.3.10 into 4.3.8, we obtain

$$\widetilde{N}_{n,H}^\xi(q) = \sum_{H \le S \le \boldsymbol{\mu}_\mathbf{2}^n} (-1)^{\mathrm{rk}(S)-\mathrm{rk}(H)}\, 2^{\binom{\mathrm{rk}(S)-\mathrm{rk}(H)}{2}} \prod_{i=1}^{2^{\mathrm{rk}(S)}} N_{a_i^{\mathrm{rk}(S)}}^{\Pi_i(\xi)}(q). \tag{4.3.11}$$

Define

$$\widetilde{E}_{n,H}^\xi(q) := \frac{\widetilde{N}_{n,H}^\xi(q)}{(q-1)^n}. \tag{4.3.12}$$

and

$$E_n^\xi(q) := \frac{N_n^\xi(q)}{(q-1)^n}. \tag{4.3.13}$$

Plugging 4.3.11 into 4.3.13, we obtain

$$\widetilde{E}_{n,H}^\xi(q) = \sum_{H \le S \le \boldsymbol{\mu}_2^n} (-1)^{\mathrm{rk}(S)-\mathrm{rk}(H)} 2^{\binom{\mathrm{rk}(S)-\mathrm{rk}(H)}{2}} \prod_{i=1}^{2^{\mathrm{rk}(S)}} E_{\mathrm{rk}(S)}^{\Pi_i(\xi)}(q) \tag{4.3.14}$$

Now, we make this fundamental assumption:

Claim 4.3.3. *For any $k \in \mathbb{N}$ such that $k \le n$ and for any possible choice of a generic element ξ, $E_k^\xi(q)$ is a polynomial in q with integral coefficients.*

Then, we are able to prove the following

Theorem 4.3.4. *Assume that Claim 4.3.3 is true. Then for any $Z(G) \le H \le \boldsymbol{\mu}_2^n$, the variety $\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}$ has polynomial count and its E-polynomial satisfies*

$$E\left(\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}; q\right) = \widetilde{E}_{n,H}^\xi(q). \tag{4.3.15}$$

Proof. Let us take the spreading out $\mathfrak{X}_H$ over the algebra R defined in 4.3.2 of $\widetilde{\mathcal{U}}_{n,H}^\xi/\mathbb{C}$ considered in Proposition 4.3.2. For every homomorphism

$$\phi : R \longrightarrow \mathbb{F}_q \tag{4.3.16}$$

the image $\phi(\zeta)$ is a primitive $2m$-root of unity in $\mathbb{F}_q$, because the identity

$$\prod_{i=1}^{2m-1} \left(1 - \zeta^i\right) = 2m$$

guarantees that $1 - \zeta^i$ is a unit in R for $i = 1, \dots, 2m-1$, and therefore cannot be zero in the image. Moreover, since 3 is invertible in R, its image in $\mathbb{F}_q$ cannot be zero too. Hence the number of rational points of the scheme $\mathfrak{X}_{H,\phi}(\mathbb{F}_q)$ obtained from $\mathfrak{X}_H$ by the extension of scalars in 4.3.16 is given by $|\mathfrak{X}_{H,\phi}(\mathbb{F}_q)| = \widetilde{N}_{n,H}^\xi(q)$.

Now take an $\mathbb{F}_q$-point of the scheme $\mathfrak{Y}_{H,\phi}$ obtained from the spreading out $\mathfrak{Y}_H$ of $\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}$ defined in 4.3.3 by the extension of scalars in 4.3.16. By [Ka80, Lemma 3.2], the fiber over it in $\mathfrak{X}_{H,\phi}(\mathbb{F}_q)$ is non empty and an orbit of $(T/H)(\mathbb{F}_q)$ and one can easily show that $(T/H)(\mathbb{F}_q)$ acts freely on $\mathfrak{X}_{H,\phi}(\mathbb{F}_q)$. Consequently

$$|\mathfrak{Y}_{H,\phi}(\mathbb{F}_q)| = \frac{|\mathfrak{X}_{H,\phi}(\mathbb{F}_q)|}{|(T/H)(\mathbb{F}_q)|} = \frac{\widetilde{N}_{n,H}^\xi(q)}{(q-1)^n} = \widetilde{E}_{n,H}^\xi(q). \tag{4.3.17}$$

Thus by 4.4.2, the assumption on the validity of Claim 4.3.3 tells us that $\widetilde{\mathcal{M}}_{n,H}^\xi/\mathbb{C}$ has polynomial count. Now the theorem follows Theorem 4.2.3. □

Remark 4.3.5. Notice that in the second equality in 4.3.17, we used the fact that $|(T/H)(\mathbb{F}_q)| = (q-1)^n$. This depends on the finiteness of the group H.

Corollary 4.3.6. *If Claim 4.3.3 is true, then the E-polynomial of $\mathcal{M}_n^\xi/\mathbb{C}$ satisfies*

$$E\left(\mathcal{M}_n^\xi/\mathbb{C}; q\right) = E_n^\xi(q). \tag{4.3.18}$$

Proof. Specializing 4.3.7 at $H = Z(G)$, we have

$$N_n^\xi(q) = \sum_{Z(G)\leq S\leq \boldsymbol{\mu}_2^n} \widetilde{N}_{n,S}^\xi(q). \tag{4.3.19}$$

Thus plugging 4.3.15 into 4.3.1, we obtain

$$E\left(\mathcal{M}_n^\xi/\mathbb{C};q\right) = \sum_{Z(G)\leq S\leq \boldsymbol{\mu}_2^n} \frac{\widetilde{N}_{n,S}^\xi(q)}{(q-1)^n} \overset{4.3.19}{=} \frac{N_n^\xi(q)}{(q-1)^n} = E_n^\xi(q)$$

and we are done. □

So we reduce to prove Claim 4.3.3. Since by definition

$$N_n^\xi(q) = \left|\left\{(A_1,B_1,\ldots,A_g,B_g)\in \mathrm{Sp}(2n,\mathbb{F}_q)^{2g} \mid \prod_{i=1}^g [A_i:B_i]=\xi\right\}\right|$$

then by Frobenius formula 2.2.5 we have that

$$E_n^\xi(q) = \frac{1}{(q-1)^n} \sum_{\chi\in \mathrm{Irr}(\mathrm{Sp}(2n,\mathbb{F}_q))} \left(\frac{|\mathrm{Sp}(2n,\mathbb{F}_q)|}{\chi(1)}\right)^{2g-1} \chi(\xi) \tag{4.3.20}$$

4.3.3 The case $n=1$

In this case we are dealing with parabolic character varieties defined over $\mathrm{Sp}(2,\mathbb{C})=\mathrm{SL}(2,\mathbb{C})$. Thus a generic element is of the form

$$\xi = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1}\end{pmatrix} \tag{4.3.21}$$

where φ a primitive m-th root of unity with $m>2$. We denote in the same way its counterpart over finite fields.

We can use the well known character table of $\mathrm{SL}(2,\mathbb{F}_q)$ to compute $E_1^\xi(q)$. This can be found for instance in [DM91, pag.157]. From there, one can see that if $\chi\in \mathrm{Irr}(\mathrm{SL}(2,\mathbb{F}_q))$ then:

$$\chi(\xi)=0 \text{ unless } \chi \text{ is a principal series of } \mathrm{SL}(2,\mathbb{F}_q).$$

Thus we can restrict the range of the summation 4.3.20 to the set $\mathcal{R}$ of the principal series of $\mathrm{SL}(2,\mathbb{F}_q)$. Let us extract the information that we need from the character table and collect them in the following table:

Table 4.1: Principal series table of $\mathrm{SL}(2,\mathbb{F}_q)$.

classes	I_2	ξ
$R_T^G(\theta)$ $\theta\in\widehat{\mathbb{F}_q^\times}$ $\theta\sim k,\ k\in Q$	$q+1$	$\varphi^k+\varphi^{-k}$
$\chi_\epsilon^\pm$	$\frac{q+1}{2}$	$(-1)^{\frac{q-1}{m}}$
1_{G^F}	1	1
St_{G^F}	q	1

where G^F denote $\mathrm{SL}(2,\mathbb{F}_q)$ and Q is as in Proposition 2.2.64:3. Since

$$\left|G^F\right| = q\,(q-1)\,(q+1) \tag{4.3.22}$$

plugging the values of the table 4.1 and 4.3.22 in 4.3.20 for $n=1$, the character sum 4.3.20 becomes:

$$\begin{aligned}E_1^\xi(q) = \frac{1}{(q-1)}\Bigg[&\sum_{k=1}^{\frac{q-3}{2}}\left(\varphi^k+\varphi^{-k}\right)\left(q^2-q\right)^{2g-1}\\ &+2^{2g}\,(-1)^{\frac{q-1}{m}}\left(q^2-q\right)^{2g-1}\\ &+\left(q^2-1\right)^{2g-1}\\ &+\left(q^3-q\right)^{2g-1}\Bigg]\end{aligned} \tag{4.3.23}$$

where different lines match the corresponding rows in the table 4.1. Collecting terms of the same degree and simplifying the sums, 4.3.23 reduces to

$$E_1^\xi(q) = \frac{1}{(q-1)}\Bigg[\Big((-1)^{\frac{q-1}{m}}\left(2^{2g}-1\right)-1\Big)\left(q^2-q\right)^{2g-1} + \left(q^2-1\right)^{2g-1} + \left(q^3-q\right)^{2g-1}\Bigg] \tag{4.3.24}$$

and finally to the quasi-polynomial:

$$E_1^\xi(q) = \Big((-1)^{\frac{q-1}{m}}\left(2^{2g}-1\right)-1\Big)\, q\left(q^2-q\right)^{2g-2} + (q+1)\left(q^2-1\right)^{2g-2} + \left(q^2+q\right)\left(q^3-q\right)^{2g-2} \tag{4.3.25}$$

which for $q \underset{2m}{\equiv} 1$ becomes:

$$E_1^\xi(q) = \left(2^{2g}-2\right) q\left(q^2-q\right)^{2g-2} + (q+1)\left(q^2-1\right)^{2g-2} + \left(q^2+q\right)\left(q^3-q\right)^{2g-2}. \tag{4.3.26}$$

Thus Claim 4.3.3 is proved in this case, so we have that $E_1^\xi(q)$ as in 4.3.26 is the E-polynomial of $\mathcal{M}_1^\xi/\mathbb{C}$.

Remark 4.3.7. It turns out that the E-polynomial of $\mathcal{M}_1^\xi/\mathbb{C}$ does not depend on the choice of the generic element ξ. So we write $E_1(q)$ instead of $E_1^\xi(q)$.

Remark 4.3.8. If merely $q \underset{m}{\equiv} 1$, then we only get the quasi-polynomial 4.3.25. This motivates the requirement 4.3.4.

Remark 4.3.9. We see that the coefficient of the leading term of the E-polynomial found in 4.3.26 is attained at the trivial character of $\mathrm{SL}(2, \mathbb{F}_q)$ and it is equal to 1. So $E\Big(\mathcal{M}_1^\xi/\mathbb{C}; q\Big)$ is monic of degree $6g-4$ and since by Corollary 3.1.19 $\mathcal{M}_1^\xi/\mathbb{C}$ is equidimensional, the principal coefficient of the E-polynomial counts the number of the connected components of $\mathcal{M}_1^\xi/\mathbb{C}$ by Remark 4.2.4, so the character variety $\mathcal{M}_1^{\xi}/\mathbb{C}$ is connected.

Remark 4.3.10. It is easy to see that the E-polynomial $E\left(\mathcal{M}_1^{\xi}/\mathbb{C};q\right) = E_1(q)$ is palindromic, i.e., it does not change when its coefficients are reversed. Being palindromic for a polynomial $P \in \mathbb{Z}[q]$ means that $P(q) = q^{\deg P} P\left(q^{-1}\right)$. In our case, the polynomial $E_1(q)$ has degree $6g-4$, as we have already noticed, and verifies:

$$\begin{aligned} E_1\left(q^{-1}\right) = &\left(2^{2g}-2\right) q^{-1}\left(q^{-2}-q^{-1}\right)^{2g-2} \\ &+\left(q^{-1}+1\right)\left(q^{-2}-1\right)^{2g-2} \\ &+\left(q^{-2}+q^{-1}\right)\left(q^{-3}-q^{-1}\right)^{2g-2}. \end{aligned} \tag{4.3.27}$$

multiplying both sides of 4.3.27 by q^{6g-4}

$$\begin{aligned} q^{6g-4}E_1\left(q^{-1}\right) = &\left(2^{2g}-2\right) q\left(q^3\left(q^{-2}-q^{-1}\right)\right)^{2g-2} \\ &+\left(q^2+q\right)\left(q^3\left(q^{-2}-1\right)\right)^{2g-2} \\ &+(q+1)\left(q^3\left(q^{-3}-q^{-1}\right)\right)^{2g-2} \\ = &\left(2^{2g}-2\right) q\left(q^2-q\right)^{2g-2} \\ &+\left(q^2+q\right)\left(q^3-q\right)^{2g-2} \\ &+(q+1)\left(q^2-1\right)^{2g-2}. \end{aligned}$$

whose right-hand side is $E_1(q)$.

Remark 4.3.11. The Euler characteristic of $\mathcal{M}_1^{\xi}/\mathbb{C}$ is calculated evaluating at $q=1$ the polynomial $E_1(q) = E\left(\mathcal{M}_1^{\xi}/\mathbb{C};q\right)$. When $g>1$, we can isolate in $E_1(q)$ a factor equal to $(q-1)^{2g-2}$, so $E_1(1)=0$. When $g=1$, $E_1(q)$ assumes the following form:

$$E_1^{\xi}(q) = 2q+q+1+q^2+q = q^2+4q+1$$

so in this case

$$E_1^{\xi}(1) = q^2 + 4q + 1\big|_{q=1} = 1 + 4 + 1 = 6.$$

Remark 4.3.12. The resulting E-polynomial $E\left(\mathcal{M}_1^{\xi}/\mathbb{C}; q\right) = E_1(q)$ agrees with the one computed in [MM15, Theorem 2] for any generic ξ.

4.3.4 The case $n = 2$

In this case we are dealing with parabolic character varieties defined over $\mathrm{Sp}(4, \mathbb{C})$. Thus a generic element is of the form

$$\xi = \begin{pmatrix} \varphi^{m_1} & & & \\ & \varphi^{m_2} & & \\ & & \varphi^{-m_2} & \\ & & & \varphi^{-m_1} \end{pmatrix} \tag{4.3.28}$$

where φ is a primitive m-th root of unity for some sufficiently large natural m and m_1 and m_2 satisfy the following conditions

$$\begin{aligned} \varphi^{m_1}, \varphi^{m_2} &\neq -1 \\ m_1 &\neq \pm m_1 \\ m_2 &\neq \pm m_2 \\ m_1 &\neq \pm m_2. \end{aligned} \tag{4.3.29}$$

Again, we denote in the same way its counterpart in finite fields. We can use the known character table of $\mathrm{Sp}(2, \mathbb{F}_q)$ computed in [Sr68] to calculate $E_2^{\xi}(q)$. From there, one can see as in the previous case that if $\chi \in \mathrm{Irr}(\mathrm{Sp}(4, \mathbb{F}_q))$ then:

$$\chi(\xi) = 0 \text{ unless } \chi \text{ is a principal series of } \mathrm{Sp}(4, \mathbb{F}_q).$$

Thus we can restrict again the range of the summation 4.3.20 to the set $\mathcal{R}$ of the principal series of $\mathrm{Sp}(4, \mathbb{F}_q)$. Extracting the information that we need in the character table of $\mathrm{Sp}(4, \mathbb{F}_q)$, keeping the same notation in [Sr68], we construct the following table:

Table 4.2: Principal series table of $\mathrm{Sp}(4, \mathbb{F}_q)$.

classes	I_4	ξ
$\chi_3(k,l)$ $k, l \in Q$; $k \lneq l$	$(q+1)^2\left(q^2+1\right)$	$\alpha_{m_1k}\alpha_{m_2l} + \alpha_{m_2k}\alpha_{m_1l}$
$\chi_8(k)$ $k \in Q$	$(q+1)\left(q^2+1\right)$	$\alpha_{m_1k}\alpha_{m_2k}$
$\chi_9(k)$ $k \in Q$	$q\,(q+1)\left(q^2+1\right)$	$\alpha_{m_1k}\alpha_{m_2k}$
$\xi_3(k)$ $k \in Q$	$(q+1)\left(q^2+1\right)$	$\alpha_{m_1k} + \alpha_{m_2k}$
$\xi_3'(k)$ $k \in Q$	$q\,(q+1)\left(q^2+1\right)$	$\alpha_{m_1k} + \alpha_{m_2k}$
$\xi_{41}(k), \xi_{41}'(k)$ $k \in Q$	$\frac{1}{2}\left(q^2+1\right)(q+1)^2$	$(-1)^{t_1}\alpha_{m_1k} + (-1)^{t_2}\alpha_{m_2k}$
Φ_5, Φ_6	$\frac{1}{2}\left(q^2+1\right)(q+1)$	$(-1)^{t_1} + (-1)^{t_2}$
Φ_7, Φ_8	$\frac{1}{2}q\left(q^2+1\right)(q+1)$	$(-1)^{t_1} + (-1)^{t_2}$
Φ_9	$q\left(q^2+1\right)$	$2\,(-1)^{t_1+t_2}$
θ_1, θ_2	$\frac{1}{2}q^2\left(q^2+1\right)$	$(-1)^{t_1+t_2}$
θ_3, θ_4	$\frac{1}{2}\left(q^2+1\right)$	$(-1)^{t_1+t_2}$
θ_9	$\frac{1}{2}q\,(q+1)^2$	2
θ_{11}, θ_{12}	$\frac{1}{2}q\left(q^2+1\right)$	1
St_{G^F}	q^4	1
1_{G^F}	1	1

where Q is again as in Proposition 2.2.64:3, $G^F = \mathrm{Sp}(4, \mathbb{F}_q)$, $t_i := m_i \frac{q-1}{m}$ for

$i = 1, 2$, $\alpha_{sk} := \varphi^{sk} + \varphi^{-sk}$ for $k \in Q$ and $s \in \{m_1, m_2\}$. Since

$$\left|G^F\right| = q^4 \left(q^2 + 1\right) \left(q - 1\right)^2 \left(q + 1\right)^2 \tag{4.3.30}$$

plugging the values of the table 4.2 and 4.3.30 in 4.3.20 for $n = 2$, the character sum 4.3.20 becomes:

$$
\begin{aligned}
E_2^{\xi}(q) = \frac{1}{(q-1)^2}\Bigg[& \sum_{\substack{k,l=1\\ k\neq l}}^{\frac{q-3}{2}} \alpha_{m_1k}\alpha_{m_2l}\left(q^4(q-1)^2\right)^{2g-1} \\
& + \sum_{k=1}^{\frac{q-3}{2}} \alpha_{m_1k}\alpha_{m_2k}\left(q^4(q+1)(q-1)^2\right)^{2g-1} \\
& + \sum_{k=1}^{\frac{q-3}{2}} \alpha_{m_1k}\alpha_{m_2k}\left(q^3(q+1)(q-1)^2\right)^{2g-1} \\
& + \sum_{k=1}^{\frac{q-3}{2}} (\alpha_{m_1k}+\alpha_{m_2k})\left(q^4(q+1)(q-1)^2\right)^{2g-1} \\
& + \sum_{k=1}^{\frac{q-3}{2}} (\alpha_{m_1k}+\alpha_{m_2k})\left(q^3(q+1)(q-1)^2\right)^{2g-1} \\
& + \sum_{k=1}^{\frac{q-3}{2}} 2^{2g}\left((-1)^{t_1}\alpha_{m_1k}+(-1)^{t_2}\alpha_{m_2k}\right)\left(q^4(q-1)^2\right)^{2g-1} \\
& + 2^{2g}\left((-1)^{t_1}+(-1)^{t_2}\right)\left(q^4(q+1)(q-1)^2\right)^{2g-1} \\
& + 2^{2g}\left((-1)^{t_1}+(-1)^{t_2}\right)\left(q^3(q+1)(q-1)^2\right)^{2g-1} \\
& + 2(-1)^{t_1+t_2}\left(q^3(q+1)^2(q-1)^2\right)^{2g-1} \\
& + 2^{2g}(-1)^{t_1+t_2}\left(q^2(q+1)^2(q-1)^2\right)^{2g-1} \\
& + 2^{2g}(-1)^{t_1+t_2}\left(q^4(q+1)^2(q-1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^3\left(q^2+1\right)(q-1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^3(q+1)^2(q-1)^2\right)^{2g-1} \\
& + \left(\left(q^2+1\right)(q+1)^2(q-1)^2\right)^{2g-1} \\
& + \left(q^4\left(q^2+1\right)(q-1)^2(q+1)^2\right)^{2g-1}\Bigg]
\end{aligned} \tag{4.3.31}
$$

where different lines match the corresponding rows in the table 4.2. Define $a_m := 1 + (-1)^{\frac{q-1}{m}}$. Collecting terms of the same degree and simplifying the sums, 4.3.31 reduces to

$$\begin{aligned}
E_2^\xi(q) = \frac{1}{(q-1)^2}\Bigg[& a_m \left(4 - 2^{2g}\left((-1)^{t_1} + (-1)^{t_1}\right)\right)\left(q^4 (q-1)^2\right)^{2g-1} \\
& + \left(2^{2g}\left((-1)^{t_1} + (-1)^{t_2}\right) - 4a_m\right)\left(q^4 (q+1)(q-1)^2\right)^{2g-1} \\
& + \left(2^{2g}\left((-1)^{t_1} + (-1)^{t_2}\right) - 4a_m\right)\left(q^3 (q+1)(q-1)^2\right)^{2g-1} \\
& + \left(2^{2g} + 2(-1)^{t_1+t_2}\right)\left(q^3 (q+1)^2 (q-1)^2\right)^{2g-1} \\
& + 2^{2g}(-1)^{t_1+t_2}\left(q^2 (q+1)^2 (q-1)^2\right)^{2g-1} \\
& + 2^{2g}(-1)^{t_1+t_2}\left(q^4 (q+1)^2 (q-1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^3 \left(q^2+1\right)(q-1)^2\right)^{2g-1} \\
& + \left(\left(q^2+1\right)(q+1)^2 (q-1)^2\right)^{2g-1} \\
& + \left(q^4 \left(q^2+1\right)(q-1)^2 (q+1)^2\right)^{2g-1}\Bigg]
\end{aligned} \tag{4.3.32}$$

and finally to the quasi-polynomial:

$$\begin{aligned}E_2^\xi(q) = \Bigg[& a_m\left(4 - 2^{2g}\left((-1)^{t_1} + (-1)^{t_2}\right)\right) q^{8g-4} \\ & + \left(2^{2g}\left((-1)^{t_1} + (-1)^{t_2}\right) - 4a_m\right)\left(q^4(q+1)\right)^{2g-1} \\ & + \left(2^{2g}\left((-1)^{t_1} + (-1)^{t_2}\right) - 4a_m\right)\left(q^3(q+1)\right)^{2g-1} \\ & + \left(2^{2g} + 2(-1)^{t_1+t_2}\right)\left(q^3(q+1)^2\right)^{2g-1} \\ & + 2^{2g}(-1)^{t_1+t_2}\left(q^2(q+1)^2\right)^{2g-1} \\ & + 2^{2g}(-1)^{t_1+t_2}\left(q^4(q+1)^2\right)^{2g-1} \\ & + 2^{2g}\left(q^3\left(q^2+1\right)\right)^{2g-1} \\ & + \left(\left(q^2+1\right)(q+1)^2\right)^{2g-1} \\ & + \left(q^4\left(q^2+1\right)(q+1)^2\right)^{2g-1}\Bigg](q-1)^{4g-4}\end{aligned} \tag{4.3.33}$$

which for $q \underset{2m}{\equiv} 1$ becomes:

$$
\begin{aligned}
E_2^\xi(q) = \Bigg[& \left(8 - 2^{2g+2}\right)\left(q^4\right) 2g - 1 \\
& + \left(2^{2g+1} - 8\right)\left(q^4 (q+1)\right)^{2g-1} \\
& + \left(2^{2g+1} - 8\right)\left(q^3 (q+1)\right)^{2g-1} \\
& + \left(2^{2g} + 2\right)\left(q^3 (q+1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^2 (q+1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^4 (q+1)^2\right)^{2g-1} \\
& + 2^{2g}\left(q^3 \left(q^2+1\right)\right)^{2g-1} \\
& + \left(\left(q^2+1\right)(q+1)^2\right)^{2g-1} \\
& + \left(q^4\left(q^2+1\right)(q+1)^2\right)^{2g-1}\Bigg] (q-1)^{4g-4}
\end{aligned} \tag{4.3.34}
$$

Thus Claim 4.3.3 is proved in this case too, also thanks to what we proved for the case $n = 1$, so we have that $E_2^\xi(q)$ as in 4.3.34 is the E-polynomial of $\mathcal{M}_2^\xi/\mathbb{C}$.

Remark 4.3.13. As in the case $n = 1$, we have that

1. E-polynomial of $\mathcal{M}_2^\xi/\mathbb{C}$ does not depend on the choice of the generic element ξ. So we write $E_2(q)$ instead of $E_2^\xi(q)$.

2. If merely $q \underset{m}{\equiv} 1$, then we only get the quasi-polynomial 4.3.33.

3. $E\left(\mathcal{M}_2^\xi/\mathbb{C}; q\right)$ is monic of degree $10\,(2g-1) - 2$ and since by Corollary 3.1.19 $\mathcal{M}_2^\xi/\mathbb{C}$ is equidimensional, the character variety $\mathcal{M}_2^\xi/\mathbb{C}$ is connected.

Remark 4.3.14. The E-polynomial $E\left(\mathcal{M}_2^\xi/\mathbb{C}; q\right) = E_2(q)$ is palindromic. $E_2(q)$

has degree $10\,(2g-1)-2=20g-12$, as we have already noticed, and verifies:

$$\begin{aligned}
E_2\left(q^{-1}\right) = \Bigg[& \left(8-2^{2g+2}\right)\left(q^{-4}\right)^{2g-1} \\
& + \left(2^{2g+1}-8\right)\left(q^{-4}\left(q^{-1}+1\right)\right)^{2g-1} \\
& + \left(2^{2g+1}-8\right)\left(q^{-3}\left(q^{-1}+1\right)\right)^{2g-1} \\
& + \left(2^{2g}+2\right)\left(q^{-3}\left(q^{-1}+1\right)^{2}\right)^{2g-1} \\
& + 2^{2g}\left(q^{-2}\left(q^{-1}+1\right)^{2}\right)^{2g-1} \\
& + 2^{2g}\left(q^{4}\left(q^{-1}+1\right)^{2}\right)^{2g-1} \\
& + 2^{2g}\left(q^{-3}\left(q^{-2}+1\right)\right)^{2g-1} \\
& + \left(\left(q^{-2}+1\right)\left(q^{-1}+1\right)^{2}\right)^{2g-1} \\
& + \left(q^{-4}\left(q^{-2}+1\right)\left(q^{-1}+1\right)^{2}\right)^{2g-1}\Bigg]\left(q^{-1}-1\right)^{4g-4}
\end{aligned} \tag{4.3.35}$$

multiplying both sides of 4.3.27 by $q^{10(2g-1)-2}$

$$\begin{aligned}
q^{20g-12}E_2\left(q^{-1}\right) = &\Bigg[\left(8-2^{2g+2}\right)\left(q^8q^{-4}\right)^{2g-1} \\
&+\left(2^{2g+1}-8\right)\left(q^8q^{-4}\left(q^{-1}+1\right)\right)^{2g-1} \\
&+\left(2^{2g+1}-8\right)\left(q^8q^{-3}\left(q^{-1}+1\right)\right)^{2g-1} \\
&+\left(2^{2g}+2\right)\left(q^8q^{-3}\left(q^{-1}+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^8q^{-2}\left(q^{-1}+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^8q^{4}\left(q^{-1}+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^8q^{-3}\left(q^{-2}+1\right)\right)^{2g-1} \\
&+\left(q^8\left(q^{-2}+1\right)\left(q^{-1}+1\right)^2\right)^{2g-1} \\
&+\left(q^{-4}\left(q^{-2}+1\right)\left(q^{-1}+1\right)^2\right)^{2g-1}\Bigg] \\
&\cdot\left(\left(q^{-1}-1\right)q\right)^{4g-4} \qquad (4.3.36)\\
= &\Bigg[\left(8-2^{2g+2}\right)\left(q^4\right)^{2g-1} \\
&+\left(2^{2g+1}-8\right)\left(q^3\left(q+1\right)\right)^{2g-1} \\
&+\left(2^{2g+1}-8\right)\left(q^4\left(q+1\right)\right)^{2g-1} \\
&+\left(2^{2g}+2\right)\left(q^3\left(q+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^4\left(q+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^2\left(q+1\right)^2\right)^{2g-1} \\
&+2^{2g}\left(q^3\left(q^2+1\right)\right)^{2g-1} \\
&+\left(q^4\left(q^2+1\right)\left(q+1\right)^2\right)^{2g-1} \\
&+\left(\left(q^2+1\right)\left(q+1\right)^2\right)^{2g-1}\Bigg]\left(q-1\right)^{4g-4}
\end{aligned}$$

whose right-hand side is $E_2(q)$.

Remark 4.3.15. The Euler characteristic of $\mathcal{M}_2^{\xi}/\mathbb{C}$ is equal to $E_2(1)$. In formula 4.3.34, we put in evidence a factor equal to $(q-1)^{4g-4}$, so when $g > 1$, $E_2(1) = 0$. When $g = 1$, $E_2(q)$ assumes the following form:

$$
\begin{aligned}
E_2(q) = &- 8q^4 + 6q^3 (q+1)^2 + 4q^2 (q+1)^2 + 4q^4 (q+1)^2 \\
&+ 4q^3 \left(q^2+1\right) + \left(q^2+1\right)(q+1)^2 + q^4 \left(q^2+1\right)(q+1)^2
\end{aligned}
$$

so in this case

$$
\begin{aligned}
E_2(1) = &\left. - 8q^4 + 6q^3 (q+1)^2 + 4q^2 (q+1)^2 + 4q^4 (q+1)^2 \right|_{q=1} \\
&\left. + 4q^3 \left(q^2+1\right) + \left(q^2+1\right)(q+1)^2 + q^4 \left(q^2+1\right)(q+1)^2 \right|_{q=1} \\
= &-8 + 24 + 16 + 16 + 8 + 8 + 8 = 72.
\end{aligned}
$$

4.3.5 The general case

The computation of the formula 4.3.20 requires to evaluate the irreducible characters of $G^F = \mathrm{Sp}(2n, \mathbb{F}_q)$ at the generic element ξ. In order to do this, we recall the following

Proposition 4.3.16. ([DL76, (7.6.2)]). *Let $s \in G^F$ be a regular diagonal element, $\chi \in \mathrm{Irr}\left(G^F\right)$, T^F the maximal torus of diagonal symplectic matrices. Then*

$$
\chi(s) = \sum_{\theta \in \widehat{T^F}} \theta\left(s^{-1}\right) \left\langle R_T^G(\theta), \chi \right\rangle_{G^F}. \tag{4.3.37}
$$

Thus since ξ is a regular semisimple matrix, the range of the summation in 4.3.20 restricts to the set of the principal series $\mathcal{R}$ of G^F and by Proposition 2.2.84, we can collect principal series according to the type τ, so combining with 2.2.64, we obtain that

$$E_n^\xi(q) = \frac{1}{(q-1)^n} \sum_\tau (H_\tau(q))^{2g-1} C_\tau$$

where

$$C_\tau := \sum_{\tau(\chi)=\tau} \chi_\tau(\xi).$$

Our next task is to compute C_τ; we will find that it is an integer constant for any type τ. In particular, since the number of all possible types does not depend on q, by Remark 2.2.91 this will show that $E_n^\xi(q) \in \mathbb{Z}[q]$, proving Claim 4.3.3.

We refer to the notations used in 2.2.5. If $\tau = (\lambda, \alpha_1, \alpha_\epsilon, \beta)$, with $c = |\lambda|$, $l := l(\lambda)$, and $\beta \in \mathrm{Irr}(S_{\lambda,\alpha_1,\alpha_\epsilon})$, then combining Remark 2.2.63 and Proposition 2.2.78 in formula 4.3.37 we have

$$\chi(\xi) = \frac{1}{|S_{\lambda,\alpha_1,\alpha_\epsilon}|} \sum_{w \in W_{B_n}} \theta^w(\xi^{-1})\beta(1)$$

where $\theta \in \widehat{T^F}$ is of the form 2.2.46. Define, for $i = 1, \ldots, n$ and $k \in Q$,

$$\gamma_{km_i} := \varphi^{km_i} + \varphi^{-km_i}.$$

Then, after some little algebra, we obtain the following expression for C_τ:

$$C_\tau = \frac{n!\beta(1)}{\prod_i m_i(\lambda)! \prod_{i=1}^{l} \lambda_i!\alpha_1!\alpha_\varepsilon!} \sum_{\substack{k_1,\ldots,k_l \in Q \\ k_s \neq k_t \\ s \neq t}} \left(\prod_{j=1}^{l} \prod_{s \in I_j} \gamma_{k_j m_s} \right) \tag{4.3.38}$$

where $\pi := \coprod_{j=1}^{l} I_j$ is the partition of $[c]$ such that

$$I_j = \left\{ \sum_{i=1}^{j-1} \lambda_i + 1, \dots, \sum_{i=1}^{j} \lambda_i \right\}$$

for any $j = 1, \dots, l$. If

$$\sigma := \coprod_{j=1}^{l(\sigma)} I'_j \in \Pi_c$$

let us consider the sets Σ_σ and Σ'_σ as in 2.1.12, replacing $[x]$ with Q, and define

$$\Psi(\sigma) := \sum_{h \in \Sigma_\sigma} \left(\prod_{j=1}^{l(\sigma)} \prod_{s \in I'_j} \gamma_{h(I'_j) m_s} \right) \tag{4.3.39}$$

$$\Phi(\sigma) := \sum_{h \in \Sigma'_\sigma} \left(\prod_{j=1}^{l(\sigma)} \prod_{s \in I'_j} \gamma_{h(I'_j) m_s} \right) \tag{4.3.40}$$

It is evident that

$$C_\tau = \frac{n!\beta(1)}{\prod_i m_i(\lambda)! \prod_{i=1}^{l} \lambda_i! \alpha_1! \alpha_\varepsilon!} \Phi(\pi) \tag{4.3.41}$$

and that $\Psi(\pi) = \sum_{\pi \preceq \sigma} \Phi(\sigma)$. By Möbius inversion formula 2.1.9 applied on the poset of set-partitions of $[c]$, we have

$$\Phi(\pi) = \sum_{\pi \preceq \sigma} \mu(\pi, \sigma) \Psi(\sigma). \tag{4.3.42}$$

Interchanging sum and product in 4.3.39, we get

$$\Psi(\sigma) = \prod_{j=1}^{l(\sigma)} \Delta_j$$

with

$$\Delta_j := \sum_{k\in Q}\left(\prod_{s\in I'_j}\gamma_{km_s}\right).$$

Since $\varphi^{m_1},\ldots,\varphi^{m_n}$ satisfy 3.1.1, together with Remark 3.1.2 we deduce that

$$\prod_{s\in I'_j}\gamma_{km_s} = \sum_{i=1}^{2^{\lambda'_j}}\varphi_i^k \tag{4.3.43}$$

where $\lambda'_j = \left|I'_j\right|$ and φ_i's are primitive k_i-th roots of unity with $k_i \gneq 1$, $k_i|m$. Now, condition 4.3.4 implies that

$$|Q| = \frac{q-3}{2} \underset{m}{\equiv} -1$$

so by 4.3.43, $\Delta_j = -2^{\lambda'_j}$ and then

$$\Psi(\sigma) = (-1)^{l(\sigma)}\,2^c. \tag{4.3.44}$$

Plugging 4.3.44 into 4.3.42 and using 2.1.14, we get

$$\Phi(\pi) = 2^c\,(-1)^l\,l!. \tag{4.3.45}$$

Therefore, plugging 4.3.45 into 4.3.41, we finally get

$$C_\tau = \frac{n!\beta(1)2^c\,(-1)^l\,l!}{\prod_i m_i(\lambda)!\prod_{i=1}^{l}\lambda_i!\alpha_1!\alpha_\varepsilon!} = \frac{(-1)^l\,l!\,[W_{B_n}:S_{\lambda,\alpha_1,\alpha_\epsilon}]\,\beta(1)}{\prod_i m_i(\lambda)!} \tag{4.3.46}$$

that is an integer number.

Remark 4.3.17. It turns out that the value of the coefficient C_τ does not depend on the choice of the generic element ξ for any type τ, hence $E_n^{\xi}(q)$ does not depend on the generic ξ too, so we can write $E_n(q)$.

Remark 4.3.18. From the formula 4.3.46, it is evident that $C_\tau = C_{\tau'}$ for any possible type τ.

To summarize, we have proved Claim 4.3.3 and by Corollary 4.3.6 we get

Theorem 4.3.19. *The E-polynomial of* $\mathcal{M}_n^{\xi}/\mathbb{C}$ *satisfies*

$$E\left(\mathcal{M}_n^{\xi}/\mathbb{C}; q\right) = E_n(q) = \frac{1}{(q-1)^n} \sum_{\tau} (H_\tau(q))^{2g-1} C_\tau. \tag{4.3.47}$$

Example 4.3.20. Let us compute again the E-polynomial of $\mathcal{M}_1^{\xi}/\mathbb{C}$ using the formula 4.3.47. In this case, the possible types are the following:

$$\tau_1 = ((0), 1, 0, 1_{\boldsymbol{\mu_2}}), \tau_2 = ((0) 1, 0, \varepsilon);$$
$$\tau_3 = ((0), 0, 1, \varepsilon), \tau_4 = ((0), 0, 1, 1_{\boldsymbol{\mu_2}});$$
$$\tau_5 = ((1), 0, 0, \mathbb{I})$$

where ε is the sign character of $\boldsymbol{\mu_2}$ and $\mathbb{I}$ is the trivial character of the trivial group. Using formula 4.3.46 we get

$$\begin{aligned} C_{\tau_1} &= C_{\tau_2} = \left[W_{B_1} : S_{(0),0,1}\right] = 1; \\ C_{\tau_3} &= C_{\tau_4} = \left[W_{B_1} : S_{(0),0,1}\right] = 1; \\ C_{\tau_5} &= -\left[W_{B_1} : S_{(1),0,0}\right] = -2. \end{aligned} \tag{4.3.48}$$

On the other hand, let us compute the degrees of the characters of the same type τ. We refer to the same notation of subsection 2.2.5.

If $\tau = \tau_1, \tau_2$, $W_{S_{(0),0,1}}$ is trivial, so restricting $1_{\boldsymbol{\mu_2}}$ and ε to $W_{S_{(0),0,1}}$ we obtain $\mathbb{I}$, hence using formula 2.2.61, we have

$$\chi_{\tau_1}(1) = \chi_{\tau_2}(1) = \frac{P_{B_1}(q)}{2} = \frac{q+1}{2}. \tag{4.3.49}$$

If $\tau = \tau_3, \tau_4$, we have to use formula 2.2.60 to compute $\chi_{\tau_3}(1)$ and $\chi_{\tau_4}(1)$. From there, it turns out that

$$\chi_{\tau_3}(1) = d_{\varepsilon}(q),\ \chi_{\tau_4}(1) = d_{1_{\boldsymbol{\mu_2}}}(q).$$

By the correspondence 2.1.8, we see that $\varepsilon \sim (\{1\}, \{1,0\})$ and $1_{\boldsymbol{\mu_2}} \sim (\emptyset, \{1\})$, so using formula 2.2.36 we get

$$\begin{aligned} d_{\varepsilon}(q) &= \frac{q\,(q-1)\,(q+1)\,(q-1)\,2q\,(q+1)}{2q\,(q-1)\,(q+1)\,(q-1)\,(q+1)} = q; \\ d_{1_{\boldsymbol{\mu_2}}}(q) &= \frac{(q-1)\,(q+1)}{(q-1)\,(q+1)} = 1 \end{aligned} \tag{4.3.50}$$

If $\tau = \tau_5$, there is just one irreducible character of type τ_5 and its degree is given by

$$\chi_{\tau_5}(1) = q+1. \tag{4.3.51}$$

Now, plugging 4.3.49, 4.3.50 and 4.3.51 into 2.2.64 for $n = 1$ we get:

$$\begin{aligned} H_{\tau_1}(q) = H_{\tau_2}(q) &= \frac{|\mathrm{SL}(2,\mathbb{F}_q)|}{\frac{q+1}{2}} = 2q\,(q-1)\,; \\ H_{\tau_3}(q) &= \frac{|\mathrm{SL}(2,\mathbb{F}_q)|}{q} = (q-1)\,(q+1)\,, \\ H_{\tau_4}(q) &= \frac{|\mathrm{SL}(2,\mathbb{F}_q)|}{1} = q\,(q-1)\,(q+1)\,; \\ H_{\tau_5}(q) &= \frac{|\mathrm{SL}(2,\mathbb{F}_q)|}{q+1} = q\,(q-1)\,. \end{aligned} \tag{4.3.52}$$

So finally, plugging 4.3.52 and 4.3.48 into 4.3.47, we obtain

$$\begin{aligned}
E\Big(\mathcal{M}_1^{\xi}/\mathbb{C};q\Big) &= \frac{1}{(q-1)}\Bigg[(2q\,(q-1))^{2g-1}+(2q\,(q-1))^{2g-1} \\
&\qquad ((q+1)\,(q-1))^{2g-1}+(q\,(q-1)\,(q+1))^{2g-1} \\
&\qquad -2\,(q\,(q-1))^{2g-1}\Bigg] \\
&= \left(2^{2g}-2\right)q\left(q^2-q\right)^{2g-2}+(q+1)\left(q^2-1\right)^{2g-2} \\
&\quad +\left(q^2+q\right)\left(q^3-q\right)^{2g-2}
\end{aligned}$$

which recovers formula 4.3.26, confirming the validity of Theorem 4.3.19.

4.4 Topological properties of $\mathcal{M}_n^{\xi}/\mathbb{C}$

In this final section, we deduce some important topological information on $\mathcal{M}_n^{\xi}/\mathbb{C}$ encoded in its E-polynomial $E\Big(\mathcal{M}_n^{\xi}/\mathbb{C};q\Big)$. According to Remark 4.2.4, we have that

Corollary 4.4.1. *The E-polynomial of $\mathcal{M}_n^{\xi}/\mathbb{C}$ is palindromic and monic of degree $d_n=(2g-1)\,n\,(2n+1)-n$. In particular, $\mathcal{M}_n^{\xi}/\mathbb{C}$ is connected.*

Proof. By 4.3.47, it is sufficient to prove that $E_n(q)$ is palindromic and monic. First of all, the degree of $E_n(q)$ is equal to $d_n=(2g-1)\,n\,(2n+1)-n$ for Remark 4.1.5 and Corollary 3.1.19. Next, we have to prove that

$$q^{d_n}E_n\left(q^{-1}\right)=E_n(q).$$

The polynomial $E_n(q)$ verifies:

$$\begin{aligned}
E_n\left(q^{-1}\right) &= \frac{1}{\left(q^{-1}-1\right)^n}\sum_\tau \left(H_\tau\left(q^{-1}\right)\right)^{2g-1} C_\tau \\
&\overset{2.2.66}{=} \frac{1}{\left(q^{-1}-1\right)^n}\sum_\tau \left(\frac{(-1)^n}{q^{n(2n+1)}} H_{\tau'}(q)\right)^{2g-1} C_\tau \\
&= \frac{q^n}{(q-1)^n}\frac{1}{q^{(2g-1)n(2n+1)}}\sum_\tau \left(H_{\tau'}(q)\right)^{2g-1} C_\tau \\
&\overset{\text{Rem. } 4.3.18}{=} \frac{q^n}{(q-1)^n}\frac{1}{q^{(2g-1)n(2n+1)}}\sum_\tau \left(H_{\tau'}(q)\right)^{2g-1} C_{\tau'}
\end{aligned} \tag{4.4.1}$$

multiplying both sides of 4.4.1 by $q^{(2g-1)n(2n+1)-n}$

$$\begin{aligned}
q^{(2g-1)n(2n+1)-n} E_n\left(q^{-1}\right) &= \frac{q^{(2g-1)n(2n+1)}}{q^n}\frac{q^n}{q^{(2g-1)n(2n+1)}}\frac{1}{(q-1)^n} \\
&\quad \sum_\tau \left(H_{\tau'}(q)\right)^{2g-1} C_{\tau'} \\
&= \frac{1}{(q-1)^n}\sum_\tau \left(H_{\tau'}(q)\right)^{2g-1} C_{\tau'}
\end{aligned}$$

whose right-hand side is $E_n(q)$.

Finally, Corollary 3.1.19 says that $\mathcal{M}_n^\xi$ is equidimensional. Thus, by Remark 4.2.4, the leading coefficient of $E\left(\mathcal{M}_n^\xi/\mathbb{C}; q\right)$ is the number of connected components of $\mathcal{M}_n^\xi$. By 4.3.47, the top degree term in $E_n(q)$ corresponds to the biggest $H_\tau(q)$, that is attained by the trivial character 1_{G^F} because of 2.2.64 and Remark 2.2.87. Thus, the leading coefficient of $E_n(q)$ is equal to $C_{(\hat{0},n,0,\beta)}$ with $\beta \in \mathrm{Irr}(W_{B_n})$. Now, using 2.2.42, we get

$$\beta(1) \overset{2.2.51}{=} \left\langle 1_{G^F}, R_T^G(1_{T^F})\right\rangle_{G^F} \overset{2.2.3}{=} \langle 1_{B^F}, 1_{B^F}\rangle_{B^F} = 1$$

so $C_{(\hat{0},n,0,\beta)} = 1$ by 4.3.46 and we are done. □

Corollary 4.4.2. *The Euler characteristic $\chi\Big(\mathcal{M}_n^\xi/\mathbb{C}\Big)$ of $\mathcal{M}_n^\xi/\mathbb{C}$ vanishes for $g>1$. For $g=1$, we have*

$$\sum_{n\geq 0}\frac{\chi\Big(\mathcal{M}_n^\xi/\mathbb{C}\Big)}{|W_{B_n}|}T^n=\prod_{k\geq 1}\frac{1}{(1-T^k)^3}=1+3T+9T^2+\cdots.$$

Proof. By 4.3.47 and Remark 4.2.4, the Euler characteristic of $\mathcal{M}_n^\xi/\mathbb{C}$ equals

$$E_n(1)=\sum_\tau \frac{(H_\tau(q))^{2g-1}}{(q-1)^n}\Bigg|_{q=1} C_\tau. \tag{4.4.2}$$

it follows from 2.2.65, 2.2.64 and Remark 2.2.91 that $(q-1)^{n(2g-2)}$ divides $\frac{(H_\tau(q))^{2g-1}}{(q-1)^n}$, so $E_n(1)=0$ when $g>1$, and this proves the first assertion.

When $g=1$, plugging 2.2.65 in 2.2.64 and using 2.2.57 and Remark 2.2.54:2, we get

$$\frac{H_\tau(q)}{(q-1)^n}\Bigg|_{q=1}=\frac{|W_{B_n}|}{[W_{B_n}:S_{\lambda,\alpha_1,\alpha_\epsilon}]\,\beta(1)} \tag{4.4.3}$$

if $\tau=(\lambda,\alpha_1,\alpha_\epsilon,\beta)$ where $\lambda\vdash c$, $c+\alpha_1+\alpha_\epsilon=n$ and $\beta\in \mathrm{Irr}(S_{\lambda,\alpha_1,\alpha_\epsilon})$. Thus plugging 4.4.3 and 4.3.46 in 4.4.2 for $g=1$ and summing over β, we have

$$E_n(1)=|W_{B_n}|\sum_{\lambda,\alpha_1,\alpha_\epsilon}\frac{(-1)^{l(\lambda)}\,l(\lambda)!}{\prod_i m_i(\lambda)!}\,|\mathrm{Irr}(S_{\lambda,\alpha_1,\alpha_\epsilon})|. \tag{4.4.4}$$

Since $S_{\lambda,\alpha_1,\alpha_\epsilon}=\left(\prod_{i=1}^{l(\lambda)}W_{A_{\lambda_i-1}}\right)\times W_{B_{\alpha_1}}\times W_{B_{\alpha_\epsilon}}$, it follows that

$$\mathrm{Irr}\left(S_{\lambda,\alpha_1,\alpha_\epsilon}\right)=\prod_{i=1}^{l(\lambda)}p(\lambda_i)\left|\mathrm{Irr}\big(W_{B_{\alpha_1}}\big)\right|\left|\mathrm{Irr}\big(W_{B_{\alpha_\epsilon}}\big)\right|$$

where $p(\lambda_i)$ is the number of partitions of λ_i, for $i=1,\ldots,l(\lambda)$. Thus, if we collect the partitions of the same size and length, summing over α_1 and α_ϵ, equation 4.4.4 becomes

$$E_n(1) = |W_{B_n}| \sum_{c=0}^{n} a_c b_{n-c} \tag{4.4.5}$$

with

$$a_c := \sum_{l\geq 0} \sum_{\substack{\lambda\vdash c \\ l(\lambda)=l}} \frac{(-1)^l\, l!}{\prod_i m_i(\lambda)!} \prod_{i=1}^{l} p(\lambda_i)$$

and

$$b_{n-c} := \sum_{\substack{\alpha_1,\alpha_\epsilon\geq 0 \\ \alpha_1+\alpha_\epsilon=n-c}} \left|\mathrm{Irr}\left(W_{B_{\alpha_1}}\right)\right| \left|\mathrm{Irr}\left(W_{B_{\alpha_\epsilon}}\right)\right|$$

so

$$\sum_{n\geq 0} \frac{E_n(1)}{|W_{B_n}|} T^n = \left(\sum_{c\geq 0} a_c T^c\right)\left(\sum_{m\geq 0} b_m T^m\right) \tag{4.4.6}$$

Now, it is easy to see that

$$\sum_{c\geq 0} a_c T^c = \sum_{l\geq 0} \left(-\sum_{n\geq 1} p(n) T^n\right)^l = \frac{1}{\sum_{n\geq 0} p(n) T^n}$$

so by an identity of Euler, we get

$$\sum_{c\geq 0} a_c T^c = \prod_{k\geq 1} \left(1 - T^k\right). \tag{4.4.7}$$

On the other hand, by the correspondence 2 and 4.4.7, we can deduce that

$$\sum_{n\geq 0} |\mathrm{Irr}(W_{B_n})|\, T^n = \prod_{k\geq 1} \frac{1}{(1-T^k)^2}$$

hence

$$\sum_{m\geq 0} b_m T^m = \prod_{k\geq 1} \frac{1}{(1-T^k)^4}. \tag{4.4.8}$$

Thus the second assertion of the corollary follows by plugging 4.4.7 and 4.4.8 in 4.4.6. □

Bibliography

[Bo91] Borel, A.: *Linear algebraic groups*. Springer-Verlag, New York (1991).

[Ca85] Carter, R.W.: *Finite groups of Lie type: conjugacy classes and complex characters*. Wiley, New York (1985). (Reprinted 1993 as Wiley Classics Library Edition)

[Co88] Corlette, K.: Flat G-bundles with canonical metrics. *J. Diff. Geom.* **28**, 361-382 (1988).

[CIK72] Curtis, C.W., Iwahori, N., Kilmoyer, R.W.: Hecke algebras and characters of parabolic type of finite groups with BN-pairs. *Inst. Hautes Études Sci. Publ. Math.* **40**, 81-116 (1972).

[DK86] Danilov, V. I. and Khovanskiĭ, A. G.: Newton polyhedra and algorithm for calculating Hodge-Deligne numbers. *Izv. Akad. Nauk. SSSR Ser. Mat.* **50**(5), 925-945 (1986).

[De70] Deligne, P.: *Équations différentielles á points singuliers réguliers*. Lecture Notes in Mathematics, vol. 163, Springer-Verlag, Berlin (1970).

[De71] Deligne, P.: Théorie de Hodge II. *Inst. Hautes Études Sci. Publ. Math.* **40**, 5-47, (1971).

[De74] Deligne, P.: Théorie de Hodge III. *Inst. Hautes Études Sci. Publ. Math.* **44**, 5-77 (1974).

[DL76] Deligne, P., Lusztig, G.: Representations of reductive groups over finite fields. *Annals Math.* **103**, 103-161 (1976).

[DM91] Digne, F., Michel, J.: *Representations of finite groups of Lie type*. London Math. Soc. Student Texts, vol. 21. Cambridge University Press, Cambridge (1991).

[FQ93] Freed, D.S., Quinn, F.: Chern-Simons theory with finite gauge group. *Commun. Math. Phys.* **156**(3), 435-472 (1993).

[FS906] Frobenius, G., Schur, I.: *Über die reellen Darstellungen der endlichen Gruppen*. Reimer, London (1906).

[FH91] Fulton, W., Harris, J.: *Representation theory*. Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York (1991).

[Fu93] Fulton, W.: *Introduction to toric varieties*. Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ (1993).

[GHS14] García-Prada, O., Heinloth, J., Schmitt, A.: On the motives of moduli of chains and Higgs bundles. *J. Eur. Math. Soc.*, **16**(12), 2617-2668 (2014).

[GP00] Geck, M., Pfeiffer, G.: *Characters of finite Coxeter groups and Iwahori-Hecke algebras*. London Math. Soc. Monographs, New Series 21, Oxford University Press, New York (2000).

[Go94] Gothen, P.B.: The Betti numbers of the moduli space of rank 3 Higgs bundles. *Internat. J. Math.*, **5**, 861-875 (1994).

[GWZ17] Groechenig, M., Wyss, D., Ziegler, P.: Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration. *arXiv e-prints, arXiv:1707.06417* (2017).

[HLRV11] Hausel, T., Letellier, E., Rodriguez-Villegas, F.: Arithmetic harmonic analysis on character and quiver varieties. *Duke Math. J.*, **160**(2), 323-400 (2011).

[HRV08] Hausel, T., Rodriguez-Villegas, F.: Mixed Hodge polynomials of character varieties. *Invent. Math.*, **174**(3), 555-624 (2008). (With an appendix by Nicholas M. Katz)

[HT03] Hausel, T., Thaddeus, M.: Mirror symmetry, Langlands duality and Hitchin systems. *Invent. Math.*, **153**, 197-229 (2003).

[Hi87] Hitchin, N. J.: The self-duality equations on a Riemann surface. *Proc. Lond. Math. Soc.* (3) **55**(1), 59-126 (1987).

[Hi87] Hitchin, N. J.: Stable bundles and integrable systems. *Duke Math. J.*, **54**, 91-114 (1987).

[Ho12] Hoskins, V.: *Geometric invariant theory and symplectic quotients.* http://userpage.fu-berlin.de/hoskins/GITnotes.pdf (2012).

[HK80] Howlett, R.B., Kilmoyer, R.W.: Principal series representations of finite groups with split BN-pairs. *Comm. in Algebra* **8**(6), 543-583 (1980).

[Hu75] Humphreys, J.E.: *Linear algebraic groups.* Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg (1975).

[Ka80] Kac, V.: Infinite root systems, representations of graphs and invariant theory. *Invent. Math.* **56**(1), 57-92 (1980).

[Ki69] Kilmoyer, R.W.: *Some irreducible complex representations of a finite group with a BN-pair.* Ph.D. dissertation, M.I.T (1969).

[Kl16] Klingsberg, P.: *Probability: Möbius inversion.* http://people.sju.edu/~pklingsb/moebinv.pdf (2016).

[LMN13] Logares, M., Muñoz, V., Newstead, P.E.: Hodge polynomials of $\mathrm{SL}(2,\mathbb{C})$-charcter varieties for curves of small genus. *Rev. Mat. Complut.* **26**(2), 635 (2013).

[Lu77] Lusztig, G.: Irreducible representations of finite classical groups. *Invent. Math.*, **43**, 125-176 (1977).

[MM15] Martínez, J., Muñoz, V.: E-polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character varieties of surface groups. *Int. Math. Res. Notices.* **2016**(3), 923-961 (2015).

[McG82] McGovern, K.: Multiplicities of principal series representations of finite groups with split BN-pairs. *J. of Algebra* **77**, 419-442 (1982).

[Med78] Mednyh, A.D.: Determination of the number of non equivalent coverings over a compact Riemann surface. *Dokl. Akad. Nauk SSSR* **239**(2), 269-271 (1978).

[Mel17] Mellit, A.: Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures). *arXiv preprint, arXiv:1707.04214* (2017).

[Mer15] Mereb, M.: On the E-polynomials of a family of Sl_n-character varieties. *Math. Ann.* **363**, 857-892 (2015).

[MS14] Mozgovoy, S., Schiffmann, O.: Counting Higgs bundles. *arXiv preprint arXiv:1411.2101* (2014).

[PS08] Peters, C.A.M., Steenbrink, J.H.M.: *Mixed Hodge structures.* Ergebnisse der Mathematik und Ihrer Grenzgebiete. Springer, Berlin (2008).

[Sch14] Schiffmann, O.: Indecomposable vector bundles and stable Higgs bun- dles over smooth projective curves. *Annals Math.*, **183**(1), 297-362 (2016).

[Ser77] Serre, J.-P.: *Linear representations of finite groups.* Graduate Texts in Mathematics, vol. 42, Spriger-Verlag, New York (1977).

[Se77] Seshadri, C.S.: Geometric reductivity over arbitrary base. *Advances in Math.* **26**(3), 225-274 (1977).

[Si92] Simpson, C.: Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.* **75**, 5-95 (1992).

[Si95] Simpson, C.: Moduli space of representations of the fundamental group of a smooth projective variety II. *Inst. Hautes Études Sci. Publ. Math.* **80**, 5-79 (1995).

[Si90] Simpson, C.: Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.*, **3**, 713-770 (1990).

[Sr68] Srinivasan, B.: The Characters of the Finite Symplectic Group $\mathrm{Sp}(4, q)$. *Trans. Amer. Math. Soc.*, **131**(2), 488-525 (1968).

[St12] Stanley, R.P.: *Enumerative Combinatorics. Vol 1.* Cambridge Studies in Advanced Mathematics, **49**. Cambridge University Press, Cambridge (2012).

[SYZ96] Strominger, A., Yau, S., Zaslow, E.: Mirror Symmetry is T-duality. *Nucl. Phys. B.* **479**(1-2), 243-259 (1996).