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Doctoral Thesis

Geometry and Analytic Theory of Semisimple Coalescent Frobenius Structures

An Isomonodromic approach to Quantum Cohomology and Helix structures in Derived Categories

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Giordano Cotti

Aristippus philosophus
Socraticus, naufragio cum
eiectus ad Rhodiensium litus
animadvertisset geometrica
schemata descripta,
exclamavisse ad comites
ita dicitur:
“Bene speremus! Hominum
enim vestigia video”.

Marcus Vitruvius Pollio,
De Architectura, Liber VI

Abstract

In this Thesis we study geometrical and analytic aspects of semisimple points of Frobenius manifolds presenting a phenomenon of coalescence of canonical coordinates. Particular attention is given to the isomonodromic description of these resonances as well as to their (still conjectural) relationships with the derived geometry of Fano varieties.

This Thesis contains the work done by the candidate during the doctoral programme at SISSA under the supervision of Boris A. Dubrovin and Davide Guzzetti. This consists in the following publications:

- In [**Cot16**], contained in Part 1 of the Thesis, the occurrence and frequency of a phenomenon of resonance (namely the coalescence of some Dubrovin canonical coordinates) in the locus of Small Quantum Cohomology of complex Grassmannians is studied. It is shown that surprisingly this frequency is strictly subordinate and highly influenced by the distribution of prime numbers. Two equivalent formulations of the Riemann Hypothesis are given in terms of numbers of complex Grassmannians without coalescence: the former as a constraint on the disposition of singularities of the analytic continuation of the Dirichlet series associated to the sequence counting non-coalescing Grassmannians, the latter as asymptotic estimate (whose error term cannot be improved) for their distribution function.
- In [**CDG17b**], contained in Part 2 of the Thesis, we consider an $n \times n$ linear system of ODEs with an irregular singularity of Poincaré rank 1 at $z = \infty$, holomorphically depending on parameter t within a polydisc in $\mathbb{C}^n$ centred at $t = 0$. The eigenvalues of the leading matrix at $z = \infty$ coalesce along a locus Δ contained in the polydisc, passing through $t = 0$. Namely, $z = \infty$ is a *resonant irregular singularity for* $t \in \Delta$. We analyse the case when the leading matrix remains diagonalisable at Δ. We discuss the existence of fundamental matrix solutions, their asymptotics, Stokes phenomenon and monodromy data as t varies in the polydisc, and their limits for t tending to points of Δ. When the deformation is isomonodromic away from Δ, it is well known that a fundamental matrix solution has singularities at Δ. When the system also has a Fuchsian singularity at $z = 0$, we show under minimal vanishing conditions on the residue matrix at $z = 0$ that isomonodromic deformations can be extended to the whole polydisc, including Δ, in such a way that the fundamental matrix solutions and the constant monodromy data are well defined in the whole polydisc. These data can be computed just by considering the system at fixed $t = 0$. Conversely, if the t-dependent system is isomonodromic in a small domain contained in the polydisc not intersecting Δ, if the entries of the Stokes matrices with indices corresponding to coalescing eigenvalues vanish, then we show that Δ is not a branching locus for the fundamental matrix solutions. The importance of these results for the analytic theory of Frobenius Manifolds is explained. An application to Painlevé equations is discussed.
- In [**CDG17c**], which is in preparation and it is contained in Part 3 of the Thesis, we extend the analytic theory of Frobenius manifold at semisimple points with coalescing eigenvalues of the operator of multiplication by the Euler vector field. We clarify which freedom and mutual constraints are allowed in the definition of monodromy data, in view of their importance for conjectural relationships between Frobenius manifolds and derived categories. Detailed examples and applications are taken from singularity and quantum cohomology theories. We explicitly compute the monodromy data at points of the Maxwell Stratum of the A_3-Frobenius manifold, as well as at the small quantum cohomology of the Grassmannian $\mathbb{G}(2, 4)$. In this last case, we analyse in details the action of the braid group on the computed monodromy

data. This proves that these data can be expressed in terms of characteristic classes of mutations of Kapranov's exceptional 5-block collection, as conjectured by B. Dubrovin.

- In [**CDG17a**], which is in preparation and it is contained in the final Part 4 of this Thesis, we address a conjecture formulated by B. Dubrovin in occasion of the 1998 ICM in Berlin ([**Dub98**]). This conjecture states the equivalence, for a Fano variety X, of the semisimplicity condition for the quantum cohomology $QH^\bullet(X)$ with the existence condition of exceptional collections in the derived category of coherent sheaves $\mathcal{D}^b(X)$. Furthermore, in its quantitative formulation, the conjecture also prescribes an explicit relationship between the monodromy data of $QH^\bullet(X)$ and characteristic classes of X and objects of the exceptional collections. In [**CDG17a**] we reformulate a refinement of [**Dub98**], which corrects the *ansatz* of [**Dub13**], [**GGI16, GI15**] for what concerns the conjectural expression of the central connection matrix. Through an explicit computation of the monodromy data, and a detailed analysis of the action of the braid group on both the monodromy data and the set of exceptional collections, we prove the validity of our refined conjecture for all complex Grassmannians $\mathbb{G}(r,k)$. Finally, a property of *quasi-periodicity* of the Stokes matrices of complex Grassmannians, along the locus of the small quantum cohomology, is described.

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Contents

CHAPTER 1

Introduction

1.1. General presentation of the Thesis

This Thesis is devoted to the study of two mutually stimulating problems:

- a **foundational problem** in the theory of Frobenius manifolds, consisting in the description of semisimple coalescent Frobenius structures in terms of certain «monodromy local moduli»;
- an explicit and analytical **conjectural relationship**, in literature known as *Dubrovin's conjecture* ([**Dub98**]), between the enumerative geometry of smooth projective varieties, admitting semisimple Quantum Cohomology, and the study of exceptional collections and helices in their derived category of coherent sheaves.

In itinere, some contributions to the general analytic theory of Isomonodromic deformations have been given. Furthermore, we exhibit an unexpected and direct connection between the theory of Frobenius manifolds, more precisely the Gromov-Witten and Quantum Cohomologies theories, and open problems about the distribution of prime numbers. The results are presented in fifteen Chapters, divided in four Parts, which we are going to describe in details.

1.2. Background Materials

Before explaining the results of the Thesis in more detail, we briefly recall preliminary basic facts.

Born in the last decades of the XX-th century, in the middle of the creative impetus for a mathematically rigorous foundations of Mirror Symmetry, the theory of Frobenius Manifolds ([**Dub96**], [**Dub98**], [**Dub99b**], [**Man99**], [**Her02**], [**Sab08**]) seems to be characterized by a sort of *universality* (see [**Dub04**]): this theory, in some sense, is able to unify in a unique, rich, geometrical and analytical description many aspects and features shared by the theory of Integrable Systems, Singularity Theory, Gromov-Witten Invariants, the theory of Isomondromic Deformations and Riemann-Hilbert Problems, as well as the theory of special functions like Painlevé Transcendents.

A Frobenius manifold M is a complex manifold, of finite dimension n, endowed with a structure of associative, commutative algebra with product $\circ_p$ and unit on each tangent space T_pM, analytically depending on the point $p\in M$; in order to be *Frobenius* the algebra must also satisfy an invariance property with respect to a non-degenerate symmetric bilinear form η on TM, called *metric*, whose corresponding Levi-Civita connection ∇ is *flat*:

$$\eta(a\circ_p b,c)=\eta(a,b\circ_p c)\quad \text{for all } a,b,c\in T_pM,\ p\in M.$$

The unit vector field is assumed to be ∇-flat. Furthemore, the above structure is required to be compatible with a $\mathbb{C}^*$-action on M (the so-called quasi-homogeneity assumption, see the precise definition in Chapter 2): this translates into the existence of a second distinguished ∇-flat vector field E, the *Euler vector-field*, which is Killing-conformal and whose flow preserves the tensor of structural constants of the algebras, i.e. it satisfies $\mathfrak{L}_E(\circ)=\circ$.

The geometry of a Frobenius manifold is (almost) equivalent to the flatness condition for an extended connection $\widehat{\nabla}$ defined on the pull-back tangent vector bundle π^*TM along the projection map $\pi\colon \mathbb{C}^*\times M\to M$. Consequently, we can look for n holomorphic functions $\tilde{t}^1,\dots,\tilde{t}^n\colon \mathbb{C}^*\times M\to\mathbb{C}$ such that $(z,\tilde{t}^1,\dots,\tilde{t}^n)$ are $\widehat{\nabla}$-flat coordinates. In ∇-flat coordinates $t=(t^1,...,t^n)$, the $\widehat{\nabla}$-flatness condition $\widehat{\nabla}d\tilde{t}(z,t)=0$ for a single function $\tilde{t}$ reads

$$\frac{\partial\zeta}{\partial z}=\left(\mathcal{U}(t)+\frac{1}{z}\mu\right)\zeta, \tag{1.1}$$

$$\frac{\partial\zeta}{\partial t^\alpha}=z\mathcal{C}_\alpha(t)\zeta,\quad \alpha=1,\dots,n. \tag{1.2}$$

Here the entries of the column vector $\zeta(z,t)$ are the components of the η-gradient of $\tilde{t}$

$$\operatorname{grad}\tilde{t}:=\zeta^\alpha(z,t)\frac{\partial}{\partial t^\alpha},\qquad \zeta^\alpha(z,t):=\eta^{\alpha\nu}\frac{\partial\tilde{t}}{\partial t^\nu},\qquad \eta_{\alpha\beta}:=\eta\left(\frac{\partial}{\partial t^\alpha},\frac{\partial}{\partial t^\beta}\right), \tag{1.3}$$

and $\mathcal{C}_\alpha(t)$, $\mathcal{U}(t)$ and $\mu:=\operatorname{diag}(\mu_1,\dots,\mu_n)$ are $n\times n$ matrices described in Section 2.1, satisfying $\eta\mathcal{U}=\mathcal{U}^T\eta$ and $\eta\mu+\mu^T\eta=0$. In particular, $\mathcal{U}(t)$ represents the operator of multiplication by the Euler vector field at a point $p\in M$ having ∇-flat coordinates $t=(t^1,...,t^n)$. A fundamental matrix solution of (1.1)-(1.2) provides n independent $\widehat{\nabla}$-flat coordinates $(\tilde{t}^1,\dots,\tilde{t}^n)$.

For fixed t, the equation (1.1) is an ordinary linear differential system with rational coefficients, with a *Fuchsian* singularity at $z=0$ and a *singularity of the second kind* of Poincaré rank 1 at $z=\infty$ (which is irregular).

A point $p\in M$ is called *semisimple* if the Frobenius algebra $(T_pM,\circ_p)$ is semisimple, i.e. without nilpotents. A Frobenius manifold is *semisimple* if it contains an open dense subset M_{ss} of semisimple points. In [**Dub96**] and [**Dub99b**], it is shown that if the matrix $\mathcal{U}$ is diagonalizable at p with *pairwise distinct* eigenvalues, then $p\in M_{ss}$. This condition is not necessary: there exist semisimple points $p\in M_{ss}$ where $\mathcal{U}$ has not a simple spectrum. In this case, if we move in M_{ss} along a curve terminating at p, then some eigenvalues of $\mathcal{U}(t)$ coalesce as $t\to t(p)$.

The eigenvalues $u:=(u_1,\dots,u_n)$ of the operator $\mathcal{U}$, with chosen labelling, define a local system of coordinates $p\mapsto u=u(p)$, called *canonical*, in a sufficiently small neighborhood of any semisimple point p. In canonical coordinates, we set

$$\operatorname{grad}\tilde{t}^\alpha(u,z)\equiv\sum_i Y^i_\alpha(u,z)f_i(u),\quad f_i(u):=\frac{1}{\eta(\frac{\partial}{\partial u_i}|_u,\frac{\partial}{\partial u_i}|_u)^{\frac12}}\left.\frac{\partial}{\partial u_i}\right|_u. \tag{1.4}$$

Then, the equations $\widehat{\nabla}d\tilde{t}^\alpha(u,z)=0$, $\alpha=1,\dots,n$, namely equations (1.1), (1.2), are equivalent to the following system:

$$\frac{\partial Y}{\partial z}=\left(U+\frac{V(u)}{z}\right)Y, \tag{1.5}$$

$$\frac{\partial Y}{\partial u_k}=(zE_k+V_k(u))Y,\quad 1\le k\le n, \tag{1.6}$$

where $(E_k)_{ij}:=\delta_{ik}\delta_{jk}$, $U=\operatorname{diag}(u_1,\dots,u_n)$, V is skew-symmetric and

$$U:=\Psi\mathcal{U}\Psi^{-1},\quad V:=\Psi\mu\Psi^{-1},\quad V_k(u):=\frac{\partial\Psi(u)}{\partial u_k}\Psi(u)^{-1}.$$

Here, $\Psi(u)$ is a matrix defined by the change of basis between $(\frac{\partial}{\partial t^1},\dots,\frac{\partial}{\partial t^n})$ and the normalized canonical vielbein $(f_1,\dots,f_n)$

$$\frac{\partial}{\partial t^\alpha}=\sum_{i=1}^n\Psi_{i\alpha}f_i.$$

The compatibility conditions of the equations (1.5)-(1.6) are

$$[U, V_k] = [E_k, V], \tag{1.7}$$

$$\frac{\partial V}{\partial u_k} = [V_k, V]. \tag{1.8}$$

When $u_i \neq u_j$ for $i \neq j$, equations (1.7) coincide with the Jimbo-Miwa-Ueno *isomonodromy deformation equations* for system (1.5), with deformation parameters $(u_1, \dots, u_n)$ ([**JMU81**], [**JM81a**], [**JM81b**]). This isomonodromic property allows to classify germs of semisimple Frobenius manifolds by *locally constant monodromy data* of (1.5). Conversely, such *local invariants* allow to reconstruct the Frobenius structure by means of an inverse Riemann-Hilbert problem [**Dub96**], [**Dub99b**], [**Guz01**]. Below, we briefly recall how they are defined in [**Dub96**], [**Dub99b**].

In [**Dub96**], [**Dub99b**] it was shown that system (1.5) has a fundamental solution near $z = 0$ in *Levelt normal form*

$$Y_0(z, u) = \Psi(u)\Phi(z, u)\ z^{\mu} z^R, \qquad \Phi(z, u) := \mathbb{1} + \sum_{k=1}^{\infty} \Phi_k(u) z^k, \tag{1.9}$$

satisfying the orthogonality condition

$$\Phi(-z, u)^T \eta\ \Phi(z, u) = \eta \quad \text{ for all } z \in \mathcal{R}, u \in M. \tag{1.10}$$

Since $z = 0$ is a regular singularity, $\Phi(z, u)$ is convergent.

If $u = (u_1, \dots, u_n)$ are pairwise distinct, so that U has distinct eigenvalues, then the system (1.5) admits a formal solution of the form

$$Y_{\text{formal}}(z, u) = G(z, u) e^{zU}, \quad G(z, u) = \mathbb{1} + \sum_{k=1}^{\infty} G_k(u) \frac{1}{z^k}, \quad G(-z, u)^T G(z, u) = \mathbb{1}. \tag{1.11}$$

Although Y_{formal} in general does not converge, it aways defines the asymptotic expansion of a unique genuine solution on any sectors in the universal covering $\mathcal{R} := \widetilde{\mathbb{C}\setminus\{0\}}$ of the punctured z-plane, having central opening angle $\pi + \varepsilon$, for $\varepsilon > 0$ sufficiently small.

The choice of a ray $\ell_+(\phi) := \{z \in \mathcal{R} \colon \arg z = \phi\}$, with directional angle $\phi \in \mathbb{R}$, induces a decomposition of the Frobenius manifold into disjoint chambers.[1] An *ℓ-chamber* is defined (see Definition 2.14) to be any connected component of the open dense subset of points $p \in M$, such that the eigenvalues of $\mathcal{U}$ at p are all distinct (so, in particular, they are points of M_{ss}), and the ray $\ell_+(\phi)$ does not coincide with any Stokes rays at p, namely $\Re(z(u_i(p) - u_j(p))) \neq 0$ for $i \neq j$ and $z \in \ell_+(\phi)$. Let p belong to an ℓ-chamber, and let $u = (u_1, ..., u_n)$ be the canonical coordinates in a neighbourhood of p, contained in the chamber. Then, there exist unique solutions $Y_{\text{left/right}}(z, u)$ such that

$$Y_{\text{left/right}}(z, u) \sim Y_{\text{formal}}(z, u) \qquad \text{for } z \to \infty,$$

respectively in the sectors

$$\Pi^{\varepsilon}_{\text{right}}(\phi) := \{z \in \mathcal{R} \colon \phi - \pi - \varepsilon < \arg z < \phi + \varepsilon\}, \quad \Pi^{\varepsilon}_{\text{left}}(\phi) := \{z \in \mathcal{R} \colon \phi - \varepsilon < \arg z < \phi + \pi + \varepsilon\}. \tag{1.12}$$

The two solutions $Y_{\text{left/right}}(z, u)$ are connected by the multiplication by two invertible matrices S, S_-, called *Stokes matrices*, a priori depending on u:

$$Y_{\text{left}}(z, u) = Y_{\text{right}}(z, u) S(u), \quad Y_{\text{left}}(e^{2\pi i} z, u) = Y_{\text{right}}(z, u) S_-(u) \quad \text{ for all } z \in \mathcal{R}, \quad .$$

Moreover, there exists a *central connection matrix* C, a priori depending on u, such that

$$Y_{\text{right}}(z, u) = Y_0(z, u) C(u), \quad \text{ for all } z \in \mathcal{R}.$$

[1]This definition does not appear in [**Dub96**] [**Dub99b**].

It can be shown that, because of the antisymmetry of $V(u)$, we have that $S_-(u) = S(u)^T$ (see also Section 1.5.1). By applying the classical results of the isomonodromy deformation theory of [JMU81], in [Dub96] and [Dub99b] it is shown that coefficients Φ_k's and G_k's are holomorphic at any point of any ℓ-chamber, an that the *monodromy data* μ, R, S, C are *constant* over a ℓ-chamber (the Isomonodromy Theorem I and II of [Dub99b] , cf. Theorem 2.4 and 2.12 below). They define local invariants of the semisimple Frobenius manifold M. In this sense, there is a local identification of a semisimple Frobenius manifold with the space of isomonodromy deformation parameters $(u_1, ..., u_n)$ of the equation (1.5).

1.3. Results of Part 1

1.3.1. Ambiguity in associating Monodromy Data with a point of the Manifold.

From the above discussion, we see that with a point $p \in M_{ss}$ such that $u_1(p), ..., u_n(p)$ are pairwise distinct, we associate the monodromy data (μ, R, S, C). These data are constant on the whole ℓ-chamber containing p. Nevertheless, there is not a unique choice of (μ, R, S, C) at p. The understanding of this issue is crucial in order to undertake a meaningful and well-founded study of the conjectured relations with derived categories.

The starting point is the observation that a normal form (1.9) is not unique, because of some freedom in the choice of Φ and R (in particular, even for a fixed R, there is freedom in Φ). The description of this freedom was given in [Dub99b], with a minor imprecision, to be corrected below. Let us identify all tangent spaces T_pM, for $p \in M$, using the Levi-Civita connection on M, with a n-dimensional complex vector space V, so that $\mu \in \mathrm{End}(V)$. Let $\mathcal{G}(\eta, \mu)$ be the complex (η, μ)-parabolic orthogonal Lie group, consisting of all endomorphisms $G\colon V \to V$ of the form $G = \mathbb{1}_V + \Delta$, with Δ a μ-nilpotent endomorphism, and such that $\eta(e^{i\pi\mu}Ga, Gb) = \eta(e^{i\pi\mu}a, b)$ for any $a, b \in V$ (see Section 2.1.2 and Definition 2.3). We denote by $\mathfrak{g}(\eta, \mu)$ its Lie algebra.

Theorem 1.1 (cf. Sections 2.1.1, 2.1.2). [2]

Given a fundamental matrix solution of system (1.5) in Levelt form (1.9) near $z = 0$, holomorphically depending on $(u_1, \dots, u_n)$ and satisfying the orthogonality condition (1.10), with $\mu = \Psi(u)^{-1}V\Psi(u)$ constant and diagonal, then the holomorphic function $R = R(u)$ takes values in the Lie algebra $\mathfrak{g}(\eta, \mu)$. Moreover,

(1) *All other solutions in Levelt form near $z = 0$ are $Y_0(z, u)G(u)$, where G is a holomorphic function with values in $\mathcal{G}(\eta, \mu)$; the Levelt normal form of $Y_0(z, u)G(u)$ has again the structure (1.9) with $R(u)$ substituted by $\widetilde{R}(u) := G(u)R(u)G(u)^{-1}$ (cf. Theorem 2.3).*

(2) *Because of the compatibility of (1.5) and (1.6), $G(u)$ can be chosen so that $\widetilde{R}$ is independent of u (Isomonodromy Theorem I in [Dub99b], Theorem 2.4).*

(3) *For a fixed $R \in \mathfrak{g}(\eta, \mu)$, the isotropy subgroup $\mathcal{G}(\eta, \mu)_R$ of transformations $G \in \mathcal{G}(\eta, \mu)$, such that $GRG^{-1} = R$, coincides with the group*[3]

$$\widetilde{\mathcal{C}}_0(\mu, R) := \left\{ \begin{aligned} G \in GL(n, \mathbb{C})\colon\ & P_G(z) := z^\mu z^R G z^{-R} z^{-\mu} \text{ is a matrix-valued polynomial} \\ & \text{such that } P_G(0) = \mathbb{1}, \text{ and } P_G(-z)^T \eta P_G(z) = \eta \end{aligned} \right\}.$$

If $G \in \widetilde{\mathcal{C}}_0(\mu, R)$ and $Y_0(z, u) = \Psi(u)\Phi(z, u)z^\mu z^R$, then $Y_0(z, u)G = \Psi(u)\Phi(z, u)P_G(z)z^\mu z^R$.

The refinement introduced here is the condition

$$P_G(-z)^T \eta P_G(z) = \eta,$$

which does not appear in [**Dub99b**], but is essential to preserve (1.10) and the constraints (1.14) below.

Let us now summarize the freedom in assigning the monodromy data (μ, R, S, C) to a given semisimple point p of the Frobenius manifold. It has various origins: it can come from a re-ordering of the canonical coordinates $(u_1(p), \ldots u_n(p))$, from changing signs of the normalized idempotents, from changing the Levelt fundamental solution at $z = 0$ and, last but not least, from changing the slope of the oriented line $\ell_+(\phi)$. Taking into account all these possibilities we have the following

Theorem 1.2 (cf. Section 2.3). *Let $p \in M_{ss}$ be such that $(u_1(p), ..., u_n(p))$ are pairwise distinct. If (μ, R, S, C) is a set of monodromy data computed at p, then with a different labelling of the eigenvalues, different signs, different choice of $Y_0(z, u)$ and different ϕ, another set of monodromy data can be computed at* the same *p, which lies in the orbit of (μ, R, S, C) under the following actions:*

- *the action of the group of permutations $\mathfrak{S}_n$*

$$S \longmapsto P\ S\ P^{-1}, \qquad C \longmapsto C\ P^{-1},$$

which corresponds to a relabelling $(u_1, \ldots, u_n) \mapsto (u_{\tau(1)}, \ldots, u_{\tau(n)})$, where $\tau \in \mathfrak{S}_n$ and the invertible matrix P has entries $P_{ij} = \delta_{j\tau(i)}$. For a suitable choice of the permutation[4], $P\ S\ P^{-1}$ is in upper-triangular form

- *the action of the group $(\mathbb{Z}/2\mathbb{Z})^{\times n}$*

$$S \longmapsto \mathcal{I}\ S\ \mathcal{I}, \qquad C \longmapsto C\ \mathcal{I},$$

where $\mathcal{I}$ is a diagonal matrix with entries equal to 1 or -1, which corresponds to a change of signs of the square roots in (1.4);

- *the action of the group $\widetilde{\mathcal{C}}_0(\mu, R)$*

$$S \longmapsto S, \qquad C \longmapsto G\ C, \qquad G \in \widetilde{\mathcal{C}}_0(\mu, R),$$

which corresponds to a change $Y_0(z, u) \mapsto Y_0(z, u)G^{-1}$ as in Theorem 1.1.

- *the action of the braid group $\mathcal{B}_n$, as in formulae (2.40) and (2.41),*

$$S \mapsto A^\beta(S) \cdot S \cdot (A^\beta(S))^T, \qquad C \mapsto C \cdot (A^\beta(S))^{-1}, \tag{1.13}$$

where β is a specific braid associated with a translation of ϕ, corresponding to a rotation of $\ell_+(\phi)$. More details are in Section 2.3.

Any representative of μ, R, S, C in the orbit of the above actions satisfies the monodromy identity

$$CS^T S^{-1} C^{-1} = e^{2\pi i \mu} e^{2\pi i R},$$

---

[2] In the description of the monodromy phenomenon of solutions of the system (1.5) near $z = 0$, the assumption of semisimplicity is not used. This will be crucial only for the description of solutions near $z = \infty$. Theorem 1.1 can be formulated for system (1.1), having the fundamental solutions $\Xi_0 = \Psi^{-1} Y_0$.

[3] In [**Dub99b**] neither the η-orthogonality conditions appeared in the definition of the group $\mathcal{C}_0(\mu, R)$, nor this group was identified with the isotropy subgroup of R w.r.t. the adjoint action of $\mathcal{G}(\eta, \mu)$ on its Lie algebra $\mathfrak{g}(\eta, \mu)$. These η-orthogonality conditions are crucial for preserving the constraints of all monodromy data (μ, R, S, C) (see below and Theorem 2.11).

and the constraints

$$S = C^{-1}e^{-\pi i R}e^{-\pi i\mu}\eta^{-1}(C^T)^{-1}, \qquad S^T = C^{-1}e^{\pi i R}e^{\pi i\mu}\eta^{-1}(C^T)^{-1}. \tag{1.14}$$

We stress again that the freedoms in Theorem 1.2 must be kept into account when we want to investigate the relation between monodromy data and similar object in the theory of derived categories

1.3.2. Coalescence Phenomenon of Quantum Cohomology of Grassmannians. Originally introduced by physicists ([**Vaf91**]), in the context of $N = 2$ Supersymmetric Field Theories and mirror phenomena, the Quantum Cohomology of a complex projective variety X (or more in general a symplectic manifold [**MS12**]) is a family of deformations of its classical cohomological algebra structure defined on $H^\bullet(X) := \bigoplus_k H^k(X;\mathbb{C})$, and parametrized over an open (nonempty) domain $\mathcal{D} \subsetneq H^\bullet(X)$: the fiber over $p \in \mathcal{D}$ is identified with the tangent space $T_p\mathcal{D} \cong H^\bullet(X)$. This is exactly the prototype of a Frobenius manifolds, the flat metric being the Poincaré pairing. The structure constants of the quantum deformed algebras are given by (third derivatives of) a generating function F_0^X of Gromov-Witten Invariants of genus 0 of X, supposed to be *convergent* on $\mathcal{D}$: if $(T_1,\dots T_n)$ denotes a $\mathbb{C}$-basis of $H^\bullet(X)$, and if $(t^1,\dots,t^n)$ denotes the associated (flat) coordinates, we set

$$F_0^X(t) := \sum_{n=0}^{\infty}\sum_{\beta\in\mathrm{Eff}(X)}\sum_{\alpha_1,\dots,\alpha_n=0}^{N}\frac{t^{\alpha_1}\dots t^{\alpha_n}}{n!}\int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{\mathrm{vir}}}\bigcup_{i=1}^{n}\mathrm{ev}_i^*\,T_{\alpha_i}, \tag{1.15}$$

where $\overline{\mathcal{M}}_{0,n}(X,\beta)$ denotes the Deligne-Mumford stack of stable maps of genus 0, with n-marked points and degree β with target manifold X, and where the evaluations maps ev_i's are tautologically defined as

$$\mathrm{ev}_i\colon \overline{\mathcal{M}}_{0,n}(X,\beta)\to X\colon ((C,\mathbf{x});f)\mapsto f(x_i).$$

The Gromov-Witten invariants of X, namely the rational numbers appearing in (1.15) as integrals on $\overline{\mathcal{M}}_{0,n}(X,\beta)$, morally "count" (modulo parametrizations) algebraic/pseudo-holomorphic curves of genus 0 on X, with a fixed degree β, and intersecting some fixed subvarieties of X. The reader can find in Chapter 3 more details on Gromov-Witten Theory.

In almost all studied cases of quantum cohomology, the Frobenius structure of the manifold is explicitly known only at the locus of *small quantum cohomology*, i.e. the locus $\mathcal{D}\cap H^2(X;\mathbb{C})$, where only the deformation contributions due to the three-points genus zero Gromov-Witten invariants are taken into account. Along this locus, a *coalescence phenomenon* may occur. By this we mean that the operator $\mathcal{U}$ of multiplication by the Euler vector field[5]does not have simple spectrum at some points where nevertheless the Frobenius algebra is semisimple.

Definition 1.1. A point $p\in M_{ss}$ such that the eigenvalues of $\mathcal{U}$ at p are not pairwise distinct is called a *semisimple coalescence point* (or *semisimple bifurcation point*).

The classical isomonodromy deformation results, exposed in Section 1.2 above, apply if U has distinct eigenvalues. If two or more eigenvalues coalesce, as it happens at semisimple coalescence points of Definition 1.1, then a priori solutions $Y_{\mathrm{left/right/F}}(z,u)$ are expected to become singular and monodromy data must be redefined. Therefore, if we want to compute monodromy data, we can only rely on the information available at coalescence points. Thus, we need to extend the analytic theory of Frobenius manifolds, in order to include this case, showing that the monodromy data are well defined at a semisimple coalescence point, and locally constant. Moreover, from these data we must be able

[4]Unique choice at a semisimple non-coalescing point, but not unique at a semisimple colescence one.

[5]Along the small quantum cohomology, the Euler vector field coincide with the first Chern class $c_1(X)$.

to reconstruct the data for the whole manifold. We stress that this extension of the theory is essential in order to study the conjectural links to derived categories.

Before addressing this problem, we focus on the case of complex Grassmannians $\mathbb{G}(k, n)$ of k-planes in $\mathbb{C}^n$, and we study the occurrence and frequency of this coalescence phenomenon. For simplicity, we will call *coalescing* a Grassmannian for which some canonical coordinates coalesce along the locus of small quantum cohomology.

The questions, to which we answer in Chapter 4, are the following:

(i) For which k, n the Grassmannian $\mathbb{G}(k, n)$ is coalescing?
(ii) How frequent is the phenomenon of coalescence among all Grassmannians?

By looking at Grassmannians as symplectic (or, if You prefer, GIT) quotients, and by applying the so called *abelian-nonabelian correspondence* ([**Mar00, BCFK08, CFKS08**]), which allows to establish a relationship between the enumerative geometry of a symplectic quotient $V /\!\!/ G$ and that of the "abelianized quotient" $V /\!\!/ T$, with T maximal torus of G, we reformulate the above question (i) in terms of vanishing sums of roots of unity (see Section 4.3.2). We show that the occurrence and frequency of this coalescence phenomenon is surprisingly related to the distribution of prime numbers. This relation is so strict that it leads to (at least) two equivalent formulations of the celebrated Riemann Hypothesis: the former is given as a constraint on the disposition of the singularities of a generating function of the numbers of Grassmannians not presenting the resonance, the latter as an (essentially optimal) asymptotic estimate for a distribution function of the same kind of Grassmannians. Besides their geometrical-enumerative meaning, three point genus zero Gromov-Witten invariants of complex Grassmannians implicitly contain information about the distribution of prime numbers. This mysterious relation deserves further investigations. Let us summarize some of the main results obtained (cf. Theorems 4.6, 4.7, 4.8, 4.9 and Corollaries 4.7, 4.6 for more details). Here, $P_1(n)$ denotes the smallest prime number which divides n.

Theorem 1.3 (cf. Theorem 4.6). *The complex Grassmannian $\mathbb{G}(k, n)$ is coalescing if and only if*

$$P_1(n) \leq k \leq n - P_1(n).$$

In particular, all Grassmannians of proper subspaces of $\mathbb{C}^p$, with p prime, are not coalescing.

In Section 4.3.3, we introduce the sequence $(\tilde{\text{ℶ}}_n)_{n\geq 2}$, where $\tilde{\text{ℶ}}_n$ denotes the number of non-coalescing Grassmannians of proper subspaces of $\mathbb{C}^n$, i.e.

$$\tilde{\text{ℶ}}_n := \operatorname{card}\{k \colon \mathbb{G}(k, n) \textit{ is not coalescing}\}.$$

In order to study properties of the sequence $(\tilde{\text{ℶ}}_n)_{n\geq 2}$, we collect these numbers into a Dirichlet series generating function

$$\tilde{\text{ℶ}}(s) := \sum_{n=2}^{\infty} \frac{\tilde{\text{ℶ}}_n}{n^s},$$

and we study its analytical properties. In particular, we deduce the following result.

Theorem 1.4 (cf. Theorem 4.7). *The function $\tilde{\text{ℶ}}(s)$ is absolutely convergent in the half-plane* $\operatorname{Re}(s) > 2$, *where it can be represented by the infinite series*

$$\tilde{\text{ℶ}}(s) = \sum_{p \text{ prime}} \frac{p-1}{p^s}\left(\frac{2\zeta(s)}{\zeta(s, p-1)} - 1\right),$$

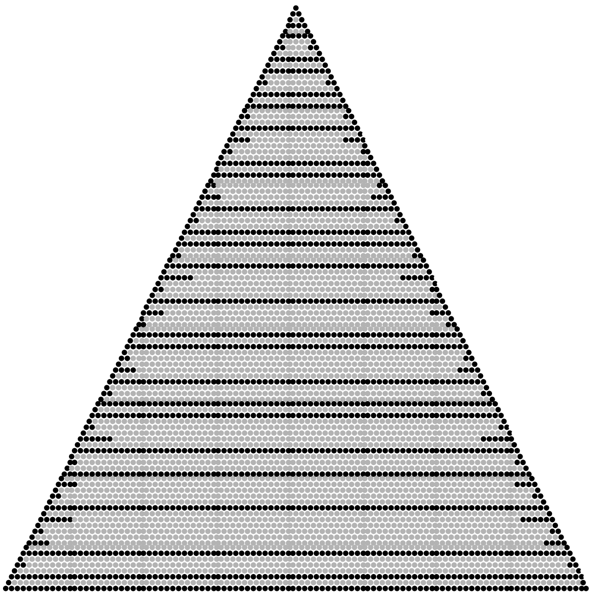

FIGURE 1.1. In this figure we represent complex Grassmannians as disposed in a Tartaglia-Pascal triangle: the k-th element (from the left) in the n-th row (from the top of the triangle) represents the Grassmannian $\mathbb{G}(k, n+1)$, where $n \leq 102$. The dots colored in black represent *non-coalescing* Grassmannians, while the dots colored in gray the *coalescing* ones. The reader can note that black dots are *rare* w.r.t. the gray ones, and that the black lines correspond to Grassmannians of subspaces in $\mathbb{C}^p$ with p prime.

involving the Riemann zeta function $\zeta(s)$ *and the truncated Euler products*

$$\zeta(s,k) := \prod_{\substack{p \leq k \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

By analytic continuation, $\widetilde{\Pi}(s)$ *can be extended to (the universal cover of) the punctured half-plane*

$$\{s \in \mathbb{C} \colon \operatorname{Re}(s) > \overline{\sigma}\} \setminus \left\{ s = \frac{\rho}{k} + 1 \colon \begin{array}{c} \rho \textit{ pole or zero of } \zeta(s), \\ k \textit{ squarefree positive integer} \end{array} \right\},$$

$$\overline{\sigma} := \limsup_{n \to \infty} \frac{1}{\log n} \cdot \log \left(\sum_{\substack{k \leq n \\ k \text{ composite}}} \widetilde{\pi}_k \right), \quad 1 \leq \overline{\sigma} \leq \frac{3}{2},$$

having logarithmic singularities at the punctures.

In particular, we have the equivalence of the following statements:

- (RH) *all non-trivial zeros of the Riemann zeta function* $\zeta(s)$ *satisfy* $\operatorname{Re}(s) = \frac{1}{2}$*;*
- *the derivative* $\widetilde{\Pi}'(s)$ *extends, by analytic continuation, to a meromorphic function in the half-plane* $\frac{3}{2} < \operatorname{Re}(s)$ *with a single pole of oder one at* $s = 2$*.*

At the point $s = 2$ the following asymptotic estimate holds

$$\widetilde{\text{ת}}(s) = \log\left(\frac{1}{s-2}\right) + O(1), \qquad s \to 2, \qquad \operatorname{Re}(s) > 2.$$

As a consequence, we have that

$$\sum_{k=2}^{n} \tilde{\text{ת}}_k \sim \frac{1}{2}\frac{n^2}{\log n},$$

which means that non-coalescing Grassmannians are rare.

In Section 4.4, we also introduce a cumulative function for the number of vector spaces $\mathbb{C}^n$, $2 \le n \le x$, having more than $2x^{\frac{1}{2}}$ non-coalescing Grassmannians of proper subspaces: more precisely, for $x \in \mathbb{R}_{\ge 4}$ we define

$$\widehat{\mathscr{H}}(x) := \operatorname{card}\left\{n\colon \quad 2 \le n \le x \text{ is such that } \tilde{\text{ת}}_n > 2x^{\frac{1}{2}}\right\}.$$

Before stating the results, let us introduce the *prime* Riemann zeta function, together with its truncations

$$\zeta_P(s) := \sum_{p \text{ prime}} \frac{1}{p^s}, \quad \zeta_{P,k}(s) := \sum_{\substack{p \text{ prime} \\ p \le k}} \frac{1}{p^s}.$$

Theorem 1.5 (cf Theorem 4.8, Theorem 4.9). *We have the following results:*

(a) *for any $\kappa > 1$, the following integral representations*[6] *hold*

$$\widehat{\mathscr{H}}(x) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\left[\left(\frac{\zeta(s)}{\zeta\left(s, x^{\frac{1}{2}}+1\right)} - 1\right) - \zeta_{P,2x^{\frac{1}{2}}+1}(s) + \zeta_{P,x^{\frac{1}{2}}+1}(s)\right]\frac{x^s}{s}ds,$$

$$\widehat{\mathscr{H}}(x) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\left[\left(\frac{\zeta(s)}{\zeta\left(s, x^{\frac{1}{2}}+1\right)} - 1\right)x^s + \zeta_P(s)\left((x^{\frac{1}{2}}+1)^s - (2x^{\frac{1}{2}}+1)^s\right)\right]\frac{ds}{s},$$

both valid for $x \in \mathbb{R}_{\ge 2} \setminus \mathbb{N}$, and where $\Lambda_\kappa := \{\kappa + it\colon t \in \mathbb{R}\}$ is the line oriented from $t = -\infty$ to $t = +\infty$.

(b) *The function $\widehat{\mathscr{H}}$ admits the following asymptotic estimate:*

$$\widehat{\mathscr{H}}(x) = \int_0^x \frac{dt}{\log t} + O\left(x^{\Theta}\log x\right), \quad \textit{where } \Theta := \sup\left\{\operatorname{Re}(\rho)\colon \zeta(\rho) = 0\right\}.$$

Hence, it is clear the equivalence of (RH) *with the (essentially optimal) estimate with $\Theta = \frac{1}{2}$.*

Question (i) has already been addressed in [**GGI16**] (Remark 6.2.9): it is claimed, but not proved, that the condition $\gcd(\min(k, n-k)!, n) > 1$ (which is equivalent to the condition $P_1(n) \le k \le n - P_1(n)$) is a necessary condition for coalescence of some canonical coordinates in the small quantum locus of $\mathbb{G}(k, n)$.

[6] The integral must be interpreted as a Cauchy Principal Value.

1.4. Results of Part 2

In Part 2, which is the analytical core of this Thesis, we address to the problem of extending the Theory of Isomonodromic Deformations in order to include in the treatment also linear differential systems which violate one of the main assumptions[7] of the work of M. Jimbo, T. Miwa and K. Ueno [**JMU81**].

1.4.1. General description of the problem. In Part 2 we study deformations of linear differential systems, playing an important role in applications, with a resonant irregular singularity at $z=\infty$. The $n\times n$ linear (deformed) system depends on parameters $t=(t_1,...,t_m)\in\mathbb{C}^m$, (here $n,m\in\mathbb{N}\backslash\{0\}$) and has the following form:

$$\frac{dY}{dz}=\widehat{A}(z,t)Y,\qquad \widehat{A}(z,t)=\Lambda(t)+\sum_{k=1}^{\infty}\widehat{A}_k(t)z^{-k}, \tag{1.16}$$

$$\Lambda(t):=\operatorname{diag}(u_1(t),...,u_n(t)), \tag{1.17}$$

with singularity of Poincaré rank 1 at $z=\infty$, and where $\widehat{A}_k(t)$, $k\geq 1$, and $\Lambda(t)$ are holomorphic matrix valued functions on an open connected domain of $\mathbb{C}^m$.

The deformation theory is well understood when $\Lambda(t)$ has distinct eigenvalues $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ for t in the domain. On the other hand, there are important cases for applications (see below) when two or more eigenvalues may coalesce when t reaches a certain locus Δ in the t-domain, called the *coalescence locus*. This means that $u_a(t)=u_b(t)$ for some indices $a\neq b\in\{1,...,n\}$ whenever t belongs to Δ, while $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ are pairwise distinct otherwise[8]. Points of Δ will be called *coalescence points*. To the best of our knowledge, this case is missing from the existing literature, as we will shortly review later. This is the main problem which we address in Part 2 of the present Thesis, both in the non-isomonodromic and isomonodromic cases. The main results of this part of the Thesis are contained in:

- Theorem 7.1, Corollaries 7.3 and 7.4, and in Theorem 7.2, for the non-isomonodromic case;
- Theorem 1.6 (Th. 8.2), Corollary 1.1 (Corol. 8.3) and Theorem 1.7, for the isomonodromic case.

For the sake of the local analysis at coalescence points, we can restrict to the case when the domain is a polydisk

$$\mathcal{U}_{\epsilon_0}(0):=\{t\in\mathbb{C}^m\quad\text{such that}\quad |t|\leq\epsilon_0\},\qquad |t|:=\max_{1\leq i\leq m}|t_i|$$

for suitable $\epsilon_0>0$, being $t=0$ a point of the coalescence locus. We will again denote by Δ the coalescence locus in $\mathcal{U}_{\epsilon_0}(0)$.

When Δ is not empty, the dependence on t of fundamental solutions of (1.16) near $z=\infty$ is quite delicate. If $t\notin\Delta$, then the system (1.16) has a unique formal solution (see [**HS66**]),

$$Y_F(z,t):=\Big(I+\sum_{k=1}^{\infty}F_k(t)z^{-k}\Big)z^{B_1(t)}e^{\Lambda(t)z},\qquad B_1(t):=\operatorname{diag}(\widehat{A}_1(t)), \tag{1.18}$$

where the matrices $F_k(t)$ are uniquely determined by the equation and are holomorphic on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

In order to find actual solutions, and their domain of definition in the space of parameters t, one can refer to the local existence results of Sibuya [**Sib62**] [**HS66**] (see Theorems 5.1 and 5.3 below),

[7] See [**JMU81**], page 312, assumption that the eigenvalues of $A_{\nu-r_\nu}$ are distinct. See also condition (2) at page 133 of [**FIKN06**].

[8] Δ is a discrete set for $m=1$, otherwise it is a continuous locus for $m\geq 2$. For example, for the matrix $\operatorname{diag}(t_1,t_2,...,t_n)$, the coalescence locus is the union of the *diagonals* $t_i=t_j$, $i\neq j\in\{1,2,...,n\}$.

which guarantees that, given $t_0 \in \mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, there exists a sector and a fundamental solution $Y(z,t)$ holomorphic for $|z|$ large and $|t-t_0|<\rho$, where ρ is sufficiently small, such that $Y(z,t) \sim Y_F(z,t)$ for $z\to\infty$ in the sector. The condition $|t-t_0|$ is restrictive, since ρ is expected to be very small. In the present Thesis, we prove this result for t in a wider domain $\mathcal{V}\subset\mathcal{U}_{\epsilon_0}(0)$, extending $|t-t_0|<\rho$. $\mathcal{V}$ is constructed as follows. Let $t=0$ and consider the *Stokes rays* associated with the matrix $\Lambda(0)$, namely rays in the universal covering $\mathcal{R}$ of the z-punctured plane $\mathbb{C}\backslash\{0\}$, defined by the condition that $\Re e[(u_a(0)-u_b(0))z]=0$, with $u_a(0)\neq u_b(0)$ $(1\le a\neq b\le n)$. Then, consider an *admissible ray*, namely a ray in $\mathcal{R}$, with a certain direction $\widetilde{\tau}$, that does not contain any of the Stokes rays above, namely $\Re e[(u_a(0)-u_b(0))z]\neq 0$ for any $u_a(0)\neq u_b(0)$ and $\arg z=\widetilde{\tau}$. Define the locus $X(\widetilde{\tau})$ to be the set of points $t\in\mathcal{U}_{\epsilon_0}(0)$ such that some Stokes rays $\{z\in\mathcal{R}\mid \Re e[(u_a(t)-u_b(t))z]=0\}$ associated with $\Lambda(t)$, $t\notin\Delta$, coincide with the admissible ray $\arg z=\widetilde{\tau}$. Finally, define a *$\widetilde{\tau}$-cell* to be any connected component of $\mathcal{U}_{\epsilon_0}(0)\backslash(\Delta\cup X(\widetilde{\tau}))$ (see Section 7.2 for a thorough study of the cells). Then, we take an open connected open domain $\mathcal{V}$ such that its closure $\overline{\mathcal{V}}$ is contained in a $\widetilde{\tau}$-cell.

Definition 1.2. The deformation of the linear system (1.20), such that t varies in an open connected domain $\mathcal{V}\subset\mathcal{U}_{\epsilon_0}(0)$, such that $\overline{\mathcal{V}}$ is contained in a $\widetilde{\tau}$-cell, is called an **admissible deformation**[9]. For simplicity, we will just say that t is an admissible deformation.

By definition, an admissible deformation means that as long as t varies within $\overline{\mathcal{V}}$, then *no Stokes rays of* $\Lambda(t)$ *cross the admissible ray* of direction $\widetilde{\tau}$.

If t belongs to a domain $\mathcal{V}$ as above, then we prove in Section 7.5 that there is a family of actual fundamental solutions $Y_r(z,t)$, labelled by $r\in\mathbb{Z}$, uniquely determined by the canonical asymptotic representation

$$Y_r(z,t)\sim Y_F(z,t),$$

for $z\to\infty$ in suitable sectors $\mathcal{S}_r(\overline{\mathcal{V}})$ of the universal covering $\mathcal{R}$ of $\mathbb{C}\backslash\{0\}$. Each $Y_r(z,t)$ is holomorphic in $\{z\in\mathcal{R}\mid |z|\ge N\}\times\mathcal{V}$, for a suitably large N. The asymptotic series $I+\sum_{k=1}^{\infty}F_k(t)z^{-k}$ is uniform in $\overline{\mathcal{V}}$.

The sectors $\mathcal{S}_r(\overline{\mathcal{V}})$ are constructed as follows: take for example the "half plane" $\Pi_1:=\{z\in\mathcal{R}\mid \widetilde{\tau}-\pi<\arg z<\widetilde{\tau}\}$. The open sector containing Π_1 and extending up to the closest Stokes rays of $\Lambda(t)$ outside Π_1 will be called $\mathcal{S}_1(t)$. Then, we define $\mathcal{S}_1(\overline{\mathcal{V}}):=\bigcap_{t\in\overline{\mathcal{V}}}\mathcal{S}_1(t)$. Analogously, we consider the "half-planes" $\Pi_r:=\{z\in\mathcal{R}\mid \widetilde{\tau}+(r-3)\pi<\arg z<\widetilde{\tau}+(r-1)\pi\}$ and repeat the same construction for $\mathcal{S}_r(\overline{\mathcal{V}})$. The sectors $\mathcal{S}_r(\overline{\mathcal{V}})$ have central opening angle greater than π and their successive intersections do not contain Stokes rays $\Re e[(u_a(t)-u_b(t))z]=0$ associated with the eigenvalues of $\Lambda(t)$, $t\in\overline{\mathcal{V}}$. The sectors $\mathcal{S}_r(\overline{\mathcal{V}})$ for $r=1,2,3$ are represented in Figure 1.2. An admissible ray $\arg z=\widetilde{\tau}$ in $\mathcal{S}_1(\overline{\mathcal{V}})\cap\mathcal{S}_2(\overline{\mathcal{V}})$ is also represented.

If the t-analytic continuation of $Y_r(z,t)$ exists outside $\mathcal{V}$, then the delicate points emerge, as follows.

- The expression $\Re e\,[(u_a(t)-u_b(t))z]$, $1\le a\neq b\le n$, has constant sign in the $\widetilde{\tau}$-cell containing $\mathcal{V}$, but it vanishes when a Stokes ray $\Re e\,[(u_a(t)-u_b(t))z]=0$ crosses the admissible direction $\widetilde{\tau}$. This corresponds to the fact that t crosses the boundary of the cell. Then, it changes sign for t outside of the cell. Hence, the asymptotic representation $Y_r(z,t)\sim Y_F(z,t)$ for $z\to\infty$ in $\mathcal{S}_r(\overline{\mathcal{V}})$ does no longer hold for t outside the $\widetilde{\tau}$-cell containing $\mathcal{V}$.
- The coefficients $F_k(t)$ are in general divergent at Δ.
- The locus Δ is expected to be a locus of singularities for the $Y_r(z,t)$'s (see Example 5.4 below).

[9]The definition of *admissible deformation* of a linear system is in accordance with the definition given in [**FIKN06**].

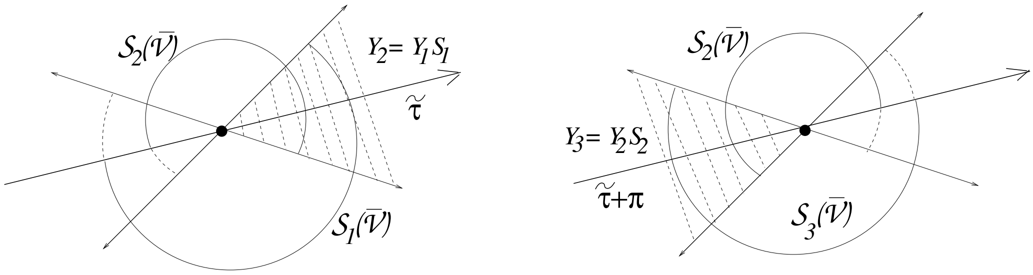

FIGURE 1.2. Stokes phenomenon of formula (1.19). In the left figure is represented the sheet of the universal covering $\widetilde{\tau} - \pi < \arg z < \widetilde{\tau} + \pi$ containing $\mathcal{S}_1(\overline{\mathcal{V}}) \cap \mathcal{S}_2(\overline{\mathcal{V}})$, and in the right figure the sheet $\widetilde{\tau} < \arg z < \widetilde{\tau} + 2\pi$ containing $\mathcal{S}_2(\overline{\mathcal{V}}) \cap \mathcal{S}_3(\overline{\mathcal{V}})$. The rays $\arg z = \widetilde{\tau}$ and $\widetilde{\tau} + \pi$ (and then $\widetilde{\tau} + k\pi$ for any $k \in \mathbb{Z}$) are *admissible rays*, such that $\Re e\Big[(u_a(0) - u_b(0))z\Big] \neq 0$ along these rays, for any $u_a(0) \neq u_b(0)$. Moreover, $\Re e\Big[(u_a(t) - u_b(t))z\Big] \neq 0$ for any $t \in \mathcal{V}$ and any $1 \leq a \neq b \leq n$.

- The *Stokes matrices* $\mathbb{S}_r(t)$, defined for $t \in \mathcal{V}$ by the relations (see Figure 1.2)
$$Y_{r+1}(z,t) = Y_r(z,t)\ \mathbb{S}_r(t), \tag{1.19}$$
are expected to be singular as t approaches Δ.

REMARK 1.1. It is well known that, in order to completely describe the Stokes phenomenon, it suffices to consider only three fundamental solutions, for example $Y_r(z,t)$ for $r = 1, 2, 3$, and $\mathbb{S}_1(t)$, $\mathbb{S}_2(t)$.

The matrix $\widehat{A}(z,t)$ may have other singularities at finite values of z. In the isomonodromic case, we will consider $\widehat{A}(z,t)$ with a simple pole at $z = 0$, namely
$$\frac{dY}{dz} = \widehat{A}(z,t)Y, \qquad \widehat{A}(z,t) = \Lambda(t) + \frac{\widehat{A}_1(t)}{z}. \tag{1.20}$$
An isomonodromic system of type (1.20) with antisymmetric $\widehat{A}_1(t)$, is at the core of the analytic approach to semisimple Frobenius manifolds [**Dub96**] [**Dub98**] [**Dub99b**] (see also [**Sai93**] [**Sai83**] [**SYS80**] [**Man99**] [**Sab08**]). Its monodromy data play the role of local moduli. Coalescence of eigenvalues of $\Lambda(t)$ occurs in important cases, such as quantum cohomology (see [**Cot16**] [**CDG17c**] and Section 1.5.4 below). For $n = 3$, a special case of system (1.20) gives an isomonodromic description of the general sixth Painlevé equation, according to [**Maz02**] (see also [**Har94**]). This description was given also in [**Dub96**] [**Dub99b**] for a sixth Painlevé equation associated with Frobenius manifolds. Coalescence occurs at the critical points of the Painlevé equation (see Section 10.3).

For given t, a matrix $G^{(0)}(t)$ puts $\widehat{A}_1(t)$ in Jordan form
$$J^{(0)}(t) := (G^{(0)}(t))^{-1}\widehat{A}_1(t)\ G^{(0)}(t).$$
Close to the Fuchsian singularity $z = 0$, and for a given t, the system (1.20) has a fundamental solution
$$Y^{(0)}(z,t) = G^{(0)}(t)\left(I + \sum_{l=1}^{\infty}\Psi_l(t)z^l\right) z^{D^{(0)}(t)} z^{S^{(0)}(t)+R^{(0)}(t)}, \tag{1.21}$$
in standard Birkhoff–Levelt normal form, whose behaviour in z and t is not affected by the coalescence phenomenon. The matrix coefficients $\Psi_l(t)$ of the convergent expansion are constructed by a recursive

procedure. $D^{(0)}(t) = \mathrm{diag}(d_1(t), ..., d_n(t))$ is a diagonal matrix of integers, piecewise constant in t, $S^{(0)}(t)$ is a Jordan matrix whose eigenvalues $\rho_1(t), ..., \rho_n(t)$ have real part in $[0, 1[$, and the nilpotent matrix $R^{(0)}(t)$ has non-vanishing entries only if some eigenvalues of $\widehat{A}_1(t)$ differ by non-zero integers. If some eigenvalues differ by non-zero integers, we say that $\widehat{A}_1(t)$ is *resonant*. The sum

$$J^{(0)}(t) = D^{(0)}(t) + S^{(0)}(t)$$

is the Jordan form of $\widehat{A}_1(t)$ above. Under the assumptions of our Theorem 1.6 below, the solution (1.21) turns out to be holomorphic in $t \in \mathcal{U}_{\epsilon_0}(0)$.

Chosen a solution $Y^{(0)}(z, t)$ with normal form (1.21), a *central connection matrix* $C^{(0)}$ is defined by the relation

$$Y_1(z, t) = Y^{(0)}(z, t)\ C^{(0)}(t), \qquad z \in \mathcal{S}_1(\overline{\mathcal{V}}). \tag{1.22}$$

Then, the *essential monodromy data* of the system (1.20) are defined to be

$$\mathbb{S}_1(t), \quad \mathbb{S}_2(t), \quad B_1(t) = \mathrm{diag}(\widehat{A}_1(t)), \quad C^{(0)}(t), \quad J^{(0)}(t), \quad R^{(0)}(t). \tag{1.23}$$

Now, when t tends to a point $t_\Delta \in \Delta$, the limits of the above data may not exist. If the limits exist, they *do not in general give the monodromy data of the system* $\widehat{A}(z, t_\Delta)$. The latter have in general different nature, as it is clear from the results of [**BJL79c**], and from Section 5.3 below.[10]

Definition 1.3. If the deformation is admissible in a domain $\mathcal{V}$, as in Definition 1.2, we say that it is **isomonodromic in** $\mathcal{V}$ if the essential monodromy data (1.23) do not depend on $t \in \mathcal{V}$.

When this definition holds, the classical theory of Jimbo-Miwa-Ueno [**JMU81**] applies.[11] We are interested in extending the deformation theory to the whole $\mathcal{U}_{\epsilon_0}(0)$, including the coalescence locus Δ.

1.4.2. Main Results. A) The non-isomonodromic case of system 1.16 In Chapter 5, Chapter 6 and Chapter 7, we study system (1.16) without requiring that the deformation is isomonodromic. Referring the reader to the main body of the Thesis for more details, we just mention the main results obtained:

(1) necessary and sufficient conditions for the holomorphy at Δ of the formal solutions (1.18) are given (Proposition 5.3).
(2) In Section 7.5 we prove that the fundamental solutions $Y_r(z, t)$, $r \in \mathbb{Z}$, of (1.16) can be t-analytically continued to a whole $\widetilde{\tau}$-cell containing the domain $\mathcal{V}$ of Definition 1.2, preserving the asymptotic representation (1.18).
(3) In Theorem 7.1 we give sufficient conditions for the holomorphy at Δ of fundamental solutions $Y_r(z, t)$, together with their Stokes matrices $\mathbb{S}_r(t)$, so that *the asymptotic representation* $Y_r(z, t) \sim Y_F(z, t)$ *continues to hold* in a wider sector $\widehat{\mathcal{S}}_r$ containing $\mathcal{S}_r(\overline{\mathcal{V}})$, to be introduced below (Section 7.6.2).
(4) eventually, we show that in this case, the limits

$$\lim_{t \to t_\Delta} \mathbb{S}_r(t), \qquad t_\Delta \in \Delta, \quad \text{exist and are finite,} \tag{1.24}$$

and they coincide with the Stokes matrices of system 1.16 with matrix coefficient $\widehat{A}(z, t_\Delta)$ (see Corollary 7.3 and 7.4).

[10] See for example the solution (5.37), where it is evident that the monodromy datum L, defined at $t = 0$, is not the limit for $t \to 0$ of $B_1(t)$ as in (1.18).

[11] Notice that in [**JMU81**] it is also assumed that $A_1(t)$ is diagonalisable with eigenvalues not differing by integers. We do not make this assumption here.

In the analysis of the above issues, wall crossing phenomena and cell decompositions of $\mathcal{U}_{\epsilon_0}(0)$ will be studied. Another result on the analytic ocntinuation of fundamental solutions, with vanishing conditions on the Stokes matrices, is given in Theorem 7.2.

B) Isomonodromic case of system (1.20). Let the deformation be isomonodromic in $\mathcal{V}$, as in Definition 1.3, so that the classical theory of Jimbo-Miwa-Ueno applies. As a result of [**JMU81**], the eigenvalues can be chosen as the independent deformation parameters. This means that we can assume[12] linearity in $t \in \mathcal{U}_{\epsilon_0}(0)$, as follows:

$$u_a(t) = u_a(0) + t_a, \quad 1 \leq a \leq n \quad \Longrightarrow \quad m = n. \tag{1.25}$$

Therefore,

$$\Lambda(t) = \Lambda(0) + \mathrm{diag}(t_1, \ \ldots, \ t_n)$$

with

$$\Lambda(0) = \Lambda_1 \oplus \cdots \oplus \Lambda_s, \quad s < n, \qquad \Lambda_i = \lambda_i I_{p_i}, \tag{1.26}$$

where λ_1, ..., λ_s are the $s < n$ *distinct* eigenvalues of $\Lambda(0)$, of respectively multiplicities $p_1, \ldots, p_s$ ($p_1 + \cdots + p_s = n$). Here, I_{p_i} is the $p_i \times p_i$ identity matrix. Now, the size ϵ_0 of $\mathcal{U}_{\epsilon_0}(0)$ is taken sufficiently small so that we can write

$$\Lambda(t) = \Lambda_1(t) \oplus \cdots \oplus \Lambda_s(t), \tag{1.27}$$

with the properties that $\lim_{t\to 0} \Lambda_j(t) = \lambda_j I_{p_j}$, and that $\Lambda_i(t)$ and $\Lambda_j(t)$ have no common eigenvalues for $i \neq j$. The following result extends the isomonodromy deformation theory from $\mathcal{V}$ to the whole $\mathcal{U}_{\epsilon_0}(0)$ in this case.

Theorem 1.6 (cf. Theorem 8.2). *Consider the system (1.20), with eigenvalues of $\Lambda(t)$ linear in t as in (1.25), and with $A_1(t)$ holomorphic on a closed polydisc $\mathcal{U}_{\epsilon_0}(0)$ centred at $t = 0$, with sufficiently small radius ϵ_0 as specified in Section 7.6.1. Let Δ be the coalescence locus in $\mathcal{U}_{\epsilon_0}(0)$, passing through $t = 0$. Let the dependence on t be isomonodromic in a domain $\mathcal{V}$ as in Definition 1.3.*

If the matrix entries of $\widehat{A}_1(t)$ satisfy in $\mathcal{U}_{\epsilon_0}(0)$ the vanishing conditions

$$\left(\widehat{A}_1(t)\right)_{ab} = \mathcal{O}(u_a(t) - u_b(t)), \qquad 1 \leq a \neq b \leq n, \tag{1.28}$$

whenever $u_a(t)$ and $u_b(t)$ coalesce as t tends to a point of Δ, then the following results hold:

- *The formal solution $Y_F(z,t)$ of (1.20) as given in (1.18) is holomorphic on the whole $\mathcal{U}_{\epsilon_0}(0)$.*
- *The three fundamental matrix solutions $Y_r(z,t)$, $r = 1, 2, 3$, of the system of (1.20), which are defined on $\mathcal{V}$, with asymptotic representation $Y_F(z,t)$ for $z \to \infty$ in sectors $\mathcal{S}_r(\overline{\mathcal{V}})$ introduced above, can be t-analytically continued as single-valued holomorphic functions on $\mathcal{U}_{\epsilon_0}(0)$, with asymptotic representation*

$$Y_r(z,t) \sim Y_F(z,t), \qquad z \to \infty \text{ in } \widehat{\mathcal{S}}_r,$$

for any $t \in \mathcal{U}_{\epsilon_1}(0)$, any $0 < \epsilon_1 < \epsilon_0$, and where $\widehat{\mathcal{S}}_r$ are wider sectors, containing $\mathcal{S}_r(\overline{\mathcal{V}})$, to be introduced in Section 7.6.2. In particular, they are defined at any $t_\Delta \in \Delta$ with asymptotic representation $Y_F(z, t_\Delta)$. The fundamental matrix solution $Y^{(0)}(z,t)$ is also t-analytically continued as a single-valued holomorphic function on $\mathcal{U}_{\epsilon_0}(0)$

[12] This assumption will be used in the Thesis starting from Section 7.6.2.

- *The constant Stokes matrices* $\mathbb{S}_1$, $\mathbb{S}_2$, *and a central connection matrix* $C^{(0)}$, *initially defined for* $t \in \mathcal{V}$, *are actually* globally defined on $\mathcal{U}_{\epsilon_0}(0)$. *They coincide with the Stokes and connection matrices of the fundamental solutions* $Y_r(z,0)$ *and* $Y^{(0)}(z,0)$ *of the system*

$$\frac{dY}{dz} = \widehat{A}(z,0)Y, \qquad \widehat{A}(z,0) = \Lambda(0) + \frac{\widehat{A}_1(0)}{z}. \tag{1.29}$$

Also the remaining t*-independent monodromy data in (1.23) coincide with those of (1.29).*

- *The entries* (a,b) *of the Stokes matrices are characterised by the following vanishing property:*

$$(\mathbb{S}_1)_{ab} = (\mathbb{S}_1)_{ba} = (\mathbb{S}_2)_{ab} = (\mathbb{S}_2)_{ba} = 0 \quad \textit{whenever } u_a(0) = u_b(0), \quad 1 \le a \ne b \le n. \tag{1.30}$$

Theorem 1.6 allows to holomorphically define the fundamental solutions and the monodromy data on the whole $\mathcal{U}_{\epsilon_0}(0)$, under the only condition (1.28). This fact is remarkable. Indeed, according to [**Miw81**], in general the solutions $Y^{(0)}(z,t)$, $Y_r(z,t)$ and $\widehat{A}(z,t)$, $t \in \mathcal{V}$, of monodromy preserving deformation equations can be analytically continued as meromorphic matrix valued functions on the universal covering of $\mathbb{C}^n \backslash \Delta_{\mathbb{C}^n}$, where $\Delta_{\mathbb{C}^n} = \bigcup_{a \ne b}^n \{u_a(t) = u_b(t)\}$ is the coalescence locus in $\mathbb{C}^n$. They have fixed singularities at the branching locus $\Delta_{\mathbb{C}^n}$, and so at $\Delta \subset \Delta_{\mathbb{C}^n}$. Moreover, the t-analytic continuation on $\mathcal{U}_{\epsilon_0}(0)$ of a the solutions $Y_r(z,t)$ are expected to lose their asymptotic representation $Y_r(z,t) \sim Y_F(z,t)$ in $\mathcal{S}_r(\overline{\mathcal{V}})$, when t moves sufficiently far away from $\mathcal{V}$, namely when Stokes rays cross and admissible ray of direction $\widetilde{\tau}$. Under the assumptions of Theorem 1.6 these singular behaviours do not occur.

Then, under the assumptions of Theorem 1.6, the system (1.29) has a formal solutions (here we denote objects Y, $\mathbb{S}$ and C referring to the system (1.29) with the symbols $\mathring{Y}$, $\mathring{\mathbb{S}}$ and $\mathring{C}$) with behaviour[13]

$$\mathring{Y}_F(z) = \Big(I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k}\Big) z^{B_1(0)} e^{\Lambda(0)z}, \qquad B_1(0) = \mathrm{diag}(\widehat{A}_1(0)). \tag{1.31}$$

The matrix-coefficients $\mathring{F}_k$ are recursively constructed from the equation (1.29), but not uniquely determined. Actually, there is a family of formal solutions as above, depending on a finite number of complex parameters. To each element of the family, there correspond unique actual solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$ such that $\mathring{Y}_r(z) \sim \mathring{Y}_F(z)$ for $z \to \infty$ in a sector $\mathcal{S}_r \supset \mathcal{S}_r(\overline{\mathcal{V}})$, $r = 1,2,3$, with Stokes matrices defined by

$$\mathring{Y}_{r+1}(z) = \mathring{Y}(z)\, \mathring{\mathbb{S}}_r, \qquad r = 1,2.$$

Only one element of the family of formal solutions (1.31) satisfies the condition $\mathring{F}_k = F_k(0)$ for any $k \ge 1$, and by Theorem 1.6 the relations $\mathbb{S}_r = \mathring{\mathbb{S}}_r$ hold. Let us choose a solution $\mathring{Y}^{(0)}(z)$ close to $z = 0$ in the Birkhoff-Levelt normal form, and define the corresponding central connection matrix $\mathring{C}^{(0)}$ such that

$$\mathring{Y}_1(z) = \mathring{Y}^{(0)}(z)\, \mathring{C}^{(0)}.$$

Corollary 1.1 (cf. Corollary 8.3). *Let the assumptions of Theorem 1.6 hold. If the diagonal entries of* $\widehat{A}_1(0)$ *do not differ by non-zero integers, then there is a unique formal solution (1.31) of the system (1.29), whose coefficients necessarily satisfy the condition*

$$\mathring{F}_k \equiv F_k(0).$$

[13] If the vanishing condition (1.28) fails, formal solutions are more complicated (see Theorem 5.2).

Hence, (1.29) only has at $z=\infty$ canonical fundamental solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$, which coincide with the canonical solutions $Y_1(z,t)$, $Y_2(z,t)$, $Y_3(z,t)$ of (1.20) evaluated at $t=0$, namely:

$$Y_1(z,0)=\mathring{Y}_1(z),\quad Y_2(z,0)=\mathring{Y}_2(z),\quad Y_3(z,0)=\mathring{Y}_3(z).$$

Moreover, for any $\mathring{Y}^{(0)}(z)$ there exists $Y^{(0)}(z,t)$ such that $Y^{(0)}(z,0)=\mathring{Y}^{(0)}(z)$. The following equalities hold:

$$\mathbb{S}_1=\mathring{\mathbb{S}}_1,\quad \mathbb{S}_2=\mathring{\mathbb{S}}_2,\quad C^{(0)}=\mathring{C}^{(0)}.$$

Corollary 1.1 has a practical computational importance: the *constant monodromy data* (1.23) of the system (1.20) on the whole $\mathcal{U}_{\epsilon_0}(0)$ are computable just by considering the system (1.29) at the coalescence point $t=0$. This is useful for applications in the following two cases.

a) When $\widehat{A}_1(t)$ is known in a whole neighbourhood of a coalescence point, but the computation of monodromy data, which is highly transcendental, can be explicitly done (only) at a coalescence point, where (1.20) simplifies due to (1.28). An example is given in Chapter 10 for the A_3-Frobenius manifold, which in Section 10.3 will be recast in terms of the sixth Painlevé equation PVI_μ.

b) When $\widehat{A}_1(t)$ is *explicitly known only* at a coalescence point. This may happen in the case of Frobenius manifolds, as already explained in Section 1.3.2. Our result is at the basis of the extension of the theory, as it will be thoroughly exposed in Chapter 9. Theorem 1.6 and Corollary 1.1 allows the computation of local moduli (monodromy data) of a semisimple Frobenius manifold just by considering a coalescence point. The link between the notations of Part 2 (and of [**CDG17b**]) and the usual ones of the general theory of Frobenius manifolds, used in Part 3 (resp. [**CDG17c**]), will be established in Section 1.5.1.

In Part 2 of this Thesis, we also prove Theorem 1.7 below, which is the converse of Theorem 1.6. Assume that the system is isomonodromic on a simply connected domain $\mathcal{V}\subset\mathcal{U}_{\epsilon_0}(0)$ as in Definition 1.2. As a result of [**Miw81**], the fundamental solutions $Y_r(z,t)$, $r=1,2,3$, and $\widehat{A}_1(t)$ can be analytically continued as meromorphic matrix valued functions on the universal covering of $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, with movable poles at the Malgrange divisor [**Pal99**], [**Mal83a**], [**Mal83b**], [**Mal83c**]. The coalescence locus Δ is in general a fixed branching locus. Moreover, although for $t\in\mathcal{V}$ the fundamental solutions $Y_r(z,t)$ have in $\mathcal{S}_r(\overline{\mathcal{V}})$ the canonical asymptotic behavior $Y_F(z,t)$ as in (1.18), in general this is no longer true when t moves sufficiently far away from $\mathcal{V}$.

Nevertheless, if the vanishing condition (1.30) on Stokes matrices holds, then we can prove that the fundamental solutions $Y_r(z,t)$ and $\widehat{A}_1(t)$ have *single-valued* meromorphic continuation on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, so that Δ *is not a branching locus*. Moreover, the asymptotic behaviour is preserved, according to the following

THEOREM 1.7 (cf. Section 8.5). *Let ϵ_0 be as small as in Section 7.6.1. Consider the system (1.20). Let the matrix $\widehat{A}_1(t)$ be holomorphic on an open simply connected domain $\mathcal{V}\subset\mathcal{U}_{\epsilon_0}(0)$ such that the deformation is admissible and isomonodromic as in Definitions 1.2 and 1.3. Assume that the entries of the constant Stokes matrices satisfy the vanishing condition*

$$(\mathbb{S}_1)_{ab}=(\mathbb{S}_1)_{ba}=(\mathbb{S}_2)_{ab}=(\mathbb{S}_2)_{ba}=0\quad \textit{whenever } u_a(0)=u_b(0),\quad 1\le a\neq b\le n.$$

Then, as functions of t, the fundamental solutions $Y_r(z,t)$ and $\widehat{A}_1(t)$ admit single-valued meromorphic continuation on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. Moreover, for any $t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ which is not a pole of $Y_r(z,\tilde{t})$ (i.e. which *is not a point of the Malgrange divisor), we have*

$$Y_r(z,t) \sim Y_F(z,t) \quad \textit{for } z\to\infty \textit{ in } \widehat{\mathcal{S}}_r(t), \qquad r=1,2,3,$$

and

$$Y_{r+1}(z,t)=Y_r(z,t)\ \mathbb{S}_r, \quad r=1,2.$$

The $\widehat{\mathcal{S}}_r(t)$'s are the wide sectors introduced in Section 7.6.2.

Remark 1.2. In the main body of Part 2, the matrices Y_r, sectors $\mathcal{S}_r$ and Stokes matrices $\mathbb{S}_r$ will be labelled differently as $Y_{\nu+(r-1)\mu}$, $\mathcal{S}_{\nu+(r-1)\mu}$ and $\mathbb{S}_{\nu+(r-1)\mu}$, $\nu,\mu\in\mathbb{Z}$. This labelling will be explained.

For a detailed comparison of our results with the ones available in literature, the reader can see Section 7.8 and Section 8.6.

1.5. Results of Part 3

In Part 3 of the Thesis, we apply the results obtained in the previous Part 2 to the isomonodromic linear differential systems associated with semisimple Frobenius manifolds. The result is an extension of the Isomonodromy Theorems also at semisimple coalescence points. Furthermore, we apply this result in two examples, the first one for Frobenius structure related to singularity theory, the second one for quantum cohomology of Grassmannians.

1.5.1. Notational Dictionary. First of all, let us establish a translation from notations of Section 1.2 with the ones of Section 1.4. Notice that the system (1.5) is of type (1.20), and if we write $u=u(t)$ as in (1.25), then the following identification holds

$$U\equiv\Lambda(t), \qquad V(u(t))\equiv\widehat{A}_1(t).$$

It is *crucial* for our discussion that the matrix $\Psi(u)$, which gives a change of basis between flat coordinate vector fields and normalized idempotents, is always holomorphic and invertible at semisimple points, also when U has coalescing eigenvalues there. The proof of this fact is given in Chapter 2. Therefore, the matrices $V_k(u)$ of (1.6) are holomorphic at semisimple points. $\Psi(u)$ diagonalises $V(u)$, with constant eigenvalues $\mu_1,\dots,\mu_n$ independent of the point of the manifold (see [**Dub96**],[**Dub99b**]):

$$V(u)=\Psi(u)\ \mu\ \Psi(u)^{-1}, \qquad \mu:=\operatorname{diag}(\mu_1,\mu_2,\dots,\mu_n).$$

Therefore, $V(u)$ is holomorphically similar to μ at semisimple points.

The system (1.5) admits a normal form at $z=0$ such that the corresponding fundamental matrix, denoted

$$Y_0(z,u)=\Big(\Psi(u)+\sum_{l=1}^{\infty}\Phi_l(u)z^l\Big)z^{\mu}z^{R}, \tag{1.32}$$

has monodromy exponent R independent of the point of the manifold. $Y_0(z,u)$ is holomorphic of u on the domain where $V(u)$ is holomorphic. In our notations, $R\equiv R^{(0)}$, and $Y_0\equiv Y^{(0)}$, as in (1.21).

Next, we establish the translation between Stokes and central connection matrices as defined in Section 1.2 and Section 1.4. As in Section 1.2, we consider an oriented ray $\ell_+(\phi):=\{z\in\mathcal{R}\mid \arg z=\phi\}$ and (for $\epsilon>0$ small) the two sectors $\Pi^{\epsilon}_{\text{left/right}}(\phi)$ of (1.12). As explained before, the choice of such a line gives a ℓ-chamber decomposition of the Frobenius manifold. Let $\mathcal{V}$ be an open connected domain such that $\overline{\mathcal{V}}$ is contained in an ℓ-chamber. For suitable ϵ, we can identify[14]

$$e^{-2\pi i}\Pi^{\epsilon}_{\text{left}}(\phi)=\mathcal{S}_1(\overline{\mathcal{V}}),\qquad \Pi^{\epsilon}_{\text{right}}(\phi)=\mathcal{S}_2(\overline{\mathcal{V}}),\qquad \Pi^{\epsilon}_{\text{left}}(\phi)=\mathcal{S}_3(\overline{\mathcal{V}}),\tag{1.33}$$

where $e^{-2\pi i}\Pi^{\epsilon}_{\text{left}}(\phi):=\{z\in\mathcal{R}\ |z=\zeta e^{-2\pi i},\ \zeta\in\Pi^{\epsilon}_{\text{left}}(\phi)\}$, and $\mathcal{S}_r(\overline{\mathcal{V}})$ is defined in the previous Section 1.4. Let $Y_{\text{left}}(z,u)$, $Y_{\text{right}}(z,u)$ be the unique fundamental matrix solutions having the canonical asymptotics $Y_F(z,u)=(I+O(1/z))e^{zU}$ in $\Pi^{\epsilon}_{\text{left}}(\phi)$ and $\Pi^{\epsilon}_{\text{right}}(\phi)$ respectively. The Stokes matrices S and S_- of [**Dub99b**] are defined by the relations,

$$Y_{\text{left}}(z,u)=Y_{\text{right}}(z,u)S,\qquad Y_{\text{left}}(e^{2\pi i}z,u)=Y_{\text{right}}(z,u)S_-,\qquad z\in\mathcal{R}.\tag{1.34}$$

The symmetries of the system (9.4) imply that $S_-=S^T$. In our notations as in (1.19), the Stokes matrices are defined by

$$Y_3(z,u)=Y_2(z,u)\mathbb{S}_2,\qquad Y_2(z,u)=Y_1(z,u)\mathbb{S}_1.\tag{1.35}$$

We identify

$$Y_3(z,u)=Y_{\text{left}}(z,u),\qquad Y_2(z,u)=Y_{\text{right}}(z,u).\tag{1.36}$$

Let B_1 denote the exponent of formal monodromy[15] at $z=\infty$, so that the relation $Y_1(ze^{-2\pi i},u)=Y_3(z,u)e^{-2\pi i B_1}$ holds.[16] Since V is skew symmetric and $B_1=$ diag $(V)=0$, the above relation reduces to

$$Y_1(ze^{-2\pi i},u)=Y_{\text{left}}(z,u).$$

Therefore (1.35) coincides with (1.34), with

$$S_-=\mathbb{S}_1^{-1},\qquad S=\mathbb{S}_2.$$

The central connection matrix such that $Y_1=Y^{(0)}C^{(0)}$ was defined in (1.22) (see also Definition 8.1). In the theory of Frobenius manifolds, such as in [**CDG17c**], the central connection matrix is denoted by C, defined by

$$Y_{\text{right}}(z,u)=Y_0(z,u)C.$$

Since $Y_0=Y^{(0)}$, $Y_{\text{right}}=Y_2$, $Y_2=Y_1\mathbb{S}_1$, and $\mathbb{S}_1^{-1}=S^T$, then

$$C^{(0)}=C\mathbb{S}_1^{-1}=CS^T.$$

Summarising, monodromy data of a semisimple Frobenius manifold are μ, R, S, C, versus the monodromy data $\mu_1,...,\mu_n$, $R^{(0)}$, $\mathbb{S}_1,\mathbb{S}_2$, $C^{(0)}$ of Part 2.

1.5.2. Isomonodromy Theorem at semisimple coalescence points. Coalescence points for U in (1.5) are singular points for the monodromy preserving deformation equations (1.7)-(1.8). Their study is at the core of the analytic continuation of Frobenius structures. Our Theorem 1.6 allows to extend the isomonodromic approach to Frobenius manifolds at coalescence points if the manifold is

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[14] In the notation used in the main body of Part 2,

$$e^{-2\pi i}\Pi_{\text{left}}=\mathcal{S}_\nu(\overline{\mathcal{V}}),\qquad \Pi_{\text{right}}=\mathcal{S}_{\nu+\mu}(\overline{\mathcal{V}}),\qquad \Pi_{\text{left}}=\mathcal{S}_{\nu+2\mu}(\overline{\mathcal{V}}),\qquad \text{for } \tau_\nu<\widetilde{\tau}<\tau_{\nu+1}.$$

[15] In general, a formal solution is $(I+\sum_{k=1}^{\infty}F_k(u)z^{-k})z^{B_1}e^{zU}$, but in case of Frobenius manifolds $B_1=0$.

[16] In the notation of the main body of the Thesis, $Y_r\mapsto Y_{\nu+(r-1)\mu}$, $r=1,2,3$, $\mathbb{S}_1\mapsto\mathbb{S}_\nu$, $\mathbb{S}_2\mapsto\mathbb{S}_{\nu+\mu}$ and $Y_\nu(z_{(\nu)})=Y_{\nu+2\mu}(z_{(\nu+2\mu)})e^{-2\pi i L}$, where $z_{(\nu+(r-1)\mu)}\in\mathcal{S}_{\nu+(r-1)\mu}(\overline{\mathcal{V}})$ is seen as a point of $\mathcal{R}$ and not of $\mathbb{C}$.

semisimple at these points. Let $u^{(0)}=(u_1^{(0)},...,u_n^{(0)})$ denote a coalescence point. By a change $Y\mapsto PY$ in (1.5), given by a permutation matrix P, there is no loss of generality in assuming that

$$\begin{aligned} u_1^{(0)}=\cdots=u_{p_1}^{(0)} &=: \lambda_1\\ u_{p_1+1}^{(0)}=\cdots=u_{p_1+p_2}^{(0)} &=: \lambda_2\\ &\vdots\\ u_{p_1+\cdots+p_{s-1}+1}^{(0)}=\cdots=u_{p_1+\cdots+p_{s-1}+p_s}^{(0)} &=: \lambda_s, \end{aligned}$$

where $p_1,...,p_s$ are integers such that $p_1+\cdots+p_s=n$, and $\lambda_j\neq\lambda_k$ for $j\neq k$. In order to have a correspondence with [Dub99b], as in formula (1.33) and (1.36), we take the ray $\ell_+(\phi)$ with

$$\phi=\widetilde{\tau}+\pi \quad \mod 2\pi, \tag{1.37}$$

where $\widetilde{\tau}$ is the direction of an admissible ray for U at the point $u^{(0)}$, i.e. not containing any Stokes rays. Similarly to what done in Section 1.4.2, we consider a sufficiently small positive number ϵ_0 (specified in Section 9.1), and we introduce the neighbourhood (polydisc) of $u^{(0)}$ defined by

$$\mathcal{U}_{\epsilon_0}(u^{(0)}):=\Big\{u\in\mathbb{C}^n\ \Big|\ |u-u_0|\leq\epsilon_0\Big\}$$

and denote by Δ the coalescence locus passing through $u^{(0)}$, namely

$$\Delta:=\{u(p)\in\mathcal{U}_{\epsilon_0}(u^{(0)})\mid u_i=u_j \text{ for some } i\neq j\}.$$

If $u^{(0)}$ is a semisimple coalescence point, then the Frobenius Manifold M is semisimple in $\mathcal{U}_{\epsilon_0}(u^{(0)})$ for sufficiently small ϵ_0 (if necessary, we further restrict ϵ_0). Given the above assumption of semisimplicity, then $\Psi(u)$ is holomorphic at Δ and this implies that $V(u)$ is holomorphically similar to μ. Equation (1.7) for $k=i$ is $V_{ij}=(u_i-u_j)(V_i)_{ij}$, which implies that $V_{ij}(u)=0$ for $i\neq j$ and $u_i=u_j$. Therefore, recalling that $V(u^{(0)})$ corresponds to $\widehat{A}_1(0)$, we conclude that the vanishing condition (1.28) holds true and then our Theorem 1.6 applies. We note that diag $\Big(V(u^{(0)})\Big)=0$, then the diagonal entries of $\widehat{A}_1(0)$ do not differ by non-zero integers, so that also Corollary 1.1 applies. Then, the following result holds:

THEOREM 1.8 (cf. Theorem 9.1).

(1) *System (1.5) at the* fixed *value* $u=u^{(0)}$ *admits a unique formal solution, which we denote with* $\mathring{Y}_{\rm formal}(z)$*, having the structure* (1.11)*, namely* $\mathring{Y}_{\rm formal}(z)=\Big(\mathbb{1}+\sum_{k=1}^{\infty}\mathring{G}_kz^{-k}\Big)e^{zU}$*; moreover, it admits unique fundamental solutions, which we denote with* $\mathring{Y}_{\rm left/right}(z)$*, having asymptotic representation* $\mathring{Y}_{\rm formal}(z)$ *in sectors* $\Pi^{\varepsilon}_{\rm left/right}(\phi)$*, for suitable* $\varepsilon>0$ *(precisely quantified in the main body of Cahpter 9). Let* $\mathring{S}$ *be the Stokes matrix such that*
$$\mathring{Y}_{\rm left}(z)=\mathring{Y}_{\rm right}(z)\mathring{S}.$$

(2) *The coefficients* $G_k(u)$*,* $k\geq 1$*, in* (1.11) *are holomorphic over* $\mathcal{U}_{\epsilon_0}(u^{(0)})$*, and* $G_k(u^{(0)})=\mathring{G}_k$*; moreover* $Y_{\rm formal}(z,u^{(0)})=\mathring{Y}_{\rm formal}(z)$*.*

(3) $Y_{\text{left}}(z,u)$, $Y_{\text{right}}(z,u)$, *computed in a neighbourhood of a point* $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})\backslash\Delta$, *can be u-analytically continued as single-valued holomorphic functions on the whole* $\mathcal{U}_{\epsilon_0}(u^{(0)})$. *Moreover*

$$Y_{\text{left/right}}(z,u^{(0)}) = \mathring{Y}_{\text{left/right}}(z).$$

(4) *For any* $\epsilon_1 < \epsilon_0$, *the asymptotic relations*

$$Y_{\text{left/right}}(z,u)e^{-zU} \sim I + \sum_{k=1}^{\infty} G_k(u), \qquad \text{for } z \to \infty \text{ in } \Pi^{\varepsilon}_{\text{left/right}}(\phi),$$

hold uniformly in $u \in \mathcal{U}_{\epsilon_1}(u^{(0)})$. *In particular they also hold at* $u \in \Delta$.

(5) *Denote with* $\mathring{Y}_0(z)$ *a solution of system (1.5) with* fixed *value* $u = u^{(0)}$, *in Levelt form* $\mathring{Y}_0(z) = \Psi(u^{(0)})(\mathbb{1} + O(z))z^{\mu}z^{\mathring{R}}$, *having monodromy data* μ *and* $\mathring{R}$. *For any such* $\mathring{Y}_0(z)$ *there exists a fundamental solution* $Y_0(z,u)$ *in Levelt form* (1.9), *holomorphic in* $\mathcal{U}_{\epsilon_0}(u^{(0)})$, *such that its monodromy data* μ *and* R *are independent of* u *and*

$$Y_0(z,u^{(0)}) = \mathring{Y}_0(z), \qquad R = \mathring{R}. \tag{1.38}$$

Let $\mathring{C}$ *be the central connection matrix for* $\mathring{Y}_0$ *and* $\mathring{Y}_{\text{right}}$; *namely*

$$\mathring{Y}_{\text{right}}(z) = \mathring{Y}_0(z)\mathring{C}.$$

(6) *For any* $\epsilon_1 < \epsilon_0$, *the monodromy data* μ, R, S, C *of system* (1.5) *are defined and constant in the whole* $\mathcal{U}_{\epsilon_1}(u^{(0)})$, *namely the system is isomonodromic in* $\mathcal{U}_{\epsilon_1}(u^{(0)})$. *They coincide with the data* μ, $\mathring{R}$, $\mathring{S}$ *and* $\mathring{C}$ *associated to fundamental solutions* $\mathring{Y}_{\text{left/right}}(z)$ *and* $\mathring{Y}_0(z)$ *above. In particular, the entries of* $S = (S_{ij})_{i,j=1}^n$ *with indices corresponding to coalescing canonical coordinates vanish, namely:*

$$S_{ij} = S_{ji} = 0 \quad \text{for all } i \neq j \text{ such that } \quad u_i^{(0)} = u_j^{(0)}. \tag{1.39}$$

We recall that the monodromy data for the whole manifold can be computed by an action of the braid group (see [**Dub96**], [**Dub99b**] and [**CDG17c**]) staring from the data obtained in $\mathcal{U}_{\epsilon_1}(u^{(0)})$. Hence, our result allows to obtain the monodromy data for the whole manifold from the data computed at a coalescence point. This relevant fact is important in the following two cases:

a) The Frobenius structure (i.e. $V(u)$ in (1.5)) is known everywhere, but the computation of monodromy data is extremely difficult – or impossible – at generic semisimple points where $U = \text{diag}(u_1,...,u_n)$ has distinct eigenvalues. On the other hand, the system (1.5) at a coalescence point simplifies, so that we may be able to explicitly solve it in terms of special functions and compute S and C.

b) The Frobenius structure is explicitly known *only at points where* U *has two or more non-distinct eigenvalues.*

In Chapters 10 and 11 we give two explicit and detailed examples of applications of the above Theorem 1.8 to both these cases. The case a) will be exemplified through a detailed study of the Maxwell Stratum of the A_3-Frobenius manifold, whereas the computation of the monodromy data at points of the small quantum cohomology of $\mathbb{G}(2,4)$ will exemplify the case b).

1.5.3. The Maxwell Stratum of A_3-Frobenius Manifold. The first example, in Chapter 10, is the analysis of the monodromy data at the points of one of the two irreducible components of the bifurcation diagram (namely, the Maxwell stratum) of the Frobenius manifold associated to the Coxeter group A_3. This is the simplest polynomial Frobenius structure in which semisimple coalescence points appear. The whole structure is globally and explicitly known, and the system (1.5) at generic points is solvable in terms of oscillatory integrals. At semisimple coalescence points, however, the system considerably simplifies, and it reduces to a Bessel equation. Thus, the asymptotic analysis of its solutions can be easily completed using Hankel functions, and S and C can be computed. By Theorem 1.8 above, these are monodromy data of points in a whole neighbourhood of the coalescence point. We will explicitly show that the fundamental solutions expressed by means of oscillatory integrals converge to those expressed in terms of Hankel functions at a coalescence point, and that the computation done away from the coalescence point provides the same S and C, as Theorem 1.8 predicts. In particular, the Stokes matrix S computed invoking Theorem 1.8 is in agreement with both the well-known results of [**Dub96**], [**Dub99b**], stating that $S + S^T$ coincides with the Coxeter matrix of the group $W(A_3)$ (group of symmetries of the regular tetrahedron), and with the analysis of [**DM00**] for monodromy data of the the algebraic solutions of PVI_μ corresponding to A_3 (see also [**CDG17b**] for this last point).

1.5.4. Quantum Cohomology of $\mathbb{G}(2,4)$. In Chapter 11, we consider the Frobenius structure on $QH^\bullet(\mathbb{G}(2,4))$. The *small quantum ring* – or *small quantum cohomology* – of Grassmannians has been one of the first cases of quantum cohomology rings to be studied both in physics ([**Wit95**]) and mathematical literature ([**ST97**], [**Ber97**]), so that a quantum extension of the classical Schubert calculus has been obtained ([**Buc03**]). However, the ring structure of the big quantum cohomology is not explicitly known, and the computation of the monodromy data can only be done at the small quantum cohomology locus. As explained in Theorem 1.3, Theorem 1.4, Theorem 1.5, it happens that the small quantum locus of *almost all* Grassmannians $\mathbb{G}(k,n)$ is made of semisimple coalescence points. The case of $\mathbb{G}(2,4)$ is the simplest case where this phenomenon occurs. Therefore, in order to compute the monodromy data, we invoke Theorem 1.8 above. For brevity, we will set $\mathbb{G} := \mathbb{G}(2,4)$.

In Section 11.2, we carry out the asymptotic analysis of the system (1.5) *at the coalescence locus*, corresponding to $t = 0 \in QH^\bullet(\mathbb{G})$, and we explicitly compute the monodromy data (see (11.3-(11.20)) for μ and R; see (11.33) and Appendix B, with $v = 6$ for S and C). For the computation of S, we take an admissible[17] line $\ell := \{z \in \mathbb{C} \ :\ z = e^{i\phi}\}$ with the slope $0 < \phi < \frac{\pi}{4}$. The signs in the square roots in (1.4) and the labelling of $(u_1, ..., u_6)$ are chosen in Section 11.1.2. As the fundamental solution (1.38) of (1.5) with fixed $t = 0$, we choose the restriction of the *topological-enumerative* fundamental solution[18] $Y_0(z) := \Psi|_{t=0}\, \Phi(z)\ z^\mu z^R$, whose coefficients are the 2-points genus 0 Gromov-Witten invariants with descendants

$$\Phi(z)^\alpha_\beta = \delta^\alpha_\beta + \sum_{n=0}^{\infty}\sum_{\lambda}\sum_{\nu\in \mathrm{Eff}(\mathbb{G})\setminus\{0\}} \langle \tau_n T_\beta, T_\lambda\rangle^{\mathbb{G}}_{0,2,\nu}\eta^{\lambda\alpha} z^{n+1},$$

$$\text{with}\quad \langle \tau_n T_\beta, T_\lambda\rangle^{\mathbb{G}}_{0,2,\nu} := \int_{[\overline{\mathcal{M}}_{0,2}(\mathbb{G},\nu)]^{\mathrm{vir}}} \psi_1^n \cup \mathrm{ev}_1(T_\beta)\cup \mathrm{ev}_2(T_\nu),\qquad \psi_1 := c_1(\mathcal{L}_1)$$

where $\mathcal{L}_1$ is the first tautological cotangent bundle on $\overline{\mathcal{M}}_{0,2}(\mathbb{G},\nu)$, and $(\eta^{\mu\nu})$ denotes the inverse of the Poincaré metric. This solution will be precisely described in Section 3.3 (cf. Proposition 3.2).

Summarizing, let S and C be the data we have concretely computed by means of the asymptotic analysis of Chapter 11. Then, let us denote by S' and C' the data obtained from S and C by a suitable action

$$S \longmapsto \mathcal{I}PS(\mathcal{I}P)^{-1} =: S'$$

[17] Namely, $\ell_+(\phi)$ defined above is an admissible ray.

[18] This is the solution $\Psi(0)Y(z,0) = \Psi(0)H(z,0)z^\mu z^R$ in Proposition 3.2, where Φ is called H.

$$C \longmapsto C(\mathcal{I}P)^{-1} \longmapsto G^{-1}C(\mathcal{I}P)^{-1} =: C',$$

of the groups of Theorem 1.2, with $G = A$ or $G = AB \in \widetilde{\mathcal{C}}_0(\mu, R)$ as in (1.40), (1.41) below (P and $\mathcal{I}$ are explicitly given in Theorem 11.2), corresponding to

- an appropriate re-ordering of the canonical coordinates $u_1, \dots, u_6$ near $0 \in QH^\bullet(\mathbb{G})$, yielding the Stokes matrix in upper-triangular form.
- another determination of signs in the square roots of (1.4) of the normalized idempotents vector fields $(f_i)_i$
- another choice of the fundamental solution of the equation (1.5) in Levelt-normal form (1.9), obtained from the topological-enumerative solution by the action $Y_0 \mapsto Y_0 G$ of $\widetilde{\mathcal{C}}_0(\mu, R)$.

Given these explicit data, we prove Theorem 1.9 below, which cast new light for $\mathbb{G}(2, 4)$ the conjecture, formulated by B. Dubrovin in [**Dub98**], and then refined in [**Dub13**], relating the enumerative geometry of a Fano manifold with its derived category. More details and new more general results about this conjecture are the contents of the final Part 4 of this Thesis.

THEOREM 1.9 (cf. Theorem 11.2). *The Stokes matrix and the central connection matrix at $t = 0 \in QH^\bullet(\mathbb{G})$ are related to a full exceptional collection $(E_1, \dots, E_6)$ in the derived category of coherent sheaves $\mathcal{D}^b(\mathbb{G})$ in the following way.*

The central connection matrix C', obtained in the way explained above, is equal to the matrix (one for both choices of sign $\pm$) associated to the following $\mathbb{C}$-linear morphism

$$\mathfrak{X}^{\pm}_{\mathbb{G}} \colon K_0(\mathbb{G}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^\bullet(\mathbb{G}; \mathbb{C})$$

$$[E] \mapsto \frac{1}{(2\pi)^2 c^{\frac{1}{2}}} \widehat{\Gamma}^{\pm}(\mathbb{G}) \cup \mathrm{Ch}(E)$$

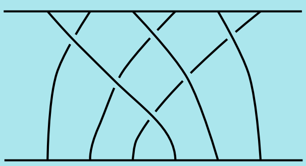
computed w.r.t. the basis $([E_1], \dots, [E_6])$ of $K_0(\mathbb{G})$, obtained by projection of an exceptional collection from the derived category $\mathcal{D}^b(\mathbb{G})$ of coherent sheaves on the Grassmannian, and the Schubert basis $(T_0, T_1, T_2, T_3, T_4, T_5) = (1, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2})$ of $H^\bullet(\mathbb{G}; \mathbb{C})$ normalized so that

$$\int_{\mathbb{G}} \sigma_{2,2} = c \in \mathbb{C}^*.$$

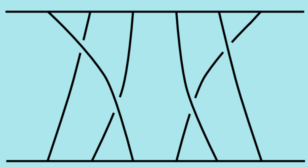
The exceptional collection $(E_1, \dots, E_6)$ is a 5-block[19]*, obtained from the Kapranov exceptional 5-block collection*

$$\left(\mathbb{S}^0 \mathcal{S}^*, \quad \mathbb{S}^1 \mathcal{S}^*, \quad \begin{matrix} \mathbb{S}^2 \mathcal{S}^* \\ \mathbb{S}^{1,1} \mathcal{S}^* \end{matrix}, \quad \mathbb{S}^{2,1} \mathcal{S}^*, \quad \mathbb{S}^{2,2} \mathcal{S}^* \right),$$

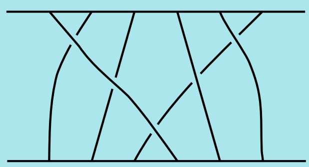
by mutation[20]*under the* inverse *of any one of the following braids*[21]*in $\mathcal{B}_6$*



$\beta_{34}\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$



$\beta_{12}\beta_{56}\beta_{23}\beta_{45}$



$\beta_{12}\beta_{56}\beta_{23}\beta_{45}\beta_{34}$

Here, $\mathcal{S}$ denotes the tautological bundle on $\mathbb{G}$ and $\mathbb{S}^\lambda$ is the Schur functor associated to the Young diagram λ. β_{34} acts just as a permutation of the third and fourth elements of the block.

More precisely:

- *the matrix representing* $\mathfrak{X}^-_{\mathbb{G}}$ *w.r.t. the basis* $([E_1],\dots,[E_6])$ *of* $K_0(\mathbb{G})$ *above is equal to the central connection matrix* C' *computed w.r.t. the solution* $Y_0(z)\cdot A^{-1}$*, where* $A\in\widetilde{\mathcal{C}}_0(\mu,R)$ *is*

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 2i\pi & 1 & 0 & 0 & 0 & 0\\ -2\pi^2 & 2i\pi & 1 & 0 & 0 & 0\\ -2\pi^2 & 2i\pi & 0 & 1 & 0 & 0\\ -\frac{1}{3}\left(8i\pi^3\right) & -4\pi^2 & 2i\pi & 2i\pi & 1 & 0\\ \frac{4\pi^4}{3} & -\frac{1}{3}\left(8i\pi^3\right) & -2\pi^2 & -2\pi^2 & 2i\pi & 1\end{pmatrix}; \tag{1.40}$$

- *the matrix representing* $\mathfrak{X}^+_{\mathbb{G}}$ *is equal to the central connection matrix* C' *computed w.r.t. the solution* $Y_0(z)\cdot(A\cdot B)^{-1}$*, where* $B\in\widetilde{\mathcal{C}}_0(\mu,R)$ *is*

$$B=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -8\gamma & 1 & 0 & 0 & 0 & 0\\ 32\gamma^2 & -8\gamma & 1 & 0 & 0 & 0\\ 32\gamma^2 & -8\gamma & 0 & 1 & 0 & 0\\ \frac{8}{3}\left(\zeta(3)-64\gamma^3\right) & 64\gamma^2 & -8\gamma & -8\gamma & 1 & 0\\ \frac{64}{3}\left(16\gamma^4-\gamma\zeta(3)\right) & \frac{8}{3}\left(\zeta(3)-64\gamma^3\right) & 32\gamma^2 & 32\gamma^2 & -8\gamma & 1\end{pmatrix}. \tag{1.41}$$

In both cases $(\pm)$*,* $(S')^{-1}$ *coincides with the Gram matrix* $(\chi(E_i,E_j))_{i,j=1}^n$.

The Stokes and the central connection matrices

- *at all other points of small quantum cohomology,*
- *and/or computed w.r.t. other possible admissible lines* ℓ*,*

satisfy the same properties as above w.r.t other full exceptional 5*-block collections, obtained from* $(E_1,\dots,E_6)$ *by alternate mutation under the braids*

$$\omega_1:=\beta_{12}\beta_{56},\quad \omega_2:=\beta_{23}\beta_{45}\beta_{34}\beta_{23}\beta_{45}.$$

In particular, the Kapranov 5*-block exceptional collection* does not *appear neither at* $t=0$ *nor anywhere else along the locus of the small quantum cohomology.*

The monodromy data in any other chamber of $QH^\bullet(\mathbb{G})$ *are obtained from the data* S',C' *(or from* PSP^{-1} *and* CP^{-1}*) computed at* $0\in QH^\bullet(\mathbb{G})$*, by the action (1.13) of the braid group.*

Here, for any smooth projective variety X, the class $\Gamma_X^\pm$ denotes the multiplicative characteristic class of the tangent bundle TX, obtained through Hirzebruch's construction ([**Hir78**]) starting with the formal Taylor series, centered at $z=0$, of the function $\Gamma(1\pm z)$, $\Gamma(z)$ being the classical Euler's

[19] This means that $\chi(E_3,E_4)=\chi(E_4,E_3)=0$ and thus that both $(E_1,E_2,E_3,E_4,E_5,E_6)$ and $(E_1,E_2,E_4,E_3,E_5,E_6)$ are exceptional collections: we will write

$$\left(E_1,\quad E_2,\quad \begin{matrix}E_3\\E_4\end{matrix},\quad E_5,\quad E_6\right)$$

if we consider the exceptional collection with an unspecified order. Passing from one to the other reflects the passage from one ℓ-cell to the other one, decomposing a sufficiently small neighborhood of $0\in QH^\bullet(\mathbb{G})$.

[20] The definition of the action of the braid group on the set of exceptional collections will be given in Section 11.3.4, slightly modifying (by a shift) the classical definitions that the reader can find e.g. in [**GK04**]. Our convention for the composition of action of braids is the following: braids act on an exceptional collection/monodromy datum *on the right.*

[21] Curiously, these braids show a mere "mirror symmetry": notice that they are indeed equal to their specular reflection. Any contingent geometrical meaning of this fact deserves further investigations.

Γ-function, namely

$$\Gamma(1 \pm z) = \exp\left\{\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n}(\mp z)^n\right\}.$$

In other words, if we denote by $\alpha_1, \dots, \alpha_d$ the Chern roots of the tangent bundle TX, we set

$$\widehat{\Gamma}_X^{\pm} := \prod_{j=1}^{d} \Gamma(1 \pm \alpha_j).$$

Moreover, we defined the *graded* Chern character of an object $V^\bullet \in \mathrm{Ob}(\mathcal{D}^b(X))$ as follows:

$$\mathrm{Ch}(V) := \sum_k e^{2\pi i x_k}, \quad x_k\text{'s are the Chern roots of a vector bundle } V,$$

$$\mathrm{Ch}(V^\bullet) := \sum_j (-1)^j \,\mathrm{Ch}(V^j) \quad \text{for a bounded complex } V^\bullet.$$

1.6. Results of Part 4

In the fourth and final part of this Thesis, we address the study of the conjecture, formulated by B. Dubrovin in occasion of the the 1998 ICM in Berlin [**Dub98**]. The original and genuine aim of the conjecture is a characterization of smooth projective Fano varieties admitting semisimple quantum cohomology in terms of derived geometry.

The original version of Dubrovin's conjecture can be described in two different parts, a *qualitative* and a *quantitative* one. In the first qualitative part, the semisimplicity condition for a smooth projective Fano variety X is conjectured to be equivalent to the existence of full exceptional collections in the derived category of coherent sheaves on X. These consist in ordered collections of objects $\mathfrak{E} = (E_1, \dots, E_n)$ in $\mathcal{D}^b(X)$ satisfying the semi-orthogonality conditions

$$\mathrm{Hom}^\bullet(E_i, E_i) \cong \mathbb{C}$$

$$\mathrm{Hom}^\bullet(E_j, E_i) \cong 0, \quad \text{if } j > i.$$

Furthermore, in order to be *full*, the collection $\mathfrak{E}$ must generate the category $\mathcal{D}^b(X)$ as a triangulated one. For a detailed discussion of geometrical properties of exceptional collections, their *mutations* under the action of the braid group, and the interesting non-symmetric orthogonal geometry which they induce on the Grothendieck group $K_0(X)$, we refer the reader to Chapters 12 and 13.

The second quantitative (and maybe most astonishing) part of the conjecture predicates an *analytic* and *explicit* relationship between the monodromy data of the quantum cohomology $QH^\bullet(X)$ and algebro-geometric data of the objects of $\mathfrak{E}$. Remarkably, the Stokes matrix S, computed at any point $p \in QH^\bullet(X)$ w.r.t. any choice of signs in equation (1.4) defining normalized idempotents, a suitable order of canonical coordinates (the *lexicographical one*, see Definition 2.16) and any oriented line $\ell(\phi)$ in the complex plane, was conjectured to be equal to the Gram matrix of the Euler-Poincaré product

$$\chi(E, F) := \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}^i(E, F), \quad E, F \in \mathrm{Ob}(\mathcal{D}^b(X)),$$

associated with some exceptional collection $\mathfrak{E}$. For what concerns the central connection matrix C, in the original formulation of the conjecture, it was not completely identified its geometrical counterpart in the derived category $\mathcal{D}^b(X)$. The only observation appearing in [**Dub98**] is an *ansatz* for the general structure of the central connection matrix: it was originally conjectured to be of the form

$$C = C' \cdot C'',$$

where C'' is a matrix whose column entries are the components of the graded Chern character $\mathrm{Ch}(E_i)$ of the objects of $\mathfrak{E}$, and where C' is a matrix only required to commute with the operator of classical $\cup$-multiplication

$$c_1(X) \cup (-) \colon H^\bullet(X; \mathbb{C}) \to H^\bullet(X; \mathbb{C}).$$

1.6.1. Refinement of Dubrovin's conjecture. In Part 4, strong evidences are given in favor of the following general conjecture, which refines the last part of Dubrovin's conjecture.

CONJECTURE 1.1 (cf. Conjecture 14.2). *Let X be a smooth Fano variety of Hodge-Tate type.*

(1) *The quantum cohomology $QH^\bullet(X)$ is semisimple if and only if there exists a full exceptional collection in the derived category of coherent sheaves $\mathcal{D}^b(X)$.*
(2) *If $QH^\bullet(X)$ is semisimple, then for any oriented line ℓ (of slope $\phi \in [0; 2\pi[$) in the complex plane there is a correspondence between ℓ-chambers and founded helices, i.e. helices with a marked foundation, in the derived category $\mathcal{D}^b(X)$.*
(3) *The monodromy data computed in a ℓ-chamber Ω_ℓ, in lexicographical order, are related to the following geometric data of the corresponding exceptional collection $\mathfrak{E}_\ell = (E_1, \dots, E_n)$ (the marked foundation):*
    (a) *the Stokes matrix is equal to the* inverse *of the Gram matrix of the Grothendieck-Poincaré-Euler product on $K_0(X)_\mathbb{C} := K_0(X) \otimes_\mathbb{Z} \mathbb{C}$, computed w.r.t. the exceptional basis $([E_i])_{i=1}^n$*
    $$S^{-1}_{ij} = \chi(E_i, E_j);$$
    (b) *the Central Connection matrix C, connecting the solution Y_{right} of equations (1.5)-(1.6), described in Section 1.2, with the topological-enumerative solution $Y_{\text{top}} = \Psi \cdot Z_{\text{top}}$, coincides with the matrix associated to the $\mathbb{C}$-linear morphism*
    $$\text{Д}^-_X \colon K_0(X)_\mathbb{C} \to H^\bullet(X; \mathbb{C}) \colon E \mapsto \frac{i^{\bar{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}^-_X \cup \exp(-\pi i c_1(X)) \cup \mathrm{Ch}(E),$$
    where $d = \dim_\mathbb{C} X$, and $\bar{d}$ is the residue class $d \pmod 2$. The matrix is computed w.r.t. the exceptional basis $([E_i])_{i=1}^n$ and any pre-fixed basis $(T_\alpha)_\alpha$ in cohomology (see Section 3.1.1).

Here, by "topological-enumerative solution" $Z_{\text{top}}(z,t)$ we mean the fundamental systems of solutions of (1.1)-(1.2) whose coefficients are given by Gromov-Witten invariants with descendants:

$$Z_{\text{top}}(z,t) := \Theta_{top}(z,t) \cdot z^\mu z^{c_1(X)\cup},$$

$$\Theta_{top}(z,t)^\gamma_\lambda := \delta^\gamma_\lambda + \sum_{k,n=0}^\infty \sum_{\beta \in \mathrm{Eff}(X)} \sum_{\alpha_1,\dots,\alpha_k} \frac{h^\gamma_{\lambda,k,n,\beta,\underline{\alpha}}}{k!} \cdot t^{\alpha_1} \dots t^{\alpha_k} \cdot z^{n+1},$$

$$h^\gamma_{\lambda,k,n,\beta,\underline{\alpha}} := \sum_\delta \eta^{\delta\gamma} \int_{[\overline{\mathcal{M}}_{0,k+2}(X,\beta)]^{\text{virt}}} c_1(\mathcal{L}_1)^n \cup \mathrm{ev}_1^* \sigma_\lambda \cup \mathrm{ev}_2^* \sigma_\delta \cup \prod_{j=1}^k \mathrm{ev}^*_{j+2} \sigma_{\alpha_j},$$

$\mathcal{L}_1$ being the 1-st tautological cotangent bundle on the Deligne-Mumford stack $\overline{\mathcal{M}}_{0,k+2}(X,\beta)$ of stable maps of genus 0, with degree β and $(k+2)$-punctures and target space X.

Remarkably, our Theorem 1.8 suggest the validity of a constraint on the kind of exceptional collections associated with the monodromy data in a neighborhood of a semisimple coalescing point of the quantum cohomology $QH^\bullet(X)$ of a smooth projective variety X. If the eigenvalues u_i's coalesce, at some semisimple point t_0, to $s<n$ values $\lambda_1,\dots,\lambda_s$ with multiplicities $p_1,\dots,p_s$ (with $p_1+\dots+p_s=n$, here n is the sum of the Betti numbers of X), then the corresponding monodromy data can be expressed in terms of Gram matrices and characteristic classes of objects of a *full s-block exceptional collection*, i.e. a collection of the type

$$\mathcal{E}:=(\underbrace{E_1,\dots,E_{p_1}}_{\mathcal{B}_1},\underbrace{E_{p_1+1},\dots,E_{p_1+p_2}}_{\mathcal{B}_2},\dots,\underbrace{E_{p_1+\dots+p_{s-1}+1},\dots,E_{p_1+\dots+p_s}}_{\mathcal{B}_s}),\quad E_j\in \operatorname{Obj}\left(\mathcal{D}^b(X)\right),$$

where for each pair (E_i,E_j) in a same block $\mathcal{B}_k$ the orthogonality conditions hold

$$\operatorname{Ext}^\ell(E_i,E_j)=0,\quad \text{for any } \ell.$$

In particular, any reordering of the objects inside a single block $\mathcal{B}_j$ preserves the exceptionality of $\mathcal{E}$.

Recently, the interests and feelings of necessity for a deeper understanding and refinement of Dubrovin's conjecture increased. In such a direction, two main contributions require to be mentioned.

(1) In [**Dub13**], Dubrovin suggested that the column entries of the central connection matrix C should be equal to the components of the characteristic classes

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^-_X\cup \operatorname{Ch}(E_i),\quad d=\dim_{\mathbb{C}}X, \tag{1.42}$$

E_i being objects of an exceptional collection.

(2) Almost contemporarily to Dubrovin, in [**GGI16**] and [**GI15**] S. Galkin, V. Golyshev and H. Iritani proposed a set of conjectures, called Γ-conjectures (I and II) describing the exponential asymptotic behaviour of flat sections of the *quantum connection* (namely, Dubrovin's extended deformed connection $\widehat{\nabla}$ defined on $QH^\bullet(X)$). It is claimed that Γ-conjecture II refines Dubrovin's conjecture and it prescribes that the column entries of the central connection matrix, defined as above, are the components of the characteristic classes

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^+_X\cup \operatorname{Ch}(E_i),\quad d=\dim_{\mathbb{C}}X, \tag{1.43}$$

E_i being objects of an exceptional collection.

Our explicit computations for the simple case of $\mathbb{G}(2,4)$, described in Section 1.5.4, suggest that both proposals of the conjecture formulated in [**Dub13**] and [**GGI16, GI15**] require some refinements, at least as far as the central connection matrix C is concerned. Indeed, Theorem 1.9 claims that the connection matrix for $\mathbb{G}(2,4)$ can be of both the forms (1.42) and (1.43) (which belong to the same $\widetilde{\mathcal{C}}_0(\mu,R)$-orbit, with $R:=c_1(\mathbb{G})\cup(-)\in \operatorname{End}(H^\bullet(\mathbb{G}(2,4);\mathbb{C}))$ if computed w.r.t. two different solutions in Levelt normal form at $z=0$, *no one of the two coinciding with the topological-enumerative one*. Our Conjecture 1.1 both clarifies what is the precise form of the central connection matrix, and also explains the geometrical meaning of the matrix $A\in\widetilde{\mathcal{C}}_0(\mu,R)$ in Theorem 1.9.

In Chapter 14 we also show that the identifications between the monodromy data and the geometry of the derived category can be further enriched, according to the following result.

THEOREM 1.10 (cf. Theorem 14.1). *Let X be a smooth Fano variety of Hodge-Tate type for which Conjecture 1.1 holds true. Then, all admissible operations on the monodromy data have a geometrical counterpart in the derived category $\mathcal{D}^b(X)$, as summarized in Table 1.1 at the end of this Introduction. In particular, we have the following:*

(1) *Mutations of the monodromy data (S,C) correspond to mutations of the exceptional basis.*

(2) *The monodromy data* $(S, C^{(k)})$ *computed w.r.t. the others solutions* $Y^{(k)}_{\text{left/right}}$*, having the prescribed asymptotic expansion in rotated sectors*

$$Y^{(k)}_{\text{left/right}}(z,t) \sim Y_{\text{formal}}(z,t), \quad z \in e^{2\pi i k}\Pi^{\varepsilon}_{\text{left/right}}(\phi), \quad |z| \to \infty, \quad k \in \mathbb{Z},$$

uniformly in t*, are associated as in points* (3a)-(3b) *of Conjecture 1.1, with different foundations of the helix, related to the marked one by an iterated application of the Serre functor* $(\omega_X \otimes -)[\dim_{\mathbb{C}} X]\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$*.*

(3) *The group* $\widetilde{\mathcal{C}}_0(X) := \widetilde{\mathcal{C}}_0(\mu, R)$*, with* $R = c_1(X) \cup (-)$*, is isomorphic to a subgroup of the identity component of the isometry group* $\mathrm{Isom}_{\mathbb{C}}(K_0(X)_{\mathbb{C}}, \chi)$*: more precisely, the morphism*

$$\widetilde{\mathcal{C}}_0(X) \to \mathrm{Isom}_{\mathbb{C}}(K_0(X)_{\mathbb{C}}, \chi)_0\colon A \mapsto \left(Д^-_X\right)^{-1} \circ A \circ Д^-_X$$

defines a monomorphism. In particular, $\widetilde{\mathcal{C}}_0(X)$ *is abelian.*

(4) *The monodromy matrix* $M_0 := e^{2\pi i \mu} e^{2\pi i R}$ *has spectrum contained in* $\{-1, 1\}$*.*

1.6.2. Results for complex Projective Spaces. In Chapter 15, we focus on the case of complex Projective Spaces $\mathbb{P}^{k-1}_{\mathbb{C}}$. There we prove the validity of Conjecture 1.1, we explicitly compute the central connection matrix at points of the small quantum cohomology, and we carry on a detailed analysis of the braid group on the monodromy data and on the corresponding exceptional collections. in particular we complete the study initiated by D. Guzzetti in [**Guz99**]. Let us summarize the main results obtained.

THEOREM 1.11 (cf. Theorem 15.2, Corollary 15.2). *Conjecture 1.1 is true for all complex Projective Spaces* $\mathbb{P}^{k-1}_{\mathbb{C}}$*,* $k \geq 2$*. More precisely, the central connection matrix computed at* $0 \in QH^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}})$ *w.r.t. an oriented line* ℓ *of slope* $\phi \in]0; \frac{\pi}{k}[$ *coincide with the matrix associated with the morphism*

$$Д^-_{\mathbb{P}^{k-1}_{\mathbb{C}}}\colon K_0(\mathbb{P}^{k-1}_{\mathbb{C}})_{\mathbb{C}} \to H^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}}; \mathbb{C})$$

computed w.r.t. the exceptional bases obtained by projecting on the K_0*-group suitable shifts of the following exceptional collections:*

CASE k **EVEN:**

$$\left(\mathcal{O}\left(\frac{k}{2}\right), \bigwedge^1 \mathcal{T}\left(\frac{k}{2}-1\right), \mathcal{O}\left(\frac{k}{2}+1\right), \bigwedge^3 \mathcal{T}\left(\frac{k}{2}-2\right), \dots, \mathcal{O}(k-1), \bigwedge^{k-1} \mathcal{T}\right);$$

CASE k **ODD:**

$$\left(\mathcal{O}\left(\frac{k-1}{2}\right), \mathcal{O}\left(\frac{k+1}{2}\right), \bigwedge^2 \mathcal{T}\left(\frac{k-3}{2}\right), \mathcal{O}\left(\frac{k+3}{2}\right), \bigwedge^4 \mathcal{T}\left(\frac{k-5}{2}\right), \dots, \mathcal{O}\left(k-1\right), \bigwedge^{k-1} \mathcal{T}\right).$$

Here, we denote by $\mathcal{O}$ *and* $\mathcal{T}$ *the structural and the tangent sheaf of* $\mathbb{P}^{k-1}_{\mathbb{C}}$ *repsectively, and more in general by* $\bigwedge^p \mathcal{T}(q)$ *the tensor product*

$$\left(\bigwedge^p \mathcal{T}\right) \otimes \mathcal{O}(q).$$

To the best of our knowledge, the result above is the first *explicit* description of the exceptional collections that actually arise from the monodromy data according to Dubrovin's conjecture. We remark that the exceptional collections appearing in Theorem 1.11 are in the same $\mathcal{B}_k$-orbit of the

Beilinson exceptional collection $\mathfrak{B} := (\mathcal{O}, \dots, \mathcal{O}(k-1))$. Hence, it is worthy understanding for which Projective Spaces there exists suitable choices of sings for the Ψ-matrix, and oriented lines ℓ for which the monodromy data computed along the small quantum locus $H^2(\mathbb{P}^{k-1}_{\mathbb{C}}; \mathbb{C})$ are associated with the Beilinson exceptional collection $\mathfrak{B}$. The following result gives us the answer.

THEOREM 1.12 (cf Theorem 15.3, Corollary 15.3). *The Beilinson exceptional collection $\mathfrak{B}$ arise from the monodromy data of $QH^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}})$ along the small quantum cohomology if and only if $k = 2, 3$.*

Potentially, this result could give us information about some region of the *big* quantum cohomology of complex Projective Spaces: if we were able to reconstruct the solution of the Riemann-Hilbert problem associated with the monodromy data corresponding to $\mathfrak{B}$, this could lead to an explicit representation of the analytic continuation of the genus 0 Gromov-Witten potential of $\mathbb{P}^{k-1}_{\mathbb{C}}$.

In order to prove Theorem 1.12, a careful analysis of the hidden symmetries of the Stokes phenomenon is carried on. By using symmetries of the regular polygons (which represent the spectrum of the operator $\mathcal{U}$ along the small quantum locus), and studying properties of all Stokes factors, we obtain the following result.

THEOREM 1.13 (cf. Theorem 15.5). *The monodromy data computed at any other point of the small quantum cohomology of $\mathbb{P}^{k-1}_{\mathbb{C}}$ with $k \geq 2$, w.r.t. any other choice of oriented line ℓ, are obtained from those computed at $0 \in QH^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}})$ w.r.t. a line of slope $\phi \in]0; \frac{\pi}{k}[$ by acting with a braid of the form*

$$\omega_{1,k}\omega_{2,k}\omega_{1,k}\omega_{2,k}\dots,$$

where

- **if k is even** *we set*

$$\omega_{1,k} := \prod_{\substack{i=2 \\ i \text{ even}}}^{k} \beta_{i-1,i}, \quad \omega_{2,k} := \prod_{\substack{i=3 \\ i \text{ odd}}}^{k-1} \beta_{i-1,i};$$

- **if k is odd** *we set*

$$\omega_{1,k} := \prod_{\substack{i=3 \\ i \text{ odd}}}^{k} \beta_{i-1,i}, \quad \omega_{2,k} := \prod_{\substack{i=2 \\ i \text{ even}}}^{k-1} \beta_{i-1,i}.$$

The corresponding exceptional collections are obtained (up to shifts) by acting with the above braids on the collections of Theorem 1.11.

Moreover, if we denote by $S(p, \phi)$ the Stokes matrix computed at a point $p \in H^2(\mathbb{P}^{k-1}_{\mathbb{C}}; \mathbb{C})$, w.r.t. a line $\ell(\phi)$ of slope $\phi \in \mathbb{R}$, and in ℓ-lexicographical order, then the following facts hold.

(1) *If σ denotes the generator of $H^2(\mathbb{P}^{k-1}_{\mathbb{C}}; \mathbb{C})$, then the Stokes matrix has the following functional form*

$$S(t\sigma, \phi) = S(\mathrm{Im}(t) + k\phi), \quad t \in \mathbb{C}.$$

(2) *The Stokes matrix satisfies the* quasi-periodicity *condition*

$$S(p, \phi) \sim S\left(p, \phi + \frac{2\pi i}{k}\right),$$

where $A \sim B$ means that the matrices A, B are in the same $(\mathbb{Z}/2\mathbb{Z})^k$-orbit w.r.t. the action of Theorem 1.2.

(3) *The entries*

$$S(p,\phi)_{i,i+1} \text{ and } S\left(p,\phi+\frac{\pi i}{k}\right)_{i,i+1}$$

differ for some signs for all $p\in H^2(\mathbb{P}^{k-1}_{\mathbb{C}};\mathbb{C})$, $\phi\in\mathbb{R}$ *and for any* $i=1,\dots k-1$. *In particular, the* $(k-1)$*-tuple*

$$(|S(p,\phi)_{1,2}|\,,|S(p,\phi)_{2,3}|\,,\dots,|S(p,\phi)_{k-1,k}|)$$

does not depend on p *and* ϕ, *and it is equal to*

$$\left(\binom{k}{1},\dots,\binom{k}{k-1}\right).$$

Finally, we also obtained some results concerning the group $\widetilde{\mathcal{C}}_0(\mathbb{P}^{k-1}_{\mathbb{C}})$, refining point (3) of Theorem 1.10.

THEOREM 1.14 (cf. Theorem 15.1, Corollary 15.1). *The group* $\widetilde{\mathcal{C}}_0(\mathbb{P}^{k-1}_{\mathbb{C}})$ *is an abelian unipotent algebraic group of dimension* $[\frac{k}{2}]$. *In particular, the exponential map defines an isomorphism*

$$\widetilde{\mathcal{C}}_0(\mathbb{P})\cong\underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{[\frac{k}{2}]\text{ copies}}.$$

With respect to the basis $(1,\sigma,\dots,\sigma^{k-1})$ *of* $H^\bullet(\mathbb{P};\mathbb{C})$, *the group* $\widetilde{\mathcal{C}}_0(\mathbb{P})$ *is described as follows*

$$\widetilde{\mathcal{C}}_0(\mathbb{P})=\left\{C\in GL(k,\mathbb{C})\colon C=\sum_{i=0}^{k-1}\alpha_iJ_i,\quad \alpha_0=1,\quad 2\alpha_{2n}+\sum_{\substack{i+j=2n\\1\le i,j}}(-1)^i\alpha_i\alpha_j=0,\quad 2\le 2n\le k-1\right\}.$$

In particular, the group $\widetilde{\mathcal{C}}_0(\mathbb{P}^{k-1}_{\mathbb{C}})$ *is isomorphic to the identity component of the isometry group* $\mathrm{Isom}_{\mathbb{C}}(K_0(\mathbb{P}^{k-1}_{\mathbb{C}})_{\mathbb{C}},\chi)$.

1.6.3. Results for complex Grassmannians. As a final application of the abelian-nonabelian correspondence, described in Chapter 4 for the specific case of complex Grassmannians, in Chapter 16 we explicitly compute the monodromy data of $QH^\bullet(\mathbb{G}(r,k))$ at points of the small quantum cohomology. Notice that these data are well defined for any r and k by our Theorem 1.8, and that their exact values is deduced from the corresponding monodromy data for the Projective Space $\mathbb{P}^{k-1}_{\mathbb{C}}$. In the following statement, we denote by $\bigwedge^r A$ the r-th exterior power of a matrix $A\in M_k(\mathbb{C})$ (also called r-th *compound matrix* of A), namely the matrix of all $r\times r$ minors of A, ordered in lexicographical order. Let us summarize the main results.

THEOREM 1.15 (cf. Theorem 16.1, Corollary 16.2, Theorem 16.2). *Let* ℓ *be an oriented line of slope* $\phi\in]0;\frac{\pi}{k}[$, *admissible at both points*

$$p=t\sigma_1\in H^2(\mathbb{G}(r,k),\mathbb{C})\quad\text{and}\quad \hat{p}:=(t+(r-1)\pi i)\sigma\in H^2(\mathbb{P}^{k-1}_{\mathbb{C}};\mathbb{C}),$$

σ and σ_1 being the Schubert classes generating the second comology groups of $\mathbb{P}^{k-1}_{\mathbb{C}}$ and $\mathbb{G}(r,k)$ respectively. For a suitable choice of signs of the Ψ-matrices, the monodromy data of $\mathbb{G}(r,k)$ are given by

$$S_{\mathbb{G}(r,k)}(p,\phi)=\bigwedge^r S_{\mathbb{P}^{k-1}_{\mathbb{C}}}(\hat{p},\phi),\quad C_{\mathbb{G}(r,k)}:=i^{-\binom{k}{r}}\left(\bigwedge^r C_{\mathbb{P}^{k-1}_{\mathbb{C}}}(\hat{p},\phi)\right)e^{\pi i(r-1)\sigma_1\cup(-)}.$$

In particular, Conjecture 1.1 holds true for the Grassmannian $\mathbb{G}(r,k)$. The exceptional collections associated with its monodromy data are (modulo shifts) in the same orbit of the twisted Kapranov exceptional collection

$$\left(\mathbb{S}^\lambda\mathcal{S}^\vee\otimes\mathscr{L}\right)_\lambda,\quad \mathscr{L}:=\det\left(\bigwedge^2\mathcal{S}^\vee\right),$$

where $\mathbb{S}^\lambda$ denotes the λ-th Schur functor and $\mathcal{S}$ the tautological bundle on $\mathbb{G}(r,k)$. Furthermore, the Stokes matrices satisfies the following conditions:

(1) *it has the following functional form*
$$S_{\mathbb{G}(r,k)}(t\sigma_1,\phi)=S(\mathrm{Im}t+k\phi);$$
(2) *it is* quasi-periodic *along the small quantum locus, in the sense that*
$$S_{\mathbb{G}(r,k)}(p,\phi)\sim S_{\mathbb{G}(r,k)}\left(p,\phi+\frac{2\pi i}{k}\right),$$
where $A\sim B$ means that the matrices A and B are in the same orbit under the action of $(\mathbb{Z}/2\mathbb{Z})^{\binom{k}{r}}$;
(3) *the upper-diagonal entries*
$$S_{\mathbb{G}(r,k)}(p,\phi)_{i,i+1},\quad S_{\mathbb{G}(r,k)}\left(p,\phi+\frac{\pi i}{k}\right)_{i,i+1}$$
differ for some signs, and we have that
$$|S_{\mathbb{G}(r,k)}(p,\phi)_{i,i+1}|\in\left\{\binom{k}{1},\dots,\binom{k}{k-1}\right\}\cup\{0\}.$$

Corollary 1.2 (cf. Corollary 16.3). *The Kapranov exceptional collection $(\mathbb{S}^\lambda\mathcal{S}^\vee)_\lambda$, twisted by a suitable line bundle, is associated with the monodromy data of $\mathbb{G}(r,k)$ at points of the small quantum locus if and only if $(r,k)=(1,2),(1,3),(2,3)$. In this cases, the line bundle is trivial, and the Kapranov collection coincides with the Beilinson one*[22].

[22] Notice that $\mathbb{G}(2,3)\cong\mathbb{P}((\mathbb{C}^3)^\vee)\cong\mathbb{P}^2_{\mathbb{C}}$ by duality.

TABLE 1.1. Identifications enriching Dubrovin's Conjecture

Frobenius Manifolds $QH^\bullet(X)$	Derived category $\mathcal{D}^b(X)$	Grothendieck group $K_0(X)_\mathbb{C}$
Stokes matrix S		inverse of the Gram matrix $G_{ij} := \chi(E_i, E_j)$ for an exceptional basis $([E_i])_i$
Central connection matrix C		matrix associated with the morphism $Д_X^-\colon K_0(X)_\mathbb{C} \to H^\bullet(X;\mathbb{C})$
action of the braid group $\mathcal{B}_n$	action of $\mathcal{B}_n$ on the set of exceptional collections	action of $\mathcal{B}_n$ on exceptional bases
action of the group $(\mathbb{Z}/2\mathbb{Z})^{\times n}$	shifts of exceptional collections	projected shifts, i.e. change of signs, of exceptional bases
action of the group $\widetilde{\mathcal{C}}_0(X)$	action of a subgroup of autoequivalences $\mathrm{Aut}(\mathcal{D}^b(X))$	action of a subgroup of the identity component of $\mathrm{Isom}_\mathbb{C}(K_0(X)_\mathbb{C}, \chi)$
complete ccw rotation of the line ℓ, action of the generator of the center $Z(\mathcal{B}_n)$, action of the matrix $M_0 \in \widetilde{\mathcal{C}}_0(X)$, $M_0 := \exp(2\pi i\mu)\exp(2\pi iR)$	action of the Serre functor $(\omega_X \otimes -)[\dim_\mathbb{C} X]$ on the set of exceptional collections	action of the canonical operator $\kappa\colon K_0(X)_\mathbb{C} \to K_0(X)_\mathbb{C}$ on the set of exceptional bases

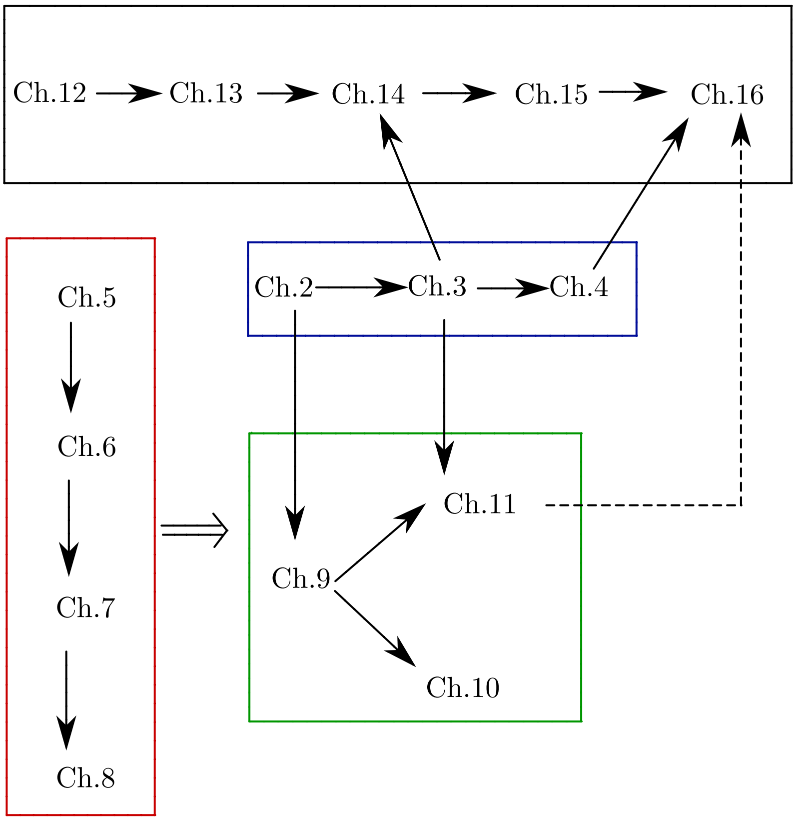


FIGURE 1.3. Interdependence of the Chapters of the Thesis

Part 1

Coalescence Phenomenon of Quantum Cohomology of Grassmannians and the Distribution of Prime Numbers

Anche l'amore della meraviglia par che si debba ridurre all'amore dello straordinario e all'odio della noia ch'è prodotta dall'uniformità.

Giacomo Leopardi,
Zibaldone di Pensieri

CHAPTER 2

Frobenius Manifolds and their Monodromy Local Moduli

Abstract. In this Chapter we review the analytic theory of Frobenius manifolds, their monodromy data and the Isomonodromy Theorems, according to [**Dub98**], [**Dub96**], [**Dub99b**]. After recalling the main definitions and properties of Frobenius Manifolds, we correct minor imprecisions which are found in *loc. cit.*: in particular, we characterise the freedom in the choice of systems of deformed flat coordinates, introducing the group $\widetilde{\mathcal{C}}_0(\mu, R)$. The Spectrum of a Frobenius Manifold is defined and, under the assumption of semisimplicity, a set of monodromy local invariants, which play the role of local moduli, is introduced. We define a chamber-decomposition of the manifold, depending on the choice of an oriented line ℓ in the complex plane. Finally, the ambiguities and freedom, up to which the monodromy data are defined, are discussed in details. The discontinuous jumps of the monodromy data from one chamber to another one, encoded in the action of the braid group, are presented as a wall-crossing phenomenon.

2.1. Introduction to Frobenius Manifolds

We denote with $\odot$ the symmetric tensor product of vector bundles, and with $(-)^\flat$ the standard operation of lowering the index of a $(1,k)$-tensor using a fixed inner product.

Definition 2.1. A *Frobenius manifold* structure on a complex manifold M of dimension n is defined by giving

(FM1) a symmetric non-degenerate $\mathcal{O}(M)$-bilinear metric tensor $\eta \in \Gamma\left(\bigodot^2 T^*M\right)$, whose corresponding Levi-Civita connection ∇ is flat;

(FM2) a $(1,2)$-tensor $c \in \Gamma\left(TM \otimes \bigodot^2 T^*M\right)$ such that
- the induced multiplication of vector fields $X \circ Y := c(-, X, Y)$, for $X, Y \in \Gamma(TM)$, is *associative*,
- $c^\flat \in \Gamma\left(\bigodot^3 T^*M\right)$,
- $\nabla c^\flat \in \Gamma\left(\bigodot^4 T^*M\right)$;

(FM3) a vector field $e \in \Gamma(TM)$, called the *unity vector field*, such that
- the bundle morphism $c(-, e, -)\colon TM \to TM$ is the identity morphism,
- $\nabla e = 0$;

(FM4) a vector field $E \in \Gamma(TM)$, called the *Euler vector field*, such that
- $\mathfrak{L}_E c = c$,
- $\mathfrak{L}_E \eta = (2-d)\cdot\eta$, where $d \in \mathbb{C}$ is called the *charge* of the Frobenius manifold.

Since the connection ∇ is flat, there exist local flat coordinates, that we denote $(t^1, ..., t^n)$, w.r.t. which the metric η is constant and the connection ∇ coincides with partial derivatives $\partial_\alpha = \partial/\partial t^\alpha$, $\alpha = 1, ..., n$. Because of flatness and the conformal Killing condition, the Euler vector field is affine, i.e.

$$\nabla\nabla E = 0, \qquad \text{so that } \; E = \sum_{\alpha=1}^n \left((1-q_\alpha)t^\alpha + r_\alpha\right)\frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.$$

Following [**Dub96, Dub98, Dub99b**], we choose flat coordinates so that $\frac{\partial}{\partial t^1} \equiv e$ and $r_\alpha \neq 0$ only if $q_\alpha = 1$ (this can always be done, up to an affine change of coordinates). In flat coordinates, let $\eta_{\alpha\beta} = \eta(\partial_\alpha, \partial_\beta)$, and $c^\gamma_{\alpha\beta} = c(dt^\gamma, \partial_\alpha, \partial_\beta)$, so that $\partial_\alpha \circ \partial_\beta = c^\gamma_{\alpha\beta}\partial_\gamma$. Condition (FM2) means that $c_{\alpha\beta\gamma} := \eta_{\alpha\rho}c^\rho_{\beta\gamma}$ and $\partial_\alpha c_{\beta\gamma\delta}$ are symmetric in all indices. This implies the local existence of a function F such that

$$c_{\alpha\beta\gamma} = \partial_\alpha\partial_\beta\partial_\gamma F.$$

The associativity of the algebra is equivalent to the following conditions for F, called WDVV-equations

$$\partial_\alpha\partial_\beta\partial_\gamma F\ \eta^{\gamma\delta}\partial_\delta\partial_\epsilon\partial_\nu F = \partial_\nu\partial_\beta\partial_\gamma F\ \eta^{\gamma\delta}\partial_\delta\partial_\epsilon\partial_\alpha F,$$

while axiom (FM4) is equivalent to

$$\eta_{\alpha\beta} = \partial_1\partial_\alpha\partial_\beta F, \quad \mathfrak{L}_E F = (3-d)F + Q(t),$$

with $Q(t)$ a quadratic expression in t_α's. Conversely, given a solution of the WDVV equations, satisfying the quasi-homogeneity conditions above, a structure of Frobenius manifold is naturally defined on open subset of the space of parameters t^α's.

Let us consider the canonical projection $\pi\colon \mathbb{P}^1_{\mathbb{C}} \times M \to M$, and the pull-back of the tangent bundle TM:

$$\begin{array}{ccc} \pi^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \\ \mathbb{P}^1_{\mathbb{C}} \times M & \xrightarrow{\pi} & M \end{array}$$

We will denote by

(1) $\mathscr{T}_M$ the sheaf of sections of TM,
(2) $\pi^*\mathscr{T}_M$ the pull-back sheaf, i.e. the sheaf of sections of π^*TM
(3) $\pi^{-1}\mathscr{T}_M$ the sheaf of sections of π^*TM constant on the fiber of π.

Introduce two (1,1)-tensors $\mathcal{U}, \mu$ on M defined by

$$\mathcal{U}(X) := E \circ X, \quad \mu(X) := \frac{2-d}{2}X - \nabla_X E \tag{2.1}$$

for all $X \in \Gamma(TM)$. In flat coordinates $(t^\alpha)^n_{\alpha=1}$ chosen as above, the operator μ is *constant* and in *diagonal* form

$$\mu = \ \mathrm{diag}(\mu_1, ..., \mu_n), \qquad \mu_\alpha = q_\alpha - \frac{d}{2} \in \mathbb{C}.$$

All the tensors $\eta, e, c, E, \mathcal{U}, \mu$ can be lifted to π^*TM, and their lift will be denoted with the same symbol. So, also the Levi-Civita connection ∇ is lifted on π^*TM, and it acts so that

$$\nabla_{\partial_z} Y = 0 \quad \text{for } Y \in (\pi^{-1}\mathscr{T}_M)(M).$$

Let us now twist this connection by using the multiplication of vectors and the operators $\mathcal{U}, \mu$.

Definition 2.2. Let $\widehat{M} := \mathbb{C}^* \times M$. The *deformed connection* $\widehat{\nabla}$ on the vector bundle $\pi^*TM|_{\widehat{M}} \to \widehat{M}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + z \cdot X \circ Y,$$

$$\widehat{\nabla}_{\partial_z} Y = \nabla_{\partial_z} Y + \mathcal{U}(Y) - \frac{1}{z}\mu(Y)$$

for $X, Y \in (\pi^*\mathscr{T}_M)(\widehat{M})$.

The crucial fact is that the deformed extended connection $\widehat{\nabla}$ is *flat*.

THEOREM 2.1 ([**Dub96**],[**Dub99b**]). *The flatness of $\widehat{\nabla}$ is equivalent to the following conditions on M*

- $\nabla c^\flat$ *is completely symmetric,*
- *the product on each tangent space of M is associative,*
- $\nabla\nabla E = 0$,
- $\mathfrak{L}_E c = c$.

Because of this integrabiilty condition, we can look for *deformed flat coordinates* $(\tilde{t}^1, ..., \tilde{t}^n)$, with $\tilde{t}^\alpha = \tilde{t}^\alpha(t, z)$. These coordinates are defined by n independent solutions of the equation

$$\widehat{\nabla} d\tilde{t} = 0.$$

Let ξ denote a column vector of components of the differential $d\tilde{t}$. The above equation becomes the linear system

$$\begin{cases} \partial_\alpha \xi = z\mathcal{C}_\alpha^T(t)\xi, \\ \\ \partial_z \xi = \left(\mathcal{U}^T(t) - \frac{1}{z}\mu^T\right)\xi, \end{cases} \tag{2.2}$$

where $\mathcal{C}_\alpha$ is the matrix $(\mathcal{C}_\alpha)^\beta_\gamma = c^\beta_{\alpha\gamma}$. We can rewrite the system in the form

$$\begin{cases} \partial_\alpha \zeta = z\mathcal{C}_\alpha\zeta, \\ \\ \partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta, \end{cases} \tag{2.3}$$

where $\zeta := \eta^{-1}\xi$. In order to obtain (2.3), we have also used the invariance of the product, encoded in the relations

$$\begin{aligned} \eta^{-1}\mathcal{C}_\alpha^T\eta &= \mathcal{C}_\alpha, \\ \mathcal{U}^T\eta &= \eta\mathcal{U}, \end{aligned} \tag{2.4}$$

and the η-skew-symmetry of μ

$$\mu\eta + \eta\mu = 0. \tag{2.5}$$

Geometrically, ζ is the η-gradient of a deformed flat coordinate as in (1.3). Monodromy data of system (2.3) define local invariants of the Frobenius manifold, as explained below.

2.1.1. Monodromy at $z = 0$. Let us fix a point t of the Frobenius manifold M, and let us focus on the associated equation

$$\partial_z \zeta = \left(\mathcal{U}(t) + \frac{1}{z}\mu(t)\right)\zeta. \tag{2.6}$$

REMARK 2.1. If ζ_1, ζ_2 are solution of the equation (2.6), then the two products

$$\langle \zeta_1, \zeta_2 \rangle_\pm := \zeta_1^T(e^{\pm\pi i}z)\eta\zeta_2(z)$$

are independent of z. Indeed we have

$$\begin{aligned} \partial_z\left(\zeta_1^T(e^{\pm\pi i}z)\eta\zeta_2(z)\right) &= \partial_z\left(\zeta_1^T(e^{\pm\pi i}z)\right)\eta\zeta_2(z) + \zeta_1^T(e^{\pm\pi i}z)\eta\partial_z\zeta_2(z) \\ &= \zeta_1^T(e^{\pm\pi i}z)\left[\eta\mathcal{U} - \mathcal{U}^T\eta + \frac{1}{z}(\mu\eta + \eta\mu)\right]\zeta_2(z) \\ &= 0 \quad \text{by (2.4) and (2.5).} \end{aligned}$$

THEOREM 2.2 (Normal Form,[**Dub99b**]). *There exists a formal gauge transformation*

$$\tilde{\zeta} := G(z)\zeta, \qquad G(z) = \mathbb{1} + \sum_{k=1}^{\infty} G_k z^k, ,$$

satisfying

$$G(-z)^T \eta G(z) = \eta, \tag{2.7}$$

which transforms the system (2.6) *into a normal – or canonical – form*

$$\partial_z \tilde{\zeta} = \left(\frac{1}{z}\mu + R_1 + zR_2 + z^2 R_3 + \dots\right)\tilde{\zeta} \tag{2.8}$$

such that the matrices R_k *satisfy*

$$R_k^T = (-1)^{k+1}\eta R_k \eta^{-1}, \qquad k = 1, 2, \dots, \tag{2.9}$$

$$(R_k)^\alpha_\beta \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k, \qquad 1 \leq \alpha, \beta \leq n, \tag{2.10}$$

where μ_α, μ_β *are entries of* $\mu = \ diag(\mu_1, ..., \mu_n)$. *Hence, there is only a finite number of nonzero matrices* R_k. *The equation* (2.8) *is called* normal form *of the system* (2.6).
A fundamental matrix solution of the normal form (2.8) *is*

$$\tilde{\zeta} := z^\mu z^R, \quad R := \sum R_k.$$

Moreover, the matrix R *satisfies the following relation (not explicitly given in* [**Dub96**], [**Dub98**]*):*

$$z^{R^T} \eta e^{\pm i\pi\mu} z^R = \eta e^{\pm i\pi\mu}. \tag{2.11}$$

PROOF. The first part is proved in [**Dub99b**]. Here we prove (2.11). Observe that (2.9) implies

$$z^{R^T} = \eta\left(z^{R_1 - R_2 + R_3 - R_4 + \dots}\right)\eta^{-1}.$$

Moreover, from (2.10) we deduce that

$$e^{\mp i\pi\mu} R_k e^{\pm i\pi\mu} = (-1)^k R_k.$$

So, we conclude that

$$\begin{aligned} z^{R^T} \eta e^{\pm i\pi\mu} z^R &= \eta\left(z^{R_1 - R_2 + R_3 - R_4 + \dots}\right)\left(e^{\pm i\pi\mu} z^R e^{\mp i\pi\mu}\right) e^{\pm i\pi\mu} \\ &= \eta\left(z^{R_1 - R_2 + R_3 - R_4 + \dots}\right)\left(z^{-R_1 + R_2 - R_3 + R_4 - \dots}\right) e^{\pm i\pi\mu} \\ &= \eta e^{\pm i\pi\mu}. \end{aligned}$$

□

Since $z = 0$ is a Fuchsian singularity, the series of $G(z)$ is *convergent* (see [**CL85**]). Hence, the matrix

$$\Phi(z) z^\mu z^R, \qquad \Phi(z) := G(z)^{-1} = \mathbb{1} + \Phi_1 z + \Phi_2 z^2 + \dots, \tag{2.12}$$

is a genuine fundamental solution of system (2.6). It follows from (2.7) that it satisfies the constraint

$$\Phi(-z)^T \eta\, \Phi(z) = \eta. \tag{2.13}$$

If $\mu_\alpha - \mu_\beta \notin \mathbb{N}^*$, for any $1 \leq \alpha, \beta \leq n$, then $R_k = 0$ for any $k = 1, 2, ..$ and the Frobenius manifold is said to be *non-resonant*. Otherwise, it is called *resonant*. The reduction of the system (2.6) to its normal form is not unique and consequently the matrix R is not uniquely determined; moreover, even for fixed matrix R, there is still a freedom to choose different solutions of the form (2.12), saisfying (2.13).

THEOREM 2.3 ([Dub99b]). *Suppose that the system (2.6) can be reduced to two normal forms of Theorem 2.2:*

$$\partial_z \zeta = \left(\frac{1}{z}\mu + R_1 + zR_2 + z^2 R_3 + \dots\right)\zeta \tag{2.14}$$

$$\partial_z \tilde{\zeta} = \left(\frac{1}{z}\mu + \tilde{R}_1 + z\tilde{R}_2 + z^2 \tilde{R}_3 + \dots\right)\tilde{\zeta}. \tag{2.15}$$

Then, they necessarily are related by a gauge transformation

$$\tilde{\zeta} := G(z)\zeta \tag{2.16}$$

where

$$G(z) := \mathbb{1} + z\Delta_1 + z^2\Delta_2 + \dots \tag{2.17}$$

$$G(-z)^T \eta G(z) = \eta. \tag{2.18}$$

Moreover, defining $G := G(1) = \mathbb{1} + \Delta_1 + \Delta_2 + \dots$, we have that

$$\tilde{R} = GRG^{-1}, \quad \text{where } \tilde{R} := \sum_k \tilde{R}_k, \quad R := \sum_k R_k.$$

DEFINITION 2.3. The set of matrices

$$G := \mathbb{1} + \Delta_1 + \Delta_2 + \dots$$

such that

$$(\Delta_n)^\alpha_\beta = 0 \text{ unless } \mu_\alpha - \mu_\beta = n,$$

$$\left(\mathbb{1} - \Delta_1^T + \Delta_2^T - \dots\right)\eta\,(\mathbb{1} + \Delta_1 + \Delta_2 + \dots) = \eta$$

is a group under matrix multiplication, called (η,μ)-*parabolic orthogonal group*, denoted $\mathcal{G}(\eta,\mu)$.

Notice that if the Frobenius manifold is non-resonant, then the parabolic orthogonal group $\mathcal{G}(\eta,\mu)$ is trivial, and $R = 0$.

REMARK 2.2. In the proof of Theorem 2.3 it is shown that $G \in \mathcal{G}(\eta,\mu)$ satisfies the property

$$G(z) := z^\mu z^{\tilde{R}} G z^{-R} z^{-\mu} = \mathbb{1} + \Delta_1 z + \dots + \Delta_j z^j \quad \text{is polynomial}$$

and

$$G(-z)^T \eta G(z) = \eta.$$

The monodromy matrix of the system (2.6) at the singularity $z = 0$ is expressed in terms of μ and R

$$M_0 = e^{2\pi i\mu} e^{2\pi i R}.$$

DEFINITION 2.4 ([Dub96][Dub99b]). We call *monodromy data of the system (2.6) at $z = 0$* :

- the matrix μ,
- the class of equivalence $[R]$ under the action of the (η,μ)-parabolic orthogonal group $\mathcal{G}(\eta,\mu)$.

So far, we have defined the monodromy data at fixed point of the manifold. Now let us vary the point t in system (2.6), so that a fundamental solution $\Phi(z,t)z^\mu z^{R(t)}$, as in (2.12), depends on t.

THEOREM 2.4 (Isomonodromy Theorem I, [Dub99b]). *The monodromy data* $\mu, [R]$ *at the origin of the system* (2.6) *do not depend on the point* $p \in M$. *More precisely, there exists a* t*-independent representative in the class* $[R]$.

Thanks to the above theorem, the following definition is well given.

DEFINITION 2.5. The monodromy data of the system (2.6) at $z = 0$ as in Definition 2.4 are called *monodromy data of the Frobenius manifold at* $z = 0$.

REMARK 2.3. In the general case, although not related to Frobenius manifolds, when μ is not diagonalizable and has a non-trivial nilpotent part, analogous results can be proved. However, the normal form becomes a little more complicated: e.g. it is no more defined by requiring that some entries of matrices R_k are nonzero, but that some *blocks* are. For a detailed analysis of such case, we recommend the book by F.R. Gantmacher [Gan60].

In the following, we will be interested in choosing a specific value of R. This choice does not fix a fundamental solution. Indeed, if $\Phi^{(i)}(z,t)z^{\mu}z^{R}$, $i = 1, 2$, are both solutions of (2.3) of the form (2.12) and (2.13), *with the same* R, satisfying

$$\Phi^{(i)}(z,t) = \mathbb{1} + \sum_{k=1}^{\infty} \Phi_k^{(i)}(t)z^k, \quad \Phi^{(i)}(-z,t)^T \eta\, \Phi^{(i)}(z,t) = \eta, \quad i = 1, 2,$$

then there exists a constant invertible matrix G, which in general is non-trivial, such that

$$\Phi^{(2)}(z,t)z^{\mu}z^{R} = \Phi^{(1)}(z,t)z^{\mu}z^{R}G,$$

so that

$$\Phi^{(1)}(z,t)^{-1}\Phi^{(2)}(z,t) = z^{\mu}z^{R}Gz^{-R}z^{-\mu}.$$

Thus, the r.h.s.

$$P(z) := z^{\mu}z^{R}Gz^{-R}z^{-\mu}$$

must be analytic at $z = 0$, and in fact a matrix-valued polynomial of the form $\mathbb{1}+A_1z+A_2z^2+\cdots+A_jz^j$. Moreover, by imposing the orthogonality relation, we must have

$$P(-z)^T\eta P(z) = \eta. \tag{2.19}$$

DEFINITION 2.6. We define $\widetilde{\mathcal{C}}_0(\mu, R)$ to be the group of all invertible matrices G such that $P(z) := z^{\mu}z^{R}Gz^{-R}z^{-\mu}$ is a polynomial of the form $1 + A_1z + A_2z^2 + \cdots + A_jz^j$, satisfying the orthogonality condition (2.19).

It follows from Remark 2.2 that $\widetilde{\mathcal{C}}_0(\mu, R)$ *is the subgroup of* $\mathcal{G}(\eta, \mu)$ made of those elements such that $GRG^{-1} \equiv R$. Notice that if $G \in \mathcal{G}(\eta, \mu)$ is such that $\tilde{R} = GRG^{-1}$ and $C \in \widetilde{\mathcal{C}}_0(\mu, R)$, then $GCG^{-1} \in \widetilde{\mathcal{C}}_0(\mu, \tilde{R})$.

The freedom in the choice of the normal form $\Phi(z,t)z^{\mu}z^{R}$ with fixed R is then regulated by the group $\widetilde{\mathcal{C}}_0(\mu, R)$. As already anticipated in the Introduction, this freedom was studied in [Dub99b] and identified with the group of connection matrices G such that $P(z) = z^{\mu}z^{R}Gz^{-R}z^{-\mu}$ is a matrix-valued polynomial. This group, called $\mathcal{C}_0(\mu, R)$ in [Dub99b], is *too large*. It is crucial to restrict it to $\widetilde{\mathcal{C}}_0(\mu, R)$ by imposing also the orthogonality relation (2.19).

We conclude this section with a result giving sufficient conditions on solutions of the system (2.3) for *resonant* Frobenius manifolds in order that they satisfy the η-orthogonality condition (2.13). In its essence, this result is stated and proved in [GGI16], in the specific case of quantum cohomologies of Fano manifolds.

PROPOSITION 2.1. *Let M be a* resonant *Frobenius manifold, and $t_0 \in M$ a fixed point.*

(1) *Suppose that there exists a fundamental solution of (2.3) of the form*

$$Z(z,t) = \Phi(z,t)z^{\mu}z^{R}, \quad \Phi(t) = \mathbb{1} + \sum_{j=1}^{\infty}\Phi_j(t)z^j,$$

with R satisfying all the properties of the Theorem 2.2, such that

$$H(z) := z^{-\mu}\Phi(z,t_0)z^{\mu}$$

is a holomorphic function at $z=0$ and $H(0) \equiv \mathbb{1}$. Then $\Phi(z,t)$ satisfies the constraint

$$\Phi(-z,t)^T\eta\;\Phi(z,t) = \eta$$

for all points $t \in M$.

(2) *If a solution with the properties above exists, then it is unique.*

PROOF. From Remark 2.1, we already know that the following bracket must be independent of z:

$$\begin{aligned}\langle Z(z,t_0), Z(z,t_0)\rangle_+ &= \left(\Phi(-z,t_0)(e^{i\pi}z)^{\mu}(e^{i\pi}z)^{R}\right)^T\eta\left(\Phi(z,t_0)z^{\mu}z^{R}\right)\\ &= \left((e^{i\pi}z)^{\mu}H(-z)(e^{i\pi}z)^{R}\right)^T\eta\left(z^{\mu}H(z)z^{R}\right)\\ &= e^{i\pi R^T}z^{R^T}H(-z)^Te^{i\pi\mu}z^{\mu}\eta z^{\mu}H(z)z^{R}\\ &= e^{i\pi R^T}z^{R^T}H(-z)^Te^{i\pi\mu}\eta H(z)z^{R}.\end{aligned}$$

By taking the first term of the Taylor expansion in z of the r.h.s., and using (2.11), we get

$$\langle Z(z,t_0), Z(z,t_0)\rangle_+ = e^{i\pi R^T}e^{i\pi\mu}\eta.$$

So, using again the equation $z^{\mu^T}\eta z^{\mu} = \eta$ and (2.11), we can conclude that

$$\Phi(-z,t_0)^T\eta\Phi(z,t_0) = \left((e^{i\pi}z)^{\mu}(e^{i\pi}z)^{R}\right)^{-T}\langle Z(z,t_0), Z(z,t_0)\rangle_+(z^{\mu}z^{R})^{-1} = \eta.$$

Because of (2.3) and the property of η-compatibility of the Frobenius product, we have that

$$\frac{\partial}{\partial t^{\alpha}}\left(\Phi(-z,t)^T\eta\Phi(z,t)\right) = z\cdot\Phi(-z,t)^T\cdot\left(\eta\mathcal{C}_{\alpha} - \mathcal{C}_{\alpha}^T\eta\right)\cdot\Phi(z,t) = 0.$$

This concludes the proof of (1). Let us now suppose that there are two solutions

$$\Phi_1(z,t)z^{\mu}z^{R}, \quad \Phi_2(z,t)z^{\mu}z^{R}$$

such that

$$z^{-\mu}\Phi_1(z,t_0)z^{\mu} = \mathbb{1} + zK_1 + z^2K_2 + \dots, \tag{2.20}$$

$$z^{-\mu}\Phi_2(z,t_0)z^{\mu} = \mathbb{1} + zK_1' + z^2K_2' + \dots. \tag{2.21}$$

The two solutions must be related by

$$\Phi_2(z,t)z^{\mu}z^{R} = \Phi_1(z,t)z^{\mu}z^{R}\cdot C$$

for some matrix $C \in \widetilde{\mathcal{C}}_0(\mu, R)$. This implies that $\Phi_2(z,t) = \Phi_1(z,t) \cdot P(z)$, where $P(z)$ is a matrix valued polynomial of the form

$$P(z) = \mathbb{1} + z\Delta_1 + z^2\Delta_2 + \dots, \quad \text{with } (\Delta_k)^\alpha_\beta = 0 \text{ unless } \mu_\alpha - \mu_\beta = k, \quad \text{and } P(1) \equiv C.$$

We thus have $z^{-\mu}\Phi_1^{-1}\Phi_2 z^\mu = z^{-\mu}P(z)z^\mu$, and

$$(z^{-\mu}P(z)z^\mu)^\alpha_\beta = \delta^\alpha_\beta + \sum_k (\Delta_k)^\alpha_\beta \, z^{k-\mu_\alpha+\mu_\beta} = \delta^\alpha_\beta + \sum_k (\Delta_k)^\alpha_\beta \equiv C.$$

Then, from formulae (2.20), (2.21) it immediately follows that $C = \mathbb{1}$, which proves that $\Phi_1 = \Phi_2$. □

2.1.2. Spectrum of a Frobenius Manifold. We give an intrinsic description of the relation between $\mathcal{G}(\eta, \mu)$ and $\widetilde{\mathcal{C}}_0(\mu, R)$, by introducing the concept of *spectrum* of a Frobenius manifold. Let (V, η, μ) be the datum of

- an n-dimensional complex vector space V,
- a bilinear symmetric non-degenerate form η on V,
- a diagonalizable endomorphism $\mu\colon V \to V$ which is η-antisymmetric

$$\eta(\mu a, b) + \eta(a, \mu b) = 0 \quad \text{for any } a, b \in V.$$

Let *spec*$(\mu) = (\mu_1, ..., \mu_n)$ and let V_{μ_α} be the eigenspace of a μ_α. We say that an endomorphism $A\colon V \to V$ is *μ-nilpotent* if

$$AV_{\mu_\alpha} \subseteq \bigoplus_{m \geq 1} V_{\mu_\alpha + m} \quad \text{for any } \mu_\alpha \in \operatorname{spec}(\mu).$$

In particular such an operator is nilpotent in the usual sense. We will also decompose a μ-nilpotent operator A in components A_k, $k \geq 1$, such that

$$A_k V_{\mu_\alpha} \subseteq V_{\mu_\alpha + k} \quad \text{for any } \mu_\alpha \in \operatorname{spec}(\mu),$$

so that the following identities hold:

$$z^\mu A z^{-\mu} = A_1 z + A_2 z^2 + A_3 z^3 + \dots, \quad [\mu, A_k] = kA_k \quad \text{for } k = 1, 2, 3, \dots.$$

The set of all endomorphisms $G\colon V \to V$ of the form

$$G = \mathbb{1}_V + \Delta$$

with Δ a μ-nilpotent operator and such that

$$\{Ga, Gb\} = \{a, b\} \quad \text{for any } a, b \in V$$

where

$$\{a, b\} := \eta\left(e^{i\pi\mu}a, b\right)$$

is a Lie group $\mathcal{G}(\eta, \mu)$ called (η, μ)-*parabolic orthogonal group*. Its Lie algebra $\mathfrak{g}(\eta, \mu)$ coincides with the set of all μ-nilpotent operators R which are also *μ-skew-symmetric* in the sense that

$$\{Rx, y\} + \{x, Ry\} = 0.$$

In particular, any such matrix R commutes with the operator $e^{2\pi i\mu}$. Writing all operators by matrices w.r.t. a basis of eigenvectors of μ we find that any μ-nilpotent operator A is of the form

$$(A_k)^\alpha_\beta = 0 \quad \text{unless } \mu_\alpha - \mu_\beta = k \quad \text{for } k = 1, 2, 3, \dots$$

and moreover an element of the algebra $\mathfrak{g}(\eta, \mu)$ satisfies also the constraints

$$R_k^T = (-1)^{k+1}\eta R_k \eta^{-1}.$$

Observe that the parabolic orthogonal group acts canonically on its Lie algebra by the adjoint representation.

Starting from a given Frobenius manifold we can canonically associate to it a triple (V, η, μ): indeed, using the Levi-Civita connection all tangent spaces can be identified. In this way, the choice of a normal form of equation (2.6) corresponds to the choice of an element $R \in \mathfrak{g}(\eta, \mu)$, and moreover the whole equivalence class of normal forms coincides with the orbit of R w.r.t. the action of $\mathcal{G}(\eta, \mu)$ on $\mathfrak{g}(\eta, \mu)$ given by the adjoint representation

$$R \mapsto GRG^{-1}.$$

However, this action of $\mathcal{G}(\eta, \mu)$ on $\mathfrak{g}(\eta, \mu)$ is not free, and the isotropy group of R is nothing else than the group $\widetilde{C}_0(\mu, R)$ introduced in the previous section. Indeed, if $R = GRG^{-1}$ we have that

$$\begin{aligned} z^\mu z^R &= z^\mu z^{GRG^{-1}} \\ &= z^\mu G z^R G^{-1} \\ &= (\mathbb{1} + \Delta_1 z + \Delta_2 z^2 + \dots) z^\mu z^R G^{-1}, \end{aligned}$$

where $\Delta_1, \Delta_2, \dots$ are the components of G. This proves our assertion. As a consequence, the isomorphism class of the group $\widetilde{C}_0(\mu, R)$ depends only on the orbit $[R]$ w.r.t. the $\mathcal{G}(\eta, \mu)$-action, two isotropy groups of two elements of the same orbit being related by a conjugation.

2.2. Semisimple Frobenius Manifolds

Definition 2.7. A commutative and associative $\mathbb{K}$-algebra A with unit is called *semisimple* if there is no nonzero nilpotent element, i.e. an element $a \in A \setminus \{0\}$ such that $a^k = 0$ for some $k \in \mathbb{N}$.

In what follows we will always assume that the ground field is $\mathbb{C}$.

Theorem 2.5. *Let A be a $\mathbb{C}$-Frobenius algebra of dimension n. The following are equivalent:*

1. *A is semisimple;*
2. *A is isomorphic to $\mathbb{C}^{\oplus n}$;*
3. *A has a basis of idempotents, i.e. elements $\pi_1, \dots \pi_n$ such that*
$$\pi_i \circ \pi_j = \delta_{ij} \pi_i,$$
$$\eta(\pi_i, \pi_j) = \eta_{ii} \delta_{ij};$$
4. *there is a vector $\mathcal{E} \in A$ such that the multiplication operator $\mathcal{E}\circ \colon A \to A$ has n pairwise distinct eigenvalues.*

Proof. The equivalence between (1) and (2) is the well-known Wedderburn-Artin Theorem applied to commutative algebras (see [ASS06]). An elementary proof can be found in the Lectures notes [Dub99b]. The fact that (2) and (3) are equivalent is trivial. Let us prove that (3) and (4) are equivalent. If (3) holds it is sufficient just to take

$$\mathcal{E} = \sum_k k \pi_k.$$

So $\mathcal{E}\circ$ has spectrum $\{1, \dots, n\}$. Let us now suppose that (4) holds. Because of the commutativity of the algebra, all operators $a\circ \colon A \to A$ are commuting. Consequently, they are all diagonalizable since they preserve the one dimensional eigenspaces of $\mathcal{E}\circ$. For a well known theorem, these operators are simultaneously diagonalizable. So idempotents are easily constructed by suitable scaling eigevectors of $\mathcal{E}\circ$. □

Definition 2.8 (Semisimple Frobenius Manifolds). A point p of a Frobenius manifold M is *semisimple* if the corresponding Frobenius algebra T_pM is semisimple. If there is an open dense subset of M of semisimple points, then M is called a *semisimple Frobenius manifold.*

It is evident from point (4) of the Theorem 2.5 that semisimplicity is an *open property*: if p is semisimple, then all points in a neighborhood of p are semisimple.

Definition 2.9 (Caustic and Bifurcation Set). Let M be a semisimple Frobenius manifold. We call *caustic* the set

$$\mathcal{K}_M := M \setminus M_{ss} = \{p \in M \colon T_pM \text{ is not a semisimple Frobenius algebra}\}.$$

We call *bifurcation set* of the Frobenius manifold the set

$$\mathcal{B}_M := \{p \in M \colon \operatorname{spec}(E\circ_p \colon T_pM \to T_pM) \text{ is not simple}\}.$$

By Theorem 2.5, we have $\mathcal{K}_M \subseteq \mathcal{B}_M$. Semisimple points in $\mathcal{B}_M \setminus \mathcal{K}_M$ are called *semisimple coalescence points*, or *semisimple bifurcation points.*

The bifurcation set $\mathcal{B}_M$ and the caustic $\mathcal{K}_M$ are either empty or an hypersurface, invariant w.r.t. the unit vector field e (see [**Her02**]). For Frobenius manifolds defined on the base space of semiuniversal unfoldings of a singularity, these sets coincide with the *bifurcation diagram* and the *caustic* as defined in the classical setting of singularity theory ([**AGLV93, Arn90**]). In this context, the set $\mathcal{B}_M \setminus \mathcal{K}_M$ is called *Maxwell stratum.* Remarkably, all these subsets typically admit a naturally induced Frobenius submanifold structure ([**Str01, Str04**]). In what follows we will assume that the semisimple Frobenius manifold M admits nonempty bifurcation set $\mathcal{B}_M$, caustic $\mathcal{K}_M$ and set of semisimple coalescence points $\mathcal{B}_M \setminus \mathcal{K}_M$.

At each point p in the open dense semisimple subset $M_{ss} \subseteq M$, there are n idempotent vectors

$$\pi_1(p), \dots, \pi_n(p) \ \in T_pM,$$

unique up to a permutation. By Theorem 2.5 there exists a suitable local vector field $\mathcal{E}$ such that $\pi_1(p), \dots, \pi_n(p)$ are eigenvectors of the multiplication $\mathcal{E}\circ$, with simple spectrum at p and consequently in a whole neighborhood of p. Using the results exposed in [**Kat82, Kat95**] about analytic deformation of operators with simple spectrum w.r.t. one complex parameter, in particular the results stating analyticity of eigenvectors and eigenprojections, and extending them to the case of more parameters using Hartogs' Theorem, we deduce the following

Lemma 2.1. *The idempotent vector fields are holomorphic at a semisimple point p, in the sense that, chosen and ordering $\pi_1(p), \dots, \pi_n(p)$, there exist a neighborhood of p where the resulting local vector fields are holomorphic.*

Notice that, although the idempotents are defined (and unique up to a permutation) at each point of M_{ss}, it is not true that there exist n globally well-defined holomorphic idempotent vector fields. Indeed, the caustic $\mathcal{K}_M$ is in general a locus of algebraic branch points: if we consider a semisimple point p and a close loop $\gamma\colon [0,1] \to M$, with base point p, encircling $\mathcal{K}_M$, along which a coherent ordering is chosen, then

$$(\pi_1(\gamma(0)), \dots, \pi_n(\gamma(0))) \quad \text{and} \quad (\pi_1(\gamma(1)), \dots, \pi_n(\gamma(1)))$$

may differ by a permutation. Thus, the idempotent vector fields are holomorphic and single-valued on simply connected open subsets not containing points of the caustic.

REMARK 2.4. More generally, the idempotents vector fields define single-valued and holomorphic local sections of the tangent bundle TM on any connected open set $\Omega \subseteq M \setminus \mathcal{K}_M = M_{ss}$ satisfying the following property: for any $z \in \Omega$ the inclusions

$$\Omega \overset{\alpha}{\hookrightarrow} M_{ss} \overset{\beta}{\hookrightarrow} M$$

induce morphisms in homotopy

$$\pi_1(\Omega, z) \xrightarrow{\alpha_*} \pi_1(M_{ss}, z) \xrightarrow{\beta_*} \pi_1(M, z)$$

such that $\mathrm{im}(\alpha_*) \cap \ker(\beta_*) = \{0\}$. Moreover, this means that the structure group of the tangent bundle of M_{ss} is reduced to the symmetric group $\mathfrak{S}_n$, and that the *local* isomorphism of $\mathcal{O}_{M_{ss}}$-algebras

$$\mathscr{T}_{M_{ss}} \cong \mathcal{O}_{M_{ss}}^{\oplus n},$$

existing everywhere, can be replaced by a global one by considering a Frobenius structure prolonged to an unramified covering of degree at most $n!$ (see [**Man99**]).

THEOREM 2.6 ([**Dub92**], [**Dub96**], [**Dub99b**]). *Let $p \in M_{ss}$ be a semisimple point, and $(\pi_i(p))_{i=1}^n$ a basis of idempotents in T_pM. Then*

$$[\pi_i, \pi_j] = 0;$$

as a consequence there exist local coordinates $u_1, \dots, u_n$ such that

$$\pi_i = \frac{\partial}{\partial u_i}.$$

DEFINITION 2.10 (Canonical Coordinates [**Dub96**], [**Dub99b**]). Let M a Frobenius manifold and $p \in M$ a semisimple point. The coordinates defined in a neighborhood of p of Theorem 2.6 are called *canonical coordinates*.

Canonical coordinates are defined only up to permutations and shifts. They are holomorphic local coordinates in a simply connected neighbourhood of a semisimple point not containing points of the caustic $\mathcal{K}_M$, or more generally on domains with the property of Remark 2.4. Holomorphy holds also at semisimple coalescence points.

THEOREM 2.7 ([**Dub99b**]). *If $u_1, \dots, u_n$ are canonical coordinates near a semisimple point of a Frobenius manifold M, then (up to shifts) the following relations hold*

$$\frac{\partial}{\partial u_i} \circ \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}, \qquad e = \sum_{i=1}^n \frac{\partial}{\partial u_i}, \qquad E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}.$$

In this thesis we will fix the shifts of canonical coordinates so that they coincide with the eigenvalues of the (1,1)-tensor $E\circ$.

DEFINITION 2.11 (Matrix Ψ). Let M be a semisimple Frobenius manifold, $t^1, \dots, t^n$ be local flat coordinates such that $\frac{\partial}{\partial t^1} = e$ and $u_1, \dots, u_n$ be canonical coordinates. Introducing the orthonormal basis

$$f_i := \frac{1}{\eta\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}\right)^{\frac{1}{2}}} \frac{\partial}{\partial u_i} \tag{2.22}$$

for arbitrary choices of signs in the square roots, we define a matrix Ψ (depending on the point of the Frobenius manifold) whose elements $\Psi_{i\alpha}$ (i-th row, α-th column) are defined by the relation

$$\frac{\partial}{\partial t^\alpha}=\sum_{i=1}^n \Psi_{i\alpha} f_i \quad \alpha=1,\dots,n.$$

LEMMA 2.2. *The matrix Ψ is a single-valued holomorphic function on any simply connected open subset not containing points of the caustic $\mathcal{K}_M$, or more generally on any open domain Ω as in Remark 2.4. Moreover, it satisfies the following relations:*

$$\Psi^T\Psi=\eta, \quad \Psi_{i1}=\eta\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_i}\right)^{\frac{1}{2}},$$

$$f_i=\sum_{\alpha,\beta=1}^n \Psi_{i1}\Psi_{i\beta}\eta^{\beta\alpha}\frac{\partial}{\partial t^\alpha}, \quad c_{\alpha\beta\gamma}=\sum_{i=1}^n \frac{\Psi_{i\alpha}\Psi_{i\beta}\Psi_{i\gamma}}{\Psi_{i1}}.$$

If $\mathcal{U}$ is the operator of multiplication by the Euler vector field, then Ψ diagonalizes it:

$$\Psi\mathcal{U}\Psi^{-1}=U:=diag(u_1,\dots,u_n).$$

PROOF. The first assertion is a direct consequence of the analogous property of the idempotents vector fields, as in Lemma 2.1. All the other relations follow by computations (see [**Dub99b**]). □

We stress that Ψ and the coordinates u_i's are holomorphic also at semisimple coalescence points, due to the same property of the idempotents.

2.2.1. Monodromy Data for a Semisimple Frobenius Manifold. Monodromy data at $z=\infty$ are defined in [**Dub98**],[**Dub96**] and [**Dub99b**] at point of a semisimple Frobenius manifold *not belonging to the bifurcation set*. In the present section we review these issues, and we enlarge the definition to *all semisimple points*, including the bifurcation ones, namely the semisimple coalescence points of Definition 1.1.

In this section, we fix an open subset $\Omega\subseteq M_{ss}$ satisfying the property of Remark 2.4, so that we can choose and fix on Ω

- an ordering for idempotent vector fields and canonical local coordinates $p\mapsto u(p)$, $p\in\Omega$,
- a determination for the square roots in the definition of normalized idempotent vector fields f_i's, and hence a determination of the matrix Ψ.

In this way, system (2.3) and system (2.24) below, are determined. In the idempotent frame

$$y=\Psi\zeta, \tag{2.23}$$

system (2.3) becomes

$$\begin{cases}\partial_i y=(zE_i+V_i)y\\ \partial_z y=\left(U+\frac{1}{z}V\right)y\end{cases} \tag{2.24}$$

where $(E_i)^\alpha_\beta=\delta^\alpha_i\delta^\beta_i$ and

$$V:=\Psi\mu\Psi^{-1}, \quad V_i:=\partial_i\Psi\cdot\Psi^{-1}, \tag{2.25}$$
$$U:=\Psi\mathcal{U}\Psi^{-1}=\mathrm{diag}(u_1,\dots,u_n),$$

with not necessarily $u_i\neq u_j$ when $i\neq j$. By Lemma 2.2, $\Psi(u)$, $V(u)$ and $V_i(u)$'s are holomorphic on Ω.

LEMMA 2.3. *The matrix $V=\Psi\mu\Psi^{-1}$ is antisymmetric, i.e. $V^T+V=0$. Moreover,*

$$\textit{if } u_i=u_j, \textit{ then } V_{ij}=V_{ji}=0.$$

PROOF. Antisymmetry is an easy consequence of (2.5) and the η-orthogonality of Ψ (see [**Dub99b**]). Moreover, compatibility conditions of the system (2.24) imply that

$$[E_i, V] = [V_i, U].$$

Reading this equation for entries at place (i, j), we find that

$$V_{ij} = (u_j - u_i)(V_i)_{ij}.$$

Now, $(V_i)_{ij}$ is holomorphic, by and Lemma 2.2 and (2.25), so that if $i \neq j$, but $u_i = u_j$, then $V_{ij} = 0$. ☐

We focus on the second linear system

$$\partial_z y = \left(U + \frac{1}{z}V\right) y, \tag{2.26}$$

and study it at *a fixed* point $p \in \Omega$, namely for u fixed.

THEOREM 2.8. *Let $\Omega \subseteq M_{ss}$ as in Remark 2.4. At a (fixed) point $p \in \Omega$, there exists a unique formal (in general divergent) series*

$$F(z) := \mathbb{1} + \sum_{k=1}^{\infty} \frac{A_k}{z^k}$$

with

$$F^T(-z)F(z) = \mathbb{1},$$

such that the transformation $\tilde{y} = F(z)y$ reduces the corresponding system (2.26) at p to the one with constant coefficients

$$\partial_z \tilde{y} = U\tilde{y}.$$

Hence, system (2.26) has a unique formal solution

$$Y_{\text{formal}}(z) = G(z)e^{zU}, \qquad G(z) := F(z)^{-1} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{G_k}{z^k}. \tag{2.27}$$

PROOF. By a direct substitution, one finds the following recursive equations for the coefficients A_k:

$$[U, A_1] = V, \qquad [U, A_{k+1}] = A_k V - kA_k, \quad k = 1, 2, \dots.$$

If (i, j) is such that $u_i \neq u_j$, then we can determine $(A_{k+1})^i_j$ by the second equation in terms of entries of A_k; if $u_i = u_j$, then we can determine $(A_{k+1})^i_j$ from the successive equation:

$$[U, A_{k+2}] = A_{k+1}V - (k + 1)A_{k+1}$$

the (i, j)-entry of the l.h.s. is 0, and by Lemma 2.3 $(A_{k+1}V)^i_j$ is a linear combination of entries $(A_{k+1})^i_h$, with $u_i \neq u_h$, already determined. In such a way we can construct $F(z)$. Let us now prove that $F^T(-z)F(z) = \mathbb{1}$. Let us take any solution Y of the original system, and pose

$$A := Y(e^{-i\pi}z)^T Y(z).$$

A is a constant matrix, since it does not depend on z. Thus, for an appropriate constant matrix C we have

$$F(z)Y(z) = e^{zU}C,$$

from which we deduce that

$$F(z)^{-1} = Y(z)C^{-1}e^{zU}, \qquad F(-z)^{-T} = e^{-zU}C^{-T}Y(e^{-i\pi}z)^T.$$

So

$$F(-z)^{-T}F(z)^{-1} = e^{-zU}C^{-T}AC^{-1}e^{zU}.$$

Comparing constant terms of the expansion of the r.h.s and the l.h.s we conclude that $C^{-T}AC^{-1} = \mathbb{1}$. □

Notice in the proof above that $[U, A_{k+1}] = A_kV - kA_k$, namley $(u_i - u_j)(A_{k+1})^i_j = (A_kV - kA_k)^i_j$, implies that if we let p vary in Ω, then the G_k's define holomorphic matrix valued functions $G_k(u)$ at points u, lying in $u(\Omega)$, such that $u_i \neq u_j$ for $i \neq j$. Accordingly, the formal matrix solution

$$Y_{\text{formal}}(z, u) = G(z, u)e^{zU}, \qquad G(z, u) = \mathbb{1} + \sum_{k=1}^{\infty} \frac{G_k(u)}{z^k}, \tag{2.28}$$

is well defined and holomorphic w.r.t $u = u(p)$ away from semisimple coalescence points in Ω. In Theorem 9.1 below, we will show that $Y_{\text{formal}}(z, u)$ extends holomorphically also at semisimple coalescence points.

Remark 2.5. The proof of Theorem 2.8 is based on a simple computation, which holds both at a coalescence and a non-coalescence semisimple point. The statement can also be deduced from the more general results of [**BJL79c**] (see also [**CDG17b**]). A similar computation can be found also in [**Tel12**] and [**GGI16**]. Notice however that this computation does not provide any information about the analiticity of $G(u)$ in case of coalescence $u_i \to u_j$, $i \neq j$. The analiticity of $Y_{\text{formal}}(z, u)$ – and of actual fundamental solutions – at a semisimple coalescence point follows from the results proved in [**CDG17b**], and will be the content of Theorem 9.1 below.

In order to study actual solutions at $p \in \Omega$, we introduce Stokes rays. In what follows, we denote with $\mathrm{pr}: \mathcal{R} \to \mathbb{C}\backslash\{0\}$ the covering map. For pairs (u_i, u_j) such that $u_i \neq u_j$, we take the determination α_{ij} of $\arg(u_i - u_j)$ in the interval $[0; 2\pi[$, and we let

$$\tau_{ij} := \frac{3\pi}{2} - \alpha_{ij}.$$

Definition 2.12 (Stokes rays). We call *Stokes rays* of the system (2.26) the rays in the universal covering $\mathcal{R}$ defined by

$$R_{ij,k} := \{z \in \mathcal{R}\colon \arg z = \tau_{ij} + 2k\pi\}$$

for any $k \in \mathbb{Z}$. The projections on the $\mathbb{C}$-plane

$$R_{ij} := \mathrm{pr}\,(R_{ij,k})$$

will also be called *Stokes rays*.

Observe that the projected Stokes rays coincide with the ones defined in [**Dub99b**], namely

$$R_{ij} := \{z \in \mathbb{C}\colon z = -i\rho(\overline{u_i} - \overline{u_j}),\ \rho > 0\}. \tag{2.29}$$

Stokes rays have a natural orientation from 0 to ∞. Their characterisation is that $z \in R_{ij,k}$ if and only if

$$\mathrm{Re}((u_i - u_j)z) = 0, \qquad \mathrm{Im}((u_i - u_j)z) < 0.$$

For $z \in \mathbb{C}$ we have

$$\begin{aligned} |e^{zu_i}| &= |e^{zu_j}| \quad \text{if } z \in R_{ij}, \\ |e^{zu_i}| &> |e^{zu_j}| \quad \text{if } z \text{ is on the left of } R_{ij}, \\ |e^{zu_i}| &< |e^{zu_j}| \quad \text{if } z \text{ is on the right of } R_{ij}. \end{aligned}$$

DEFINITION 2.13 (Admissible Rays and Line). Let $\phi \in \mathbb{R}$ and let us define the rays in $\mathcal{R}$

$$\ell_+(\phi) := \{z \in \mathcal{R} \colon \arg z = \phi\},$$

$$\ell_-(\phi) := \{z \in \mathcal{R} \colon \arg\ z = \phi - \pi\}.$$

We will say that these rays are *admissible at* u, for the system (2.26), if they do not coincide with any Stokes rays $R_{ij,k}$ for any i,j s.t. $u_i \neq u_j$ and any $k \in \mathbb{Z}$. Moreover, a line $\ell(\phi) := \{z = \rho e^{i\phi},\ \rho \in \mathbb{R}\}$ of the complex plane, with the natural orientation induced by $\mathbb{R}$, is called *admissible at* u for the system (2.26) if

$$\mathrm{Re}\ z(u_i - u_j)|_{z \in \ell \setminus 0} \neq 0$$

for any i,j s.t. $u_i \neq u_j$. In other words, a line is admissible if it does not contain (projected) Stokes rays R_{ij}.

Notice that $\mathrm{pr}\,(\ell_\pm(\phi))$ are contained in an admissible line $\ell(\phi)$, and that the natural orientation is such that the positive part of $\ell(\phi)$ is $\mathrm{pr}\,(\ell_+(\phi))$.

DEFINITION 2.14 (ℓ-Chambers). Given a semisimple Frobenius manifold M, and fixed an oriented line $\ell(\phi) = \{z = \rho e^{i\phi},\ \rho \in \mathbb{R}\}$ in the complex plane, consider the open dense subset of points $p \in M$ such that

- the eigenvalues of U at p are pairwise distinct,
- the line ℓ is admissible at $u(p) = (u_1(p), ..., u_n(p))$.

We call *ℓ-chamber* any connected component Ω_ℓ of this set.

The definition is well posed, since it does not depend on the ordering of the idempotents (i.e. the labelling of the canonical coordinates) and on the signs in the square roots defining Ψ. Any ℓ-chamber satisfies the property of Remark 2.4: hence, idempotent vector fields and canonical coordinates are single-valued and holomorphic on any ℓ-chamber. The topology of an ℓ-chamber in M can be highly non-trivial (it should not be confused with the simple topology in $\mathbb{C}^n$ of an *ℓ-cell* of Definition 9.1 below). For example, in [**Guz05**] the analytic continuation of the Frobenius structure of the Quantum Cohomology of $\mathbb{P}^2$ is studied: it is shown that there exist points $(u_1, u_2, u_3) \in \mathbb{C}^3$, with $u_i \neq u_j$, which do not correspond to any true geometric point of the Frobenius manifold. This is due to singularities of the change of coordinates $u \mapsto t$.

For a fixed $\phi \in \mathbb{R}$, we define the sectors

$$\Pi_{\text{right}}(\phi) := \{z \in \mathcal{R} \colon \phi - \pi < \arg\ z < \phi\},$$

$$\Pi_{\text{left}}(\phi) := \{z \in \mathcal{R} \colon \phi < \arg\ z < \phi + \pi\}.$$

THEOREM 2.9. *Let $\Omega \subset M_{ss}$ be as in Remark 2.4 and let system (2.24) be determined as in the beginning of this section. Let $\phi \in \mathbb{R}$ be fixed. Then the following statements hold.*

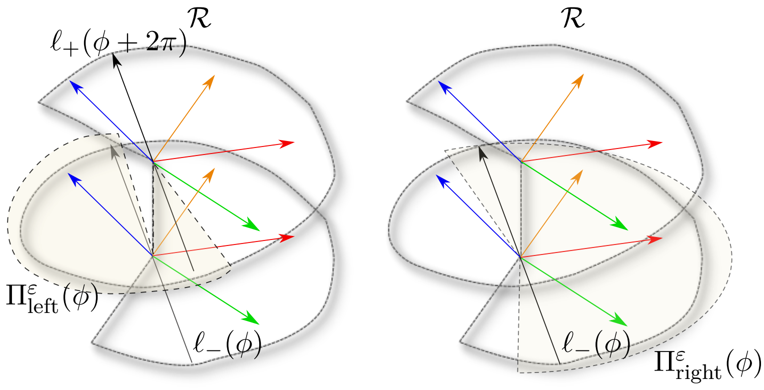


FIGURE 2.1. The figure shows $\Pi^{\varepsilon}_{\text{right}}(\phi), \Pi^{\varepsilon}_{\text{left}}(\phi)$ as dashed sectors, $\ell_{\pm}(\phi)$ in (black) and Stokes rays (in color).

(1) *At any $p \in \Omega$ such that $\ell(\phi)$ is admissible at $u(p)$, and for any $k \in \mathbb{Z}$, there exists two fundamental matrix solutions $Y^{(k)}_{\textit{left/right}}(z)$, uniquely determined by the asymptotic condition*

$$Y^{(k)}_{\textit{left/right}}(z) \sim Y_{\text{formal}}(z), \quad |z| \to \infty, \quad z \in e^{2\pi i k}\Pi_{\textit{left/right}}(\phi).$$

(2) *The above solutions $Y^{(k)}_{\textit{left/right}}$ satisfy*

$$Y^{(k)}_{\textit{left/right}}\left(e^{2\pi i k}z\right) = Y^{(0)}_{\textit{left/right}}(z), \quad z \in \mathcal{R}. \tag{2.30}$$

(3) *In case $\Omega \equiv \Omega_\ell$ is an $\ell(\phi)$-chamber, if p varies in Ω_ℓ, then the solutions $Y^{(k)}_{\textit{left/right}}(z)$ define holomorphic functions*

$$Y^{(k)}_{\textit{left/right}}(z,u),$$

w.r.t. to $u = u(p)$. Moreover, the asymptotic expansion

$$Y^{(k)}_{\textit{left/right}}(z,u) \sim Y_{\text{formal}}(z,u), \quad |z| \to \infty, \quad z \in e^{2\pi i k}\Pi_{\textit{left/right}}(\phi), \tag{2.31}$$

holds uniformly in u for p varying in Ω_ℓ. Here $Y_{\text{formal}}(z,u)$ is the u-holomorphic formal solution (2.28).

PROOF. The proof of (1) and (2) away from coalescence points is standard (see [**Was65**], [**BJL79a**], [**Dub99b**], [**Dub04**]), while at coalescence points it follows from the results of [**CDG17b**] and [**BJL79c**]. Point (3) is stated in [**Dub99b**], [**Dub04**], though the name "ℓ-chamber" does not appear there. □

REMARK 2.6. The holomorphic properties at point (3) of Theorem 2.9 hold in a ℓ-chamber, where there are no coalescence points. In our Theorem 9.1 below, we will see that point (3) actually holds in a set $\Omega \subset M_{ss}$ as in Remark 2.4, no matter if it contains semisimple coalescence points or not. The only requirement is that $\ell(\phi)$ is admissible at $u = u(p)$ for any $p \in \Omega$.

REMARK 2.7. The asymptotic relation (2.31) means that

$$\forall K \Subset \Omega_\ell,\ \forall h \in \mathbb{N},\ \forall \overline{\mathcal{S}} \subsetneq e^{2\pi k i}\Pi_{\text{right/left}}(\phi),\ \exists C_{K,h,\overline{\mathcal{S}}} > 0 \text{ such that if } z \in \overline{\mathcal{S}} \setminus \{0\} \text{ then}$$

$$\sup_{u\in K}\left\|Y^{(k)}_{\text{right/left}}(z,u)\cdot\exp(-zU)-\sum_{m=0}^{h-1}\frac{G_m(u)}{z^m}\right\|<\frac{C_{K,h,\overline{\mathcal{S}}}}{|z|^h}.$$

Here $\overline{\mathcal{S}}$ denotes any unbounded closed sector of $\mathcal{R}$ with vertex at 0. Actually, the solutions $Y^{(k)}_{\text{right/left}}(z,u)$ maintain their asymptotic expansion (2.31) in sectors wider than $e^{2\pi ik}\Pi_{\text{right/left}}(\phi)$, extending at least up to the nearest Stokes rays outside $e^{2\pi ik}\Pi_{\text{right/left}}(\phi)$. In particular, for any $p\in K\Subset\Omega_\ell$ and suitably small $\varepsilon=\varepsilon(K)>0$, then the asymptotics holds in $e^{2\pi ik}\Pi^{\varepsilon}_{\text{right/left}}(\phi)$, where

$$\Pi^{\varepsilon}_{\text{right}}(\phi):=\{z\in\mathcal{R}\colon \phi-\pi-\varepsilon<\arg\ z<\phi+\varepsilon\},\quad \Pi^{\varepsilon}_{\text{left}}(\phi):=\{z\in\mathcal{R}\colon \phi-\varepsilon<\arg\ z<\phi+\pi+\varepsilon\}.$$

The positive number ε is chosen small enough in such a way that, as p varies in the compact set K, no Stokes ray is contained in the following sectors:

$$\Pi^{\varepsilon}_{+}(\phi):=\{z\in\mathcal{R}\colon \phi-\varepsilon<\arg\ z<\phi+\varepsilon\},\quad \Pi^{\varepsilon}_{-}(\phi):=\{z\in\mathcal{R}\colon \phi-\pi-\varepsilon<\arg\ z<\phi-\pi+\varepsilon\}.$$

LEMMA 2.4. *In the assumptions of Theorem 2.9, for any $k\in\mathbb{Z}$ and any $z\in\mathcal{R}$ the following orthogonality relation holds:*

$$Y^{(k)}_{\text{left}}(e^{i\pi}z)^TY^{(k)}_{\text{right}}(z)=\mathbb{1}.$$

PROOF. From Remark 2.1 we already know that the product above is independent of $z\in\mathcal{R}$. According to Remark 2.7, if $\varepsilon>0$ is a sufficiently small positive number, then

$$Y^{(k)}_{\text{left/right}}(z)\sim Y_{\text{formal}}(z),\quad |z|\to\infty,\quad z\in e^{2\pi ik}\Pi^{\varepsilon}_{\text{left/right}}(\phi)$$

Consequently,

$$Y^{(k)}_{\text{left}}(e^{i\pi}z)\sim G(-z)e^{-zU},\quad Y^{(k)}_{\text{right}}(z)\sim G(z)e^{zU},\quad |z|\to\infty,\quad z\in e^{2\pi ik}\Pi^{\varepsilon}_{+}(\phi).$$

Thus, $Y^{(k)}_{\text{left}}(e^{i\pi}z)^TY^{(k)}_{\text{right}}(z)=\mathbb{1}$ for all $z\in e^{2\pi ik}\Pi^{\varepsilon}_{+}(\phi)$, and by analytic continuation for all $z\in\mathcal{R}$. □

Let $Y_0(z,u)$ be a fundamental solution of (2.26) near $z=0$ of the form (1.9), i.e.

$$Y_0(z,u)=\Psi(u)\Phi(z,u)z^\mu z^R,\qquad \Phi(z,u)=\mathbb{1}+\sum_{k=1}^{\infty}\Phi_k(u)z^k,\qquad \Phi(-z,u)^T\eta\,\Phi(z,u)=\eta,\tag{2.32}$$

with $\Psi^T\Psi=\eta$, obtained from (2.12) and (2.13) through the constant gauge (2.23). This solution is not affected by coalescence phenomenon and since μ and R are independent of $p\in\Omega$, it is holomorphic w.r.t. u (see [**Dub99b**], [**Dub04**]). Recall that $Y_0(z,u)$ is not uniquely determined by the choice of R.

DEFINITION 2.15 (Stokes and Central Connection Matrices). Let $\Omega\subset M_{ss}$ be as in Remark 2.4 and let the system (2.24) be determined as in the beginning of this section. Let $\phi\in\mathbb{R}$ be fixed. Let $p\in\Omega$ be such that $\ell(\phi)$ is admissible at $u(p)$. Finally, let $Y^{(0)}_{\text{right/left}}(z)$ be the fundamental solutions of Theorem 2.9 at p. The matrices S and S_- defined at $u(p)$ by the relations

$$Y^{(0)}_{\text{left}}(z)=Y^{(0)}_{\text{right}}(z)\ S,\quad z\in\mathcal{R},\tag{2.33}$$

$$Y^{(0)}_{\text{left}}(e^{2\pi i}z)=Y^{(0)}_{\text{right}}(z)\ S_-,\quad z\in\mathcal{R}\tag{2.34}$$

are called *Stokes matrices* of the system (2.26) at the point p w.r.t. the line $\ell(\phi)$. The matrix C such that

$$Y^{(0)}_{\text{right}}(z)=Y_0(z,u(p))\ C,\quad z\in\mathcal{R}\tag{2.35}$$

is called *central connection matrix* of the system (2.24) at p, w.r.t. the line ℓ and the fundamental solution Y_0.

THEOREM 2.10. *The Stokes matrices S, S_- and the central connection matrix C of Definition 2.15 at a point $p \in \Omega$ satisfy the following properties, for all $k \in \mathbb{Z}$ and all $z \in \mathcal{R}$:*

(1)

$$\begin{aligned} Y^{(k)}_{\text{left}}(z) &= Y^{(k)}_{\text{right}}(z)\ S, \\ Y^{(k)}_{\text{left}}(z) &= Y^{(k+1)}_{\text{right}}(z)\ S_-, \\ Y^{(k)}_{\text{right}}(z) &= Y_0(z, u(p))\ M_0^{-k}\ C, \end{aligned}$$

where $M_0 = \exp(2\pi i \mu)\exp(2\pi i R)$*;*

(2)

$$\begin{aligned} Y^{(k)}_{\text{right}}(e^{2\pi i} z) &= Y^{(k)}_{\text{right}}(z)\ \left(S_-\ S^{-1}\right), \\ Y^{(k)}_{\text{left}}(e^{2\pi i} z) &= Y^{(k)}_{\text{right}}(z)\ \left(S^{-1}\ S_-\right); \end{aligned}$$

(3)

$$\begin{gathered} S_- = S^T, \\ S_{ii} = 1, \quad i = 1, \dots, n, \\ S_{ij} \neq 0 \textit{ with } i \neq j \textit{ only if } u_i \neq u_j \textit{ and } R_{ij} \subset \operatorname{pr}\left(\Pi_{\text{left}}(\phi)\right). \end{gathered}$$

PROOF. The first and second identities of (1) follow from equation (2.30). For the third note that

$$Y^{(k)}_{\text{right}}(z) = Y^{(0)}_{\text{right}}(e^{-2ik\pi} z) = Y_0(e^{-2ik\pi} z) C = Y_0(z) M_0^{-k} C.$$

Point (2) follows easily from the vanishing of the exponent of formal monodromy ($\operatorname{diag}(V) = 0$). By definition of Stokes matrices we have that

$$Y^{(0)}_{\text{left}}(e^{i\pi} z) = Y^{(0)}_{\text{right}}(e^{-i\pi} z) S_-, \quad Y^{(0)}_{\text{right}}(z) = Y^{(0)}_{\text{left}}(z) S^{-1},$$

and by Lemma 2.4

$$S_-^T \underbrace{Y^{(0)}_{\text{right}}(z)^T Y^{(0)}_{\text{left}}(e^{i\pi} z)}_{\mathbb{1}} S^{-1} \equiv \mathbb{1}.$$

We conclude $S_-^T = S$. If we consider the sector $\Pi^{\varepsilon}_{+}(\phi)$ fo sufficiently small $\varepsilon > 0$ as in proof of Lemma 2.4, them from the relation $Y^{(0)}_{\text{left}}(z) = Y^{(0)}_{\text{right}}(z) S$, we deduce that

$$e^{z(u_i - u_j)} S_{ij} \sim \delta_{ij}, \quad |z| \to \infty, \quad z \in \Pi^{\varepsilon}_{+}(\phi).$$

So, if $u_i = u_j$ we deduce $S_{ij} = \delta_{ij}$. If $i \neq j$ are such that $u_i \neq u_j$, then if $R_{ij} \subset \operatorname{pr}\left(\Pi_{\text{right}}(\phi)\right)$ we have

$$|e^{z(u_i - u_j)}| \to \infty \quad \text{for } |z| \to \infty, \quad z \in \Pi^{\varepsilon}_{+}(\phi),$$

and hence necessarily $S_{ij} = 0$. For the opposite ray $R_{ji} \subset \operatorname{pr}\left(\Pi_{\text{left}}\right)$ we have

$$|e^{z(u_i - u_j)}| \to 0 \quad \text{for } |z| \to \infty, \quad z \in \Pi^{\varepsilon}_{+}(\phi),$$

so S_{ij} need not to be 0. This proves (3).

□

The monodromy data must satisfy some important constraints, summarised in the following theorem, whose proof is *not* found in [**Dub98**], [**Dub99b**].

THEOREM 2.11. *The monodromy data μ, R, S, C at apoint $p \in \Omega$ as in Definition 2.15 satisfy the identities:*

(1) $CS^TS^{-1}C^{-1} = M_0 = e^{2\pi i\mu}e^{2\pi i R}$,
(2) $S = C^{-1}e^{-\pi i R}e^{-\pi i\mu}\eta^{-1}(C^T)^{-1}$,
(3) $S^T = C^{-1}e^{\pi i R}e^{\pi i\mu}\eta^{-1}(C^T)^{-1}$.

PROOF. The first identity has a simple topological motivation: loops around the origin in $\mathbb{C}^*$ are homotopic to loops around infinity. So, one easily obtains the relation using Theorem 2.10, and the definition of central connection matrix. Using the orthogonality relations for solutions, equation (2.11) and the fact that

$$z^{\mu^T}\eta z^{\mu} = \eta$$

(μ being diagonal and η-antisymmetric), we can now prove the identities (2) and (3). By Lemma 2.4 we have that

$$\begin{aligned}\mathbb{1} &= Y^{(0)}_{\text{right}}(z)^T Y^{(0)}_{\text{left}}(e^{i\pi}z)\\ &= Y^{(0)}_{\text{right}}(z)^T Y^{(0)}_{\text{right}}(e^{i\pi}z)S = C^T Y_0(z)^T Y_0(e^{i\pi}z)CS.\end{aligned}$$

Now we have

$$\begin{aligned}Y_0(z)^T Y_0(e^{i\pi}z) &= z^{R^T}z^{\mu^T}\left(\Phi(z)^T\Psi^T\Psi\Phi(-z)\right)(e^{i\pi}z)^{\mu}(e^{i\pi}z)^R\\ &= z^{R^T}z^{\mu^T}\eta z^{\mu}e^{i\pi\mu}z^R e^{i\pi R}\\ &= \eta e^{i\pi\mu}e^{i\pi R}.\end{aligned}$$

This shows the first identity. For the second one, we have that

$$\begin{aligned}\mathbb{1} &= Y^{(0)}_{\text{right}}(z)^T Y^{(0)}_{\text{left}}(e^{i\pi}z)\\ &= Y^{(0)}_{\text{right}}(z)^T Y^{(0)}_{\text{right}}(e^{-i\pi}z)S^T\\ &= C^T Y_0(z)^T Y_0(e^{-i\pi}z)CS^T.\end{aligned}$$

Again, we have

$$\begin{aligned}Y_0(z)^T Y_0(e^{-i\pi}z) &= z^{R^T}z^{\mu^T}\left(\Phi(z)^T\Psi^T\Psi\Phi(-z)\right)(e^{-i\pi}z)^{\mu}(e^{-i\pi}z)^R\\ &= z^{R^T}z^{\mu^T}\eta z^{\mu}e^{-i\pi\mu}z^R e^{-i\pi R}\\ &= \eta e^{-i\pi\mu}e^{-i\pi R}.\end{aligned}$$

□

It follows from point (3) of Theorem 2.9 that S and C depend holomorphically on p varying in an ℓ-chamber Ω_ℓ, namely they define analytic matrix valued functions $S(u)$ and $C(u)$, $u = u(p)$. Moreover, due to the compatibility conditions $[E_i, V] = [V_i, U]$ and $\partial_i\Psi = V_i\Psi$, the system (2.24) is isomonodromic. Therefore $\partial_i S = \partial_i C = 0$. Indeed, the following holds:

THEOREM 2.12 (Isomonodromy Theorem, II, [Dub96, Dub98, Dub99b]). *The Stokes matrix S and the central connection matrix C, computed w.r.t. a line ℓ, are independent of p varying in an ℓ-chamber. The values of S, C in two different ℓ-chambers are related by an action of the braid group of Section 2.3.*

2.3. Freedom of Monodromy Data and Braid Group action

In associating the data (μ, R, S, C) to $p \in M$, several choices have been done, all preserving the constraints of Theorem 2.11

$$S = C^{-1} e^{-i\pi R} e^{-i\pi\mu} \eta^{-1} (C^{-1})^T, \tag{2.36}$$

$$S^T = C^{-1} e^{i\pi R} e^{i\pi\mu} \eta^{-1} (C^{-1})^T. \tag{2.37}$$

While the operator μ is completely fixed by the choice of flat coordinates as in Section 2.1, R is determined only up to conjugacy class of the (η, μ)-parabolic orthogonal group $\mathcal{G}(\eta, \mu)$ as in Theorem 2.3. Suppose now that R has been chosen in this class. The remaining local invariants S, C are subordinate to the following choices:

(1) an oriented line $\ell(\phi) = \{z = \rho e^{i\phi},\ \rho \in \mathbb{R}\}$ in the complex plane;
(2) for given $\phi \in \mathbb{R}$, the change of determination $\phi \mapsto \phi - 2k\pi$, $k \in \mathbb{Z}$, or dually, for fixed ϕ, the change $Y^{(0)}_{\text{left/right}}(z) \mapsto Y^{(k)}_{\text{left/right}}(z)$;
(3) the choice of a ordering of canonical coordinates on each ℓ-chamber Ω_ℓ;
(4) the choice of the branch of the square roots (2.22) defining the matrix Ψ on each ℓ-chamber Ω_ℓ;
(5) the choice of different solutions Y_0 in Levelt normal form corresponding to the same exponent R.

The transformations of the data depending on the choice of ℓ in (1) will be studied in the next Section. Here we describe how the freedoms in (2),(3),(4) and (5) affect the data (S, C):

- *Action of the additive group* $\mathbb{Z}$: according to formula (2.30), S remains invariant and
$$C \mapsto M_0^{-k} \cdot C, \quad k \in \mathbb{Z}, \quad M_0 = e^{2\pi i\mu} e^{2\pi i R}, \quad t \in \Omega_\ell.$$
- *Action of the group of permutations* $\mathfrak{S}_n$: if τ is a permutation, we can reorder the canonical coordinates:
$$(u_1, \dots, u_n) \mapsto (u_{\tau(1)}, \dots, u_{\tau(n)}).$$
The system (2.26) is changed to $U \mapsto PUP^{-1} = \operatorname{diag}(u_{\tau(1)}, \dots, u_{\tau(n)})$, $V \mapsto PVP^{-1}$. The fundamental matrices change as follows: $Y^{(0)}_{\text{left/right}} \mapsto PY^{(0)}_{\text{left/right}} P^{-1}$ and $Y_0 \longmapsto PY_0$. Therefore
$$S \mapsto PSP^{-1}, \quad C \mapsto CP^{-1}. \tag{2.38}$$
- *Action of the group* $(\mathbb{Z}/2\mathbb{Z})^{\times n}$: by choosing opposite signs for the normalized idempotents (matrix Ψ), we can change the sign of the entries of the matrices S and C. If $\mathcal{I}$ is a diagonal matrix with 1's or (-1)'s on the diagonal, the system (2.26) is changed to $U \mapsto \mathcal{I}U\mathcal{I} \equiv U$, $V \mapsto \mathcal{I}V\mathcal{I}$. Correspondingly, $Y_{\text{left/right}} \mapsto \mathcal{I}Y_{\text{left/right}}\mathcal{I}$, $Y_0 \mapsto \mathcal{I}Y_0$. Therefore
$$S \mapsto \mathcal{I}S\mathcal{I}, \quad C \mapsto C\mathcal{I}.$$

- *Action of the group* $\widetilde{\mathcal{C}}_0(\mu, R)$: for chosen R, the choice of a fundamental system at the origin, having the form (2.32), is determined up to $Y_0 \mapsto Y_0 G$, where $G \in \widetilde{\mathcal{C}}_0(\mu, R)$ of Definition 2.6. The corresponding left action on C is

$$C \longmapsto GC, \qquad G \in \widetilde{\mathcal{C}}_0(\mu, R).$$

Among all possible ordering of the canonical coordinates, a particularly useful one is the *lexicographical order* w.r.t an admissible line $\ell(\phi)$, defined as follows. Let us consider the rays starting from the points $u_1, \dots, u_n$ in the complex plane

$$L_j := \left\{ u_j + \rho e^{i\left(\frac{\pi}{2} - \phi\right)} : \rho \in \mathbb{R}_+ \right\}, \quad j = 1, \dots, n,$$

and for any complex number z_0 let us define the oriented line

$$L_{z_0,\phi} := \left\{ z_0 + \rho e^{-i\phi} : \rho \in \mathbb{R} \right\}$$

where the orientation is induced by $\mathbb{R}$. In this way we have a natural total order $\preceq$ on the points of $L_{z_0,\phi}$. We can choose z_0, with $|z_0|$ sufficiently large, so that the intersections $L_j \cap L_{z_0,\phi} =: \{p_j\}$ are non-empty.

Definition 2.16 (Lexicographical order). The canonical coordinates u_j's are in *ℓ-lexicographical order* if

$$p_1 \preceq p_2 \preceq p_3 \preceq \cdots \preceq p_n.$$

The definition does not depend on the choice of $z_0 \in \mathbb{C}$, with $|z_0|$ sufficiently large.

Observe that if $u_1, \dots, u_n$ are in lexicographical order w.r.t. the admissible line $\ell(\phi)$, then:

(1) the Stokes matrix is in upper triangular form;
(2) the nearest Stokes rays to the positive half-line $\mathrm{pr}(\ell_+(\phi))$ are of the form

$$R_{i,i+1} \subseteq \mathrm{pr}\left(\Pi_{\mathrm{left}}(\phi)\right), \qquad R_{j,j-1} \subseteq \mathrm{pr}\left(\Pi_{\mathrm{right}}(\phi)\right).$$

In general, condition (1) alone does not imply that the canonical coordinates are in lexicographical order: it does if and only if the number of nonzero entries of the Stokes matrix S is maximal (and equal to $\frac{n(n+1)}{2}$). In this case, by Theorem 2.10, necessarily $u_i \neq u_j$ for $i \neq j$. On the other hand, if there are some vanishing entries $S_{ij} = S_{ji} = 0$ for $i \neq j$, and S is upper triangular, then also PSP^{-1} in (2.38) is upper triangular for any permutation exchanging u_i and u_j corresponding to $S_{ij} = S_{ji} = 0$. For example, this happens at a coalescence point: by Theorem 2.10, the entries S_{ij} with $i \neq j$ are 0 corresponding to coalescing values $u_i = u_j$, $i \neq j$.

Definition 2.17 (Triangular order). We say that $u_1, ..., u_n$ are in triangular order w.r.t. the line ℓ whenever S is upper triangular.

It follows from the preceding discussion that at a semisimple coalescence point there are more than one triangular orders. Moreover, *any* of them is also lexicographical. For further comments, see Remark 9.1.

2.3.1. Action of the braid group $\mathcal{B}_n$. In this section, canonical coordinates are pairwise distinct, corresponding to a non-coalescence semisimple points lying in ℓ-chambers. The braid group is

$$\mathcal{B}_n = \pi_1\left((\mathbb{C}^n \setminus \Delta)/\mathfrak{S}_n\right),$$

where Δ stands for the union of all diagonals in $\mathbb{C}^n$. It is generated by $n-1$ elementary braids $\beta_{12}, \beta_{23}, \dots, \beta_{n-1,n}$ with the relations

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1} \quad \text{for } i+1 \neq j, j+1 \neq i,$$

$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2}.$$

The action of the braid group $\mathcal{B}_n$ on the monodromy data manifests whenever some Stokes' rays and the chosen line ℓ cross under mutation. This can happens in two ways:

- First: we let vary the point of the Frobenius manifold at which we compute the data, keeping fixed the line ℓ; this is the case if, starting from the data computed in a ℓ-chamber, we want to compute the data in neighboring ℓ-chamber, or even more in general if we want to analyze properties of the analytic continuation of the whole Frobenius structure by letting varying the coordinates $(u_1, \dots, u_n)$ in the universal cover $\widetilde{\mathbb{C}^n \setminus \Delta}$.
- Second: we fix the point at which we compute the data and change the admissible line ℓ by a rotation.

In the first case the ℓ-chambers are fixed, in the second case they change: indeed, the fixed point of the Frobenius manifold is in two different chambers before and after the rotation of ℓ. In both cases, we will always label the canonical coordinates $(u_1, \dots, u_n)$ in lexicographical order w.r.t. ℓ both before and after the transformation (so that, in particular, any Stokes matrix is always in upper triangular form).

Any continuous deformation of the n-tuple $(u_1, \dots, u_n)$, represented as a deformation of n points in $\mathbb{C}$ never colliding, can be decomposed into *elementary* ones. If we restrict to the case of a continuous deformation which ends exactly with the same initial ordered pattern of points, then we can identify an elementary deformation with a generator of the *pure* braid group, i.e. $\pi_1(\mathbb{C}^n \setminus \Delta)$. Otherwise, by allowing permutations, we can identify an elementary deformation with a generator of the braid group $\mathcal{B}_n$. In particular, an elementary deformation which will be denoted by $\beta_{i,i+1}$ consists in a counter-clockwise rotation of u_i w.r.t. u_{i+1}, so that the two exchange. All other points u_j's are subjected to a sufficiently small perturbation, so that the corresponding Stokes' rays almost do not move. $\beta_{i,i+1}$ corresponds to

- clockwise rotation of the Stokes' ray $R_{i,i+1}$ crossing the line ℓ,
- or, dually, counter-clockwise rotation of the line ℓ crossing the Stokes' ray $R_{i,i+1}$

This determines the following mutation of the monodromy data, as shown in [Dub96] and [Dub99b]:

$$S^{\beta_{i,i+1}} := A^{\beta_{i,i+1}}(S)\ S\ A^{\beta_{i,i+1}}(S)^T \tag{2.39}$$

where

$$\left(A^{\beta_{i,i+1}}(S)\right)_{hh} = 1, \qquad h = 1, \dots, n \quad h \neq i, i+1$$

$$\left(A^{\beta_{i,i+1}}(S)\right)_{i+1,i+1} = -s_{i,i+1},$$

$$\left(A^{\beta_{i,i+1}}(S)\right)_{i,i+1} = \left(A^{\beta_{i,i+1}}(S)\right)_{i+1,i} = 1.$$

For a generic braid β, which is a product of N elementary braids $\beta = \beta_{i_1,i_1+1} \dots \beta_{i_N,i_N+1}$, the action is

$$S \mapsto S^\beta := A^\beta(S) \cdot S \cdot A^\beta(S)^T \tag{2.40}$$

where

$$A^\beta(S) = A^{\beta_{i_N,i_N+1}}\left(S^{\beta_{i_{N-1},i_{N-1}+1}}\right) \cdot \dots \cdot A^{\beta_{i_2,i_2+1}}\left(S^{\beta_{i_1,i_1+1}}\right) \cdot A^{\beta_{i_1,i_1+1}}(S).$$

The action on the central connection matrix (in lexicographical order) is

$$C \mapsto C^\beta := C\ (A^\beta)^{-1}. \tag{2.41}$$

Now, let us consider a complete counter-clockwise 2π-rotation of the admissible line ℓ, and observe the following:

(1) in the generic case (i.e. when the canonical coordinates u_j's are in general position) there are $n(n-1)$ distinct projected Stokes' rays R_{jk}. An elementary braid acts any time the line ℓ crosses a Stokes ray. So, in total, we expect that a complete rotation of ℓ correspond to the product of $n(n-1)$ elementary braids $\beta_{i,i+1}$'s.
(2) Since the formal monodromy is vanishing, the effect of the rotation of ℓ on the Stokes matrix is trivial, while the central connection matrix C is transformed to $M_0^{-1}C$, M_0 being the monodromy at the origin (point (1) of Theorem 2.10). As a consequence, the complete rotation of the line ℓ can be viewed as a deformation of points u_j's *commuting* with any other braid.

From point (2) we deduce that the braid corresponding to the complete rotation of ℓ is an element of the center

$$Z(\mathcal{B}_n) = \left\{ (\beta_{12}\beta_{23}\dots\beta_{n-1,n})^{kn} : k \in \mathbb{Z} \right\}.$$

From point (1) and from the fact that ℓ rotates counter-clockwise, we deduce that $k = 1$. In this way we have proved the following

Lemma 2.5. *The braid corresponding to a complete counter-clockwise 2π-rotation of ℓ is the braid*

$$(\beta_{12}\beta_{23}\dots\beta_{n-1,n})^n,$$

and its acts on the monodromy data as follows:

- *trivially on Stokes matrices,*
- *the central connection matrix is transformed as $C \mapsto M_0^{-1}C$.*

CHAPTER 3

Gromov-Witten Invariants, Gravitational Correlators and Quantum Cohomology

ABSTRACT. In this Chapter basic notions of Gromov-Witten and Quantum Cohomology theories are presented. After recalling the main definitions and properties of Gromov-Witten invariants (and of the more general gravitational correlators) for a smooth projective variety X, for any genus $g \geq 0$ we introduce a generating function for these numbers (the total descendant potential of genus g). Under an assumption of analytical convergence for $g = 0$, we introduce the Quantum Cohomology of X, and we describe its Frobenius manifold structure. In Section 3.3 we introduce and describe in details the so called *topological-enumerative solution* for the system defining deformed flat coordinates: in the case X is Fano, this solution is completely characterized.

3.1. Gromov-Witten Theory

3.1.1. Notations and preliminaries. Let X be a smooth projective complex variety. In order not to deal with Frobenius superstructures, we will suppose for simplicity that the variety X has vanishing odd cohomology, i.e. $H^{2k+1}(X;\mathbb{C}) \cong 0$ for $0 \leq k$. Let us fix a homogeneous basis $(T_0, T_1, \dots, T_N)$ of $H^\bullet(X;\mathbb{C}) = \bigoplus_k H^{2k}(X;\mathbb{C})$ such that

- $T_0 = 1$ is the unity of the cohomology ring;
- $T_1, \dots, T_r$ span $H^2(X;\mathbb{C})$.

We will denote by $\eta\colon H^\bullet(X;\mathbb{C}) \times H^\bullet(X;\mathbb{C}) \to \mathbb{C}$ the Poincaré metric

$$\eta(\xi,\zeta) := \int_X \xi \cup \zeta,$$

and in particular

$$\eta_{\alpha\beta} := \int_X T_\alpha \cup T_\beta.$$

If $\beta \in H_2(X;\mathbb{Z})/\text{torsion}$, we denote by $\overline{\mathcal{M}}_{g,n}(X,\beta)$ the Kontsevich-Manin moduli stack of n-pointed, genus g stable maps to X of degree β, which parametrizes equivalence classes of pairs $((C_g, \mathbf{x}); f)$, where:

- $(C_g, \mathbf{x})$ is an n-pointed algebraic curve of genus g, with at most nodal singularities and with n marked points $\mathbf{x} = (x_1, \dots, x_n)$, and $f\colon C_g \to X$ is a morphism such that $f_*[C_g] \equiv \beta$. Two pairs $((C_g, \mathbf{x}); f)$ and $((C'_g, \mathbf{x}'); f')$ are defined to be equivalent if there exists a bianalytic map $\varphi\colon C_g \to C'_g$ such that $\varphi(x_i) = x'_i$, for all $i = 1, \dots, n$, and $f' = \varphi \circ f$.
- The morphisms f are required to be *stable*: if f is constant on any irreducible component of C_g, then that component should have only a finite number of automorphisms as pointed curves (in other words, it must have at least 3 distinguished points, i.e. points that are either nodes or marked ones).

We will denote by $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta)\to X\colon ((C_g,\mathbf{x});f)\mapsto f(x_i)$ the naturally defined evaluations maps, and by $\psi_i\in H^2(\overline{\mathcal{M}}_{g,n}(X,\beta);\mathbb{Q})$ the Chern classes of tautological cotangent line bundles

$$\mathcal{L}_i\to\overline{\mathcal{M}}_{g,n}(X,\beta),\quad \mathcal{L}_i|_{((C_g,\mathbf{x});f)}=T^*_{x_i}C_g,\quad \psi_i:=c_1(\mathcal{L}_i).$$

Using the construction of [BF97] of a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}$ in the Chow ring $CH_\bullet(\overline{\mathcal{M}}_{g,n}(X,\beta))$, and of degree equal to the expected dimension

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}\in CH_D(\overline{\mathcal{M}}_{g,n}(X,\beta)),\quad D=(1-g)(\dim_{\mathbb{C}}X-3)+n+\int_\beta c_1(X),$$

a good theory of intersection is allowed on the Kontsevich-Manin moduli stack.

Definition 3.1. We define the *Gromov-Witten invariants (with descendants) of genus g, with n marked points and of degree β* of X as the integrals (whose values are rational numbers)

$$\langle\tau_{d_1}\gamma_1,\dots,\tau_{d_n}\gamma_n\rangle^X_{g,n,\beta}:=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}}\prod_{i=1}^n \mathrm{ev}_i^*(\gamma_i)\cup\psi_i^{d_i},\tag{3.1}$$

$$\gamma_i\in H^\bullet(X;\mathbb{C}),\quad d_i\in\mathbb{N},\quad i=1,\dots,n.$$

If some $d_i\neq 0$, the invariants (3.1) are also called *gravitational correlators*.

Since by *effectiveness* (for an axiomatic treatment of the Gromov-Witten invariants we follow [Man99], [KM94] and [CK99]) the integral is non-vanishing only for effective classes $\beta\in\mathrm{Eff}(X)\subseteq H_2(X;\mathbb{Z})$, the generating function of rational numbers (3.1), called *total descendent potential* (or also *gravitational Gromov-Witten potential*, or even *Free Energy*) of *genus g* is defined as the formal series

$$\mathcal{F}^X_g(\gamma,\mathbf{Q}):=\sum_{n=0}^\infty\sum_{\beta\in\mathrm{Eff}(X)}\frac{\mathbf{Q}^\beta}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n\text{ times}}\rangle^X_{g,n,\beta},\tag{3.2}$$

where we have introduced (infinitely many) coordinates $\mathbf{t}:=(t^{\alpha,p})_{\alpha,p}$

$$\gamma=\sum_{\alpha,p}t^{\alpha,p}\tau_pT_\alpha,\quad \alpha=0,\dots,N,\ p\in\mathbb{N},$$

and formal parameters

$$\mathbf{Q}^\beta:=Q_1^{\int_\beta T_1}\cdot\dots\cdot Q_r^{\int_\beta T_r},\quad Q_i\text{'s elements of the Novikov ring }\Lambda:=\mathbb{C}[\![Q_1,\dots,Q_r]\!].$$

The free energy $\mathcal{F}^X_g\in\Lambda[\![\mathbf{t}]\!]$ can be seen a function on the *large phase-space*, and restricting the free energy to the *small phase space* (naturally identified with $H^\bullet(X;\mathbb{C})$),

$$F^X_g(t^{1,0},\dots,t^{N,0}):=\mathcal{F}^X_g(\mathbf{t})|_{t^{\alpha,p}=0,\ p>0},$$

one obtains the generating function of the Gromov-Witten invariants of genus g.

3.2. Quantum Cohomology

3.2.1. Quantum cohomology as a Frobenius manifold. By the Divisor Axiom, the genus 0 Gromov-Witten potential $F^X_0(t)$, can be seen as an element of the ring $\mathbb{C}[\![t^0,Q_1e^{t^1},\dots,Q_re^{t^r},t^{r+1},\dots,t^N]\!]$: in what follows we will be interested in cases in which F^X_0 is the analytic expansion of an analytic function, i.e.

$$F^X_0\in\mathbb{C}\left\{t^0,Q_1e^{t^1},\dots,Q_re^{t^r},t^{r+1},\dots,t^N\right\}.$$

Without loss of generality, we can put $Q_1 = Q_2 = \cdots = Q_r = 1$, and $F_0^X(t)$ defines an analytic function in an open neighborhood $\mathcal{D} \subseteq H^\bullet(X;\mathbb{C})$ of the point

$$t^i = 0, \quad i = 0, r+1, \dots, N, \tag{3.3}$$

$$\operatorname{Re} t^i \to -\infty, \quad i = 1, 2, \dots, r. \tag{3.4}$$

Remark 3.1. At the classical limit point (3.3), (3.4), the algebra structure on the tangent spaces coincide with the classical cohomological algebra structure. Indeed, the following is the structure of the potential:

(1) by Point Mapping Axiom, the Gromov-Witten potential can be decomposed into a *classical* term and a *quantum* correction as follows

$$\begin{aligned} F_0^X(\gamma) &= F_{\text{classical}} + F_{\text{quantum}} \\ &= \frac{1}{6}\int_X \gamma^3 + \sum_{k=0}^{\infty} \sum_{\beta \in \operatorname{Eff}(X)\setminus\{0\}} \frac{1}{k!} \langle \underbrace{\gamma, \dots, \gamma}_{k \text{ times}} \rangle_{0,k,\beta}^X, \quad \text{where } \gamma = \sum_{\alpha=0}^N t^\alpha T_\alpha; \end{aligned}$$

(2) the variable t^0 appears only in the classical term of F_0^X;

(3) because of the Divisor axiom, the variables corresponding to cohomology degree 2 (i.e. $t^1, \dots, t^r$) appear in the exponential form in the quantum term; the Frobenius structure is $2\pi i$-periodic in the 2-nd cohomology directions: the structure can be considered as defined on an open region of the quotient $H^\bullet(X;\mathbb{C})/2\pi i H^2(X;\mathbb{Z})$.

The function F_0^X is a solution of WDVV equations (for a proof see [**KM94**], [**Man99**], [**CK99**]), and thus it defines an analytic Frobenius manifold structure on $\mathcal{D}$ ([**Dub92**, **Dub96**, **Dub98**, **Dub99b**, **CDG17c**]). Note that

- the flat metric is given by the Poincaré metric η;
- the unity vector field is $T_0 = 1$, using the canonical identifications of tangent spaces

$$T_p\mathcal{D} \cong H^\bullet(X;\mathbb{C}) \colon \partial_{t^\alpha} \mapsto T_\alpha;$$

- the Euler vector field is

$$E := c_1(X) + \sum_{\alpha=0}^N \left(1 - \frac{1}{2}\deg T_\alpha\right) t^\alpha T_\alpha. \tag{3.5}$$

More precisely, by the Point Mapping Axiom, the Gromov-Witten potential can be decomposed into a *classical* term and a *quantum* correction as follows

$$\begin{aligned} F_0^X(\gamma) &= F_{\text{classical}} + F_{\text{quantum}} \\ &= \frac{1}{6}\int_X \gamma^3 + \sum_{k=0}^{\infty} \sum_{\beta \in \operatorname{Eff}(X)\setminus\{0\}} \frac{1}{k!} \langle \underbrace{\gamma, \dots, \gamma}_{k \text{ times}} \rangle_{0,k,\beta}^X, \quad \text{where } \gamma = \sum_{\alpha=0}^N t^\alpha T_\alpha. \end{aligned}$$

Consequently, the product of each algebra $(T_p\mathcal{D}, \circ_p)$ defined by

$$T_\alpha \circ_p T_\beta := \sum_{\gamma,\delta} \left.\frac{\partial^3 F_0^X}{\partial t^\alpha \partial t^\beta \partial t^\gamma}\right|_p \eta^{\gamma\delta} T_\delta, \quad p \in \mathcal{D}, \tag{3.6}$$

defines a deformation of the classical cohomological $\cup$-product. The associativity of $\circ_p$ is equivalent to the validity of the WDVV equations

$$\frac{\partial^3 F_0^X}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^X}{\partial t^\delta \partial t^\epsilon \partial t^\mu} = \frac{\partial^3 F_0^X}{\partial t^\mu \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^X}{\partial t^\delta \partial t^\epsilon \partial t^\alpha},$$

and it is easily seen that

$$\frac{\partial^3 F_0^X}{\partial t^0 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta},$$

the variable t^0 appearing only in the classical term of F_0^X. Hence, the resulting algebras $(T_p\mathcal{D}, \circ_p)$ are Frobenius. Furthermore, the Gromov-Witten potential F_0^X satisfies also the quasi-homogeneity condition

$$\mathfrak{L}_E F_0^X = (3 - \dim_{\mathbb{C}} X) \cdot F_0^X.$$

DEFINITION 3.2. The Frobenius manifold structure defined on the domain of convergence $\mathcal{D}$ of the Gromov-Witten potential F_0^X, solution of the WDVV problem, is called *Quantum Cohomology* of X, and denoted by $QH^\bullet(X)$. By the expression *small quantum cohomology of* X (or *small quantum locus*) we denote the Frobenius structure attached to points in $\mathcal{D} \cap H^2(X;\mathbb{C})$. In case convergence, the potential F_0^X (and hence the whole Frobenius structure) can be maximally analytically continued to an unramified covering of the domain $\mathcal{D} \subseteq H^\bullet(X;\mathbb{C})$. We refer to this global Frobenius structure as the *big quantum cohomology of* X, and it will be still denoted by $QH^\bullet(X)$.

Although no general results guarantee the convergence of the Gromov-Witten potential F_0^X for a generic smooth projective variety X, for some classes of varieties it is known that the sum defining F_0^X at points of the small quantum cohomology (at which $t^0 = t^{r+1} = \cdots = t^N = 0$) is finite. This is the case for

- Fano varieties,
- varieties admitting a transitive action of a semisimple Lie group.

For a proof see [**CK99**]. Notice that for these varieties the small quantum locus coincide with the whole space $H^2(X;\mathbb{C})$. Conjecturally, for Calabi-Yau manifolds the series defining F_0^X is convergent in a neighborhood of the classical limit point (see [**CK99**], [**KM94**]).

3.2.2. Semisimplicity of Quantum Cohomology. In this thesis we will focus on smooth projective varieties X whose (big) quantum cohomology is a semisimple Frobenius manifold. A point $p \in QH^\bullet(X)$ whose associated Frobenius algebra is semisimple will be called a *semisimple point*, for short. Note that the classical Frobenius cohomological algebra $(H^\bullet(X;\mathbb{C}), \cup)$, corresponding to the limit point (3.3)-(3.4), is not semisimple, since it clearly contains nilpotent elements. By quantum deformation of the $\cup$-product, it may happen that the semisimplicity condition is satisfied. The problem of characterizing smooth projective varieties with semisimple quantum cohomology is far from being solved. The following result shows that the assumption on X considered above, of having odd-vanishing cohomology $H^{\text{odd}}(X;\mathbb{C}) \cong 0$, is a necessary condition in order to have semisimplicity of the quantum cohomology $QH^\bullet(X)$.

THEOREM 3.1 ([**HMT09**]). *If X is a smooth projective variety whose quantum cohomology $QH^\bullet(X)$ is a semisimple analytic Frobenius manifold, then X is of Hodge-Tate type, i.e.*

$$h^{p,q}(X) = 0, \quad \text{if } p \neq q.$$

In particular, X is with odd-vanishing cohomology.

For some classes of varieties, such as some Fano threefolds [**Cio04**], toric varieties [**Iri07**], and some homogeneous spaces [**CMP10**], it has been proved that points of the small quantum cohomology are all semisimple. Grassmannians are among these varieties. More general homogeneous spaces may have non-semisimple small quantum cohomology ([**CMP10**], [**CP11**], [**GMS15**]). Some sufficient conditions for other Fano varieties are given in [**Per14**].

REMARK 3.2. Remarkably, under the assumptions of convergence of the genus 0 Gromov-Witten potential F_0^X and semisimplicity of the quantum cohomology $QH^\bullet(X)$, it can be shown ([CI15]) that there exist two real positive constants C, ε such that, for any $g \geq 0$, the power series (3.2) defining the genus g total descendant potential $\mathcal{F}_g^X$ is convergent on the infinite-dimensional polydisc

$$|t^{\alpha,p}| < \varepsilon \frac{p!}{C^p} \quad \text{for } \alpha = 0, \dots, N, \text{ and } p \in \mathbb{N},$$
$$|Q_i| < \varepsilon \quad \text{for } i = 1, \dots, r.$$

3.3. Topological-Enumerative Solution

For quantum cohomologies of smooth projective varieties, a fundamental system of solutions of the equation for gradients of deformed flat coordinates

$$\begin{cases} \partial_\alpha \zeta = z\mathcal{C}_\alpha \zeta, \\ \partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta, \end{cases} \tag{3.7}$$

can be expressed in *enumerative-topological* terms, namely the genus 0 correlations functions.

PROPOSITION 3.1. *For a sufficiently small $R > 0$, it is defined an analytic function*

$$\Theta \colon B_{\mathbb{C}}(0; R) \times \Omega \to \mathrm{End}(H^\bullet(X; \mathbb{C}))$$

with series expansion

$$\Theta(z, t) := \mathrm{Id} + \sum_{\alpha=0}^N \left\langle\!\!\left\langle \frac{z \cdot (-)}{1 - z\psi}, T_\alpha \right\rangle\!\!\right\rangle_0 (t) T^\alpha$$
$$= \mathrm{Id} + \sum_{n=0}^\infty z^{n+1} \sum_{\alpha=0}^N \langle\!\langle \tau_n(-), T_\alpha \rangle\!\rangle_0 (t) T^\alpha.$$

This function Θ satisfies the following properties:

(1) *for any $\phi \in H^\bullet(X; \mathbb{C})$, the vector field*

$$\Theta_\phi := \Theta(z, t)\phi = \phi + \sum_{\alpha=0}^N \left\langle\!\!\left\langle \frac{z\phi}{1 - z\psi}, T_\alpha \right\rangle\!\!\right\rangle_0 (t) T^\alpha$$
$$= \phi + \sum_{n=0}^\infty z^{n+1} \sum_{\alpha=0}^N \langle\!\langle \tau_n \phi, T_\alpha \rangle\!\rangle_0 (t) T^\alpha$$

satisfies the equations

$$\partial_\alpha \Theta_\phi = z \partial_\alpha * \Theta_\phi;$$

(2) *when restricted to the small quantum locus $\Omega \cap H^2(X; \mathbb{C})$, i.e. $t^i = 0$ for $i = 0, r+1, \dots, N$, then*

$$\Theta_\phi = e^{z\delta} \cup \phi + \sum_{\beta \neq 0} \sum_{\alpha=0}^N e^{\int_\beta \delta} \left\langle \frac{z e^{z\delta} \cup \phi}{1 - z\psi}, T_\alpha \right\rangle_{0,2,\beta}^X T^\alpha, \quad \delta := \sum_{i=1}^r t^i T_i \in H^2(X; \mathbb{C});$$

(3) *for any $\phi_1, \phi_2 \in H^\bullet(X; \mathbb{C})$ we have*

$$\eta\left(\Theta(-z, t)\phi_1, \Theta(z, t)\phi_2\right) = \eta\left(\phi_1, \phi_2\right);$$

(4) *for any $\phi \in H^\bullet(X;\mathbb{C})$, the vector field*

$$(Z_{\rm top})_\phi := \left(\Theta(z,t)\circ z^\mu z^{c_1(X)\cup(-)}\right)\phi$$

is a solution of the system (3.7), *i.e.*

$$\partial_\alpha(Z_{\rm top})_\phi = z\partial_\alpha * (Z_{\rm top})_\phi, \qquad \partial_z(Z_{\rm top})_\phi = \left(\mathcal{U}+\frac{1}{z}\mu\right)(Z_{\rm top})_\phi.$$

Thus, the vector fields $(Z_{\rm top})_{T_\alpha}$'s are gradients of deformed flat coordinates: if $(\Theta^\alpha_\beta)_{\alpha,\beta}, ((Z_{\rm top})^\alpha_\beta)_{\alpha,\beta}$ are the matrices representing the two $\mathrm{End}(H^\bullet(X;\mathbb{C}))$-valued functions Θ and $Z_{\rm top}$ w.r.t. the basis $(T_\alpha)_\alpha$, i.e.

$$\Theta(z,t)T_\beta = \sum_{\alpha=0}^N \Theta^\alpha_\beta(z,t)T_\alpha, \quad Z_{\rm top}(z,t)T_\beta = \sum_{\alpha=0}^N (Z_{\rm top})^\alpha_\beta(z,t)T_\alpha,$$

then there exist analytic functions $(\tilde{t}_\alpha(z,t))_\alpha, (h_\alpha(z,t))_\alpha$ on $B_{\mathbb{C}}(0;R)\times\Omega$ such that

$$(Z_{\rm top})^\alpha_\beta(z,t) = (\mathrm{grad}\,\tilde{t}_\beta(z,t))^\alpha, \quad (\tilde{t}_0,\tilde{t}_1,\dots,\tilde{t}_N) = (h_0,h_1,\dots,h_N)\cdot z^\mu z^R,$$

$$\Theta^\alpha_\beta(z,t) = (\mathrm{grad}\,h_\beta(z,t))^\alpha, \quad \Theta^T(-z,t)\eta\Theta(z,t)=\eta,$$

$$h_\alpha(z,t) := \sum_{p=0}^\infty h_{\alpha,p}(t)z^p, \quad h_{\alpha,0}(t) = t_\alpha \equiv t^\lambda\eta_{\lambda\alpha}.$$

Proof. Notice that

$$Y(z,t) := H(z,t)z^\mu z^R, \quad H(z,t) = \sum_{p=0}^\infty H_p(t)z^p, \quad H_0(t)\equiv \mathbb{1}$$

is a fundamental solution of (3.7) if and only if $H(z,t)$ is a solution of the system

$$\begin{cases}\partial_\alpha H = z\mathcal{C}_\alpha H,\\ \partial_z H = \mathcal{U}H + \frac{1}{z}[\mu,H] - HR.\end{cases}$$

Because of the symmetry of $c_{\alpha\beta\gamma}$, the columns of H are the components w.r.t. $(\partial_\alpha)_\alpha$ of the gradients of some functions:

$$h_\alpha(z,t) := \sum_{p=0}^\infty h_{\alpha,p}(t)z^p, \quad h_{\alpha,0}(t) = t_\alpha,$$

$$H^\alpha_\beta(z,t) = (\mathrm{grad}\,h_\beta)^\alpha, \quad H^\alpha_{\beta,p}(z,t) = (\mathrm{grad}\,h_{\beta,p})^\alpha.$$

The above system for H is equivalent to the following recursion relations on $h_{\alpha,p}$'s functions:

$$\partial_\alpha\partial_\beta h_{\gamma,p}(t) = c^\nu_{\alpha\beta}\partial_\nu h_{\gamma,p-1}(t), \quad p\geq 1, \tag{3.8}$$

$$\mathfrak{L}_E(\mathrm{grad}\,h_{\alpha,p}) = \left(p+\frac{\dim_{\mathbb{C}}X-2}{2}+\mu_\alpha\right)\mathrm{grad}\,h_{\alpha,p} + \sum_{\beta=0}^N(\mathrm{grad}\,h_{\beta,p-1})R^\beta_\alpha, \quad p\geq 1.$$

The last equation is equivalent to the recursion relations on the differentials

$$\mathfrak{L}_E(dh_{\alpha,p}) = \left(p-\frac{\dim_{\mathbb{C}}X-2}{2}+\mu_\alpha\right)dh_{\alpha,p} + \sum_{\beta=0}^N dh_{\beta,p-1}R^\beta_\alpha, \quad p\geq 1. \tag{3.9}$$

In our case we have

$$H(z,t) = (\Theta^\alpha_\beta(z,t))_{\alpha,\beta}, \quad \partial_\alpha h_{\beta,p}(t) = \langle\!\langle \tau_{p-1}T_\beta, T_\alpha\rangle\!\rangle_0(t).$$

The recursion relations (3.8) then reads

$$\langle\langle T_\alpha, T_\beta, T_\gamma\rangle\rangle_0 = \langle\langle T_\alpha, T_\beta, T^\nu\rangle\rangle_0 \eta_{\nu\gamma} \quad \text{for } p = 1,$$

$$\langle\langle T_\alpha, \tau_{p-1}T_\gamma, T_\beta\rangle\rangle_0 = \langle\langle T_\alpha, T_\beta, T^\nu\rangle\rangle_0 \langle\langle \tau_{p-2}T_\gamma, T_\nu\rangle\rangle_0 \quad \text{for } p \geq 2.$$

These are exactly the *topological recursion relations in genus 0*.

Let us now prove that also the recursion relations (3.9) hold. K. Hori ([**Hor95**], see also [**EHX97**]) proved that, for any $\omega \in H^2(X;\mathbb{C})$ we have the following constraint on the genus g free energy

$$\begin{aligned}\omega^\alpha \frac{\partial \mathcal{F}_g^X}{\partial t^{\alpha,0}} = \sum_{\beta\in \mathrm{Eff}(X)} \left(\int_\beta \omega\right)\mathcal{F}_{g,\beta}^X + \sum_{\substack{n,\alpha,\\ \sigma,\nu}} \omega^\sigma c_{\sigma\alpha}^\nu t^{\alpha,n}\frac{\partial \mathcal{F}_g^X}{\partial t^{\nu,n-1}}\\ + \frac{\delta_g^0}{2}\sum_{\alpha,\nu,\sigma}\omega^\sigma c_{\sigma\alpha\nu}t^{\alpha,0}t^{\nu,0} - \frac{\delta_g^1}{24}\int_X \omega\cup c_{\dim X-1}(X),\end{aligned} \tag{3.10}$$

where

$$\omega \cup T_\alpha = C_{\omega\alpha}^\nu T_\nu, \quad C_{\omega\alpha\nu} := \eta_{\nu\gamma}C_{\omega\alpha}^\gamma,$$

and $\mathcal{F}_{g,\beta}^X$ is the (g,β)-free energy

$$\mathcal{F}_{g,\beta}^X := \sum_{n=0}^\infty \frac{1}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n\text{ times}}\rangle_{g,n,\beta}^X.$$

By dimensional consideration, one obtains also the selection rule

$$\sum_{n,\alpha}(n+q_\alpha-1)t^{\alpha,n}\frac{\partial \mathcal{F}_g^X}{\partial t^{\alpha,n}} = \sum_{\beta\in\mathrm{Eff}(X)}\left(\int_\beta\omega\right)\mathcal{F}_{g,\beta}^X + (3-\dim X)(g-1)\mathcal{F}_g^X. \tag{3.11}$$

If we introduce the perturbed first Chern class

$$\mathcal{E}(\mathbf{t}) := c_1(X) + \sum_{m,\sigma}(1-q_\sigma-m)t^{\sigma,m}\tau_m(T_\sigma) - \sum_{m,\sigma}t^{\sigma,m}\tau_{m-1}(c_1(X)\cup T_\sigma),$$

and using the selection rule (3.11), the Hori's constraint (3.10) (specialized to $g=0$ and $\omega = c_1(X)$) can be reformulated as

$$\langle\langle\mathcal{E}\rangle\rangle_0 = (3-\dim X)\mathcal{F}_0^X + \frac{1}{2}t^{\sigma,0}t^{\rho,0}\int_X c_1(X)\cup T_\sigma\cup T_\rho.$$

Taking the derivative w.r.t. $t^{\alpha,n}, t^{\beta,0}$ we obtain

$$\begin{aligned}\langle\langle\mathcal{E},\tau_nT_\alpha,T_\beta\rangle\rangle_0 - (n+q_\alpha+q_\beta-2)\langle\langle\tau_nT_\alpha,T_\beta\rangle\rangle_0 - \langle\langle\tau_{n-1}(c_1(X)\cup T_\alpha),T_\beta\rangle\rangle_0\\ = (3-\dim X)\langle\langle\tau_nT_\alpha,T_\beta\rangle\rangle_0 + \delta_{n,0}\int_X c_1(X)\cup T_\alpha\cup T_\beta.\end{aligned}$$

These recursion relations, restricted to the small phase space, are easily seen to be equivalent to (3.9). This proves (1), (4) and the convergence of $\Theta(z,t)$ for $|z|$ small enough, because of the regular feature of the singularity $z=0$. The proof of (2) can be found in [**CK99**]. Condition (3) follows from WDVV and string equation, as shown in [**Giv98a**]. □

3.3.1. Characterization in the Fano case. In the case of Fano manifolds, we have the following analytic characterization of the fundamental solution Z_{top}. Furthermore, because of Proposition 2.1, we obtain another proof of (3) in the previous Proposition.

PROPOSITION 3.2. *If X is a Fano manifold, among all fundamental matrix solutions of the system (3.7) for deformed flat coordinates,*[1] *there exists a unique solution such that, on the small quantum locus (i.e. $t^i = 0$ for $i = 0, r+1, \dots, N$) the function $z^{-\mu}H(z,t)z^{\mu}$ is holomorphic at $z = 0$, with series expansion*

$$z^{-\mu}H(z,t)z^{\mu} = e^{t\cup} + zK_1(t) + z^2K_2(t) + \dots, \qquad t^i = 0 \text{ for } i = 0, r+1, \dots, N,$$

This solution coincides with the solution $\left((Z_{\text{top}})^{\alpha}_{\beta}(z,t)\right)_{\alpha,\beta}$.

PROOF. We already know from Proposition 2.1 that such a solution is unique. Let us now prove the main statement. In what follows, we will denote the degree $\deg T_\alpha$ just by $|\alpha|$ for brevity. By point (2) of Proposition 3.1, we have that

$$z^{-\mu}\left(\Theta_{(z^{\mu}\phi)}\right) = z^{-\mu}\left(e^{z\delta} \cup z^{\mu}\phi + \sum_{\beta\neq 0}\sum_{\alpha=0}^{N} e^{\int_\beta \delta}\left\langle \frac{ze^{z\delta}\cup z^{\mu}\phi}{1-z\psi}, T_\alpha\right\rangle^X_{0,2,\beta} T^{\alpha}\right),$$

with $\delta := \sum_{i=1}^r t^i T_i \in H^2(X;\mathbb{C})$. Specialising to $\phi = T_\sigma$, we have

$$z^{-\mu}\left(\Theta_{(z^{\mu}T_\sigma)}\right) = e^{\delta}\cup T_\sigma + \sum_{\beta\neq 0}\sum_{\alpha,\lambda=0}^{N}\sum_{n,k=0}^{\infty}\frac{e^{\int_\beta \delta}}{k!} z^{n+1+k+\mu_\sigma-\mu_\lambda}\langle \tau_n(\delta^{\cup k}\cup T_\sigma), T_\alpha\rangle^X_{0,2,\beta}\eta^{\alpha\lambda}T_\lambda.$$

In the second addend, we have non-zero terms only if

- $|\alpha| + |\lambda| = 2\dim_{\mathbb{C}} X$,
- $2n + 2k + |\sigma| + |\alpha| = \text{vir}\dim_{\mathbb{R}} X_{0,2,\beta}$.

By putting together these conditions, we obtain

$$n + 1 + k + \frac{1}{2}(|\sigma| - |\lambda|) = -\int_\beta \omega_X.$$

The assumption of being Fano is equivalent to the requirement that the functional $\beta \mapsto -\int_\beta \omega_X$ is positive on the closure of the effective cone. This proves the Proposition, the l.h.s. being exactly the exponents of z which appear in the series expansion above. ∎

[1] Throughout Chapter 2, $Y(z,t) = H(z,t)z^{\mu}z^R$ has been denoted $Y(z,t) = \Phi(z,t)z^{\mu}z^R$.

CHAPTER 4

Abelian-Nonabelian Correspondence and Coalescence Phenomenon of $QH^\bullet(\mathbb{G}(r,k))$

Abstract. In this Chapter, as an example of Abelian-Nonabelian Correspondence, both classical and quantum cohomologies of the complex Grassmannians $\mathbb{G}(r,k)$ are studied using an identification with r-exterior powers of classical/quantum cohomologies of Projective Spaces $\mathbb{P}^{k-1}_{\mathbb{C}}$. After obtaining an explicit description of the spectrum of the quantum multiplication by the Euler vector field at points of the small quantum cohomology of $\mathbb{G}(r,k)$, a phenomenon of coalescence of canonical coordinates is studied. Recasting the problem in terms of sums of roots of unity, a complete characterization of *coalescing* Grassmannians is obtained. It is shown that surprisingly the frequency of this coalescence phenomenon is strictly subordinate and highly influenced by the distribution of prime numbers. Two equivalent formulations of the Riemann Hypothesis are given in terms of numbers of complex Grassmannians without coalescence: the former as a constraint on the disposition of singularities of the analytic continuation of the Dirichlet series associated to the sequence counting non-coalescing Grassmannians, the latter as asymptotic estimate (whose error term cannot be improved) for their distribution function.

4.1. Notations

In what follows

- r,k will be natural numbers such that $1 \le r < k$.
- We will denote by $\mathbb{P}$ the complex projective space $\mathbb{P}^{k-1}_{\mathbb{C}}$;
- $\mathbb{G}$ will be the complex Grassmannian $\mathbb{G}(r,k)$ of r-planes in $\mathbb{C}^k$;
- Π will denote the cartesian product

$$\underbrace{\mathbb{P}\times\cdots\times\mathbb{P}}_{r \text{ times}}.$$

- $\sigma \in H^2(\mathbb{P};\mathbb{C})$ will be the generator of the cohomology of $\mathbb{P}$, normalized so that

$$\int_{\mathbb{P}} \sigma^{k-1} = 1.$$

We will denote the power σ^h, with $h \in \mathbb{N}$, by σ_h.

Moreover, in this Chapter, we will also use the following notations for number theoretical functions:

- $P_1(n) := \min\{p \in \mathbb{N} \colon p \text{ is prime and } p|n\},\ n \ge 2;$
- for real positive x,y we define

$$\Phi(x,y) := \operatorname{card}\left(\{n \le x \colon n \ge 2,\ P_1(n) > y\}\right);$$

- $\pi_\alpha(n) := \sum_{\substack{p \text{ prime} \\ p\le n}} p^\alpha, \quad \alpha \ge 0;$
- $\zeta(s)$ is the Riemann ζ-function;

- $\zeta(s,k)$ will denote the truncated Euler product

$$\zeta(s,k) := \prod_{\substack{p \text{ prime} \\ p\leq k}} \left(1-\frac{1}{p^s}\right)^{-1}, \quad k\in\mathbb{R}_{>0},\ s\in\mathbb{C}\setminus\{0\}.$$

- $\zeta_P(s)$ is the Riemann prime ζ-function, defined on the half-plane $\mathrm{Re}(s)>1$ by the series

$$\zeta_P(s) := \sum_{p \text{ prime}} \frac{1}{p^s};$$

- $\zeta_{P,k}(s)$ will denote the partial sums

$$\zeta_{P,k}(s) := \sum_{\substack{p \text{ prime} \\ p\leq k}} \frac{1}{p^s};$$

- $\omega\colon \mathbb{R}_{\geq 1}\to\mathbb{R}$ is the Buchstab function ([**Buc37**]), i.e. the unique continuous solution of the delay differential equation

$$\frac{d}{du}(u\omega(u)) = \omega(u-1), \quad u\geq 2,$$

  with the initial condition

$$\omega(u) = \frac{1}{u}, \quad \text{for } 1\leq u\leq 2.$$

If $f,g\colon\mathbb{R}_+\to\mathbb{R}$, with g definitely strictly positive, we will write

- $f(x)=\Omega_+(g(x))$ to denote

$$\limsup_{x\to\infty}\frac{f(x)}{g(x)}>0;$$

- $f(x)=\Omega_-(g(x))$ to denote

$$\liminf_{x\to\infty}\frac{f(x)}{g(x)}<0;$$

- $f(x)=\Omega_\pm(g(x))$ if both $f(x)=\Omega_+(g(x))$ and $f(x)=\Omega_-(g(x))$ hold.

4.2. Quantum Satake Principle

The quantum cohomology of Grassmannians has been one of the first cases that both physicists [**Wit95**] and mathematicians (see e.g. [**Ber96**], [**Ber97**], [**Buc03**]) studied in details. In this section we expose an identification, valid both in the classical ([**Mar00**]) and in the quantum setup ([**GM**], [**BCFK05**], [**GGI16**]), of the cohomology of Grassmannians with an alternate product of the cohomology of Projective Spaces. This identification has been well known to physicists for long time: e.g. the reader can find an analogue description of the supersymmetric σ-model of $\mathbb{G}(r,k)$ in Section 8.3 and Appendix A of the paper [**CV93**], on the classification of $N=2$ Supersymmetric Field Theories. In the context of the theory of Frobenius manifolds, such an identification has been generalized and axiomatized in [**KS08**] in the notion of *alternate product of Frobenius manifolds*.

4.2.1. Results on classical cohomology of Grassmannians. A classical reference for cohomology of Grassmannians is [GH78]. Let us introduce the following notations, used only in this section, to denote the (products of) complex flag manifolds

$$\mathbb{P} := \mathbb{P}^{k-1}_{\mathbb{C}}, \quad \Pi := \underbrace{\mathbb{P} \times \cdots \times \mathbb{P}}_{r \text{ times}}$$

$$\mathbb{G} := \mathbb{G}(r,k), \quad \mathbb{F} := \mathrm{Fl}(1,2,\dots,r,k),$$

with $k \geq 2$, and $1 \leq r < k$.

The complex Grassmannian $\mathbb{G}$ can be seen as a symplectic quotient. Let us consider the complex vector space $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k)$ endowed with its standard symplectic structure: if we introduce on $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k)$ coordinates $a_{ij} = x_{ij} + \sqrt{-1}y_{ij}$, for $1 \leq i \leq k$ and $1 \leq j \leq r$, then the standard symplectic structure is

$$\omega := \sum_{i,j} dx_{ij} \wedge dy_{ij}.$$

Let us consider the action of $U(r)$ on $\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k)$ defined by $g \cdot A := A \circ g^{-1}$: this action is hamiltonian and a moment map $\mu_{U(r)} \colon \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) \to \mathfrak{u}(r)$ is given by

$$\mu_{U(r)}(A) := A^\dagger A - \mathbb{1}.$$

Since the subset $\mu^{-1}_{U(r)}(0)$ is the set of unitary r-frames in $\mathbb{C}^k$, we have clearly the identification

$$\mathbb{G} \cong \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) /\!\!/ U(r) := \mu^{-1}_{U(r)}(0)/U(r).$$

If $\mathbb{T} \subseteq U(r)$ is the subgroup of diagonal matrices, then $\mathbb{T} \cong U(1)^{\times r}$ is a maximal torus. Denoting by $\mu_{\mathbb{T}} \colon \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) \to \mathfrak{u}(1)^{\times r}$ the composition of $\mu_{U(r)}$ and the canonical projection $\mathfrak{u}(r) \to \mathfrak{u}(1)^{\times r}$, we have that $\mu^{-1}_{\mathbb{T}}(0)$ is the set of matrices $A \in M_{k,r}(\mathbb{C})$ whose columns have unit length. Hence, we have

$$\Pi \cong \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) /\!\!/ \mathbb{T} := \mu^{-1}_{\mathbb{T}}(0)/\mathbb{T}.$$

Moreover, the quotient

$$\mu^{-1}_{U(r)}(0)/\mathbb{T}$$

can be identified with the flag manifold $\mathbb{F} := \mathrm{Fl}(1,2,\dots,r,k)$ (for the identification we have to choose an hermitian metric on $\mathbb{C}^k$, e.g. the standard one, compatible with the standard symplectic structure). Because of the inclusion $\mu^{-1}_{U(r)}(0) \subseteq \mu^{-1}_{\mathbb{T}}(0)$, we have the following quotient diagram:

$$\begin{tikzcd} & \mathbb{F} \arrow[dl, "p"'] \arrow[dr, hook, "\iota"] & \\ \mathbb{G} & & \Pi \end{tikzcd}$$

where p is the canonical projecton, and ι the inclusion. Note that in this way there is also a natural rational map «*taking the span*»

$$\Pi \dashrightarrow \mathbb{G} \colon \ (\ell_1, \dots, \ell_r) \mapsto \mathrm{span}\langle \ell_1, \dots, \ell_r \rangle \ ,$$

whose domain is the image of ι. On the manifold Π we have r canonical line bundles, denoted $\mathfrak{L}_j$ for $j = 1, \dots, r$, defined as the pull-back of the bundle $\mathcal{O}(1)$ on the j-th factor $\mathbb{P}$. If we denote $\mathfrak{V}_1 \subseteq \mathfrak{V}_2 \subseteq \cdots \subseteq \mathfrak{V}_r$ the tautological bundles over $\mathbb{F}$, we have that

$$\iota^*\mathfrak{L}_j \cong (\mathfrak{V}_j/\mathfrak{V}_{j-1})^\vee.$$

Denoting with the same symbol x_i the Chern class $c_1(\mathfrak{L}_i)$ on Π and its pull-back $c_1(\iota^*\mathfrak{L}_i) = \iota^* c_1(\mathfrak{L}_i)$ on $\mathbb{F}$, we have

$$H^\bullet(\Pi;\mathbb{C}) \cong H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r} \cong \frac{\mathbb{C}[x_1,\dots,x_r]}{\langle x_1^k,\dots x_r^k\rangle} \quad \text{(by Künneth Theorem)},$$

$$H^\bullet(\mathbb{F};\mathbb{C}) \cong \frac{\mathbb{C}[x_1,\dots,x_r]}{\langle h_{k-r+1},\dots,h_k\rangle},$$

where h_j stands for the j-th complete symmetric polynomial in $x_1,\dots,x_r$. Since the classes $x_1,\dots,x_r$ are the Chern roots of the dual of the tautological bundle $\mathfrak{V}_r$, we also have

$$H^\bullet(\mathbb{G};\mathbb{C}) \cong \frac{\mathbb{C}[e_1,\dots,e_r]}{\langle h_{n-k+1},\dots,h_n\rangle} \cong \frac{\mathbb{C}[x_1,\dots,x_k]^{\mathfrak{S}_k}}{\langle h_{n-k+1},\dots,h_n\rangle},$$

where the e_j's are the elementary symmetric polynomials in $x_1,\dots,x_r$. This is the classical representation of the cohomology ring of the Grassmannian $\mathbb{G}$ with generators the Chern classes of the dual of the tautological vector bundle $\mathcal{S}$, and relations generated by the Segre classes of $\mathcal{S}$.
From this ring representation, it is clear that any cohomology class of $\mathbb{G}$ can be *lifted* to a cohomology class of Π: we will say that $\tilde{\gamma} \in H^\bullet(\Pi;\mathbb{C})$ is the lift of $\gamma \in H^\bullet(\mathbb{G};\mathbb{C})$ if $p^*\gamma = \iota^*\tilde{\gamma}$. The following integration formula allow us to express the cohomology pairings on $H^\bullet(\mathbb{G};\mathbb{C})$ in terms of the cohomology pairings on $H^\bullet(\Pi;\mathbb{C})$.

Theorem 4.1 ([Mar00]). *If $\gamma \in H^\bullet(\mathbb{G};\mathbb{C})$ admits the lift $\tilde{\gamma} \in H^\bullet(\Pi;\mathbb{C})$, then*

$$\int_{\mathbb{G}} \gamma = \frac{(-1)^{\binom{r}{2}}}{r!} \int_\Pi \tilde{\gamma} \cup_\Pi \Delta^2, \tag{4.1}$$

where

$$\Delta := \prod_{1\le i<j\le r} (x_i - x_j).$$

Corollary 4.1 ([ES89]). *The linear morphism*

$$\vartheta \colon H^\bullet(\mathbb{G};\mathbb{C}) \to H^\bullet(\Pi;\mathbb{C}) \colon \gamma \mapsto \tilde{\gamma} \cup_\Pi \Delta$$

is injective, and its image is the subspace of anty-simmetric part of $H^\bullet(\Pi,\mathbb{C})$ w.r.t. the $\mathfrak{S}_r$-action. Moreover

$$\vartheta(\alpha \cup_{\mathbb{G}} \beta) = \vartheta(\alpha) \cup_\Pi \tilde{\beta} = \tilde{\alpha} \cup_\Pi \vartheta(\beta).$$

Proof. If $\vartheta(\gamma) = 0$, then

$$\int_{\mathbb{G}} \gamma \cup \gamma' = \frac{(-1)^{\binom{r}{2}}}{r!} \int_\Pi (\tilde{\gamma} \cup \Delta) \cup (\tilde{\gamma}' \cup \Delta) = 0$$

for all $\gamma' \in H^\bullet(\mathbb{G};\mathbb{C})$. Then $\gamma = 0$. Being clear that $\vartheta(\gamma)$ is anti-symmetric, observe that any anti-symmetric class is of the form $\tilde{\gamma} \cup \Delta$ with $\tilde{\gamma}$ symmetric in $x_1,\dots,x_r$. The last statement follows from the fact that the lift of a cup product is the cup product of the lifts. $\square$

We can identify the anti-symmetric part of $H^\bullet(\Pi;\mathbb{C}) \cong H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}$ with $\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$, using the identifications i, j illustrated in the following diagram

$$\begin{tikzcd} H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r} \arrow[r, "\pi"] & \bigwedge^r H^\bullet(\mathbb{P};\mathbb{C}) \arrow[dl, "i"'] \\ {[H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}]^{\mathrm{ant}}} \arrow[u, hook] \arrow[ur, bend right, "j"'] & \end{tikzcd}$$

where

$$\pi\colon H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r} \to \bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})\colon \alpha_1\otimes\dots\otimes\alpha_r \mapsto \alpha_1\wedge\dots\wedge\alpha_r,$$

$$i\colon \bigwedge^r H^\bullet(\mathbb{P};\mathbb{C}) \to [H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}]^{\mathrm{ant}}\colon \alpha_1\wedge\dots\wedge\alpha_r \mapsto \sum_{\rho\in\mathfrak{S}_r}\varepsilon(\rho)\alpha_{\rho(1)}\otimes\dots\otimes\alpha_{\rho(r)},$$

together with its inverse

$$j\colon [H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}]^{\mathrm{ant}} \to \bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})\colon \alpha_1\otimes\dots\otimes\alpha_r \mapsto \frac{1}{r!}\alpha_1\wedge\dots\wedge\alpha_r.$$

The Poincaré pairing $g^{\mathbb{P}}$ on $H^\bullet(\mathbb{P};\mathbb{C})$ induces a metric $g^{\otimes\mathbb{P}}$ on $H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}$ and a metric $g^{\wedge\mathbb{P}}$ on $\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$ given by

$$g^{\otimes\mathbb{P}}(\alpha_1\otimes\dots\otimes\alpha_r, \beta_1\otimes\dots\otimes\beta_r) := \prod_{i=1}^r g^{\mathbb{P}}(\alpha_i,\beta_i),$$

$$g^{\wedge\mathbb{P}}(\alpha_1\wedge\dots\wedge\alpha_r, \beta_1\wedge\dots\wedge\beta_r) := \det\left(g^{\mathbb{P}}(\alpha_i,\beta_j)\right)_{1\le i,j\le r}.$$

Using the identifications above, when $g^{\otimes\mathbb{P}}$ is restricted on the subspace $[H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}]^{\mathrm{ant}}$ it coincides with $r!g^{\wedge\mathbb{P}}$ on $\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$. From the integration formula (4.1), we deduce the following result.

Corollary 4.2. *The isomorphism*

$$j\circ\vartheta\colon \left(H^\bullet(\mathbb{G};\mathbb{C}), g^{\mathbb{G}}\right) \to \left(\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C}), (-1)^{\binom{r}{2}}g^{\wedge\mathbb{P}}\right)$$

is an isometry.

An additive basis of $H^\bullet(\mathbb{G};\mathbb{C})$ is given by the Schubert classes (Poincaré-dual to the Schubert cycles), given in terms of $x_1,\dots,x_r$ by the Schur polynomials

$$\sigma_\lambda := \frac{\det\begin{pmatrix} x_1^{\lambda_1+r-1} & x_1^{\lambda_2+r-2} & \dots & x_1^{\lambda_r} \\ x_2^{\lambda_1+r-1} & x_2^{\lambda_2+r-2} & \dots & x_2^{\lambda_r} \\ & & \vdots & \\ x_r^{\lambda_1+r-1} & x_r^{\lambda_2+r-2} & \dots & x_r^{\lambda_r}\end{pmatrix}}{\det\begin{pmatrix} x_1^{r-1} & x_1^{r-2} & \dots & 1 \\ x_2^{r-1} & x_2^{r-2} & \dots & 1 \\ & & \vdots & \\ x_r^{r-1} & x_r^{r-2} & \dots & 1\end{pmatrix}}$$

where λ is a partition whose corresponding Young diagram is contained in in a $r\times(k-r)$ rectangle. The lift of each Schubert class to $H^\bullet(\Pi;\mathbb{C})$ is the Schur polynomial in $x_1,\dots,x_r$ (indeed each x_i in the Schur polynomial has exponent at most $k-r<k$). Thus, under the identification above, the class $j\circ\vartheta(\sigma_\lambda)$ is $\sigma_{\lambda_1+r-1}\wedge\cdots\wedge\sigma_{\lambda_r}\in\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$, σ being the generator of $H^2(\mathbb{P};\mathbb{C})$.

Using the Künneth isomorphism $H^\bullet(\Pi;\mathbb{C})\cong H^\bullet(\mathbb{P};\mathbb{C})^{\otimes r}$, the cup product $\cup_\Pi$ is expressed in terms of $\cup_\mathbb{P}$ as follows:

$$\left(\sum_i \alpha_1^i\otimes\cdots\otimes\alpha_r^i\right)\cup_\Pi\left(\sum_j \beta_1^j\otimes\cdots\otimes\beta_r^j\right)=\sum_{i,j}(\alpha_1^i\cup_\mathbb{P}\beta_1^j)\otimes\cdots\otimes(\alpha_r^i\cup_\mathbb{P}\beta_r^j).$$

If $\gamma\in H^\bullet(\Pi;\mathbb{C})^{\mathfrak{S}_r}$, then $\gamma\cup_\Pi(-)\colon H^\bullet(\Pi;\mathbb{C})\to H^\bullet(\Pi;\mathbb{C})$ leaves invariant the subspace of antysymmetric classes. Thus, $\gamma\cup_\Pi(-)$ induces an endomorphism $A_\gamma\in\operatorname{End}\left(\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})\right)$ that acts on decomposable elements $\alpha=\alpha_1\wedge\cdots\wedge\alpha_r$ as follows

$$A_\gamma(\alpha)=j(\gamma\cup_\Pi i(\alpha))=\frac{1}{r!}\sum_{i,\rho}\varepsilon(\rho)(\gamma_1^i\cup_\mathbb{P}\alpha_{\rho(1)})\wedge\cdots\wedge(\gamma_r^i\cup_\mathbb{P}\alpha_{\rho(r)}),\tag{4.2}$$

where $\gamma_j^i\in H^\bullet(\mathbb{P};\mathbb{C})$ are such that

$$\gamma=\sum_i\gamma_1^i\otimes\cdots\otimes\gamma_r^i.$$

As an example, in the following Proposition we reformulate in $\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$ the classical Pieri formula, expressing the multiplication by a special Schubert class σ_ℓ in $H^\bullet(\mathbb{G};\mathbb{C})$

$$\sigma_\ell\cup_\mathbb{G}\sigma_\mu=\sum_\nu\sigma_\nu,$$

where the sum is on all partitions ν which belong to the set $\mu\otimes\ell$ (the set of partitions obtained by adding ℓ boxes to μ, at most one per column) and which are contained in the rectangle $r\times(k-r)$, in terms of the multiplication by $\sigma_\ell=(\sigma)^\ell\in H^\bullet(\mathbb{P};\mathbb{C})$. We also make explicit the operation of multiplication by the classes $p_\ell\in H^\bullet(\mathbb{G};\mathbb{C})$ defined in terms of the special Schubert classes by

$$p_\ell:=-\left(\sum_{\substack{n_1+2n_2+\cdots+rn_r=\ell\\ n_1,\dots,n_r\geq 0}}\frac{\ell(n_1+\cdots+n_r-1)!}{n_1!\dots n_r!}\prod_{i=1}^r(-\sigma_i)^{n_i}\right),\qquad \ell=0,\dots,k-1,$$

because of the nice form of their lifts $\tilde{p}_\ell\in H^\bullet(\Pi;\mathbb{C})$.

Proposition 4.1. *If $\sigma_\mu\in H^\bullet(\mathbb{G};\mathbb{C})$ is a Schubert class, then*

- *the product $\sigma_\ell\cup_\mathbb{G}\sigma_\mu$ with a special Schubert class σ_ℓ is given by*

$$j\circ\vartheta(\sigma_\ell\cup_\mathbb{G}\sigma_\mu)=\frac{1}{r!}\left(\sum_{\substack{i_1+\cdots+i_r=\ell\\ i_1,\dots,i_r\geq 0}}\sum_{\rho\in\mathfrak{S}_r}\bigwedge_{h=1}^r\sigma_{i_{\rho(h)}}\cup_\mathbb{P}\sigma_{\mu_h+r-h}\right);$$

- *the product $p_\ell\cup_\mathbb{G}\sigma_\mu$ is given by*

$$j\circ\vartheta(p_\ell\cup_\mathbb{G}\sigma_\mu)=\sum_{i=1}^r\sigma_{\mu_1+r-1}\wedge\cdots\wedge(\sigma_{\mu_i+r-i}\cup_\mathbb{P}\sigma_\ell)\wedge\cdots\wedge\sigma_{\mu_r}.$$

Proof. From Corollary (4.1) we have

$$\vartheta(\sigma_\ell\cup_\mathbb{G}\sigma_\mu)=\tilde{\sigma}_\ell\cup_\Pi\vartheta(\sigma_\mu)$$

If $\gamma = \tilde{\sigma}_\ell$ is the lift of the special Schubert class $\sigma_\ell \in H^\bullet(\mathbb{G};\mathbb{C})$, then

$$\tilde{\sigma}_\ell = h_\ell(x_1,\dots,x_r) = \sum_{\substack{i_1+\dots+i_r=\ell\\ i_1,\dots,i_r\geq 0}} \sigma_{i_1}\otimes\dots\otimes\sigma_{i_r},$$

and using (4.2) we easily conclude. Analogously, we have that

$$\tilde{p}_\ell = \sum_{i=1}^r x_i^\ell = \sum_{i=1}^r 1\otimes\dots\otimes \underset{i\text{-th}}{\sigma_\ell}\otimes\dots\otimes 1,$$

and

$$A_{\tilde{p}_\ell}(\alpha) = \sum_{i=1}^r \alpha_1\wedge\dots\wedge(\sigma_\ell\cup_{\mathbb{P}}\alpha_i)\wedge\dots\wedge\alpha_r.$$

□

Corollary 4.3. *For any $z\in\mathbb{C}^*$, any $t^2\sigma_1\in H^2(\mathbb{G};\mathbb{C})$, and any Schubert class $\sigma_\lambda\in H^\bullet(\mathbb{G};\mathbb{C})$, the following identity holds:*

$$j\circ\vartheta(z^{t^2\sigma_1}\cup\sigma_\lambda) = \bigwedge_{h=1}^r z^{t^2\sigma}\cup\sigma_{\lambda_h+r-h}.$$

Proof. We have that

$$\begin{aligned}
\bigwedge_{h=1}^r z^{t^2\sigma}\cup\sigma_{\lambda_h+r-h} &= \bigwedge_{h=1}^r\sum_{k_h=0}^\infty \frac{(\log z)^{k_h}}{k_h!}(t^2\sigma)^{k_h}\cup\sigma_{\lambda_h+r-h}\\
&= \sum_{k_1=0}^\infty\cdots\sum_{k_r=0}^\infty \frac{(\log z)^{k_1+\dots+k_r}}{k_1!\dots k_r!}\bigwedge_{h=1}^r (t^2\sigma)^{k_h}\cup\sigma_{\lambda_h+r-h}\\
&= \sum_{k=0}^\infty\frac{(\log z)^k}{k!}\sum_{k_1+\dots+k_r=k}\binom{k}{k_1\dots k_r}\bigwedge_{h=1}^r (t^2\sigma)^{k_h}\cup\sigma_{\lambda_h+r-h}\\
&= j\circ\vartheta\left(\sum_{k=0}^\infty\frac{(\log z)^k}{k!}((t^2\sigma_1)^k\cup\sigma_\lambda)\right)\\
&= j\circ\vartheta(z^{t^2\sigma_1}\cup\sigma_\lambda).
\end{aligned}$$

□

Proposition 4.2. *If $\mu^{\mathbb{P}}\in\mathrm{End}(H^\bullet(\mathbb{P};\mathbb{C}))$ and $\mu^{\mathbb{G}}$ denotes the grading operator for the Projective Space and the Grassmannian respectively, defined as in Section 2.1, then for all Schubert classes $\sigma_\lambda\in H^\bullet(\mathbb{G};\mathbb{C})$ the following identities hold:*

$$j\circ\vartheta(\mu^{\mathbb{G}}\sigma_\lambda) = \sum_{h=1}^r \sigma_{\lambda_1+r-1}\wedge\dots\wedge\mu^{\mathbb{P}}\sigma_{\lambda_h+r-h}\wedge\dots\wedge\sigma_{\lambda_r},$$

$$j\circ\vartheta(z^{\mu^{\mathbb{G}}}\sigma_\lambda) = \bigwedge_{h=1}^r z^{\mu^{\mathbb{P}}}\sigma_{\lambda_h+r-h},\quad z\in\mathbb{C}^*.$$

Proof. For the first identity notice that

$$\begin{aligned}
&(j\circ\vartheta)^{-1}\left(\sum_{h=1}^{r}\sigma_{\lambda_1+r-1}\wedge\cdots\wedge\left(\lambda_h+r-h-\frac{k-1}{2}\right)\sigma_{\lambda_h+r-h}\wedge\cdots\wedge\sigma_{\lambda_r}\right)\\
&\qquad= \sigma_\lambda\cdot\left(\left(\sum_{h=1}^{r}\lambda_h\right)+r^2-\frac{r(r+1)}{2}-\frac{(k-1)r}{2}\right)\\
&\qquad= \sigma_\lambda\cdot\left(\left(\sum_{h=1}^{r}\lambda_h\right)-\frac{r(k-r)}{2}\right)\\
&\qquad= \mu^{\mathbb{G}}(\sigma_\lambda).
\end{aligned}$$

For the second identity, we have that

$$\begin{aligned}
\bigwedge_{h=1}^{r} z^{\mu^{\mathbb{P}}}\sigma_{\lambda_h+r-h} &= \bigwedge_{h=1}^{r} z^{\mu^{\mathbb{P}}_{\lambda_h+r-h}}\cdot\sigma_{\lambda_h+r-h}\\
&= \exp\left(\log(z)\cdot\sum_{h=1}^{r}\mu^{\mathbb{P}}_{\lambda_h+r-h}\right)\cdot\bigwedge_{h=1}^{r}\sigma_{\lambda_h+r-h}\\
&= j\circ\vartheta\left(z^{\mu^{\mathbb{G}}}\sigma_\lambda\right).
\end{aligned}$$

□

4.2.2. Quantum Cohomology of $\mathbb{G}(r,k)$. The identification in the classical cohomology setting of $H^\bullet(\mathbb{G};\mathbb{C})$ with the wedge product $\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})$, exposed in the previous section, has been extended also to the quantum case in [BCFK05], [BCFK08], [CFKS08], and [KS08].

The following isomorphism of the (small) quantum cohomology algebra of Grassmannians at a point $t\sigma_1=\log q\in H^2(\mathbb{G};\mathbb{C})$ is well-known

$$QH^\bullet_q(\mathbb{G})\cong\frac{\mathbb{C}[x_1,\dots,x_r]^{\mathfrak{S}_r}[q]}{\langle h_{k-r+1},\dots,h_k-(-1)^{r-1}q\rangle},$$

while for the (small) quantum cohomology algebra of Π, being equal to the r-fold tensor product of the quantum cohomology algebra of $\mathbb{P}$, we have

$$QH^\bullet_{q_1,\dots,q_r}(\Pi)\cong\frac{\mathbb{C}[x_1,\dots,x_r][q_1,\dots,q_r]}{\langle x_1^k-q_1,\dots,x_r^k-q_r\rangle}.$$

Following [BCFK05], and interpreting now the parameters q's just as formal parameters, if we denote by $\overline{QH}^\bullet_q(\Pi)$ the quotient of $QH^\bullet_{q_1,\dots,q_r}(\Pi)$ obtained by substituing $q_i=(-1)^{r-1}q$, and denoting the canonical projection by

$$[-]_q\colon QH^\bullet_{q_1,\dots,q_r}(\Pi)\to\overline{QH}^\bullet_q(\Pi),$$

we can extend by linearity the morphisms ϑ,j of the previous section to morphisms

$$\overline{\vartheta}\colon QH^\bullet_q(\mathbb{G})\to\overline{QH}^\bullet_q(\Pi),$$

$$\overline{j}\colon\left[\overline{QH}^\bullet_q(\Pi)\right]^{\mathrm{ant}}\to\left(\bigwedge^r H^\bullet(\mathbb{P};\mathbb{C})\right)\otimes_{\mathbb{C}}C[q].$$

Notice that the image under $\overline{\vartheta}$ of any Schubert class σ_λ is equal to the classical product $\tilde{\sigma}_\lambda\cup_\Pi\Delta$, the exponents of x_i's in the product $\sigma_\lambda(x)\prod_{i<j}(x_i-x_j)$ being less than k; as a consequence, the image of

$\overline{\vartheta}$ is equal to the anti-symmetric part w.r.t. the natural $\mathfrak{S}_r$ action (permuting the x_i's)

$$\left[\overline{QH}^{\bullet}_{q}(\Pi)\right]^{\rm ant} \cong [H^\bullet(\Pi;\mathbb{C})]^{\rm ant}\otimes_{\mathbb{C}}\mathbb{C}[q].$$

The following result, is a quantum generalization of Corollary 4.1.

THEOREM 4.2 ([**BCFK05**]). *For any Schubert classes $\sigma_\lambda,\sigma_\mu\in H^\bullet(\mathbb{G};\mathbb{C})$ we have*

$$\overline{\vartheta}(\sigma_\lambda *_{\mathbb{G},q}\sigma_\mu)=\left[\vartheta(\sigma_\mu)*_{\Pi,q_1,\dots,q_r}\tilde{\sigma}_\lambda\right]_q.$$

PROOF. The essence of the result is the following identity between 3-point Gromov-Witten invariants of genus 0 of the grassmannian $\mathbb{G}$ and Π:

$$\langle\sigma_\mu,\sigma_\nu,\sigma_\rho\rangle^{\mathbb{G}}_{0,3,d\sigma_1^\vee}=\frac{(-1)^{\binom{r}{2}}}{r!}\sum_{d_1+\dots+d_r=d}(-1)^{d(r-1)}\langle\sigma_\mu\Delta,\sigma_\nu,\sigma_\rho\Delta\rangle^{\Pi}_{0,3,d_1(\sigma_1^{(1)})^\vee+\dots+d_r(\sigma_1^{(r)})^\vee},$$

where $d,d_1,\dots,d_r\geq 0$ and $\sigma_1^\vee$ (resp. $(\sigma_1^{(i)})^\vee$) is the Poincaré dual homology class of σ_1 (resp. $\sigma_1^{(i)}$). This is easily proved using the Vafa-Intriligator residue formula (see [**Ber96**]). $\square$

Using the identification $\overline{\jmath}$, we can deduce from the previous result the following generalization to Proposition 4.1.

COROLLARY 4.4. *If $\sigma_\mu\in H^\bullet(\mathbb{G};\mathbb{C})$ is a Schubert class, then*

$$\overline{\jmath}\circ\overline{\vartheta}(\sigma_\mu *_{\mathbb{G},q} p_\ell)=\sum_{i=1}^{r}\sigma_{\mu_1+r-1}\wedge\dots\wedge\sigma_{\mu_i+r-i}*_{\mathbb{P},(-1)^{r-1}q}\sigma_\ell\wedge\dots\wedge\sigma_{\mu_r}. \tag{4.3}$$

From this identity, it immediately follows that:

(1) *At the point $p=t^2\sigma_1\in H^2(\mathbb{G};\mathbb{C})$ of the small quantum cohomology of $\mathbb{G}$, the eigenvalues of the operator*
$$\mathcal{U}^{\mathbb{G}}_p:=c_1(\mathbb{G})*_q(-)\colon H^\bullet(\mathbb{G};\mathbb{C})\to H^\bullet(\mathbb{G};\mathbb{C})$$
are given by the sums
$$u_{i_1}+\dots+u_{i_r},\quad 1\leq i_1<\dots<i_r\leq k,$$
where $u_1,\dots,u_k$ are the eigenvalues of the corresponding operator $\mathcal{U}^{\mathbb{P}}$ for projective spaces at the point $\hat{p}:=t^2\sigma_1+(r-1)\pi i\sigma_1\in H^2(\mathbb{P};\mathbb{C})$, i.e.
$$\mathcal{U}^{\mathbb{P}}_{\hat{p}}:=c_1(\mathbb{P})*_{(-1)^{r-1}q}(-)\colon H^\bullet(\mathbb{P};\mathbb{C})\to H^\bullet(\mathbb{P};\mathbb{C}).$$

(2) *If $\pi_1,\dots,\pi_n$ denote the idempotents of the small quantum cohomology of the projective space $\mathbb{P}$ at the point $\hat{p}:=t^2\sigma_1+(r-1)\pi i\sigma_1\in H^2(\mathbb{P};\mathbb{C})$, then*
   - *the idempotents of the small quantum cohomology of $\mathbb{G}$ at $p=t^2\sigma_1\in H^2(\mathbb{G};:\mathbb{C})$ are*
   $$(\overline{\jmath}\circ\overline{\vartheta})^{-1}(\varkappa_I\cdot\pi_I),\quad \pi_I:=\pi_{i_1}\wedge\dots\wedge\pi_{i_r},$$
   with $1\leq i_1<\dots<i_r\leq k$, and where
   $$\varkappa_I:=\frac{1}{g^{\wedge\mathbb{P}}(\pi_I,\pi_I)}\det\begin{pmatrix}g^{\mathbb{P}}(\pi_{i_1},\sigma_{r-1}) & \dots & g^{\mathbb{P}}(\pi_{i_1},\sigma_0)\\ \vdots & \ddots & \vdots\\ g^{\mathbb{P}}(\pi_{i_r},\sigma_{r-1}) & \dots & g^{\mathbb{P}}(\pi_{i_r},\sigma_0)\end{pmatrix};$$

- *the normalized idempotents are given by*

$$(\bar{j}\circ\bar{\vartheta})^{-1}\left(i^{\binom{r}{2}}\cdot f_I\right),\quad f_I:=\frac{\pi_I}{g^{\wedge\mathbb{P}}(\pi_I,\pi_I)^{\frac{1}{2}}}.$$

Proof. Equation (4.3) is an immediate consequence of Theorem 4.2. Notice that, for $\ell=1$, the equation (4.3) can be rewritten in the form

$$\begin{aligned}\bar{j}\circ\bar{\vartheta}(\sigma_\mu *_{\mathbb{G},q}\sigma_1)=&\sum_{i=1}^r\sigma_{\mu_1+r-1}\wedge\dots\wedge\underset{i\text{-th}}{\sigma_{\mu_i+r-i+1}}\wedge\dots\wedge\sigma_{\mu_r}\\ &+q(-1)^{r-1}\delta_{n-1,\mu_1+r-1}\sigma_0\wedge\sigma_{\mu_2+r-2}\wedge\dots\wedge\sigma_{\mu_r}.\end{aligned}$$

The first term coincides with the classical one, whereas the second term is the quantum correction dictated by the Quantum Pieri Formula (see [Ber97]). From this equality and from the value of the first Chern class $c_1(\mathbb{G})=k\sigma_1$, one obtains point (1). The semisimplicity of the small quantum cohomology of the Grassmannian $\mathbb{G}$ is well known (see [Abr00], [CMP10]), so that the existence of the idempotent vectors is guaranteed. By Theorem 4.2 we deduce that the image of the idempotents $\alpha_1,\dots,\alpha_{\binom{k}{r}}$ of the small quantum cohomology of $\mathbb{G}$ under the map $\bar{j}\circ\bar{\vartheta}$ are scalar multiples of $\pi_I:=\pi_{i_1}\wedge\dots\wedge\pi_{i_r}$, with $1\le i_1<\dots<i_r\le k$, i.e. are of the form $\varkappa_I\cdot\pi_I$, for some constants $\varkappa_I\in\mathbb{C}^*$. Using Corollary 4.2, from the equality $g^{\mathbb{G}}(\alpha_i,\alpha_i)=g^{\mathbb{G}}(\alpha_i,1)$, we find that necessarily

$$\varkappa_I^2\cdot g^{\wedge\mathbb{P}}(\pi_I,\pi_I)=\varkappa_I\cdot g^{\wedge\mathbb{P}}(\pi_I,\sigma_{r-1}\wedge\dots\wedge\sigma_0),$$

and one concludes. By normalization, one obtains the expression for normalized idempotents. □

4.3. Frequency of Coalescence Phenomenon in $QH^\bullet(\mathbb{G}(r,k))$

Given $1\le r<k$, the canonical coordinates of the quantum cohomology of the projective space $\mathbb{P}_{\mathbb{C}}^{k-1}$ at the point $\hat{p}=t^2\sigma_1+(r-1)\pi i\sigma_1$ in the small locus $H^2(\mathbb{P}_{\mathbb{C}}^{k-1};\mathbb{C})$ are

$$u_h=k\exp\left(\frac{t^2+(r-1)\pi i}{k}\right)\zeta_k^{h-1},\quad \zeta_k:=e^{\frac{2\pi i}{k}},\quad h=1,\dots,k. \tag{4.4}$$

Consequently, by Corollary 4.4, the canonical coordinates of the quantum cohomology of the Grassmannian $\mathbb{G}(r,k)$ at the point $p=t^2\sigma_1$ are given by the sums

$$k\exp\left(\frac{t^2+(r-1)\pi i}{k}\right)\sum_{j=1}^r\zeta_k^{i_j}, \tag{4.5}$$

for all possible combinations $0\le i_1<i_2<\dots<i_r\le k-1$. This means that, although general results guarantees the semisimplicity of the small quantum cohomology of Grassmannians (see Section 3.2.2), it may happens that some Dubrovin canonical coordinates coalesce (i.e. the spectrum of the operator $c_1(\mathbb{G}(r,k))*_p(-)$ is not simple). More precisely, if there is a point $p\in H^2(\mathbb{G}(r,k);\mathbb{C})$ with coalescing canonical coordinates then all points of the small quantum locus have this property. In such a case, we will simply say that the Grassmannian $\mathbb{G}(r,k)$ is *coalescing*. In this and in the next sections, we want to answer to the following

Question 1. *For which r and k the Grassmannian $\mathbb{G}(r,k)$ is coalescing?*

Question 2. *How much frequent is this phenomenon of coalescence among all Grassmannians?*

For the answers we need some preliminary results.

4.3.1. Results on vanishing sums of roots of unity. In this section we collect some useful notions and results concerning the problem of vanishing sums, and more general linear relations among roots of unity. The interested reader can find more details and historical remarks in [Man65], [CJ76], [Len78], [Zan89], [Zan95] and the references therein. Following [Man65] and the survey [Len78], we will say that a relation

$$\sum_{\nu=1}^{r} a_\nu z_\nu = 0, \quad a_\nu \in \mathbb{Q}, \tag{4.6}$$

and z_ν's are roots of unity is *irreducible* if no proper sub-sum vanishes; this means that there is no relation

$$\sum_{\nu=1}^{r} b_\nu z_\nu = 0, \quad \text{with } b_\nu(a_\nu - b_\nu) = 0 \text{ for all } \nu = 1, \dots, r$$

with at least one but all $b_\nu = 0$.

THEOREM 4.3 (H.B. Mann, [Man65]). *Let $z_1, \dots, z_v$ be roots of unity, and $a_1, \dots, a_v \in \mathbb{N}^*$ such that*

$$\sum_{i=1}^{v} a_i z_i = 0.$$

Moreover, suppose that such a vanishing relation is irreducible. *Then, for any $i, j \in \{1, \dots, v\}$ we have*

$$\left(\frac{z_i}{z_j}\right)^m = 1, \quad m := \prod_{\substack{p \text{ prime} \\ p \le v}} p.$$

Let $G = \langle a \rangle$ be a cyclic group of order m, and let ζ_m be a fixed primitive m-th root of unity. There is a well defined natural morphism of ring

$$\phi \colon \mathbb{Z}G \to \mathbb{Z}[\zeta_m] \colon a \mapsto \zeta_m,$$

so that, we have the following identification

$$\ker \phi \equiv \left\{ \begin{matrix} \mathbb{Z}\text{-linear relations among} \\ \text{the } m\text{-th roots of unity} \end{matrix} \right\}.$$

Let us also introduce

- the function $\varepsilon_0 \colon \mathbb{Z}G \to \mathbb{Z}$, defined by
$$\varepsilon_0\left(\sum_{g \in G} x_g g\right) := \operatorname{card}(\{g \colon x_g \neq 0\});$$
- a natural partial ordering on $\mathbb{Z}G$, by declaring that given two sums
$$x = \sum_{g \in G} x_g g, \ y = \sum_{g \in G} y_g g,$$
we have $x \geq y$ if and only if $x_g \geq y_g$ for all $g \in G$.

We define $\mathbb{N}G := \{x \in \mathbb{Z}G \colon x \geq 0\}$.

THEOREM 4.4 (T.Y. Lam, K.H. Leung, [LL00]). *Suppose that G is a cyclic group of order $m = p_1p_2\dots p_v$, with $p_1 < p_2 < \dots < p_v$ primes and $v \geq 2$. Let $\phi\colon \mathbb{Z}G \to \mathbb{Z}[\zeta_m]$ be the natural map, and let $x, y \in \mathbb{N}G$ such that $\phi(x) = \phi(y)$. If $\varepsilon_0(x) \leq p_1 - 1$, then we have*

(A) *either $y \geq x$,*
(B) *or $\varepsilon_0(y) \geq (p_1 - \varepsilon_0(x))(p_2 - 1)$.*

In case (A), we have $\varepsilon_0(y) \geq \varepsilon_0(x)$, and in case (B) we have $\varepsilon_0(y) > \varepsilon_0(x)$.

COROLLARY 4.5. *In the same hypotheses of the previous Theorem, let us suppose that $\varepsilon_0(x) = \varepsilon_0(y)$. Then $x = y$.*

PROOF. We necessarily have case (A), and by symmetry of x and y, we conclude. □

DEFINITION 4.1. Let $k \geq 2$, and $0 \leq r \leq k$. We will say that k is *r-balancing* if there exists a combination of integers $1 \leq i_1 < \dots < i_r \leq k$ such that

$$\zeta_k^{i_1} + \dots + \zeta_k^{i_r} = 0, \quad \zeta_k := e^{\frac{2\pi i}{k}}.$$

In other words, there are r distinct k-roots of unity whose sum is 0.

THEOREM 4.5 (G. Sivek, [Siv10]). *If $k = p_1^{\alpha_1}\dots p_v^{\alpha_r}$, with p_i's prime and $\alpha_i > 0$, then k is r-balancing if and only if*

$$\{r, k-r\} \subseteq \mathbb{N}p_1 + \dots + \mathbb{N}p_v.$$

4.3.2. Characterization of coalescing Grassmannians. Using the results exposed above on vanishing sums of roots of unity, we want to study and quantify the occurrence and the frequency of the coalescence of Dubrovin canonical coordinates in small quantum cohomologies of Grassmannians. Our first aim is to explicitly describe the following sets, defined for $k \geq 2$:

$$\mathfrak{A}_k := \{h\colon 0 < h < n \text{ s.t. } \mathbb{G}(h,k) \text{ is coalescing}\},$$

together with their complements

$$\widetilde{\mathfrak{A}}_k := \{h\colon 0 < h < k \text{ s.t. } \mathbb{G}(h,k) \text{ is not coalescing}\}.$$

We need some previous Lemmata.

LEMMA 4.1. *The following conditions are equivalent*

- *$r \in \mathfrak{A}_k$;*
- *there exist two combinations*

$$1 \leq i_1 < \dots < i_r \leq k \quad \textit{and} \quad 1 \leq j_1 < \dots < j_r \leq k,$$

with $i_h \neq j_h$ for at least one $h \in \{1, \dots, r\}$, such that

$$\zeta_k^{i_1} + \dots + \zeta_k^{i_r} = \zeta_k^{j_1} + \dots + \zeta_k^{j_r}, \quad \zeta_k := e^{\frac{2\pi i}{k}}.$$

PROOF. It is an immediate consequence of Corollary 4.4, and formulae (4.4), (4.5). □

LEMMA 4.2.

(1) *If k is prime, then $\mathfrak{A}_k = \emptyset$.*
(2) *If $r \in \{2, \dots, k-2\}$ is such that $r \in \mathfrak{A}_k$, then $\{\min(r, k-r), \dots, \max(r, k-r)\} \subseteq \mathfrak{A}_k$.*
(3) *If k is r-balancing (with $2 \leq r \leq k-2$), then $r \in \mathfrak{A}_k$. Thus, if $P_1(k) \leq k-2$, we have $\{P_1(k), \dots, k - P_1(k)\} \subseteq \mathfrak{A}_k$.*

PROOF. Point (1) follows from Corollary 4.5. For the point (2), notice that given a linear relation as in Lemma 4.1 with r roots on both l.h.s. and r.h.s. we can obtain a relation with more terms, by adding to both sides the same roots. For point (3), if we have $\zeta_k^{i_1} + \dots + \zeta_k^{i_r} = 0$, then also $\zeta_k \cdot (\zeta_k^{i_1} + \dots + \zeta_k^{i_r}) = 0$, and Lemma 4.1 applies. The last statement follows from the previous Theorem 4.5 and point (1). $\square$

PROPOSITION 4.3. *If $P_1(k) \leq k-2$, then $\min \mathfrak{A}_k = P_1(k)$.*

PROOF. Let $r := \min \mathfrak{A}_k$. We subdivide the proof in several steps.

- Step 1. Let us suppose that k is *squarefree*. By a straightforward application of Corollary 4.5, from an equality like
$$\zeta_k^{i_1} + \dots + \zeta_k^{i_v} = \zeta_k^{j_1} + \dots + \zeta_k^{j_v},$$
and $v < P_1(k)$ we deduce that necessarily $i_h = j_h$ for all $h = 1, \dots, v$. Thus $r = P_1(k)$. This proves the Proposition if k is squarefree.
- Step 2. From now on, k is not supposed to be squarefree. We suppose, by contradiction, that $r < P_1(k)$. Because of the minimality condition on r, in an equality
$$\zeta_k^{i_1} + \dots + \zeta_k^{i_r} = \zeta_k^{j_1} + \dots + \zeta_k^{j_r}, \tag{4.7}$$
we have that $i_h \neq j_h$ for all $h = 1, \dots, r$. Multiplying, if necessary, by the inverse of one root of unity, we can suppose that one root appearing in (4.7) is 1. Moreover, we can rewrite equation (4.7) as a vanishing sum
$$\sum_{i=1}^{2r} \alpha_i z_i = 0, \quad \alpha_i \in \{-1, +1\} \tag{4.8}$$
and where $z_1, \dots, z_{2r}$ are distinct k-roots of unity.
- Step 3. We show that the vanishing sum (4.8) is irreducible. Indeed, if we consider the smallest (i.e. with the least number of terms) proper vanishing sub-sum, then it must have at most r addends, otherwise its complement w.r.t. (4.8) would be a vanishing proper sub-sum with less terms. By application of Theorem 4.3 to this smallest sub-sum, we deduce that for all roots z_i's appearing in it, we must have
$$\left(\frac{z_i}{z_j}\right)^m = 1, \quad m := \prod_{\substack{p \text{ prime} \\ p \leq r}} p.$$
Under the assumption $r < P_1(k)$, we have that $\gcd(m, k) = 1$, and since also
$$\left(\frac{z_i}{z_j}\right)^k = 1, \quad \text{we deduce } \frac{z_i}{z_j} = 1,$$
which is absurd by minimality of r. Thus (4.8) is irreducible.

- Step 4. We now show that the order of any roots appearing in (4.8) must be a squarefree number. By application of Theorem 4.3, we know that for all i, j

$$\left(\frac{z_i}{z_j}\right)^m = 1, \quad m := \prod_{\substack{p \text{ prime} \\ p \leq 2r}} p.$$

Since for one root in (4.8) we have $z_j = 1$, we deduce that $z_i^m = 1$ for any roots in (4.8), and that any orders, being divisors of m, must be squarefree.
- Step 5. By applying the argument of Step 1, we conclude. □

THEOREM 4.6. *The complex Grassmannian $\mathbb{G}(r,k)$ is coalescing if and only if*

$$P_1(k) \leq r \leq k - P_1(k).$$

In particular, all Grassmannians of proper subspaces of $\mathbb{C}^p$, with p prime, are not coalescing.

PROOF. The proof directly follows from Lemma 4.2 and Proposition 4.3. □

4.3.3. Dirichlet series associated to non-coalescing Grassmannians, and their rareness. Let us now define the sequence

$$\tilde{\text{л}}_n := \text{card}\left(\widetilde{\mathfrak{A}}_n\right), \quad n \geq 2.$$

Introducing the Dirichlet series

$$\widetilde{\text{Л}}(s) := \sum_{n=2}^{\infty} \frac{\tilde{\text{л}}_n}{n^s},$$

we want deduce information about $(\tilde{\text{л}}_n)_{n\geq 2}$ studying properties of the generating function $\widetilde{\text{Л}}(s)$.

THEOREM 4.7. *The Dirichlet series $\widetilde{\text{Л}}(s)$ associated to the sequence $(\tilde{\text{л}}_n)_{n\geq 2}$ is absolutely convergent in the half-plane* $\text{Re}(s) > 2$, *where it can be represented by the infinite series*

$$\widetilde{\text{Л}}(s) = \sum_{p \text{ prime}} \frac{p-1}{p^s}\left(\frac{2\zeta(s)}{\zeta(s,p-1)} - 1\right).$$

The function defined by $\widetilde{\text{Л}}(s)$ can be analytically continued into (the universal cover of) the punctured half-plane

$$\{s \in \mathbb{C} \colon \text{Re}(s) > \overline{\sigma}\} \setminus \left\{ s = \frac{\rho}{k} + 1 \colon \begin{array}{c} \rho \text{ pole or zero of } \zeta(s), \\ k \text{ squarefree positive integer} \end{array} \right\},$$

$$\overline{\sigma} := \limsup_{n\to\infty} \frac{1}{\log n} \cdot \log\left(\sum_{\substack{k \leq n \\ k \text{ composite}}} \tilde{\text{л}}_k\right), \quad 1 \leq \overline{\sigma} \leq \frac{3}{2},$$

having logarithmic singularities at the punctures. In particular, at the point $s = 2$ the following asymptotic estimate holds

$$\widetilde{\text{Л}}(s) = \log\left(\frac{1}{s-2}\right) + O(1), \quad s \to 2, \quad \text{Re}(s) > 2. \tag{4.9}$$

PROOF. Let σ_a be the abscissa of (absolute) convergence for $\widetilde{\text{Л}}(s)$. Since

$$\inf\{\alpha \in \mathbb{R} \colon \tilde{\text{л}}_n = O(n^\alpha)\} = 1,$$

we have $1 \leq \sigma_a \leq 2$. Moreover, the sequence $(\tilde{\text{л}}_n)_{n\geq 2}$ being positive, by a Theorem of Landau ([Cha68], [Ten15]) the point $s = \sigma_a$ is a singularity for $\widetilde{\text{Л}}(s)$. For $\mathrm{Re}(s) > \sigma_a$, we have (by Theorem 4.6)

$$\widetilde{\text{Л}}(s) = \sum_{p \text{ prime}} \frac{p-1}{p^s} + \sum_{n \text{ composite}} \frac{2(P_1(n)-1)}{n^s}. \tag{4.10}$$

Note that

$$\begin{aligned}
\sum_{n \text{ composite}} \frac{2(P_1(n)-1)}{n^s} &= \sum_{p \text{ prime}} \sum_{\substack{m\geq 2\\ P_1(m)\geq p}} \frac{2(p-1)}{(pm)^s}\\
&= 2\sum_{p \text{ prime}} \frac{p-1}{p^s}\left(-1 + \sum_{\substack{m\geq 1\\ P_1(m)\geq p}} \frac{1}{m^s}\right)\\
&= 2\sum_{p \text{ prime}} \frac{p-1}{p^s}\left(-1 + \prod_{\substack{q \text{ prime}\\ q\geq p}} \sum_{k=0}^{\infty} \frac{1}{q^{ks}}\right)\\
&= 2\sum_{p \text{ prime}} \frac{p-1}{p^s}\left(-1 + \zeta(s)\prod_{\substack{q \text{ prime}\\ q< p}} \left(1-\frac{1}{q^s}\right)\right).
\end{aligned}$$

From this and equation (4.10) it follows that

$$\widetilde{\text{Л}}(s) = \sum_{p \text{ prime}} \frac{p-1}{p^s}\left(\frac{2\zeta(s)}{\zeta(s,p-1)} - 1\right).$$

Since for any s with $\mathrm{Re}(s) > 1$ we have $\lim_n \frac{\zeta(s)}{\zeta(s,p_n-1)} = 1$, by asymptotic comparison we deduce that the half-plane of absolute convergence of $\widetilde{\text{Л}}(s)$ coincides with the half-plane of $\zeta_P(s-1) - \zeta_P(s)$, hence $\sigma_a = 2$ ([Frö68]).

The second Dirichlet series in (4.10) defines an holomorphic function in the half-plane of absolute convergence $\mathrm{Re}(s) > \overline{\sigma}$, where ([HR15])

$$\overline{\sigma} := \limsup_{n\to\infty} \frac{1}{\log n} \cdot \log\left(\sum_{\substack{k\leq n\\ k \text{ composite}}} \tilde{\text{л}}_k\right).$$

From the elementary and optimal inequality $P_1(n) \leq n^{\frac{1}{2}}$, valid for any composite number n, we deduce that $\frac{1}{2} \leq \overline{\sigma} \leq \frac{3}{2}$. Thus, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ defined by

$$\alpha_n := \frac{1}{\log 2n} \cdot \log\left(\sum_{\substack{k\leq 2n\\ k \text{ composite}}} \tilde{\text{л}}_k\right),$$

is bounded: by Bolzano-Weierstrass Theorem, we can extract a subsequence converging to a positive real number r and, by characterization of the superior limit, we necessarily have $r \leq \overline{\sigma}$. Notice that we have the trivial estimate

$$\sum_{\substack{k\leq 2n\\ k \text{ composite}}} \tilde{\text{Л}}_k = \left(\sum_{\substack{4\leq k\leq 2n\\ k \text{ even}}} \tilde{\text{Л}}_k\right) + \left(\sum_{\substack{k\leq 2n\\ k \text{ odd composite}}} \tilde{\text{Л}}_k\right) > 2(n-1),$$

and we deduce that $1 \leq r$. In conclusion, $1 \leq \overline{\sigma} \leq \frac{3}{2}$.

As a consequence, the function $\widetilde{\text{Л}}(s)$ can be extended by analytic continuation at least up to the half-plane $\mathrm{Re}(s) > \overline{\sigma}$, and it inherits from the function $\zeta_P(s-1)-\zeta_P(s)$ some logarithmic singularities in the strip $\overline{\sigma} < \mathrm{Re}(s) \leq 2$: they correspond to the points of the form

$$\frac{\rho}{k} + 1, \quad 0 < \mathrm{Re}(\rho) \leq 1,$$

where $\rho = 1$ or $\zeta(\rho) = 0$, and k is a squarefree positive integer. This follows from the well known representation

$$\zeta_P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns),$$

μ being the Möbius arithmetic function (see [**Gle91**], [**Frö68**] and [**THB86**]).

For $\rho = k = 1$, we find again that $s = 2$ is a logarithmic singularity for $\widetilde{\text{Л}}(s)$: the asymptotic expansion (4.9) follows from

$$\zeta_P(s) = \log\left(\frac{1}{s-1}\right) + O(1), \quad s \to 1, \ \mathrm{Re}(\rho) > 1.$$

This completes the proof. □

Corollary 4.6. *The following statements are equivalent:*

(1) (RH) *all non-trivial zeros of the Riemann zeta function* $\zeta(s)$ *satisfy* $\mathrm{Re}(s) = \frac{1}{2}$*;*

(2) *the derivative* $\widetilde{\text{Л}}'(s)$ *extends by analytic continuation to a meromorphic function in the half-plane* $\frac{3}{2} < \mathrm{Re}(s)$ *with a single pole of oder one at* $s = 2$.

Remark 4.1. The analytic continuation of the function $\widetilde{\text{Л}}(s)$ beyond the line $\mathrm{Re}(s) = \overline{\sigma}$ is highly influenced by the analytic continuation of the series

$$\sum_{n \text{ composite}} \frac{\tilde{\text{Л}}_n}{n^s}$$

in the strip $1 < \mathrm{Re}(s) < \overline{\sigma}$. In particular, if in this strip it does not have enough logarithmic singularities annihilating those of $\zeta_P(s-1) - \zeta_P(s)$, then the line $\mathrm{Re}(s) = 1$ is necessarily a natural boundary for $\widetilde{\text{Л}}(s)$: indeed, the singularities of $\zeta_P(s-1)$ cluster near all points of this line ([**LW20**]). Notice that $s = \overline{\sigma}$ is necessarily a singularity for $\widetilde{\text{Л}}(s)$, by Landau Theorem.

Remark 4.2. If we introduce the sequence $\text{л}_n := \mathrm{card}\,(\mathfrak{A}_n)$, for $n \geq 2$, and the corresponding generating function

$$\text{Л}(s) := \sum_{n=2}^{\infty} \frac{\text{л}_n}{n^s},$$

the following identity holds:

$$\text{Л}(s) + \widetilde{\text{Л}}(s) = \zeta(s-1) + \zeta(s).$$

In this sense, $\text{Л}(s)$ is "dual" to $\widetilde{\text{Л}}(s)$.

COROLLARY 4.7. *The following asymptotic expansion holds*

$$\sum_{k=2}^{n} \tilde{\text{л}}_k \sim \frac{1}{2}\frac{n^2}{\log n}.$$

In particular, the non-coalescing Grassmannians are rare*:*

$$\lim_{n} \frac{2}{n^2-n} \sum_{k=2}^{n} \tilde{\text{л}}_k = 0.$$

PROOF. Since the function $\widetilde{\text{Л}}(s)$ is holomorphic at all points of the line $\mathrm{Re}(s) = 2$ but $s \neq 2$, and the asymptotic expansion (4.9) holds, an immediate application of Ikehara-Delange Tauberian Theorem for the case of singularities of mixed-type (involving both monomial and logarithmic terms in their principal parts) for Dirichlet series, gives the result (see [**Del54**] Theorem IV, and [**Ten15**] pag. 350).
Another more elementary (and maybe less elegant) proof is the following: from Theorem 4.6 we have that

$$\sum_{k=2}^{n} \tilde{\text{л}}_k = 2(1-n) + \pi_0(n) - \pi_1(n) + 2\sum_{j=2}^{n} P_1(j),$$

and recalling the following asymptotic estimates (see [**SZ68**], [**KL12**] or [**Jak13**])

$$\pi_\alpha(n) \sim \frac{n^{1+\alpha}}{(1+\alpha)\log n}, \quad \alpha \geq 0,$$

$$\sum_{j=2}^{n} P_1(n)^m \sim \frac{1}{m+1}\frac{n^{m+1}}{\log n}, \quad m \geq 1,$$

one concludes. □

4.4. Distribution functions of non-coalescing Grassmannians, and equivalent form of the Riemann Hypothesis

In this section we want to obtain some more fine results about the distribution of these rare not coalescing Grassmannians. Thus, let us introduce the following

DEFINITION 4.2. For all real numbers $x, y \in \mathbb{R}_{\geq 2}$, with $x \geq y$, define the function

$$\mathscr{H}(x,y) := \operatorname{card}\left(\{n \leq x \colon n \geq 2,\ \tilde{\text{л}}_n > y\}\right).$$

In other words, $\mathscr{H}$ is the cumulative number of vector spaces $\mathbb{C}^n$, $2 \leq n \leq x$, having more than y non-coalescing Grassmannians of proper subspaces. For $x \in \mathbb{R}_{\geq 4}$ we will define also the restriction

$$\widehat{\mathscr{H}}(x) := \mathscr{H}(x, 2x^{\frac{1}{2}}).$$

In the following result, we describe some analytical properties of the function $\mathscr{H}$.

THEOREM 4.8.
(1) *For any $\kappa > 1$, the following integral representation*[1] *holds*

$$\mathscr{H}(x,y) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\left[\left(\frac{\zeta(s)}{\zeta\left(s,\frac{y}{2}+1\right)}-1\right)-\zeta_{P,y+1}(s)+\zeta_{P,\frac{y}{2}+1}(s)\right]\frac{x^s}{s}ds,$$

valid for $x \in \mathbb{R}_{\geq 2}\setminus\mathbb{N}$, $\quad y \in \mathbb{R}_{\geq 2}$ (with $y \leq x$), and where $\Lambda_\kappa := \{\kappa + it\colon t \in \mathbb{R}\}$ is the line oriented from $t = -\infty$ to $t = +\infty$.
(2) *For any $\kappa > 1$, the following integral representation holds*

$$\mathscr{H}(x,y) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\left[\left(\frac{\zeta(s)}{\zeta\left(s,\frac{y}{2}+1\right)}-1\right)x^s+\zeta_P(s)\left(\frac{(y+2)^s}{2^s}-(y+1)^s\right)\right]\frac{ds}{s},$$

valid for $x, y \in \mathbb{R}_{\geq 2}\setminus\mathbb{N}$ (with $y \leq x$), and where $\Lambda_\kappa := \{\kappa + it\colon t \in \mathbb{R}\}$ is the line oriented from $t = -\infty$ to $t = +\infty$.
(3) *The following asymptotic estimate holds uniformly in the range $x \geq y \geq 2$*

$$\mathscr{H}(x,y) = \frac{x}{\zeta\left(1,\frac{1}{2}y+1\right)}\left(e^\gamma\omega\left(\frac{\log x}{\log y}\right)+O\left(\frac{1}{\log y}\right)\right)+O\left(\frac{y}{\log y}\right),$$

where ω is the Buchstab function.

PROOF. The crucial observation is the following: if we consider, for fixed x and y, the sets

$$\mathcal{A} := \{n\colon 2 \leq n,\ \tilde{\pi}_n > y\},$$
$$\mathcal{B} := \{n\colon 2 \leq n,\ 2P_1(n) - 2 > y\},$$
$$\mathcal{C} := \{p \text{ prime } : p - 1 \leq y,\ 2p - 2 > y\},$$

then we have $\mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{A} \equiv \mathcal{B}\setminus\mathcal{C}$. In this way:

- the Dirichlet series associated to the sequence $\mathbb{1}_{\mathcal{A}}(n)$ (indicator function of $\mathcal{A}$) is the difference of the Dirichlet series associated to $\mathbb{1}_{\mathcal{B}}(n)$ and $\mathbb{1}_{\mathcal{C}}(n)$. The first one is given by (see e.g. [Ten15])
$$\frac{\zeta(s)}{\zeta\left(s,\frac{y}{2}+1\right)}-1,$$
while the second one is given by the difference of partial sums
$$\zeta_{P,y+1}(s) - \zeta_{P,\frac{y}{2}+1}(s).$$
An application of Perron Formula for x not integer gives the integral representation (1) of $\sum_{n\leq x}\mathbb{1}_A(n)$.
- Moreover, we also get the identity
$$\mathscr{H}(x,y) = \Phi\left(x,\frac{y}{2}+1\right) - \pi_0(y+1) + \pi_0\left(\frac{y}{2}+1\right). \tag{4.11}$$
For x and y not integer, we can apply Perron Formula separately for the three terms:
$$\Phi\left(x,\frac{y}{2}+1\right) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\left(\frac{\zeta(s)}{\zeta\left(s,\frac{y}{2}+1\right)}-1\right)\frac{x^s}{s}ds,$$
$$\pi_0(y+1) = \frac{1}{2\pi i}\int_{\Lambda_\kappa}\zeta_P(s)\,(y+1)^s\,\frac{ds}{s},$$

$$\pi_0\left(\frac{y}{2}+1\right)=\frac{1}{2\pi i}\int_{\Lambda_\kappa}\zeta_P(s)\left(\frac{y}{2}+1\right)^s\frac{ds}{s}.$$

The sum of the three terms gives the second integral representation (2).

- Form equation (4.11), by applying the well known de Bruijn's asymptotic estimate ([**dB50**], [**SMC06**]), we obtain the estimate (3).

□

THEOREM 4.9. *The function* $\widehat{\mathscr{H}}$ *admits the following asymptotic estimate:*

$$\widehat{\mathscr{H}}(x)=\int_0^x\frac{dt}{\log t}+O\left(x^\Theta\log x\right),\quad \textit{where } \Theta:=\sup\{\mathrm{Re}(\rho)\colon\zeta(\rho)=0\}.$$

Hence the following statements are equivalent:

(1) (RH) *all non-trivial zeros of the Riemann zeta function* $\zeta(s)$ *satisfy* $\mathrm{Re}(s)=\frac{1}{2}$*;*
(2) *for a sufficiently large* x*, the following (essentially optimal) estimate holds*

$$\widehat{\mathscr{H}}(x)=\int_0^x\frac{dt}{\log t}+O\left(x^{\frac{1}{2}}\log x\right).\tag{4.12}$$

PROOF. Using the elementary fact that for any composite number n we have $P_1(n)\leq n^{\frac{1}{2}}$, we obtain the estimate

$$\Phi(x,x^{\frac{1}{2}})=\pi_0(x)+O\left(x^{\frac{1}{2}}\right).$$

Hence, from the equation (4.11) specialized to the case $y=2x^{\frac{1}{2}}$, and by invoking the Prime Number Theorem, we obtain that

$$\widehat{\mathscr{H}}(x)=\pi_0(x)+O\left(x^{\frac{1}{2}}\right).$$

It is well known (see e.g. [**Ten15**] pag. 271) that

$$\pi_0(x)=\int_0^x\frac{dt}{\log t}+O\left(x^\Theta\log x\right),\quad \Theta:=\sup\{\mathrm{Re}(\rho)\colon\zeta(\rho)=0\}.$$

Since we have $\Theta\geq\frac{1}{2}$ (Hardy proved in 1914 that $\zeta(s)$ has an infinity of zeros on $\mathrm{Re}(s)=\frac{1}{2}$; see [**Har14**], and also [**THB86**] pag. 256), the estimate for $\widehat{\mathscr{H}}(x)$ follows. The equivalence with RH is evident. The optimality of the estimate (4.12) (within a factor of $(\log x)^2$) is a consequence of Littlewood's result ([**Lit14**]; see also [**HL18**] and [**MV07**], Chapter 15) on the oscillation for the error terms in the Prime Number Theorem:

$$\pi_0(x)-\int_0^x\frac{dt}{\log t}=\Omega_\pm\left(\frac{x^{\frac{1}{2}}\log\log\log x}{\log x}\right).$$

This completes the proof. □

Part 2

Isomonodromy Deformations at an Irregular Singularity with Coalescing Eigenvalues

«А ведь, голубчик: нет сильнее тех двух воинов, *терпение* и *время.*»

Л. Н. Толстой, *Война и Мир*,
Том III, Часть II, Глава XVI

CHAPTER 5

Structure of Fundamental Solutions

Abstract. In this Chapter we study the structure of both formal and fundamental systems of solutions for a (not necessarily isomonodromic) family of equations on the complex domain, holomorphically depending on a parameter $t \in \mathcal{U}_{\epsilon_0}(0) \subseteq \mathbb{C}^m$, where $\mathcal{U}_{\epsilon_0}(0)$ is a polydisc centered at 0, and admitting an irregular singularity at $z = \infty$ of Poincaré rank 1. The singularity is assumed to be resonant for t varying in some *coalescence locus* $\Delta \subseteq \mathbb{C}^m$. After recalling classical results of Y. Sibuya [Sib62] about gauge equivalence transformations, holomorphic w.r.t. t, the study of formal solutions (their existence, uniqueness and their structure) and of genuine fundamental solutions (their existence in sufficiently narrow sectors) is carried on, both at the coalescence locus Δ and away of it (Section 5.4 and Section 5.5, respectively). Under a holomorphically diagonalizability assumption for the coefficients (Assumption 5.1), we give necessary and sufficient conditions for a formal solution, computed away from coalescence points, to admit holomorphic continuation to the coalescence locus (Proposition 5.3).

5.1. Conventions and Notations

If $\alpha < \beta$ are real numbers, an open sector and a closed sector with central opening angle $\beta - \alpha > 0$ are respectively denoted by

$$S(\alpha, \beta) := \{ z \in \mathcal{R} \mid \ \alpha < \arg z < \beta \ \}, \qquad \overline{S}(\alpha, \beta) := \{ z \in \mathcal{R} \mid \ \alpha \leq \arg z \leq \beta \ \}.$$

The rays with directions α and β will be called *the right and left boundary rays* respectively. If $\overline{S}(\theta_1, \theta_2) \subset S(\alpha, \beta)$, then $\overline{S}(\theta_1, \theta_2)$ is called a *proper (closed) subsector*.

Given a function $f(z)$ holomorphic on a sector containing $\overline{S}(\alpha, \beta)$, we say that it admits an asymptotic expansion $f(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}$ for $z \to \infty$ in $\overline{S}(\alpha, \beta)$, if for any $m \geq 0$, $\lim_{z\to\infty} z^m (f(z) - \sum_{k=0}^{m} a_k z^{-k}) = 0$, $z \in \overline{S}(\alpha, \beta)$. If f depends on parameters t, the asymptotic representation $f(z, t) \sim \sum_{k=0}^{\infty} a_k(t) z^{-k}$ is said to be uniform in t belonging to a compact subset $K \subset \mathbb{C}^m$, if the limits above are uniform in K. In case the sector is open, we write $f(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}$ as $z \to \infty$ in $S(\alpha, \beta)$ if the limits above are zero *in every proper closed subsector* of $S(\alpha, \beta)$. When we take the limits above for matrix valued functions $A = (A_{ij}(z, t))_{i,j=1}^{n}$, we use the norm $|A| := \max_{ij} |A_{ij}|$.

5.2. Deformation of a Differential System with Singularity of the Second Kind

Let us consider the following system of differential equations

$$\frac{dY}{dz} = A(z, t)Y, \qquad t = (t_1, t_2, ..., t_m) \in \mathbb{C}^m, \tag{5.1}$$

depending on m complex parameters[1] t. Since we want to develop a local study, without loss of generality, we assume that the deformation parameters vary in a polydisc

$$\mathcal{U}_{\epsilon_0}(0) := \{ t \in \mathbb{C}^m \colon |t| \leq \epsilon_0 \}, \quad |t| := \max_{i=1,\dots,m} |t^i|,$$

[1] Later, we will take $n = m$, as in (1.25).

for some sufficiently small positive constant ϵ_0. The $n\times n$ matrix $A(z,t)$ is assumed to be holomorphic in (z,t) for $|z|\geq N_0>0$ for some positive constant N_0, and $|t|\leq\epsilon_0$, with uniformly convergent Taylor expansion

$$A(z,t)=\sum_{j=0}^{\infty}A_j(t)z^{-j}. \tag{5.2}$$

The coefficients $A_j(t)$ are holomorphic for $|t|\leq\epsilon_0$. We assume that $A_0(0)$ is diagonalisable, with *distinct eigenvalues* λ_1, ..., λ_s, $s\leq n$. We are interested in the case when s is strictly less than n. Up to a constant gauge transformation, there is no loss of generality in assuming that

$$A_0(0)=\Lambda:=\Lambda_1\oplus\cdots\oplus\Lambda_s,\qquad \Lambda_i:=\lambda_i I_{p_i},\quad i=1,2,...,s\leq n, \tag{5.3}$$

being I_{p_i} the $p_i\times p_i$ identity matrix. In Section 5.5 below, we will assume that $A_0(t)$ is *holomorphically similar* to $\Lambda(t)$ (see Assumption 5.1 below): in such a case $\Lambda=\Lambda(0)$. However, at this stage of the discussion we do not assume holomorphic similarity, so we keep the notation Λ instead of $\Lambda(0)$.

Remark 5.1. A result due to Kostov [**Kos99**] states that, if system (5.1) is such that $A(z,0)=A_0(0)+A_1(0)/z$, and if the matrix $A_1(0)$ has no eigenvalues differing by a non-zero integers, than there exists a gauge transformation $Y=W(z,t)\widetilde{Y}$, with $W(z,t)$ holomorphic at $z=\infty$ and $t=0$, such that (5.1) becomes a system of the form

$$\frac{d\widetilde{Y}}{dz}=\left(\widetilde{A}_0(t)+\frac{\widetilde{A}_1(t)}{z}\right)\widetilde{Y}. \tag{5.4}$$

Nevertheless, since $A_0(0)$ has non-distinct eigenvalues, we cannot find in general a gauge transformation holomorphic at $z=\infty$ which transforms $A(z,0)$ of the system (5.1) into $A_0(0)+A_1(0)/z$ (see also [**Bol94**] and references therein). Therefore the system (5.1) is more general than system (5.4).

5.2.1. Sibuya's Theorem. General facts about eigenvalues and eigenvectors of a matrix $M(t)$, depending holomorphically on t in a domain $\mathcal{D}\subset\mathbb{C}^m$, such that $M(0)$ has eigenvalues λ_1, ..., λ_s, $s\leq n$, can be found in [**Lax07**] and at page 63-87 of [**Kat95**]. If s is strictly smaller than n, then $t=0$ is a coalescence point. For $\mathcal{D}\subset\mathbb{C}^m$ and $m=1$ the coalescence points are isolated, while for $m\geq 2$ they form the *coalescence locus*. Except for the special case when $M(t)$ is holomorphically similar to a Jordan form $J(t)$, which means that there exists an invertible holomorphic matrix $G_0(t)$ on $\mathcal{D}$ such that $(G_0(t))^{-1}M(t)G_0(t)=J(t)$, in general the eigenvectors of $M(t)$ are holomorphic in the neighborhood of a non-coalescence point, but their analytic continuation is singular at the coalescence locus.

Example 5.1. For example,

$$M(t)=\begin{pmatrix}0&1\\t&0\end{pmatrix},\qquad t\in\mathbb{C},$$

has eigenvalues $\lambda_\pm=\pm\sqrt{t}$, which are branches of $f(t)=t^{1/2}$, with ramification at $\Delta=\{t=0\}$. The eigenvectors can be chosen to be either

$$\vec{\xi}_\pm=(\pm 1/\sqrt{t},1),\quad \text{ or }\ \vec{\xi}_\pm=(\pm 1,\sqrt{t}).$$

The matrix $G_0(t):=[\vec{\xi}_+(t),\vec{\xi}_-(t)]$ puts $M(t)$ in diagonal form $G_0(t)^{-1}A_0(t)G_0(t)=\mathrm{diag}(\sqrt{t},-\sqrt{t})$, for $t\neq 0$, while $M(0)$ is in Jordan non-diagonal form. Either $G_0(t)$ or $G_0(t)^{-1}$ is singular at $t=0$. The branching could be eliminated by changing deformation parameter to $s=t^{1/2}$. Nevertheless, this would not cure the singularity of G_0 or G_0^{-1} at $s=0$.

EXAMPLE 5.2. Another example is

$$M(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad t \in \mathbb{C}.$$

The eigenvalues $u_1 = u_2 = 1$ are always coalescing. The Jordan types at $t \neq 0$ and $t = 0$ are different. Indeed, $M(0) = \text{diag}(1,1)$, while for $t \neq 0$,

$$G_0(t)^{-1} M(t)\ G_0(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad G_0(t) := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, $G_0(t)$ is not invertible and $G_0(t)^{-1}$ diverges at $t = 0$.

In the above examples, the Jordan type of $M(t)$ changes. In the next example, the Jordan form remains diagonal, and nevertheless $G_0(t)$ is singular.

EXAMPLE 5.3. Consider

$$M(t) = \begin{pmatrix} 1+t_1 & t_2 \\ 0 & 1-t_2 \end{pmatrix}, \qquad t = (t_1, t_2) \in \mathbb{C}^2.$$

The eigenvalues coalesce at $t = 0$, where $M(0) = I$. Moreover, there exists a diagonalizing matrix $G_0(t)$ such that

$$G_0(t)^{-1} M(t) G_0(t) = \begin{pmatrix} 1+t_1 & 0 \\ 0 & 1-t_2 \end{pmatrix} \text{ is diagonal}, \quad G_0(t) = \begin{pmatrix} a(t) & -t_2\, b(t) \\ 0 & (t_1+t_2)\, b(t) \end{pmatrix},$$

for arbitrary non-vanishing holomorphic functions $a(t), b(t)$. At $t = 0$ the matrix $G_0(t)$ has zero determinant and $G_0(t)^{-1}$ diverges.

Although $M(t)$ is not in general holomorphically similar to a Jordan form, holomorphic similarity can always be realised between $M(t)$ and a block-diagonal matrix $\widehat{M}(t)$ having the same block structure of a Jordan form of $M(0)$, as follows.

LEMMA 5.1. **[LEMMA 1 of [Sib62]]:** *Let $M(t)$ be a $n \times n$ matrix holomorphically depending on $t \in \mathbb{C}^m$, with $|t| \leq \epsilon_0$, where ϵ_0 is a positive constant. Let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of $M(0)$, with multiplicities $p_1, p_2, ..., p_s$, so that $p_1 + p_2 + \cdots + p_s = n$. Assume that $M(0)$ is in Jordan form*

$$M(0) = M_1(0) \oplus \cdots \oplus M_s(0)$$

where

$$M_j(0) = \lambda_j I_{p_j} + \mathcal{H}_j, \qquad \mathcal{H}_j = \begin{bmatrix} 0 & \mathfrak{h}_{j1} & & & \\ & 0 & \mathfrak{h}_{j2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathfrak{h}_{jp_j-1} \\ & & & & 0 \end{bmatrix}, \quad 1 \leq j \leq s,$$

$\mathfrak{h}_{jk}$ being equal to 1 or 0. Then, for sufficiently small $0 < \epsilon \leq \epsilon_0$ there exists a matrix $G_0(t)$, holomorphic in t for $|t| \leq \epsilon$, such that

$$G_0(0) = I,$$

and $\widehat{M}(t) = (G_0(t))^{-1} M(t) G_0(t)$ has block diagonal form

$$\widehat{M}(t) = \widehat{M}_1(t) \oplus \cdots \oplus \widehat{M}_s(t), \tag{5.5}$$

where $\widehat{M}_j(t)$ are $p_j \times p_j$ matrices. For $|t| \leq \epsilon$, $\widehat{M}_i(t)$ and $\widehat{M}_j(t)$ have no common eigenvalues for any $i \neq j$.

Remark 5.2. The lemma also holds when $t \in \mathbb{R}^m$ in the continuous (not necessarily holomorphic) setting.

Lemma 5.1 can be applied to $M(t) \equiv A_0(t)$ in (5.2), with $A_0(0) = \Lambda$. Therefore[2]

$$\widehat{A}_0(t) := G_0(t)^{-1} A_0(t) G_0(t) = \widehat{A}^{(0)}_{11}(t) \oplus \cdots \oplus \widehat{A}^{(0)}_{ss}(t), \tag{5.6}$$

$$G_0(0) = I, \qquad \widehat{A}_0(0) = A_0(0) = \Lambda.$$

Remark 5.3. $G_0(t)$ is determined up to $G_0 \mapsto G_0(t)\Delta_0(t)$, where $\Delta_0(t)$ is any block-diagonal matrix solution of $[\Delta_0(t), \widehat{A}_0(t)] = 0$. Sibuya's normalization condition $G_0(0) = I$ can be softened to $G_0(0) = \Delta_0$.

We define a family of sectors $\mathcal{S}_\nu$ in $\mathcal{R}$ and state Sibuya's theorem. Let $\arg_p(\lambda_j - \lambda_k)$ be the principal determination. Let $\eta \in \mathbb{R}$ be an **admissible direction** for Λ in the λ-plane (we borrow this name and the following definition of the η_ν's and τ_ν's from [**BJL79a**] and [**BJL81**]). By definition, this means that,

$$\eta \neq \arg_p(\lambda_j - \lambda_k) \bmod (2\pi), \quad \forall\ 1 \leq j \neq k \leq s.$$

Introduce another determination $\widehat{\arg}$ as follows:

$$\eta - 2\pi < \widehat{\arg}(\lambda_j - \lambda_k) < \eta, \qquad 1 \leq j \neq k \leq s. \tag{5.7}$$

Let 2μ, $\mu \in \mathbb{N}$, be the number of values $\widehat{\arg}(\lambda_j - \lambda_k)$, when (j,k) spans all the indices $1 \leq j \neq k \leq s$.[3] Denote the 2μ values of $\widehat{\arg}(\lambda_j - \lambda_k)$ with $\eta_0, \eta_1, ..., \eta_{2\mu-1}$, according to the following ordering:

$$\eta > \eta_0 > \ \cdots \ > \eta_{\mu-1} > \eta_\mu > \ \cdots \ > \eta_{2\mu-1} > \eta - 2\pi. \tag{5.8}$$

Clearly

$$\eta_{\nu+\mu} = \eta_\nu - \pi, \qquad \nu = 0, 1, ..., \mu - 1. \tag{5.9}$$

Consider the following directional angles in the z-plane

$$\tau := \frac{3\pi}{2} - \eta, \qquad \tau_\nu := \frac{3\pi}{2} - \eta_\nu, \qquad 0 \leq \nu \leq 2\mu - 1. \tag{5.10}$$

From (5.8) if follows that,

$$\tau < \tau_0 < \ \cdots \ < \tau_{\mu-1} < \tau_\mu < \ \cdots \ < \tau_{2\mu-1} < \tau + 2\pi. \tag{5.11}$$

From (5.9) if follows that,

$$\tau_{\nu+\mu} = \tau_\nu + \pi, \qquad \nu = 0, 1, ..., \mu - 1.$$

The extension of the above to directions in $\mathcal{R}$ is obtained by the following definition:

$$\tau_{\nu+k\mu} := \tau_\nu + k\pi, \qquad k \in \mathbb{Z}.$$

This allows to speak of directions τ_ν for any $\nu \in \mathbb{Z}$.

Definition 5.1 (Sector $\mathcal{S}_\nu$). We define the following sectors of central opening angle greater than π:

$$\mathcal{S}_\nu := S(\tau_\nu - \pi, \tau_{\nu+1}) \equiv S(\tau_{\nu-\mu}, \tau_{\nu+1}), \qquad \nu \in \mathbb{Z}. \tag{5.12}$$

[2]Given a $n \times n$ matrix A_0, partitioned into s^2 blocks ($s \leq n$), we use the notation $A^{(0)}_{ij}$, $1 \leq i,j \leq s$, to denote the block in position (i,j). Such a block has dimension $p_i \times p_j$, with $p_1 + ... + p_n = n$.

[3] $2\mu \leq s(s-1)$, with "=" occurring when $\arg(\lambda_j - \lambda_k) \neq \arg(\lambda_r - \lambda_s) \bmod 2\pi$ for any $(j,k) \neq (r,s)$.

THEOREM 5.1 (Sibuya [Sib62] [HS66]). *Let $A(z,t)$ be holomorphic in (z,t) for $|z| \geq N_0 > 0$ and $|t| \leq \epsilon_0$ as in (5.2), such that $A_0(0) = \Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_s$, as in (5.3). Pick up a sector $\mathcal{S}_\nu$ as in (5.12). Then, for any proper closed subsector $\overline{S}(\alpha,\beta) = \{z \mid \tau_\nu - \pi < \alpha \leq \arg z \leq \beta < \tau_{\nu+1}\} \subset \mathcal{S}_\nu$, there exist a sufficiently large positive number $N \geq N_0$, a sufficiently small positive number $\epsilon \leq \epsilon_0$, and matrices $G_0(t)$ and $G(z,t)$ with the following properties:*

i) $G_0(t)$ is holomorphic for $|t| \leq \epsilon$ and

$$G_0(0) = I, \qquad \widehat{A}_0(t) := G_0(t)^{-1} A_0(t) G_0(t) \text{ is block-diagonal as in (5.6).}$$

ii) $G(z,t)$ is holomorphic in (z,t) for $|z| \geq N$, $z \in \overline{S}(\alpha,\beta)$, $|t| \leq \epsilon$;

iii) $G(z,t)$ has a uniform asymptotic expansion for $|t| \leq \epsilon$, with holomorphic coefficients $G_k(t)$:

$$G(z,t) \sim I + \sum_{k=1}^{\infty} G_k(t) z^{-k}, \quad z \to \infty, \quad z \in \overline{S}(\alpha,\beta),$$

iv) The gauge transformation

$$Y(z,t) = G_0(t) G(z,t) \widetilde{Y}(z,t),$$

reduces the initial system to a block diagonal form

$$\frac{d\widetilde{Y}}{dz} = B(z,t)\widetilde{Y}, \qquad B(z,t) = B_1(z,t) \oplus \cdots \oplus B_s(z,t), \tag{5.13}$$

where $B(z,t)$ is holomorphic in (z,t) in the domain $|z| \geq N$, $z \in \overline{S}(\alpha,\beta)$, $|t| \leq \epsilon$, and has a uniform asymptotic expansion for $|t| \leq \epsilon$, with holomorphic coefficients $B_k(t)$,

$$B(z,t) \sim \widehat{A}_0(t) + \sum_{k=1}^{\infty} B_k(t) z^{-k}, \quad z \to \infty, \quad z \in \overline{S}(\alpha,\beta). \tag{5.14}$$

In particular, setting $\widehat{A}_1(t) := G_0^{-1}(t) A_1(t) G_0(t)$, then $B_1(t) = \widehat{A}_{11}^{(1)}(t) \oplus \cdots \oplus \widehat{A}_{ss}^{(1)}(t)$.

REMARK 5.4. In the theorem above, ϵ is such that $\widehat{A}_{ii}^{(0)}(t)$ and $\widehat{A}_{jj}^{(0)}(t)$ have no common eigenvalues for any $i \neq j$ and $|t| \leq \epsilon$. Observe that one can always choose $\beta - \alpha > \pi$.

REMARK 5.5. $\mathcal{S}_\nu$ coincides with a sector $\{\, z \in \mathcal{R} \ |-3\pi/2 - \omega_- < r \arg z < 3\pi/2 - \omega_+\}$, introduced by Sibuya in [HS66]. A closed subsector $\overline{S}(\alpha,\beta)$ is a sector $\mathcal{D}(N,\gamma)$ introduced by Sibuya in [Sib62].

REMARK 5.6. If $\Lambda = \lambda_1 I$, Theorem 5.1 gives no new information, being $G_0(t) = G(z,t) \equiv I$ and $\mathcal{S}_\nu = \mathcal{R}$.

– *A Short Review of the Proof:* The z-constant gauge transformation $Y(z,t) = G_0(t)\widehat{Y}(z,t)$ transforms (5.1) into

$$\frac{d\widehat{Y}}{dz} = \widehat{A}(z,t)\, \widehat{Y}, \qquad \widehat{A}(z,t) = \sum_{i=0}^{\infty} \widehat{A}_i(t) z^{-i}, \qquad \widehat{A}_i(t) := G_0^{-1}(t) A_i(t) G_0(t). \tag{5.15}$$

Another gauge transformation $\widehat{Y}(z,t) = G(z,t)\widetilde{Y}(z,t)$ yields (5.13). Substitution into (5.15) gives the differential equation

$$G' + GB = \widehat{A}(z,t) G, \tag{5.16}$$

with unknowns $G(z,t), B(z,t)$. If formal series $G(z,t) = I + \sum_{j=1}^{\infty} G_j(t) z^{-j}$ and $B(z,t) = \widehat{A}_0(t) + \sum_{j=1}^{\infty} B_j(t) z^{-j}$ are inserted into (5.16), the following recursive equations (t is understood) are found:

For $l = 0$: $B_0(t) = \widehat{A}_0(t)$.

For $l = 1$:

$$\widehat{A}_0 G_1 - G_1 \widehat{A}_0 = -\widehat{A}_1 + B_1. \tag{5.17}$$

For $l \geq 2$:

$$\widehat{A}_0 G_l - G_l \widehat{A}_0 = \Big[\sum_{j=1}^{l-1} \Big(G_j B_{l-j} - \widehat{A}_{l-j} G_j \Big) - \widehat{A}_l \Big] \; - (l-1) G_{l-1} \; + B_l. \tag{5.18}$$

Once $G_0(t)$ has been fixed, the recursion equations can be solved. A solution $\{G_l(t)\}_{l=1}^{\infty}$, $\{B_l(t)\}_{l=1}^{\infty}$ is not unique in general. The following choice is possible:

$$G_{jj}^{(l)}(t) = 0, \qquad 1 \leq j \leq s, \qquad \text{[diagonal blocks are zero]}, \tag{5.19}$$

and

$$B_l(t) = B_1^{(l)}(t) \oplus \cdots \oplus B_s^{(l)}(t), \qquad \text{[off-diagonal blocks are zero]}. \tag{5.20}$$

Then, the $G_l(t)$'s and $B_l(t)$'s are determined by the recursion relations, because for a diagonal block $[j, j]$ the l.h.s of (5.17) and (5.18) is equal to 0 and the r.h.s determines the only unknown variable $B_{jj}^{(l)}$. For off-diagonal blocks $[i, j]$ there is no unknown in the r.h.s while in the l.h.s the following expression appears

$$\widehat{A}_{ii}^{(0)}(t) G_{ij}^{(l)} - G_{ij}^{(l)} \widehat{A}_{jj}^{(0)}(t), \qquad 1 \leq i \neq j \leq s.$$

For $|t| \leq \epsilon$ small enough, $\widehat{A}_{ii}^{(0)}(t)$ and $\widehat{A}_{jj}^{(0)}(t)$ have no common eigenvalues, so the equation is solvable for $G_{ij}^{(l)}$. With the above choice, Sibuya [**Sib62**] proves that there exist actual solutions $G(z, t)$ and $B(z, t)$ of (5.16) with asymptotic expansions $I + \sum_j G_j(t) z^{-j}$ and $\widehat{A}_0 + \sum_j B_j(t) z^{-j}$ respectively. We remark that the proof relies on the above choice. It is evident that this choice also ensures that all the coefficients $G_j(t)$'s and $B_j(t)$'s are holomorphic where the $\widehat{A}_j(t)$'s are. Note that (5.17) yields $B_1(t) = \widehat{A}_{11}^{(1)} \oplus \cdots \oplus \widehat{A}_{ss}^{(1)}(t)$. $\square$

5.3. Fundamental Solutions of (5.13)

The system (5.13) admits block-diagonal fundamental solutions $\widetilde{Y}(z, t) = \widetilde{Y}_1(z, t) \oplus \cdots \oplus \widetilde{Y}_s(z, t)$. Here, $\widetilde{Y}_i(z, t)$ is a $p_i \times p_i$ fundamental matrix of the i-th diagonal block of (5.13). The problem is reduced to solving a system whose leading matrix has only one eigenvalue. The case when $A_0(t)$ has distinct eigenvalues for $|t|$ small is well known (see [**HS66**], and also [**BJL79a**] for the t-independent case). The case when $A_0(0) = \Lambda$ is diagonalisable, with $s \leq n$ distinct eigenvalues, will be studied here and in the subsequent sections.
We do another gauge transformation

$$\widetilde{Y}(z, t) = e^{\Lambda z} \, Y_{red}(z, t), \tag{5.21}$$

where the subscript *red* stand for “rank reduced". We substitute into (5.13) and find

$$\cancel{e^{\Lambda z}} (\Lambda Y_{red} + Y'_{red}) = B(z, t) \cancel{e^{\Lambda z}} Y_{red}.$$

The exponentials cancel because $B(z, t)$ is block diagonal with the same structure as Λ. Thus, we obtain

$$\frac{dY_{red}}{dz} = \frac{1}{z} B_{red}(z, t) \; Y_{red}, \tag{5.22}$$

with

$$B_{red}(z,t) := z(B(z,t) - \Lambda) \ = B_1^{(red)}(z,t) \oplus \cdots \oplus B_s^{(red)}(z,t), \tag{5.23}$$

$$B_{red}(z,t) \sim z(\widehat{A}_0(t) - \Lambda) + \sum_{k=1}^{\infty} B_k(t) z^{-k+1}. \tag{5.24}$$

Fundamental solutions can be taken with block diagonal structure,

$$Y_{red}(z,t) = Y_1^{(red)}(z,t) \oplus \cdots \oplus Y_s^{(red)}(z,t).$$

where $Y_i^{(red)}(z,t)$ solves

$$\frac{dY_i^{(red)}}{dz} = \frac{1}{z} B_i^{(red)}(z,t) \ Y_i^{(red)}.$$

The exponential $e^{\Lambda z}$ commutes with the above matrices, hence a fundamental solution of (5.1) exists in the form

$$Y(z,t) = G_0(t) G(z,t) Y_{red}(z,t) \ e^{\Lambda z}.$$

We proceed as follows. In Section 5.4 we describe the structure of fundamental solutions of (5.1) for $t = 0$ fixed. In Section 5.5 we describe the structure of fundamental solutions at other points $t \in \mathcal{U}_{\epsilon_0}(0)$.

5.4. A Fundamental Solution of (5.1) at $t = 0$

At $t = 0$, *the rank is reduced*, since the system (5.22) becomes a Fuchsian system in $\overline{S}(\alpha,\beta)$,

$$\frac{dY_{red}}{dz} = \frac{1}{z} B_{red}(z,0) \ Y_{red}, \tag{5.25}$$

with $B_{red}(z,0) \sim \sum_{k=1}^{\infty} B_k(0) z^{-k+1}$ for $z \to \infty$ in $\overline{S}(\alpha,\beta)$. Let J_i be a Jordan form of the i-th block $B_i^{(1)}(0) = \widehat{A}_{ii}^{(1)}(0) \equiv A_{ii}^{(1)}(0)$, $1 \le i \le s$. Following [**Was65**], we choose J_i arranged into $h_i \le p_i$ Jordan blocks $J_1^{(i)}$, ..., $J_{h_i}^{(i)}$

$$J_i = J_1^{(i)} \oplus \cdots \oplus J_{h_i}^{(i)}. \tag{5.26}$$

Each block $J_j^{(i)}$, $1 \le j \le h_i$, has dimension $r_j \times r_j$, with $r_j \ge 1$, $r_1 + \cdots + r_{h_i} = p_i$. Each $J_j^{(i)}$ has only one eigenvalue $\mu_j^{(i)}$, with structure,

$$J_j^{(i)} = \mu_j^{(i)} I_{r_j} + H_{r_j}, \qquad I_{r_j} = \ r_j \times r_j \text{ identity matrix},$$

$$H_{r_j} = 0 \ \text{ if } r_j = 1, \qquad H_{r_j} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \ \text{ if } r_j \ge 2.$$

Note that $\mu_1^{(i)}$, ..., $\mu_{h_i}^{(i)}$ are not necessarily distinct. One can choose a t-independent matrix $\Delta_0 = \Delta_1^{(0)} \oplus \cdots \oplus \Delta_s^{(0)}$, in the block-diagonal of Remark 5.3, such that $(\Delta_i^{(0)})^{-1} \widehat{A}_{ii}^{(1)}(0) \Delta_i^{(0)} = J_i$. Hence,

$$\Delta_0^{-1} \widehat{A}_1(0) \Delta_0 \equiv \Delta_0^{-1} A_1(0) \Delta_0 = \begin{bmatrix} J_1 & * & * & * \\ * & J_2 & & * \\ * & * & \ddots & * \\ * & * & * & J_s \end{bmatrix}.$$

The transformation $Y_{red} = \Delta_0 X_{red}$ of the system (5.25) yields[4]

$$\frac{dX_{red}}{dz} = \frac{1}{z}\mathcal{B}_{red}(z)\ X_{red}, \qquad \mathcal{B}_{red}(z) := \Delta_0^{-1} B_{red}(z,0)\Delta_0, \tag{5.27}$$

$$\mathcal{B}_{red}(z) \sim J + \sum_{k=1}^{\infty} \frac{\mathcal{B}_{k+1}}{z^k}, \qquad \mathcal{B}_k = \Delta_0^{-1} B_k(0)\Delta_0.$$

The system (5.27) has block-diagonal fundamental solutions $X_{red} = X_1^{(red)} \oplus \cdots \oplus X_1^{(red)}$, each block satisfying

$$\frac{dX_i^{(red)}}{dz} = \frac{1}{z}\mathcal{B}_i^{(red)}(z)\ X_i^{(red)}, \qquad 1 \le i \le s. \tag{5.28}$$

Now, J_i has the unique decomposition

$$J_i = D_i + S_i, \qquad D_i = \text{ diagonal matrix of integers}, \tag{5.29}$$

$$S_i = \text{ Jordan form with diagonal elements of real part} \in [0,1). \tag{5.30}$$

For $i = 1, 2, ..., s$, let $m_i \ge 0$ be the maximum integer difference between couples of eigenvalues of J_i ($m_i = 0$ if eigenvalues do not differ by integers). Let $\overline{m} := \max_{i=1,..,s} m_i$. The general theory of Fuchsian systems assures that (5.28) has a fundamental matrix solution

$$X_i^{(red)}(z) = K_i(z)\ z^{D_i} z^{L_i}, \qquad K_i(z) \sim I + \sum_{j=1}^{\infty} K_j^{(i)} z^{-j}, \qquad z \to \infty \text{ in } \overline{S}(\alpha,\beta).$$

Here $L_i := S_i + R_i$, where the matrix R_i is a sum $R_i = R_{(1),i} + \cdots R_{(m_i),i}$, whose terms satisfy

$$[R_{(l),i}]_{block\ a,b} \ne 0 \quad \text{ only if } \quad \mu_b^{(i)} - \mu_a^{(i)} = l > 0 \text{ integer}. \tag{5.31}$$

Let

$$D := D_1 \oplus \cdots \oplus D_s, \qquad S := S_1 \oplus \cdots \oplus S_s, \qquad R := R_1 \oplus \cdots \oplus R_s, \qquad L := R + S. \tag{5.32}$$

Observe now that R has a sum decomposition

$$R = R_{(1)} + R_{(2)} + \cdots + R_{(\overline{m})}, \tag{5.33}$$

where $R_{(l)} = R_{(l),1} \oplus \cdots \oplus R_{(l),s}$. Here it is understood that $R_{(l),i} = 0$ if $m_i < l \le \overline{m}$. We conclude that

$$X_{red}(z) = K(z)\ z^D z^L, \qquad K(z) \sim I + \sum_{j=1}^{\infty} K_j z^{-j}, \qquad z \to \infty \text{ in } \overline{S}(\alpha,\beta),$$

$$K(z) := K_1(z) \oplus \cdots \oplus K_s(z), \qquad K_j = K_1^{(j)} \oplus \cdots \oplus K_s^{(j)}.$$

Hence, there is a fundamental solution of (5.1) at $t = 0$, of the form

$$\mathring{Y}(z) := G(z,0)\ \Delta_0 K(z)\ z^D z^L\ e^{\Lambda z}.$$

This is rewritten as,

$$\mathring{Y}(z) = \Delta_0 \mathcal{G}(z)\ z^D z^L\ e^{\Lambda z},$$

[4] The gauge transformation $\widetilde{Y}(z,0) = \Delta_0 X(z)$, of the system (5.13) at $t = 0$ yields,

$$\frac{dX}{dz} = \mathcal{B}(z)\widetilde{X}, \qquad \mathcal{B}(z) := \Delta_0^{-1} B(z,0)\Delta_0, \qquad \mathcal{B}(z) \sim \Lambda + \frac{J}{z} + \sum_{k=2}^{\infty} \frac{\mathcal{B}_k}{z^k}, \qquad \mathcal{B}_k := \Delta_0^{-1} B_k(0)\Delta_0.$$

where $\mathcal{G}(z) := \Delta_0^{-1} G(z,0)\Delta_0 K(z)$. Clearly,

$$\mathcal{G}(z) \sim I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k} := \Big(I + \sum_{k=1}^{\infty} \Delta_0^{-1} G_k(0)\Delta_0\Big)\ \Big(I + \sum_{k=1}^{\infty} K_k z^{-k}\Big), \qquad z \to \infty \text{ in } \overline{S}(\alpha,\beta). \tag{5.34}$$

The results above can be summarized in the following theorem:

THEOREM 5.2. *Consider the system (5.1) satisfying the assumptions of Theorem 5.1. There exist an invertible block-diagonal matrix Δ_0 and a matrix $\mathcal{G}(z)$, holomorphic for $|z| > N$, $z \in \bar{S}(\alpha,\beta)$, with asymptotic expansion*

$$\mathcal{G}(z) \sim I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k}, \quad z \to \infty, \quad z \in \bar{S}(\alpha,\beta), \tag{5.35}$$

such that the gauge transformation $Y(z,0) = \Delta_0 \mathcal{G}(z)\mathcal{Y}(z)$ transforms (5.1) at $t = 0$ into a blocked-diagonal system

$$\frac{d\mathcal{Y}}{dz} = \left[\Lambda + \frac{1}{z}\left(J + \frac{R_{(1)}}{z} + \cdots + \frac{R_{(\overline{m})}}{z^{\overline{m}}}\right)\right]\mathcal{Y}, \qquad J = J_1 \oplus \cdots \oplus J_s, \tag{5.36}$$

where J_i is a Jordan form of $A_{ii}^{(1)}(0) = \widehat{A}_{ii}^{(1)}(0)$, $1 \le i \le s$, and the $R_{(l)}$, $1 \le l \le \overline{m}$ are defined in (5.31)-(5.33). The system (5.36) has a fundamental solution $\mathcal{Y}(z) = z^D z^L\, e^{\Lambda z}$, hence (5.1) restricted at $t = 0$ has a fundamental solution,

$$\mathring{Y}(z) = \Delta_0 \mathcal{G}(z)\ z^D z^L\ e^{\Lambda z}. \tag{5.37}$$

The matrices D, L are defined in (5.29), (5.30) and (5.32). The matrix Δ_0 satisfies

$$\Delta_0^{-1} A_1(0)\ \Delta_0 = \begin{bmatrix} J_1 & * & * & * \\ * & J_2 & & * \\ * & * & \ddots & * \\ * & * & * & J_s \end{bmatrix}.$$

REMARK 5.7. Observe that (5.37) does not solve (5.1) for $t \neq 0$.

DEFINITION 5.2. The matrix

$$\mathring{Y}_F(z) := \Delta_0 F(z)\ z^D z^L\ e^{\Lambda z}, \qquad F(z) := I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k}\ , \tag{5.38}$$

is called a **formal solution** of (5.1) for $t = 0$ and $A_0(0) = \Lambda$.

Notice that we use the notation $\mathring{Y}$ for solutions of the system with $t = 0$. For fixed Δ_0, D, L and Λ the formal solution is in general not unique. See Corollary 5.1.

We note that (5.38) can be transformed into a formal solution with the structure described in [**BJL79c**], but the specific form (5.38) is more refined and is obtainable by an explicit construction from the differential system (see also Section 5.4.1 below).

5.4.1. Explicit computation of the $\mathring{F}_k$'s and R of (5.35) and (5.36). Uniqueness of Formal Solutions. We present the computation of the $\mathring{F}_k$'s in (5.35) and R in (5.33). This serves for two reasons. First, the details of the computation in itself will be used later, starting from section

5.4.2. Second, it yields the Corollary 5.1 below concerning the (non-)uniqueness of formal solutions. Consider the gauge transformation $Y = \Delta_0 \widehat{X}$ at $t = 0$, which transforms (5.1) into

$$\frac{d\widehat{X}}{dz} = \Big(\Delta_0^{-1} A(z,0)\Delta_0\Big)\widehat{X}(z),$$
$$\Delta_0^{-1} A(z,0)\Delta_0 = \Lambda + \sum_{j=1}^{\infty} \mathcal{A}_j z^{-j}, \qquad \mathcal{A}_j := \Delta_0^{-1} A_j(0)\Delta_0.$$

The recurrence equations (5.17), (5.18) become (using F_l instead of G_l),

$$\Lambda F_1 - F_1\Lambda = -\mathcal{A}_1 + B_1, \qquad \text{with } \ \mathrm{diag}(\mathcal{A}_1) = J, \tag{5.39}$$

$$\Lambda F_l - F_l\Lambda = \Big[\sum_{j=1}^{l-1}\Big(F_j B_{l-j} - \mathcal{A}_{l-j}F_j\Big) - \mathcal{A}_l\Big] \ - (l-1)F_{l-1} \ + B_l. \tag{5.40}$$

Proposition 5.1. *(5.39)-(5.40) admit a solution $\{F_k\}_{k\geq 1}$, $\{B_k\}_{k\geq 1}$ which satisfies,*

$$B_1 = J,$$
$$B_2 = R_{(1)}, \quad \dots, \quad B_{\overline{m}+1} = R_{(\overline{m})},$$
$$B_k = 0 \ \text{ for any } k \geq \overline{m}+2,$$

where $R_{(l)} = R_{(l),1} \oplus \cdots \oplus R_{(l),s}$, and each $R_{(l),i}$ is as in (5.31). The F_k's so obtained are exactly the coefficients $\mathring{F}_k$ of the asymptotic expansion of the gauge transformation (5.35), which yields (5.36).

Proof: Let $\mathcal{K}_l := \Big[\sum_{j=1}^{l-1}\Big(F_j B_{l-j} - \mathcal{A}_{l-j}F_j\Big) - \mathcal{A}_l\Big]$, and rewrite (5.39) and (5.40) in blocks i, j:

- For $l = 1$ ($[i,j]$ is the block index, $1 \leq i, j \leq s$):

$$\Lambda F_1 - F_1\Lambda = -\mathcal{A}_1 + B_1 \quad \Longrightarrow \quad (\lambda_i - \lambda_j)F_{ij}^{(1)} = -\mathcal{A}_{ij}^{(1)} \ + B_{ij}^{(1)}.$$

- For $l \geq 2$:

$$\Lambda F_l - F_l\Lambda = \mathcal{K}_j - (l-1)F_{l-1} + B_l \quad \Longrightarrow \quad (\lambda_i - \lambda_j)F_{ij}^{(l)} = \mathcal{K}_{ij}^{(l)} - (l-1)F_{ij}^{(l-1)} \ + B_{ij}^{(l)}.$$

- For $l = 1$ we find:

– If $i = j$:

$$B_{ii}^{(1)} = \mathcal{A}_{ii}^{(1)} \equiv J_i, \qquad F_{ii}^{(1)} \text{ not determined.}$$

– If $i \neq j$:

$$F_{ij}^{(1)} = -\frac{\mathcal{A}_{ij}^{(1)}}{\lambda_i - \lambda_j}, \qquad B_{ij}^{(1)} = 0.$$

- For $l \geq 2$ we find:

– If $i \neq j$:

$$F_{ij}^{(l)} = (\lambda_i - \lambda_j)^{-1}\Big(\mathcal{K}_{ij}^{(l)} - (l-1)F_{ij}^{(l-1)}\Big), \qquad B_{ij}^{(l)} = 0.$$

In the r.h.s. matrix entries of F_1, ..., F_{l-1} appear, therefore the equation determines $F_{ij}^{(l)}$.

– If $i=j$:

$$0=\mathcal{K}_{ii}^{(l)}-(l-1)F_{ii}^{(l-1)}\ +B_{ii}^{(l)}. \tag{5.41}$$

We observe that in $\mathcal{K}_{ii}^{(l)}$ the matrix entries of F_1, ..., F_{l-1} appear, including the entry $F_{ii}^{(l-1)}$. Keeping into account that $B_1=\mathcal{A}_{11}^{(1)}\oplus\cdots\oplus\mathcal{A}_{ss}^{(1)}$, we explicitly write (5.41):

$$(l-1)F_{ii}^{(l-1)}=\sum_{k=1}^{s}\Big(F_{ik}^{(l-1)}B_{ki}^{(1)}-\mathcal{A}_{ik}^{(1)}F_{ki}^{(l-1)}\Big)+\sum_{j=1}^{l-2}\Big(F_jB_{l-j}-\mathcal{A}_{l-j}F_j\Big)_{[i,i]}\ -\mathcal{A}_{ii}^{(l)}+B_{ii}^{(l)}=$$

$$=F_{ii}^{(l-1)}\mathcal{A}_{ii}^{(1)}-\mathcal{A}_{ii}^{(1)}F_{ii}^{(l-1)}-\sum_{k\neq i}\mathcal{A}_{ik}^{(1)}F_{ki}^{(l-1)}+\sum_{j=1}^{l-2}\Big(F_jB_{l-j}-\mathcal{A}_{l-j}F_j\Big)_{[i,i]}\ -\mathcal{A}_{ii}^{(l)}+B_{ii}^{(l)}.$$

Thus, keeping into account that $\mathcal{A}_{ii}^{(1)}=J_i$, the above is rewritten as follows:

$$\Big(J_i+l-1\Big)F_{ii}^{(l-1)}-F_{ii}^{(l-1)}J_i=-\sum_{k\neq i}\mathcal{A}_{ik}^{(1)}F_{ki}^{(l-1)}+\sum_{j=1}^{l-2}\Big(F_jB_{l-j}-\mathcal{A}_{l-j}F_j\Big)_{[i,i]}\ -\mathcal{A}_{ii}^{(l)}+B_{ii}^{(l)}. \tag{5.42}$$

In the r.h.s. every term is determined by previous steps (diagonal elements $F_{jj}^{(k)}$ appear up to $k\le l-2$), except for $B_{ii}^{(l)}$, which is still undetermined. (5.42) splits into the blocks inherited from $J_i=J_1^{(i)}\oplus\cdots\oplus J_{h_i}^{(i)}$. Let the eigenvalues of J_i be $\mu_1^{(i)}$, ..., $\mu_{h_i}^{(i)}$, $h_i\le p_i$. Then (for $l\ge 2$),

$$\Big(\mu_a^{(i)}+l-1+H_{r_a}\Big)[F_{ii}^{(l-1)}]_{ab}-[F_{ii}^{(l-1)}]_{ab}(\mu_b^{(i)}+H_{r_b})=$$

$$=\left[-\sum_{k\neq i}\mathcal{A}_{ik}^{(1)}F_{ki}^{(l-1)}+\sum_{j=1}^{l-2}\Big(F_jB_{l-j}-\mathcal{A}_{l-j}F_j\Big)_{[i,i]}\ -\mathcal{A}_{ii}^{(l)}+B_{ii}^{(l)}\right]_{ab}. \tag{5.43}$$

Here $[\cdots]_{ab}$ denotes a block, with $1\le a,b\le h_i$.

• If $\underline{\mu_b^{(i)}-\mu_a^{(i)}=l-1}$, the l.h.s. of (5.43) is $H_{r_a}[F_{ii}^{(l-1)}]_{ab}-[F_{ii}^{(l-1)}]_{ab}H_{r_b}$. The homogeneous equation $H_{r_a}[F_{ii}^{(l-1)}]_{ab}-[F_{ii}^{(l-1)}]_{ab}H_{r_b}=0$ has non trivial solutions, depending on parameters, since the matrices H_{r_a} and H_{r_b} have common eigenvalue. One can then choose F_{ii} to be a solution of the homogeneous equation, and determine $[B_{ii}^{(l)}]_{ab}\neq 0$ by imposing that the r.h.s. of (5.43) is equal to 0.

• If $\underline{\mu_b^{(i)}-\mu_a^{(i)}\neq l-1}$, the choice $[B_{ii}^{(l)}]_{ab}=0$ is possible and $[F_{ii}^{(l-1)}]_{ab}$ is determined.

We conclude that

$$[B_{ii}^{(l+1)}]_{ab}\neq 0 \quad \text{only if} \quad \mu_b^{(i)}-\mu_a^{(i)}=l>0 \quad \text{integer}.$$

This means that $[B_{ii}^{(l+1)}]_{ab}=[R_{(l),i}]_{ab}$. □

Corollary 5.1 (Uniqueness of Formal Solution at $t=0$). *A formal solution (5.38) with given Δ_0, D, L, Λ is unique if and only if for any $1\le i\le s$ the eigenvalues of $\widehat{A}_{ii}^{(1)}(0)$ do not differ by a non-zero integer.*

Proof: Computations above show that $\{F_k\}_{k=1}^{\infty}$ is not uniquely determined if and only if some $\mu_b^{(i)}-\mu_a^{(i)}=l-1$, for some $l\ge 2$, some $i\in\{1,2,...,s\}$, and some a,b. □

5.4.2. Special sub-case with $R = 0$, J diagonal, $\Delta_0 = I$. A sub-case is very important for the discussion to come, occurring when $\Delta_0 = I$ and $A_{ii}^{(1)}(0)$ is diagonal. Clearly, if $\Delta_0 = I$, then $J_i = A_{ii}^{(1)}(0)$. Hence, if $\Delta_0 = I$, then J is diagonal if and only if $\left(\widehat{A}_{ii}^{(1)}(0)\right)_{pq} = 0$ for any $1 \leq p \neq q \leq p_i$.

PROPOSITION 5.2. *There exists a fundamental solution (5.37) at $t = 0$ in a simpler form*

$$\mathring{Y}(z) = \mathcal{G}(z) z^{B_1(0)} e^{\Lambda z}, \tag{5.44}$$

with $\Delta_0 = I$, $J = B_1(0) = \operatorname{diag}(A_1(0))$ diagonal, and

$$\mathcal{G}(z) \sim I + \sum_{k=1}^{\infty} \mathring{F}_k z^{-k}, \qquad z \to \infty \quad \text{in } \overline{S}(\alpha, \beta), \tag{5.45}$$

if and only if the following conditions hold:

- *For every $i \in \{1, 2, ..., s\}$, and every p, q, with $1 \leq p \neq q \leq p_i$, then*

$$\left(\widehat{A}_{ii}^{(1)}(0)\right)_{pq} = 0. \tag{5.46}$$

- *If $\left(\widehat{A}_{ii}^{(1)}(0)\right)_{pp} - \left(\widehat{A}_{ii}^{(1)}(0)\right)_{qq} + l - 1 = 0$, for some $l \geq 2$, some $i \in \{1, 2, ..., s\}$, and some diagonal entries $\left(\widehat{A}_{ii}^{(1)}(0)\right)_{pp}$, $\left(\widehat{A}_{ii}^{(1)}(0)\right)_{qq}$, then*

$$\sum_{k \neq i}^{s} \left(\widehat{A}_{ik}^{(1)}(0)\ \mathring{F}_{ki}^{(l-1)}\right)_{pq} + \sum_{j=1}^{l-2} \sum_{k=1}^{s} \left(\widehat{A}_{ik}^{(l-j)}(0)\ \mathring{F}_{ki}^{(j)}\right)_{pq} + \left(\widehat{A}_{ii}^{(l)}(0)\right)_{pq} = 0, \tag{5.47}$$

for those values of l, i, p and q.

Proof: We only need to clarify (5.47), while (5.46) has already been motivated. We solve (5.39), (5.40) when $\Delta_0 = I$, namely (recall that $\widehat{A}_j(0) \equiv A_j(0)$) (we write F_l, as in (5.39), (5.40), but it is clear that the result of the computation will be the $\mathring{F}_l$ appearing in (5.45)):

$$\Lambda F_1 - F_1 \Lambda = -\widehat{A}_1(0) + B_1,$$

$$\Lambda F_l - F_l \Lambda = \left[\sum_{j=1}^{l-1} \left(F_j B_{l-j} - \widehat{A}_{l-j}(0) F_j\right) - \widehat{A}_l(0)\right] - (l-1) F_{l-1} + B_l.$$

At level $l = 1$:

$$B_1 = \operatorname{diag} \widehat{A}_1(0), \qquad F_{ij}^{(1)} = -\frac{\widehat{A}_{ij}(0)}{\lambda_i - \lambda_j}.$$

At level $l \geq 2$,

$$F_{ij}^{(l)} = \frac{\mathcal{K}_{ij}^{(l)} - (l-1) F_{ij}^{(l-1)}}{\lambda_i - \lambda_j}, \qquad B_{ij}^{(l)} = 0,$$

where $\mathcal{K}_l = \left[\sum_{j=1}^{l-1} \left(F_j B_{l-j} - \widehat{A}_{l-j}(0) F_j\right) - \widehat{A}_l(0)\right]$. Formula (5.43) reads

$$\left(\mu_a^{(i)} - \mu_b^{(i)} + l - 1\right) [F_{ii}^{(l-1)}]_{ab} = \left[-\sum_{k \neq i} \widehat{A}_{ik}^{(1)}(0) F_{ki}^{(l-1)} + \sum_{j=1}^{l-2} \left(F_j B_{l-j} - \widehat{A}_{l-j}(0) F_j\right)_{[i,i]} - \widehat{A}_{ii}^{(l)}(0) + B_{ii}^{(l)}\right]_{ab}.$$

Indices above are block indices. The above can be re-written in terms of the matrix entries,

$$\Big((\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+l-1\Big)(F^{(l-1)}_{ii})_{pq}=$$
$$=\left[-\sum_{k\neq i}\widehat{A}^{(1)}_{ik}(0)F^{(l-1)}_{ki}+\sum_{j=1}^{l-2}\Big(F_jB_{l-j}-\widehat{A}_{l-j}(0)F_j\Big)_{[i,i]}-\widehat{A}^{(l)}_{ii}(0)+B^{(l)}_{ii}\right]_{entry\ pq}.$$

- If $(\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+l-1\neq 0$, choose $B^{(l)}_{ii}=0$ and determine $(F^{(l-1)}_{ii})_{pq}$.
- If $(\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+l-1=0$, by induction assume that the $B_{l-j}=0$. Then the equation is satisfied for any $(F^{(l-1)}_{ii})_{pq}$ and for

$$(B^{(l)}_{ii})_{pq}=\left[\sum_{k\neq i}\widehat{A}^{(1)}_{ik}(0)F^{(l-1)}_{ki}+\sum_{j=1}^{l-2}\Big(\widehat{A}_{l-j}(0)F_j\Big)_{block\ [i,i]}+\widehat{A}^{(l)}_{ii}(0)\right]_{entry\ pq}.$$

Then, if we impose that $(B^{(l)}_{ii})_{pq}=0$ we obtain the necessary and sufficient condition (5.47). The proof by induction is justified because at the first step, namely $l=2$, we need to solve

$$\Big((\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+1\Big)\,(F^{(1)}_{ii})_{pq}=-\sum_{k\neq i}^{n}(\widehat{A}^{(1)}_{ik}(0)\ F^{(1)}_{ki})_{pq}-(\widehat{A}^{(2)}_{ii}(0))_{pq}+(B^{(2)}_{ii})_{pq}. \tag{5.48}$$

If $(\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+1\neq 0$, the above has a unique solution for any choice of $(B^{(2)}_{ii})_{pq}$. We choose $(B^{(2)}_{ii})_{pq}=0$. If $\widehat{A}^{(1)}_{ii}(0))_{pp}-(\widehat{A}^{(1)}_{ii}(0))_{qq}+1=0$, the equation leaves the choice of $(F^{(1)}_{ii})_{pq}$ free, and determines

$$(B^{(2)}_{ii})_{pq}=\sum_{k\neq i}^{n}(\widehat{A}^{(1)}_{ik}(0)\ F^{(1)}_{ki})_{pq}+(\widehat{A}^{(2)}_{ii}(0))_{pq}=-\sum_{k\neq i}^{n}\frac{(\widehat{A}^{(1)}_{ik}(0)\widehat{A}^{(1)}_{ki}(0))_{pq}}{\lambda_k-\lambda_i}+(\widehat{A}^{(2)}_{ii}(0))_{pq}.$$

We can choose $(B^{(2)}_{ii})_{pq}=0$ if and only if

$$(\widehat{A}^{(2)}_{ii}(0))_{pq}=\sum_{k\neq i}^{n}\frac{(\widehat{A}^{(1)}_{ik}(0)\widehat{A}^{(1)}_{ki}(0))_{pq}}{\lambda_k-\lambda_i}, \tag{5.49}$$

which is precisely (5.47) for $l=2$. □

5.5. Solutions for $t\in\mathcal{U}_{\epsilon_0}(0)$ with $A_0(t)$ Holomorphically Diagonalisable.

In the previous section, we have constructed fundamental solutions at the coalescence point $t=0$. Now, we let t vary in $\mathcal{U}_{\epsilon_0}(0)$. In Sibuya Theorem, $\widehat{A}_0(t)=\widehat{A}^{(0)}_{11}(t)\oplus\cdots\oplus\widehat{A}^{(0)}_{ss}(t)$ is neither diagonal nor in Jordan form, except for $t=0$. $A_0(t)$ admits a Jordan form at each point of $\mathcal{U}_{\epsilon_0}(t)$, but in general this similarity is not realizable by a holomorphic transformation. In order to procede, we need the following fundamental assumption (implicitly supposed to hold true in the Introduction).

Assumption 5.1. For $|t|\leq\epsilon_0$ sufficiently small and such that Lemma 5.1 and Theorem 5.1 apply, we assume that $A_0(t)$ is holomorphically similar to a diagonal form $\Lambda(t)$, namely there exists a holomorphic invertible $G_0(t)$ for $|t|\leq\epsilon_0$ such that

$$G_0(t)^{-1}A_0(t)\ G_0(t)=\Lambda(t)\equiv\mathrm{diag}\big(u_1(t),u_2(t),...,u_n(t)\big),$$

with $A_0(0)=\Lambda$, $G_0(0)=I$.

REMARK 5.8. Assumption 5.1 is equivalent to the assumption that $A_0(t)$ is holomorphically similar to its Jordan form. The requirement implies by continuity that the Jordan form is diagonal, being equal to $\Lambda = \Lambda(0)$ at $t = 0$.

With Assumption 5.1, we can represent the eigenvalues as well defined holomorphic functions $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ such that

$$u_1(0) = \cdots = u_{p_1}(0) \quad = \lambda_1, \tag{5.50}$$
$$u_{p_1+1}(0) = \cdots = u_{p_1+p_2}(0) \quad = \lambda_2, \tag{5.51}$$
$$\vdots \tag{5.52}$$
$$u_{p_1+\cdots+p_{s-1}+1}(0) = \cdots = u_{p_1+\cdots+p_{s-1}+p_s}(0) \quad = \lambda_s. \tag{5.53}$$

Moreover,

$$\Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t) \oplus \ \cdots \ \oplus \Lambda_s(t),$$

where $\Lambda_1(t)$, ..., $\Lambda_s(t)$ are diagonal matrices of dimensions respectively p_1, ..., p_s, such that $\Lambda_j(t) \to \lambda_j I_{p_j}$ for $t \to 0$, $j = 1, ..., s$. For example, $\Lambda_1(t) = \mathrm{diag}(u_1(t), ..., u_{p_1}(t))$, and so on. Any two matrices $\Lambda_i(t)$ and $\Lambda_j(t)$ have no common eigenvalues for $i \neq j$ and small ϵ_0.

The coalescence locus in $\mathcal{U}_{\epsilon_0}(0)$ is explicitly written as follows

$$\Delta := \bigcup_{\substack{a \neq b \\ a,b = 1,...,m}} \Big\{t \in \mathbb{C}^m \text{ such that: } |t| \leq \epsilon_0 \text{ and } u_a(t) = u_b(t)\Big\}.$$

We can also write

$$\Delta = \bigcup_{i=1}^{s} \Delta_i,$$

where Δ_i is the coalescence locus of $\Lambda_i(t)$. For $m = 1$, Δ is a finite set of isolated points.

IMPROVEMENT OF THEOREM 5.1: *With the same assumptions and notations as of Theorem 5.1, if Assumption 5.1 holds, then*

$$B(z,t) \sim \Lambda(t) + \sum_{k \geq 1} B_k(t) z^{-k}, \qquad z \to \infty \ \textit{in} \ \overline{S}(\alpha, \beta).$$

With Assumption 5.1, we can replace the gauge trasfromation (5.21) with

$$\widetilde{Y}(z,t) = e^{\Lambda(t)z} \, Y_{red}(z,t).$$

Since $\widehat{A}_0(t) = \Lambda(t)$, then $B_{red}(z,t) \sim \sum_{k=1}^{\infty} B_k(t) z^{-k+1}$. Hence the reduced system (5.22) is Fuchsian also for $t \neq 0$. The recursive relations (5.17) and (5.18) become $B_0(t) = \Lambda(t)$ for $l = 0$, and:

For $l = 1$:

$$\Lambda(t)G_1 - G_1\Lambda(t) = -\widehat{A}_1(t) + B_1. \tag{5.54}$$

For $l \geq 2$:

$$\Lambda(t)G_l - G_l\Lambda(t) = \Big[\sum_{j=1}^{l-1} \Big(G_j B_{l-j} - \widehat{A}_{l-j}(t) G_j\Big) - \widehat{A}_l(t)\Big] \ - (l-r)G_{l-r} \ + B_l. \tag{5.55}$$

As for Theorem 5.1, the choice which yields holomorphic $G_l(t)$'s and $B_l(t)$'s is (5.19) and (5.20). Generally speaking, it is not possible to choose the $B_l(t)$'s diagonal for $l\geq 2$, because such a choice would give $G_k(t)$'s *diverging* at the locus Δ.

5.5.1. Fundamental Solution in a neighbourhood of $t_0\notin\Delta$, with Assumption 5.1. Let Assumption 5.1 hold. Theorem 5.1 has been formulated in a neighbourhood of $t=0$, with block partition of $A_0(0)=\Lambda_1\oplus\cdots\oplus\Lambda_s$. Theorem 5.1 can also be formulated in a neighbourhood (polydisc) of a point $t_0\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, of the form

$$\mathcal{U}_{\rho_0}(t_0):=\{t\in\mathbb{C}\mid |t-t_0|\leq\rho_0\}\subset\mathcal{U}_{\epsilon_0}(0),$$

$$\mathcal{U}_{\rho_0}(t_0)\cap\Delta=\emptyset,$$

where $\Lambda(t)$ has distinct eigenvalues, provided that $\rho_0>0$ is small enough. In order to do this, we need to introduce sectors. To this end, consider a fixed point t_* in $\mathcal{U}_{\epsilon_0}(0)$, and the eigenvalues $u_1(t_*)$, ..., $u_n(t_*)$ of $\Lambda(t_*)$. We introduce an admissible direction $\eta^{(t_*)}$ such that

$$\eta^{(t_*)}\neq\arg_p\Big(u_a(t_*)-u_b(t_*)\Big)\mod(2\pi),\quad\forall\ 1\leq a\neq b\leq n. \tag{5.56}$$

There are $2\mu_{t_*}$ determinations satisfying $\eta^{(t_*)}-2\pi<\widehat{\arg}(u_a(t_*)-u_b(t_*))<\eta^{(t_*)}$. They will be numbered as

$$\eta^{(t_*)}>\eta_0^{(t_*)}>\ \cdots\ >\eta_{2\mu^{(t_*)}-1}>\eta^{(t_*)}-2\pi.$$

Correspondingly, we introduce the directions

$$\tau^{(t_*)}:=\frac{3\pi}{2}-\eta^{(t_*)},\qquad \tau_\nu^{(t_*)}=\frac{3\pi}{2}-\eta_\nu^{(t_*)},\qquad 0\leq\nu\leq 2\mu_{t_*}-1,$$

satisfying

$$\tau^{(t_*)}<\tau_0^{(t_*)}<\tau_1^{(t_*)}<\cdots<\tau_{2\mu_{t_*}-1}^{(t_*)}<\tau^{(t_*)}+2\pi.$$

The following relation defines $\tau_\sigma^{(t_*)}$ for any $\sigma\in\mathbb{Z}$, represented as $\sigma=\nu+k\mu_{t_*}$:

$$\tau_{\nu+k\mu_{t_*}}:=\tau_\nu^{(t_*)}+k\pi,\qquad \nu\in\{0,1,...,\mu_{t_*}-1\},\qquad k\in\mathbb{Z}.$$

Finally, we introduce the sectors

$$\mathcal{S}_\sigma^{(t_*)}:=S(\tau_\sigma^{(t_*)}-\pi,\tau_{\sigma+1}^{(t_*)}),\qquad \sigma\in\mathbb{Z}.$$

Theorem 5.1 in a neighbourhood of t_0 becomes:

THEOREM 5.3. *Let Assumption 5.1 hold and let $t_0\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. Pick up a sector $\mathcal{S}_\sigma^{(t_0)}=S(\tau_\sigma^{(t_0)}-\pi,\tau_{\sigma+1}^{(t_0)})$, $\sigma\in\mathbb{Z}$, as above. For any closed sub-sector*

$$\overline{S}^{(t_0)}(\alpha,\beta):=\Big\{z\in\mathcal{R}\mid \tau_\sigma^{(t_0)}-\pi<\alpha\leq\arg z\leq\beta<\tau_{\sigma+1}^{(t_0)}\Big\}\subset\mathcal{S}_\sigma^{(t_0)},$$

there exist a sufficiently large positive number N, a sufficiently small positive number ρ and an invertible matrix valued function $G(z,t)$ with the following properties:

i) $G(z,t)$ is holomorphic in (z,t) for $|z|\geq N$, $z\in\overline{S}^{(t_0)}(\alpha,\beta)$, $|t-t_0|\leq\rho$;

ii) $G(z,t)$ has uniform asymptotic expansion for $|t-t_0|\leq\rho$, with holomorphic coefficients $G_k(t)$:

$$G(z,t)\sim I+\sum_{k=1}^\infty G_k(t)z^{-k},\quad z\to\infty,\quad z\in\overline{S}^{(t_0)}(\alpha,\beta),$$

iii) The gauge transformation

$$Y(z,t) = G_0(t)G(z,t)\widetilde{Y}(z,t),$$

reduces the initial system (5.1) to

$$\frac{d\widetilde{Y}}{dz} = B(z,t)\widetilde{Y},$$

where $B(z,t)$ is a diagonal *holomorphic matrix function of (z,t) in the domain $|z| \geq N$, $z \in \overline{S}(\alpha,\beta)$, $|t-t_0| \leq \rho$, with uniform asymptotic expansion and holomorphic coefficients:*

$$B(z,t) \sim \Lambda(t) + \sum_{k=1}^{\infty} B_k(t) z^{-k}, \quad z \to \infty, \quad z \in \overline{S}^{(t_0)}(\alpha,\beta).$$

In particular, $B_1(t) = \operatorname{diag} \widehat{A}_1(t)$.

Remark 5.9. $\overline{S}^{(t_0)}(\alpha,\beta)$ is not the same $\overline{S}(\alpha,\beta)$ of Theorem 5.1 (the latter should be denoted $\overline{S}^{(0)}(\alpha,\beta)$ for consistency of notations). The matrices $G(z,t)$ and $B(z,t)$ *are not the same* of Theorem 5.1. On the other hand, $G_0(t)$ is the same, by Assumption 5.1.

As before, we let $B_{red}(z,t) = z(B(z,t) - \Lambda(t))$. Then the system (5.1) has a fundamental matrix solution

$$Y(z,t) = G_0(t)\mathcal{G}(z,t) z^{B_1(t)} e^{\Lambda(t)z},$$

where $\mathcal{G}(z,t) = G(z,t)K(z,t)$, and

$$K(z,t) = \exp\left\{\int_{\infty}^{z} \frac{B_{red}(\zeta,t) - B_1(t)}{\zeta}\, d\zeta\right\} \sim \exp\left\{\sum_{k=2}^{\infty} B_k(t) \frac{z^{-k+1}}{-k+1}\right\} = I + \sum_{j=1}^{\infty} K_j(t) z^j,$$

$z \to \infty$ in $\overline{S}(\alpha,\beta)$. This result is well known, see [**HS66**]. This proves the first part of the following

Corollary 5.2. *The analogue of Theorem 5.3 holds with a new gauge transformation $\mathcal{G}(z,t)$, enjoying the same asymptotic and analytic properties, such that $Y(z,t) = G_0(t)\mathcal{G}(z,t)\widetilde{Y}(z)$ transforms the system (5.1) into*

$$\frac{d\widetilde{Y}}{dz} = \left(\Lambda(t) + \frac{B_1(t)}{z}\right)\widetilde{Y}, \qquad B_1(t) = \operatorname{diag}\widehat{A}_1(t). \tag{5.57}$$

With the above choice, the system (5.1) has a fundamental solution,

$$Y(z,t) = G_0(t)\mathcal{G}(z,t) z^{B_1(t)} e^{\Lambda(t)z}. \tag{5.58}$$

and $\mathcal{G}(z,t)$ is holomorphic for $z \in \overline{S}^{(t_0)}(\alpha,\beta)$, $|z| \geq N$ and $|t-t_0| \leq \rho$, with expansion

$$\mathcal{G}(z,t) \sim I + \sum_{k=1}^{\infty} F_k(t) z^{-k}, \tag{5.59}$$

for $z \to \infty$ in $\overline{S}^{(t_0)}(\alpha,\beta)$, uniformly in $|t-t_0| \leq \rho$. The coefficients $F_k(t)$ are uniquely determined and holomorphic on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

Proof: The statement is clear from the previous construction. It is only to be justified that the $F_k(t)$'s, $k \geq 1$, are holomorphic functions of $t \notin \Delta$ and uniquely determined. We solve (5.54) and (5.55) for the $F_k(t)$'s, namely

$$\Lambda(t)F_1 - F_1\Lambda(t) = -\widehat{A}_1(t) + B_1,$$
$$\Lambda(t)F_l - F_l\Lambda(t) = \Big[\sum_{j=1}^{l-1}\Big(F_jB_{l-j} - \widehat{A}_{l-j}(t)F_j\Big) - \widehat{A}_l(t)\Big] - (l-1)F_{l-1} + B_l.$$

It is convenient to use the notation $u_1(t)$, ..., $u_n(t)$ for the distinct eigenvalues. Matrix entries are here denoted a, $b \in \{1, 2, ..., n\}$. For $l = 1$,

$$(F_1)_{ab}(t) = -\frac{(\widehat{A}_1)_{ab}(t)}{u_a(t) - u_b(t)}, \qquad (B_1(t))_{ab} = 0, \quad a \neq b.$$
$$(B_1)_{aa}(t) = (\widehat{A}_1)_{aa}(t), \quad \Longrightarrow \quad B_1(t) = \mathrm{diag}(\widehat{A}_1(t)).$$

Now, *impose* that $B_l(t) = 0$ for any $l \geq 2$. Hence, at level $l = 2$ we get:

$$(F_1)_{aa}(t) = -\sum_{b\neq a}(\widehat{A}_1)_{ab}(t)(F_1)_{ba}(t) - (\widehat{A}_2)_{aa}(t).$$

For any $l \geq 2$, we find:

$$\begin{aligned}(F_l)_{ab}(t) \qquad &= -\frac{1}{u_a(t) - u_b(t)}\Big\{\Big[(\widehat{A}_1)_{aa}(t) - (\widehat{A}_1)_{bb}(t) + l - 1\Big](F_{l-1})_{ab}(t) + \\ &+ \sum_{\gamma\neq a}(\widehat{A}_1)_{a\gamma}(t)(F_{l-1})_{\gamma b}(t) + \sum_{j=1}^{l-2}\Big(\widehat{A}_{l-j}(t)F_j(t)\Big)_{ab} + (\widehat{A}_l)_{ab}(t)\Big\}, \quad a \neq b. \\ (l-1)(F_{l-1})_{aa}(t) \quad &= -\sum_{b\neq a}(\widehat{A}_1)_{ab}(t)(F_{l-1})_{ba}(t) - \sum_{j=1}^{l-2}\Big(\widehat{A}_{l-j}(t)F_j(t)\Big)_{aa} - (\widehat{A}_l)_{aa}(t).\end{aligned}$$

The above formulae show that the $F_l(t)$ are uniquely determined, and holomorphic away from Δ. □

The above result has two corollaries:

Proposition 5.3. *The coefficients $F_k(t)$ in the expansion (5.59) are holomorphic at a point $t_\Delta \in \Delta$ if and only if there exists a neighbourhood of t_Δ where*

$$(\widehat{A}_1)_{ab}(t) \tag{5.60}$$

and

$$\Big[(\widehat{A}_1)_{aa}(t) - (\widehat{A}_1)_{bb}(t) + l - 1\Big](F_{l-1})_{ab}(t) + \sum_{\gamma\neq a}(\widehat{A}_1)_{a\gamma}(t)(F_{l-1})_{\gamma b}(t) + \sum_{j=1}^{l-2}\Big(\widehat{A}_{l-j}(t)F_j(t)\Big)_{ab} + (\widehat{A}_l)_{ab}(t) \tag{5.61}$$

vanish as fast as $\mathcal{O}(u_a(t) - u_b(t))$ in the neighbourhood, for those indexes $a, b \in \{1, 2, ..., n\}$ such that $u_a(t)$ and $u_b(t)$ coalesce when t approaches a point of Δ in the neighbourhood. In particular, the $F_k(t)$'s are holomorphic in the whole $\mathcal{U}_{\epsilon_0}(0)$ if and only if (5.60) and (5.61) are zero along Δ.

Remarkably, in the isomonodromic case, we will prove that if we just require vanishing of $(A_1)_{ab}(t)$ then all the complicated expressions (5.61) also vanish consequently.

PROPOSITION 5.4. *If the holomorphic conditions of Proposition 5.3 hold at $t = 0$, then (5.46) and (5.47) are satisfied, with the choice*

$$\mathring{F}_k = F_k(0), \quad k \geq 1.$$

If moreover $\left(\widehat{A}_1(0)\right)_{aa} - \left(\widehat{A}_1(0)\right)_{bb} + l - 1 \neq 0$ for every $l \geq 2$, then the above is the unique choice of the $\mathring{F}_k$'s, according to Corollary 5.1.

Expression (5.61) is a rational function of the matrix entries of $\widehat{A}_1(t)$, ..., $\widehat{A}_l(t)$, since $F_1(t)$,...,$F_{l-1}(t)$ are expressed in terms of $\widehat{A}_1(t)$, ..., $\widehat{A}_l(t)$. For example, for $l = 2$, (5.61) becomes

$$\left((\widehat{A}_1)_{bb}(t) - (\widehat{A}_1)_{aa}(t) - 1\right)\frac{(\widehat{A}_1)_{ab}(t)}{u_a(t) - u_b(t)} + (\widehat{A}_2)_{ab}(t) - \sum_{\gamma \neq a} \frac{(\widehat{A}_1)_{a\gamma}(t)(\widehat{A}_1)_{\gamma b}(t)}{u_\gamma(t) - u_b(t)}. \tag{5.62}$$

EXAMPLE 5.4. The following system does not satisfy the vanishing conditions of Proposition 5.3

$$\widehat{A}(z,t) = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 1 & 0 \\ t & 2 \end{pmatrix}, \quad \Delta = \{t \in \mathbb{C} \mid t = 0\} \equiv \{0\} \tag{5.63}$$

It has a fundamental solution

$$Y(z,t) = \begin{bmatrix} 1 & 0 \\ w(z,t) & 1 \end{bmatrix}\begin{pmatrix} z & 0 \\ 0 & z^2 e^{tz} \end{pmatrix},$$

with

$$w(z,t) := t^2 z e^{tz}\mathrm{Ei}(tz) - t \sim \sum_{k=1}^{\infty} \frac{(-1)^k k!}{t^{k-1}} z^{-k}, \quad z \to \infty, \quad -3\pi/2 < \arg(tz) < 3\pi/2.$$

The above solution has asymptotic representation (5.59), namely (1.18). Now, $t = 0$ is a branch point of logarithmic type, since $\mathrm{Ei}(zt) = -\ln(zt)+$ holomorphic function of zt. Moreover, the coefficients $F_k(t)$ diverge when $t \to 0$. The reader can check that the system has also fundamental solutions which are holomorphic at $t = 0$, but without the standard asymptotic representation $Y_F(z,t)$. We also notice a peculiarity of this particular example, namely that $Y(z,t)$ and $Y(ze^{-2\pi i}, t)$ are connected by a Stokes matrix $\mathbb{S} = \begin{bmatrix} 1 & 0 \\ 2\pi i t^2 & 1 \end{bmatrix}$, which is holomorphic also at $t = 0$ and coincides with the trivial Stokes matrix I of the system $\widehat{A}(z, t = 0)$. $\square$

5.5.2. Fundamental Solution in a neighbourhood of $t_\Delta \in \Delta$, with Assumption 5.1. Let Assumption 5.1 hold. Let $t_\Delta \in \Delta$. Since the case $t_\Delta = 0$ has already been discussed in detail, suppose that $t_\Delta \neq 0$. Then $t_\Delta \in \Delta_i$, for some $i \in \{1, 2, ..., s\}$.

Directions $\tau_\sigma^{(t_\Delta)}$, $\sigma \in \mathbb{Z}$, and sectors $\mathcal{S}_\sigma^{(t_\Delta)}$ have been defined in section 5.5.1 (just put $t_* = t_\Delta$). We leave to the reader the task to adjust the statement of Theorem 5.1 reformulated in a neighbourhood of t_Δ, with the block partition of $\Lambda(t_\Delta)$, which is finer than that of $\Lambda(0)$. The closed sector in the theorem will be denoted $\overline{S}^{(t_\Delta)}(\alpha, \beta) \subset \mathcal{S}_\sigma^{(t_\Delta)}$. A solution analogous to (5.37) is constructed at $t = t_\Delta$, with finer block partition than (5.37). Special cases as in Section 5.4.2 are very important for us, hence we state the following.

Proposition 5.2 generalised at t_Δ: For $t = t_\Delta$, the fundamental solution analogous to (5.37) reduces to an analogous to (5.44), namely

$$Y_{(t_\Delta)}(z) = G_0(t_\Delta)\mathcal{G}_{(t_\Delta)}(z) z^{B_1(t_\Delta)} e^{\Lambda(t_\Delta) z}, \qquad \text{with} \qquad B_1(t_\Delta) = \mathrm{diag}(\widehat{A}_1(t_\Delta)),$$

$$\mathcal{G}_{(t_\Delta)}(z) \sim I + \sum_{k=1}^{\infty} F_{(t_\Delta);k} z^{-k}, \qquad z \to \infty \quad \text{in } \overline{\mathcal{S}}^{(t_\Delta)}(\alpha,\beta),$$

if and only if the following conditions generalising (5.47) hold. For those $a \neq b \in \{1, ..., n\}$ such that $u_a(t_\Delta) = u_b(t_\Delta)$,

$$\left(\widehat{A}_1(t_\Delta)\right)_{ab} = 0, \tag{5.64}$$

and if also $\left(\widehat{A}_1(t_\Delta)\right)_{aa} - \left(\widehat{A}_1(t_\Delta)\right)_{bb} + l - 1 = 0$ for some $l \geq 2$, the following further conditions must hold:

$$\sum_{\substack{\gamma \in \{1,...,n\}, \\ u_\gamma(t_\Delta) \neq (u_a(t_\Delta) = u_b(t_\Delta))}} \left(\widehat{A}_1(t_\Delta)\right)_{a\gamma} \left(F_{(t_\Delta);l-1}\right)_{\gamma b} + \sum_{j=1}^{l-2} \left(\widehat{A}_{l-j}(t_\Delta) F_{(t_\Delta);j}\right)_{ab} + \left(\widehat{A}_l(t_\Delta)\right)_{ab} = 0. \tag{5.65}$$

In the notation used here, then $\mathring{Y}(z)$ in (5.44) is $Y_{(0)}(z)$, while $\mathcal{G}(z)$ in (5.45) is $\mathcal{G}_{(0)}(z)$. Finally, $\mathring{F}_k$ in (5.38) is $F_{(0);k}$. Keeping into account that $(\widehat{A}_1)_{a\gamma}$ vanishes in (5.61) for $t \to t_\Delta$ and $u_\gamma(t_\Delta) = u_a(t_\Delta) = u_b(t_\Delta)$, it is immediate to prove the following,

Proposition 5.4 generalised: If the vanishing conditions for (5.60) and (5.61) of Proposition 5.3 hold for $t \to t_\Delta \in \Delta$, then (5.64) and (5.65) at $t = t_\Delta$ are satisfied with the choice

$$F_{(t_\Delta);k} = F_k(t_\Delta), \quad k \geq 1. \tag{5.66}$$

If moreover $\left(\widehat{A}_1(t_\Delta)\right)_{aa} - \left(\widehat{A}_1(t_\Delta)\right)_{bb} + l - 1 \neq 0$ for every $l \geq 2$, the above (5.66) is the unique choice. Namely, for the system with $t = t_\Delta$ there is only the unique formal solution

$$\Big(I + \sum_{k=1}^{\infty} F_k(t_\Delta)\Big) z^{B_1(t_\Delta)} z^{\Lambda(t_\Delta)}, \qquad B_1(t_\Delta) = \mathrm{diag}(\widehat{A}_1(t_\Delta)).$$

CHAPTER 6

Stokes Phenomenon

Abstract. In this Chapter the Stokes phenomenon at $z = \infty$ for the system (5.1) is studied, both at coalescence and non-coalescence points (Sections 6.1-6.2 and Section 6.3, respectively). Assuming Assumption 5.1, we show that also at coalescence points there exist genuine fundamental solutions uniquely characterized in sufficiently wide sectors by an asymptotic expansion, prescribed by the formal solution found in the previous Chapter. All the instruments needed for the description of the Stokes phenomenon (Stokes rays, admissible rays, Canonical Sectors, complete sets of Stokes matrices etc.) are introduced at coalescence points, and their properties are described in details. The classical description of the Stokes phenomenon at non-coalescence points is summarized.

When Assumption 5.1 holds, the system (5.1) is gauge equivalent to (5.15) (i.e. system (1.16) in the Introduction) with $G_0(t)$ diagonalizing $A_0(t)$, namely

$$\frac{d\widehat{Y}}{dz} = \widehat{A}(z,t)\ \widehat{Y}, \qquad \widehat{A}(z,t) := G_0^{-1}(t)A(z,t)G_0(t) = \Lambda(t) + \sum_{k=1}^{\infty} \widehat{A}_k(t)z^{-k}. \tag{6.1}$$

At $t_0 \notin \Delta$, $\Lambda(t_0)$ has distinct eigenvalues, the Stokes phenomenon is studied as in [**BJL79a**]. We describe below the analogous results at $t = 0$ and $t_\Delta \in \Delta$, namely the existence and uniqueness of fundamental solutions with given asymptotics (5.38) in wide sectors. The results could be derived from the general construction of [**BJL79b**], especially from Theorem V and VI therein[1]. Nevertheless, it seems to be more natural to us to derive them in straightforward way, which we present below. First, we concentrate on the most degenerate case $\Lambda = \Lambda(0)$, for $t = 0$, so that $A(z,0) = \widehat{A}(z,0)$ and the systems (5.1) and (6.1) coincide. In Section 6.2 we consider the case of any other $t_\Delta \in \Delta$.

6.1. Stokes Phenomenon at $t = 0$

6.1.1. Stokes Rays of $\Lambda = \Lambda(0)$.

Definition 6.1. The **Stokes rays** associated with the pair of eigenvalues (λ_j, λ_k), $1 \leq j \neq k \leq n$, of Λ are the infinitely many rays contained in the universal covering $\mathcal{R}$ of $\mathbb{C}\backslash\{0\}$, oriented outwards from 0 to ∞, defined by

$$\Re\Big((\lambda_j - \lambda_k)z\Big) = 0, \quad \Im\Big((\lambda_j - \lambda_k)z\Big) < 0, \qquad z \in \mathcal{R}.$$

The definition above implies that for a couple of eigenvalues (λ_j, λ_k) the associated rays are

$$R(\theta_{jk} + 2\pi N) := \Big\{\ z \in \mathcal{R}\ \Big|\ z = \rho e^{i(\theta_{jk}+2\pi N)}, \quad \rho > 0\ \Big\}, \qquad N \in \mathbb{Z}. \tag{6.2}$$

where

$$\theta_{jk} := \frac{3\pi}{2} - \arg_p(\lambda_j - \lambda_k). \tag{6.3}$$

[1]Note that notations here and in [**BJL79b**] are similar, but they indicate objects that are slightly different (for example Stokes rays τ_ν and sectors $\mathcal{S}_\nu$ are not defined in the same way).

• **Labelling:** We enumerate Stokes rays with $\nu \in \mathbb{Z}$, using directions τ_ν introduced in Section 5.2. Indeed, by Definition 6.1, Stokes rays have directions $\arg z = \tau_\nu$, ordered in counter-clockwise sense as ν increases. For any sector of central angle π in $\mathcal{R}$, whose boundaries are not Stokes rays, there exists a $\nu_0 \in \mathbb{Z}$ such that the μ Stokes rays $\tau_{\nu_0-\mu+1} < \cdots < \tau_{\nu_0-1} < \tau_{\nu_0}$ are contained in the sector. All other Stokes rays have directions

$$\arg z = \tau_{\nu+k\mu} := \tau_\nu + k\pi, \qquad k \in \mathbb{Z}, \qquad \nu \in \{\nu_0 - \mu + 1, ..., \nu_0 - 1, \nu_0\}. \tag{6.4}$$

Rays $\tau_{\nu_0-\mu+1} < \cdots < \tau_{\nu_0-1} < \tau_{\nu_0}$ are called a set of **basic Stokes rays**, because they generate the others [2].

• **Sectors $\mathcal{S}_\nu$:** Consider a sector S of central opening less than π, with boundary rays which are not Stokes rays. The first rays encountered outside S upon moving clockwise and anti-clockwise, will be called **the two nearest Stokes rays outside** S. If S contains in its interior a set of basic rays, say $\tau_{\nu+1-\mu}, \tau_{\nu+2-\mu}, ..., \tau_\nu$, then the two nearest Stokes rays outside S are $\tau_{\nu-\mu}$ and $\tau_{\nu+1}$, namely the boundaries rays of $\mathcal{S}_\nu$ in (5.12), and obviously $S \subset \mathcal{S}_\nu$.

• **Projections onto $\mathbb{C}$:** If R is any of the rays in $\mathcal{R}$, its projection onto $\mathbb{C}$ will be denoted PR. For example, let $\overline{\lambda}_j$ be the complex conjugate of λ_j, then for any N the projection of (6.2) is

$$PR(\theta_{jk} + 2\pi N) = \{z \in \mathbb{C} \mid z = -i\rho(\overline{\lambda}_j - \overline{\lambda}_k),\ \rho > 0\}.$$

Definition 6.2. An **admissible ray for** $\Lambda(0)$ is a ray $R(\widetilde{\tau}) := \{z \in \mathcal{R} \mid z = \rho e^{i\widetilde{\tau}}, \quad \rho > 0\}$ in $\mathcal{R}$, of direction $\widetilde{\tau} \in \mathbb{R}$, which does not coincide with any of the Stokes rays of $\Lambda(0)$. Let

$$l_+(\widetilde{\tau}) := PR(\widetilde{\tau} + 2k\pi), \quad l_-(\widetilde{\tau}) := PR(\widetilde{\tau} + (2k+1)\pi), \qquad k \in \mathbb{Z},$$

$$l(\widetilde{\tau}) := l_-(\widetilde{\tau}) \cup \{0\} \cup l_+(\widetilde{\tau}).$$

We call the oriented line $l(\widetilde{\tau})$ an **admissible line for** $\Lambda(0)$. Its positive part is $l_+(\widetilde{\tau})$.

Observe that there exists a suitable ν such that $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$, which implies

$$R(\widetilde{\tau}) \subset \mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}, \qquad R(\widetilde{\tau} + \pi) \subset \mathcal{S}_{\nu+\mu} \cap \mathcal{S}_{\nu+2\mu}.$$

In particular, if τ is as in (5.10), then $\tau_{-1} < \tau < \tau_0$, and $l(\tau)$ is an admissible line.

6.1.2. Uniqueness of the Fundamental Solution with given Asymptotics. In case of distinct eigenvalues, it is well known that there exists a unique fundamental solution, determined by the asymptotic behaviour given by the formal solution, on a sufficiently large sector. This fact must now be proved also at coalescence points.

Let the diagonal form $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_s$ of A_0 be fixed. Let a formal solution $\mathring{Y}_F(z) = \Delta_0 F(z) z^D z^L e^{\Lambda z}$ be chosen in the class of formal solutions with given Δ_0, D, L, Λ, as in Definition 5.2. As a consequence of Theorem 5.1 and Theorem 5.2, there exists at least one actual solution as in (5.37), namely

$$\mathring{Y}(z) = \Delta_0 \mathcal{G}(z) z^D z^L\, e^{\Lambda z}, \qquad \mathcal{G}(z) \sim F(z), \qquad z \to \infty, \quad z \in \overline{S}(\alpha, \beta). \tag{6.5}$$

Observe that $\overline{S}(\alpha, \beta)$ can be chosen in Theorem 5.1 so that *it contains the set of basic Stokes rays of* $\mathcal{S}_\nu$, namely $\tau_{\nu+1-\mu}, ..., \tau_{\nu-1}, \tau_\nu$. The asymptotic relation in (6.5) is conventionally written as follows,

$$\mathring{Y}(z) \sim \mathring{Y}_F(z), \qquad z \to \infty, \quad z \in \overline{S}(\alpha, \beta).$$

Now, $\mathcal{G}(z)$ is holomorphic for $|z|$ sufficiently big in $\overline{S}(\alpha, \beta)$. Since $A(z)$ has no singularities for $|z| \geq N_0$ large, except the point at infinity, then $\mathring{Y}(z)$ *and* $\mathcal{G}(z)$ *have analytic continuation on* $\mathcal{R} \cap \{|z| \geq N_0\}$.

[2] Although notations are similar to [BJL79b], definitions are slightly different here.

Lemma 6.1. *Let $C \in GL(n,\mathbb{C})$, and S an arbitrary sector. Then*

$$z^D z^L C z^{-L} z^{-D} \sim I, \quad z \to \infty \text{ in } S \quad \Longleftrightarrow \quad z^D z^L C z^{-L} z^{-D} = I \quad \Longleftrightarrow \quad C = I.$$

The simple proof is left as an exercise.

Lemma 6.2 (Extension Lemma). *Let $\mathring{Y}(z)$ be a fundamental matrix solution with asymptotic behaviour,*

$$\mathring{Y}(z) \sim \mathring{Y}_F(z), \qquad z \to \infty, \quad z \in S,$$

in a sector S of a non specified central opening angle. Suppose that there is a sector $\widetilde{S}$ not containing Stokes rays, such that $S \cap \widetilde{S} \neq \emptyset$. Then,

$$\mathring{Y}(z) \sim \mathring{Y}_F(z), \qquad z \to \infty, \qquad \text{for } \ z \in S \cup \tilde{S}.$$

Proof: $\widetilde{S}$ has central opening angle less than π, because it does not contain Stokes rays. Therefore, by Theorem 5.1, there exists a fundamental matrix solution $\widetilde{Y}(z) = \Delta_0 \tilde{\mathcal{G}}(z) z^D z^L e^{\Lambda z}$, with asymptotic behaviour $\tilde{Y}(z) \sim \mathring{Y}_F(z)$, for $z \to \infty$, $z \in \widetilde{S}$. The two fundamental matrices are connected by an invertible matrix C, namely $\mathring{Y}(z) = \widetilde{Y}(z)\ C$, $z \in S \cap \widetilde{S}$. Therefore,

$$\widetilde{\mathcal{G}}^{-1}(z)\ \mathcal{G}(z) = z^D z^L e^{\Lambda z}\ C\ e^{-\Lambda z} z^{-L} z^{-D}.$$

Since $\mathcal{G}(z)$ and $\widetilde{\mathcal{G}}^{-1}(z)$ have the same asymptotic behaviour in $S \cap \widetilde{S}$, the l.h.s has asymptotic series equal to the identity matrix I, for $z \to \infty$ in $z \in S \cap \widetilde{S}$. Thus, so must hold for the r.h.s. The r.h.s has diagonal-block structure inherited from Λ. We write the block $[i,j]$, $1 \leq i,j \leq s$, of C with simple notation C_{ij}. The block $[i,j]$ in r.h.s. is then, $e^{(\lambda_i - \lambda_j)z} z^{D_i} z^{L_i}\ C_{ij}\ z^{-L_j} z^{-D_j}$. Hence, the following must hold,

$$e^{(\lambda_i - \lambda_j)z} z^{D_i} z^{L_i}\ C_{ij}\ z^{-L_j} z^{-D_j} \sim \delta_{ij}\ I_i, \qquad z \to \infty, \quad z \in S \cap \widetilde{S}.$$

Here I_i is the $p_i \times p_i$ identity matrix.

– For $i \neq j$: Since there are no Stokes rays in $\widetilde{S}$, the sign of $\Re(\lambda_i - \lambda_j)z$ does not change in $\widetilde{S}$. This implies that $e^{(\lambda_i - \lambda_j)z} z^{D_i} z^{L_i}\ C_{ij}\ z^{-L_j} z^{-D_j} \sim 0$ for $z \to \infty$ in $\widetilde{S}$.

– For $i = j$: We have $z^{D_i} z^{L_i}\ C_{ii}\ z^{-L_i} z^{-D_i} \sim I_i$ for $z \to \infty$ in $S \cap \widetilde{S}$. From Lemma 6.1 it follows that $z^{D_i} z^{L_i}\ C_{ii}\ z^{-L_i} z^{-D_i} = I_i$. This holds on the whole $\widetilde{S}$.

The above considerations imply that $z^D z^L e^{\Lambda z}\ C\ e^{-\Lambda z} z^{-L} z^{-D} \sim I$ for $z \to \infty$ in $\widetilde{S}$. From the fact that $\widetilde{\mathcal{G}}(z) \sim I + \sum_{k \geq 1} \mathring{F}_k z^{-k}$ in $\widetilde{S}$, we conclude that also $\mathcal{G}(z) \sim I + \sum_{k \geq 1} \mathring{F}_k z^{-k}$ for $z \to \infty$ in $\widetilde{S}$. Therefore, $\mathcal{G}(z) \sim I + \sum_{k \geq 1} \mathring{F}_k z^{-k}$ in $S \cup \widetilde{S}$. $\square$

The extension Lemma immediately implies the following:

Theorem 6.1 (Extension Theorem). *Let $\mathring{Y}(z)$ be a fundamental matrix solution such that $\mathring{Y}(z) \sim \mathring{Y}_F(z)$ in a sector S, containing a set of μ basic Stokes rays, and no other Stokes rays. Then, the asymptotics $\mathring{Y}(z) \sim \mathring{Y}_F(z)$ holds on the open sector which extends up to the two nearest Stokes rays outside S. This sector has central opening angle greater than π and is a sector $\mathcal{S}_\nu$ for a suitable ν.*

Important Remark: The above extension theorem has the important consequence that in the statement of Theorem 5.2 and Proposition 5.2, the matrix $\mathcal{G}(z)$, which has analytic continuation in $\mathcal{R}$ for $|z| \geq N_0$, has the prescribed asymptotic expansion in *any* proper closed subsector of $\mathcal{S}_\nu$. Hence, *by definition, the asymptotics holds in the open sector $\mathcal{S}_\nu$.*

THEOREM 6.2 (Uniqueness Theorem). *A fundamental matrix $\mathring{Y}(z)$ as (5.37) such that $\mathring{Y}(z) \sim \mathring{Y}_F(z)$, for $z \to \infty$ in a sector S containing a set of basic Stokes rays, is unique. In particular, this applies if $\overline{S}(\alpha, \beta)$ of Theorem 5.1 contains a set of basic Stokes rays.*

Proof: Suppose that there are two solutions $\mathring{Y}(z)$ and $\widetilde{Y}(z)$ with asymptotic representation $\mathring{Y}_F(z)$ in a sector S, which contains μ basic Stokes rays. Then, there exists an invertible matrix C such that $\mathring{Y}(z) = \widetilde{Y}(z)\ C$, namely

$$\widetilde{\mathcal{G}}^{-1}(z)\ \mathcal{G}(z) = z^D z^L e^{\Lambda z}\ C\ e^{-\Lambda z} z^{-L} z^{-D}.$$

The l.h.s. has asymptotic series equal to I as $z \to \infty$ in S. Therefore, for the block $[i, j]$, the following must hold,

$$e^{(\lambda_i - \lambda_j)z} z^{D_i} z^{L_i}\ C_{ij}\ z^{-L_j} z^{-D_j}\ \sim\ \delta_{ij}\ I_i, \qquad \text{for}\ \ z \to \infty\ \ \text{in}\ \ S.$$

Since S contains a set of basic Stokes rays, $\Re(\lambda_i - \lambda_j)z$ changes sign at least once in S, for any $1 \le i \ne j \le s$. Thus, $e^{(\lambda_i - \lambda_j)z}$ diverges in some subsector of S. For $i \ne j$ this requires that $C_{ij} = 0$ for $i \ne j$. For $i = j$, we have $z^{D_i} z^{L_i} C_{ii} z^{-L_i} z^{-D_i} \sim I_i$. Lemma 6.1 assures that $C_{ii} = I_i$. Thus, $C = I$. $\Box$

• **[The notation $\mathring{Y}_\nu(z)$]:** There exist $\nu \in \mathbb{Z}$ such that a sector S of Theorem 6.2 contains the basic rays $\tau_{\nu+1-\mu}$, ..., $\tau_{\nu-1}$, τ_ν. Hence $S \subset \mathcal{S}_\nu$. The unique fundamental solution of Theorem 6.2, with asymptotics extended to $\mathcal{S}_\nu$ according to Theorem 6.1, will be denoted $\mathring{Y}_\nu(z)$.

6.1.3. Stokes Matrices. The definition of Stokes matrices is standard. Recall that the Stokes rays associated with (λ_j, λ_k) are (6.2). Consider also the rays

$$R(\theta_{jk} + 2\pi N + \delta) = \left\{\ z \in \mathcal{R}\ \middle|\ z = \rho e^{i(\theta_{jk} + 2\pi N + \delta)}, \quad \rho > 0\ \right\}, \qquad N \in \mathbb{Z}.$$

The sign of $\Re(\lambda_j - \lambda_k)z$ for $z \in R_N(\theta_{jk} + \delta)$ is:

$$\begin{cases} \Re(\lambda_j - \lambda_k)z < 0, & \text{for} \quad -\pi < \delta < 0 \quad \bmod 2\pi \\ \Re(\lambda_j - \lambda_k)z > 0, & \text{for} \quad 0 < \delta < \pi \quad \bmod 2\pi \\ \Re(\lambda_j - \lambda_k)z = 0, & \text{for} \quad \delta = 0,\ \pi,\ -\pi \quad \bmod 2\pi \end{cases}$$

DEFINITION 6.3 (Dominance relation). In a sector where $\Re(\lambda_j - \lambda_k)z > 0$, λ_j is said to be dominant over λ_k in that sector, and we write $\lambda_j \succ \lambda_k$. In a sector where $\Re(\lambda_j - \lambda_k)z < 0$, λ_j is said to be sub-dominant, or dominated by λ_k, and we write $\lambda_j \prec \lambda_k$.

If a sector S does not contain Stokes rays in its interior, it is well defined a dominance relation in S, which *determines an* **ordering relation** *among eigenvalues,* **referred to the sector** S.

Denote by

$$\mathring{Y}_\nu(z) \ \ \text{and} \ \ \mathring{Y}_{\nu+\mu}(z)$$

the unique fundamental solutions (5.37) with asymptotic behaviours $\mathring{Y}_F(z)$ on $\mathcal{S}_\nu$ and $\mathcal{S}_{\nu+\mu}$ respectively, as in Theorem 6.2. Observe that $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu} = S(\tau_\nu, \tau_{\nu+1})$ is not empty and does not contain Stokes rays.

DEFINITION 6.4. For any $\nu \in \mathbb{Z}$, the **Stokes matrix** $\mathring{\mathbb{S}}_\nu$ is the connection matrix such that

$$\mathring{Y}_{\nu+\mu}(z) = \mathring{Y}_\nu(z)\mathring{\mathbb{S}}_\nu, \qquad z \in \mathcal{R}. \tag{6.6}$$

PROPOSITION 6.1. *Let $\prec$ be the dominance relation referred to the sector $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}$. Then, the Stokes matrix $\mathring{\mathbb{S}}_\nu$ has the following block-triangular structure:*

$$\mathring{\mathbb{S}}^{(\nu)}_{jj} = I_{p_j},$$
$$\mathring{\mathbb{S}}^{(\nu)}_{jk} = 0 \quad \text{for } \lambda_j \succ \lambda_k \text{ in } \mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}, \qquad j,k \in \{1,2,...,s\}.$$

Proof: We re-write (6.6) as,

$$\mathcal{G}_\nu^{-1}(z)\ \mathcal{G}_{\nu+\mu}(z) = z^D z^L e^{\Lambda z}\ \mathring{\mathbb{S}}_\nu\ e^{-\Lambda z} z^{-L} z^{-D}.$$

For $z \in \mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}$, the l.h.s. has asymptotic expansion equal to I. Hence, the same must hold for the r.h.s. Recalling that no Stokes rays lie in $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}$, we find:

- For $j \neq k$, we have $e^{(\lambda_j-\lambda_k)z} z^{D_j} z^{L_j} \mathring{\mathbb{S}}^{(\nu)}_{jk} z^{-L_k} z^{-D_k} \sim 0$ in $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}$ if and only if $\mathring{\mathbb{S}}^{(\nu)}_{jk} = 0$ for $\lambda_j \succ \lambda_k$, where the dominance relation is referred to the sector $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu}$.

- For $j = k$, we have $z^{D_j} z^{L_j} \mathring{\mathbb{S}}^{(\nu)}_{jj} z^{-L_j} z^{-D_j} \sim I_{p_j}$ if and only if $\mathring{\mathbb{S}}^{(\nu)}_{jj} = I_{p_j}$, by Lemma 6.1. This proves the Proposition. $\square$

6.1.4. Canonical Sectors, Complete Set of Stokes Matrices, Monodromy Data. There are no Stokes rays in the intersection of successive sectors $\mathcal{S}_{\nu+k\mu}$ and $\mathcal{S}_{\nu+(k+1)\mu}$ (recall that $\tau_\nu + k\pi = \tau_{\nu+k\mu}$ for any $k \in \mathbb{Z}$). Therefore, we can introduce the unique fundamental matrix solutions

$$\mathring{Y}_{\nu+k\mu}(z) \tag{6.7}$$

with asymptotic behaviour $\mathring{Y}_F(z)$ in $\mathcal{S}_{\nu+k\mu}$, and the Stokes matrices $\mathring{\mathbb{S}}_{\nu+k\mu}$ connecting them,

$$\mathring{Y}_{\nu+(k+1)\mu}(z) = \mathring{Y}_{\nu+k\mu}(z)\ \mathring{\mathbb{S}}_{\nu+k\mu}, \qquad z \in \mathcal{R}.$$

From Proposition 6.1, it follows that the blocks $[j,k]$ and $[k,j]$ satisfy

$$\mathring{\mathbb{S}}^{(\nu)}_{jk} = 0 \text{ for } \lambda_j \succ \lambda_k \text{ in } \mathcal{S}_\nu \cap \mathcal{S}_{\nu+\mu} \quad \Longleftrightarrow \quad \mathring{\mathbb{S}}^{(\nu+\mu)}_{kj} = 0 \text{ for the same } (j,k).$$

We call $\mathcal{S}_\nu$, $\mathcal{S}_{\nu+\mu}$, $\mathcal{S}_{\nu+2\mu}$ the **canonical sectors associated with** τ_ν.

Given a formal solution, a simple computation (recall that $[L,\Lambda] = 0$) yields $\mathring{Y}_F(e^{2\pi i} z) = \mathring{Y}_F(z)\, e^{2\pi i L}$. L is called **exponent of formal monodromy**.

THEOREM 6.3. *We introduce the notation $z_{(\nu)}$ if $z \in \mathcal{S}_\nu$. Thus $z_{(\nu+2\mu)} = e^{2\pi i} z_{(\nu)}$. The following equalities hold*

$$(i) \quad \mathring{Y}_{\nu+2\mu}(z_{(\nu+2\mu)}) = \mathring{Y}_\nu(z_{(\nu)})\ e^{2\pi i L},$$
$$(ii) \quad \mathring{Y}_{\nu+2\mu}(z) = \mathring{Y}_\nu(z)\ \mathring{\mathbb{S}}_\nu\ \mathring{\mathbb{S}}_{\nu+\mu}, \qquad z \in \mathcal{R},$$
$$(iii) \quad \mathring{Y}_\nu(e^{2\pi i} z) = \mathring{Y}_\nu(z)\ e^{2\pi i L}\left(\mathring{\mathbb{S}}_\nu\ \mathring{\mathbb{S}}_{\nu+\mu}\right)^{-1}, \qquad z \in \mathcal{R}.$$

where $|z| \geq N_0$ is sufficiently large, in such a way that any other singularity of $A(z)$ is contained in the ball $|z| < N_0$.

Proof: As in the case of distinct eigenvalues. Alternatively, one can adapt Proposition 4 of [**BJL79b**] to the present case.[3]. $\square$.

[3]With the warning that notations are similar but objects are slightly different here and in [**BJL79b**].

The equality (iii) provides the **monodromy matrix** $M_\infty^{(\nu)}$ **of** $\mathring{Y}_\nu(z)$ **at** $z = \infty$:

$$M_\infty{}^{(\nu)} := \left(\mathring{\mathbb{S}}_\nu \; \mathring{\mathbb{S}}_{\nu+\mu}\right) \; e^{-2\pi i L}. \tag{6.8}$$

corresponding to a clockwise loop with $|z| \geq N_0$ large, in such a way that all other singularities of $A(z)$ are inside the loop.

The two Stokes matrices $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$, and the matrix L generate all the other Stokes matrices $\mathring{\mathbb{S}}_{\nu+k\mu}$, according to the following proposition

PROPOSITION 6.2. *For any $\nu \in \mathbb{Z}$, the following holds: $\mathring{\mathbb{S}}_{\nu+2\mu} = e^{-2\pi i L} \; \mathring{\mathbb{S}}_\nu \; e^{2\pi i L}$.*

Proof: For simplicity, take $\nu = 0$. A point in $z \in \mathcal{S}_{2\mu} \cap \mathcal{S}_{3\mu}$ can represented both as $z_{(2\mu)}$ and $z_{(3\mu)}$, and a point in $\mathcal{S}_0 \cap \mathcal{S}_\mu$ is represented both as $z_{(0)}$ and $z_{(\mu)}$. Therefore, the l.h.s. of the equality $\mathring{Y}_{3\mu}(z) = \mathring{Y}_{2\mu}(z) \; \mathring{\mathbb{S}}_{2\mu}$ is $\mathring{Y}_{3\mu}(z_{(3\mu)}) = \mathring{Y}_\mu(z_{(\mu)}) \; e^{2\pi i L} = \mathring{Y}_0(z_{(0)})\mathring{\mathbb{S}}_0 \; e^{2\pi i L}$. The r.h.s. is $\mathring{Y}_{2\mu}(z_{(2\mu)}) \; \mathring{\mathbb{S}}_{2\mu} = \mathring{Y}_0(z_{(0)})e^{2\pi i L} \quad \mathring{\mathbb{S}}_{2\mu}$. Thus $\mathring{Y}_0(z_{(0)}) \; \mathring{\mathbb{S}}_0 \; e^{2\pi i L} = \mathring{Y}_0(z_{(0)})e^{2\pi i L} \quad \mathring{\mathbb{S}}_{2\mu}$. This proves the proposition. □

The above proposition implies that $\mathring{\mathbb{S}}_{\nu+k\mu}$ are generated by $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$, which therefore form a **complete set of Stokes matrices**. A complete set of Stokes matrices and the exponent of formal monodromy are necessary and sufficient to obtain the monodromy at $z = \infty$, through formula (6.8). This justifies the following definition.

DEFINITION 6.5. For a chosen ν, $\{\mathring{\mathbb{S}}_\nu, \; \mathring{\mathbb{S}}_{\nu+\mu}, \; L\}$ is a set of **monodromy data** at $z = \infty$ of the system (5.1) with $t = 0$.

REMARK 6.1. By a factorization into Stokes factors, as in the proof of Theorem 7.2 below, it can be shown that $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$ suffice to generate $\mathring{\mathbb{S}}_{\nu+1}$, ..., $\mathring{\mathbb{S}}_{\nu+\mu-1}$. Hence, $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$ are really sufficient to generate all Stokes matrices. This technical part will be omitted.

6.2. Stokes Phenomenon at fixed $t_\Delta \in \Delta$

The results of Section 6.1 apply to any other $t_\Delta \in \Delta$. By a permutation matrix P we arrange $P^{-1}\Lambda(t_\Delta)P$ in blocks, in such a way that each block has only one eigenvalue and two distinct blocks have different eigenvalues. This is achieved by the transformation $\widehat{Y}(z,t) = P \; \widetilde{Y}(z,t)$ applied to the system (6.1). Then, the procedure is exactly the same of Section 6.1, applied to the system

$$\frac{d\widetilde{Y}}{dz} = P^{-1}\widehat{A}(z,t_\Delta)P \; \widetilde{Y}. \tag{6.9}$$

The block partition of all matrices in the computations and statements is that inherited from $P^{-1}\Lambda(t_\Delta)P$. The Stokes rays are defined in the same way as in Definition 6.1, using the eigenvalues of $\Lambda(t_\Delta)$, namely

$$\begin{aligned} &\Re\Big((u_a(t_\Delta) - u_b(t_\Delta))z\Big) = 0, \qquad \Im\Big((u_a(t_\Delta) - u_a(t_\Delta))z\Big) < 0, \qquad z \in \mathcal{R}, \\ &\text{for} \quad 1 \leq a \neq b \leq n \quad \underline{\text{and}} \quad u_a(t_\Delta) \neq u_a(t_\Delta). \end{aligned}$$

Hence, the Stokes rays associated with $u_a(t_\Delta), u_b(t_\Delta)$ are the infinitely many rays with directions

$$\arg z = \frac{3\pi}{2} - \arg_p(u_a(t_\Delta) - u_b(t_\Delta)) + 2N\pi, \quad N \in \mathbb{Z}.$$

The rays associated with $u_b(t_\Delta), u_a(t_\Delta)$ are opposite to the above, having directions

$$\arg z = \frac{3\pi}{2} - \arg_p(u_b(t_\Delta) - u_a(t_\Delta)) + 2N\pi.$$

We conclude that all Stokes rays have directions

$$\arg z = \tau_\sigma^{(t_\Delta)}, \qquad \sigma \in \mathbb{Z},$$

analogous to (6.4), with directions $\tau_\sigma^{(t_\Delta)}$ defined in Section 5.5.1. Once the Stokes matrices for the above system are computed, in order to go back to the original arrangement corresponding to $\Lambda(t_\Delta)$ we just apply the inverse permutation. Namely, if $\mathbb{S}$ is a Stokes matrix of (6.9), then $P\mathbb{S}P^{-1}$ is a Stokes matrix for (6.1) with $t = t_\Delta$.

6.3. Stokes Phenomenon at $t_0 \notin \Delta$

The results of Section 6.1 (extension theorem, uniqueness theorem, Stokes matrices, etc) apply *a fortiori* if the eigenvalues are distinct, namely at a point $t_0 \notin \Delta$ such that Theorem 5.3 and Corollary 5.2 apply. The block partition of $\Lambda(t_0)$ is into one-dimensional blocks, being the eigenvalues all distinct, and we are back to the well known case of [**BJL79a**]. The Stokes rays are defined in the same way as in Definition 6.1, using the eigenvalues of $\Lambda(t_0)$, namely

$$\Re\Big((u_a(t_0) - u_b(t_0))z\Big) = 0, \qquad \Im\Big((u_a(t_0) - u_b(t_0))z\Big) < 0, \qquad z \in \mathcal{R}, \qquad \forall\ 1 \le a \ne b \le n.$$

Since and $u_a(t_0) \ne u_b(t_0)$ for any $a \ne b$, the above definition holds for any $1 \le a \ne b \le n$. Hence, the Stokes rays associated with $u_a(t_0), u_b(t_0)$ are the infinitely many rays with directions

$$\arg z = \frac{3\pi}{2} - \arg_p(u_a(t_0) - u_b(t_0)) + 2N\pi, \qquad N \in \mathbb{Z}. \tag{6.10}$$

The rays associated with $u_b(t_0), u_a(t_0)$ are opposite to the above, having directions

$$\arg z = \frac{3\pi}{2} - \arg_p(u_b(t_0) - u_a(t_0)) + 2N\pi. \tag{6.11}$$

We conclude that all Stokes rays have directions

$$\arg z = \tau_\sigma^{(t_0)}, \qquad \sigma \in \mathbb{Z},$$

analogous to (6.4), being the directions $\tau_\sigma^{(t_0)}$ defined in Section 5.5.1. We stress that t_0 is fixed here. The Stokes phenomenon is studied in the standard way. The canonical sectors are the sectors $\mathcal{S}_\sigma^{(t_0)}$ of Theorem 5.3. The sector $\mathcal{S}_\sigma^{(t_0)}$ contains the set of *basic Stokes rays*

$$\tau^{(t_0)}_{\sigma+1-\mu_{t_0}}, \quad \tau^{(t_0)}_{\sigma+2-\mu_{t_0}}, \quad \dots, \quad \tau^{(t_0)}_{\sigma}, \tag{6.12}$$

which serve to generate all the other rays by adding multiples of π. The rays $\tau^{(t_0)}_{\sigma-\mu_{t_0}}$ and $\tau^{(t_0)}_{\sigma+1}$ are the nearest Stokes rays, boundaries of $\mathcal{S}_\sigma^{(t_0)}$. The Stokes matrices connect solutions of Corollary 5.2, having the prescribed canonical asymptotics on successive sectors, for example $\mathcal{S}_\sigma^{(t_0)}$, $\mathcal{S}_{\sigma+\mu_{t_0}}^{(t_0)}$, $\mathcal{S}_{\sigma+2\mu_{t_0}}^{(t_0)}$, etc.

Our purpose is now to show how the Stokes phenomenon can be described in a consistent "holomorphic" way as t varies. The definition of Stokes matrices for varying t will require some steps.

CHAPTER 7

Cell Decomposition, t-analytic Stokes Matrices

Abstract. In this Chapter, under Assumption 5.1, we discuss the analytic continuation of fundamental solutions of (5.1). We show that $\mathcal{U}_{\epsilon_0}(0)$ splits into topological cells, determined by the fact that Stokes rays associated with $\Lambda(t)$ cross a fixed admissible ray. In Theorem 7.1 and Corollary 7.3 we give sufficient conditions such that fundamental solutions can be analytically continued to the whole $\mathcal{U}_{\epsilon_0}(0)$, preserving their asymptotic representation, so that the Stokes matrices admit the limits at coalescence points. In Section 7.7 we prove a partial converse of Theorem 7.1, by showing that a vanishing conditions on the entries of Stokes matrices at coalescence points implies that Δ is not a branch locus for fundamental solutions.

7.1. Stokes Rays rotate as t varies

At $t=0$, Stokes rays have directions $3\pi/2-\arg_p(\lambda_i-\lambda_j)+2N\pi$, $1\le i\neq j\le s$. For t away from $t=0$, the following occurs:

1) [Splitting] For $1\le i\neq j\le s$, there are rays of directions $3\pi/2-\arg_p(u_a(t)-u_b(t)) \bmod(2\pi)$, with $u_a(0)=\lambda_i$, $u_b(0)=\lambda_j$. These rays are the splitting of $3\pi/2-\arg_p(\lambda_i-\lambda_j) \bmod(2\pi)$ into more rays.

2) [Unfolding] For any $i=1,2,...,s$, new rays appear, with directions $3\pi/2-\arg_p(u_a(t)-u_b(t))$, $u_a(0)=u_b(0)=\lambda_i$. These rays are due to the *unfolding* of λ_i.

The cardinality of a set of basic Stokes rays is maximal away from the coalescence locus Δ, minimal at $t=0$, and intermediate at $t_\Delta\in\Delta\backslash\{0\}$.

If $t\notin\Delta$, then $u_a(t)\neq u_b(t)$ for any $a\neq b$. The direction of every Stokes ray (6.10) or (6.11) is a continuous functions of $t\notin\Delta$. As t varies in $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, each one of the rays (6.10) or (6.11) rotates in $\mathcal{R}$.

Remark 7.1. **Problems with enumeration of moving Stokes rays.** Apparently, we cannot assign a coherent labelling to the rotating rays as t moves in $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. At a given $t_0\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, the rays are enumerated according to the choice of an admissible direction $\eta^{(t_0)}$, as in formula (5.56) with $t_*=t_0$. If t is very close to t_0, we may choose $\eta^{(t_0)}=\eta^{(t)}$, and we can label the rays in such a way that $\tau_\sigma^{(t)}$, $\sigma\in\mathbb{Z}$, is the result of the continuous rotation of $\tau_\sigma^{(t_0)}$. Nevertheless, if t moves farther in $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, then some rays, while rotating, may cross with each other and cross the rays $R(\tau^{(t_0)}+k\pi)$, $k\in\mathbb{Z}$, which are admissible for $\Lambda(t_0)$. This phenomenon destroys the ordering. Hence, labellings are to be taken independently at t_0 and at any other $t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, with respect to independent admissible directions $\eta^{(t_0)}$ and $\eta^{(t)}$. In this way, $\tau_\sigma^{(t)}$ will not be the deformation of a $\tau_\sigma^{(t_0)}$ with the same σ.

This complication in assigning a coherent numeration to rays and sectors as t varies will be solved in Section 7.3, by introducing a new labelling, valid for almost all $t\in\mathcal{U}_{\epsilon_0}(0)$, induced by the labelling at $t=0$. Before that, we need some topological preparation.

7.2. Ray Crossing, Wall Crossing and Cell Decomposition

We consider an oriented admissible ray $R(\widetilde{\tau})$ for $\Lambda(0)$, with direction $\widetilde{\tau}$, as in Definition 6.2 and we project $\mathcal{R}$ onto $\mathbb{C}\backslash\{0\}$. For $t \in \mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, some projected rays associated with $\Lambda(t)$ will be to the left of $l(\widetilde{\tau})$ and some to the right. Moreover, some projected ray may lie exactly on $l(\widetilde{\tau})$, in which case we improperly say that *"the ray lies on $l(\widetilde{\tau})$"*. Suppose we start at a value $t_* \in \mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ such that no rays associated with $\Lambda(t_*)$ lie on $l(\widetilde{\tau})$. If t moves away from t_* in $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, then the directions of Stokes rays change continuously and the projection of two or more rays[1] may cross $l(\widetilde{\tau})$ as t varies, in which case we say that *"two or more rays cross $l(\widetilde{\tau})$"*. Let

$$\widetilde{\eta} := \frac{3\pi}{2} - \widetilde{\tau}.$$

Two or more Stokes rays cross $l(\widetilde{\tau})$ for t belonging to the following **crossing locus**

$$X(\widetilde{\tau}) := \bigcup_{1\leq a<b\leq n} \Big\{ t \in \mathcal{U}_{\epsilon_0}(0) \ \Big|\ u_a(t) \neq u_b(t), \quad \arg_p(u_a(t)-u_b(t)) = \widetilde{\eta} \bmod \pi \Big\}.$$

Let

$$W(\widetilde{\tau}) := \Delta \cup X(\widetilde{\tau}).$$

Definition 7.1. A $\widetilde{\tau}$**-cell** is every connected component of the set $\mathcal{U}_{\epsilon_0}(0)\backslash W(\widetilde{\tau})$.

$W(\widetilde{\tau})$ is the *"wall" of the cells.* For t in a $\widetilde{\tau}$-cell, $\Lambda(t)$ is diagonalisable with distinct eigenvalues, and the Stokes rays projected onto $\mathbb{C}$ lie either to the left or to the right of $l(\widetilde{\tau})$. If t varies and hits $W(\widetilde{\tau})$, then either some Stokes rays disappear (when $t \in \Delta$), or some rays cross the admissible line $l(\widetilde{\tau})$ (when $t \in X(\widetilde{\tau})$). Notice that

$$\Delta \cap X(\widetilde{\tau}) \neq \emptyset.$$

A cell is open, by definition. If the eigenvalues are linear in t, as in (1.25), we will show in Section 7.2.1 that a cell is simply connected and convex, namely it is a topological cell, so justifying the name. Explicit examples and figures are given in the Appendix A.

7.2.1. Topology of $\widetilde{\tau}$-cells and hyperplane arrangements. In order to study the topology of the $\widetilde{\tau}$-cells, it is convenient to first extend their definition to $\mathbb{C}^n$. A $\widetilde{\tau}$-cells in $\mathbb{C}^n$ can be proved to be homeomorphic to an open ball, therefore it is a *cell* in the topological sense. A $\widetilde{\tau}$-cell in $\mathbb{C}^n$ is defined to be a connected component of $\mathbb{C}^n\backslash(\Delta_{\mathbb{C}^n} \cup X_{\mathbb{C}^n}(\widetilde{\tau}))$, where

$$\Delta_{\mathbb{C}^n} := \bigcup_{1\leq a<b\leq n} \Big\{ u \in \mathbb{C}^n \ \Big|\ u_a = u_b \Big\},$$

$$X_{\mathbb{C}^n}(\widetilde{\tau}) := \bigcup_{1\leq a<b\leq n} \Big\{ u \in \mathbb{C}^n \ \Big|\ u_a - u_b \neq 0 \text{ and } \arg_p(u_a - u_b) = \widetilde{\eta} \bmod \pi \Big\}.$$

Recall that $\widetilde{\eta} = \frac{3\pi}{2} - \widetilde{\tau}$.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. A point $u = (u_1, ..., u_n)$ is identified with $(\mathbf{x}, \mathbf{y}) = (x_1, ..., x_n,\ y_1, ..., y_n)$, by $u_a = x_a + iy_a$, $1 \leq a \leq n$. Therefore

a) $\Delta_{\mathbb{C}^n}$ is identified with

$$A := \bigcup_{1\leq a<b\leq n} \Big\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n} \ \Big|\ x_a - x_b = y_a - y_b = 0 \Big\}.$$

[1] Crossing involves always at least two opposite projected rays, which have directions differing by π. One projection crosses the positive part $l_+(\widetilde{\tau})$ of $l(\widetilde{\tau})$, and one projection crosses the negative part $l_-(\widetilde{\tau}) = l_+(\widetilde{\tau} \pm \pi)$.

b) $X_{\mathbb{C}^n}(\widetilde{\tau})$ is identified with

$$B := \bigcup_{1\le a<b\le n} \Big\{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n} \;\Big|\; (x_a,y_a) \neq (x_b,y_b) \textit{ and } L_{ab}(\mathbf{x},\mathbf{y}) = 0\Big\}$$

where $L_{ab}(\mathbf{x},\mathbf{y})$ is a linear function

$$L_{ab}(\mathbf{x},\mathbf{y}) = (y_a - y_b) - \tan\widetilde{\eta}\,(x_a - x_b), \qquad \text{for } \widetilde{\eta} \neq \frac{\pi}{2} \bmod \pi, \tag{7.1}$$

$$L_{ab}(\mathbf{x},\mathbf{y}) = x_a - x_b, \qquad \text{for } \widetilde{\eta} = \frac{\pi}{2} \bmod \pi. \tag{7.2}$$

Hence $A \cup B$ is a *union of hyperplanes* H_{ab}:

$$A \cup B = \bigcup_{1\le a<b\le n} H_{ab}, \qquad H_{ab} := \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n} \mid L_{ab}(\mathbf{x},\mathbf{y}) = 0\}.$$

Note that $L_{ab}(\mathbf{x},\mathbf{y}) = 0$ if and only if $L_{ba}(\mathbf{x},\mathbf{y}) = 0$, namely $H_{ab} = H_{ba}$. The set $\mathcal{A} = \{H_{ab}\}_{a<b}$ is known as a *hyperplane arrangement* in $\mathbb{R}^{2n}$. We have proved the following lemma

LEMMA 7.1. *Let $u \in \mathbb{C}^n$ be represented as $u = \mathbf{x} + i\mathbf{y}$, $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n}$. Then, $\Delta_{\mathbb{C}^n} \cup X_{\mathbb{C}^n}(\widetilde{\tau})$ is the union of hyperplanes $H_{ab} \in \mathcal{A}$ defined by the linear equations $L_{ab}(\mathbf{x},\mathbf{y}) = 0$, $1 \le a < b \le$, as in (7.1), (7.2).*

Properties of finite hyperplane arrangements in $\mathbb{R}^{2n}$ are well knows. In particular, consider the set

$$\mathbb{R}^{2n} - \bigcup_{1\le a<b\le n} H_{ab}.$$

A connected component of the above set is called a *region* of $\mathcal{A}$. It is well known that every region of $\mathcal{A}$ is open and convex, and hence homeomorphic to the interior of an $2n$-dimensional ball of $\mathbb{R}^{2n}$. It is therefore *a cell* in the proper sense. We have proved the following

PROPOSITION 7.1. *A $\widetilde{\tau}$-cell in $\mathbb{C}^n$ is a cell, namely an open and convex subset of $\mathbb{C}^n$, homeomorphic to the open ball $\{u \in \mathbb{C}^n \mid |u_1|^2 + \cdots + |u_n|^2 < 1\} = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2n} \mid x_1^2 + \cdots + y_n^2 < 1\}$.*

REMARK 7.2. Three hyperplanes with one index in common intersect. Indeed, let b be the common index. Then,

$$\begin{cases} L_{ab}(\mathbf{x},\mathbf{y}) = 0 \\ L_{bc}(\mathbf{x},\mathbf{y}) = 0 \end{cases} \qquad \Longrightarrow \qquad L_{ac}(\mathbf{x},\mathbf{y}) = 0.$$

Hence,

$$H_{ab} \cap H_{bc} \subset H_{ac}, \quad H_{bc} \cap H_{ac} \subset H_{ab}, \quad H_{ac} \cap H_{ab} \subset H_{bc}.$$

Equivalently

$$H_{ab} \cap H_{bc} \cap H_{ac} = H_{ab} \cap H_{bc} = H_{ab} \cap H_{ac} = H_{bc} \cap H_{ac}.$$

We now consider $\widetilde{\tau}$-cells in $\mathcal{U}_{\epsilon_0}(0)$ in case the eigenvalues of $\Lambda(t)$ are *linear in t* as in (1.25). The arguments above apply to this case, since $u_a = u_a(0) + t_a$ is a linear translation. Let $u(0) = (u_1(0), ..., u_n(0))$ be as in (5.50)-(5.53), so that $u(t) = u(0) + t$. Let us split $u(t)$ into real ($\Re$) and imaginary ($\Im$) parts:

$$u(0) = \mathbf{x}_0 + i\mathbf{y}_0, \quad t = \Re t + i\Im t \qquad \Longrightarrow \qquad u(t) = \Big(\mathbf{x}_0 + i\mathbf{y}_0\Big) + \Big(\Re t + i\Im t\Big).$$

Here, $\Re t := (\Re t_1, ..., \Re t_n) \in \mathbb{R}^n$ and $\Im t := (\Im t_1, ..., \Im t_n) \in \mathbb{R}^n$. Define the hyperplanes

$$H'_{ab} := \Big\{(\Re t, \Im t) \in \mathbb{R}^n \;\Big|\; L_{ab}(\Re t, \Im t) + L_{ab}(\mathbf{x}_0, \mathbf{y}_0) = 0\Big\}, \qquad 1 \le a \neq b \le n, \tag{7.3}$$

and

$$\widetilde{H}_{ab} := H'_{ab} \cap \mathcal{U}_{\epsilon_0}(0). \tag{7.4}$$

Then,

$$\Delta \cup X(\widetilde{\tau}) = \bigcup_{1\leq a<b\leq n} \widetilde{H}_{ab}.$$

Note that $L_{ab}(\mathbf{x}_0, \mathbf{y}_0) = 0$ for any $a \neq b$ corresponding to a coalescence $u_a(t) - u_b(t) \to 0$ for $t \to 0$.

Corollary 7.1. *If the eigenvalues of $\Lambda(t)$ are linear in t as in (1.25), then a $\widetilde{\tau}$-cell in $\mathcal{U}_{\epsilon_0}(0)$ is simply connected.*

Proof: Any of the regions of a the hyperplane arrangement with hyperplanes (7.3) is open and convex. $\mathcal{U}_{\epsilon_0}(0)$ is a polydisc, hence it is convex. The intersection of a region and $\mathcal{U}_{\epsilon_0}(0)$ is then convex and simply connected. $\square$

Remark 7.3. The $\widetilde{H}$'s enjoy the same properties of hyperplanes H's as in Remark 7.2. In other words, if a Stokes ray associated with the pair $u_a(t), u_b(t)$ and a Stokes ray associated with $u_b(t), u_c(t)$ cross an admissible direction $R(\widetilde{\tau} \bmod \pi)$ at some point t, then also a ray associated with $u_a(t), u_c(t)$ does.

Remark 7.4. We anticipate the fact that if ϵ_0 is sufficiently small as in Section 7.6.1, then $\widetilde{H}_{ab} \cap \mathcal{U}_{\epsilon_0}(0) = \emptyset$ for any $a \neq b$ such that for $t \to 0$, $u_a(t) \to \lambda_i$ and $u_b(t) \to \lambda_j$ with $1 \leq i \neq j \leq s$ (i.e. $u_a(0) \neq u_b(0)$). See below Remark 7.8 for explanations.

7.3. Sectors $\mathcal{S}_\nu(t)$ and $\mathcal{S}_\nu(K)$

We introduce $t-$dependent sectors, which serve to define Stokes matrices of $Y(z,t)$ of Corollary 5.2 in a consistent way w.r.t. matrices of $\mathring{Y}(z)$ of Theorem 5.2.

Definition 7.2 (Sectors $\mathcal{S}_{\nu+k\mu}(t)$). Let $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$, and $k \in \mathbb{Z}$. Let $t \in \mathcal{U}_{\epsilon_0}(0)\backslash X(\widetilde{\tau})$. We define $\mathcal{S}_{\nu+k\mu}(t)$ to be the sector containing the closed sector $\overline{S}(\widetilde{\tau} - \pi + k\pi, \widetilde{\tau} + k\pi)$, and extending up to the nearest Stokes rays of $\Lambda(t)$ outside $\overline{S}(\widetilde{\tau} - \pi + k\pi, \widetilde{\tau} + k\pi)$.

The definition implies that

$$\mathcal{S}_{\nu+k\mu}(t) \subset \mathcal{S}_{\nu+k\mu}, \qquad \mathcal{S}_{\nu+k\mu}(0) = \mathcal{S}_{\nu+k\mu}.$$

For simplicity, put $k = 0$. Note that $\mathcal{S}_\nu(t)$ is uniquely defined and contains the set of basic Stokes rays of $\Lambda(t)$ lying in $S(\widetilde{\tau} - \pi, \widetilde{\tau})$. We point out the following facts:

• Due to the continuous dependence on t of the directions of Stokes rays for $t \notin \Delta$, then $\mathcal{S}_\nu(t)$ continuously deforms as t varies in a $\widetilde{\tau}$ cell.

• $\mathcal{S}_\nu(t)$ is "discontinuous" at Δ, by which we mean that some Stokes rays disappear at points of Δ.

• $\mathcal{S}_\nu(t)$ is "discontinuous" at $X(\widetilde{\tau})$, because one or more Stokes rays cross the admissible ray $R(\widetilde{\tau})$ (this is why $\mathcal{S}_\nu(t)$ has not been defined at $X(\widetilde{\tau})$). More precisely, consider a continuous monotone curve $t = t(x)$, x belonging to a real interval, which for one pair (a,b) intersects $\widetilde{H}_{ab}\backslash\Delta$ at $x = x_*$ (recall that $\widetilde{H}_{ab}$ is define in (7.4)). Hence, the curve passes from one cell to another cell, which are separated by $\widetilde{H}_{ab}$. A Stokes ray associated with $(u_a(t), u_b(t))$ crosses $R(\widetilde{\tau})$ when $t = t(x_*)$. Then $\mathcal{S}_\nu(t(x))$ has a discontinuous jump at x_*.

The above observations assure that the following definition is well posed.

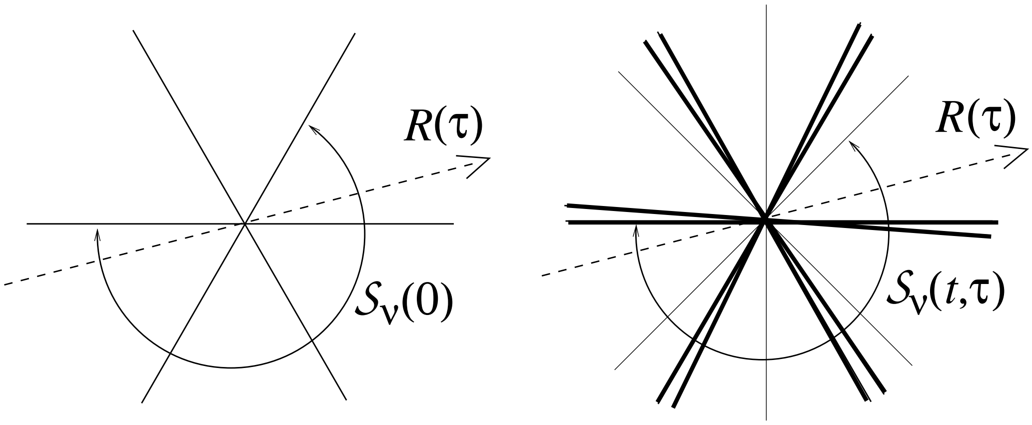


FIGURE 7.1. In the left figure $t=0$ and the sector $\mathcal{S}_\nu \equiv \mathcal{S}_\nu(0)$ is represented in a sheet of the universal covering $\mathcal{R}$. The dashed line represents $R(\widetilde{\tau}) \cup R(\widetilde{\tau}-\pi)$. The arrow is that of the oriented ray $R(\widetilde{\tau})$. The rays are the Stokes rays associated with couples λ_i, λ_j, $1 \le i \neq j \le s$. In the right figure t slightly differs from $t=0$; the rays in bold are small deformations of the rays appearing in the left figure, associated with couples $u_a(t), u_b(t)$ s.t. $u_a(0)=\lambda_i$, $u_b(0)=\lambda_j$ with $i \neq j$. The rays in finer tone are the rays associated with couples such that $u_a(0)=u_b(0)=\lambda_i$. The sector $\mathcal{S}_\nu(t)=\mathcal{S}_\nu(t,\widetilde{\tau})$ is represented.

DEFINITION 7.3 (Sector $\mathcal{S}_\nu(K)$). Let K be a compact subset of a $\widetilde{\tau}$-cell. We define

$$\mathcal{S}_\nu(K) := \bigcap_{t\in K} \mathcal{S}_\nu(t) \ \subset \mathcal{S}_\nu .$$

By the definitions, $\mathcal{S}_\nu(t)$ and $\mathcal{S}_\nu(K)$ have the angular width strictly greater than π and they contain the admissible ray $R(\widetilde{\tau})$ of Definition 6.2. Moreover $\mathcal{S}_\nu(K_1) \supset \mathcal{S}_\nu(K_2)$ for $K_1 \subset K_2$, and $\mathcal{S}_\nu(K_1 \cup K_2) = \mathcal{S}_\nu(K_1) \cap \mathcal{S}_\nu(K_2)$. Below in the Chapter we will consider a simply connected subset $\mathcal{V}$ of a $\widetilde{\tau}$-cell, such that the closure $\overline{\mathcal{V}}$ is also contained in the cell, and take

$$K=\overline{\mathcal{V}}.$$

REMARK 7.5. A more precise notation could be used as follows:

$$\mathcal{S}_\nu(t)=\mathcal{S}_\nu(t;\widetilde{\tau}) \ , \tag{7.5}$$

to keep track of $\widetilde{\tau}$, because for given ν and two different choices of $\widetilde{\tau} \in (\tau_\nu, \tau_{\nu+1})$, then the resulting $\mathcal{S}_\nu(t)$'s may be different. Figures 7.1 and 7.2 show two different $\mathcal{S}_\nu(t)$, according to two choices of $\widetilde{\tau}$. As a consequence, while in Definition 7.2 we could well define $\mathcal{S}_{\nu+k\mu}(t) \subset \mathcal{S}_{\nu+k\mu}$, for any $k \in \mathbb{Z}$, we cannot define sectors $\mathcal{S}_{\nu+1}(t)$, $\mathcal{S}_{\nu+2}(t)$, ..., $\mathcal{S}_{\nu+\mu-1}(t)$.

7.4. Fundamental Solutions $Y_\nu(z,t)$ and Stokes Matrices $\mathbb{S}_\nu(t)$

Let $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$. We show that, if $t_0 \notin \Delta$ belongs to a $\widetilde{\tau}$-cell, we can extend the asymptotic behaviour (5.59) of Corollary 5.2 from $\overline{S}^{(t_0)}(\alpha,\beta)$ to $\mathcal{S}_\nu(t)$. The fundamental matrix of Corollary 5.2 will then be denoted by $Y_\nu(z,t)$.

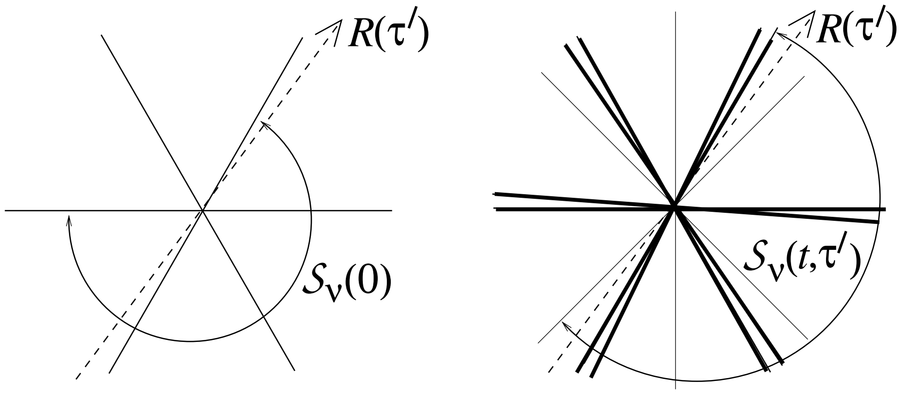


FIGURE 7.2. The explanation for this figure is the same as for Figure 7.1, but $\widetilde{\tau}' \neq \widetilde{\tau}$. $\mathcal{S}_\nu \equiv \mathcal{S}_\nu(0)$ is the same, but $\mathcal{S}_\nu(t) = \mathcal{S}_\nu(t,\widetilde{\tau}')$ differs from $\mathcal{S}_\nu(t,\widetilde{\tau})$ of figure 7.1.

PROPOSITION 7.2 (Solution $Y_\nu(z,t)$ with asymptotics on $\mathcal{S}_\nu(t)$, $t \in \mathcal{U}_\rho(t_0)$). *Let Assumption 5.1 hold for the system (5.1). Let t_0 belong to a $\widetilde{\tau}$-cell. For any $\nu \in \mathbb{Z}$ there exists $\mathcal{U}_\rho(t_0)$ contained in the cell of t_0 and a unique fundamental solution of the system (5.1) as in Corollary 5.2 of the form*

$$Y_\nu(z,t) = G_0(t)\mathcal{G}_\nu(z,t)z^{B_1(t)}e^{\Lambda(t)z}, \tag{7.6}$$

holomorphic in $(z,t) \in \{z \in \mathcal{R} \mid |z| \geq N\} \times \mathcal{U}_\rho(t_0)$, with asymptotic behaviour (5.59) extended to $\mathcal{S}_\nu(t)$, $t \in \mathcal{U}_\rho(t_0)$. Namely $\forall\ t \in \mathcal{U}_\rho(t_0)$ the following asymptotic expansion holds:

$$\mathcal{G}_\nu(z,t) \sim I + \sum_{k=1}^{\infty} F_k(t)z^{-k}, \qquad z \to \infty, \quad z \in \mathcal{S}_\nu(t). \tag{7.7}$$

The asymptotics (7.7) restricted to $z \in \mathcal{S}_\nu(\mathcal{U}_\rho(t_0))$ is uniform in the compact polydisc $\mathcal{U}_\rho(t_0)$.

Note: Recall that by definition of asymptotics, the last sentence of the above Proposition means that the asymptotics (7.7) is uniform in the compact polydisc $\mathcal{U}_\rho(t_0)$ when $z \to \infty$ in any *proper closed subsector* of $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0))$.

Proof: In Theorem 5.3 choose $\overline{S}^{(t_0)}(\alpha,\beta) = \overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau})$. This contains a set of basic Stokes rays of $\Lambda(t_0)$ and of $\Lambda(t)$ for any t in the cell of t_0. Then, Sibuya's Theorem 5.3 and Corollary 5.2 apply, with fundamental solution $Y(z,t)$ defined for t in some $\mathcal{U}_\rho(t_0)$. It is always possible to restrict ρ so that $\mathcal{U}_\rho(t_0)$ is all contained in the cell.

• [Extension to $\mathcal{S}_\nu(t)$] For $t \in \mathcal{U}_\rho(t_0)$, the sector containing $S(\widetilde{\tau}-\pi,\widetilde{\tau})$ and extending up to the nearest Stokes rays outside is $\mathcal{S}_\nu(t)$, by definition. Hence there exists a labelling as in Section 5.5.1, and a $\sigma \in \mathbb{Z}$, such that $\mathcal{S}_\nu(t_0) = \mathcal{S}_\sigma^{(t_0)}$. The Extension Theorem and the Uniqueness Theorem can be applied to $Y(z,t)$ for any fixed t, because $S(\widetilde{\tau}-\pi,\widetilde{\tau})$ contains a set of basic Stokes rays. Hence, for any $t \in \mathcal{U}_\rho(t_0)$ the solution $Y(z,t)$ is unique with the asymptotic behaviour (5.59) for $z \to \infty$ in $\mathcal{S}_\nu(t)$.

• [Uniformity in $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0))$] Clearly, $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0)) \supset \overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau})$. Since $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0)) \subset \mathcal{S}_\nu(t)$ for any $t \in \mathcal{U}_\rho(t_0)$, the asymptotics (7.7) holds also in $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0))$. Moreover, the asymptotics is uniform in $\mathcal{U}_\rho(t_0)$ if $z \to \infty$ in $\overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau})$, by Theorem 5.3 and Corollary 5.2. We apply the same proof of the Extension Lemma 6.2 as follows. Let θ_L and θ_R be the directions of the left and right boundary rays of $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0))$ (i.e. $\overline{\mathcal{S}}_\nu(\mathcal{U}_\rho(t_0)) = \overline{S}(\theta_R,\theta_L)$). Let $\overline{S}_1 := \overline{S}(\phi,\psi)$, for $\theta_R + \pi < \phi < \psi < \theta_L$, and

$\overline{S}_2 := \overline{S}(\phi',\psi')$ for $\theta_R < \phi' < \psi' < \theta_L - \pi$. Let us consider $\overline{S}_1$. By construction, $\overline{S}_1$ does not contain Stokes rays of $\Lambda(t)$ for any $t \in \mathcal{U}_\rho(t_0)$, and so, by Theorem 5.3 now applied with a $\overline{S}^{(t_0)} = \overline{S}_1$, there exists $\widetilde{Y}(z,t) \sim Y_F(z,t)$, for $z \to \infty$ in $\overline{S}_1$, uniformly in $|t-t_0| \le \rho_1$, for suitable $\rho_1 > 0$. Moreover, $Y(z,t) = \widetilde{Y}(z,t)C(t)$, where $C(t)$ is an invertible holomorphic matrix in $|t-t_0| \le \min(\rho,\rho_1)$. The matrix entries satisfy $e^{(u_a(t)-u_b(t))z}C_{ab}(t) = \widetilde{\mathcal{G}}(z,t)^{-1}\mathcal{G}(z,t) \sim \delta_{ab}$, $a,b = 1,\dots,n$, for $|t-t_0| \le \min(\rho,\rho_1)$ and $z \to \infty$, $z \in \overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau}) \cap \overline{S}_1$. Since $\Re((u_a(t)-u_b(t))z)$ does not change sign for t in the cell and $z \in \overline{S}_1$, then $Y(z,t) \sim Y_F(z,t)$ also for $z \in \overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau}) \cup \overline{S}_1$, uniformly in $|t-t_0| \le \min(\rho,\rho_1)$. The same arguments for $\overline{S}_2$ allow to conclude that $Y(z,t) \sim Y_F(z,t)$ for $z \in \overline{S}(\widetilde{\tau}-\pi,\widetilde{\tau}) \cup \overline{S}_1 \cup \overline{S}_2$, uniformly in $|t-t_0| \le \min(\rho,\rho_1,\rho_2)$. Finally, from the proof given by Sibuya of Theorem 5.3 (cf. [**Sib62**], especially from page 44 on) it follows that ρ_1 and ρ_2 are greater or equal to ρ. The proof is concluded. We denote $Y(z,t)$ with $Y_\nu(z,t)$. $\square$

Definition 7.4 (Stokes matrices $\mathbb{S}_{\nu+k\mu}(t)$). The **Stokes matrix** $\mathbb{S}_{\nu+k\mu}(t)$, $k \in \mathbb{Z}$, is defined for $t \in \mathcal{U}_\rho(t_0)$ of Proposition 7.2 by,

$$Y_{\nu+(k+1)\mu}(z,t) = Y_{\nu+k\mu}(z,t)\mathbb{S}_{\nu+k\mu}(t), \qquad z \in \mathcal{R},$$

where the $Y_{\nu+k\mu}(z,t)$ and $Y_{\nu+(k+1)\mu}(z,t)$ are as in Proposition 7.2.

$\mathbb{S}_{\nu+k\mu}(t)$ is holomorphic in $t \in \mathcal{U}_\rho(t_0)$, because so are $Y_{\nu+(k+1)\mu}(z,t)$ and $Y_{\nu+k\mu}(z,t)$.

7.5. Analytic Continuation of $Y_\nu(z,t)$ on a Cell preserving the Asymptotics

Proposition 7.3 (Continuation of $Y_\nu(z,t)$ preserving the asymptotics, along a curve in a cell). *Let Assumption 5.1 hold for the system (5.1). The fundamental solution $Y_\nu(z,t)$ of Proposition 7.2 holomorphic in $t \in \mathcal{U}_\rho(t_0)$ admits t-analytic continuation along any curve contained in the $\widetilde{\tau}$-cell of t_0, and maintains its asymptotics (7.7) for $z \to \infty$, $z \in \mathcal{S}_\nu(t)$, for any t belonging to a neighbourhood of the curve. The asymptotics is uniform in a closed tubular neighbourhood U of the curve for $z \to \infty$ in (any proper subsector of) $\mathcal{S}_\nu(U)$.*

Proof: Let $Y_\nu(z,t)$, $t \in \mathcal{U}_\rho(t_0)$ be as in Proposition 7.2. Join t_0 to a point t_{final}, belonging to the $\widetilde{\tau}$-cell of t_0 and not belonging to $\mathcal{U}_\rho(t_0)$, by a curve whose support is contained in the $\widetilde{\tau}$-cell. Let $t_1 \in \partial\mathcal{U}_\rho(t_0)$ be the intersection point with the curve. Theorem 5.3 and its Corollary 5.2 can be applied at t_1, with sector $\mathcal{S}_\sigma^{(t_1)} \equiv \mathcal{S}_\nu(t_1)$, by definition. By Proposition 7.2, there exists a unique fundamental solution, which we temporarily denote $Y_\nu^{(1)}(z,t)$, with asymptotics (7.7) for $z \to \infty$, $z \in \mathcal{S}_\nu(t)$, $t \in \mathcal{U}_{\rho_1}(t_1)$. Here ρ_1 is possibly restricted so that $\mathcal{U}_{\rho_1}(t_1)$ is contained in the cell. The asymptotics is uniform in $\mathcal{U}_{\rho_1}(t_1)$ for $z \to \infty$ in $\mathcal{S}_\nu(\mathcal{U}_{\rho_1}(t_1))$. Now, when $t \in \mathcal{U}_\rho(t_0) \cap \mathcal{U}_{\rho_1}(t_1)$, both $Y_\nu(z,t)$ and $Y_\nu^{(1)}(z,t)$ are defined, with the same asymptotic behaviour (7.7) for $z \to \infty$, $z \in \mathcal{S}_\nu(\mathcal{U}_\rho(t_0)) \cap \mathcal{S}_\nu(\mathcal{U}_{\rho_1}(t_1))$, uniform in $t \in \mathcal{U}_\rho(t_0) \cap \mathcal{U}_{\rho_1}(t_1)$. Moreover, $\mathcal{S}_\nu(\mathcal{U}_\rho(t_0)) \cap \mathcal{S}_\nu(\mathcal{U}_{\rho_1}(t_1))$ has central opening angle strictly greater than π because both $\mathcal{U}_\rho(t_0)$ and $\mathcal{U}_{\rho_1}(t_1)$ are contained in the cell. By uniqueness it follows that $Y_\nu(z,t) = Y_\nu^{(1)}(z,t)$ for $t \in \mathcal{U}_\rho(t_0) \cap \mathcal{U}_{\rho_1}(t_1)$. This gives the t-analytic continuation of $Y_\nu(z,t)$ on $\mathcal{U}_\rho(t_0) \cup \mathcal{U}_{\rho_1}(t_1)$. The procedure can be repeated for a sequence of neighbourhoods $\mathcal{U}_{\rho_n}(t_n)$, $n = 1,2,3,\dots$ (t_n is point of intersection of the curve with $\mathcal{U}_{\rho_{n-1}}(t_{n-1})$). Consider $U := \bigcup_n \mathcal{U}_{\rho_n}(t_n)$. If t_{final} is an internal point of $\in U$, the proof is completed and $\mathcal{U}_{\rho_n}(t_n)$ is a finite sequence. If not, the point t_* of intersection of ∂U with the curve either precedes t_{final}, or $t_* = t_{final} \in \partial U$. Since t_* belongs to the cell, Proposition 7.2 can be applied. The sector $\mathcal{S}^{(t_*)}_{\sigma_*}$, $\sigma_* \in \mathbb{Z}$, prescribed by Theorem 5.3 and Corollary 5.2 coincides with $\mathcal{S}_\nu(t_*)$, by definition. Therefore, the analytic continuation is feasible in a $\mathcal{U}_{\rho^*}(t_*)$, as in the construction above. We can add $\mathcal{U}_{\rho^*}(t_*)$ to U. In this way, t_{final} is always reached by a finite sequence, and U is compact. By construction, the asymptotics is uniform in any compact subset $K \subset U$, including also $K \equiv U$, for $z \to \infty$, $z \in \mathcal{S}_\nu(K)$. $\Box$

COROLLARY 7.2. **(Analytic continuation of $Y_\nu(z,t)$ preserving the asymptotics on the whole cell – case of eigenvalues (1.25))**. *Let Assumption 5.1 hold for the system (5.1). If the eigenvalues of $\Lambda(t)$ are linear in t as in (1.25) then $Y_\nu(z,t)$ of Proposition 7.2 is holomorphic on the whole $\widetilde{\tau}$-cell, with asymptotics (7.7) for $z \to \infty$ in $\mathcal{S}_\nu(t)$, for any t in the cell. For any compact subset K of the cell, the asymptotics (7.7) for $z \to \infty$, $z \in \mathcal{S}_\nu(K)$, is uniform in $t \in K$.*

Proof: If the eigenvalues of $\Lambda(t)$ are linear in t as in (1.25), then any $\widetilde{\tau}$-cell is simply connected (see Corollary 7.1). Hence, the continuation of $Y_\nu(z,t)$ is independent of the curve. $\Box$

- **Notation:** If c is the $\widetilde{\tau}$-cell of Corollary 7.2, the following notation will be used

$$Y_\nu(z,t) = Y_\nu(z,t;\widetilde{\tau},c), \qquad t \in c. \tag{7.8}$$

7.5.1. Analytic continuation of $Y_\nu(z,t;\widetilde{\tau},c)$ preserving the asymptotics beyond ∂c. Let the eigenvalues of $\Lambda(t)$ be linear in t as in (1.25). The analytic continuation of Corollary 7.2 and the asymptotics (7.7) can be extended to values of t a little bit outside the cell. This is achieved by a small variation $\widetilde{\tau} \mapsto \widetilde{\tau} \pm \varepsilon$, for $\varepsilon > 0$ sufficiently small.

Recall that the Stokes rays in $\mathcal{R}$ associated with the pair $(u_a(t), u_b(t))$ and $(u_b(t), u_a(t))$, $a \neq b$, have respectively directions

$$\arg z = \frac{3\pi}{2} - \arg_p(u_a(t) - u_b(t)) + 2N\pi \quad \text{and} \quad \arg z = \frac{3\pi}{2} - \arg_p(u_b(t) - u_a(t)) + 2N\pi, \qquad N \in \mathbb{Z}.$$

Thus, their projections onto $\mathbb{C}$ are the following opposite rays

$$PR_{ab}(t) := \{z \in \mathbb{C} \mid z = -i\rho(\overline{u}_a(t) - \overline{u}_b(t))\}, \qquad PR_{ba}(t) := \{z \in \mathbb{C} \mid z = -i\rho(\overline{u}_b(t) - \overline{u}_a(t))\}. \tag{7.9}$$

For $t \notin W(\widetilde{\tau})$, a ray $PR_{ab}(t)$ lies either in the half plane to the left or to the right of the oriented admissible line $l(\widetilde{\tau})$. For $t \notin W(\widetilde{\tau})$, the *finite* set of projected rays is the union of the two disjoint subsets of (projected) rays to the left and to the right of $l(\widetilde{\tau})$ respectively. Now, for t varying inside a cell c, the projected rays never cross $l(\widetilde{\tau})$. On the other hand, if t and t' belong to different cells c and c', then the two subsets of rays to the right and the left of $l(\widetilde{\tau})$ which are associated with t do not coincide with the two subsets associated with t'. These simple considerations imply the following:

PROPOSITION 7.4. *A $\widetilde{\tau}$-cell is uniquely characterised by the subset of projected rays which lie to the left of $l(\widetilde{\tau})$.*

DEFINITION 7.5. A point $t_* \in \widetilde{H}_{ab} \backslash \Delta$ is **simple** if $t_* \notin \widetilde{H}_{ab} \cap \widetilde{H}_{a'b'}$ for any $(a',b') \neq (a,b)$.

If t varies along a curve crossing the boundary ∂c of a cell c at a simple point belonging to $\widetilde{H}_{ab} \backslash \Delta$, for some $a \neq b$, the ray $PR_{ab}(t)$ crosses either $l_+(\widetilde{\tau})$ or $l_-(\widetilde{\tau})$, while $PR_{ba}(t)$ crosses either $l_-(\widetilde{\tau})$ or $l_+(\widetilde{\tau})$. Since only $PR_{ab}(t)$ and $PR_{ba}(t)$ have crossed $l(\widetilde{\tau})$, then by Proposition 7.4 there is only one

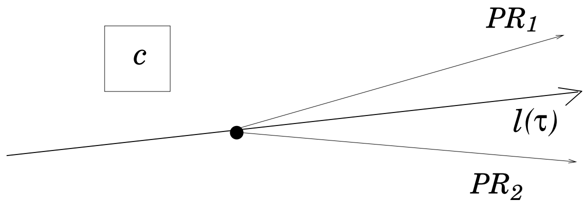


FIGURE 7.3. Configuration of rays corresponding to the cell c of figures 7.10 and 7.11.

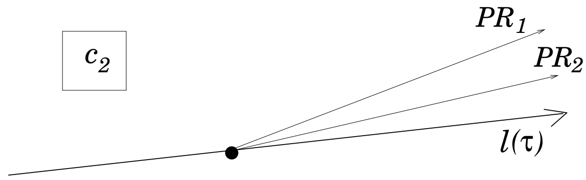


FIGURE 7.4. Configuration of rays corresponding to the cell c_2 of figures 7.10 and 7.11.

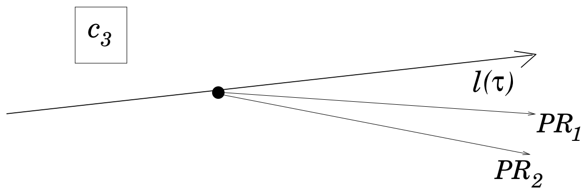


FIGURE 7.5. Configuration of rays corresponding to the cell c_3 of figures 7.10 and 7.11.

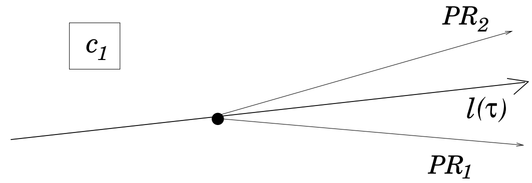


FIGURE 7.6. Configuration of rays corresponding to the cell c_1 of figures 7.10 and 7.11.

neighbouring cell c' sharing the boundary $\widetilde{H}_{ab}$ with c. On the other hand, if the curve crosses $\partial c\backslash\Delta$ at a non simple point, then two or more rays simultaneously cross $l_+(\widetilde{\tau})$ (and the opposite ones cross $l_-(\widetilde{\tau})$). For example, if the crossing occurs at $(\widetilde{H}_{ab}\cap\widetilde{H}_{a'b'})\backslash\Delta$ then there are three cells, call them c_1, c_2, c_3, sharing common boundary $(\widetilde{H}_{ab}\cap\widetilde{H}_{a'b'})\backslash\Delta$ with c. Looking at the configuration of Stokes rays as in the figures 7.3, 7.4, 7.5, 7.6, we conclude that out of the three cells c_1, c_2, c_3, there is one, say it is c_1, such that the transition from c to c_1 occurs with a double crossing of Stokes rays (figure 7.6), namely at a non-simple point; while for the remaining c_2 and c_3 the transition occurs at simple points. In figures 7.3, 7.4, 7.5, 7.6, PR_1 stands for $PR_{ab}(t)$ (or $PR_{ba}(t)$) and PR_2 stands for $PR_{a'b'}(t)$ (or $PR_{b'a'}(t)$). The transition between figure 7.3 and 7.6 is between c and c_1 of figure 7.10, through non simple points of $(\widetilde{H}_{ab}\cap\widetilde{H}_{a'b'})\backslash\Delta$.

REMARK 7.6. Recall that for any $a\neq b$, $\widetilde{H}_{ab}\cap\Delta\neq\emptyset$. Therefore, when we discuss analytic continuation, this requires crossing of "hyperplanes" $\widetilde{H}_{ab}\backslash\Delta$.

PROPOSITION 7.5 (Continuation slightly beyond the cell, preserving asymptotics). *Let the assumptions of Corollary 7.2 hold. Let c and c' be $\widetilde{\tau}$-cells such that $\partial c\cap\partial c'\neq\emptyset$. If $\partial c\cap\partial c'$ does not coincide with the multiple intersection of two or more $\widetilde{H}_{ab}$'s, then $Y_\nu(z,t;\widetilde{\tau},c)$ has analytic continuation, with asymptotics (7.7) in $\mathcal{S}_\nu(t)$, for t slightly beyond $\partial c\backslash\Delta$ into c'. The asymptotics for $z\to\infty$ in $\mathcal{S}_\nu(K)$ is uniform in any compact subset K of the extended cell. Equivalently, $Y_\nu(z,t;\widetilde{\tau},c)$ can be analytically continued along any curve crossing $\partial c\backslash\Delta$ at a simple point and ending slightly beyond $\partial c\backslash\Delta$ in the neighbouring cell c'.*

Proof: Let U be an open connected subset of the $\widetilde{\tau}$-cell c, such that $\overline{U}$ is contained in c. There exists a small $\vartheta=\vartheta(U)>0$ such that for any $t\in\overline{U}$ the projected Stokes rays of $\Lambda(t)$ lie outside the two closed sectors containing $l(\widetilde{\tau})$ and bounded by $l(\widetilde{\tau}+\theta)$ and $l(\widetilde{\tau}-\theta)$, as in figure 7.7. Let $\varepsilon\in[0,\vartheta]$. All

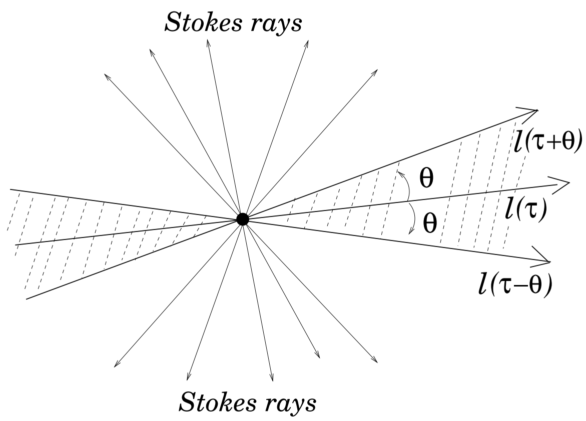


FIGURE 7.7. The two closed sectors of amplitude 2ϑ, not containing Stokes rays when $t \in \overline{U}$.

lines $l(\widetilde{\tau} \pm \varepsilon)$ are admissible for the Stokes rays, when $t \in \overline{U}$. Consider the subset of projected Stokes rays to the left of $l(\widetilde{\tau})$. It uniquely identifies (cf. Proposition 7.4) the $(\widetilde{\tau}+\varepsilon)$-cell and the $(\widetilde{\tau}-\varepsilon)$-cell obtained by deforming the boundaries of c when $\widetilde{\tau} \mapsto \widetilde{\tau}+\varepsilon$ and $\widetilde{\tau} \mapsto \widetilde{\tau}-\varepsilon$ respectively (recall that L_{ab} in (7.3) depends on $\widetilde{\eta} = 3\pi/2 - \widetilde{\tau}$). Call these cells c_ε and $c_{-\varepsilon}$. By construction

$$\begin{aligned} &\overline{U} \subset c \cap c_{\pm\varepsilon}, \qquad \varepsilon \in [0,\vartheta], \\ &Y_\nu(z,t;\widetilde{\tau},c) = Y_\nu(z,t;\widetilde{\tau}\pm\varepsilon, c_{\pm\varepsilon}), \quad t \in U. \end{aligned}$$

The last equality follows from the definition of Y_ν, its uniqueness and Corollary 7.2. Indeed, the analytic continuation explained in the proof of Proposition 7.3 can be repeated for the function $Y_\nu(z,t;\widetilde{\tau}\pm\varepsilon,c_{\pm\varepsilon})$ initially defined in a neighbourhood of t_0 contained in $\overline{U}$, but *with cell partition determined by* $\widetilde{\tau}\pm\varepsilon$. Moreover, by uniqueness of solutions with asymptotics, it follows that $Y_\nu(z,t;\widetilde{\tau},c) = Y_\nu(z,t;\widetilde{\tau}\pm\varepsilon,c_{\pm\varepsilon})$ for $t \in U$. Therefore, $Y_\nu(z,t;\widetilde{\tau},c)$ has analytic continuation to $c_{\pm\varepsilon}$. Now,

$$c_{\pm\varepsilon} \cap \{\text{ union of cells sharing boundary with } c\ \} \neq \emptyset.$$

Then, the analytic continuation of $Y_\nu(z,t;\widetilde{\tau},c)$ obtained above is actually defined in a t-domain bigger than c. We characterise this domain, showing that it intersect any cell c' which is a neighbour of c, and such that $\partial c \cap \partial c'$ does not coincide with the multiple intersection of two or more hyperplanes. Thus, we need to show that $c_{\pm\varepsilon} \cap c' \neq \emptyset$. Notice that $\partial c \cap \partial c' = \widetilde{H}_{ab}$ for suitable a,b. Then, suppose without loss of generality that $PR_{ab}(t)$ crosses $l_+(\widetilde{\tau})$ clockwise when t crosses $\widetilde{H}_{ab}\backslash\Delta$ moving along a curve from c to c'. An example of this crossing is the transition from figure 7.3 to figure 7.5, with the identification $c' = c_3$ of Figure 7.10, and $PR_1 = PR_{ab}$. Then, for the small deformation $\widetilde{\tau} \mapsto \widetilde{\tau}-\varepsilon$ the above discussion applies. Namely, $c_{-\epsilon} \cap c' \neq \emptyset$. See figures 7.8 and 7.9. $\Box$

If $\partial c \cap \partial c' = \widetilde{H}_{ab} \cap \widetilde{H}_{a'b'}$ for some $(a',b') \neq (a,b)$, there is multiple crossing of $l(\widetilde{\tau})$. The proof does not work if the crossing corresponds to a transition such as that from figure 7.3 to figure 7.6, with the identification $c' = c_1$. Since PR_1 and PR_2 cross simultaneously $l_+(\widetilde{\tau})$ from opposite sides, any deformation $\widetilde{\tau} \mapsto \widetilde{\tau}\pm\varepsilon$ produces a cell $c_{\pm\varepsilon}$ which does not intersect c_1. In other words, the deformation prevents points of $c_{\pm\varepsilon}$ from getting close to $\widetilde{H}_{ab} \cap \widetilde{H}_{a'b'}$. The schematic figure 7.10 shows the 4 cells corresponding to the figures from 7.3 to 7.6. It is shown that $Y_\nu(z,t;\widetilde{\tau},c)$ can be continued slightly inside c_2 and c_3, but not inside $c' = c_1$. It is worth noticing that both $Y_\nu(z,t;\widetilde{\tau},c_2)$ and $Y_\nu(z,t;\widetilde{\tau},c_3)$ can be continued beyond $\widetilde{H}_{ab} \cap \widetilde{H}_{a'b'}$. See figure 7.11 for $Y_\nu(z,t;\widetilde{\tau},c_3)$.

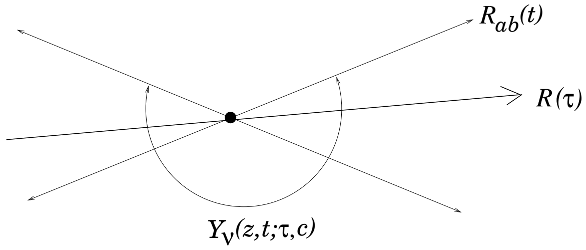


FIGURE 7.8. $Y_\nu(z,t;\widetilde{\tau},c)$ for $t \in c$. The sector where $Y_\nu(z,t;\widetilde{\tau},c)$ has the canonical asymptotic behaviour is represented.

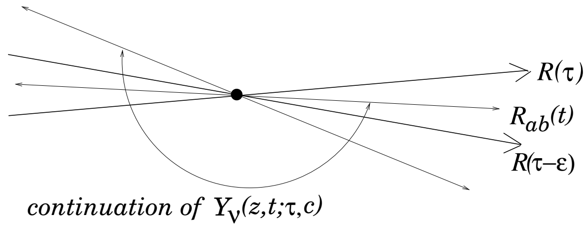


FIGURE 7.9. Analytic continuation of $Y_\nu(z,t;\widetilde{\tau},c)$ for t in the neighbouring cell c' just after the crossing of $\partial c\backslash\Delta$, namely just after $R_{ab}(t)$ has crossed $R(\widetilde{\tau})$. The sector where $Y_\nu(z,t;\widetilde{\tau},c)$ has the canonical asymptotic behaviour is represented.

REMARK 7.7. If the eigenvalues are linear in t as in (1.25), the results of this section assures that the fundamental solutions $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$'s are holomorphic in a $\widetilde{\tau}$-cell c and a little beyond, that they maintain the asymptotic behaviour, and then the corresponding Stokes matrices $\mathbb{S}_{\nu+k\mu}(t)$'s are defined and holomorphic in the whole $\widetilde{\tau}$-cell c and a little bit beyond.

7.6. Fundamental Solutions $Y_\nu(z,t)$ and Stokes Matrices $\mathbb{S}_\nu(t)$ holomorphic at Δ

If the fundamental solutions $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$'s of (5.1) (with Assumption 5.1) have analytic continuation to the whole $\mathcal{U}_{\epsilon_0}(0)$, in this section we give sufficient conditions such that the continuations are c-indendent solutions $Y_{\nu+k\mu}(z,t)$'s, which maintain the asymptotic behaviour in large sectors $\widehat{\mathcal{S}}_\nu$ defined below, so that the Stokes matrices $\mathbb{S}_{\nu+k\mu}(t)$ are well defined in the whole $\mathcal{U}_{\epsilon_0}(0)$. Moreover, we show that $Y_{\nu+k\mu}(z,0)\equiv \mathring{Y}_{\nu+k\mu}(z)$ and $\mathbb{S}_{\nu+k\mu}(0)\equiv\mathring{\mathbb{S}}_{\nu+k\mu}$, where $\mathring{Y}_{\nu+k\mu}(z)$, $\mathring{\mathbb{S}}_{\nu+k\mu}$ have been defined in Section 6.1 for the system at fixed $t=0$.

7.6.1. Restriction of ϵ_0. So far, ϵ_0 has been taken so small that $\Lambda_i(t)$ and $\Lambda_j(t)$, $1\le i\neq j\le s$, have no common eigenvalues for $t\in\mathcal{U}_{\epsilon_0}(0)$. If $\Lambda=\Lambda(0)$ has at least two distinct eigenvalues, we consider a further restriction of ϵ_0. Let $\widetilde{\eta}=3\pi/2-\widetilde{\tau}$ be the admissible direction associated with the direction $\widetilde{\tau}$ of the admissible ray $R(\widetilde{\tau})$. Let δ_0 be a small positive number such that

$$\delta_0<\min_{1\le i\neq j\le s}\delta_{ij}, \tag{7.10}$$

where δ_{ij} is $1/2$ of the distance between two parallel lines of angular direction $\widetilde{\eta}$ in the λ-plane, one passing through λ_i and one through λ_j; namely

$$\delta_{ij}:=\frac{1}{2}\min\left\{\left|\lambda_i-\lambda_j+\rho e^{i\widetilde{\eta}}\right|,\ \rho\in\mathbb{R}\right\},\qquad i\neq j=1,2,...,s. \tag{7.11}$$

Clearly, δ_0 depends on the choice of $\widetilde{\eta}$ (see also Remark 7.9). Let $\overline{B}(\lambda_i;\delta_0)$ be the closed ball in $\mathbb{C}$ with center λ_i and radius δ_0. Then, we choose ϵ_0 so small that the eigenvalues $u_1(t)$, ..., $u_n(t)$ for $t\in\mathcal{U}_{\epsilon_0}(0)$ satisfy

$$(u_1(t),...,u_n(t))\in\ \overline{B}(\lambda_1;\delta_0)^{\times p_1}\times\ \cdots\ \times\overline{B}(\lambda_s;\delta_0)^{\times p_s}.$$

As t varies in $\mathcal{U}_{\epsilon_0}(0)$ above, the Stokes rays continuously move, but the directions of the rays associated with a $u_a\in\overline{B}(\lambda_i;\delta_0)$ and a $u_b\in\overline{B}(\lambda_j;\delta_0)$, $i\neq j$, never cross the values $\widetilde{\eta}$ and $\widetilde{\eta}-\pi$ (mod 2π), so that the projected rays $PR_{ab}(t)$ and $PR_{ba}(t)$ never cross the admissible line $l(\widetilde{\tau})$. It follows that

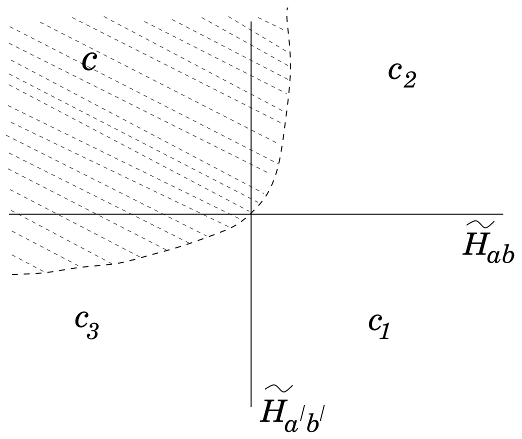


FIGURE 7.10. The cells of complex dimension n (real dimension $2n$) are schematically and improperly depicted in real dimension 2. Boundaries $\widetilde{H}_{ab}$ and $\widetilde{H}_{a'b'}$ are represented as lines, their intersection as a point (understanding that it is not in Δ). The domain of the analytic continuation of $Y_\nu(z,t;\widetilde{\tau},c)$ beyond the boundary of c is the dashed region. The analytic continuation does not go beyond $\widetilde{H}_{ab}\cap\widetilde{H}_{a'b'}$, because the transition from figure 7.3 to figure 7.6 is obtained by a simultaneous crossing of $l(\widetilde{\tau})$ by PR_1 and PR_2 from *opposite sides* of $l(\widetilde{\tau})$.

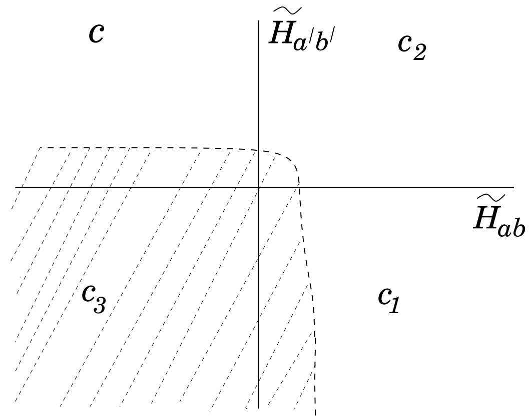


FIGURE 7.11. Analytic continuation of $Y_\nu(z,t;\widetilde{\tau},c_3)$ beyond the boundary of c. The continuation goes up to the 3 neighbouring cells. This corresponds to the fact that the three transitions form figure 7.5 to figures 7.3 and 7.6 occur when PR_1 and PR_2 respectively cross $l(\tau)$, while the transition from figure 7.5 to figure 7.4 occurs when PR_1 and PR_2 simultaneously cross $l(\tau)$, coming from *the same side* of $l(\widetilde{\tau})$ (moving in anticlockwise sense).

the cell decomposition only depends on the Stokes rays associated with couples $(u_a(t),\, u_b(t))$ such that $u_a(0)=u_b(0)=\lambda_i,\ i=1,\ldots,s$.

For eigenvalues *linear* in t as in (1.25), we can take $\epsilon_0=\delta_0$ and

$$\mathcal{U}_{\epsilon_0}(0)\equiv\overline{B}(0;\delta_0)^{\times p_1}\times\ \cdots\ \times\overline{B}(0;\delta_0)^{\times p_s},\qquad \epsilon_0=\delta_0. \tag{7.12}$$

REMARK 7.8. If t moves from one $\widetilde{\tau}$-cell to another, the only Stokes rays which may cross admissible rays $R(\widetilde{\tau}+k\pi)$, $k\in\mathbb{Z}$, are those associated with pairs $u_a(t),u_b(t)$ with $u_a(0)=u_b(0)=\lambda_i$, $i=1,\ldots,s$.

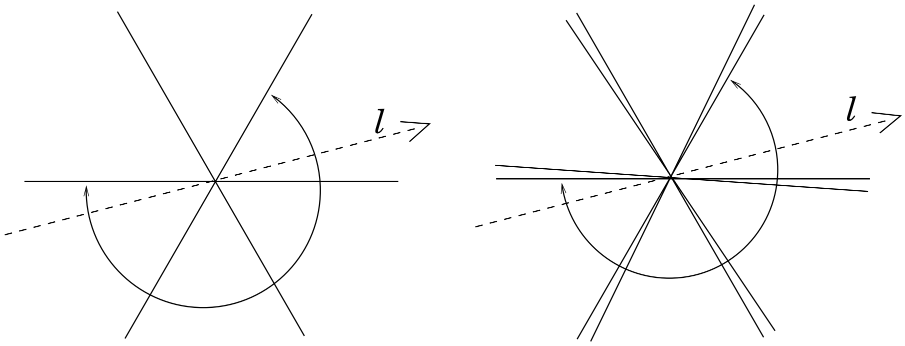


FIGURE 7.12. In the left figure $t=0$ and the sector $\mathcal{S}_\nu$ is represented. The explanation is as for the left part of Figure 7.1. In the right figure, $t\neq 0$. Represented are only the rays associated with couples $u_a(t), u_b(t)$ with $u_a(0)=\lambda_i$, $u_b(0)=\lambda_j$, for $i\neq j$, together with the sector $\widehat{\mathcal{S}}_\nu(t)$.

Therefore, the boundaries of the cells are only the $\widetilde{H}_{ab}$'s such that $u_a(0)=u_b(0)$. In this case, $L_{ab}(\mathbf{x}_0,\mathbf{y}_0)=0$, so that

$$H'_{ab} := \Big\{(\Re t, \Im t)\in\mathbb{R}^{2n}\ \Big|\ L_{ab}(\Re t,\Im t)=0\Big\}.$$

Remark 7.4 follows from the above observations.

7.6.2. The Sectors $\widehat{\mathcal{S}}_\nu(t)$ and $\widehat{\mathcal{S}}_\nu$. Let $\Lambda(t)$ be of the form (1.17) with eigenvalues (1.25). Let $\epsilon_0=\delta_0$ be as in subsection 7.6.1. We define a subset $\mathfrak{R}(t)$ of the set of Stokes rays of $\Lambda(t)$ as follows: $\mathfrak{R}(t)$ contains only those Stokes rays $\{z\in\mathcal{R}\mid \Re(z(u_a(t)-u_b(t)))=0\}$ which are associated with pairs $u_a(t)$, $u_b(t)$ satisfying the condition $u_a(0)\neq u_b(0)$ (namely, $u_a(0)=\lambda_i$, $u_b(0)=\lambda_j$, $i\neq j$; see (5.50)-(5.53)). The reader may visualise the rays in $\mathfrak{R}(t)$ as being originated by the splitting of Stokes rays of $\Lambda(0)$. See figure 7.12.

$\mathfrak{R}(t)$ has the following important property: if t varies in $\mathcal{U}_{\epsilon_0}(0)$, the rays in $\mathfrak{R}(t)$ continuously move, but since $\epsilon_0=\delta_0$, they never cross any admissible ray $R(\widetilde{\tau}+k\pi)$, $k\in\mathbb{Z}$.

DEFINITION 7.6 (Sectors $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$). We define $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$ to be the unique sector containing $S(\widetilde{\tau}-\pi+k\pi,\widetilde{\tau}+k\pi)$ and extending up to the nearest Stokes rays in $\mathfrak{R}(t)$, $t\in\mathcal{U}_{\epsilon_0}(0)$.

Any $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$ contains a set of basic Stokes rays of $\mathfrak{R}$. Moreover,

$$R(\widetilde{\tau})\subset\widehat{\mathcal{S}}_\nu(t)\cap\widehat{\mathcal{S}}_{\nu+\mu}(t)\subset S(\tau_\nu,\tau_{\nu+1}),$$

and

$$\mathcal{S}_\nu(t)\subset\widehat{\mathcal{S}}_\nu(t),\qquad \widehat{\mathcal{S}}_\nu(0)\equiv\mathcal{S}_\nu.$$

In case $\Lambda(0)=\lambda_1 I$, then $\widehat{\mathcal{S}}_\nu(t)$ is unbounded, namely it coincides with $\mathcal{R}$.

DEFINITION 7.7 (Sectors $\widehat{\mathcal{S}}_\nu(K)$). For any compact $K\subset\mathcal{U}_{\epsilon_0}(0)$ we define

$$\widehat{\mathcal{S}}_\nu(K):=\bigcap_{t\in K}\widehat{\mathcal{S}}_\nu(t).$$

If $K_1\subset K_2$, then $\widehat{\mathcal{S}}_\nu(K_2)\subset\widehat{\mathcal{S}}_\nu(K_1)$. For any K_1, K_2, we have $\widehat{\mathcal{S}}_\nu(K_1\cup K_2)=\widehat{\mathcal{S}}_\nu(K_1)\cap\widehat{\mathcal{S}}_\nu(K_2)$.

DEFINITION 7.8 (Sectors $\widehat{\mathcal{S}}_\nu$). If $K=\mathcal{U}_{\epsilon_0}(0)$, we define

$$\widehat{\mathcal{S}}_\nu:=\widehat{\mathcal{S}}_\nu(\mathcal{U}_{\epsilon_0}(0)).$$

Since $\epsilon_0 = \delta_0$, $\widehat{\mathcal{S}}_\nu$ has angular opening greater than π and

$$\widehat{\mathcal{S}}_\nu \subset \widehat{\mathcal{S}}_\nu(0) \equiv \mathcal{S}_\nu,$$
$$R(\widetilde{\tau}) \subset \widehat{\mathcal{S}}_\nu \cap \widehat{\mathcal{S}}_{\nu+\mu} \subset S(\tau_\nu, \tau_{\nu+1}).$$

REMARK 7.9. Notice that $\widetilde{\tau} \in (\tau_\nu, \tau_{\nu+1})$ determines δ_0 through (7.11) and (7.10). Let $\widetilde{\tau}' \in (\tau_\nu, \tau_{\nu+1})$ and let δ_0' be obtained through (7.11) and (7.10). Let $\epsilon_0 = \min\{\delta_0, \delta_0'\}$. We temporarily denote by $\widehat{\mathcal{S}}_\nu[\widetilde{\tau}]$ the sector $\widehat{\mathcal{S}}_\nu$ of Definition 7.8 obtained starting from $\widetilde{\tau}$. Then for the above ϵ_0 we have

$$\widehat{\mathcal{S}}_\nu[\widetilde{\tau}] = \widehat{\mathcal{S}}_\nu[\widetilde{\tau}'].$$

7.6.3. Fundamental group of $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ and generators. Let the eigenvalues of $\Lambda(t)$ be linear in t as in (1.25), $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$ and $\widetilde{\eta} = 3\pi/2 - \widetilde{\tau}$.

The fundamental group $\pi_1(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta, t_{base})$ is generated by loops γ_{ab}, $1 \le a \ne b \le n$, which are homotopy classes of simple curves encircling the component $\{t \in \mathcal{U}_{\epsilon_0}(0) \mid u_a(t) = u_b(t)\}$ of Δ. The choice of the base point is free, because $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ is path-wise connected, since Δ is a braid arrangement in $\mathcal{U}_{\epsilon_0}(0)$ and the hyperplanes are *complex*.

For $\epsilon_0 = \delta_0$ of Section 7.6.1, Stokes rays in $\mathfrak{R}(t)$ never cross the admissible rays $R(\widetilde{\tau} + k\pi)$, $k \in \mathbb{Z}$, when t goes along any loop in $\mathcal{U}_{\epsilon_0}(0)$ (see Remark 7.8). Therefore, as far as the analytic continuation of $Y_\nu(z,t)$ is concerned, it is enough to consider $u_a(t)$ and $u_b(t)$ coming from the unfolding of an eigenvalue λ_i of $\Lambda(0)$ (see the beginning of Section 7.1), namely

$$u_a(t) = \lambda_i + t_a, \qquad u_b(t) = \lambda_i + t_b. \tag{7.13}$$

If we represent t_a and t_b in the same complex plane, so that $t_a - t_b$ is a complex number, a representative of γ_{ab}, which we also denote γ_{ab} with abuse of notation, is represented by the following loop around $t_a - t_b = 0$,

$$t_a - t_b \longmapsto (t_a - t_b)e^{2\pi i}. \tag{7.14}$$

$|t_a - t_b|$ will be taken small. The Stokes rays associated with $u_a(t)$ and $u_b(t)$ have directions

$$\frac{3\pi}{2} - \arg(t_a - t_b) \mod(2\pi), \qquad \frac{3\pi}{2} - \arg(t_b - t_a) \mod(2\pi). \tag{7.15}$$

The projection of these rays onto $\mathbb{C}$ are the two opposite rays $PR_{ab}(t)$ and $PR_{ba}(t)$, as in (7.9) . Along the loop (7.14), each of these rays rotate clockwise and crosses the line $l(\widetilde{\tau})$ twice (recall Definition 6.2), once passing over the positive half line and once over the negative half line, returning to the initial position at the end of the loop. Hence, the support of γ_{ab} is contained in at least two cells, but generally in more than two, as follows.

• There exists a representative contained in only two cells if only $PR_{ab}(t)$ and its opposite $PR_{ba}(t)$ cross $l(\widetilde{\tau})$, each twice. For example, in figure 7.13 the ball $\overline{B}(\lambda_i;\epsilon_0)$ is represented with the loop (7.14). The dots represent other points $u_\gamma(t) \in \overline{B}(\lambda_i;\epsilon_0)$, $\gamma \ne a, b$. $PR_{ab}(t)$ and $PR_{ba}(t)$ cross $l(\widetilde{\tau})$ when $u_a(t)$ and $u_b(t)$ are aligned with the admissible direction $\widetilde{\eta}$. Along the loop, no other u_γ aligns with $u_a(t)$ and $u_b(t)$.

• In general, other (projected) rays cross $l(\widetilde{\tau})$ along any possible representative of γ_{ab}. For example, the representative of (7.14) in figure 7.14 is contained in three cells. Indeed, also $PR_{a\gamma}(t)$ and $PR_{\gamma a}(t)$ cross $l(\widetilde{\tau})$ when u_a and u_γ get aligned with $\widetilde{\eta}$. Alignment corresponds to the passage from one cell to another.

7.6.4. Holomorphic conditions such that $Y_\nu(z,t) \to \mathring{Y}_\nu(z)$ and $\mathbb{S}_\nu(t) \to \mathring{\mathbb{S}}_\nu$ for $t \to 0$, in case of linear eigenvalues (1.25). The following theorem is one of the central results of the Chapter, and it will be used to prove Theorem 1.6.

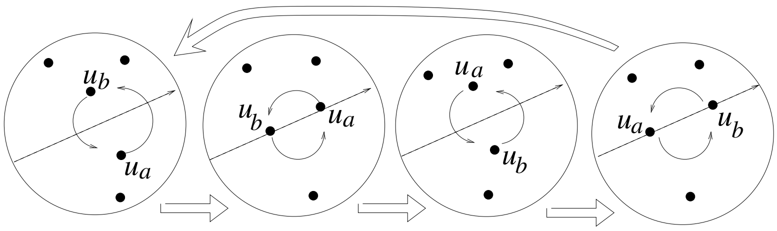


FIGURE 7.13. Loop γ_{ab} represented in $\overline{B}(\lambda_i;\epsilon_0)$. The dashed oriented line is the direction $\widetilde{\eta}$. Along the loop, u_a and u_b get aligned with $\widetilde{\eta}$ twice, in the second and fourth figures. The second figure corresponds to the passage from one initial cell c to a neighbouring cell c' (while PR_{ab} crosses clockwise a half line of $l(\widetilde{\tau})$) and the fourth figure to the return to c (while PR_{ab} crosses clockwise the opposite half line of $l(\widetilde{\tau})$). Other dots represent other eigenvalues $u_\gamma(t)$ in $\overline{B}(\lambda_i;\epsilon_0)$.

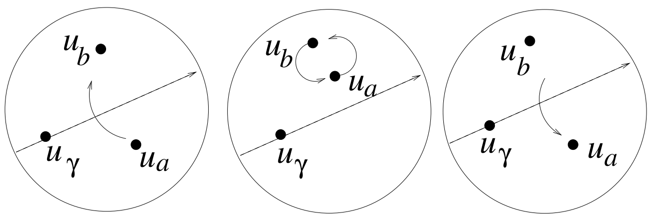


FIGURE 7.14. Loop γ_{ab} represented in $\overline{B}(\lambda_i;\epsilon_0)$. The dashed oriented line is the direction $\widetilde{\eta}$. In the first figure, u_a moves close to u_b. Along the way it gets aligned with u_γ. At this alignment, $PR_{a\gamma}$ crosses clockwise a half line of $l(\widetilde{\tau})$ and t passes from the initial cell c to a cell c'. The second figure is figure 7.13. Here t passes from c' to another cell c'' and then back to c'. In the third figure, u_a moves to the initial position. Along the way it gets aligned with u_γ, $PR_{a\gamma}$ crosses anti-clockwise the same half line of $l(\widetilde{\tau})$ and t returns to the cell c. In this example, γ_{ab} has support contained in three cells.

THEOREM 7.1. *Consider the system (5.1) and let Assumption 5.1 hold, so that (5.1) is holomorphically equivalent to the system (6.1). Let $\Lambda(t)$ be of the form (1.17), with eigenvalues (1.25) and $\epsilon_0=\delta_0$ as in subsection 7.6.1. Let $\widetilde{\tau}$ be the direction of an admissible ray $R(\widetilde{\tau})$, satisfying $\tau_\nu<\widetilde{\tau}<\tau_{\nu+1}$. Suppose that:*

1) For every integer $j\geq 1$, the $F_j(t)$'s are holomorphic in $\mathcal{U}_{\epsilon_0}(0)$ (so necessary and sufficient conditions of Proposition 5.3 hold);

2) For any $\widetilde{\tau}$-cell c of $\mathcal{U}_{\epsilon_0}(0)$ and any $k\in\mathbb{Z}$, the fundamental solution $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$ has analytic continuation as a single-valued holomorphic function on the whole $\mathcal{U}_{\epsilon_0}(0)$. Denote the analytic continuation with the same symbol $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$, $t\in\mathcal{U}_{\epsilon_0}(0)$.

Then:

- *For any $\widetilde{\tau}$-cells c and c',*

$$Y_{\nu+k\mu}(z,t;\ \widetilde{\tau},c)=Y_{\nu+k\mu}(z,t;\ \widetilde{\tau},c'),\qquad t\in\mathcal{U}_{\epsilon_0}(0).$$

Therefore, we can simply write $Y_{\nu+k\mu}(z,t;\widetilde{\tau})$.

• *Let* $\mathcal{G}_{\nu+k\mu}(z,t;\widetilde{\tau}) := G_0(t)^{-1}Y_{\nu+k\mu}(z,t;\widetilde{\tau})z^{-B_1(t)}e^{-\Lambda(t)z}$*. For any* $\epsilon_1<\epsilon_0$ *the following asymptotic expansion holds:*

$$\mathcal{G}_{\nu+k\mu}(z,t;\widetilde{\tau}) \sim I+\sum_{k=0}^{\infty}F_k(t)z^{-k}, \qquad z\to\infty, \quad z\in\widehat{\mathcal{S}}_{\nu+k\mu}, \quad t\in\mathcal{U}_{\epsilon_1}(0). \tag{7.16}$$

The asymptotic expansion is uniform in t *in* $\mathcal{U}_{\epsilon_1}(0)$ *and uniform in* z *in any closed subsector of* $\widehat{\mathcal{S}}_{\nu+k\mu}$*.*

• *For any* $t\in\mathcal{U}_{\epsilon_1}(0)$*, the diagonal blocks of any Stokes matrix* $\mathbb{S}_{\nu+k\mu}(t)$ *are the identity matrices* I_{p_1}*,* I_{p_2}*, ...,*I_{p_s}*. Namely*

$$(\mathbb{S}_{\nu+k\mu})_{ab}(t)=(\mathbb{S}_{\nu+k\mu})_{ba}(t)=0 \qquad \text{whenever } u_a(0)=u_b(0).$$

REMARK 7.10 (Continuation of Remark 7.9). Since $Y_{\nu+k\mu}(z,t;\ \widetilde{\tau},c)=Y_{\nu+k\mu}(z,t;\ \widetilde{\tau},c')\equiv Y_{\nu+k\mu}(z,t;\ \widetilde{\tau})$, only the choice of $\widetilde{\tau}$ is relevant. If $\widetilde{\tau}$ and $\widetilde{\tau}'$ are as in Remark 7.9 , then

$$Y_{\nu+k\mu}(z,t;\widetilde{\tau})=Y_{\nu+k\mu}(z,t;\widetilde{\tau}'),$$

because the rays in $\mathfrak{R}(t)$, $t\in\mathcal{U}_{\epsilon_0}(0)$, neither cross the admissible rays $R(\widetilde{\tau}+m\pi)$ nor the rays $R(\widetilde{\tau}'+m\pi)$, $m\in\mathbb{Z}$. In other words, $Y_{\nu+k\mu}(z,t;\widetilde{\tau})$ depends on $\widetilde{\tau}$ only through ϵ_0. Hence, we can restore the notation

$$Y_{\nu+k\mu}(z,t), \qquad t\in\mathcal{U}_{\epsilon_0}(0).$$

COROLLARY 7.3. *Let the assumptions of Theorem 7.1 hold. Let* $\mathring{Y}_{\nu+k\mu}(z)$*,* $k\in\mathbb{Z}$*, denote the unique fundamental solution (6.7) of the form (5.44), namely*

$$\mathring{Y}_{\nu+k\mu}(z)=\mathcal{G}_{\nu+k\mu}(z)z^{B_1(0)}e^{\Lambda z},$$

with the asymptotics (5.45)

$$\mathcal{G}_{\nu+k\mu}(z)\sim I+\sum_{j=1}^{\infty}\mathring{F}_jz^{-j}, \qquad z\to\infty, \quad z\in\mathcal{S}_{\nu+k\mu},$$

corresponding to the particular choice $\mathring{F}_j=F_j(0)$*,* $j\geq 1$*. Then,*

$$\mathcal{G}_{\nu+k\mu}(z,0)=\mathcal{G}_{\nu+k\mu}(z),$$
$$Y_{\nu+k\mu}(z,0)=\mathring{Y}_{\nu+k\mu}(z).$$

Proof: Observe that $Y_{\nu+k\mu}(z,0)$ is defined at $t=0$. Now, $\widehat{\mathcal{S}}_\nu\subset\mathcal{S}_\nu$ and both sectors have central opening angle greater than π. Hence, the solution with given asymptotics in $\widehat{\mathcal{S}}_\nu$ is unique, namely $\mathcal{G}_\nu(z)=\mathcal{G}_\nu(z,0)$. □

COROLLARY 7.4. *Let the assumptions of Theorem 7.1 hold. Let* $\mathbb{S}_\nu(t)$*,* $\mathbb{S}_{\nu+\mu}(t)$ *be a complete set of Stokes matrices associated with fundamental solutions* $Y_\nu(z,t)$*,* $Y_{\nu+\mu}(z,t)$*,* $Y_{\nu+2\mu}(z,t)$*, with canonical asymptotics, for* t *in a* $\widetilde{\tau}$*-cell of* $\mathcal{U}_{\epsilon_0}(0)$*, in sectors* $\mathcal{S}_\nu(t)$*,* $\mathcal{S}_{\nu+\mu}(t)$ *and* $\mathcal{S}_{\nu+2\mu}(t)$ *respectively, which by Theorem 7.1 extend to* $\widehat{\mathcal{S}}_\nu$*,* $\widehat{\mathcal{S}}_{\nu+\mu}$ *and* $\widehat{\mathcal{S}}_{\nu+2\mu}$ *respectively for* $t\in\mathcal{U}_{\epsilon_1}(0)$*. Then there exist*

$$\lim_{t\to0}\mathbb{S}_\nu(t)=\mathring{\mathbb{S}}_\nu, \qquad \lim_{t\to0}\mathbb{S}_{\nu+\mu}(t)=\mathring{\mathbb{S}}_{\nu+\mu},$$

where $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$ is a complete set of Stokes matrices for the system at $t=0$, referred to three fundamental solutions $\mathring{Y}_{\nu+k\mu}(z)$, $k=0,1,2$, of Corollary 7.3 having asymptotics in sectors $\mathcal{S}_{\nu+k\mu}$, with $\mathring{F}_j = F_j(0)$, $j\geq 1$.

Proof: The analyticity of $Y_{\nu+k\mu}(z,t)$ in assumption 2) of Theorem 7.1 implies that the Stokes matrices are holomorphic in $\mathcal{U}_{\epsilon_0}(0)$. Hence, for $k=1,2$, there exists

$$\mathbb{S}_{\nu+k\mu}(0)=\lim_{t\to 0}\Big(Y_{\nu+(k+1)\mu}(z,t)^{-1}Y_{\nu+k\mu}(z,t)\Big)=\mathring{Y}_{\nu+(k+1)\mu}(z)^{-1}\mathring{Y}_{\nu+k\mu}(z)=\mathring{\mathbb{S}}_{\nu+k\mu}. \qquad \square$$

7.6.5. Proof of Theorem 7.1.

LEMMA 7.2. *Let Assumption 5.1 hold for the system (5.1). Let the eigenvalues of $\Lambda(t)$ be linear in t as in (1.25). Suppose that $Y_\nu(z,t;\widetilde{\tau},c)$ has t-analytic continuation on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, with $\epsilon_0=\delta_0$ as in subsection 7.6.1. Temporarily call $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ the continuation. Also suppose that*

$$Y_\nu^{cont}(z,t;\widetilde{\tau},c)\Big|_{t\in c'}=Y_\nu(z,t;\widetilde{\tau},c').$$

Then:

a) Any $Y_\nu(z,t;\widetilde{\tau},c')$ has analytic continuation on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, coinciding with $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$. Due to the independence of c, we denote this continuation by

$$Y_\nu(z,t;\widetilde{\tau}).$$

b) $\mathcal{G}_\nu(z,t;\widetilde{\tau}):=G_0(t)^{-1}Y_\nu(z,t;\widetilde{\tau})z^{-B_1(t)z}e^{-\Lambda(t)z}$ has asymptotic expansion

$$\mathcal{G}_\nu(z,t;\widetilde{\tau})\sim I+\sum_{k=0}^{\infty}F_k(t)z^{-k},\qquad z\to\infty,\quad z\in\widehat{\mathcal{S}}_\nu(t),\quad t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta.$$

The asymptotics for $z\to\infty$ in $\widehat{\mathcal{S}}_\nu(K)$ is uniform on any compact subset $K\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

Proof of Lemma 7.2: a) is obvious. We prove b), dividing the proof into two parts.

Part 1 (in steps). Chosen an arbitrary cell c (all cells are equivalent, by a)) and any $\check{t}\in c$, we prove that the sector where $Y_\nu(z,\check{t};\widetilde{\tau})$ has canonical asymptotics can be extended from $\mathcal{S}_\nu(\check{t})$ to $\widehat{\mathcal{S}}_\nu(\check{t})$. For clarity in the discussion below, let us still write $Y_\nu(z,\check{t};\widetilde{\tau},c)$.

Step 1. At $\check{t}$, consider the Stokes rays in $\widehat{\mathcal{S}}_\nu(\check{t})\backslash S(\widetilde{\tau}-\pi,\widetilde{\tau})$ associated with the unfolding of the λ_i's. Those with direction greater than $\widetilde{\tau}$ will be labelled in anticlockwise sense as $R_1(\check{t})$, $R_2(\check{t})$, ..., etc. Those with direction smaller than $\widetilde{\tau}-\pi$ will be labelled in clockwise sense $R_1'(\check{t})$, $R_2'(\check{t})$, etc. Therefore, $R_1(\check{t})$ is the closest to the admissible ray $R(\widetilde{\tau})$, while $R_1'(\check{t})$ is the closest to $R(\widetilde{\tau}-\pi)$. (Warning about the notation: The dependence on t is indicated in Stokes rays R_1, R_2 etc, while for the admissible ray $R(\widetilde{\tau})$, $\widetilde{\tau}$ is the direction as in Definition 6.2). See figure 7.15.

Let t vary from $\check{t}$ into a neighbouring cell c_1, in such a way that $R_1(t)$ approaches and crosses $R(\widetilde{\tau})$ clockwise. By Proposition 7.5, $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ is well defined with canonical asymptotics on a sector having left boundary ray equal to $R_1(t)$, for values of $t\in c_1$ just after the crossing.[2]

By assumption, $Y_\nu^{cont}(z,t;\widetilde{\tau},c)=Y_\nu(z,t;\widetilde{\tau},c_1)$. For $t\in c_1$ just after the crossing, $Y_\nu(z,t;\widetilde{\tau},c_1)$ has canonical asymptotics in $\mathcal{S}_\nu(t)$, which now has left boundary ray equal to $R_2(t)$. See Figures 7.16 e 7.17. This implies that $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ has canonical asymptotics extended up to $R_2(t)$, $t\in c_1$ as above. See Figure 7.18.

[2]As long as $R_1(t)$ does not reach another Stokes ray

Let t go back along the same path, so that $R_1(t)$ crosses $R(\widetilde{\tau})$ anticlockwise. Proposition 7.5 now can be applied to $Y_\nu(z,t;\widetilde{\tau},c_1)$ for this crossing.[3] Hence, $Y_\nu(z,t;\widetilde{\tau},c_1)$ has analytic continuation for t before the crossing, certainly up to $\check{t}$ (because $R_1(t)$ does not cross $R_2(t)$), with canonical asymptotics in a sector having $R_2(\check{t})$ as left boundary. See Figure 7.19. Again, by assumption, we have that $Y_\nu(z,\check{t};\widetilde{\tau},c) = Y_\nu^{cont}(z,\check{t};\widetilde{\tau},c_1)$. Hence, $Y_\nu(z,\check{t};\widetilde{\tau},c)$ has canonical asymptotics extended up to the ray $R_2(\check{t})$. See Figure 7.20. In conclusion, $R_1(t)$ has been erased.

Step 2. We repeat the arguments analogous to those of Step 1 in order to erase $R_2(t)$. Let t vary in such a way that $R_1(t)$, which is now a "virtual ray", crosses $R(\widetilde{\tau})$ clockwise, as in step 1. After the crossing, $t \in c_1$ and $Y_\nu^{cont}(z,t;\widetilde{\tau},c) = Y_\nu(z,t;\widetilde{\tau},c_1)$. Then, let t vary in such a way that also $R_2(t)$ crosses $R(\widetilde{\tau})$ clockwise. See Figures 7.21, 7.22. Just after the crossing, t belongs to another cell c_2 (clearly, $c_2 \neq c$ and c_1; see Proposition 7.4).

The same discussion done at Step 1 for $Y_\nu(z,t;\widetilde{\tau},c)$ is repeated now for $Y_\nu(z,t;\widetilde{\tau},c_1)$. Indeed, $Y_\nu^{cont}(z,t;\widetilde{\tau},c_1) = Y_\nu(z,t;\widetilde{\tau},c_2)$, for $t \in c_2$ just after $R_2(t)$ has crossed $R(\widetilde{\tau})$. The conclusion, as before, is that $Y_\nu^{cont}(z,t;\widetilde{\tau},c_1)$ has canonical asymptotics extended up to $R_3(t)$ for $t \in c_1$. See Figure 7.23.

Now, let t go back along the same path up to $\check{t}$. Also the virtual ray $R_1(t)$ comes to the initial position, and $Y_\nu(z,\check{t};\widetilde{\tau},c) = Y_\nu^{cont}(z,\check{t};\widetilde{\tau},c_1) = Y_\nu^{cont}(z,\check{t};\widetilde{\tau},c_2)$, with canonical asymptotics extended up to $R_3(\check{t})$. See figure 7.24.

Step 3. The discussion above can be repeated for all Stokes rays R_1, R_2, R_3 , etc.

Step 4. Observe that the right boundary ray R_1' of the sector where $Y_\nu(z,t;\widetilde{\tau},c)$ has asymptotics is not affected by the above construction. Once the left boundary rays R_1, R_2,... have been erased, the same discussion must be repeated considering crossings of the admissible ray $R(\widetilde{\tau}-\pi)$ by the rays R_1', R_2', etc, as in figure 7.25.

In conclusion, all rays $R_1, R_2, ..., R_1', R_2', ...$ from unfolding lying in $\widehat{\mathcal{S}}_\nu(\check{t})\backslash S(\widetilde{\tau}-\pi,\widetilde{\tau})$ are erased. Hence $Y_\nu(z,\check{t};\widetilde{\tau},c) \equiv Y_\nu(z,\check{t};\widetilde{\tau})$ has canonical asymptotics extended up to the closest Stokes rays in $\mathfrak{R}(\check{t})$ outside $S(\widetilde{\tau}-\pi,\widetilde{\tau})$, namely the asymptotics holds in $\widehat{\mathcal{S}}_\nu(\check{t})$.

The above discussion can be repeated also if one of more rays among R_1, R_2, etc. is double (i.e. it corresponds to three eigenvalues) at $\check{t}$, because as t varies the rays unfold. Thus, the above discussion holds for any $\check{t} \in c$ and any c. Therefore, $Y_\nu(z,t;\widetilde{\tau})$ has asymptotics in $\widehat{\mathcal{S}}_\nu(t)$ for any t belonging to the union of the cells.[4]

We observe that a ray $R_1(t)$, $R_2(t)$, etc, crosses $R(\widetilde{\tau})$ for t equal to a simple point t_* (see Definition 7.5). The above proof allows to conclude that $Y_\nu(z,t_*;\widetilde{\tau})$ has asymptotics in $\widehat{\mathcal{S}}_\nu(t_*)$ also when $\check{t}=t_*$.

Part 2: Points $\check{t}$ internal to cells and simple points have been considered. It remains to discuss non simple points $t_* \in (\widetilde{H}_{a_1b_1}\cap\widetilde{H}_{a_2b_2}\cap\cdots\cap\widetilde{H}_{a_l,b_l})\backslash\Delta$, for some $l \geq 2$. Consider all the Stokes rays associated with either one of $(u_{a_m}(t),u_{b_m}(t))$ or $(u_{b_m}(t),u_{a_m}(t))$, $m=1,...,l$, and lying in $S(\widetilde{\tau},\widetilde{\tau}+\pi)$. There exists a cell c, among the cells having boundary sharing the above intersection, such that these rays cross $R(\widetilde{\tau})$ clockwise and simultaneously at t_*, when t approaches t_* from c. Call these rays $R_{a_1b_1}(t)$, $R_{a_2b_2}(t)$, etc. See figures 7.26, 7.27, 7.28.

Let t start from $\check{t} \in c$ and vary, reaching t_* and penetrating into a neighbouring cell c' through $(\widetilde{H}_{a_1b_1}\cap\widetilde{H}_{a_2b_2}\cap\cdots\cap\widetilde{H}_{a_l,b_l})\backslash\Delta$. At t_* the above Stokes rays cross $R(\widetilde{\tau})$ clockwise and simultaneously, from the same side. Hence $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ has analytic continuation into c' (here the situation is

[3] In the proof, deform $\widetilde{\tau}\mapsto\widetilde{\tau}+\varepsilon$.

[4] Namely, $t\in\mathcal{U}_{\epsilon_0}(0)\backslash(\Delta\cup X(\widetilde{\tau})) = \mathcal{U}_{\epsilon_0}(0)\backslash\left(\bigcup\widetilde{H}_{ab}\right)$, a,b from unfolding.

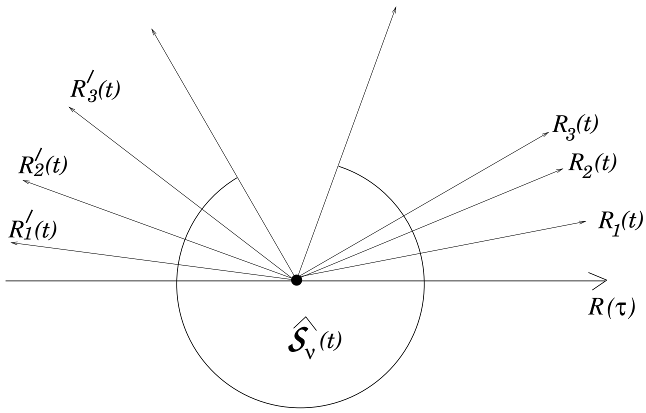


FIGURE 7.15. Rays in $\widehat{\mathcal{S}}_\nu(t)$ which are going to be erased in the proof.

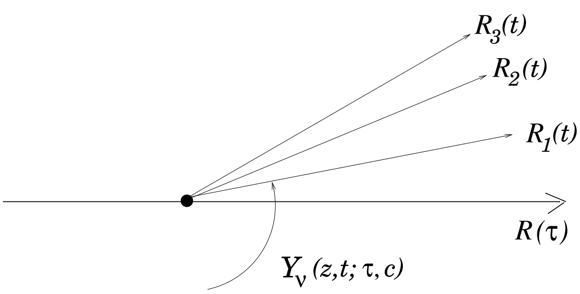


FIGURE 7.16. $Y_\nu(z,t;\widetilde{\tau},c)$ for $t \in c$, before $R_1(t)$ crosses $R(\widetilde{\tau})$. A portion of $\mathcal{S}_\nu(t)$ is represented by an arc.

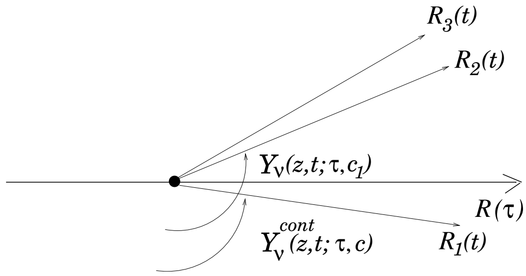


FIGURE 7.17. $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ and $Y_\nu(z,t;\widetilde{\tau},c_1)$ just after $R_1(t)$ has crossed $R(\widetilde{\tau})$. Portions of sectors where the asymptotics holds are represented.

similar to the continuation from c_3 to c_2 in figure 7.11). After the crossing, $t \in c'$ and the same discussion of Part 1 applies. Namely, $Y_\nu^{cont}(z,t;\widetilde{\tau},c) = Y_\nu(z,t;\widetilde{\tau},c')$. The canonical asymptotics is extended up to the nearest Stokes ray in $S(\widetilde{\tau},\widetilde{\tau}+\pi)$. Then,[5] as in Proposition 7.5, $Y_\nu(z,t;\widetilde{\tau},c')$ is analytically continued for t back to c, up to $\check{t}$. Therefore, the asymptotics of $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$ gets extended up to the above mentioned nearest Stokes ray in $S(\widetilde{\tau},\widetilde{\tau}+\pi)$. This fact holds also for $t = t_*$. In this way, $R_{a_1b_1}(t)$, $R_{a_2b_2}(t)$, etc, get erased also at t_*. Proceeding as in Part 1, we conclude that $Y_\nu(z,t_*;\widetilde{\tau}) \equiv Y_\nu^{cont}(z,t_*;\widetilde{\tau},c)$ has asymptotics in the sector $\widehat{\mathcal{S}}_\nu(t_*)$.

Uniformity follows from Corollary 7.2 and Proposition 7.5 applied to any $Y_\nu(z,t;\widetilde{\tau},c')$. $\square$

REMARK 7.11. If $\Lambda(0) = \lambda_1 I$, then $\widehat{\mathcal{S}}_\nu = \mathcal{R}$, so that the asymptotics extends to $\mathcal{R}$.

Proof of Theorem 7.1: We do the proof for $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$. For any other $Y_{\nu+k\mu}$, $k \in \mathbb{Z}$, the proof is the same. We compute the analytic continuation of $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ along loops γ_{ab} in $\pi_1(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta, t_{base})$, associated with $u_a(t)$ and $u_b(t)$ in (7.13). For these a,b, only one of the infinitely many rays of

[5] By a small deformation $\widetilde{\tau} \mapsto \widetilde{\tau} + \varepsilon$.

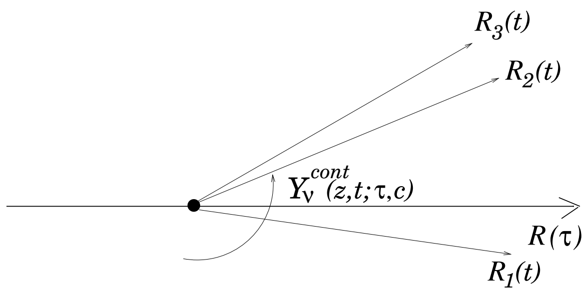


FIGURE 7.18. Extension of sector for the asymptotics of $Y_\nu^{cont}(z,t;\widetilde{\tau},c)$, $t\in c_1$.

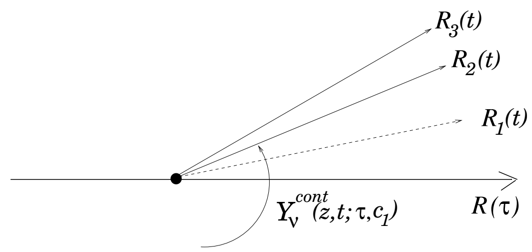


FIGURE 7.19. Continuation $Y_\nu^{cont}(z,t;\widetilde{\tau},c_1)$, $t\in c$ before the crossing. The sector of the asymptotics is represented.

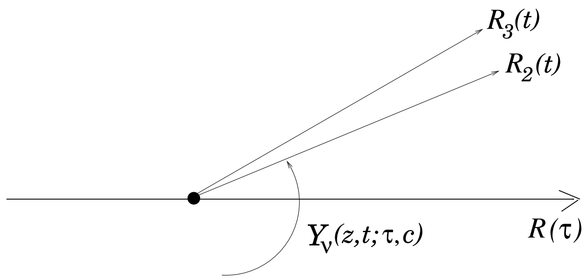


FIGURE 7.20. The sector where $Y_\nu(z,t;\widetilde{\tau},c)$ has canonical asymptotics has been extended up to $R_2(t)$, $t\in c$.

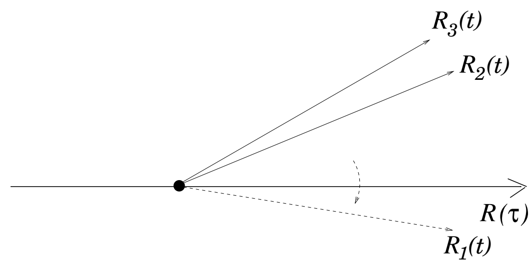


FIGURE 7.21. The dashed "virtual ray" $R_1(t)$ crosses $R(\widetilde{\tau})$, when t enters into c_1.

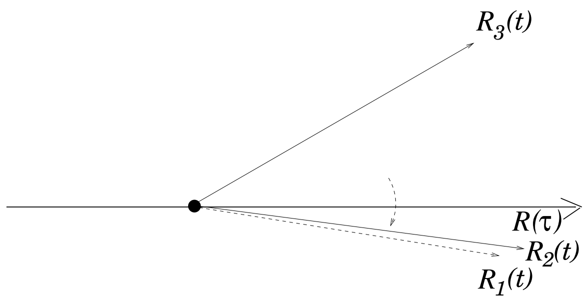


FIGURE 7.22. $R_2(t)$ crosses $R(\widetilde{\tau})$ when t enters into c_2

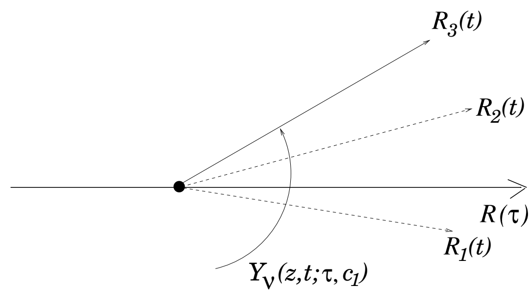


FIGURE 7.23. Extension up to $R_3(t)$ of the sector for the asymptotics of $Y_\nu(z,t;\widetilde{\tau},c_1)$, for $t\in c_1$.

directions (7.15) is contained in $S(\widetilde{\tau},\widetilde{\tau}+\pi)$ for $t\in c$. We can suppose that this is the ray

$$R_{ab}(t):=\left\{z\in\mathcal{R}\;\middle|\;\arg z=\frac{3\pi}{2}-\arg_p(t_a-t_b)+2N_c\pi\right\},$$

(recall that $\arg_p(u_a(t)-u_b(t))=\arg_p(t_a-t_b)$) where N_c is a suitable integer such that

$$\widetilde{\tau}<\frac{3\pi}{2}-\arg_p(t_a-t_b)+2N_c\pi<\widetilde{\tau}+\pi,\qquad t\in c.$$

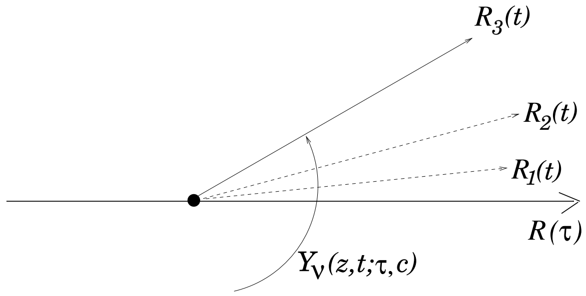


FIGURE 7.24. Extension up to $R_3(t)$ of the sector for the asymptotics of $Y_\nu(z,t;\widetilde{\tau},c)$, for $t \in c$.

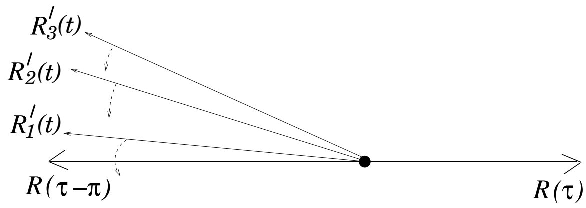


FIGURE 7.25. The extension of the sector for the asymptotics of $Y_\nu(z,t;\widetilde{\tau},c)$ must be done as above also at $R(\widetilde{\tau}-\pi)$, considering crossings as in figure.

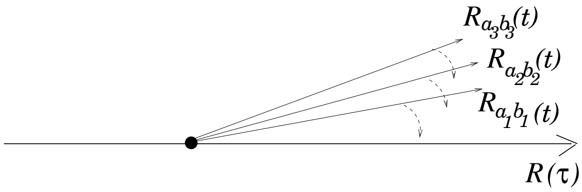


FIGURE 7.26. t belongs to a cell c whose boundary contains $\widetilde{H}_{a_1b_1} \cap \widetilde{H}_{a_2b_2} \cap \cdots \cap \widetilde{H}_{a_l,b_l}$, and such that the Stokes rays associated with these hyperplanes cross $R(\widetilde{\tau})$ simultaneously from the same side (c can be taken so that the crossing is clockwise).

FIGURE 7.27. Simultaneous crossing for $t \in (\widetilde{H}_{a_1b_1} \cap \widetilde{H}_{a_2b_2} \cap \cdots \cap \widetilde{H}_{a_l,b_l})\backslash\Delta$.

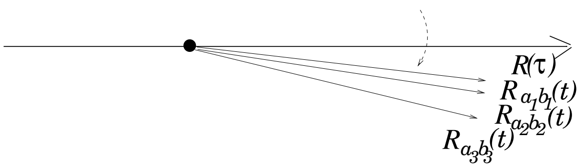


FIGURE 7.28. After the simultaneous crossing, $t \in c'$.

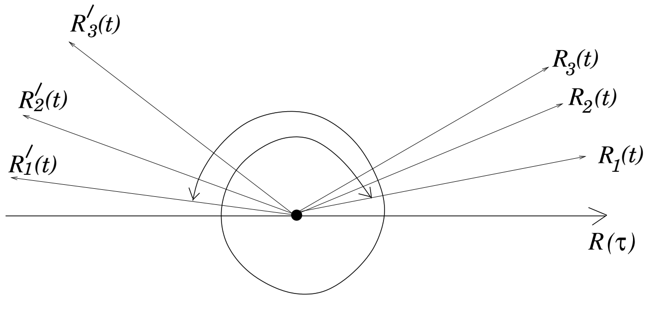


FIGURE 7.29. If $\Lambda(0) = \lambda_1 I$, the asymptotics extends to $S(\arg(R_1(\check{t})) - 2\pi, \arg(R_1'(\check{t})) + 2\pi)$.

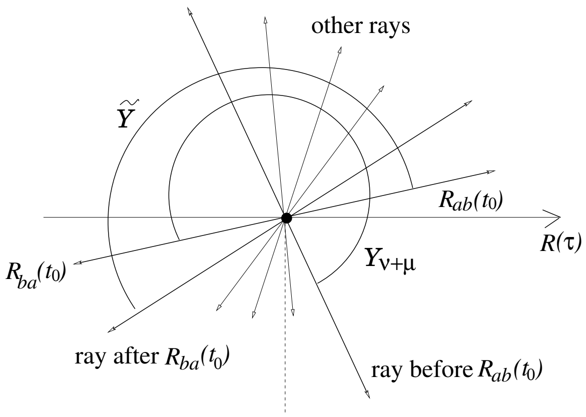


FIGURE 7.30. This and the following pictures represent the sheet $S(\widetilde{\tau}-\pi/2, \widetilde{\tau}+3\pi/2)$ (this is the meaning of the dashed vertical half-line). The Stokes rays at the starting point t_0 are represented. $Y_{\nu+\mu}$ is $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$, while $\widetilde{Y}$ is $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$

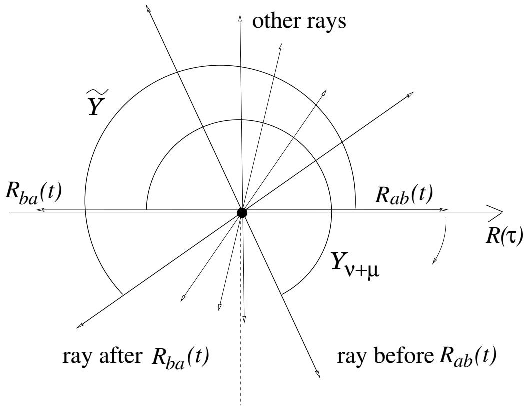


FIGURE 7.31. Crossing of $R(\widetilde{\tau})$. Note that also the other rays can move, but never cross the admissible ray $R(\widetilde{\tau})$ or $R(\widetilde{\tau}\pm\pi)$.

If it is not the above ray, then it is a ray with $\arg z = \frac{3\pi}{2} - \arg_p(t_b-t_a) + 2N'_c$ and suitable N'_c, so that the proof holds in the same way. $R_{ab}(t)$ rotates clockwise as t moves along the support of γ_{ab}.

For the sake of this proof, if a ray R has angle θ and R' has angle $\theta+\theta'$, we agree to write $R' = R+\theta'$. Hence, let

$$R_{ba}(t) := R_{ab}(t) + \pi.$$

See Figure 7.30.

Assume first that a, b are such that for $t \in c$ and $|t_a - t_b|$ sufficiently small, then no projected Stokes rays other than PR_{ab} and PR_{ba} cross $l(\widetilde{\tau})$ when t varies along γ_{ab} (the case discussed in figure 7.13). Cases when also other projected Stokes rays cross $l(\widetilde{\tau})$, as for figure 7.14, will be discussed later.

Step 1) As base point consider $t_0 \in c$, close to $\widetilde{H}_{ab}$, in such a way that $R_{ab}(t_0) \subset S(\widetilde{\tau},\widetilde{\tau}+\pi)$ is close to $R(\widetilde{\tau})$,[6] and it is the first ray in $S(\widetilde{\tau},\widetilde{\tau}+\pi)$ encountered on moving anti-clockwise from $R(\widetilde{\tau})$. $Y_{\nu+\mu}(z,t_0;\widetilde{\tau},c)$ has the canonical asymptotics in $\mathcal{S}_{\nu+\mu}(t_0)$, which contains $R(\widetilde{\tau})$. By definition, $\mathcal{S}_{\nu+\mu}(t_0)$ contains $S(\widetilde{\tau},\widetilde{\tau}+\pi)$ and extends to the closest Stokes rays outside. These rays are:

a) [left ray] the ray $R_{ba}(t_0)$.

b) [right ray] the first ray encountered on moving clockwise from $R_{ab}(t_0)$, which we call "the ray before" $R_{ab}(t_0)$ (see Figure 7.30). The name "before" means that this ray comes before $R_{ab}(t_0)$ in the natural anti-clockwise orientation of angles). This ray is to the right of $R(\widetilde{\tau})$.

Step 2) As t moves along γ_{ab}, $R_{ab}(t)$ moves clockwise and crosses $R(\widetilde{\tau})$, while $R_{ba}(t)$ crosses $R(\widetilde{\tau}+\pi)$ (see Figure 7.31). The curve γ_{ab} crosses $\widetilde{H}_{ab}\backslash\Delta$ and penetrates into another cell c'. As in Proposition 7.5, just before the intersection of the curve with $\widetilde{H}_{ab}\backslash\Delta$, also $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ is well defined with the same asymptotics as $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$, but in the sector bounded by $R_{ab}(t)$, as right ray, and the ray coming after $R_{ba}(t)$ in anti-clockwise sense, as left ray, which we call "the ray after" (see

[6] $\widetilde{\tau}$ in $R(\widetilde{\tau})$ is the direction, while t in $R_{ab}(t)$ is the dependence on t

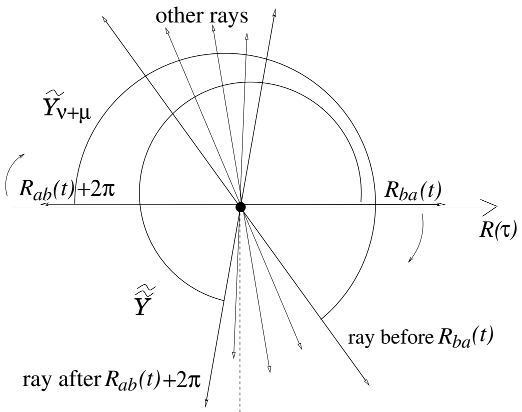


FIGURE 7.32. Second crossing. $\widetilde{Y}_{\nu+\mu}$ is $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ and $\widetilde{\widetilde{Y}}$ is $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$. The other rays represented are moving, without crossing $R(\widetilde{\tau})$ or $R(\widetilde{\tau}\pm\pi)$.

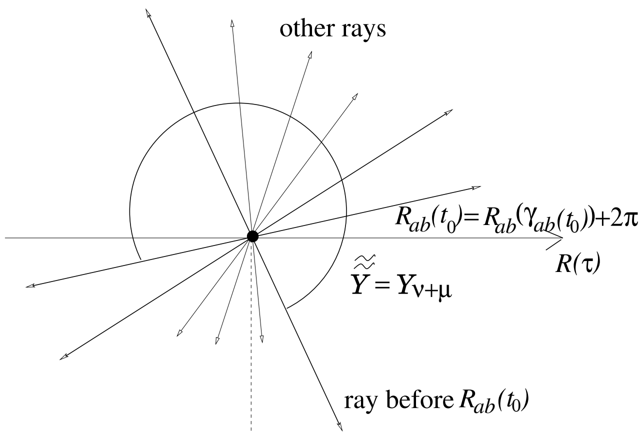


FIGURE 7.33. After the loop γ_{ab}

Figures 7.30 and 7.31). A connection matrix $\mathbb{K}^{[ab]}(t)$ (called *Stokes factor*) connects $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ and $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$,

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c') = Y_{\nu+\mu}(z,t;\widetilde{\tau},c)\ \mathbb{K}^{[ab]}(t). \tag{7.17}$$

$\mathbb{K}^{[ab]}(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, because the fundamental solutions are holomorphic by assumption 2). Again by the proof of Proposition 7.5, just after the crossing, $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ maintains its asymptotics between the ray before $R_{ab}(t)$, which has possibly only slightly moved, and $R_{ba}(t)$. Both $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ and $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ have the same asymptotics in successive sectors, and in particular they have the same asymptotics on the sector having right ray R_{ab} and left ray R_{ba}. Since $\Re[(u_a-u_b)z]>0$ on this sector, it follows from (7.17) that for t in a small open neighbourhood of the intersection point of the curve with $\widetilde{H}_{ab}\backslash\Delta$, the structure of $\mathbb{K}^{[ab]}(t)$ must be as follows

$$(\mathbb{K}^{[ab]})_{ii}=1, \qquad 1\le i\le n; \qquad (\mathbb{K}^{[ab]})_{ij}=0 \quad \forall \quad i\neq j \text{ except for } i=b, j=a.$$

The entry $(\mathbb{K}^{[ab]})_{ba}(t)$ may possibly be different from zero. Since $\mathbb{K}^{[ab]}(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, the above structure holds for every $t\in\mathcal{U}_{\epsilon_0}(0)$.

Step 3) As t moves along γ_{ab}, $R_{ab}(t)$ continues to rotate clockwise. It will cross other Stokes rays along the way, but $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ will maintain its canonical asymptotics in $\mathcal{S}_{\nu+\mu}(t)$, because $t\in c'$, until $R_{ab}(t)$ reaches $R(\widetilde{\tau}-\pi)$.

Step 4) Just before $R_{ab}(t)$ crosses $R(\widetilde{\tau}-\pi)$, $\mathcal{S}_{\nu+\mu}(t)$ has left ray equal to $R_{ab}(t)+2\pi$ and the right ray is the ray before $R_{ba}(t)$. Again by Proposition 7.5, $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ is defined with canonical asymptotics in the sector following $\mathcal{S}_{\nu+\mu}(t)$ anticlockwise (see Figure 7.32). There is a Stokes factor $\widetilde{\mathbb{K}}^{[ab]}(t)$ such that,

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c) = Y_{\nu+\mu}(z,t;\widetilde{\tau},c')\ \widetilde{\mathbb{K}}^{[ab]}(t). \tag{7.18}$$

The above relation and the common asymptotic behaviour imply that for t in a neighbourhood of the crossing point the structure must be

$$(\widetilde{\mathbb{K}}^{[ab]})_{ii}=1, \quad 1\le i\le n; \qquad (\widetilde{\mathbb{K}}^{[ab]})_{ij}=0 \quad \forall \quad i\neq j \text{ except for } i=a, j=b.$$

The entry $(\widetilde{\mathbb{K}})^{[ab]}_{ab}(t)$ may be possibly non zero. By assumption 2), $\widetilde{\mathbb{K}}^{[ab]}(t)$ is holomoprhic on $\mathcal{U}_{\epsilon_0}(0)$, so the above structure holds for any $t\in\mathcal{U}_{\epsilon_0}(0)$.

Step 5) The rotation of $R_{ab}(t)$ continues, crossing other Stokes rays. Finally, $R_{ab}(t)$ reaches the position

$$R_{ab}\Big(\gamma_{ab}(t_0)\Big)=R_{ab}(t_0)-2\pi,$$

after a full rotation of -2π. This corresponds to the full loop $t_a-t_b\mapsto(t_a-t_b)e^{2\pi i}$.

From (7.17) and (7.18) we conclude that,

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c)=Y_{\nu+\mu}(z,t;\widetilde{\tau},c)\ \mathbb{K}^{[ab]}(t)\widetilde{\mathbb{K}}^{[ab]}(t),\qquad t\in\mathcal{U}_{\epsilon_0}(0). \tag{7.19}$$

Hence

$$\mathbb{K}^{[ab]}(t)\widetilde{\mathbb{K}}^{[ab]}(t)=I,\qquad t\in\mathcal{U}_{\epsilon_0}(0).$$

This implies that $(\mathbb{K}^{[ab]})_{ba}=(\widetilde{\mathbb{K}}^{[ab]})_{ab}=0$. Therefore,

$$\mathbb{K}^{[ab]}(t)=\widetilde{\mathbb{K}}^{[ab]}(t)=I,\qquad t\in\mathcal{U}_{\epsilon_0}(0). \tag{7.20}$$

We conclude from (7.17) or (7.18) that

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c)=Y_{\nu+\mu}(z,t;\widetilde{\tau},c'),\qquad t\in\mathcal{U}_{\epsilon_0}(0). \tag{7.21}$$

The above discussion can be repeated for all loops γ_{ab} starting in c involving a simple crossing of $R(\widetilde{\tau})$.

We now turn to the case when also other projected Stokes rays, not only PR_{ab} and PR_{ba}, cross $l(\widetilde{\tau})$ along γ_{ab}. In this case, the representative of γ_{ab} can be decomposed into steps, for each of which the analytic continuation studied above and formula (7.21) hold. See for example the configuration of figure 7.14. In these occurrences, the analytic continuation is done first from c to c'. The passage from c to c' corresponds to the alignment of u_γ and u_a. Hence, $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ is continued from c to c' and (7.21) holds. Then, $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ can be used in place of $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$, applying the same proof previously explained, since for $t\in c'$, if $|t_a-t_b|$ is sufficiently small, then the crossing involves only PR_{ab} and PR_{ba}.

Concluding, (7.21) holds for any cell c' which has a boundary in common with c.

Now, we consider a cell c' which has a boundary in common with c, and we do the analytic continuation of $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ to all cells c'' which have a boundary in common with c', in the same way it was done above. In this way, we conclude that $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)=Y_{\nu+\mu}(z,t;\widetilde{\tau},c')$ and $Y_{\nu+\mu}(z,t;\widetilde{\tau},c')=Y_{\nu+\mu}(z,t;\widetilde{\tau},c'')$, for $t\in\mathcal{U}_{\epsilon_0}(0)$. With this procedures, all cells can be reached, so that (7.21) holds for any cell c and c' of $\mathcal{U}_{\epsilon_0}(0)$. For the above reasons, we are allowed to write

$$Y_{\nu+\mu}(z,t;\widetilde{\tau}),\qquad t\in\mathcal{U}_{\epsilon_0}(0), \tag{7.22}$$

in place of $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$.

The above conclusions imply that the assumptions of Lemma 7.2 hold. Lemma 7.2 assures that the asymptotics extends to the closest Stokes rays in $\mathfrak{R}(t)$ outside $S(\widetilde{\tau},\widetilde{\tau}+\pi)$. Hence the asymptotics

$$G_0(t)^{-1}Y_{\nu+\mu}(z,t;\widetilde{\tau})e^{-\Lambda(t)}z^{-B_1(t)}\sim I+\sum_{k=0}^{\infty}F_k(t)z^{-k} \tag{7.23}$$

holds for $z\to\infty$ in $\widehat{\mathcal{S}}_{\nu+\mu}(t)$, and $t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. *A fortiori*, the asymptotics holds in $\widehat{\mathcal{S}}_{\nu+\mu}=\widehat{\mathcal{S}}_{\nu+\mu}(\mathcal{U}_{\epsilon_0}(0))$. It is uniform on any compact subset $K\subset\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ for $z\to\infty$ in $\widehat{\mathcal{S}}_\nu(K)$.

The last property to be verified is that the asymptotics in $\widehat{\mathcal{S}}_{\nu+\mu}$ holds also for $t\in\Delta$. Let

$$R_k(z,t):=G_0(t)^{-1}Y_{\nu+\mu}(z,t;\widetilde{\tau})e^{-\Lambda(t)z}z^{-B_1(t)}-\left(I+\sum_{l=1}^{k-1}F_l(t)z^{-k}\right),\qquad t\in\mathcal{U}_{\epsilon_0}(0).$$

Let $(R_k(z,t))_{ls}$, $l,s=1,...,n$ be the entries of the matrix R_k. Since R_k is the k-th remainder of the asymptotic expansion, it satisfies the inequality

$$\Big|R_k(z,t)\Big| := \max_{l,s=1,\dots,n}\Big|(R_k(z,t))_{ls}\Big| \leq \frac{C(k;\overline{S};t)}{|z|^k}, \qquad t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta, \qquad z\in\overline{S}, \tag{7.24}$$

for z belonging to a proper closed subsector $\overline{S}\subset\widehat{\mathcal{S}}_{\nu+\mu}$. Here $C(k;\overline{S};t)$ is a constant depending on k, $\overline{S}$ and $t\in\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. Our goal is to prove a similar relation for $t\in\Delta$.

We consider n positive numbers $r_a\leq\epsilon_0$, $a=1,...,n$. We further require that for any $i=1,...,s$ and for any $a\neq b$, such that $u_a(0)=u_b(0)=\lambda_i$, these numbers are distinct, i.e. $r_a\neq r_b$. We introduce the polydisc $\mathcal{U}_{r_1,\dots,r_n}(0):=\{t\in\mathbb{C}^n \mid |t_a|\leq r_a,\ \ a=1,...,n\}$. Clearly, $\mathcal{U}_{r_1,\dots,r_n}(0)\subset\mathcal{U}_{\epsilon_0}(0)$. Let us denote the *skeleton* of $\mathcal{U}_{r_1,\dots,r_n}(0)$ with $\Gamma:=\{t\in\mathbb{C}^n \mid |t_a|=r_a,\ \ a=1,...,n\}$. The above choice of pairwise distinct r_a's assures that $\Gamma\cap\Delta=\emptyset$.

The inequality (7.24) holds in $\mathcal{U}_{r_1,\dots,r_n}(0)\backslash\Delta$ for any fixed $z\in\overline{S}$. Since $R_k(z,t)$ is holomorphic on the interior of $\mathcal{U}_{r_1,\dots,r_n}(0)$ and continuous up to the boundary, every matrix entry of $R_k(z,t)$ attains its maximum modulus on the *Shilov boundary* (cf. [**Sha92**], page 21-22) of $\mathcal{U}_{r_1,\dots,r_n}(0)$, which coincides with Γ. Since (7.24) holds on Γ, we conclude that

$$\Big|R_k(z,t)\Big| \leq \frac{C(k;\overline{S};\Gamma)}{|z|^k}, \qquad \forall\ t\in\mathcal{U}_{r_1,\dots,r_n}(0), \tag{7.25}$$

where $C(k;\overline{S};\Gamma)=\max_{t\in\Gamma}C(k;\overline{S};t)$. This maximum is finite, because the asymptotics is uniform on every compact subset of $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. The above estimate (7.25) means that the asymptotics (7.23) holds uniformly in t on the whole $\mathcal{U}_{r_1,\dots,r_n}(0)$, including Δ, for $z\to\infty$ in $\overline{S}$. A fortiori, the asymptotics holds in $\mathcal{U}_{\epsilon_1}(0)$, with $\epsilon_1\leq\min_a r_a<\epsilon_0$. Since (7.25) holds for any closed proper subsector $\overline{S}\subset\widehat{\mathcal{S}}_{\nu+\mu}$, by definition $G_0(t)^{-1}Y_{\nu+\mu}(z,t;\widetilde{\tau})e^{-\Lambda(t)}z^{-B_1(t)}$ is asymptotic to $I+\sum_{k=0}^{\infty}F_k(t)z^{-k}$ in $\widehat{\mathcal{S}}_{\nu+\mu}$.

It remains to comment on the structure of a Stokes matrix. In the proof above, a ray $R_{ab}(t)$ associated with a pair $u_a(t),u_b(t)$ with $u_a(0)=u_b(0)=\lambda_i$ is "invisible" as far as the asymptotics is concerned, because $\mathbb{K}^{[ab]}(t)=\widetilde{\mathbb{K}}^{[ab]}(t)=I$ for any γ_{ab}. Therefore, in the factorisation of any $\mathbb{S}_\nu(t)$, the Stokes factors associated with rays $3\pi/2-\arg(u_a(t)-u_b(t)) \bmod 2\pi$, with $u_a(0)=u_b(0)=\lambda_i$, are the identity. □

7.7. Meromorphic Continuation

In Theorem 7.1 we have assumed that for any $\widetilde{\tau}$-cell c of $\mathcal{U}_{\epsilon_0}(0)$ and any $k\in\mathbb{Z}$, the fundamental solution $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$ has analytic continuation as a single-valued holomorphic function on the whole $\mathcal{U}_{\epsilon_0}(0)$. In this section, we assume that the above fundamental matrices have continuation on the universal covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ of $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$ as meromorphic matrix-valued functions. We show that if the Stokes matrices satisfy a vanishing condition, then the continuation is actually holomorphic and single valued on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$. In particular, Δ *is not a branching locus.*

Recall that the Stokes matrices are defined by

$$Y_{\nu+(k+1)\mu}(z,t;\widetilde{\tau},c)=Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)\ \mathbb{S}_{\nu+k\mu}(t), \qquad \text{for } t\in c.$$

THEOREM 7.2. *Consider the system (5.1) with holomorphic coefficients and Assumption 5.1. Let $\Lambda(t)$ be of the form (1.17), with eigenvalues (1.25) and $\epsilon_0=\delta_0$ as in subsection 7.6.1. Let $\widetilde{\tau}$ be the direction of an admissible ray $R(\widetilde{\tau})$, satisfying $\tau_\nu<\widetilde{\tau}<\tau_{\nu+1}$.*

Assume that for any $\widetilde{\tau}$-cell c of $\mathcal{U}_{\epsilon_0}(0)$ and any $k \in \mathbb{Z}$, the fundamental solution $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$, defined for $t \in c$, has analytic continuation on the universal covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ as a meromorphic matrix-valued function. Assume that the entries of the Stokes matrices satisfy the vanishing condition

$$(\mathbb{S}_\nu(t))_{ab} = (\mathbb{S}_\nu(t))_{ba} = (\mathbb{S}_{\nu+\mu}(t))_{ab} = (\mathbb{S}_{\nu+\mu}(t))_{ba} = 0, \qquad \forall t \in c, \tag{7.26}$$

for any $1 \leq a \neq b \leq n$ such that $u_a(0) = u_b(0)$.

Then:

- *The continuation of $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$ defines a single-valued holomorphic (matrix-valued) function on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.*
- *$Y_{\nu+k\mu}(z,t;\widetilde{\tau},c) = Y_{\nu+k\mu}(z,t;\widetilde{\tau},c')$, for $t \in c$. Therefore, we write $Y_{\nu+k\mu}(z,t;\widetilde{\tau})$*
- *The asymptotics*

$$G_0^{-1}(t)Y_{\nu+k\mu}(z,t;\widetilde{\tau})e^{-\Lambda(t)z}z^{-B_1(t)} \sim I + \sum_{j\geq 1} F_j(t)z^{-j},$$

holds for $z \to \infty$ in $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$, $t \in \mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

Remark 7.12. Recall that $B_1(t) = \operatorname{diag}(\widehat{A}_1(t))$ is the exponent of formal monodromy, appearing in the fundamental solutions (7.6). The formula $\mathbb{S}_{\nu+2\mu} = e^{-2\pi i B_1}\mathbb{S}_\nu\ e^{2\pi i B_1}$, analogous to that of Proposition 6.2, implies that (7.26) holds for any $\mathbb{S}_{\nu+k\mu}$. Notice that the $F_j(t)$'s are holomorphic on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.

Proof: Without loss of generality, we label the eigenvalues as in (5.50)-(5.53), so that $\mathbb{S}_{\nu+k\mu}(t)$ is partitioned into $p_j \times p_k$ blocks $(1 \leq j,k \leq s)$ such that the $p_j \times p_j$ diagonal blocks have matrix entries $(\mathbb{S}_{\nu+k\mu}(t))_{ab}$ corresponding to coalescing eignevalues $u_a(0) = u_b(0)$.

We consider $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$. For any other $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$ the discussion is analogous. We denote the meromorphic continuation of $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ on $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ by $\mathbb{Y}_{\nu+\mu}(z,\tilde{t};\widetilde{\tau},c)$, $\tilde{t} \in \mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$. Therefore, the continuation along a loop γ_{ab} as in (7.13) and (7.14), starting in c, will be denoted by $\mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t;\widetilde{\tau},c)$, where $\tilde{t} = \gamma_{ab}t$ is the point in $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ after the loop.

We then proceed as in the proof of Theorem 7.1, up to eq. (7.19). Assume first that a,b are such that for $t \in c$ and $|t_a - t_b|$ sufficiently small, then no projected Stokes rays other than PR_{ab} and PR_{ba} cross $l(\widetilde{\tau})$ when t varies along γ_{ab} (the case discussed in figure 7.13). Cases when also other projected Stokes rays cross $l(\widetilde{\tau})$, as for figure 7.14, can be discussed later as we did in the proof of Theorem 7.1. The intermediate steps along γ_{ab}, corresponding to the formulae (7.17) and (7.18), hold. Namely:

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c') = Y_{\nu+\mu}(z,t;\widetilde{\tau},c)\ \mathbb{K}^{[ab]}(t) \tag{7.27}$$

for t in a neighbourhood of the intersection of the support of γ_{ab} with the common boundary of c and c' (i.e. $\widetilde{H}_{ab}\backslash\Delta$) corresponding to R_{ab} crossing $R(\widetilde{\tau})$. Moreover,

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c) = Y_{\nu+\mu}(z,t;\widetilde{\tau},c')\ \widetilde{\mathbb{K}}^{[ab]}(t), \tag{7.28}$$

for t in a neighbourhood of the intersection of the support of γ_{ab} with the common boundary of c and c' corresponding to R_{ab} crossing $R(\widetilde{\tau}-\pi)$. Note that to such t there corresponds a point $\tilde{t}$ in the covering, which is reached along γ_{ab}, so that $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ in the right hand-side of (7.27) becomes $\mathbb{Y}_{\nu+\mu}(z,\tilde{t};\widetilde{\tau},c)$.

$\mathbb{K}^{[ab]}(t)$, $\widetilde{\mathbb{K}}^{[ab]}(t)$ have the same structure as in the proof of Theorem 7.1, for t in a small open neighborhood of the crossing points. By assumption, $\mathbb{K}^{[ab]}(t)$, $\widetilde{\mathbb{K}}^{[ab]}(t)$ are meromorphic on $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$, so they preserve their structure.

At the end of the loop, t is back to the initial point, but in the universal covering the point $\tilde{t}=\gamma_{ab}t$ is reached and $Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ has been analytically continued to $\mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t;\widetilde{\tau},c)$. Thus, the analogous of formula (7.19) now reads as follows

$$Y_{\nu+\mu}(z,t;\widetilde{\tau},c)=\mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t;\widetilde{\tau},c)\ \mathbb{K}^{[ab]}(t)\widetilde{\mathbb{K}}^{[ab]}(t),\qquad t\in c. \tag{7.29}$$

We need to compute the only non trivial entries $(\mathbb{K}^{[ab]}(t))_{ba}$ and $(\widetilde{\mathbb{K}}^{[ab]}(t))_{ab}$. Let us consider $\mathbb{K}^{[ab]}(t)$. As it is well known, $\mathbb{S}_{\nu+\mu}$ can be factorised into Stokes factors. At the beginning of the loop γ_{ab}, just before t crosses the boundary of the cell c as in Figure 7.30, we have

$$\mathbb{S}_{\nu+\mu}=\mathbb{K}^{[ab]}\cdot\mathbb{T},$$

where $\mathbb{K}^{[ab]}$ is a Stokes factor and the matrix $\mathbb{T}$ is factorised into the remaining Stokes factors of $\mathbb{S}_{\nu+\mu}$. For simplicity, we suppose that $\mathbb{S}_{\nu+\mu}$ is upper triangular (namely $a<b$; if not, the discussion is modified in an obvious way):

$$\mathbb{S}_{\nu+\mu}=\begin{pmatrix} I_{p_1} & * & * & \cdots & * \\ 0 & I_{p_2} & * & \cdots & * \\ 0 & 0 & I_{p_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & I_{p_s}\end{pmatrix}. \tag{7.30}$$

It follows that $b<a$, namely $\mathbb{K}^{[ab]}$ has entries equal to 1's on the diagonal, 0 elsewhere, except for a non-trivial entry $m_{ba}:=(\mathbb{K}^{[ab]})_{ba}$ above the diagonal in a block corresponding to one of the I_{p_1}, ..., I_{p_s} in (7.30). Let E_{jk} be the matrix with zero entries except for $(E_{jk})_{jk}=1$. Then, $\mathbb{K}^{[ab]}=I+m_{ba}E_{ba}$, and we factorise $\mathbb{T}$ as follows:

$$\mathbb{S}_{\nu+\mu}=(I+m_{ba}E_{ba})\cdot\prod_{j<k\text{ in }V}(I+m_{jk}E_{jk})\cdot\prod_{\text{The others }j<k}(I+m_{jk}E_{jk}),$$

where V is the set of indices $j<k\in\{1,2,...,n\}$ such that $u_j(0)=u_k(0)$ and $(j,k)\neq(b,a)$ (the entries of the diagonal blocks of the matrix block partition associated with $p_1,...,p_s$).

Now, all the numbers m_{ba} and m_{jk} are uniquely determined by the entries of $\mathbb{S}_{\nu+\mu}$. This fact follows from the following result (see for example [**BJL79a**]). Let S be any upper triangular matrix with diagonal elements equal to 1. Label the upper triangular entries entries (j,k), $j<k$, in an arbitrary way,

$$(j_1,k_1),\quad (j_2,k_2),\quad \dots,\quad (j_{\frac{n(n-1)}{2}},k_{\frac{n(n-1)}{2}}).$$

Then, there exists numbers m_1, m_2, ..., $m_{\frac{n(n-1)}{2}}$ which are uniquely determined by the labelling and the entries of S, such that

$$S=(I+m_1E_{j_1,k_1})(I+m_2E_{j_2,k_2})\cdots(I+m_{\frac{n(n-1)}{2}}E_{j_{\frac{n(n-1)}{2}}k_{\frac{n(n-1)}{2}}}).$$

Indeed, a direct computation gives

$$S=I+\sum_{a=1}^{\frac{n(n-1)}{2}}m_aE_{j_ak_a}+\text{ non linear terms in the }m_a\text{'s}. \tag{7.31}$$

The commutation relations

$$E_{ij}E_{jk}=E_{ik},\qquad E_{ij}E_{lk}=0\text{ for }j\neq l,$$

imply that the non linear terms are in an upper sub-diagonal lying above the sub-diagonal where the corresponding factors appear. Hence, (7.31) gives uniquely solvable recursive relations, expressing the m_a's in terms of the entries of S.

Applying the above procedure to $S = \mathbb{S}_{\nu+\mu}$, and keeping (7.26) into account, we obtain

$$m_{ba} = 0, \quad m_{jk} = 0 \ \forall j < k \text{ in } V.$$

This proves that

$$\mathbb{K}^{[ab]}(t) = I,$$

for t in a small open neighborhood of the intersection point of the curve γ_{ab} with $\widetilde{H}_{ab} \backslash \Delta$. This structure is preserved by analytic continuation. Analogously, we factorise into Stokes factor the (lower triangular) matrix $\mathbb{S}_\nu = \widetilde{\mathbb{T}} \cdot \widetilde{\mathbb{K}}^{[ab]}$ and prove that

$$\widetilde{\mathbb{K}}^{[ab]} = I.$$

We conclude that

$$Y_{\nu+\mu}(z, t; \widetilde{\tau}, c) = \mathbb{Y}_{\nu+\mu}(z, \gamma_{ab} t; \widetilde{\tau}, c).$$

Formulae (7.27) and (7.28) also imply that

$$Y_{\nu+\mu}(z, t; \widetilde{\tau}, c') = Y_{\nu+\mu}(z, t; \widetilde{\tau}, c) \tag{7.32}$$

This discussion can be repeated for any loop and any cell, as we did in the proof of Theorem 7.1 in the paragraphs following eq. (7.21). Since $Y_{\nu+\mu}(z, t; \widetilde{\tau}, c)$ is holomorphic on c by Corollary 7.2, the above formulae imply the analyticity of $Y_{\nu+\mu}(z, t; \widetilde{\tau}, c')$ on $\mathcal{U}_{\epsilon_0}(0) \backslash \Delta$. Since (7.32) holds, the first two statements are proved.

Equation (7.32) also implies that the rays R_{ab} and R_{ba} are not the boundaries of the sector where the asymptotic behaviour of $Y_{\nu+\mu}(z, t; \widetilde{\tau})$ holds. The above discussion repeated for all a, b such that $u_a(0) = u_b(0)$ proves the third statement of the theorem. □

7.8. Comparison with results in literature

We compare our results with the existing literature, where sometimes the irregular singular point is taken at $z = 0$ (equivalent to $z = \infty$ by a change $z \mapsto 1/z$). One considers a "folded" system $A(z, 0) = z^{-k-1} \sum_{j=0}^{\infty} A_j(0) z^j$, with an irregular singularity of Poincaré rank k at $z = 0$ and studies its holomorphic unfolding $A(z, t) = p(z, t)^{-1} \sum_{j=0}^{\infty} A_j(t) z^j$, where $p(z, t) = (z - a_1(t)) \cdots (z - a_{k+1}(t))$ is a polynomial. Early studies on the relation between monodromy data of the "folded" and the "unfolded" systems were started by Garnier [Gar19], and the problem was again raised by V.I. Arnold in 1984 and studied by many authors in the '80's and '90's of the XX century, for example see [Ram89], [Duv91], [Bol94]. Under suitable conditions, some results have been recently established regarding the convergence for $t \to 0$ (t in sectors or suitable ramified domains) of fundamental solutions and monodromy data (transition or connection matrices) of the "unfolded" system to the Stokes matrices of the "folded" one [Ram89], [Duv91], [Bol94], [BV85], [Sch01], [Glu99], [Glu04], [HLR14], [LR12], [Kli13]. Nevertheless, to our knowledge, the case when $A_0(0)$ is *diagonalisable with coalescing eigenvalues* has not yet been studied. For example, in [Glu99] (see also references therein) and [HLR14] [LR12], it is assumed that the leading matrix $A_0(0)$ has distinct eigenvalues. In [Glu04], $A_0(0)$ is a single $n \times n$ Jordan block (only one eigenvalue), with a generic condition on $A(z, t)$. Moreover, the irregular singular point is required to split into non-resonant Fuchsian singularities $a_1(t), \ldots, a_{k+1}(t)$. The case when $A_0(0)$ is a 2×2 Jordan block and $k = 1$ is thoroughly described in [Kli13], again under a generic condition on $A(z, t)$, with no conditions on the polynomial $p(z, t)$. Explicit normal forms for the unfolded systems are given (including an explanation of the change of order of Borel summability when $z = 0$ becomes a resonant irregular singularity as $t \to 0$). Nevertheless, both in [Glu04] and

[**Kli13**] the system at $t = 0$ is ramified and the fundamental matrices $Y_r(z, t)$ diverge when $t \to 0$, together with the corresponding Stokes matrices. Therefore, our results (Theorem 7.1, Corollary 7.3 and Corollary 7.4) on the extension of the asymptotic representation at Δ and the existence of the limit (1.24), for a system with diagonalisable $A_0(t_\Delta)$, seem to be missing in the literature.

CHAPTER 8

Isomonodromy Deformations Theory for Systems with Resonant Irregular Singularities

ABSTRACT. In this Chapter we formulate and develop the monodromy preserving deformation theory for system (1.20) with eigenvalues identified with the deformation parameters. After recalling the structure of fundamental systems of solutions near the Fuchsian singularity $z = 0$ and the freedom of choice of their Levelt normal forms, we assume that the dependence of (1.20) is *isomonodromic* for t varying in an open connected subset $\mathcal{V} \subseteq \mathcal{U}_{\epsilon_0}(0)$, with closure contained in a cell. It is proved that Assumption 8.1 (necessary for having holomorphy of formal solutions on $\mathcal{V}$) is equivalent on a vanishing condition of entries of the residue term $\widehat{A}_1(t)$ of the coefficient, along the coalescence locus Δ. This vanishing condition is also showed to be sufficient in order to have holomorphy at Δ of the coefficients of formal solutions. Finally, we prove Theorem 1.6 , Corollary 1.1 and Theorem 1.7 of the Introduction.

We have established the theory of coalescence in $\mathcal{U}_{\epsilon_0}(0)$, and the corresponding characterisation of the limiting Stokes matrices for the system (5.1) of Section 5.2 under Assumption 5.1, or equivalently for the system (1.16). We now consider the system (5.4) under Assumption 5.1, already put in the form (1.20), namely

$$\frac{dY}{dz} = \widehat{A}(z,t)Y, \qquad \widehat{A}(z,t) = \Lambda(t) + \frac{\widehat{A}_1(t)}{z},$$

and study its isomonodromy deformations. The eigenvalues are taken to be linear in t, as in (1.25):

$$u_i(t) = u_i(0) + t_i, \quad 1 \leq i \leq n.$$

8.1. Structure of Fundamental Solutions in Levelt form at $z = 0$

At any point $t \in \mathcal{U}_{\epsilon_0}(0)$, let $\mu_1(t), \mu_2(t), ..., \mu_n(t)$ be the (non necessarily distinct) eigenvalues of $\widehat{A}_1(t)$, and let $J^{(0)}(t)$ be a Jordan form of $\hat{A}_1(t)$, with $\mathrm{diag}(J^{(0)}) = \mathrm{diag}(\mu_1, ..., \mu_n)$ (see also (8.3) below). The eigenvalues are decomposed uniquely as,

$$\mu_i(t) = d_i^{(0)}(t) + \rho_i^{(0)}(t), \qquad d_i^{(0)}(t) \in \mathbb{Z}, \quad 0 \leq \Re\rho_i^{(0)}(t) < 1.$$

Let $D^{(0)}(t) = \mathrm{diag}(d_1^{(0)}(t), ..., d_n^{(0)}(t))$, which is piecewise constant, so that

$$J^{(0)}(t) = D^{(0)}(t) + S^{(0)}(t),$$

where $S^{(0)}(t)$ is the Jordan matrix with $\mathrm{diag}(S^{(0)}) = \mathrm{diag}(\rho_1^{(0)}, ..., \rho_n^{(0)})$.

Let $\mathcal{V}$ be an open connected subset of $\mathcal{U}_{\epsilon_0}(0)$. In order to write a solution at $z = 0$ in Levelt form which is holomorphic on $\mathcal{V}$, we need the following assumption.

ASSUMPTION 8.1. We assume that $\widehat{A}_1(t)$ is *holomorphically similar* to $J^{(0)}(t)$ on $\mathcal{V}$. This means that there exists an invertible matrix $G^{(0)}(t)$ holomorphic on $\mathcal{V}$ such that

$$(G^{(0)}(t))^{-1}\widehat{A}_1(t)\, G^{(0)}(t) = J^{(0)}(t).$$

Assumption 8.1 in $\mathcal{V}$ implies that the eigenvalues $\mu_i(t)$ are holomorphic on $\mathcal{V}$. In the isomonodromic case (to be defined below), Assumption 8.1 for $\mathcal{V} = \mathcal{U}_{\epsilon_0}(0)$ turns out to be equivalent to the vanishing condition (1.28). See Proposition 8.4 below.

REMARK 8.1. In order to realise the above assumption it is not sufficient to assume, for example, that the eigenvalues of $\widehat{A}_1(t)$ are independent of t, as the example $\widehat{A}_1(t) = \begin{pmatrix} \mu & t \\ 0 & \mu \end{pmatrix}$ shows. Sufficient conditions can be found in the Wasow's book [**Was65**], Ch. VII.

With Assumption 8.1, the following fundamental solutions in Levelt form are found.

A) If $\widehat{A}_1(t)$ has distinct eigenvalues at any point of $\mathcal{V}$, it is automatically holomorphically similar to

$$\widehat{\mu}(t) := \mathrm{diag}(\mu_1(t),\ \dots,\mu_n(t)).$$

A fundamental matrix exists of the form

$$Y^{(0)}(z,t) = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big) z^{\widehat{\mu}(t)}.$$

Each matrix $\Psi_l(t)$ is holomorphic on $\mathcal{V}$, and the series $I + \sum_{l=1}^{\infty} \Psi_l(t) z^l$ is absolutely convergent for $|z|$ bounded, defining a holomorphic matrix-valued function in (z,t) on $\{|z| < r\} \times \mathcal{V}$, for any $r > 0$.

B) If $\mu_i(t) - \mu_j(t) \notin \mathbb{Z}\backslash\{0\}$ for any $i \neq j$ and any $t \in \bar{\mathcal{V}}$, then there exists a fundamental matrix

$$Y^{(0)}(z,t) = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big) z^{J^{(0)}(t)},$$

such that $G^{(0)}(t)$, $J^{(0)}(t)$ and each matrix $\Psi_l(t)$ are holomoprhic on $\mathcal{V}$, and the series $I + \sum_{l=1}^{\infty} \Psi_l(t) z^l$ is absolutely convergent for $|z|$ bounded, defining a holomorphic matrix-valued function in (z,t) on $\{|z| < r\} \times \mathcal{V}$, for any $r > 0$.

The above forms of the matrix $Y^{(0)}(z,t)$ are obtained by a recursive procedure (see [**Was65**]), aimed at constructing a gauge transformation $Y = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big)\mathcal{Y}$ that reduces the linear system to a simple form $\frac{d\mathcal{Y}}{dz} = \frac{J^{(0)}}{z}\mathcal{Y}$, whose solution $z^{J^{(0)}(t)}$ can be immediately written. In *resonant cases*, namely when $\mu_i(t) - \mu_j(t) \in \mathbb{Z}\backslash\{0\}$, this procedure yields a gauge transformation $Y = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big)\mathcal{Y}$ that reduces the system to the form

$$\frac{d\mathcal{Y}}{dz} = \frac{1}{z}\left(J^{(0)}(t) + R_1(t)z + \cdots + R_\kappa(t)z^\kappa\right)\mathcal{Y}, \tag{8.1}$$

where $1 \leq \kappa$ is the maximal integer difference of eigenvalues of $J^{(0)}$, and the $R_j(t)$'s are certain nilpotent matrices (see (8.7) below for more details). These matrix coefficients may be discontinuous in t, even if Assumption 8.1 is made. In order to avoid this, we need the following

ASSUMPTION 8.2 (Temporary, for the Resonant Case). If for some $i \neq j$ it happens that $\mu_i(t) - \mu_j(t) \in \mathbb{Z}\backslash\{0\}$ at a point $t \in \mathcal{V}$, then we require that $\mu_i(t) - \mu_j(t) = \text{constant} \in \mathbb{Z}\backslash\{0\}$ all over $\mathcal{V}$. If moreover $J^{(0)}(t)$ is not diagonal, then we require that the d_i's, $1 \leq i \leq n$, are constant on $\mathcal{V}$.

Assumptions 8.2 certainly holds if the eigenvalues μ_1, ..., μ_n are independent of t in $\mathcal{V}$, namely in the isomonodromic case of Definition 8.2 below.[1] Hence, Assumptions 8.2 is only "temporary" here, being unnecessary in the isomonodromic case.

When Assumptions 8.1 and 8.2 hold together, fundamental matrices in Levelt form can always be constructed in such a way that they are holomorphic on $\mathcal{V}$. Besides the cases **A)** and **B)** (which require only Assumption 8.1), we have the following resonant cases:

C) If $J^{(0)}(t) \equiv \widehat{\mu}(t) := \text{diag}(\mu_1(t), \mu_1(t), ..., \mu_n(t))$ (eigenvalues non necessarily distinct) then there exists a fundamental matrix

$$Y^{(0)}(z,t) = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big) z^{\widehat{\mu}(t)} z^{R^{(0)}(t)},$$

were the matrix $R^{(0)}(t) := R_1(t) + \cdots R_\kappa(t)$ has entries $R^{(0)}_{ij}(t) \neq 0$ only if $\mu_i(t) - \mu_j(t) \in \mathbb{N}\backslash\{0\}$. Moreover, $G^{(0)}(t)$, $\widehat{\mu}(t)$ $R^{(0)}(t)$ and each matrix $\Psi_l(t)$ can be chosen holomorphic on $\mathcal{V}$, and the series $I + \sum_{l=1}^{\infty} \Psi_l(t) z^l$ is absolutely convergent for $|z|$ bounded, defining a holomorphic matrix-valued function in (z,t) on $\{|z| < r\} \times \mathcal{V}$, for any $r > 0$.

D) If some $\mu_i(t) - \mu_j(t) \in \mathbb{Z}\backslash\{0\}$ and $J^{(0)}(t)$ is not diagonal, then there exists a fundamental matrix holomorphic on $\mathcal{V}$,

$$Y^{(0)}(z,t) = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big) z^{D^{(0)}} z^{L^{(0)}(t)}, \tag{8.2}$$

where

$$L^{(0)}(t) := S^{(0)}(t) + R^{(0)}(t),$$

$G^{(0)}$, $S^{(0)}$ are holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, and $R^{(0)}$ and the Ψ_l's can be chosen holomorphic on $\mathcal{V}$. The series $I + \sum_{l=1}^{\infty} \Psi_l(t) z^l$ is absolutely convergent for $|z|$ bounded, defining a holomorphic matrix-valued function in (z,t) on $\{|z| < r\} \times \mathcal{V}$, for any $r > 0$.

The structure of $R^{(0)}$ is more conveniently described if the eigenvalues $\mu_1, \mu_2, ..., \mu_n$ are re-labelled as follows. Up to a permutation $J^{(0)} \mapsto P^{-1} J^{(0)} P$, which corresponds to $G^{(0)} \mapsto G^{(0)} P$, where P is a permutation matrix, the Jordan blocks structure can be arranged as

$$J^{(0)} = J^{(0)}_1 \oplus \cdots \oplus J^{(0)}_{s_0}, \qquad s_0 \leq n. \tag{8.3}$$

For $i = 1, 2, ..., s_0$, each $J^{(0)}_i$ has dimension n_i (then $n_1 + \cdots + n_{s_0} = n$) and has only one eigenvalue $\widetilde{\mu}_i$, with structure

$$J^{(0)}_i = \widetilde{\mu}_i I_{n_i} + H_{n_i}, \qquad I_{n_i} = \ n_i \times n_i \text{ identity matrix}, \tag{8.4}$$

$$H_{n_i} = 0 \ \text{ if } n_i = 1, \qquad H_{n_i} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \ \text{if } n_i \geq 2.$$

$\widetilde{\mu}_1$, ..., $\widetilde{\mu}_{s_0}$ are not necessarily distinct. Let us partition $R^{(0)}$ according to the block structure n_1, ..., n_{s_0}. Then $[R^{(0)}]_{block\ i,j} \neq 0$ only if $\widetilde{\mu}_i - \widetilde{\mu}_j \in \mathbb{N}\backslash\{0\}$, for $1 \leq i \neq j \leq s_0$.

[1] In case we define a deformation to be isomonodromic when the monodromy matrices are constant, this is still true, namely μ_1, ..., μ_n are independent of t. See Lemma 1 of [**Bol98**].

REMARK 8.2. Also in cases **A)**, **B)** and **C)** the fundamental solution can be written in the Levelt form (8.2), with $L^{(0)} = S^{(0)}$ in **A)** and **B)**, and $L^{(0)} = S^{(0)} + R^{(0)}$ in **C)**.

8.1.1. Freedom. Let the matrix $J^{(0)}(t)$ be fixed with the convention (8.3). Let Assumptions 8.1 and 8.2 hold. The class of normal forms at the Fuchsian singularity $z = 0$ with given $J^{(0)}$ is not unique, when some eigenvalues of $\widehat{A}_1(t)$ differ by non-zero integers. Let κ be the maximal integer difference. Then, if (8.2) is a Levelt form, there are other Levelt forms

$$\begin{aligned}\widetilde{Y}^{(0)}(z,t) \qquad &= \widetilde{G}^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \widetilde{\Psi}_l(t) z^l\Big) z^{D^{(0)}(t)} z^{\widetilde{L}^{(0)}(t)} \\ &\equiv Y^{(0)}(z,t)\mathfrak{D}(t),\end{aligned}$$

where $\mathfrak{D}(t)$ is a connection matrix. From the standard theory of equivalence of Birkhoff normal forms of a given differential system with Fuchsian singularity, it follows that $\mathfrak{D}(t)$ must have the following property

$$z^{D^{(0)}(t)} z^{L^{(0)}(t)} \mathfrak{D}(t) = \mathfrak{D}_0(t)\Big(I + \mathfrak{D}_1(t) z + \cdots + \mathfrak{D}_\kappa(t) z^\kappa\Big) z^{D^{(0)}(t)} z^{\widetilde{L}^{(0)}(t)},$$

being $\mathfrak{D}_0$, ..., $\mathfrak{D}_\kappa$ *arbitrary* matrices satisfying $[\mathfrak{D}_0, J^{(0)}] = 0$, $\mathfrak{D}_{ij}^{(l)} \neq 0$ only if $\widetilde{\mu}_i - \widetilde{\mu}_j = l > 0$. The connection matrix is then

$$\mathfrak{D}(t) = \mathfrak{D}_0(t)\Big(I + \mathfrak{D}_1(t) + \cdots + \mathfrak{D}_k(t)\Big).$$

Being $\mathfrak{D}_0(t)$, ..., $\mathfrak{D}_\kappa(t)$ arbitrary, we can choose the subclass of those connection matrices $\mathfrak{D}(t)$ which are holomorphic in t. Note that $\mathfrak{D}_0$ commutes with $D^{(0)}$. The relation between matrices with $\widetilde{\ }$ and without is as follows:

$$\widetilde{G}^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \widetilde{\Psi}_l(t) z^l\Big) = G^{(0)}(t)\Big(I + \sum_{l=1}^{\infty} \Psi_l(t) z^l\Big)\ \Big[\mathfrak{D}_0(t)\Big(I + \mathfrak{D}_1(t) z + \cdots + \mathfrak{D}_\kappa(t) z^\kappa\Big)\Big]. \tag{8.5}$$

Moreover,

$$\widetilde{L}^{(0)} = \mathfrak{D}^{-1} L^{(0)} \mathfrak{D}, \qquad \widetilde{R}^{(0)} = \mathfrak{D}^{-1} R^{(0)} \mathfrak{D} + \mathfrak{D}^{-1}[S^{(0)}, \mathfrak{D}]. \tag{8.6}$$

Observe that

$$\widetilde{G}^{(0)}(t) = G^{(0)}(t) \quad \Longleftrightarrow \quad \mathfrak{D}_0(t) = I.$$

8.2. Definition of Isomonodromy Deformation of the System (1.20) with Eigenvalues (1.25)

The Stokes phenomenon at $z = \infty$ has been already described in Chapter 6. Let $\widetilde{\tau}$ be an admissible direction for $\Lambda(0)$. For the remaining part of the Chapter, $\mathcal{V}$ will denote an open simply connected subset of a $\widetilde{\tau}$-cell, such that the closure $\overline{\mathcal{V}}$ is also contained in the cell. Let the label ν satisfy $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$. The holomorphic fundamental matrices of Section 7.4, namely $Y_\sigma(z,t)$, $\sigma = \nu, \nu+\mu, \nu+2\mu$, exist and satisfy Corollary 7.2 and Proposition 7.5. Therefore, in particular, they have canonical asymptotics on $\mathcal{S}_\sigma(\overline{\mathcal{V}})$, with holomorphic on $\mathcal{V}$ Stokes matrices $\mathbb{S}_\nu(t)$ and $\mathbb{S}_{\nu+\mu}(t)$.

REMARK 8.3 (**Notations**). The notation $Y_\nu(z,t)$ of Sections 7.4-7.6 has been used for the fundamental matrix solutions of the system (5.1). We consider now the system (1.20) and use the same notation $Y_\nu(z,t)$, with the replacement $G_0(t) \mapsto I$ in all the formulae where $G_0(t)$ appears.

DEFINITION 8.1. The **central connection matrix** $C_\nu^{(0)}(t)$ is defined by

$$Y_\nu(z,t) = Y^{(0)}(z,t) C_\nu^{(0)}(t), \qquad z \in \mathcal{R}.$$

DEFINITION 8.2 (Isomonodromic Deformation in $\mathcal{V}$). Let $\mathcal{V}$ be an open connected subset of a $\widetilde{\tau}$-cell, such that $\overline{\mathcal{V}}$ is also contained the cell. A t-deformation of the system (1.20) satisfying Assumption 8.1 in $\mathcal{V}$ is said to be **isomonodromic in** $\mathcal{V}$ if the *essential monodromy data*,

$$\mathbb{S}_{\nu}, \quad \mathbb{S}_{\nu+\mu}, \quad B_1 = \mathrm{diag}(\widehat{A}_1); \qquad \{\mu_1, \ ..., \ \mu_n\},$$

are independent of $t \in \mathcal{V}$, and if there exists a fundamental solution (8.2) (see Remark 8.2), holomorphic in $t \in \mathcal{V}$, such that also the corresponding essential monodromy data

$$R^{(0)}, \quad C_{\nu}^{(0)},$$

are independent of $t \in \mathcal{V}$.

REMARK 8.4. If μ_1, ..., μ_n are independent of t as in Definition 8.2, then Assumption 8.1 in $\mathcal{V}$ implies that also Assumption 8.2 holds in $\mathcal{V}$.

The existence of a fundamental solution with constant $R^{(0)}$ implies that the system (1.20) can be reduced to a simpler form (8.1) which is independent of $t \in \mathcal{V}$, namely

$$\frac{d\mathcal{Y}}{dz} = \frac{1}{z}\left(J^{(0)} + R_1 z + \cdots + R_{\kappa} z^{\kappa}\right)\mathcal{Y}, \tag{8.7}$$

where $1 \leq \kappa$ is the maximal integer difference of eigenvalues of $J^{(0)}$, $[R_l]_{block\ i,j} \neq 0$ only if $\widetilde{\mu}_i - \widetilde{\mu}_j = l$, $R_1 + \cdots + R_{\kappa} = R^{(0)}$, with all R_l independent of $t \in \mathcal{V}$, and the $\widetilde{\mu}_i$'s are the eigenvalues of $\widehat{A}_1(t)$ as arranged in the Jordan from (8.3)-(8.4).

REMARK 8.5. There is a freedom in the isomonodromic $R^{(0)}$ and $L^{(0)}$, as in (8.6), for a t-independent $\mathfrak{D}$ such that $\widetilde{Y}^{(0)} = Y^{(0)}\mathfrak{D}$. Hence, there is a freedom in the isomonodromic central connection matrix, according to

$$C^{(0)} = \mathfrak{D}\widetilde{C}^{(0)}.$$

We call $\mathcal{C}_0(J^{(0)}, L^{(0)})$ the group of such transformations $\mathfrak{D}$ which leave $L^{(0)}$ invariant in (8.6). This notation is a slight variation of a notation introduced in [**Dub99b**] for a particular subclass of our systems (1.20), related to Frobenius manifolds.

REMARK 8.6. Definition 8.2 is given with reference to some ν. Nevertheless, it implies that it holds for any other ν' in a suitably small $\mathcal{V}' \subset \mathcal{V}$. To see this, consider another admissible $\widetilde{\tau}' \in (\tau_{\nu'}, \tau_{\nu'+1})$, and define $\mathcal{S}_{\nu'+k\mu}(t)$, $Y_{\nu'+k\mu}(z,t)$ in the usual way, for t in the intersection of $\mathcal{V}$ with a $\widetilde{\tau}'$-cell. [2] Call $\mathcal{V}'$ the intersection. Now, there is a finite product of *Stokes factors* $K_1(t)\cdots K_M(t)$ ($M \leq$ number of basic Stokes rays of $\Lambda(t)$) such that $Y_{\nu}(z,t) = Y_{\nu'}(z,t)K_1(t)\cdots K_M(t)$, $t \in \mathcal{V}'$. The Stokes matrices $\mathbb{S}_{\nu}(t)$ and $\mathbb{S}_{\nu+\mu}(t)$ are determined uniquely by their factors, and conversely a Stokes matrix determines uniquely the factors of a factorization of the prescribed structure (see the proof of Theorem 7.2, or section 4 of [**BJL79a**], point D). Moreover, the product $K_1(t)\cdots K_M(t)$ appears in the factorization of $\mathbb{S}_{\nu}$ or $\mathbb{S}_{\nu+\mu}$. Hence, if $\mathbb{S}_{\nu}$ and $\mathbb{S}_{\nu+\mu}$ do not depend on $t \in \mathcal{V}$ for a certain ν, also $\mathbb{S}_{\nu'}$ and $\mathbb{S}_{\nu'+\mu}$ do not depend on $t \in \mathcal{V}' \subset \mathcal{V}$. Thus, the same is true for $C_{\nu'}^{(0)}$.

LEMMA 8.1. *Let the deformation be isomonodromic in $\mathcal{V}$ as in Definition 8.2 (here it is not necessary to suppose that $\mathcal{V}$ is in a cell, since we are considering solutions at $z = 0$). Let Assumption 8.1 hold in $\mathcal{U}_{\epsilon_0}(0)$, namely let $\widehat{A}_1(t)$ be holomorphically equivalent to $J^{(0)}$ in $\mathcal{U}_{\epsilon_0}(0)$. Then:*

i) *$\mu_1, ..., \mu_n$, $D^{(0)}$, $S^{(0)}$ and $J^{(0)}$ are independent of t in $\mathcal{U}_{\epsilon_0}(0)$.*

[2] Note that there may be more than one choices for $\mathcal{S}_{\nu'+k\mu}$, $Y_{\nu'+k\mu}(z,t)$, depending on the neighbourhood of t considered. See Remark 7.5.

ii) *Any fundamental matrix (also non-isomonodromic ones) in Levelt form* $Y^{(0)}(z,t) = G^{(0)}(t)(I+\sum_l \Psi_l(t)z^l)z^D z^{L^{(0)}(t)}$*, which is holomorphic of* $t \in \mathcal{V}$*, is also holomorphic on the whole* $\mathcal{U}_{\epsilon_0}(0)$*.*

iii) *If* $R^{(0)}$ *(i.e* $L^{(0)}$*) is independent of* t *in* $\mathcal{V}$*, then it is independent of* t *in* $\mathcal{U}_{\epsilon_0}(0)$*.*

Proof: i) That $\mu_1, ..., \mu_n$, $D^{(0)}$, $S^{(0)}$, $J^{(0)}$ are constant in $\mathcal{U}_{\epsilon_0}(0)$ follows from the fact that $\mu_1, ..., \mu_n$ are constant in $\mathcal{V}$, and that $G^{(0)}(t)$, and so the $\mu_1, ..., \mu_n$, are holomorphic on $\mathcal{U}_{\epsilon_0}(0)$. So $\mu_1, ..., \mu_n$ are constant in $\mathcal{U}_{\epsilon_0}(0)$.

ii) Since $\mu_1, ..., \mu_n$ are constant in $\mathcal{U}_{\epsilon_0}(0)$, and $\Lambda(t)$ and $\widehat{A}_1(t)$ are holomorphic, the recursive standard procedure which yields the Birkhoff normal form at $z = 0$ allows to choose $\Psi_l(t)$'s and $R^{(0)}(t)$ holomorphic on $\mathcal{U}_{\epsilon_0}(0)$.

iii) That $R^{(0)}$ is independent of $t \in \mathcal{U}_{\epsilon_0}(0)$ follows from the fact that $R^{(0)}(t)$ is holomoprhic on $\mathcal{U}_{\epsilon_0}(0)$ and constant on $\mathcal{V}$. □

PROPOSITION 8.1. *Let the deformation of the system (1.20) be isomonodromic in* $\mathcal{V}$ *as in Definition 8.2 (here it is not necessary to assume that* $\mathcal{V}$ *is contained in a cell). Let Assumption 8.1 hold in* $\mathcal{U}_{\epsilon_0}(0)$*, namely let* $\widehat{A}_1(t)$ *be holomorphically equivalent to* $J^{(0)} = D^{(0)} + S^{(0)}$ *in* $\mathcal{U}_{\epsilon_0}(0)$*. Consider the system*

$$\frac{dY}{dz} = \widehat{A}(z,0)Y, \tag{8.8}$$

and a fundamental solution in the Levelt form

$$\mathring{Y}^{(0)}(z) = \mathring{G}^{(0)}\mathring{G}(z)z^{D^{(0)}}z^{\mathring{L}}, \qquad \mathring{G}(z) = I + \mathcal{O}(z), \tag{8.9}$$

with $\mathring{L} = S^{(0)} + \mathring{R}$*. Here* $\mathring{R}$ *is obtained by reducing (8.8) to a Birkhoff normal form at* $z = 0$*. Then, there exists an isomonodromic fundamental solution of (1.20), call it* $Y^{(0)}_{isom}(z,t)$*, with the same monodromy exponent* $\mathring{L}$ *and Levelt form*

$$Y^{(0)}_{isom}(z,t) = G^{(0)}(t)G_{isom}(z,t)z^{D^{(0)}}z^{\mathring{L}},$$

with $G_{isom}(z,t) = I + \sum_{k=1}^{\infty}\Psi_l(t)z^l$*, holomorphic on* $\mathcal{U}_{\epsilon_0}(0)$*, such that*

$$\mathring{Y}^{(0)}(z) = Y^{(0)}_{isom}(z,0).$$

Proof: We prove the proposition in two steps.

• The first step is the following

LEMMA 8.2. *Let the deformation be isomonodromic in* $\mathcal{V}$ *as in Definition 8.2 (here it is not necessary to assume that* $\mathcal{V}$ *is contained in a cell). Let* $\widehat{A}_1(t)$ *be holomorphically equivalent to* $J^{(0)}$ *in* $\mathcal{U}_{\epsilon_0}(0)$*. For any holomorphic fundamental solution in Levelt form*

$$Y(z,t) = G(t)H(z,t)z^{D^{(0)}}z^{L^{(0)}(t)}, \qquad H(z,t) = I + \sum_{l=1}^{\infty} h_l(t)z^l,$$

with monodromy exponent $L^{(0)}(t)$*, there exists an isomonodromic* $Y^{(0)}(z,t)$*, with monodromy exponent equal to* $L^{(0)}(0)$*, in the Levelt form*

$$Y^{(0)}(z,t) = G^{(0)}(t)G(z,t)z^{D^{(0)}}z^{L^{(0)}(0)}, \qquad G(z,t) = I + \sum_{l=1}^{\infty}\Psi_l(t)z^l,$$

such that $Y^{(0)}(z,0) = Y(z,0)$*.*

To prove this Lemma, consider an isomonodromic fundamental solution, which exists by assumption, say

$$\widetilde{Y}^{(0)}(z,t)=\widetilde{G}^{(0)}(t)\widetilde{G}(z,t)z^{D^{(0)}}z^{\widetilde{L}^{(0)}}, \qquad \widetilde{G}(z,t)=I+\sum_{l=1}^{\infty}\widetilde{\Psi}_l(t)z^l,$$

with t-independent monodromy exponent $\widetilde{L}^{(0)}$ and t-independent connection matrix defined by

$$Y_\nu(z,t)=\widetilde{Y}^{(0)}(z,t)\ \widetilde{C}_\nu^{(0)}.$$

Then, there exists a holomorphic invertible connection matrix $\mathfrak{D}(t)$ such that

$$Y(z,t)=\widetilde{Y}^{(0)}(z,t)\mathfrak{D}(t).$$

Hence,

$$\mathfrak{D}_0(t)\Big(I+\mathfrak{D}_1(t)z+\cdots+\mathfrak{D}_\kappa(t)z^\kappa\Big)z^{D^{(0)}}z^{L^{(0)}(t)}=z^{D^{(0)}}z^{\widetilde{L}^{(0)}}\ \mathfrak{D}(t) \tag{8.10}$$

with $\mathfrak{D}(t)=\mathfrak{D}_0(t)\Big(I+\mathfrak{D}_1(t)+\cdots+\mathfrak{D}_\kappa(t)\Big)$. Observe that $z^{D^{(0)}}z^{L^{(0)}(0)}$ and $z^{D^{(0)}}z^{\widetilde{L}^{(0)}}$ are fundamental solutions of two Birkhoff normal forms of (8.8), related by (8.10) with $t=0$, namely

$$\mathfrak{D}_0(0)(I+\mathfrak{D}_1(0)z+\cdots+\mathfrak{D}_\kappa(0)z^\kappa)z^{D^{(0)}}z^{L^{(0)}(0)}=z^{D^{(0)}}z^{\widetilde{L}^{(0)}}\ \mathfrak{D}(0).$$

Therefore, the isomonodromic fundamental solution we are looking for is

$$Y^{(0)}(z,t):=\widetilde{Y}^{(0)}(z,t)\ \mathfrak{D}(0)=Y(z,t)\mathfrak{D}(t)^{-1}\mathfrak{D}(0).$$

- Second step. Consider a fundamental solution of (8.8) in the Levelt form

$$\mathring{Y}^{(0)}(z)=\mathring{G}^{(0)}\mathring{G}(z)z^{D^{(0)}}z^{\mathring{L}},$$

where $\mathring{L}=S^{(0)}+\mathring{R}$, $\mathring{R}=\sum_{l=1}^{\kappa}\mathring{R}_l$. The $\mathring{R}_l$, $l=1,2,...,\kappa$, are coefficients of a simple gauge equivalent form(8.1), with $t=0$, of (8.8). It can be proved that there is a form (8.1) for the system (1.20), with coefficients $R_l(t)$, such that the $\mathring{R}_l$'s coincide with the values $R_l(0)$'s at $t=0$. Moreover, the $R_l(t)$'s are holomorphic on $\mathcal{U}_{\epsilon_0}(0)$. This fact follows from the recursive procedure which yileds the gauge transformation from (1.20) to (8.1). Therefore, there exists a holomorphic exponent $L^{(0)}(t)$ such that $L^{(0)}(0)=\mathring{L}$. Consider an isomonodromic fundamental solution $Y^{(0)}(z,t)$ of Lemma 8.2, with exponent $L^{(0)}(0)=\mathring{L}$. Since $Y^{(0)}(z,0)$ is a fundamental solution of (8.8), there exists an invertible and constant connection matrix C such that

$$Y^{(0)}(z,0)C=\mathring{Y}^{(0)}(z).$$

Now, $C\in\mathcal{C}_0(J^{(0)},\mathring{L})$ (cf. Remark 8.5), because $Y^{(0)}(z,0)$ and $\mathring{Y}^{(0)}(z)$ have the same monodromy exponent. This implies that

$$\begin{aligned} Y^{(0)}(z,t)C \quad &= G^{(0)}G(z,t)z^{D^{(0)}}z^{\mathring{L}}C= \\ &= G^{(0)}G(z,t)C_0(I+C_1z+\cdots+C_\kappa z^\kappa)z^{D^{(0)}}z^{\mathring{L}}, \qquad C=C_0(I+C_1+\cdots+C_\kappa). \end{aligned}$$

Moreover, also $Y^{(0)}(z,t)C$ is isomonodromic. Therefore, the solution we are looking for is $Y_{isom}^{(0)}(z,t):=Y^{(0)}(z,t)C$. $\square$

8.3. Isomonodromy Deformation Equations

Let

$$\Omega(z,t) := \sum_{k=1}^{n} \Omega_k(z,t)\ dt_k, \qquad \Omega_k(z,t) := zE_k + [F_1(t), E_k].$$

Here E_k is the matrix with all entries equal to zero, except for $(E_k)_{kk} = 1$, and $(F_1)_{ab} = -(\widehat{A}_1)_{ab}/(u_a - u_b)$, so that

$$[F_1(t), E_k] = \left(\frac{(\widehat{A}_1(t))_{ab}(\delta_{ak} - \delta_{bk})}{u_a(t) - u_b(t)} \right)_{a,b=1..n} = \begin{pmatrix} 0 & 0 & \frac{-(\widehat{A}_1)_{1k}}{u_1 - u_k} & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 \\ \frac{(\widehat{A}_1)_{k1}}{u_k - u_1} & \cdots & 0 & \cdots & \frac{(\widehat{A}_1)_{kn}}{u_k - u_n} \\ 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \frac{-(\widehat{A}_1)_{nk}}{u_n - u_k} & 0 & 0 \end{pmatrix} \tag{8.11}$$

Let $df(z,t) := \sum_{i=1}^{n} \partial f(z,t)/\partial t_i dt_i$.

Theorem 8.1. *If the deformation of the system (1.20) is isomonodromic in $\mathcal{V}$ as in Definition 8.2, then an isomonodromic $Y^{(0)}(z,t)$ and the $Y_\sigma(z,t)$'s, for $\sigma = \nu, \nu+\mu, \nu+2\mu$, satisfy the total differential system*

$$dY = \Omega(z,t)Y. \tag{8.12}$$

Conversely, if the t-deformation satisfies Assumptions 8.1 and 8.2 in $\mathcal{V}$, and if a fundamental solution $Y^{(0)}(z,t)$ in Levelt form at $z=0$, and the canonical solution $Y_\sigma(z,t)$, $\sigma = \nu, \nu+\mu, \nu+2\mu$ at $z=\infty$, satisfy the total differential system (8.12), then the deformation is isomonodromic in $\mathcal{V}$.

Proof: The proof is done in the same way as for Theorem 3.1 at page 322 in [**JMU81**]. In [**JMU81**] the proof is given for non resonant $\widehat{A}_1(t)$, but it can be repeated in our case with no changes, except for the Assumptions 8.1 and 8.2. [3] The matrix valued differential form $\Omega(z,t)$ turns out to be still as in formula (3.8) and (3.14) of [**JMU81**], which in our case becomes,

$$\Omega(z,t) = \left[\left(I + \sum_{k=1}^{\infty} F_k(t) z^{-k} \right) \ d\Lambda(t) z \ \left(I + \sum_{k=1}^{\infty} F_k(t) z^{-k} \right)^{-1} \right]_{sing},$$

where $[\cdots]_{sing}$ stands for the singular terms at infinity, namely the terms with powers z^j, $j \geq 0$, in the above formal expansion. This is

$$\Omega(z,t) = d\Lambda(t) z + [F_1(t), d\Lambda(t)].$$

Therefore,

$$\Omega_k(z,t) = \frac{\partial \Lambda(t)}{\partial t_k}\ z\ + \left[F_1(t), \frac{\partial \Lambda(t)}{\partial t_k} \right]\ = E_k + [F_1(t), E_k].$$

[3]The result was announced in [**Nob81**] and not proved. It can also be proved by the methods of [**KV06**], since the requirement that $\mu_1, \ldots, \mu_n$, $R^{(0)}$ and $C^{(0)}$ are constant is equivalent to having an isoprincipal deformation.

In the last step we have used the fact that $\Lambda(t) = \operatorname{diag}(u_1(t), u_2(t), \ldots, u_n(t))$, with eigenvalues (1.25). In the domain $\mathcal{V}$ the eigenvalues are distinct, so the off-diagonal entries of F_1 are:

$$(F_1)_{ab} = \frac{(\widehat{A}_1)_{ab}}{u_b - u_a}, \qquad 1 \leq a \neq b \leq n.$$

Hence,

$$\Omega_k(z,t) = E_k\ z\ + \left(\frac{\widehat{A}^{(1)}_{ab}}{u_b(t) - u_a(t)}\ \frac{\partial}{\partial t_k}\Big(u_b(t) - u_a(t)\Big)\right)_{a,b=1}^{n}.$$

Finally, observe that $\frac{\partial}{\partial t_k}(u_b(t) - u_a(t)) = \frac{\partial}{\partial t_k}(t_b - t_a) = \delta_{kb} - \delta_{ka}$. The proof is concluded. □

COROLLARY 8.1. *If the deformation of the system (1.20) is isomonodromic in $\mathcal{V}$ as in Definition 8.2, then $G^{(0)}(t)$ satisfies*

$$dG^{(0)} = \Theta^{(0)}(t)\ G^{(0)}, \tag{8.13}$$

where

$$\Theta^{(0)}(t) = \Omega(0,t) = \sum_k [F_1(t), E_k] dt_k.$$

More explicitly,

$$\Theta^{(0)}(t) = \left(\frac{\widehat{A}^{(1)}_{ab}}{u_a(t) - u_b(t)}\ (dt_a - dt_b)\right)_{a,b=1}^{n}.$$

Proof: Substitute $Y^{(0)}$ into (8.12) an compare coefficients of equal powers of z. Equation (8.13) comes form the coefficient of z^0. □

PROPOSITION 8.2. *If the deformation is isomonodromic in $\mathcal{V}$ as in Definition 8.2, then*

$$d\widehat{A} = \frac{\partial \Omega}{\partial z} + [\Omega, \widehat{A}]. \tag{8.14}$$

Proof: Let the deformation be isomonodromic. Then, by Theorem 8.1, equations (1.20) and (8.12) are compatible. The compatibility condition is (8.14). □

Note that (8.14) is a necessary condition of isomonodromicity, but not sufficient in case of resonances (sufficiency can be proved if the eigenvalues of $\widehat{A}_1$ do not differ by integers, cf. [**JMU81**]). Explicitly, (8.14) is

$$\begin{cases} [E_k, \widehat{A}_1] = [\Lambda, [F_1, E_k]], \quad k = 1, \ldots, n, \\[2ex] \quad d\widehat{A}_1 = [\Theta^{(0)}, \widehat{A}_1]. \end{cases}$$

The first n equations are automatically satisfied by definition of F_1. The last equation in components is

$$\frac{\partial \widehat{A}_1}{\partial t_k} = \Big[[F_1, E_k], \widehat{A}_1\Big], \tag{8.15}$$

where $[F_1, E_k]$ is in (8.11).

8.4. Holomorphic Extension of Isomonodromy Deformations to $\mathcal{U}_{\epsilon_0}(0)$ and Theorem 1.6

LEMMA 8.3. *In case the eigenvalues of $\Lambda(t)$ are as in (1.25) and $\widehat{A}_1(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, then $\Omega(z,t)$ is holomoprhic (in t) on $\mathcal{U}_{\epsilon_0}(0)$ if and only if*

$$(\widehat{A}_1)_{ab}(t) = \mathcal{O}(u_a(t) - u_b(t)) \equiv \mathcal{O}(t_a - t_b), \tag{8.16}$$

whenever $u_a(t)$ and $u_b(t)$ coalesce as t tends to a point of $\Delta \subset \mathcal{U}_{\epsilon_0}(0)$.
Also $\Theta^{(0)}(t)$ of Corollary 8.1 is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$ if and only if (8.16) holds.

Proof: By (8.11), $\Omega(z,t)$ and $\Theta^{(0)}(t)$ are continuous at $t_\Delta \in \Delta$ if and only if (8.16) holds for those $u_a(t)$, $u_b(t)$ coalescing at $t_\Delta \in \Delta$. Hence, any point of Δ is a removable singularity if and only if (8.16) holds. □

PROPOSITION 8.3. *The system*

$$\begin{aligned} &(8.14) \qquad d\widehat{A} = \frac{\partial \Omega}{\partial z} + [\Omega, \widehat{A}], \\ &(8.13) \qquad dG^{(0)} = \Theta^{(0)}(t)\ G^{(0)}, \end{aligned}$$

with $\widehat{A}_1$ holomorphic satisfying condition (8.16) on $\mathcal{U}_{\epsilon_0}(0)$, is Frobenius integrable for $t \in \mathcal{U}_{\epsilon_0}(0)$.

The proof is as in [**JMU81**]. It holds also in our case, because the algebraic relations are the same as in our case, no matter if $\widehat{A}_1$ is resonant (see e.g. Example 3.2 in [**JMU81**]).

Write $\Theta^{(0)} = \sum_k \Theta_k^{(0)} dt_k$. Since (8.13) is integrable, the compatibility of equations holds:

$$\frac{\partial \Theta_j^{(0)}}{\partial t_i} - \frac{\partial \Theta_i^{(0)}}{\partial t_j} = \Theta_i^{(0)}\Theta_j^{(0)} - \Theta_j^{(0)}\Theta_i^{(0)}. \tag{8.17}$$

PROPOSITION 8.4. *Let the deformation of the system (1.20) be isomonodromic in $\mathcal{V}$ as in Definition 8.2, with $\Lambda(t)$ is as in (1.25) and $\widehat{A}_1(t)$ holomorphic on $\mathcal{U}_{\epsilon_0}(0)$. Then, $\widehat{A}_1(t)$ is holomorphically similar to $J^{(0)}$ in the whole $\mathcal{U}_{\epsilon_0}(0)$ if and only if (8.16) holds as t tends to points of $\Delta \subset \mathcal{U}_{\epsilon_0}(0)$. In other words, if the deformation is isomonodromic in $\mathcal{V}$ with holomorphic $\widehat{A}_1(t)$, then Assumption 8.1 in the whole $\mathcal{U}_{\epsilon_0}(0)$ is equivalent to (8.16).*

Proof: Let $\widehat{A}_1(t)$ be holomorphic and let (8.16) hold, so that $\Theta^{(0)}(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$ by Lemma 8.3. The *linear* Pfaffian systems $dG^{(0)} = \Theta^{(0)}(t)G^{(0)}$ and $d[(G^{(0)})^{-1}] = -(G^{(0)})^{-1}\Theta^{(0)}(t)$ are integrable in $\mathcal{U}_{\epsilon_0}(0)$, with holomorphic coefficients $\Theta^{(0)}(t)$. Then, a solution $G^{(0)}(t)$ has analytic continuation onto $\mathcal{U}_{\epsilon_0}(0)$. We take a solution satisfying $(G^{(0)}(t))^{-1}\ \widehat{A}_1(t)\ G^{(0)}(t) = J^{(0)}$ for $t \in \mathcal{V}$, which then has analytic continuation onto $\mathcal{U}_{\epsilon_0}(0)$ as a holomorphic invertible matrix. Hence, $(G^{(0)}(t))^{-1}\ \widehat{A}_1(t)$ $G^{(0)}(t) = J^{(0)}$ holds in $\mathcal{U}_{\epsilon_0}(0)$ with holomorphic $G^{(0)}(t)$. Conversely, suppose that Assumption 8.1 holds in $\mathcal{U}_{\epsilon_0}(0)$. Then $G^{(0)}(t)$ and $G^{(0)}(t)^{-1}$ are holomorphic on $\mathcal{U}_{\epsilon_0}(0)$. Therefore, also $\Theta^{(0)}(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, because $\Theta^{(0)}(t) = dG^{(0)} \cdot (G^{(0)})^{-1}$ defines the analytic continuation of $\Theta^{(0)}(t)$ on $\mathcal{U}_{\epsilon_0}(0)$. Then (8.16) holds, by Lemma 8.3. □

Summarising, if $\Lambda(t)$ is as in (1.25) and $\widehat{A}_1(t)$ is holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, if the deformation is isomonodromic in a simply connected subset $\mathcal{V}$ of a cell, s.t. $\overline{\mathcal{V}} \subset$ cell, then it suffices to assume that $\widehat{A}_1(t)$ is holomorphically similar to a Jordan form $J^{(0)}(t)$ in $\mathcal{U}_{\epsilon_0}(0)$, or equivalently that (8.16) holds at $\Delta \subset \mathcal{U}_{\epsilon_0}(0)$, in order to conclude that the system

$$(8.12) \qquad dY = \Omega(z,t)\ Y,$$

$$(8.13) \qquad dG^{(0)} = \Theta^{(0)}(t)\ G^{(0)},$$

has holomorphic coefficients on $\mathcal{R} \times \mathcal{U}_{\epsilon_0}(0)$. The integrability/compatibility condition of (8.12) is

$$\frac{\partial \Omega_j}{\partial t_i} - \frac{\partial \Omega_i}{\partial t_j} = \Omega_i \Omega_j - \Omega_j \Omega_i. \tag{8.18}$$

If this relation is explicitly written, it turns out to be equivalent to (8.17). Hence, being (8.13) integrable, also the linear Pfaffian system (8.12) is integrable, with coefficients holomorphic in $\mathcal{U}_{\epsilon_0}(0)$. Therefore, due to linearity, any solution $Y(z,t)$ can be t-analytically continued along any curve in $\mathcal{U}_{\epsilon_0}(0)$, for z fixed.

Corollary 8.2. *Let the deformation be isomonodromic in a simply connected subset $\mathcal{V}$ of a cell, s.t. $\overline{\mathcal{V}} \subset$ cell. If $\widehat{A}_1(t)$ is holomorphically similar to a Jordan form $J^{(0)}$ in $\mathcal{U}_{\epsilon_0}(0)$, or equivalently if (8.16) holds in $\mathcal{U}_{\epsilon_0}(0)$, then the $Y_\sigma(z,t)$'s, $\sigma = \nu, \nu+\mu, \nu+2\mu$, together with an isomonodromic $Y^{(0)}(z,t)$, can be t-analytically continued as single valued holomorphic functions on $\mathcal{U}_{\epsilon_0}(0)$.*

Proof: If the deformation is isomonodromic, by Theorem 8.1 the system (1.20),(8.12) is a completely integrable linear Pfaffian system (compatibility conditions (8.14) and (8.18) hold), with common solutions $Y_\sigma(z,t)$'s, $\sigma = \nu, \nu+\mu, \nu+2\mu$, and $Y^{(0)}(z,t)$. If $\widehat{A}_1(t)$ is holomorphically similar to a Jordan form $J^{(0)}$ in $\mathcal{U}_{\epsilon_0}(0)$, or equivalently if (8.16) holds in $\mathcal{U}_{\epsilon_0}(0)$, then the coefficients are holomorphic in $\mathcal{U}_{\epsilon_0}(0)$, by Proposition 8.4. In particular, since $Y_\sigma(z,t)$'s, $\sigma = \nu, \nu+\mu, \nu+2\mu$, and $Y^{(0)}(z,t)$ solve (8.12), they can be t-analytically continued along any curve in $\mathcal{U}_{\epsilon_0}(0)$. $\Box$.

Remark 8.7. Corollary 8.2 can be compared with the result of [**Miw81**]. It is always true that the $Y_\sigma(t,z)$'s and $Y^{(0)}(t,z)$ *can be t-analytically continued on $\mathcal{T}$ as a meromorphic function*, where (in our case):

$$\mathcal{T} = \text{ universal covering of } \mathbb{C}^n \backslash \Delta_{\mathbb{C}^n}.$$

Here $\Delta_{\mathbb{C}^n}$ is the locus of $\mathbb{C}^n$ where eigenvalues of $\Lambda(t)$ coalesce. It is a locus of "fixed singularities" (including branch points and essential singularities) of $\Omega(z,t)$ and of any solution of $dY = \Omega Y$. The movable singularities of $\Omega(z,t)$, $Y_\sigma(t,z)$ and $Y^{(0)}(t,z)$ outside the locus are poles and constitute the zeros of the Jimbo-Miwa isomonodromic τ-function [**Miw81**]. Here, we have furthermore assumed that $\widehat{A}_1$ is holomorphic in $\mathcal{U}_{\epsilon_0}(0)$ and that (8.16) holds. This fact has allowed us to conclude that $Y_\sigma(z,t)$'s, $\sigma = \nu, \nu+\mu, \nu+2\mu$, and $Y^{(0)}(z,t)$ are t-holomorphic in $\mathcal{U}_{\epsilon_0}(0)$.

In order to prove Theorem 1.6, we need a last ingredient, namely the analyticity at Δ of the coefficients $F_k(t)$ of the formal solution computed away from Δ.

PROPOSITION 8.5. *Let the deformation of the system (1.20) be isomonodromic in a simply connected subset $\mathcal{V}$ of a cell, s.t. $\overline{\mathcal{V}} \subset$ cell. If $\widehat{A}_1(t)$ is holomorphically similar to a Jordan form $J^{(0)}$ in $\mathcal{U}_{\epsilon_0}(0)$, or equivalently if (8.16) holds in $\mathcal{U}_{\epsilon_0}(0)$, then the coefficients $F_k(t)$, $k \geq 1$, of a formal solution of (1.20)*

$$Y_F(z,t) = \Big(I + \sum_{k=1}^{\infty} F_k(t) z^{-k}\Big) z^{B_1} e^{\Lambda(t) z}, \tag{8.19}$$

are holomorphic on $\mathcal{U}_{\epsilon_0}(0)$.

Proof: Recall that

$$(F_1)_{ab}(t) = \frac{(\widehat{A}_1)_{ab}(t)}{u_b(t) - u_a(t)}, \qquad a \neq b,$$
$$(F_1)_{aa}(t) = -\sum_{b \neq a} (\widehat{A}_1)_{ab}(t)(F_1)_{ba}(t).$$

If by assumption (8.16) holds, the above formulas imply that $F_1(t)$ is holomorphic in $\mathcal{U}_{\epsilon_0}(0)$, because the singularities at Δ, i.e. for $u_a(t) - u_b(t) \to 0$, become removable. Since the asymptotics corresponding to (8.19) is uniform in a compact subset K of a simply connected open subset of a cell, we substitute it into $dY = \Omega(z,t)Y$, with

$$\Omega(z,t) = z d\Lambda(t) + [F_1(t), d\Lambda(t)].$$

By comparing coefficients of powers of z^{-l} we obtain

$$[F_{l+1}(t), d\Lambda(t)] = [F_1(t), d\Lambda(t)] F_l(t) - dF_l(t), \qquad l \geq 1. \tag{8.20}$$

In components of the differential d, this becomes a recursive relation (use $\partial \Lambda(t)/\partial t_i = E_i$):

$$\Big[F_{l+1}(t), E_i\Big] = \Big[F_1(t), E_i\Big] F_l(t) - \frac{\partial F_l(t)}{\partial t_i},$$

with,

$$\Big[F_{l+1}(t), E_i\Big] = \begin{pmatrix} 0 & & (F_{l+1})_{1i} & & 0 \\ & & \vdots & & \\ -(F_{l+1})_{i1} & \cdots & 0 & \cdots & -(F_{l+1})_{in} \\ & & \vdots & & \\ 0 & & (F_{l+1})_{ni} & & 0 \end{pmatrix},$$

The diagonal element (i,i) is zero. Therefore, (8.20) recursively determines F_{l+1} as a function of $F_l, F_{l-1}, ..., F_1$, except for the diagonal $\operatorname{diag}(F_{l+1})$. On the other hand, the diagonal elements are determined by the off-diagonal elements according to the already proved formula,

$$l\ (F_{l+1})_{aa}(t) = -\sum_{b \neq a} (\widehat{A}_1)_{ab}(t)(F_l)_{ba}(t). \tag{8.21}$$

Let us start with $l + 1 = 2$. Since F_1 is holomorphic, the above formulae (8.20), (8.21) imply that F_2 is holomorphic. Then, by induction the same formulae imply that all the $F_{l+1}(t)$ are holomorphic.
□

Corollary 8.2 means that assumption 2) of Theorem 7.1 applies, while Proposition 8.5 means that assumption 1) applies. This, together with Proposition 8.1, proves the following theorem, which is indeed our Theorem 1.6.

THEOREM 8.2 (Theorem 1.6). *Let $\Lambda(t)$ and $\widehat{A}_1(t)$ be holomorphic on $\mathcal{U}_{\epsilon_0}(0)$, with eigenvalues as in (1.25). If the deformation of the system (1.20) is isomonodromic on a simply connected subset $\mathcal{V}$ of a cell, such that $\overline{\mathcal{V}}$ is in the cell, and if $\widehat{A}_1(t)$ is holomorphically similar to a Jordan form $J^{(0)}$ in $\mathcal{U}_{\epsilon_0}(0)$, or equivalently the vanishing condition*

$$(\widehat{A}_1)_{ab}(t) = \mathcal{O}(u_a(t) - u_b(t)) \equiv \mathcal{O}(t_a - t_b),$$

holds at points of Δ in $\mathcal{U}_{\epsilon_0}(0)$, then Theorem 7.1 and Corollary 7.4 hold (with $G_0(t) \mapsto I$, see Remark 8.3), so that $\mathcal{G}_\sigma(z,t) = Y_\sigma(z,t)e^{\Lambda(t)}z^{-B_1(t)}$, $\sigma = \nu, \nu+\mu, \nu+2\mu$, maintains the canonical asymptotics

$$\mathcal{G}_\sigma(z,t) \sim I + \sum_{k=1}^{\infty} F_k(t) z^{-k}, \qquad z \to \infty \text{ in } \widehat{\mathcal{S}}_\sigma,$$

for any $t \in \mathcal{U}_{\epsilon_1}(0)$ and any $\epsilon_1 < \epsilon_0$. The Stokes matrices,

$$\mathbb{S}_\nu, \quad \mathbb{S}_{\nu+\mu},$$

are defined and constant on the whole $\mathcal{U}_{\epsilon_0}(0)$. They coincide with the Stokes matrices $\mathring{\mathbb{S}}_\nu$, $\mathring{\mathbb{S}}_{\nu+\mu}$ of the specific fundamental solutions $\mathring{Y}_\sigma(z)$ of the system (8.8)

$$\frac{dY}{dz} = \widehat{A}(z,0)Y,$$

which satisfy $\mathring{Y}_\sigma(z) \equiv Y_\sigma(z,0)$, according to Corollary 7.4. Any central connection matrix $C_\nu^{(0)}$ is defined and constant on the whole $\mathcal{U}_{\epsilon_0}(0)$, coinciding with a matrix $\mathring{C}_\nu^{(0)}$ defined by the relation

$$\mathring{Y}_\nu(z) = \mathring{Y}^{(0)}(z)\mathring{C}_\nu^{(0)},$$

where $\mathring{Y}^{(0)}(z)$ is a fundamental solution of (8.8) in the Levelt form (8.9), and $\mathring{Y}_\nu(z) = Y_\nu(z,0)$ as above.
The matrix entries of Stokes matrices vanish in correspondence with coalescing eigenvalues, i.e.

$$(\mathbb{S}_1)_{ij} = (\mathbb{S}_1)_{ji} = (\mathbb{S}_2)_{ij} = (\mathbb{S}_2)_{ji} = 0 \qquad \text{whenever } u_i(0) = u_j(0).$$

COROLLARY 8.3 (Corollary 1.1). *If moreover the diagonal entries of $\widehat{A}_1(0)$ do not differ by non-zero integers, Corollary 5.1 applies. Accordingly, there is a unique formal solution of the system with $t = 0$, whose coefficients are necessarily*

$$\mathring{F}_k \equiv F_k(0).$$

Hence, there exists only one choice of fundamental solutions $\mathring{Y}_\sigma(z)$'s with canonical asymptotics at $z = \infty$ corresponding to the unique formal solution, which necessarily coincide with the $Y_\sigma(z,0)$'s.

Summarizing, the monodromy data are computable from the system with fixed $t = 0$ and are:
- $J^{(0)}$ = a Jordan form of $\widehat{A}_1(0)$; $R^{(0)} = \mathring{R}$. See Proposition 8.1.
- $B_1 = \text{diag}(\widehat{A}_1(0))$.

- $\mathbb{S}_\nu = \mathring{\mathbb{S}}_\nu$, $\mathbb{S}_{\nu+\mu} = \mathring{\mathbb{S}}_{\nu+\mu}$.
- $C_\nu^{(0)} = \mathring{C}_\nu^{(0)}$.

Here, $\mathring{\mathbb{S}}_1$ and $\mathring{\mathbb{S}}_2$ are the Stokes matrices of those fundamental solutions $\mathring{Y}_1(z)$, $\mathring{Y}_2(z)$, $\mathring{Y}_3(z)$ of the system (8.8) (i.e. system (1.29)) with the specific canonical asymptotics (1.31) satisfying $\mathring{F}_k \equiv F_k(0)$, $k \geq 1$. For these solutions the identity $\mathring{Y}_r(z) = Y_r(z, 0)$ holds. In case of Lemma 8.3, only these solutions exist.

8.5. Isomonodromy Deformations with Vanishing Conditions on Stokes Matrices, Proof of Theorem 1.7

We now consider again system (1.20) with eigenvalues (1.25) coalescing at $t = 0$, but we *give up the assumption that $\widehat{A}_1(t)$ is holomorphic in the whole $\mathcal{U}_{\epsilon_0}(0)$*. We assume that $\widehat{A}_1(t)$ is holomorphic on a simply connected open domain $\mathcal{V} \subset \mathcal{U}_{\epsilon_0}(0)$, as in Definition 1.2, so that the Jimbo-Miwa-Ueno isomonodromy deformation theory[4] is well defined $\mathcal{V}$. Therefore $Y_{\nu+k\mu}(t, z)$'s ($k \in \mathbb{Z}$) and $Y^{(0)}(t, z)$ satisfy the system

$$\frac{dY}{dz} = \left(\Lambda(t) + \frac{\widehat{A}_1(t)}{z}\right) Y, \tag{8.22}$$

$$dY = \Omega(z, t)Y, \tag{8.23}$$

and $\widehat{A}_1(t)$ solves the non-linear isomonodromy deformation equations

$$d\widehat{A} = \frac{\partial \Omega}{\partial z} + [\Omega, \widehat{A}],$$
$$dG^{(0)} = \Theta^{(0)}\, G^{(0)}.$$

Here Ω and $\Theta^{(0)}$ are the same as in the previous sections, defined for $t \in \mathcal{V}$.

Since the deformation is admissible, there exists $\widetilde{\tau}$ such that $\overline{\mathcal{V}} \subset c$, where c is a $\widetilde{\tau}$-cell in $\mathcal{U}_{\epsilon_0}(0)$. The Stokes rays of $\Lambda(0)$ will be numerated so that $\tau_\nu < \widetilde{\tau} < \tau_{\nu+1}$.

As in Remark 8.7, the solutions $\widehat{A}_1(t)$, any $Y_{\nu+k\mu}(t, z)$'s and $Y^{(0)}(t, z)$ of the above isomonodromy deformation equations, initially defined in $\mathcal{V}$, can be t-analytically continued on the universal covering of $\mathbb{C}^n \backslash \Delta_{\mathbb{C}^n}$, as a meromorphic functions. The coalescence locus $\Delta_{\mathbb{C}^n}$ is a locus of fixed singularities [**Miw81**], so that it may be a branching locus for $\widehat{A}_1(t)$ and for any of the fundamental matrices $Y(z, t)$ of (8.22) (i.e. of (1.20)). Notice that our Δ is obviously contained in $\Delta_{\mathbb{C}^n}$. The movable singularities of $\widehat{A}_1(t)$, $Y_{\nu+k\mu}(t, z)$ and $Y^{(0)}(t, z)$ outside $\Delta_{\mathbb{C}^n}$ are poles and constitute, according to [**Miw81**], the locus of zeros of the Jimbo-Miwa-Ueno isomonodromic τ-function. This locus can also be called *Malgrange's divisor*, since it has been proved in [**Pal99**] that it coincides with a divisor, introduced by Malgrange (see [**Mal83a**] [**Mal83b**] [**Mal83c**]), where a certain Riemann-Hilbert problem fails to have solution (below, we formulate a Riemann-Hilbert problem in proving Lemma 8.5). This divisor has a complex co-dimension equal to 1, so it does not disconnect $\mathbb{C}^n \backslash \Delta_{\mathbb{C}^n}$ and $\mathcal{U}_{\epsilon_0}(0) \backslash \Delta$.

The fundamental solutions $Y_{\nu+k\mu}(t, z)$'s above are the unique solutions which have for $t \in \mathcal{V}$ the asymptotic behaviour

$$Y_{\nu+k\mu}(z, t) e^{-\Lambda(t)z} z^{-B_1} \sim I + \sum_{j \geq 1} F_j(t) z^{-j}, \qquad z \to \infty \text{ in } \mathcal{S}_{\nu+k\mu}(t). \tag{8.24}$$

The t-independent Stokes matrices are then defined by the relations

$$Y_{\nu+(k+1)\mu}(t, z) = Y_{\nu+k\mu}(t, z)\mathbb{S}_{\nu+k\mu}.$$

[4]The fact that $\widehat{A}_1$ may have eigenvalues differing by integers does not constitute a problem; see the proof of Theorem 8.1.

Notice that also the coefficients $F_j(t)$ are analytically continued as meromorphic multivalued matrix functions. For the sake of the proof of the Lemma 8.4 below, the analytic continuation of $Y_{\nu+k\mu}(t,z)$ will be denoted by

$$\mathbb{Y}_{\nu+k\mu}(z,\tilde{t}),$$

where $\tilde{t}$ is a point of the universal covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$, whose projection is t. The analytic continuation of $F_j(t)$ will be simply denoted by $F_j(\tilde{t})$

By arguments similar to those in Section 7.5, it is seen that as t varies in c or slightly beyond the boundary ∂c, then $Y_{\nu+k\mu}(t,z)$ maintains its asymptotic behaviour, for t away from the Malgrange's divisor. But when t moves sufficiently far form c, then the asymptotic representation (8.24) is lost. The following Lemma gives the sufficient condition such that the asymptotics (8.24) is not lost by $\mathbb{Y}_{\nu+k\mu}(z,\tilde{t})$.

LEMMA 8.4. *Assume that the Stokes matrices satisfy the vanishing condition*

$$(\mathbb{S}_\nu)_{ab} = (\mathbb{S}_\nu)_{ba} = (\mathbb{S}_{\nu+\mu})_{ab} = (\mathbb{S}_{\nu+\mu})_{ba} = 0, \tag{8.25}$$

for any $1 \leq a \neq b \leq n$ such that $u_a(0) = u_b(0)$. Then the meromorphic continuation $\mathbb{Y}_{\nu+k\mu}(z,\tilde{t})$, $k \in \mathbb{Z}$, on the universal covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ maintains the asymptotic behaviour

$$\mathbb{Y}_{\nu+k\mu}(z,\tilde{t})e^{-\Lambda(t)z}z^{-B_1} \sim I + \sum_{j\geq 1} F_j(\tilde{t})\ z^{-j},$$

for $z \to \infty$ in $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$ and any $\tilde{t} \in \mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ away from the Malgrange's divisor. Moreover,

$$\mathbb{Y}_{\nu+(k+1)\mu}(z,\tilde{t}) = \mathbb{Y}_{\nu+k\mu}(z,\tilde{t})\mathbb{S}_{\nu+k\mu}.$$

Here $\widehat{\mathcal{S}}_{\nu+k\mu}(t)$ is the sector in Definition 7.6.

REMARK 8.8. Notice that $B_1 = \mathrm{diag}(\widehat{A}_1(t))$ is independent of $t \in \mathcal{V}$ by assumption, and $\widehat{A}_1(t)$ is meromorphic, so B_1 is constant everywhere. Moreover, the relation $\mathbb{S}_{\nu+2\mu} = e^{-2\pi i B_1}\mathbb{S}_\nu e^{2\pi i B_1}$ implies that (8.25) holds for any $\mathbb{S}_{\nu+k\mu}$, $k \in \mathbb{Z}$.

Proof: Since $\overline{\mathcal{V}}$ belongs to the $\widetilde{\tau}$-cell c, then $Y_{\nu+k\mu}(z,t)$ can be denoted by $Y_{\nu+k\mu}(z,t;\widetilde{\tau},c)$, as in Theorem 7.2, for $t \in \mathcal{V}$ and for any $t \in c$ away from the Malgrange's divisor. Noticing that the Malgrange's divisor does not disconnect $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$, we proceed exactly as in the proof of Theorem 7.2. Now $\mathcal{V}$ is considered as lying on a sheet of the covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$. The relation (7.27) holds unchanged, and reads

$$Y_{\nu+\mu}(z,\tilde{t};\widetilde{\tau},c') = \mathbb{Y}_{\nu+\mu}(z,\tilde{t},\widetilde{\tau},c)\mathbb{K}^{[ab]}. \tag{8.26}$$

On the other hand, the relation (7.28) becomes

$$\mathbb{X}_{\nu+\mu}(z,\tilde{t}) = Y_{\nu+\mu}(z,\tilde{t};\widetilde{\tau},c')\ \widetilde{\mathbb{K}}^{[ab]}(t),$$

where $\mathbb{X}_{\nu+\mu}(z,\tilde{t})$ is a solution of the system (8.22) with coefficient $\widehat{A}_1(\tilde{t})$, where $\tilde{t}$ is a point of the universal covering, reached along γ_{ab} after $R_{ab}(t)$ has crossed $R(\widetilde{\tau}-\pi)$ in Figure 7.32. $\mathbb{X}_{\nu+\mu}(z,\tilde{t})$ is the unique fundamental matrix solution having asymptotic behaviour

$$\mathbb{X}_{\nu+\mu}(z,\tilde{t})e^{-\Lambda(t)z}z^{-B_1} \sim I + \sum_{j\geq 1} F_j(\tilde{t})\ z^{-j},$$

in $\mathcal{S}_{\nu+\mu}(t)$. Then (7.29) is replaced by

$$\mathbb{X}_{\nu+\mu}(z,\gamma_{ab}t) = \mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t)\ \mathbb{K}^{[ab]}\ \widetilde{\mathbb{K}}^{[ab]}, \qquad t \in c.$$

Here, $\mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t)$ is the continuation of $Y_{\nu+\mu}(z,t) \equiv Y_{\nu+\mu}(z,t;\widetilde{\tau},c)$ at

$$\tilde{t} = \gamma_{ab}t.$$

The proof that $\mathbb{K}^{[ab]}=\widetilde{\mathbb{K}}^{[ab]}=I$ holds unchanged, following from (8.25). Therefore,

$$\mathbb{X}_{\nu+\mu}(z,\gamma_{ab}t)=\mathbb{Y}_{\nu+\mu}(z,\gamma_{ab}t).$$

This proves that the analytic continuation $\mathbb{Y}_{\nu+\mu}(z,\tilde{t})$ along γ_{ab} maintains the canonical asymptotic behaviour. Moreover, the ray R_{ab} plays no role in the asymptotics, as it follows from (8.26) with $\mathbb{K}^{[ab]}=I$. Repeating the construction for all possible loops γ_{ab}, as in the proof of Theorem 7.1 and Theorem 7.2, we conclude that $\mathbb{Y}_{\nu+\mu}(z,\tilde{t})$ maintains its the canonical asymptotic representation for any $\tilde{t}$ in the universal covering ($\tilde{t}$ away from the Malgrange divisor), when $z\to\infty$ in $\widehat{S}_{\nu+\mu}(t)$. □

In Lemma 8.4, we have taken into account the fact that Δ is expected to be a branching locus, so that $\mathbb{Y}(z,\tilde{t})$ is defined on $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$, as the result of [**Miw81**] predicts. In fact, it turns out that (8.25) implies that there is no branching at Δ, as the following lemma states.

LEMMA 8.5. *If (8.25) holds, then:*

- *The meromorphic continuation on the universal covering $\mathcal{R}(\mathcal{U}_{\epsilon_0}(0)\backslash\Delta)$ of any $Y_{\nu+k\mu}(z,t)$, $k\in\mathbb{Z}$, and $Y^{(0)}(z,t)$ is single-valued on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.*
- *The meromorphic continuation of $\widehat{A}_1(t)$ is single-valued on $\mathcal{U}_{\epsilon_0}(0)\backslash\Delta$.*

In other words, Δ is not a branching locus.

The single-valued continuation of $Y_{\nu+k\mu}(z,t)$ will be simply denoted by $Y_{\nu+k\mu}(z,t)$ in the remaining part of this section, so we will no longer need the notation $\mathbb{Y}_{\nu+k\mu}(z,\tilde{t})$.

Proof of Lemma 8.5: Let $t\in\mathcal{V}$ be an admissible isomonodromic deformation and $\widehat{A}_1(t)$ be holomorphic in $\mathcal{V}$. Let $\widetilde{\tau}$ be the direction of an admissible ray for $\Lambda(0)$ such that $\mathcal{V}$ lies in a $\widetilde{\tau}$-cell. Since the linear relation (1.25)

$$u_i(t)=u_i(0)+t_i,\qquad 1\leq i\leq n,$$

holds, we will use u as variable in place of t. Accordingly, we will write $\Lambda(u)$ instead of $\Lambda(t)$ and $Y(z,u)$ instead of $Y(z,t)$. Now, the fundamental solutions $Y_{\nu+k\mu}(z,u)$ and $Y^{(0)}(z,u)$ are holomorphic functions of $u\in\mathcal{V}$. We construct a Riemann-Hilbert boundary value problem (abbreviated by R-H) satisfied by[5] $Y_{\nu-\mu}(z,u)$, $Y_{\nu}(z,u)$, $Y_{\nu+\mu}(z,u)$ and $Y^{(0)}(z,u)$.

The given data are the essential monodromy data (see Definition 8.2) $\mathbb{S}_{\nu-\mu}$, $\mathbb{S}_{\nu}$, B_1, $\mu_1,...,\mu_n$, $R^{(0)}$ and $C_\nu^{(0)}$. Instead of $\mu_1,...,\mu_n$, $R^{(0)}$, we can use $D^{(0)}$ and $L^{(0)}$ (see (8.2) and Remark 8.2). They satisfy a constraint, because the monodromy $(C_\nu^{(0)})^{-1}\ e^{2\pi i L^{(0)}}\ C_\nu^{(0)}$ at $z=0$ can be expressed in the equivalent way $e^{2\pi i B_1}(\mathbb{S}_\nu\mathbb{S}_{\nu+\mu})^{-1}$. Recalling that $\mathbb{S}_{\nu+\mu}=e^{-2\pi i B_1}\mathbb{S}_{\nu-\mu}e^{2\pi i B_1}$, the constraint is

$$\mathbb{S}_{\nu-\mu}^{-1}\ e^{2\pi i B_1}\ \mathbb{S}_{\nu}^{-1}=\ (C_\nu^{(0)})^{-1}\ e^{2\pi i L^{(0)}}\ C_\nu^{(0)}. \tag{8.27}$$

The following relations hold for fundamental solutions:

$$Y_\nu(z,u)=Y_{\nu-\mu}(z,u)\mathbb{S}_{\nu-\mu}, \tag{8.28}$$
$$Y_{\nu+\mu}(z,u)=Y_\nu(z,u)\mathbb{S}_\nu, \tag{8.29}$$
$$Y_\nu(z,u)=Y^{(0)}(z,u)C_\nu^{(0)}, \tag{8.30}$$
$$Y_{\nu+\mu}(z,u)=Y^{(0)}(z,u)C_\nu^{(0)}\mathbb{S}_\nu. \tag{8.31}$$

Since $Y_{\nu+\mu}(ze^{2\pi i})=Y_{\nu-\mu}(z)e^{2\pi i B_1}$, we can rewrite (8.28) as

$$Y_\nu(z,u)=Y_{\nu+\mu}(ze^{2\pi i},u)e^{-2\pi i B_1}\mathbb{S}_{\nu-\mu} \tag{8.32}$$

[5] Recall that $Y_{\nu+2k\mu}(ze^{2k\pi i})=Y_\nu(z)e^{2k\pi i B_1}$, $k\in\mathbb{Z}$.

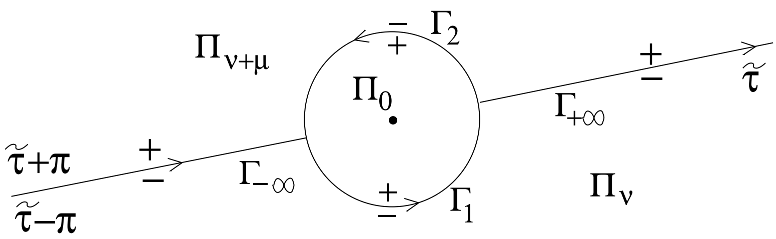


FIGURE 8.1. The contour $\Gamma_{-\infty}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_{+\infty}$ of the Riemann-Hilbert problem, which divides the plane in regions Π_ν, $\Pi_{\nu+\mu}$ and Π_0. The directional angles $\widetilde{\tau}$, $\widetilde{\tau}\pm\pi$ and the orientations are depicted.

We now write

$$Y_{\nu+k\mu}(z,u)=\mathcal{G}_{\nu+k\mu}(z,u)e^{Q(z,u)}, \qquad Q(z,u):=\Lambda(u)z+B_1\ln z,$$
$$\mathcal{G}_{\nu+k\mu}(z,u)\sim I+\sum_{j=1}^{\infty}F_j(u)z^{-j}, \qquad z\to\infty \text{ in } \mathcal{S}_{\nu+k\mu}(u), \qquad k=0,1.$$

$$Y^{(0)}(z,u)=\mathcal{G}_0(z,u)\ z^{D^{(0)}}z^{L^{(0)}}$$
$$\mathcal{G}^{(0}(z,u)=G^{(0)}(u)+O(z) \qquad \text{holomorphic at } z=0.$$

Therefore, from (8.28)-(8.32) we obtain

$$\mathcal{G}_\nu(z,u)=\mathcal{G}_{\nu+\mu}(ze^{2\pi i},u)\ e^{Q(z,u)}\mathbb{S}_{\nu-\mu}e^{-Q(z,u)}, \tag{8.33}$$
$$\mathcal{G}_{\nu+\mu}(z)=\mathcal{G}_\nu(z,u)\ e^{Q(z,u)}\mathbb{S}_\nu e^{-Q(z,u)}, \tag{8.34}$$
$$\mathcal{G}_\nu(z,u)=\mathcal{G}^{(0)}(z,u)\ z^{D^{(0)}}z^{L^{(0)}}C_\nu^{(0)}e^{-Q(z,u)}, \tag{8.35}$$
$$\mathcal{G}_{\nu+\mu}(z,u)=\mathcal{G}^{(0)}(z,u)\ z^{D^{(0)}}z^{L^{(0)}}C_\nu^{(0)}\mathbb{S}_\nu e^{-Q(z,u)}. \tag{8.36}$$

We formulate the following R-H, given the monodromy data. Consider the z-plane with the following branch cut from 0 to ∞:

$$\widetilde{\tau}-\pi<\arg z<\widetilde{\tau}+\pi.$$

Consider a circle around $z=0$ of some radius r. The oriented contour $\Gamma=\Gamma(\widetilde{\tau})$ of the R-H is the union of the following paths (see Figure 8.1):

$\Gamma_{-\infty}:\quad \arg z=\widetilde{\tau}\pm\pi,\quad |z|>r,\qquad$ half-line coming from ∞ along the branch-cut

$\Gamma_{+\infty}:\quad \arg z=\widetilde{\tau},\quad |z|>r,\qquad$ half-line going to ∞ in direction $\widetilde{\tau}$,

$\Gamma_1:\quad \widetilde{\tau}-\pi<\arg z\leq\widetilde{\tau},\quad |z|=r,\qquad$ half-circle in anti-clockwise sense,

$\Gamma_2:\quad \widetilde{\tau}\leq\arg z<\widetilde{\tau}+\pi,\quad |z|=r,\qquad$ half-circle in anti-clockwise sense.

Recalling that $\tau_\nu<\widetilde{\tau}<\tau_{\nu+\mu}$, we call:

Π_ν the unbounded domain to the right of $\Gamma_{-\infty}\cup\Gamma_1\cup\Gamma_{+\infty}$,

Π_0 the ball inside the circle $\Gamma_1\cup\Gamma_2$,

$\Pi_{\nu+\mu}$ the remaining unbounded region $\mathbb{C}\backslash\{\Pi_\nu\cup\Pi_0\cup\Gamma\}$.

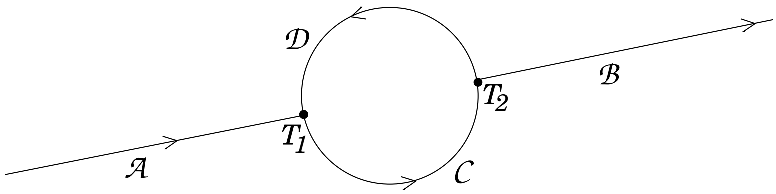


FIGURE 8.2. Jump matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ along Γ, used in step 1.

The R-H problem we need is as follows:

$$\mathcal{G}_+(\zeta) = \mathcal{G}_-(\zeta)H(\zeta,u), \qquad \zeta \in \Gamma, \tag{8.37}$$

where the jump $H(\zeta,u)$ is uniquely specified by assigning the monodromy data $\mathbb{S}_{\nu-\mu}$, $\mathbb{S}_\nu$, B_1, $C_\nu^{(0)}$, $D^{(0)}$ and $L^{(0)}$ (i.e. $\mu_1,\ldots,\mu_n$, $R^{(0)}$). Since Γ_- lies along the branch-cut, we use the symbol $\zeta_\pm$ if $\arg\zeta = \widetilde{\tau} \pm \pi$. Hence, $H(\zeta,u)$ is

$$\begin{aligned} H(\zeta,u) := \quad & e^{Q(\zeta_-,u)}\mathbb{S}_{\nu-\mu}^{-1}e^{-Q(\zeta_-,u)} \qquad \text{along } \Gamma_{-\infty},\\ & e^{Q(\zeta,u)}\mathbb{S}_\nu e^{-Q(\zeta,u)} \qquad \text{along } \Gamma_{+\infty},\\ & e^{Q(\zeta,u)}(C_\nu^{(0)})^{-1}\zeta^{-L^{(0)}}\zeta^{-D^{(0}} \qquad \text{along } \Gamma_1,\\ & e^{Q(\zeta,u)}\mathbb{S}_\nu^{-1}(C_\nu^{(0)})^{-1}\zeta^{-L^{(0)}}\zeta^{-D^{(0}} \qquad \text{along } \Gamma_2. \end{aligned}$$

We require that the solution satisfies the conditions

$$\mathcal{G}(z) \sim I + \text{series in } z^{-1}, \quad z\to\infty, \quad z\in\Pi_\nu\cup\Pi_{\nu+\mu}, \tag{8.38}$$

$$\mathcal{G}(z) \text{ holomorphic in } \Pi_0 \text{ and } \det(\mathcal{G}(0)) \neq 0. \tag{8.39}$$

By (8.33)-(8.36), our R-H has the following solution for $u \in \mathcal{V}$:

$$\mathcal{G}(z,u) = \begin{cases} \mathcal{G}_0(z,u) & \text{for } z\in\Pi_0,\\ \mathcal{G}_\nu(z,u) & \text{for } z\in\Pi_\nu,\\ \mathcal{G}_{\nu+\mu}(z,u) & \text{for } z\in\Pi_{\nu+\mu},\end{cases} \qquad \text{holomorphic of } u\in\mathcal{V}. \tag{8.40}$$

By the result of [**Miw81**], this solution can be analytically continued in u as a meromorphic function on the universal covering of $\mathbb{C}^n\backslash\Delta_{\mathbb{C}_n}$. Consider a loop around Δ, as in (7.14), involving two coalescing coordinates u_a, u_b, starting from a point in $\mathcal{V}$. We want to prove that the above continuation is single valued along this loop. As in the proof of Theorem 7.1, we just need to consider the case when $|u_a - u_b|$ is small and only PR_{ab} and PR_{ba} cross $l(\widetilde{\tau})$. Let

$$\varepsilon := u_a - u_b.$$

The lemma will be proved if we prove that $\mathcal{G}$ in (8.40) is holomorphic in a neighbourhood of $\varepsilon = 0$, except at most for a finite number of poles (the Malgrange's divisor).

In the following, we will drop u and only write the dependence on ε. For example, we write $H(\zeta,\varepsilon)$ instead of $H(\zeta,u)$. For our convenience, as in Figure 8.2 we call

$$\begin{aligned} H(\zeta,\varepsilon) \quad & =: \mathcal{A}(\zeta_-,\varepsilon) \qquad \text{along } \Gamma_{-\infty},\\ & =: \mathcal{B}(\zeta,\varepsilon) \qquad \text{along } \Gamma_{+\infty},\\ & =: \mathcal{C}(\zeta,\varepsilon) \qquad \text{along } \Gamma_1,\\ & =: \mathcal{D}(\zeta,\varepsilon) \qquad \text{along } \Gamma_2. \end{aligned}$$

$\mathcal{A}, ..., \mathcal{D}$ are *holomorphic* functions of ε. The following cyclic relations are easily verified:

$$\mathcal{A}(ze^{-2\pi i},\varepsilon)\ \mathcal{D}(z,\varepsilon)\ \mathcal{C}(ze^{-2\pi i},\varepsilon)^{-1}\ = I \qquad \mathcal{C}(z,\varepsilon)\ \mathcal{D}(z,\varepsilon)^{-1}\mathcal{B}(z,\varepsilon)^{-1}\ = I. \tag{8.41}$$

In particular, the following "smoothness condition" holds at the points T_1 and T_2 of intersection of $\Gamma_{-\infty}$ and $\Gamma_{+\infty}$ with the circle $|z| = r$ respectively:

$$\mathcal{A}(\zeta_-,\varepsilon)\ \mathcal{D}(\zeta_+,\varepsilon)\ \mathcal{C}(\zeta_-,\varepsilon)^{-1}\ = I \quad \text{ at } T_1, \qquad \mathcal{C}(\zeta,\varepsilon)\ \mathcal{D}(\zeta,\varepsilon)^{-1}\mathcal{B}(\zeta,\varepsilon)^{-1} = I \quad \text{ at } T_2.$$

Indeed,

$$\begin{aligned}
&\mathcal{A}(ze^{-2\pi i},\varepsilon)\mathcal{D}(z,\varepsilon)\mathcal{C}(ze^{-2\pi i},\varepsilon)^{-1} = \\
&= e^{Q(ze^{-2\pi i})}\mathbb{S}_{\nu-\mu}^{-1}e^{-Q(ze^{-2\pi i})}\cdot e^{Q(z)}\mathbb{S}_{\nu}^{-1}(C_\nu^{(0)})^{-1}z^{-L^{(0)}}z^{-D^{(0)}}\cdot(ze^{-2\pi i})^{D^{(0)}}(ze^{-2\pi i})^{L^{(0)}}C_\nu^{(0)}e^{-Q(ze^{-2\pi i})}\\
&= e^{-2\pi i B_1}e^{Q(z)}\ \mathbb{S}_{\nu-\mu}^{-1}\ e^{2\pi i B_1}\ \mathbb{S}_{\nu}^{-1}\ (C_\nu^{(0)})^{-1}z^{-L^{(0)}}z^{-D^{(0)}}\cdot z^{D^{(0)}}z^{L^{(0)}}e^{-2\pi i L^{(0)}}C_\nu^{(0)}\ e^{-Q(z)}\ e^{2\pi i B_1}\\
&= e^{-2\pi i B_1}e^{Q(z)}\ \left(\mathbb{S}_{\nu-\mu}^{-1}\ e^{2\pi i B_1}\ \mathbb{S}_{\nu}^{-1}\ (C_\nu^{(0)})^{-1}e^{-2\pi i L^{(0)}}C_\nu^{(0)}\right)\ e^{-Q(z)}\ e^{2\pi i B_1} = I.
\end{aligned}$$

In the last step, we have used (8.27). Moreover,

$$\mathcal{C}(\zeta,\varepsilon)\mathcal{D}(z,\varepsilon)^{-1}\mathcal{B}(z,\varepsilon)^{-1} = e^{Q(z)}(C_\nu^{(0)})^{-1}z^{-L^{(0)}}z^{-D^{(0)}}\cdot z^{D^{(0)}}z^{L^{(0)}}C_\nu^{(0)}\mathbb{S}_\nu e^{-Q(z)}\cdot e^{Q(z)}\mathbb{S}_\nu^{-1}e^{-Q(z)} = I.$$

The last result follows from simple cancellations.

In order to complete the proof, we need the theoretical background, in particular the L^p formulation of Riemann-Hilbert problems, found in the test-book [**FIKN06**], the lecture notes [**Its11**] and the papers [**Zho89**] [**DZ02**] (see also [**Dei99**] [**DKM+99b**] [**DKM+99a**] and [**CG81**] [**Pog66**] [**Vek67**]). The proof is completed in the following steps, suggested to us by Marco Bertola.

• **Step 1.** We contruct a naive solution $\mathfrak{S}(z,\varepsilon)$ to the R-H, which does not satisfy the asymptotic condition (8.38). We start by defining $\mathfrak{S}(z,\varepsilon) = I$ in Π_0. Then, keeping into account the jumps $\mathcal{C}$ and $\mathcal{B}$ along Γ_1 and $\Gamma_{+\infty}$ respecively (see Figure 8.2), we have

$$\mathfrak{S}(z,\varepsilon) = \begin{cases} I & \text{for } z\in\Pi_0,\\[2ex] \mathcal{C}(z,\varepsilon)^{-1} & \text{for } z\in\Pi_\nu,\\[2ex] \mathcal{C}(z,\varepsilon)^{-1}\mathcal{B}(z,\varepsilon) & \text{for } z\in\Pi_{\nu+\mu},\end{cases} \tag{8.42}$$

On the other hand, starting with $\mathfrak{S}(z,\varepsilon) = I$ in Π_0 and keeping into account the jump $\mathcal{D}$ at Γ_2, we must have

$$\mathfrak{S}(z,\varepsilon) = \mathcal{D}(z,\varepsilon)^{-1} \qquad \text{for } z\in\Pi_{\nu+\mu}. \tag{8.43}$$

The second relation in (8.41) ensures that (8.43) and the last expression in (8.42) coincide. Moreover, starting with $\mathfrak{S}(z,\varepsilon) = I$ in Π_0 and crossing Γ_1 and then $\Gamma_{-\infty}$ with jumps $\mathcal{C}$ and $\mathcal{A}$, we find a third representation of $\mathfrak{S}(z,\varepsilon)$ for $z\in\Pi_{\nu+\mu}$, namely

$$\mathfrak{S}(z,\varepsilon) = \mathcal{C}(ze^{-2\pi i},\varepsilon)^{-1}\mathcal{A}(ze^{-2\pi i},\varepsilon) \qquad \text{for } z\in\Pi_{\nu+\mu}. \tag{8.44}$$

Now, the first relation in (8.41) ensures that (8.43) and (8.44) coincide.

• **Step 2.** We consider an auxiliary R-H as in Figure 8.3, whose boundary contour is the union of a half line $\ell_{\mathcal{A}}$ contained in $\Gamma_{-\infty}$ from ∞ to a point P_1 preceding T_1, and a half line $\ell_{\mathcal{B}}$ contained in $\Gamma_{+\infty}$ from a point P_2 following T_2 to ∞. The jump along these half lines is $H(\zeta,\varepsilon)$ (namely, $\mathcal{A}(\zeta_-,\varepsilon)$ and $\mathcal{B}(\zeta,\varepsilon)$ on the two half lines respectively). The R-H is then

$$\begin{aligned}\Psi_+(\zeta) = \Psi_-(\zeta)H(\zeta,\varepsilon) \qquad & \zeta\in\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}},\\ \Psi(z)\sim I+\text{series in } z^{-1}, \quad z\to\infty, \quad & z\in\Pi_\nu\cup\Pi_{\nu+\mu}.\end{aligned} \tag{8.45}$$

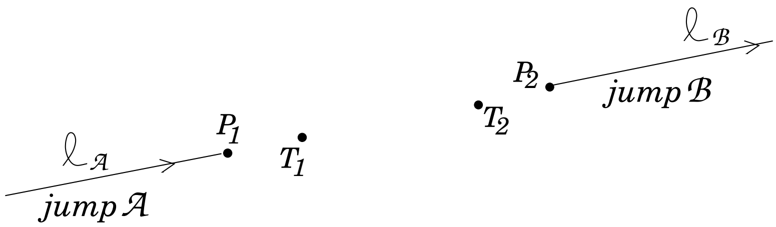


FIGURE 8.3. Step 2: the auxiliary Riemann-Hilbert problem with contour $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$.

Keeping the above asymptotics into account, the R-H is rewritten as follows:

$$\Psi(z) = I + \int_{\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} \frac{\Psi_-(\zeta)(H(\zeta,\varepsilon)-I)}{\zeta - z}\frac{d\zeta}{2\pi i}.$$

or, letting $\delta\Psi := \Psi - I$ and $\delta H := H - I$,

$$\delta\Psi(z) = \int_{\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} \frac{\delta\Psi_-(\zeta)\ \delta H(\zeta,\varepsilon)}{\zeta - z}\frac{d\zeta}{2\pi i} + \int_{\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} \frac{\delta H(\zeta,\varepsilon)}{\zeta - z}\frac{d\zeta}{2\pi i}. \tag{8.46}$$

We solve the problem by computing $\delta\Psi_-(\zeta)$, as the solution of the following integral equation (by taking the limit for $z \to z_-$ belonging to the "$-$" side of $\ell_{\mathcal{A}} \cup \ell_{\mathcal{B}}$):

$$\begin{aligned}\delta\Psi_-(z_-) &= \int_{\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} \frac{\delta\Psi_-(\zeta)\ \delta H(\zeta,\varepsilon)}{\zeta - z_-}\frac{d\zeta}{2\pi i} + \int_{\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} \frac{\delta H(\zeta,\varepsilon)}{\zeta - z_-}\frac{d\zeta}{2\pi i} \\ &= C_-\Big[\delta\Psi_-\delta H(\cdot,\varepsilon))\Big](z_-) + C_-\Big[\delta H(\cdot,\varepsilon)\Big](z_-).\end{aligned}$$

Here C_- stands for the Cauchy boundary operator. We will write $C_-[\delta\Psi_-\delta H(\cdot,\varepsilon)]$ as

$$C_-[\bullet\ \delta H(\cdot,\varepsilon)]\delta\Psi_-,$$

to represent the operator $C_-[\bullet\ \delta H(\cdot,\varepsilon)]$ acting on $\delta\Psi_-$. We observe the following facts:

1. If u is in the cell containing $\mathcal{V}$, as $\zeta \to \infty$ along $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$, the off-diagonal matrix entries of the jump are exponentially small. Indeed
$$H_{ij}(z,\varepsilon) \equiv H_{ij}(\zeta,u) = s_{ij}\exp\Big\{(u_i-u_j)\zeta + ((B_1)_{ii} - (B_1)_{jj})\ln\zeta\Big\} \longrightarrow \delta_{ij}. \tag{8.47}$$
This is due to the fact that s_{ij} is either $(\mathcal{S}_\nu)_{ij}$ or $(\mathcal{S}_{\nu-\mu}^{-1})_{ij}$. Thus, $\delta H_{ij} \in L^2(\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}, |d\zeta|)$, and $C_-[\delta H]_{ij} \in L^2(\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}, |d\zeta|)$. Hence, the problem is well posed in L^2, consisting in finding $\delta\Psi_-$ as the solution of
$$\Big(I - C_-\Big[\bullet\ \delta H(\cdot,\varepsilon)\Big]\Big)\,\delta\Psi_- = C_-\Big[\delta H(\cdot,\varepsilon)\Big]. \tag{8.48}$$
2. If u is in the cell containing $\mathcal{V}$, by assumption both the operator and the given term in (8.48) depend holomorphically on u. Along the loops $(u_i - u_j) \mapsto (u_i-u_j)e^{2\pi i}$, $1 \le i \ne j \le n$, the property (8.47) is lost, because u leaves the $\widetilde{\tau}$-cell containing $\mathcal{V}$, so that some Stokes rays cross the ray $R(\widetilde{\tau})$. On the other hand, if the vanishing condition (8.25) holds, then $s_{ab} = s_{ba} = 0$.[6] Thus, (8.47) continues to hold along the loop $\varepsilon \mapsto \varepsilon e^{2\pi i}$. It follows that $I - C_-[\bullet\ \delta H(\cdot,\varepsilon)]$ is an analytic operator in ε and the term $C_-[\delta H(\cdot,\varepsilon)]$ is also analytic, for ε belonging to a sufficiently small closed ball U centred at $\varepsilon = 0$.

[6] No difficulty arises from the fact that $\mathbb{S}_{\nu-\mu}^{-1}$ appears. If for simplicity we take the labelling (5.50)-(5.53), then $\mathbb{S}_{\nu-\mu}$ has diagonal blocks equal to $p_j \times p_j$ identity matrices. This structure persists on taking the inverse.

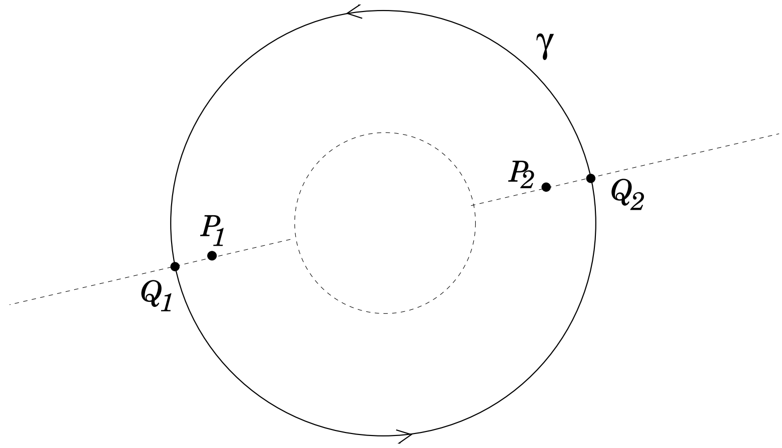


FIGURE 8.4. Step 3: the continuous Riemann-Hilbert problem on the circle γ, with jump $\Psi(\zeta,\varepsilon)\mathfrak{S}(\zeta,\varepsilon)^{-1}$.

3. If P_1 and P_2 are sufficiently far away from the origin, we can take

$$\|\delta H(\cdot,\epsilon)\|_\infty = \sup_{\zeta\in\ell_{\mathcal{A}}\cup\ell_{\mathcal{B}}} |H(\zeta,\epsilon)|$$

so small that the operator norm $\|\cdot\|$ in L^2 satisfies, for $\varepsilon\in U$,

$$\big\|C_-\big[\bullet\ \delta H(\cdot,\varepsilon)\big]\big\| \le \|C_-\|\ \|\delta H(\cdot,\epsilon)\|_\infty < 1. \tag{8.49}$$

Here, $\|C_-\|$ is the operator norm of the Cauchy operator.[7] By (8.49), the inverse exists:

$$\Big(I - C_-\Big[\bullet\ \delta H(\cdot,\varepsilon)\Big]\Big)^{-1} = \sum_{k=1}^{+\infty}\big(C_-\big[\bullet\ \delta H(\cdot,\varepsilon)\big]\big)^k. \tag{8.50}$$

The series in the r.h.s. converges in operator norm and defines an analytic operator in $\varepsilon\in U$. Using (8.50), we find the unique L^2-solution of (8.48) and then, substituting into (8.46), we find the ordinary solution $\Psi(z,\varepsilon)$ of the auxiliary problem, which is holomoprhic in $\varepsilon\in U$.

• **Step 3:** We construct a R-H along a closed contour with a *continuous* jump. Consider a "big" counter-clockwise oriented circle γ centered at the origin and intersecting $\Gamma_{-\infty}$ at a point Q_1 preceding P_1, $\Gamma_{+\infty}$ at a point Q_2 following P_2. See Figure 8.4. If $\mathcal{G}$ is the solution to the starting problem (8.37), (8.38), (8.39), we construct a matrix-valued function Φ as follows:

$$\Phi := \quad \mathcal{G}\cdot\Psi(z,\varepsilon)^{-1}, \quad \text{for } z \text{ outside } \gamma, \tag{8.51}$$
$$\mathcal{G}\cdot\mathfrak{S}(z,\varepsilon)^{-1}, \quad \text{for } z \text{ inside } \gamma. \tag{8.52}$$

By constriction, Φ only has jumps along γ:

$$\Phi_+(\zeta) = \Phi_-(\zeta)\ \widetilde{H}(\zeta,\varepsilon), \qquad \widetilde{H}(\zeta,\varepsilon) := \Psi(\zeta,\varepsilon)\mathfrak{S}(\zeta,\varepsilon)^{-1}. \tag{8.53}$$

By construction, the jump matrix $\widetilde{H}(\zeta,\varepsilon)$ is continuous in ζ along γ, and is analytic in $\varepsilon\in U$. By (8.45), then (8.38) is equivalent to

$$\Phi(z)\sim I + \text{series in } z^{-1}, \qquad z\to\infty, \qquad z\in\Pi_\nu\cup\Pi_{\nu+\mu}.$$

[7]Here we use the simple estimate $\|C_-(f\delta H)\|_{L^2}\le\|C_-\|\ \|\delta H\|_\infty\ \|f\|_{L^2}$, for any $f\in L^2$.

Therefore, the R-H for Φ is solved as in (8.48) and (8.46) by

$$\Big(I - C_-\Big[\bullet\ \delta\widetilde{H}(\cdot,\varepsilon)\Big]\Big)\,\delta\Phi_- = C_-\Big[\delta\widetilde{H}(\cdot,\varepsilon)\Big], \tag{8.54}$$

$$\delta\Phi(z) = \int_\gamma \frac{\delta\Phi_-(\zeta)\ \delta\widetilde{H}(\zeta,\varepsilon)}{\zeta - z}\frac{d\zeta}{2\pi i} + \int_\gamma \frac{\delta\widetilde{H}(\zeta,\varepsilon)}{\zeta - z}\frac{d\zeta}{2\pi i}. \tag{8.55}$$

Here C_- is Cauchy operator along γ. Since γ is a closed contour and $\widetilde{H}(\zeta,\varepsilon)$ is continuous, the procedure and results of [Zho89] [DZ02] [DKM+99b] apply. The operator $C_-\Big[\bullet\ \delta\widetilde{H}(\cdot,\varepsilon)\Big]$ is Fredholm, $I - C_-[\bullet\ \delta\widetilde{H}(\cdot,\varepsilon)]$ has index 0 and its kernel is $\{0\}$. Therefore, the "analytic Fredholm alternative" of [Zho89] holds. Namely, either $I - C_-[\bullet\ \delta\widetilde{H}(\cdot,\varepsilon)]$ can be inverted (and (8.54) can be solved) for every $\varepsilon \in U$, except for a finite number of isolated values, or is invertible for no ε. In the first case, $\Big(I - C_-[\bullet\ \delta\widetilde{H}(\cdot,\varepsilon)]\Big)^{-1}$ is meromorphic, with poles at the isolated points in U.

By (8.51)-(8.52), solvability of the R-H (8.53) is equivalent to the existence of the solution $\mathcal{G}(z,\varepsilon) \equiv \mathcal{G}(z,u)$ for the problem (8.37), (8.38), (8.39). By assumption (i.e. by the result of [Miw81]) we know that locally in u the solution $\mathcal{G}(z,u)$ exists. We therefore conclude that the "Fredholm analytic alternative" implies the existence of the solution $\Phi_-(\zeta,\varepsilon)$ of (8.54) for every $\varepsilon \in U$, except for a finite number of poles, and that (8.55) gives an ordinary solution $\Phi(z,\varepsilon)$, meromoprhic as a function of ε in U. By (8.51)-(8.52), the same conclusion holds for $\mathcal{G}(z,\varepsilon) \equiv \mathcal{G}(z,u)$. This proves the Lemma (as for $\widehat{A}_1$, it suffices to note that $\widehat{A}_1(t) = z(Y^{-1}(z,t)dY(z,t)/dz - \Lambda(t))$). $\square$

Theorem 1.7 immediately follows from Lemma 8.4 and Lemma 8.5.

8.6. Comparison with results in literature

We compare our results with the existing literature on isomonodromic deformations. The case when Δ is empty and $\widehat{A}_1(t)$ is any matrix does not add additional difficulties to the theory developed in [JMU81]. Indeed, in the definition of isomonoromic deformations given above, not only we require that the monodromy matrix at $z = 0$ is independent of t, but also the *monodromy exponents* $J^{(0)}$, $R^{(0)}$ and the connection matrix $C^{(0)}$ in (1.23) are constant (this is an *isoprincipal deformation*, in the language of [KV06]). Given these conditions on the exponents, and assuming that $\Delta = \emptyset$, one can essentially repeat the proofs given in [JMU81]. For example, the case when Δ is empty and $\widehat{A}_1(t)$ is skew-symmetric and diagonalisable has been studied in [Dub96], [Dub99b]. We also recall that in case of Fuchsian singularities only, isomonodromic deformations were completely studied[8] in [Bol98] and [KV06].

Isomonodromy deformations at irregular singular points with leading matrix admitting a Jordan form *independent of* t were studied in [BM05] (with some minor *Lidskii* generic conditions). For example, if the singularity is at $z = \infty$ as in (1.16), the results of [BM05] apply to $\widehat{A}(z,t) = z^{k-1}(J + \sum_{j=1}^{\infty}\widehat{A}_j(t)z^{-j})$, with Jordan form J and Poincaré rank $k \geq 1$. Although the eigenvalues of J have in general algebraic multiplicity greater than 1, J is "rigid", namely $u_1, \dots, u_n$ do not depend on t.

Other investigations of isomonodromy deformations at irregular singularities can be found in [Fed90] and [Bib12]. Nevertheless, these results do not apply to our coalescence problem. For example, the third admissibility conditions of definition 10 of [Bib12] is not satisfied in our case. In

[8] In [Bol98] it is only assumed that the monodromy matrices are constant. This generates non-Schlesinger deformations. On the other hand, an isopricipal deformation always leads to Schlesinger deformations [KV06].

[**Fed90**] the system with $A(z,t)=z^{r-1}B(z,t)$, $r\in\mathbb{Q}$, is considered, such that $B(\infty,t)$ has distinct eigenvalues; $z=\infty$ satisfying this condition is called a simple irregular singular point. This simplicity condition does not apply in our case.

The results of [**Kli13**], cited above, are applied in [**Kli16**] to the 3×3 isomonodromic description of the Painlevé 6 equation and its coalescence to Painlevé 5. In this case, the limiting system for $t\to 0$ has leading matrix with a 2×2 Jordan block, so that the fundamental matrices $Y_r(z,t)$ diverge.

Isomonodromic deformations of a system such as our (1.20) (with $z\mapsto 1/z$, $\widehat{A}_0\mapsto Z$, $\widehat{A}_1\mapsto f$) appears also in [**BTL13**]. Nevertheless, the deformations in Section 3 of [**BTL13**] are of a very particular kind. Indeed, the eigenvalues $u_1,...,u_n$ of the matrix Z in [**BTL13**], which is the analogue of our $\widehat{A}_0$, are deformation parameters, but always satisfy the condition

$$u_1=\cdots=u_{p_1}, \tag{8.56}$$

$$u_{p_1+1}=\cdots=u_{p_1+p_2}, \tag{8.57}$$

$$\cdots \tag{8.58}$$

$$u_{p_1+\cdots+p_{s-1}+1}=\cdots=u_{p_1+\cdots+p_s}, \tag{8.59}$$

with $p_1+\cdots+p_s=n$. Thus, no splitting of coalescences occurs, so that the deformations are always inside the same "stratum" of the coalescence locus. Moreover, the matrix $f=f(Z)$ in [**BTL13**], which is the analogue of our $\widehat{A}_1$, satisfies quite restrictively requirements that the diagonal is zero and $(\widehat{A}_1)_{ab}=0$ whenever $u_a=u_b$, $1\leq a\neq b\leq n$. These conditions are always satisfied along the deformation "stratum" of [**BTL13**]; they are a particular case of the more general conditions of Proposition 5.2 in Chapter 5. For these reasons, an adaptation of the classical Jimbo-Miwa-Ueno results [**JMU81**] (and those of [**Boa01**] for a connection on a G-bundle, with G a complex and reductive group) can be done *verbatim*, in order to describe the isomonodromicity condition for such a very particular kind of deformations. In the present Thesis, we studied *general* isomonodromic deformations of the system (1.20), not necessarily the simple decomposition of the spectrum as in (8.56)-(8.59).

Part 3

Local Moduli of Semisimple Frobenius Coalescent Structures

[...] les inventions d'inconnu
réclament des formes nouvelles.

A. Rimbaud, *Lettre du Voyant,*
à Paul Demeny, 15 mai 1871

CHAPTER 9

Application to Frobenius Manifolds

Abstract. In this Chapter we apply the main results of Part 2 to the case of the isomonodromic differential systems associated with semisimple Frobenius manifolds, in order to extend the description of their monodromy also at semisimple coalescence points. In particular, it is shown that also at semisimple coalescence points the monodormy data (which are there punctually defined as explained in Section 2.2.1, Chapter 6 and Section 8.4) are locally constant. Moreover it is deduced that both formal and asymptotically associated fundamental solutions of the system defining deformed flat coordinates are holomorphic at semisimple coalescence points.

9.1. Isomonodromy Theorem at coalescence points

So far, the monodromy data S and C of a semisimple Frobenius manifold M have been defined pointwise and then the deformation theory has been described at point (3) of Theorem 2.9 and in Theorem 2.12, away from coalescence points. In particular, S and C are constant in any ℓ-chamber, and the matrices $Y^{(k)}_{\text{left/right}}(z,u)$ are u-holomorphic in all ℓ-chambers. In this section we generalize the deformation theory to semisimple coalescence points. We show that monodromy data, which are well defined at a coalescence point, actually provide the monodromy data in a neighborhood of the point, and can be extended to the whole manifold through the action of the braid group. In this section we will use the following notation for objects computed at a coalescence point: a matrix Y, S or C will be denoted $\mathring{Y}$, $\mathring{S}$ or $\mathring{C}$.

Let $p_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$ be a semisimple coalescence point. Consider a neighbourhood $\Omega \subseteq M \setminus \mathcal{K}_M$ of p_0, satisfying the property of Remark 2.4. An ordering for canonical coordinates $(u_1, \dots, u_n)$ and a holomorphic branch of the function $\Psi\colon \Omega \to GL_n(\mathbb{C})$ can be chosen in Ω. We denote by $u(p) := (u_1(p), \dots, u_n(p))$ the value of the canonical coordinate map $u\colon \Omega \to \mathbb{C}^n$, and we define

$$\Delta_\Omega := \Big\{ u(p) = (u_1(p), ..., u_n(p)) \in \mathbb{C}^n \;\Big|\; p \in \Omega \cap \mathcal{B}_M \Big\}.$$

Therefore, if $u \in \Delta_\Omega$, then $u_i = u_j$ for some $i \neq j$. The coordinates $u(p_0)$ of p_0 will be denoted $u^{(0)} = (u_1^{(0)}, ..., u_n^{(0)})$. Δ_Ω is not empty and contains $u^{(0)}$. Let r_1, ..., r_s be the multiplicities of the eigenvalues of $U(u^{(0)}) = \operatorname{diag}(u_1^{(0)}, ..., u_n^{(0)})$, with $s < n$, $r_1 + \cdots + r_s = n$. By a permutation of $(u_1, ..., u_n)$, there is no loss in generality (cf. Section 2.3) if we assume that the entries of $u^{(0)}$ are

$$\begin{aligned} u_1^{(0)} = \cdots = u_{r_1}^{(0)} &=: \lambda_1 \\ u_{r_1+1}^{(0)} = \cdots = u_{r_1+r_2}^{(0)} &=: \lambda_2 \\ &\vdots \\ u_{r_1+\cdots+r_{s-1}+1}^{(0)} = \cdots = u_{r_1+\cdots+r_{s-1}+r_s-1}^{(0)} = u_n^{(0)} &=: \lambda_s. \end{aligned} \tag{9.1}$$

Let

$$\delta_i := \frac{1}{2}\min\left\{\left|\lambda_i - \lambda_j + \rho e^{i\left(\frac{\pi}{2}-\phi\right)}\right|,\ j \neq i,\ \rho \in \mathbb{R}\right\},$$

and let ϵ_0 be a small positive number such that

$$\epsilon_0 < \min_{1 \leq i \leq s} \delta_i. \tag{9.2}$$

We will assume that ϵ_0 is sufficiently small so that the polydisc at $u^{(0)}$, defined by[1]

$$\mathcal{U}_{\epsilon_0}(u^{(0)}) := \bigtimes_{i=1}^{s} \overline{B}(\lambda_i; \epsilon_0)^{\times r_i},$$

is completely contained in the image $u(\Omega)$ of the chart Ω. Note that, for ϵ_0 satisfying (9.2), if u varies in $\mathcal{U}_{\epsilon_0}(u^{(0)})$, the sets

$$I_1 := \{u_1,\ \dots, u_{r_1}\},\quad I_2 := \{u_{r_1+1},\ \dots, u_{r_1+r_2}\},\quad \dots,\quad I_s := \{u_{r_1+\dots+r_{s-1}+1},\ \dots, u_{r_1+\dots+r_{s-1}+r_s}\} \tag{9.3}$$

do never intersect. Thus, $u^{(0)}$ is a point of *maximal coalescence* in $\mathcal{U}_{\epsilon_0}(u^{(0)})$. We will say that a coordinate u_a *is close to a* λ_j if it belongs to I_j, which is to say that $u_a \in \overline{B}(\lambda_j; \epsilon_0)$.

Moreover, for ϵ_0 as in (9.2), the Stokes rays satisfy the following property. Let us fix $\phi \in \mathbb{R}$ so that the line $\ell = \ell(\phi)$ is admissible at p_0 (Definition 2.13). For $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$, consider the subset $\mathfrak{R}(u)$ of Stokes rays in the universal covering $\mathcal{R}$, which are associated to all couples of eigenvalues u_a and u_b, such that u_a is close to a λ_i and u_b is closed to a λ_j, with $i \neq j$. Then, the associated Stokes rays R_{ab} (projections of $R_{ab,k} \in \mathfrak{R}(u)$) continuously move, but they never cross ℓ as long as u_a varies in $\overline{B}(\lambda_i; \epsilon_0)$ and u_b in $\overline{B}(\lambda_j; \epsilon_0)$.

The choice of the line ℓ, admissible at p_0, induces a cell decomposition of $\mathcal{U}_{\epsilon_0}(u^{(0)})$, according to the following

Definition 9.1. Let ℓ be admissible at $u^{(0)}$. An ℓ-cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$ is any connected component of the open dense subset of points $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$ such that $u_1, \dots, u_n$ are pairwise distinct and ℓ is admissible a u.

According to Proposition 7.1, an ℓ-cell is a topological cell, namely it is homeomorphic to a ball. We notice that if $u(p)$ is in a ℓ-cell, then p lies in a ℓ-chamber. Thus, if $\mathcal{D}$ is an open subset whose closure is contained in a cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$, according to Theorems 2.9, point (3), the system

$$\frac{dY}{dz} = \left(U + \frac{V(u)}{z}\right) Y, \tag{9.4}$$

for $u \in \mathcal{D}$, admits two fundamental solutions $Y^{(0)}_{\text{right/left}}(z,u)$ uniquely determined by the canonical asymptotic representation $Y_{\text{formal}}(z,u)$ as in (2.28), valid in the sectors $\Pi_{\text{left/right}}(\phi)$ respectively. It follows from the proof of Theorem 2.8 that $Y_{\text{formal}}(z,u)$ is u-holomorphic in $\mathcal{U}_{\epsilon_0}(u^{(0)})\backslash\Delta_\Omega$. By Remark 2.7 actually the asymptotic representation is valid in wider sectors $\mathcal{S}_{\text{left/right}}(u)$, defined as the sectors which contain $\Pi_{\text{left/right}}(\phi)$ and extends up to the nearest Stokes rays. By Theorem 2.12 the system above with $u \in \mathcal{D}$ is isomonodromic, so that the Stokes matrices S, S_- defined in formulae (2.33),(2.34), with $S_- = S^T$, are constant.

[1] Here $\overline{B}(\lambda_i; \epsilon_0)$ is the closed ball in $\mathbb{C}$ with center λ_i and radius ϵ_0. Note that if the uniform norm $|u| = \max_i |u_i|$ is used, as in [CDG17b], then $\mathcal{U}_{\epsilon_0}(u^{(0)}) = \left\{u \in \mathbb{C}^n \;\middle|\; |u - u^{(0)}| \leq \epsilon_0\right\}$.

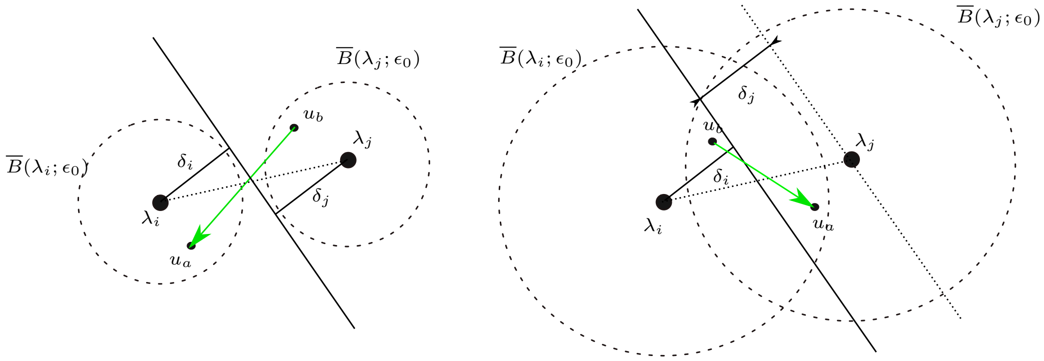


FIGURE 9.1. Points λ_i's and u_a's are represented in the same complex plane. The thick line has slope $\pi/2 - \phi$. As u varies, for values of ϵ_0 sufficiently small (left figure) the Stokes rays R_{ab} associated to u_a in the disk $\overline{B}(\lambda_i;\epsilon_0)$ and u_b in the disk of $\overline{B}(\lambda_j;\epsilon_0)$ do not cross the line ℓ. If the disks have radius exceeding $\min_{1\le i\le s}\delta_i$ as in (9.2) (see right figure), then the Stokes rays R_{ab} cross the line ℓ.

Let us now turn our attention to the coalescence point $u^{(0)}$. From the results of [**CDG17b**] – and more generally in [**BJL79c**] - it follows that there are a *unique* formal solution at $u^{(0)}$,

$$\mathring{Y}_{\text{formal}}(z) = \left(\mathbb{1} + \sum_{k=1}^{\infty}\frac{\mathring{G}_k}{z^k}\right)e^{zU},$$

and unique actual solutions $\mathring{Y}^{(0)}_{\text{left}}(z)$ and $\mathring{Y}^{(0)}_{\text{right}}(z)$, with asymptotic representation given by $\mathring{Y}_{\text{formal}}(z)$ in $\Pi_{\text{left/right}}$, and in wider sectors $\mathcal{S}_{\text{left}}(u^{(0)})$ and $\mathcal{S}_{\text{right}}(u^{(0)})$ respectively. The Stokes matrices of $\mathring{Y}^{(0)}_{\text{right}}(z)$ and $\mathring{Y}^{(0)}_{\text{left}}(z)$ are defined by

$$\mathring{Y}^{(0)}_{\text{left}}(z) = \mathring{Y}^{(0)}_{\text{right}}(z)\mathring{S},\qquad \mathring{Y}^{(0)}_{\text{left}}(e^{2\pi i}z) = \mathring{Y}^{(0)}_{\text{right}}(z)\mathring{S}_-,\qquad \mathring{S}_- = \mathring{S}^T.$$

A priori, the following problems could emerge.

(1) The asymptotic representations

$$Y^{(0)}_{\text{left/right}}(z,u) \sim Y_{\text{formal}}(z,u),\quad \text{for } |z|\to\infty \text{ and } z\in\bigcap_{u\in\mathcal{D}}\mathcal{S}_{\text{left/right}}(u) \supsetneq \Pi_{\text{left/right}}(\phi)$$

does no longer hold for u outside the cell containing $\mathcal{D}$.

(2) The coefficients $G_k(u)$'s of (2.28) may divergent at Δ_Ω.

(3) The locus Δ_Ω is expected to be a locus of singularities for the solutions $Y_{\text{formal}}(z,u)$ in (2.28) and $Y^{(0)}_{\text{left/right}}(z,u)$. $Y_{\text{formal}}(z,u)$

(4) The Stokes matrices S, S_- may differ from $\mathring{S}, \mathring{S}_-$.

We notice that the system (9.4) at $u^{(0)}$ also has a fundamental solution in Levelt form at $z=0$,

$$\mathring{Y}_0(z) = \Psi(u^{(0)})(I + \mathcal{O}(z))z^{\mu}z^{\mathring{R}}, \tag{9.5}$$

with a certain exponent $\mathring{R}$. Hence, a central connection matrix $\mathring{C}$ is defined by

$$\mathring{Y}^{(0)}_{\text{right}}(z) = \mathring{Y}_0(z)\mathring{C}.$$

We recall that $\Psi(u)$ is holomorphic in the whole $\mathcal{U}_{\epsilon_0}(u^{(0)})$, so that $V_{ij}(u)$ vanishes along Δ_Ω whenever $u_i = u_j$ (see Lemma 2.3). These are sufficient conditions to apply the main theorem of [**CDG17b**], adapted and particularised to the case of Frobenius manifolds, which becomes the following:

THEOREM 9.1. *Let M be a semisimple Frobenius manifold, $p_0 \in \mathcal{B}_M \setminus \mathcal{K}_M$ and $\Omega \subseteq M_{ss} = M \setminus \mathcal{K}_M$ an open connected neighborhood of p_0 with the property of Remark 2.4, on which a holomorphic branch for canonical coordinates $u\colon \Omega \to \mathbb{C}^n$ and $\Psi\colon \Omega \to GL_n(\mathbb{C})$ has been fixed. Let ϵ_0 be a real positive number as above, and consider the corresponding neighborhood $\mathcal{U}_{\epsilon_0}(u^{(0)})$ of $u^{(0)} = u(p_0)$. Then*

(1) *The coefficients $G_k(u)$, $k \geq 1$, in (2.28) are holomorphic over $\mathcal{U}_{\epsilon_0}(u^{(0)})$,*

$$G_k(u^{(0)}) = \mathring{G}_k \quad \text{and} \quad Y_{\text{formal}}(z, u^{(0)}) = \mathring{Y}_{\text{formal}}(z).$$

(2) *$Y^{(0)}_{\text{left}}(z,u)$, $Y^{(0)}_{\text{right}}(z,u)$, can be u-analytically continued as single-valued holomorphic functions on $\mathcal{U}_{\epsilon_0}(u^{(0)})$. Moreover*

$$Y^{(0)}_{\text{left/right}}(z, u^{(0)}) = \mathring{Y}^{(0)}_{\text{left/right}}(z).$$

(3) *For any solution $\mathring{Y}_0(z)$ as in (9.5) there exists a fundamental solution $Y_0(z,u)$ in Levelt form (2.32) such that*

$$Y_0(z, u^{(0)}) = \mathring{Y}_0(z), \qquad R = \mathring{R}.$$

(4) *For any $\epsilon_1 < \epsilon_0$, the asymptotic relations*

$$Y^{(0)}_{\text{left/right}}(z,u) \sim \left(\mathbb{1} + \sum_{k=1}^{\infty} \frac{G_k(u)}{z^k}\right) e^{zU}, \quad z \to \infty \text{ in } \Pi_{\text{left/right}}(\phi), \tag{9.6}$$

hold uniformly in $u \in \mathcal{U}_{\epsilon_1}(u^{(0)})$. In particular they hold also at points of $\Delta_\Omega \cap \mathcal{U}_{\epsilon_1}(u^{(0)})$ and at $u^{(0)}$.

(5) *For any $u \in \mathcal{U}_{\epsilon_0}(u^{(0)})$ consider the sectors $\widehat{\mathcal{S}}_{\text{right}}(u)$ and $\widehat{\mathcal{S}}_{\text{left}}(u)$ which contain the sectors $\Pi_{\text{right}}(\phi)$ and $\Pi_{\text{left}}(\phi)$ respectively, and extend up to the nearest Stokes rays in the set $\mathfrak{R}(u)$ defined above. Let*

$$\widehat{\mathcal{S}}_{\text{left/right}} = \bigcap_{u \,\in\, \mathcal{U}_{\epsilon_0}(u^{(0)})} \widehat{\mathcal{S}}_{\text{left/right}}(u).$$

Observe that for sufficiently small $\varepsilon > 0$ the sectors

$$\Pi^{\varepsilon}_{\text{right}}(\phi) := \{z \in \mathcal{R} \colon \phi - \pi - \varepsilon < \arg\ z < \phi + \varepsilon\},$$
$$\Pi^{\varepsilon}_{\text{left}}(\phi) := \{z \in \mathcal{R} \colon \phi - \varepsilon < \arg\ z < \phi + \pi + \varepsilon\},$$

are strictly contained in $\widehat{\mathcal{S}}_{\text{right}}$ and $\widehat{\mathcal{S}}_{\text{left}}$ respectively. Then, the asymptotic relations (9.6) actually hold in the sectors $\widehat{\mathcal{S}}_{\text{left/right}}$.

(6) *The monodromy data μ, R, C, S of system (9.4), defined and constant in an open subset $\mathcal{D}$ of a cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$, are actually defined and constant at any $u \in \mathcal{U}_{\epsilon_1}(u^{(0)})$, namely the system is isomonodromic in $\mathcal{U}_{\epsilon_1}(u^{(0)})$. They coincide with the data μ, $\mathring{R}$, $\mathring{C}$, $\mathring{S}$ associated to fundamental solutions $\mathring{Y}_{\text{left/right}}(z)$ and $\mathring{Y}_0(z)$ of system (9.4) at $u^{(0)}$. The entries of $S = (S_{ij})^n_{i,j=1}$ satisfy the vanishing condition (1.39), namely*

$$S_{ij} = S_{ji} = 0 \quad \text{for all } i \neq j \text{ such that} \quad u_i^{(0)} = u_j^{(0)}. \tag{9.7}$$

This Theorem allows us to obtain the monodromy data μ, R, C, S in a neighbourhood of a coalescence point just by computing them *at* the coalescence point, namely just by computiong μ, $\mathring{R}$,

$\mathring{C}$, $\mathring{S}$. Its importance has been explained in the Introduction and will be illustrated in subsequent sections.

REMARK 9.1. Suppose that S is upper triangular. By formula (9.7), it follows that in any ℓ-cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$ the order of the canonical coordinates in triangular, according to Definition 2.17, and only in one cell the order is lexicographical (Definition 2.16).

9.1.1. Reconstruction of monodromy data of the whole manifold. The monodromy data of the Frobenius manifold can be obtained from those computed in Theorem 9.1 around $u^{(0)}$. Without loss of generality, let us suppose that the ordering (9.1) is such that $\lambda_1, \dots, \lambda_s$ are in ℓ- lexicographical order. Then, the matrix S computed at the coalescence point $u^{(0)}$ is upper triangular. Therefore, by Theorem 9.1, the matrix is constant and upper triangular in the whole ploydisc $\mathcal{U}_{\epsilon_0}(u^{(0)})$. In particular, it is upper triangular in each cell of $\mathcal{U}_{\epsilon_1}(u^{(0)})$. This means that $u_1, ..., u_n$ are in triangular order (Definition 2.17) in each such cell, and in particular they are in lexicographical order in only one of these cells (Definition 2.16). Note that any permutation of canonical coordinates preserving the sets $I_1, \dots, I_s$ of (9.3) maintains the upper triangular structure of S, namely the triangular order of $u_1, ..., u_n$ in each cell of $\mathcal{U}_{\epsilon_o}(u^{(0)})$. The permutation changes the cell where the order is lexicographical. Now, each cell of the polydisc $\mathcal{U}_{\epsilon_0}(u^{(0)})$ is contained in a chamber of the manifold (identifying coordinates with points of the manifold, which is possible because of the holomorphy of canonical coordinates near *semisimple* coalescent points). Let us start from the cell of $\mathcal{U}_{\epsilon_0}(u^{(0)})$ where $u_1, ..., u_n$ are in lexicographical order. The monodromy data of Theorem 9.1 in this cell are the constant data of the chamber containing the cell (Theorems 2.4 and 2.12). Since in this chamber $u_1, ..., u_n$ are in lexicographical order (and distinct!), we can apply the action of the braid group to S and C, as dictated by formulae (2.39), (2.41). In this way, the monodromy data for any other chamber of the manifold are obtained, as explained in Section 2.3.

CHAPTER 10

Monodromy Data of the Mawell Stratum of the A_3-Frobenius Manifold

Abstract. In this Chapter, after recalling how it is defined the Frobenius structure associated with the singularities of ADE-type, we study in detail the A_3-Frobenius structure. More precisely, in order to exemplify the results of Theorem 9.1, we study the isomonodromic differential system defining deformed flat coordinates in correspondence of points of the Maxwell Stratum of A_3. We show that at these points the computation of the monodromy data can be explicitely done using asymptotic-analytical properties of Hankel special functions $H^{(1)}_{\frac{1}{4}}(z), H^{(2)}_{\frac{1}{4}}(z)$. We also compute the monodromy data in a neighborhood of the Maxwell Stratum using properties of the Pearcey Integral, and we show in this example the validity of Theorem 9.1, namely both the isomonodromicity property and the holomorphy of fundamental systems of solutions. We finally reinterpret our computations as an alternative to the M. Jimbo's procedure ([**Jim82**]) for computing the monodormy data corresponding to branches of PVI_μ-transcendents holomorphic at critical points.

With the example of A_3 Frobenius manifold, we show how Theorem 9.1 allows the computation of monodromy data in an elementary way, by means of *Hankel special functions*. Moreover, we apply the results of Section 2.3, especially showing how the braid group can be used to reconstruct the data for the whole manifold, starting from a coalescence point. The reader not interested in a general introduction to Frobenius manifolds associated to singularity theory may skip Sections 10.1 and 10.1.1 and go directly to Section 10.2.1.

10.1. Singularity Theory and Frobenius Manifolds

Let f be a quasi-homogeneous polynomial on $\mathbb{C}^m$ with an isolated simple singularity at $0 \in \mathbb{C}^m$. According to V.I. Arnol'd [**Arn72**] simple singularities are classified by simply-laced Dynkin diagrams A_n (with $n \geq 1$), D_n (with $n \geq 4$), E_6, E_7, E_8. Denoting by $(x_1 \dots, x_m)$ the coordinates in $\mathbb{C}^m$ (for singularities of type A_n we consider $m = 1$), the classification of simple singularities is summarized in Table 10.1. Let μ be the Milnor number of f (note that $\mu = n$ for A_n, D_n and E_n), and

$$f(x, a) := f(x) + \sum_{i=1}^{\mu} a_i \phi_i(x),$$

be a miniversal unfolding of f, where a varies in a ball $B \subseteq \mathbb{C}^\mu$, and $(\phi_1(x), \dots, \phi_\mu(x))$ is a basis of the Milnor ring. Using Saito's theory of *primitive forms* [**Sai83**], a flat metric and a Frobenius manifold structure can be defined on the base space B [**BV92**]. For any fixed $a \in B$, let the critical points be $x_i(a) = (x_i^{(1)}, \dots, x_i^{(m)})$, $i = 1, \dots, \mu$, defined by the condition $\partial_{x_\alpha} f(x_i, a) = 0$ for any $\alpha = 1, \dots, m$. The critical values $u_i(a) := f(x_i(a), a)$ are the canonical coordinates. The open ball B can be stratified as follows:

(1) the stratum of *generic points*, i.e. points where both critical points $x^{(i)}$'s and critical values u_i's are distinct;

	Singularity	Versal Deformation
A_n	$f(x) = x^{n+1}$	$f(x,a) = x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
D_n	$f(x) = x_1^{n-1} + x_1 x_2^2$	$f(x,a) = x_1^{n-1} + x_1 x_2^2 + a_{n-1}x_1^{n-2} + \cdots + a_1 + a_0 x_2$
E_6	$f(x) = x_1^4 + x_2^3$	$f(x,a) = x_1^4 + x_2^3 + a_6 x_1^2 x_2 + a_5 x_1 x_2 + a_4 x_1^2 + a_3 x_2 + a_2 x_1 + a_1$
E_7	$f(x) = x_1^3 x_2 + x_2^3$	$f(x,a) = x_1^3 x_2 + x_2^3 + a_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1 x_2 + a_5 x_1 + a_6 x_1^2 + a_7 x_2$
E_8	$f(x) = x_1^5 + x_2^3$	$f(x,a) = x_1^5 + x_2^3 + a_8 x_1^3 x_2 + a_7 x_1^2 x_2 + a_6 x_1^3 + a_5 x_1 x_2 + a_4 x_1^2 + a_3 x_2 + a_2 x_1 + a_1$

TABLE 10.1. Arnol'd's classification of simple singularities, and their corresponding miniversal deformations.

(2) the *Maxwell stratum*, which is the closure of the set of points with distinct critical points $x^{(i)}$'s but some coalescing critical values u_i's;
(3) the *caustic*, where some critical points coalesce.

The union of the Maxwell stratum and the caustic is called *function bifurcation diagram* Ξ of the singularity (see [**AGLV93**] and [**AGZV88**]). The complement of the caustic consists exclusively of semisimple points of the Frobenius manifold. In this section we want to show how one can reconstruct local information near semisimple points in the Maxwell stratum, by invoking Theorem 9.1. We will focus on the simplest example of A_3.

10.1.1. Frobenius structure of type A_n. ([**DVV91**], [**Dub96**], [**Dub99b**],[**Dub99a**])
Let us consider the affine space $M \cong \mathbb{C}^n$ of all polynomials

$$f(x,a) = x^{n+1} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $(a_0, \dots, a_{n-1}) \in M$ are used as coordinates. We call *bifurcation diagram* Ξ of the singularity A_n the set of polynomials in M with some coalescing critical values. The bifurcation diagram Ξ is an algebraic variety in M, which consists of two irreducible components (the derivative w.r.t. the variable x will be denoted by $(\cdot)'$):

- the caustic $\mathcal{K}$, which is the set of polynomials with degenerate critical points (i.e. solutions of the system of equations $f'(x,a) = f''(x,a) = 0$)[1];
- The Maxwell stratum $\mathcal{M}$, defined as the closure of the set of polynomials with some coalescing critical values but different critical points.

For more information about the topology and geometry of (the complement of) these strata, the reader can consult the paper [**Nek93**], and the monograph [**Vas92**]. There is a naturally defined covering map $\rho\colon \widetilde{M} \to M$ of degree $n!$, whose fiber over a point $f(x,a)$ consists of total orderings of its critical

[1]The equation of the caustic is $\Delta(f') = 0$, where $\Delta(f') := \mathrm{Res}(f', f'')$ is the discriminant of the polynomial $f'(x,a)$. The reader can consult the monograph [**GKZ94**], Chapter 12.

points. On $\widetilde{M}$, $x_1, \dots, x_n$ are well defined functions such that

$$f'(x, \rho(w)) = (n+1)\prod_{i=1}^{n}(x - x_i(w)), \quad w \in \widetilde{M}.$$

The caustic $\mathcal{K}$ is the ramification locus of the covering ρ. For any simply connected open subset $U \subseteq M \setminus \mathcal{K}$, we can choose a connected component W of $\rho^{-1}(U)$. The restriction of the functions x_1, ..., x_n on W defines single-valued functions of $a \in U$, which are local determinations of x_1, ..., x_n. For further details see [Man99].

We define on M the following structures:

(1) a free sheaf of rank n of $\mathcal{O}_M$-algebras: this is the sheaf of Jacobi-Milnor algebras

$$\frac{\mathcal{O}_M[x]}{f'(x,a)\cdot \mathcal{O}_M[x]}.$$

For fixed $a \in M$, the fiber of this sheaf is the algebra $\mathbb{C}[x]/\langle f'(x,a)\rangle$. We also define an $\mathcal{O}_M$-linear *Kodaira-Spencer* isomorphism $\kappa\colon \mathscr{T}_M \to \mathcal{O}_M[x]/\langle f'(x,a)\rangle$ which associates to a vector field ξ the class $\mathfrak{L}_\xi(f) = \xi(f) \mod f'$. In particular, for any $\alpha = 0, \dots, n-1$ the class $\partial_{a_i} f$ is associated to the vector field ∂_{a_i}. In this way we introduce a product $\circ$ of vector fields defined by

$$\xi \circ \zeta := \kappa^{-1}\left(\xi(f)\cdot\zeta(f) \mod f'\right).$$

The product $\circ$ is associative, commutative and with unit ∂_{a_0}. We call *Euler vector field* the distinguished vector field E corresponding to the class $f \mod f'$ under the Kodaira-Spencer map κ. An elementary computation shows that

$$E = \sum_{i=0}^{n-1}\frac{n+1-i}{n+1}a_i\frac{\partial}{\partial a_i}, \qquad \mathfrak{L}_E(\circ) = \circ.$$

(2) A *symmetric bilinear form* η, defined at a fixed point $a \in M$ as the Grothendieck residue

$$\eta_a(\xi,\zeta) := \frac{1}{2\pi i}\int_{\Gamma_a}\frac{\xi(f)(u,a)\cdot\zeta(f)(u,a)}{f'(u,a)}du, \tag{10.1}$$

where Γ_a is a circle, positively oriented, bounding a disc containing all the roots of $f'(u,a)$. It is a nontrivial fact that the bilinear form η is non-degenerate (for a proof, see [AGZV88]) and *flat* (explicit flat coordinates can be found in [SYS80]: notice that the natural coordinates a_i's are not flat). Notice that

$$\mathfrak{L}_E\eta = \frac{n+3}{n+1}\eta.$$

THEOREM 10.1. *The manifold M, endowed with the tensors $(\eta, \circ, \partial_{a_0}, E)$, is a Frobenius manifold of charge $\frac{n-1}{n+1}$. The caustic $\mathcal{K}_M$, defined as in Definition 2.9, coincides with the caustic $\mathcal{K}$ of the singularity A_n defined above. By analytic continuation, the semisimple Frobenius structures extends on the unramified covering space $\rho^{-1}(M\backslash\mathcal{K}) \subseteq \widetilde{M}$. Critical values define a system of canonical coordinates.*

The reader can find detailed proofs in [Dub96], [Dub99b], [Man99], [Sab08]. If a is a given point of $M \setminus \mathcal{K}$, i.e. such that $f(x,a)$ has n distinct Morse critical points $x_1, \dots, x_n$, then the elements

$$\pi_i(a) := \kappa^{-1}\left(\frac{f'(x,a)}{f''(x_i,a)(x-x_i)}\right) \quad \text{for } i = 1, \dots, n$$

are idempotents of $(T_aM, \circ_a)$. This follows from the equality $f'(x,a) = (n+1)\prod_{i=1}^{n}(x-x_i)$. Consider now a local determination $x_1(a), \dots, x_n(a)$ for critical points, with a varying in a simply connected open set away from the caustic. Let us define the functions $u_i(a) := f(x_i(a), a)$ for $i = 1, \dots, n$.

Since $\det\left(\frac{\partial u_i}{\partial a_j}\right)$ is the Vandermonde determinant of $x_i(a)$'s, the functions u_i's define a system of local coordinates. In order to see that $\pi_i \equiv \frac{\partial}{\partial u_i}$, it is sufficient to prove that $\kappa(\partial u_i)(x_j) = \delta_{ij}$, i.e. $\frac{\partial f}{\partial u_i}(x_i) = \delta_{ij}$. This follows from the equalities

$$\frac{\partial f(x_i(a),a)}{\partial a_j} = (x_i(a))^j, \quad \frac{\partial}{\partial a_j} = \sum_i (x_i(a))^j \frac{\partial}{\partial u_i}.$$

10.2. The case of A_3

10.2.1. Reduction of the system for deformed flat coordinates. We consider the space M of polynomials

$$f(x;a) = x^4 + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, a_2 \in \mathbb{C}$ are "natural" coordinates on M. The Residue Theorem implies that the metric η, defined on M as in (10.1), can be expressed as

$$\eta_a(\xi,\zeta) = -\operatorname{res}_{u=\infty} \frac{\xi(f)(u,a)\cdot\zeta(f)(u,a)}{f'(u,a)}du,$$

and consequently

$$\eta_a(\partial_i,\partial_j) = \operatorname{res}_{v=0} \frac{v^{1-i-j}}{4+2a_2v^2+a_1v^3}dv,$$

where $\partial_i = \frac{\partial}{\partial a_i}, \partial_j = \frac{\partial}{\partial a_j}$. So we find that

$$\eta_a = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & -\frac{a_2}{8} \end{pmatrix}.$$

Note that a_0, a_1, a_2 are not flat coordinates for η. The commutative and associative product defined on each tangent space T_aM, using the Kodaira-Spencer map, is given by the structural constants at a generic point $a \in M$:

$$\partial_0 \circ \partial_i = \partial_i \quad \text{for all } i,$$

$$\partial_1 \circ \partial_1 = \partial_2, \qquad \partial_1 \circ \partial_2 = -\frac{1}{2}a_2\partial_1 - \frac{1}{4}a_1\partial_0, \qquad \partial_2 \circ \partial_2 = -\frac{1}{2}a_2\partial_2 - \frac{1}{4}a_1\partial_1.$$

The Euler vector field is

$$E := \sum_{i=0}^{2} \frac{4-i}{4} a_i\partial_i = a_0\partial_0 + \frac{3}{4}a_1\partial_1 + \frac{1}{2}a_2\partial_2.$$

With such a structure M is a Frobenius manifold. The (1,1)-tensor $\mathcal{U}$ of multiplication by E is:

$$\mathcal{U}(a) = \begin{pmatrix} a_0 & -\frac{1}{8}a_1a_2 & -\frac{3}{16}a_1^2 \\ \frac{3a_1}{4} & a_0 - \frac{a_2^2}{4} & -\frac{1}{2}a_1a_2 \\ \frac{a_2}{2} & \frac{3}{4}a_1 & a_0 - \frac{a_2^2}{4} \end{pmatrix}$$

Up to a multiplicative constant, the discriminant of the characteristic polynomial of $\mathcal{U}$ is equal to

$$a_1^2\left(8a_2^3 + 27a_1^2\right)^3$$

and so the bifurcation set of the Frobenius manifold is the locus

$$\mathcal{B} = \{a_1 = 0\} \cup \left\{8a_2^3 + 27a_1^2 = 0\right\}.$$

Let us focus on the set $\{a_1 = 0\}$, and let us look for semisimple points on it. It is enough to consider the multiplication by the vector field $\lambda\partial_1 + \mu\partial_2$ ($\lambda, \mu \in \mathbb{C}$), and show that it has distinct eigenvalues. This is a (1,1)-tensor with components at points (a_0, a_1, a_2) equal to

$$\begin{pmatrix} 0 & -\frac{\mu}{4}a_1 & -\frac{\lambda}{2}a_1 \\ \lambda & -\frac{\mu}{2}a_2 & -\frac{\lambda}{2}a_2 - \frac{\mu}{4}a_1 \\ \mu & \lambda & -\frac{\mu}{2}a_2 \end{pmatrix}$$

whose characteristic polynomial, at points $(a_0, 0, a_2)$, has discriminant

$$-\frac{1}{8}\lambda^2 a_2^3\left(2\lambda^2 + \mu^2 a_2\right)^2.$$

So, the points $(a_0, 0, a_2)$ with $a_2 \neq 0$ are semisimple points of the bifurcation set, namely they belong to the Maxwell stratum. In view of Theorem 10.1, they are semisimple coalescence points of Definition 1.1. We would like to study deeper the behavior of the Frobenius structure near points $(a_0, a_1, a_2) = (0, 0, h)$ of the Maxwell stratum, with fixed $a_0 = 0$ and with $h \in \mathbb{C}^*$.

REMARK 10.1. The points $(a_0, 0, 0)$, instead, are not semisimple because we have evidently $\partial_2^2 = 0$ on them.

Let us introduce flat coordinates t_1, t_2, t_3 defined by

$$\begin{cases} a_0 = t_1 + \frac{1}{8}t_3^2, \\ a_1 = t_2, \\ a_2 = t_3 \end{cases} \qquad J = \left(\frac{\partial a_i}{\partial t_j}\right)_{i,j} = \begin{pmatrix} 1 & 0 & \frac{1}{4}t_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In flat coordinates we have:

$$\eta = \begin{pmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}, \quad \mathcal{U}(t_1,t_2,t_3) = \begin{pmatrix} t_1 & \frac{-5}{16}t_2t_3 & -\frac{3}{16}t_2^2 + \frac{1}{32}t_3^3 \\ \frac{3t_2}{4} & t_1 - \frac{t_3^2}{8} & \frac{-5}{16}t_2t_3 \\ \frac{t_3}{2} & \frac{3t_2}{4} & t_1 \end{pmatrix}, \quad \mu = \begin{pmatrix} -\frac{1}{4} & & \\ & 0 & \\ & & \frac{1}{4} \end{pmatrix}.$$

Thus, the second system in (2.2) is

$$\partial_z \xi = \left(\mathcal{U}^T - \frac{1}{z}\mu\right)\xi$$

that is

$$\begin{cases} \partial_z\xi_1 = \frac{3}{4}\xi_2 t_2 + \frac{1}{2}\xi_3 t_3 + \xi_1\left(t_1 + \frac{1}{4z}\right), \\ \partial_z\xi_2 = -\frac{5}{16}\xi_1 t_2 t_3 + \xi_2\left(t_1 - \frac{t_3^2}{8}\right) + \frac{3}{4}\xi_3 t_2, \\ \partial_z\xi_3 = \xi_1\left(-\frac{3}{16}t_2^2 + \frac{1}{32}t_3^3\right) - \frac{5}{16}\xi_2 t_2 t_3 + \xi_3\left(t_1 - \frac{1}{4z}\right). \end{cases} \tag{10.2}$$

We know that if (t_1, t_2, t_3) is a semisimple point of the Frobenius manifold, then the monodromy data are well defined, and that these are inavariant under (small) deformations of t_1, t_2, t_3 by Theorem 2.12 and Theorem 9.1. The bifurcation set is now

$$\{t_2 = 0\} \cup \left\{8t_3^3 + 27t_2^2 = 0\right\}$$

Now, if we fix $a_0 = 0$, the tensor $\mathcal{U}$ at $(0, a_1, h)$, i.e. $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, t_2, h\right)$, is

$$\mathcal{U}\left(-\frac{1}{8}h^2, t_2, h\right) = \begin{pmatrix} -\frac{h^2}{8} & -\frac{5h}{16}t_2 & \frac{1}{32}\left(h^3 - 6t_2^2\right) \\ \frac{3t_2}{4} & -\frac{h^2}{4} & -\frac{5h}{16}t_2 \\ \frac{h}{2} & \frac{3t_2}{4} & -\frac{h^2}{8} \end{pmatrix}. \tag{10.3}$$

The bifurcation locus is reached for $a_1 = t_2 = 0$. At this points

$$(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right).$$

we have

$$\mathcal{U}\left(-\frac{1}{8}h^2, 0, h\right) = \begin{pmatrix} -\frac{h^2}{8} & 0 & \frac{h^3}{32} \\ 0 & -\frac{h^2}{4} & 0 \\ \frac{h}{2} & 0 & -\frac{h^2}{8} \end{pmatrix}. \tag{10.4}$$

Remark 10.2. Note that the characteristic polynomial of the matrix (10.3) is equal to

$$p_{h,t_2}(\lambda) = \frac{1}{256}\left(-16h^4\lambda - 128h^2\lambda^2 - 256\lambda^3 - 4h^3t_2^2 - 144h\lambda t_2^2 - 27t_2^4\right)$$

whose discriminant is

$$\frac{-512h^9t_2^2 - 5184h^6t_2^4 - 17496h^3t_2^6 - 19683t_2^8}{65536}.$$

It vanishes at points in which

$$t_2 = 0, \quad t_2 = \pm\frac{2}{3}i\sqrt{\frac{2}{3}}h^{\frac{3}{2}}.$$

We are investigating the behavior near points of the first case.

Define the function

$$X(a) := \left[-9a_1 + \sqrt{3}(27a_1^2 + 8a_2^3)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

which has branch points along the caustic $\mathcal{K} = \{a_1 = a_2 = 0\} \cup \{27a_1^2 + 8a_2^3 = 0\}$. Fix a determination of X on a simply connected domain in $M \setminus \mathcal{K}$, that we also denote by $X(a)$. The critical points x_1, x_2, x_3 of $f(x, a)$ are equal to

$$x_i(a) := \frac{\vartheta_i \cdot a_2}{2\sqrt{3} \cdot X(a)} - \frac{\vartheta_i \cdot X(a)}{2 \cdot 3^{2/3}},$$

where

$$\vartheta_1 := -1, \quad \vartheta_2 := \frac{1 - i\sqrt{3}}{2}, \quad \vartheta_3 := \frac{1 + i\sqrt{3}}{2}$$

are the cubic roots of (-1). Of course, different choices of determinations of X correspond to permutations of the x_i's. After some computations, we find the following expression for Ψ:

$$\Psi(t) = \left.\begin{pmatrix} \frac{\sqrt{6x_1^2+a_2}}{2\sqrt{2}(x_1-x_2)(x_1-x_3)} & -\frac{(x_2+x_3)\sqrt{6x_1^2+a_2}}{2\sqrt{2}(x_1-x_2)(x_1-x_3)} & -\frac{\sqrt{6x_1^2+a_2}(a_2-4x_2x_3)}{8\sqrt{2}(x_1-x_2)(x_1-x_3)} \\ \frac{\sqrt{6x_2^2+a_2}}{2\sqrt{2}(x_1-x_2)(x_3-x_2)} & \frac{(x_1+x_3)\sqrt{6x_2^2+a_2}}{2\sqrt{2}(x_1-x_2)(x_2-x_3)} & \frac{\sqrt{6x_2^2+a_2}(a_2-4x_1x_3)}{8\sqrt{2}(x_1-x_2)(x_2-x_3)} \\ \frac{\sqrt{6x_3^2+a_2}}{2\sqrt{2}(x_1-x_3)(x_2-x_3)} & \frac{(x_1+x_2)\sqrt{6x_3^2+a_2}}{2\sqrt{2}(x_1-x_3)(x_3-x_2)} & \frac{(a_2-4x_1x_2)\sqrt{6x_3^2+a_2}}{8\sqrt{2}(x_1-x_3)(x_3-x_2)} \end{pmatrix}\right|_{a=a(t)},$$

where

$$a_0 = t_1 + \frac{1}{8}t_3^2, \quad a_1 = t_2, \quad a_2 = t_3.$$

The canonical coordinates are $u_i(t) = f(x_i(a(t)), a(t))$. Near the point $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$, i.e. for small t_2, we find:

$$\begin{aligned}
u_1(t_2; h) &= -\frac{t_2^2}{4h} + \frac{t_2^4}{16h^4} - \frac{t_2^6}{16h^7} + \frac{3t_2^8}{32h^{10}} + O\left(|t_2|^{10}\right), \\
u_2(t_2; h) &= -\frac{h^2}{4} + \frac{i\sqrt{h}t_2}{\sqrt{2}} + \frac{t_2^2}{8h} + \frac{it_2^3}{16\sqrt{2}h^{5/2}} - \frac{t_2^4}{32h^4} - \frac{21it_2^5}{512\sqrt{2}h^{11/2}} \\
&\quad + \frac{t_2^6}{32h^7} + \frac{429it_2^7}{8192\sqrt{2}h^{17/2}} - \frac{3t_2^8}{64h^{10}} - \frac{46189it_2^9}{524288\sqrt{2}h^{23/2}} + O\left(|t_2|^{10}\right), \\
u_3(t_2; h) &= -\frac{h^2}{4} - \frac{i\sqrt{h}t_2}{\sqrt{2}} + \frac{t_2^2}{8h} - \frac{it_2^3}{16\sqrt{2}h^{5/2}} - \frac{t_2^4}{32h^4} + \frac{21it_2^5}{512\sqrt{2}h^{11/2}} \\
&\quad + \frac{t_2^6}{32h^7} - \frac{429it_2^7}{8192\sqrt{2}h^{17/2}} - \frac{3t_2^8}{64h^{10}} + \frac{46189it_2^9}{524288\sqrt{2}h^{23/2}} + O\left(|t_2|^{10}\right),
\end{aligned} \tag{10.5}$$

$$\begin{aligned}
\Psi(t_2) &= \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt{h}} & 0 & \frac{\sqrt{h}}{4\sqrt{2}} \\ \frac{i}{2\sqrt{h}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{8}\left(i\sqrt{h}\right) \\ \frac{i}{2\sqrt{h}} & \frac{1}{2\sqrt{2}} & -\frac{1}{8}\left(i\sqrt{h}\right) \end{pmatrix} + t_2 \begin{pmatrix} 0 & -\frac{1}{2\sqrt{2}h^{3/2}} & 0 \\ -\frac{3}{8\sqrt{2}h^2} & -\frac{i}{16h^{3/2}} & -\frac{5}{32\sqrt{2}h} \\ \frac{3}{8\sqrt{2}h^2} & -\frac{i}{16h^{3/2}} & \frac{5}{32\sqrt{2}h} \end{pmatrix} \\
&+ t_2^2 \begin{pmatrix} -\frac{3}{4\sqrt{2}h^{7/2}} & 0 & \frac{1}{16\sqrt{2}h^{5/2}} \\ -\frac{39i}{128h^{7/2}} & \frac{15}{128\sqrt{2}h^3} & -\frac{41i}{512h^{5/2}} \\ -\frac{39i}{128h^{7/2}} & -\frac{15}{128\sqrt{2}h^3} & -\frac{41i}{512h^{5/2}} \end{pmatrix} + t_2^3 \begin{pmatrix} 0 & \frac{5}{8\sqrt{2}h^{9/2}} & 0 \\ \frac{303}{512\sqrt{2}h^5} & \frac{125i}{1024h^{9/2}} & \frac{265}{2048\sqrt{2}h^4} \\ -\frac{303}{512\sqrt{2}h^5} & \frac{125i}{1024h^{9/2}} & -\frac{265}{2048\sqrt{2}h^4} \end{pmatrix} + O(t_2^4).
\end{aligned}$$

Hence, at points $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$, canonical coordinates $u_i(0; h)$ are

$$(u_1, u_2, u_3) = \left(0, -\frac{h^2}{2}, -\frac{h^2}{2}\right)$$

and the system (10.2) reduces to

$$\begin{cases} \partial_z \xi_1 = \left(-\frac{h^2}{8} + \frac{1}{4z}\right)\xi_1 + \frac{h}{2}\xi_3, \\ \\ \partial_z \xi_2 = -\frac{h^2}{4}\xi_2, \\ \\ \partial_z \xi_3 = \frac{h^3}{32}\xi_1 - \left(\frac{h^2}{8} + \frac{1}{4z}\right)\xi_3. \end{cases} \tag{10.6}$$

The second equation is integrable by quadratures and yields

$$\xi_2(z) = c \cdot e^{-\frac{h^2}{4}z}, \quad c \in \mathbb{C}.$$

From the first equation we find that

$$\xi_3 = \frac{2}{h}\left(\partial_z \xi_1 + \frac{h^2}{8}\xi_1 - \frac{1}{4z}\xi_1\right), \tag{10.7}$$

and so from the third equation we obtain

$$\frac{2}{h}\xi_1''(z)+\frac{h}{2}\xi_1'(z)+\frac{3}{8z^2h}\xi_1=0.$$

Making the ansatz

$$\xi_1=z^{\frac{1}{2}}e^{-\frac{h^2z}{8}}\Lambda(z),$$

the equation for Λ becomes the following Bessel equation:

$$64z^2\Lambda''(z)+64z\Lambda'(z)-(4+z^2h^4)\Lambda(z)=0. \tag{10.8}$$

Therefore, ξ_1 is of the form

$$\xi_1=z^{\frac{1}{2}}e^{-\frac{h^2z}{8}}\left(c_1H^{(1)}_{\frac{1}{4}}\left(\frac{ih^2}{8}z\right)+c_2H^{(2)}_{\frac{1}{4}}\left(\frac{ih^2}{8}z\right)\right),\quad c_1,c_2\in\mathbb{C}$$

where $H^{(1)}_\nu(z),H^{(2)}_\nu(z)$ stand for the Hankel functions of the first and second kind of parameter $\nu=1/4$. Notice that if $\Lambda(z)$ is a solution of equation (10.8), then also $\Lambda(e^{\pm i\pi}z)$ is a solution.

10.2.2. Computation of Stokes and Central Connection matrices. In order to compute the Stokes matrix, let us fix the line ℓ to coincide with the real axis. Such a line is admissible for all points $(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,0,h\right)$ with

$$|\operatorname{Re}h|\neq|\operatorname{Im}h|,\quad h\in\mathbb{C}^*.$$

Indeed, the Stokes rays for $(u_1,u_2,u_3)=(0,-\frac{1}{4}h^2,-\frac{1}{4}h^2)$ are

$$z=i\rho\overline{h}^2\qquad\Longrightarrow\qquad\arg z=\frac{\pi}{2}-2\arg h\ (\mathrm{mod}\ \pi).$$

Thus, admissibility corresponds to $\frac{1}{2}\pi-2\arg h\neq k\pi$, $k\in\mathbb{Z}$. Let us compute the Stokes matrix in the case

$$|\operatorname{Re}h|>|\operatorname{Im}h|,\quad -\frac{\pi}{4}<\arg h<\frac{\pi}{4}.$$

The asymptotic expansion for fundamental solutions $\Xi_{\mathrm{left}},\Xi_{\mathrm{right}}$ of the system (10.6), is

$$\begin{aligned}\eta\Psi^{-1}\left(\mathbb{1}+O\left(\frac{1}{z}\right)\right)e^{zU}&=\Psi^T\left(\mathbb{1}+O\left(\frac{1}{z}\right)\right)e^{zU}\\&=\left(\mathbb{1}+O\left(\frac{1}{z}\right)\right)\begin{pmatrix}\frac{1}{\sqrt{2}\sqrt{h}}&\frac{i}{2\sqrt{h}}e^{-\frac{1}{4}(h^2z)}&\frac{i}{2\sqrt{h}}e^{-\frac{1}{4}(h^2z)}\\0&-\frac{1}{2\sqrt{2}}e^{-\frac{1}{4}(h^2z)}&\frac{1}{2\sqrt{2}}e^{-\frac{1}{4}(h^2z)}\\\frac{\sqrt{h}}{4\sqrt{2}}&-\frac{i}{8}e^{-\frac{1}{4}(h^2z)}\sqrt{h}&-\frac{i}{8}e^{-\frac{1}{4}(h^2z)}\sqrt{h}\end{pmatrix},\end{aligned}$$

$$\text{being } U:=\Psi\mathcal{U}\Psi^{-1}=\operatorname{diag}(u_1,u_2,u_3)=\operatorname{diag}\left(0,-\frac{h^2}{4},-\frac{h^2}{4}\right).$$

For the admissible line ℓ and for the above labelling of canonical coordinates the Stokes matrix must be of the form prescribed by Theorem 2.10:

$$S=\begin{pmatrix}1&0&0\\\alpha&1&0\\\beta&0&1\end{pmatrix} \tag{10.9}$$

for some constants $\alpha,\beta\in\mathbb{C}$ to be determined. This means that the last two columns of Ξ_{left} must be the analytic continuation of Ξ_{right}.

LEMMA 10.1. *The following asymptotic expansions hold:*

- *if* $m \in \mathbb{Z}$, *then*

$$H^{(1)}_{\frac{1}{4}}\left(e^{im\pi}\frac{ih^2}{8}z\right) \sim \sqrt{\frac{2}{\pi}}\left(e^{im\pi}\frac{ih^2}{8}z\right)^{-\frac{1}{2}} e^{-\frac{3i\pi}{8}} \exp\left(-e^{im\pi}\frac{h^2}{8}z\right)$$

in the sector

$$-\frac{3}{2}\pi - m\pi - \arg(h^2) < \arg z < \frac{3}{2}\pi - m\pi - \arg(h^2);$$

- *if* $m \in \mathbb{Z}$, *then*

$$H^{(2)}_{\frac{1}{4}}\left(e^{im\pi}\frac{ih^2}{8}z\right) \sim \sqrt{\frac{2}{\pi}}\left(e^{im\pi}\frac{ih^2}{8}z\right)^{-\frac{1}{2}} e^{\frac{3i\pi}{8}} \exp\left(e^{im\pi}\frac{h^2}{8}z\right)$$

in the sector

$$-\frac{5}{2}\pi - m\pi - \arg(h^2) < \arg z < \frac{\pi}{2} - m\pi - \arg(h^2).$$

PROOF. These formulae easily follow from the following well-known asymptotic expansion of Hankel functions (see [**Wat44**]):

$$H^{(1)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left(i\left(z - \frac{\nu}{2}\pi - \frac{\pi}{4}\right)\right), \qquad -\pi + \delta \le \arg z \le 2\pi - \delta,$$

δ being any positive acute angle. Analogously,

$$H^{(2)}_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left(-i\left(z - \frac{\nu}{2}\pi - \frac{\pi}{4}\right)\right), \qquad -2\pi + \delta \le \arg z \le \pi - \delta.$$

□

Using Lemma 10.1, we obtain

$$\Xi_{\text{left}}(z) = \begin{pmatrix} \xi^L_{(1),1} & \xi^L_{(2),1} & \xi^L_{(3),1} \\ 0 & -\frac{e^{-\frac{1}{4}\left(h^2 z\right)}}{2\sqrt{2}} & \frac{e^{-\frac{1}{4}\left(h^2 z\right)}}{2\sqrt{2}} \\ * & * & * \end{pmatrix}, \quad \Xi_{\text{right}}(z) = \begin{pmatrix} \xi^R_{(1),1} & \xi^R_{(2),1} & \xi^R_{(3),1} \\ 0 & -\frac{e^{-\frac{1}{4}\left(h^2 z\right)}}{2\sqrt{2}} & \frac{e^{-\frac{1}{4}\left(h^2 z\right)}}{2\sqrt{2}} \\ * & * & * \end{pmatrix} \tag{10.10}$$

where

$$\xi^L_{(2),1}(z) = \xi^L_{(3),1}(z) = \xi^R_{(2),1}(z) = \xi^R_{(3),1}(z) = \frac{i\sqrt{\pi}}{8} h^{\frac{1}{2}} e^{i\frac{5}{8}\pi} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(1)}_{\frac{1}{4}}\left(\frac{ih^2}{8}z\right),$$

with the correct required asymptotic expansion in the following sector containing both Π_{left} and Π_{right}

$$\left\{z \in \mathcal{R}\colon\ -\frac{3}{2}\pi - \arg(h^2) < \arg z < \frac{3}{2}\pi - \arg(h^2)\right\},$$

and

$$\xi^L_{(1),1}(z) = \frac{\sqrt{\pi}}{4\sqrt{2}} h^{\frac{1}{2}} e^{i\frac{\pi}{8}} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(1)}_{\frac{1}{4}}\left(e^{-i\pi}\frac{ih^2}{8}z\right),$$

$$\xi^R_{(1),1}(z) = \frac{\sqrt{\pi}}{4\sqrt{2}} h^{\frac{1}{2}} e^{-i\frac{\pi}{8}} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(2)}_{\frac{1}{4}}\left(\frac{ih^2}{8}z\right),$$

with the correct required expansion respectively in the sectors

$$\left\{z \in \mathcal{R}\colon -\frac{\pi}{2} - \arg(h^2) < \arg z < \frac{5}{2}\pi - \arg(h^2)\right\} \supseteq \Pi_{\text{left}},$$

$$\left\{z \in \mathcal{R}\colon -\frac{5}{2}\pi - \arg(h^2) < \arg z < \frac{\pi}{2} - \arg(h^2)\right\} \supseteq \Pi_{\text{right}}.$$

The entries of $\Xi_{\text{left}}, \Xi_{\text{right}}$ denoted by $*$ are reconstructed from the first rows, by applying equation (10.7).

From the second rows of $\Xi_{\text{left}}, \Xi_{\text{right}}$, we can immediately say that the entries α, β of (10.9) must be equal. Specializing the following well-known connection formula for Hankel special functions

$$\sin(\nu\pi) H_\nu^{(1)}(ze^{m\pi i}) = -\sin((m-1)\nu\pi) H_\nu^{(1)}(z) - e^{-\nu\pi i}\sin(m\nu\pi) H_\nu^{(2)}(z), \quad m \in \mathbb{Z}, \tag{10.11}$$

to the case $m = -1, \nu = \frac{1}{4}$, we easily obtain

$$\xi^L_{(1),1}(z) = \xi^R_{(1),1}(z) - \xi^R_{(2),1}(z) - \xi^R_{(3),1}(z)$$

which means that $\alpha = \beta = -1$. So, we have obtained that, at points $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$ with

$$|\operatorname{Re} h| > |\operatorname{Im} h|, \quad -\frac{\pi}{4} < \arg h < \frac{\pi}{4}$$

(and consequently in their neighborhood, by Theorem 9.1) the Stokes matrix is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \tag{10.12}$$

In order to compute the central connection matrix, we observe that the A_3 Frobenius manifold structure is $non-resonant$, i.e. the components of the tensor μ are such that $\mu_\alpha - \mu_\beta \notin \mathbb{Z}$. This implies that the (η, μ)-parabolic orthogonal group is trivial, and that the fundamental system of (10.6) near the origin $z = 0$ can be uniquely chosen in such a way that

$$\Xi_0(z) = (\eta + O(z)) z^\mu. \tag{10.13}$$

Now, let us recall the following Mellin-Barnes integral representations of Hankel functions (see [**Wat44**])

$$H_\nu^{(1)}(z) = -\frac{\cos(\nu\pi)}{\pi^{\frac{5}{2}}} e^{i(z-\pi\nu)} (2z)^\nu \int_{-\infty i}^{\infty i} \Gamma(s)\Gamma(s-2\nu)\Gamma\left(\nu + \frac{1}{2} - s\right)(-2iz)^{-s} ds,$$

$$H_\nu^{(2)}(z) = \frac{\cos(\nu\pi)}{\pi^{\frac{5}{2}}} e^{-i(z-\pi\nu)} (2z)^\nu \int_{-\infty i}^{\infty i} \Gamma(s)\Gamma(s-2\nu)\Gamma\left(\nu + \frac{1}{2} - s\right)(2iz)^{-s} ds,$$

which are valid for

- $2\nu \notin 2\mathbb{Z} + 1$,
- respectively in the sectors $|\arg(\mp iz)| < \frac{3}{2}$,
- and where the integration path separates the poles of $\Gamma(s)\Gamma(s-2\nu)$ from those of $\Gamma\left(\nu + \frac{1}{2} - s\right)$.

Specializing these integral forms to $\nu = \frac{1}{4}$, and deforming the integration path so that it reduces to positively oriented circles around the poles

$$s \in \frac{1}{2} - \frac{1}{2}\mathbb{N},$$

we immediately obtain the following expansion of the solution $\xi^{(1)}_{1,R}, \xi^{(2)}_{1,R}, \xi^{(3)}_{1,R}$ for the points $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$, with $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$, valid for small values of $|z|$:

LEMMA 10.2. *At the points* $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$, *with* $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$ *the following expansion holds:*

$$\begin{aligned}\xi^R_{(1),1}(z) =& \frac{(1+i)\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}} z^{\frac{1}{4}} + \frac{\left(\frac{1}{4} - \frac{i}{4}\right) h\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}} z^{3/4} \\ & - \frac{\left(\frac{1}{32} + \frac{i}{32}\right) h^2\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}} z^{5/4} - \frac{\left(\frac{1}{32} - \frac{i}{32}\right) h^3\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}} z^{7/4} + O\left(|z|^{9/4}\right),\end{aligned}$$

$$\xi^R_{(2),1}(z) = \xi^R_{(3),1}(z) = \frac{i\Gamma\left(\frac{5}{4}\right)}{\sqrt{\pi}} z^{\frac{1}{4}} - \frac{4ih\Gamma\left(\frac{11}{4}\right)}{21\sqrt{\pi}} z^{3/4} - \frac{ih^2\Gamma\left(\frac{5}{4}\right)}{8\sqrt{\pi}} z^{5/4} + \frac{ih^3\Gamma\left(\frac{11}{4}\right)}{42\sqrt{\pi}} z^{7/4} + O\left(|z|^{9/4}\right).$$

Moreover, using equation (10.7) *we find that*

$$\xi^R_{(1),3}(z) = \frac{\left(\frac{1}{4} - \frac{i}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}} z^{-\frac{1}{4}} - \frac{\left(\frac{1}{32} - \frac{i}{32}\right) h^2\Gamma\left(\frac{3}{4}\right)}{\sqrt{\pi}} z^{3/4} + O\left(|z|^{5/4}\right),$$

$$\xi^R_{(2),3}(z) = \xi^R_{(3),3}(z) = -\frac{4i\Gamma\left(\frac{11}{4}\right)}{21\sqrt{\pi}} z^{-\frac{1}{4}} + \frac{ih^2\Gamma\left(\frac{11}{4}\right)}{42\sqrt{\pi}} z^{3/4} + O\left(|z|^{5/4}\right).$$

PROOF. These expansion are the first term of the expressions

$$\begin{aligned}\xi^R_{(1),1}(z) =& \frac{\sqrt{\pi}}{4\sqrt{2}} h^{\frac{1}{2}} e^{-i\frac{\pi}{8}} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} \cdot \\ & \cdot \left(\frac{e^{-i\left(-\frac{\pi}{4} + \frac{ih^2z}{8}\right)}(ih^2z)^{\frac{1}{4}}}{2\pi^{5/2}} \cdot 2\pi i \sum_{n=0}^{\infty} \operatorname*{res}_{s=\frac{1}{2}-\frac{n}{2}} \left(\Gamma(s)\Gamma(s-2\nu)\Gamma\left(\nu+\frac{1}{2}-s\right)\left(e^{i\pi}\frac{h^2z}{4}\right)^{-s}\right)\right),\end{aligned}$$

and

$$\begin{aligned}\xi^R_{(2),1}(z) =& \xi^R_{(3),1}(z) = \frac{i\sqrt{\pi}}{8} h^{\frac{1}{2}} e^{i\frac{5}{8}\pi} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} \cdot \\ & \cdot \left(-\frac{e^{i\left(-\frac{\pi}{4}+\frac{1}{8}ih^2z\right)}(ih^2z)^{\frac{1}{4}}}{2\pi^{5/2}} \cdot 2\pi i \sum_{n=0}^{\infty} \operatorname*{res}_{s=\frac{1}{2}-\frac{n}{2}} \left(\Gamma(s)\Gamma(s-2\nu)\Gamma\left(\nu+\frac{1}{2}-s\right)\left(-e^{i\pi}\frac{h^2z}{4}\right)^{-s}\right)\right).\end{aligned}$$

□

By a direct comparison between these expansion of solution $\Xi_{\text{right}}(z)$ of (10.10) and the dominant term of (10.13), namely

$$\begin{pmatrix} 0 & 0 & \frac{z^{\frac{1}{4}}}{4} \\ 0 & \frac{1}{4} & 0 \\ \frac{z^{-\frac{1}{4}}}{4} & 0 & 0 \end{pmatrix},$$

we obtain the central connection matrix

$$C=\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix}(1-i)\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right)\\ 0 & -\sqrt{2\pi} & \sqrt{2\pi}\\ (1+i)\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right)\end{pmatrix}.$$

Notice that such a matrix satisfies all the constraints of Theorem 2.11.

We can put the Stokes matrix in triangular form using two different permutations of the canonical coordinates $(0,-h^2/4,-h^2/4)$, namely

- the re-labeling $(u_1,u_2,u_3)\mapsto(u_2,u_3,u_1)$, corresponding to the permutation matrix
$$P=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},$$
- or the re-labeling $(u_1,u_2,u_3)\mapsto(u_3,u_2,u_1)$, corresponding to the permutation matrix
$$P=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

In both cases these are lexicographical orders of two different ℓ-cells which divide any sufficiently small neighborhood of the point $(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,0,h\right)$, with $|\operatorname{Re}h|>|\operatorname{Im}h|$ and $-\frac{\pi}{4}<\arg h<\frac{\pi}{4}$, in which Theorem 9.1 applies. Using both permutations, the Stokes matrix becomes

$$S_{\text{lex}}=PSP^{-1}=\begin{pmatrix}1&0&-1\\0&1&-1\\0&0&1\end{pmatrix}, \tag{10.14}$$

which can be thought as in lexicographical form in one of the ℓ-cells. The central connection matrix, instead, has the following lexicographical forms in the two ℓ-cells:

$$C_{\text{lex}}=\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix}-i\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right) & (1-i)\Gamma\left(\frac{3}{4}\right)\\ \mp\sqrt{2\pi} & \pm\sqrt{2\pi} & 0\\ i\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right) & (1+i)\Gamma\left(\frac{1}{4}\right)\end{pmatrix}, \tag{10.15}$$

where we take the first sign if the lexicographical order is the relabeling $(u_1,u_2,u_3)\mapsto(u_2,u_3,u_1)$, the second if it is the re-labeling $(u_1,u_2,u_3)\mapsto(u_3,u_2,u_1)$.

10.2.3. A "tour" in the Maxwell stratum: reconstruction of neighborhing monodromy data. From the data (10.14) and (10.15), by an action of the braid group, we can compute S and C in the neighborhood of all other points $(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,0,h\right)$ with $|\operatorname{Re}h|\neq|\operatorname{Im}h|$. As an example, let us determine the Stokes matrix for points

$$(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,0,h\right),\quad \text{with } \frac{\pi}{4}<\arg h<\frac{3}{4}\pi.$$

Starting from a point in the region $-\frac{\pi}{4}<\arg h<\frac{\pi}{4}$ and moving counter-clockwise towards the region $\frac{\pi}{4}<\arg h<\frac{3}{4}\pi$, the two coalescing canonical coordinates $u_2=u_3=-\frac{1}{2}h^2$ move in the u_i's-plane

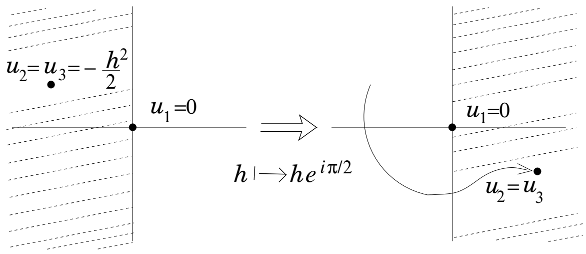


FIGURE 10.1. The triple (u_1, u_2, u_2) is represented by three points u_1, u_2, u_3 in $\mathbb{C}$. We move along $h \mapsto he^{i\frac{\pi}{2}}$, starting from $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$. The two dashed regions in the left and right figures correspond respectively to $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$ and $\frac{\pi}{4} < \arg h < \frac{3\pi}{4}$.

counter-clockwise w.r.t. $u_1 = 0$. For example, in Figure 10.1 we move along a curve $h \mapsto he^{i\frac{\pi}{2}}$, starting in $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$. At $\arg h = \frac{\pi}{4}$, the Stokes rays $R_{12} = \{z = -i\rho\overline{h}^2,\ \rho > 0\}$ and $R_{21} = \{z = i\rho\overline{h}^2,\ \rho > 0\}$ cross the real line ℓ, and a braid must act on the monodromy data.
In order to determine the braid and the transformed monodromy data, we proceed according to the prescription of Section 9.1.1, as follows.

(1) We split the coalescing canonical coordinates, for example by considering the point

$$(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, \varepsilon e^{i\varphi}, h\right), \quad \text{with } -\frac{\pi}{4} < \arg h < \frac{\pi}{4} \tag{10.16}$$

for chosen φ and ϵ, being ε small (so that $\varepsilon^2 \ll \varepsilon$). The corresponding canonical coordinates

$$u_1 = O(\varepsilon^2), \tag{10.17}$$

$$u_2 = -\frac{h^2}{4} + \varepsilon|h|^{\frac{1}{2}} \exp\left[i\left(\frac{\arg h}{2} + \varphi + \frac{\pi}{2}\right)\right] + O(\varepsilon^2), \tag{10.18}$$

$$u_3 = -\frac{h^2}{4} + \varepsilon|h|^{\frac{1}{2}} \exp\left[i\left(\frac{\arg h}{2} + \varphi - \frac{\pi}{2}\right)\right] + O(\varepsilon^2), \tag{10.19}$$

give a point (u_1, u_2, u_3) which lies in one of the two cells (Definition 9.1) which divide a polydisc centred at $(u_1, u_2, u_3) = (0, -\frac{1}{2}h^2, -\frac{1}{2}h^2)$. The Stokes rays are

$$R_{12} = \{z = -i\rho\overline{h}^2 + O(\varepsilon),\ \rho > 0\}, \qquad R_{13} = \{z = -i\rho\overline{h}^2 + O(\varepsilon),\ \rho > 0\},$$

$$R_{23} = \left\{z = \rho \exp\left[-i\left(\frac{\arg h}{2} + \varphi + \pi\right)\right] + O(\varepsilon^2),\ \rho > 0\right\}, \tag{10.20}$$

and opposite ones R_{21}, R_{31}, R_{32}. Notice that in order for the real line ℓ to remain admissible, we choose $\varphi \neq k\pi - \frac{1}{2}\arg h$, $k \in \mathbb{Z}$, $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$. The position of R_{23} w.r.t. the real line ℓ is determined by the sign of $\cos\left(\frac{\arg h}{2} + \varphi + \frac{\pi}{2}\right)$. As long as φ varies in such a way that $\operatorname{sgn}\cos\left(\frac{\arg h}{2} + \varphi + \frac{\pi}{2}\right)$ does not change, then R_{23} does not cross ℓ. See Figure 10.3. This means that (u_1, u_2, u_3) remains inside the same cell, i.e. the point corresponding to coordinates (10.16) remains inside an ℓ-chamber, where the Isomonodromy Theorem 2.12 applies.

(2) The Stokes matrix must be put in triangular form S_{lex} (10.14). In particular,

- if $\cos\left(\frac{\arg h}{2} + \varphi + \frac{\pi}{2}\right) < 0$, then R_{23} is on the *left* of ℓ, and the lexicographical order is given by the permutation $(u_1, u_2, u_3) \mapsto (u'_1, u'_2, u'_3) = (u_2, u_3, u_1)$;

- if $\cos\left(\frac{\arg h}{2}+\varphi+\frac{\pi}{2}\right)>0$, then R_{23} is on the *right* of ℓ, and the lexicographical order is given by the permutation $(u_1,u_2,u_3)\mapsto(u_1',u_2',u_3')=(u_3,u_2,u_1)$.

We choose the cell where the triangular order coincides with the lexicographical order. The passage to the other ℓ-cell is obtained by a counter-clockwise rotation of u_1' w.r.t. u_2', which corresponds to the action of the elementary braid β_{12}. Its action (2.39) is a permutation matrix, since $(S_{\rm lex})_{12}=0$; it is a trivial action on $S_{\rm lex}$, but not on $C_{\rm lex}$, as (10.15) shows.

(3) We move along a curve $h\mapsto he^{i\frac{\pi}{2}}$ in the h-plane from a point (10.16) up to a point

$$(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,\varepsilon e^{i\varphi'},h\right),\quad\text{with }\frac{\pi}{4}<\arg h<\frac{3}{4}\pi,$$

for some $\varphi'\neq k\pi-\frac{1}{2}\arg h$, $k\in\mathbb{Z}$, $\frac{\pi}{4}<\arg h<\frac{3\pi}{4}$. The transformation in Figure 10.1, due to the splitting, can substituted by the sequence of transformations in Figure 10.2, each step corresponding to an elementary braid. Each elementary braid corresponds to a Stokes ray crossing clock-wise the real line ℓ as h varies along the curve $h\mapsto he^{i\frac{\pi}{2}}$.[2] The total braid is then factored into the product of the elementary braids as in Figure 10.4, namely

$$\beta_{12}\beta_{23}\beta_{12},\quad\text{or}\quad\beta_{12}\beta_{23}\beta_{12}\beta_{23}.$$

Applying formulae (2.39),(2.41), we obtain

$$S_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}}=S_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}\beta_{23}}=\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}.\tag{10.21}$$

These are the monodromy data in the two ℓ-cells of a polydisc centred at the point

$$(t_1,t_2,t_3)=\left(-\frac{1}{8}h^2,0,h\right),\quad\text{with }\frac{\pi}{4}<\arg h<\frac{3}{4}\pi.$$

The braid β_{23} is responsible for the passage from one cell to the other. Its action $A^{\beta_{23}}(S_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}})$ is a permutation matrix, since $(S_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}})_{23}=0$, which explains the equality in (10.21). By the action (2.41), the central connection matrix (10.15), instead, assumes the following two forms (differing for a permutation of the second and third column)

$$C_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}}=\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix}(1+i)\Gamma\left(\frac{3}{4}\right)&-i\Gamma\left(\frac{3}{4}\right)&-i\Gamma\left(\frac{3}{4}\right)\\[2ex]0&\pm\sqrt{2\pi}&\mp\sqrt{2\pi}\\[2ex](1-i)\Gamma\left(\frac{1}{4}\right)&i\Gamma\left(\frac{1}{4}\right)&i\Gamma\left(\frac{1}{4}\right)\end{pmatrix},$$

$$C_{\rm lex}^{\beta_{12}\beta_{23}\beta_{12}\beta_{23}}=\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix}(1+i)\Gamma\left(\frac{3}{4}\right)&-i\Gamma\left(\frac{3}{4}\right)&-i\Gamma\left(\frac{3}{4}\right)\\[2ex]0&\mp\sqrt{2\pi}&\pm\sqrt{2\pi}\\[2ex](1-i)\Gamma\left(\frac{1}{4}\right)&i\Gamma\left(\frac{1}{4}\right)&i\Gamma\left(\frac{1}{4}\right)\end{pmatrix}.$$

[2] Notice that the ray R_{23} rotates slower than R_{12},R_{13}: namely, the angular velocity of R_{23} is approximately (i.e. modulo negligible corrections in powers of ε) equal to $\frac{1}{4}$ the one of R_{12},R_{13}.

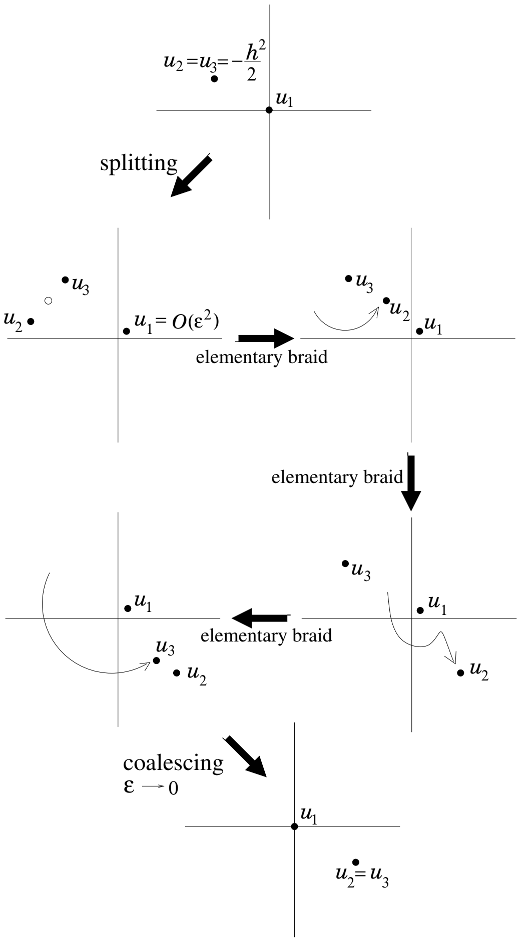


FIGURE 10.2. The transition in Figure 10.1 by splitting and elementary steps. After the splitting, we obtain a point (u_1, u_2, u_3), as in (10.17)-(10.19), lying in an ℓ-cell of the polydisc centred at $(u_1, u_2, u_3) = (0, -\frac{1}{2}h^2, -\frac{1}{2}h^2)$ of the left part of Figure 10.1. The transformation of Figure 10.1 is obtained by successive steps following the arrows. The final step is the right part of Figure 10.1. The first elementary braid is β_{12} (because $u_1' = u_2$, $u_2' = u_3$ in the the upper left figure). The second is β_{23} (after relabelling in lexicographical order, $u_2' = u_2$ and $u_3' = u_1$ in the upper right figure). The third is β_{12}.

In Table 10.2 we show the monodromy data for other values of $\arg h$, with the corresponding braid. In Figure 10.2.3 we represent the braid corresponding to the passage from $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$ to $\frac{5}{4}\pi < \arg h < \frac{7}{4}\pi$.

REMARK 10.3. The reader can re-obtain this result by direct computation observing that, for points

$$(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right), \text{ with } \frac{\pi}{4} < \arg h < \frac{3}{4}\pi,$$

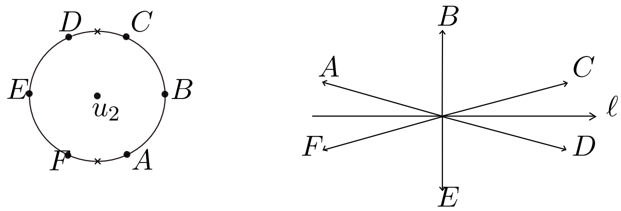


FIGURE 10.3. In the left picture we represent relative positions of u_3 w.r.t u_2 such that the real line ℓ is admissible. On the right, we represent the corresponding positions of the Stokes ray R_{23}. Notice that if we let vary u_3, by a deformation of the parameter φ, starting from A, going through B up to C, the corresponding Stokes ray does not cross the line ℓ, and no braids act. If we continue the deformation of φ from C to D, an elementary braid acts on the monodromy data.

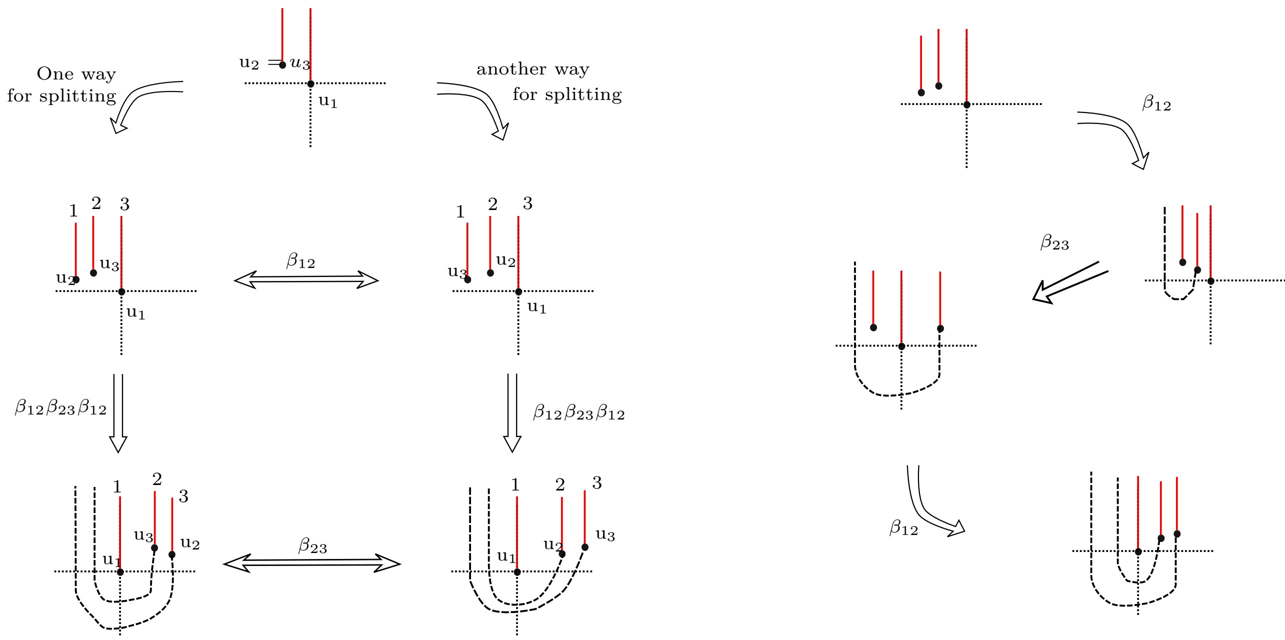


FIGURE 10.4. In the picture we represent u_1, u_2, u_3 as points in $\mathbb{C}$. On the left we describe all the braids necessary to pass from a neighborhood of $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$ with $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$ to one with $\frac{\pi}{4} < \arg h < \frac{3}{4}\pi$. Different columns of this diagram correspond to different ℓ-cells of the same neighborhood. The passage from such one cell to the other is through an action of an elementary braid (β_{12} or β_{23}) acting as a permutation matrix. In the picture on the right, we show the decomposition of the global transformation in elementary ones.

the left and right solutions of (10.6) defining the Stokes matrix[3] are of the form (10.10) with:

$$\xi^L_{(1),1} = \xi^R_{(1),1} = \frac{\sqrt{\pi}}{4\sqrt{2}} h^{\frac{1}{2}} e^{i\frac{\pi}{8}} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(1)}_{\frac{1}{4}}\left(e^{-i\pi}\frac{ih^2}{8}z\right),$$

[3] Notice that for the points with $\frac{\pi}{4} < \arg h < \frac{3}{4}\pi$ the original labelling of canonical coordinates $(u_1, u_2, u_3) = \left(0, -\frac{h^2}{4}, -\frac{h^2}{4}\right)$ already put the Stokes matrix in upper triangular form.

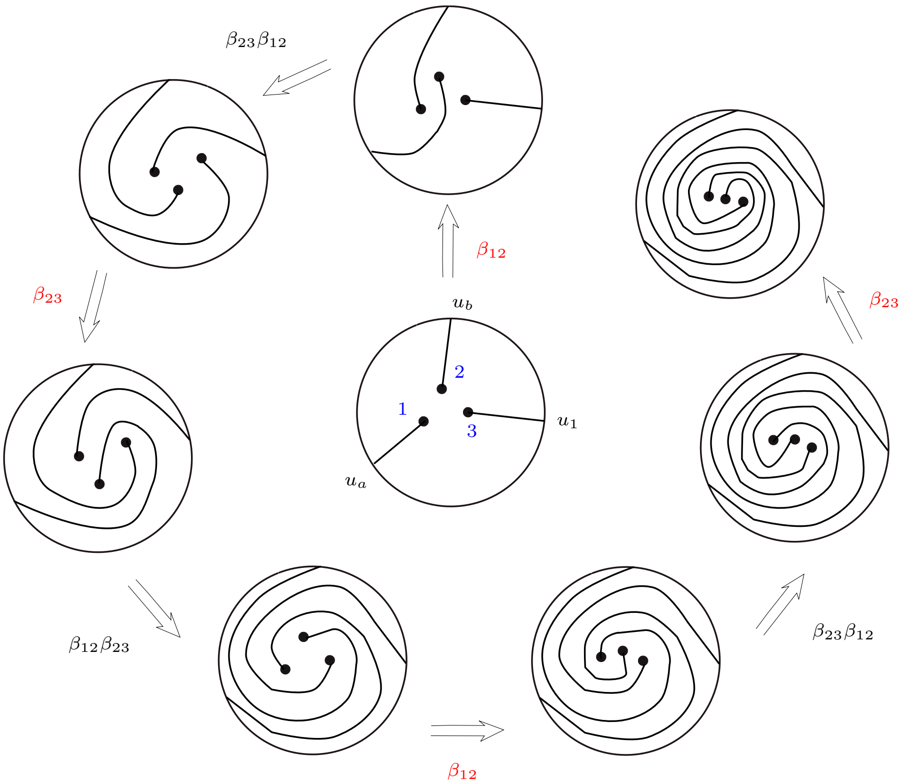


FIGURE 10.5. Using the diagram representation of the braid group as mapping class group of the punctured disk, we draw the braids acting along a curve $h \mapsto e^{\frac{3\pi i}{2}}h$, starting from the chambers close to $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$ with $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$, and reaching the ones with $\frac{5}{4}\pi < \arg h < \frac{7}{4}\pi$. The braids in red describe mutations of the split pair u_2, u_3: their action on the monodromy data is a permutation matrix. In the central disk, the blue numbers refer to the lexicographical order w.r.t. the real axis ℓ (i.e. from the left to the right). The braids are the same for both cases $(a, b) = (2, 3)$ and $(3, 2)$.

$$\xi^L_{(2),1}(z) = \xi^L_{(3),1}(z) = \frac{i\sqrt{\pi}}{8} h^{\frac{1}{2}} e^{i\frac{3}{8}\pi} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(2)}_{\frac{1}{4}}\left(e^{-3i\pi}\frac{ih^2}{8}z\right),$$
$$\xi^R_{(2),1}(z) = \xi^R_{(3),1}(z) = \frac{i\sqrt{\pi}}{8} h^{\frac{1}{2}} e^{i\frac{5}{8}\pi} z^{\frac{1}{2}} e^{-\frac{zh^2}{8}} H^{(1)}_{\frac{1}{4}}\left(\frac{ih^2}{8}z\right),$$

having the expected asymptotic expansions in suitable sectors containing Π_{left} and/or Π_{right} by Lemma 10.1. Thus, by some manipulation of formulae (10.11) and

$$\sin(\nu\pi)H^{(2)}_\nu(ze^{m\pi i}) = e^{\nu\pi i}\sin(m\nu\pi)H^{(1)}_\nu(z) + \sin((m+1)\nu\pi)H^{(2)}_\nu(z), \quad m \in \mathbb{Z},$$

one sees that

$$\xi^L_{(2),1}(z) = \xi^R_{(1),1}(z) + \xi^R_{(2),1}(z), \quad \xi^L_{(3),1}(z) = \xi^R_{(1),1}(z) + \xi^R_{(3),1}(z),$$

which are equivalent to (10.21). For the computation of the central connection matrix, one can use analogous Puiseux series expansions of the solution $\Xi_{\text{right}}(z)$, obtained from the integral representation of Hankel functions given above.

	S_{lex}	C_{lex}	Braid
$-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix} -i\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right) & (1-i)\Gamma\left(\frac{3}{4}\right) \\ \mp\sqrt{2\pi} & \pm\sqrt{2\pi} & 0 \\ i\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right) & (1+i)\Gamma\left(\frac{1}{4}\right) \end{pmatrix}$	β_{12}
$\frac{\pi}{4} < \arg h < \frac{3}{4}\pi$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix} (1+i)\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right) & -i\Gamma\left(\frac{3}{4}\right) \\ 0 & \pm\sqrt{2\pi} & \mp\sqrt{2\pi} \\ (1-i)\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right) & i\Gamma\left(\frac{1}{4}\right) \end{pmatrix}$	$\beta_{12}\beta_{23}\beta_{12}\beta_{23}$
$\frac{3}{4}\pi < \arg h < \frac{5}{4}\pi$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix} \Gamma\left(\frac{3}{4}\right) & \Gamma\left(\frac{3}{4}\right) & (1+i)\Gamma\left(\frac{3}{4}\right) \\ \mp\sqrt{2\pi} & \pm\sqrt{2\pi} & 0 \\ \Gamma\left(\frac{1}{4}\right) & \Gamma\left(\frac{1}{4}\right) & (1-i)\Gamma\left(\frac{1}{4}\right) \end{pmatrix}$	$(\beta_{12}\beta_{23})^3\beta_{12}$
$\frac{5}{4}\pi < \arg h < \frac{7}{4}\pi$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix} (-1+i)\Gamma\left(\frac{3}{4}\right) & \Gamma\left(\frac{3}{4}\right) & \Gamma\left(\frac{3}{4}\right) \\ 0 & \pm\sqrt{2\pi} & \mp\sqrt{2\pi} \\ (-1-i)\Gamma\left(\frac{1}{4}\right) & \Gamma\left(\frac{1}{4}\right) & \Gamma\left(\frac{1}{4}\right) \end{pmatrix}$	$(\beta_{12}\beta_{23})^3\beta_{12}\beta_{23}\beta_{12}\beta_{23}$
$\frac{7}{4}\pi < \arg h < \frac{9}{4}\pi$	$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{\pi^{\frac{1}{2}}}\begin{pmatrix} i\Gamma\left(\frac{3}{4}\right) & i\Gamma\left(\frac{3}{4}\right) & (-1+i)\Gamma\left(\frac{3}{4}\right) \\ \mp\sqrt{2\pi} & \pm\sqrt{2\pi} & 0 \\ -i\Gamma\left(\frac{1}{4}\right) & -i\Gamma\left(\frac{1}{4}\right) & (-1-i)\Gamma\left(\frac{1}{4}\right) \end{pmatrix}$	$(\beta_{12}\beta_{23})^6\beta_{12}$

TABLE 10.2. For different values of $\arg h$, we tabulate the monodromy data $(S_{\text{lex}}, C_{\text{lex}})$, in lexicographical order, in the two ℓ-cells which divide a sufficiently small neighborhood of the point $(t_1, t_2, t_3) = \left(-\frac{1}{8}h^2, 0, h\right)$. The difference of the data in the two ℓ-cells (just a permutation of two columns in the central connection matrix) is obtained by applying the braid written in red: if it is not applied the sign to be read is the first one, the second one otherwise. Notice that the central element $(\beta_{12}\beta_{23})^3$ acts trivially on the Stokes matrices, and by a left multiplication by $M_0^{-1} = \operatorname{diag}(i, 1, -i)$ on the central connection matrix.

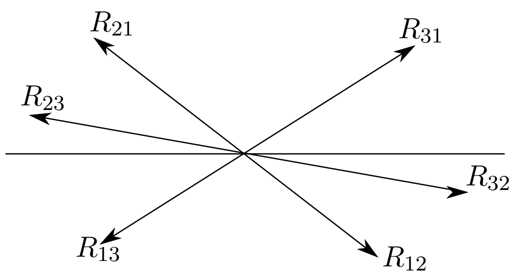


FIGURE 10.6. Disposition of the Stokes rays for a point in the chosen ℓ-chamber.

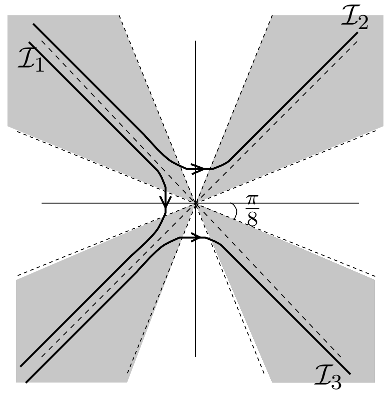


FIGURE 10.7. Integration contours $\mathcal{I}_i$ which define the functions $\mathfrak{I}_i$'s.

10.2.4. Monodromy data as computed outside the Maxwell stratum. In this section, we compute the Stokes matrix S at non-coalesce points in a neighbourhood of a coalescence one, by means of oscillatory integrals. We show that S coincides with that obtained at the coalesce point in the previous section. Moreover, we explicitly show that the fundamental matrices converge to those computed at the coalescence point, exactly as prescribed by our Theorem 9.1.

The system (10.2) admits solutions given in terms of oscillating integrals,

$$\xi_1(z,t) = z^{\frac{1}{2}} \int_\gamma \exp\{z \cdot f(x,t)\}\, dx, \tag{10.22}$$

$$\xi_2(z,t) = z^{\frac{1}{2}} \int_\gamma x \exp\{z \cdot f(x,t)\}\, dx, \tag{10.23}$$

$$\xi_3(z,t) = z^{\frac{1}{2}} \int_\gamma \left(x^2 + \frac{1}{4}t_3\right) \exp\{z \cdot f(x,t)\}\, dx, \tag{10.24}$$

where $f(x,t) = x^4 + t_3x^2 + t_2x + t_1 + \frac{1}{8}t_3^2$. Here γ is any cycle along which $\mathrm{Re}(z \cdot f(x,t)) \to -\infty$ for $|x| \to +\infty$, i.e. a relative cycle in $H_1(\mathbb{C}, \mathbb{C}_{T,z,t})$, with

$$\mathbb{C}_{T,z,t} := \{x \in \mathbb{C}\colon\ \mathrm{Re}(zf(x,t)) < -T\}, \quad \text{with } T \text{ very large positive number.}$$

First, we show that the Stokes matrix at points in ℓ-chambers near the coalescence point $(t_1,t_2,t_3) = (-\frac{1}{8}h^2, 0, h)$ coincide with the one previously computed, in accordance with Theorem 9.1. In what follows we will focus on the ℓ-chamber made of points $(t_1,t_2,t_3) = (-\frac{1}{8}h^2, \varepsilon e^{i\phi}, h)$, where $-\frac{\pi}{4} < \arg h < \frac{\pi}{4}$, and ε, ϕ are small positive numbers. For points in this ℓ-chamber, the Stokes rays are disposed as described in Figure 10.6.

Notice that in order to compute the Stokes matrix at a semisimple point with distinct canonical coordinates it is sufficient to know the first rows of $\Xi_{\text{left/right}}$. Assuming that $z \in \mathbb{R}_+$, we define the following three functions obtained from the integrals (10.22) with integration cycles $\mathcal{I}_i$ as in Figure 10.7:

$$\mathfrak{I}_i(z,t_2) := \int_{\mathcal{I}_i} \exp\left(z(x^4 + hx^2 + t_2x)\right) dx, \quad i = 1,2,3. \tag{10.25}$$

For the specified integration cycles, the integrals $\mathfrak{I}_i(z,t_2)$ are convergent in the half-plane $|\arg z| < \frac{\pi}{2}$. A continuous deformation of a path $\mathcal{I}_i$, which maintains its asymptotic directions in the shaded sectors, yields a convergent integral and defines the analytic continuation of $\mathfrak{I}_i(z,t_2)$ on the whole sector $|\arg z| < \frac{3\pi}{2}$. If we vary z (excluding $z = 0$), the shaded regions continuously rotate clockwise or counterclockwise. In order to obtain the analytic continuation of the functions $\mathfrak{I}_i(z,t_2)$ to the whole universal cover $\mathcal{R}$, we can simply rotate the integration contours $\mathcal{I}_i$. This procedure also makes it clear that the functions $\mathfrak{I}_i$ have monodromy of order 4: indeed as $\arg z$ increases or decreases by 2π, the shaded regions are cyclically permuted.

In order to obtain information about the asymptotic expansions of the functions $\mathfrak{I}_i$, we associate to any critical point x_i a relative cycle $\mathcal{L}_i$, called *Lefschetz thimble*, defined as the set of points of $\mathbb{C}$ which can be reached along the downward geodesic-flow

$$\frac{dx}{d\tau} = -\bar{z}\frac{\partial \bar{f}}{\partial \bar{x}}, \quad \frac{d\bar{x}}{d\tau} = -z\frac{\partial f}{\partial x} \tag{10.26}$$

starting at the critical point x_i for $\tau \to -\infty$. Morse and Picard-Lefschetz Theory guarantees that the cycles $\mathcal{L}_i$ are smooth one dimensional submanifolds of $\mathbb{C}$, *piecewise smoothly* dependent on the parameters z, t, and they represent a basis for the relative homology groups $H_1(\mathbb{C}, \mathbb{C}_{T,z,t})$. Moreover, the Lefschetz thimbles are steepest descent paths: namely, $\mathrm{Im}(zf(x,t))$ is constant on each connected component of $\mathcal{L}_i \setminus \{x_i\}$ and $\mathrm{Re}(zf(x,t))$ is strictly decreasing along the flow (10.26). Thus, after choosing an orientation, the paths of integration defining the functions $\mathfrak{I}_i$ can be expressed as integer combinations of the thimbles $\mathcal{L}_i$ for any value of z:

$$\mathcal{I}_i = n_1\mathcal{L}_1 + n_2\mathcal{L}_2 + n_3\mathcal{L}_3, \quad n_i \in \mathbb{Z}. \tag{10.27}$$

If we let z vary, the Lefschetz thimbles change. When z crosses a Stokes ray, Lefschetz thimbles jump discontinuosly, as shown in Figure 10.8. In particular, for z on a Stokes ray there exists a flow line of (10.26) connecting two critical points x_i's.

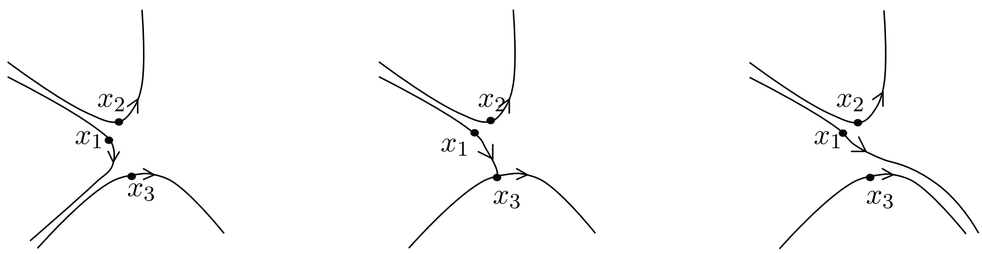


FIGURE 10.8. Discontinuous change of a Lefschetz thimbles. As z varies in $\mathcal{R}$, we pass from the configuration on the left to the one on the right. The middle configuration is realized when z is on a Stokes ray: in this case there is a downward geodesic-flow line connecting two critical points x_1 and x_3.

This discontinuous change of the thimbles implies a discontinuous change of the integer coefficients n_i in (10.27), and a discontinuous change of the leading term of the asymptotic expansions of the functions $\mathfrak{I}_i$'s. Using the notations introduced in Figure 10.9, in each configuration the following identities hold:

$$(A)\colon \begin{cases} \mathcal{I}_1 = \mathcal{L}_1, \\ \mathcal{I}_2 = \mathcal{L}_2, \\ \mathcal{I}_3 = \mathcal{L}_3, \end{cases} \qquad (B)\colon \begin{cases} \mathcal{I}_1 = \mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{I}_2 = \mathcal{L}_2, \\ \mathcal{I}_3 = \mathcal{L}_3, \end{cases} \qquad (C)\colon \begin{cases} \mathcal{I}_1 = \mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{I}_2 = \mathcal{L}_2, \\ \mathcal{I}_3 = -\mathcal{L}_1 + \mathcal{L}_3, \end{cases}$$

$$(D)\colon \begin{cases} \mathcal{I}_1 = \mathcal{L}_1 - \mathcal{L}_3, \\ \mathcal{I}_2 = \mathcal{L}_2, \\ \mathcal{I}_3 = \mathcal{L}_3, \end{cases} \qquad (E)\colon \begin{cases} \mathcal{I}_1 = \mathcal{L}_1 - \mathcal{L}_3, \\ \mathcal{I}_2 = \mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{I}_3 = \mathcal{L}_3. \end{cases}$$

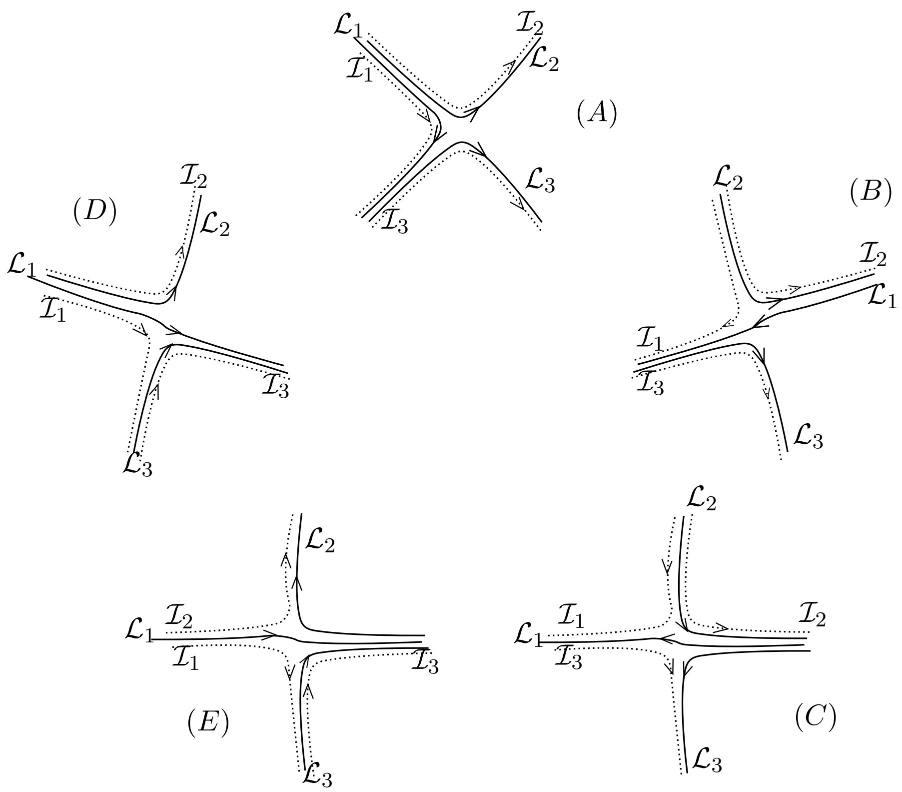


FIGURE 10.9. In this figure it is shown how the Lefschetz thimbles $\mathcal{L}_i$'s (continuous lines), and the integrations contours $\mathcal{I}_i$'s (dotted lines) change by analytic continuation with respect to the variable z. The configuration (A) corresponds to the case $\arg z = 0$. Increasing $\arg z$ the configuration (B) and (C) are reached after crossing the Stokes rays R_{31}, and R_{21} respectively. Decreasing $\arg z$, we obtain the configurations (D) and (E) after crossing the rays R_{12} and R_{13} respectively. Note that when z crosses the Stokes rays R_{32} and R_{23} no Lefschetz thimble changes, coherently with the detailed analysis done in [**CDG17b**].

By a streightforward application of the Laplace method we find that, al least for sufficiently small positive values of $\arg z$, the following asymptotic expansions hold

$$\mathfrak{I}_i(z,t_2) = \pi^{\frac{1}{2}} i z^{-\frac{1}{2}} (6x_i^2 + h)^{-\frac{1}{2}} e^{zu_i} \left(1 + O\left(\frac{1}{z}\right)\right).$$

Since the deformations of the thimbles $\mathcal{I}_2, \mathcal{I}_3$ happen for values of z for which the exponent e^{zu_1} is subdominant, we immediately conclude that the functions

$$\xi^L_{(2),1}(z,t_2) = \xi^R_{(2),1}(z,t_2) = \pm i\pi^{-\frac{1}{2}} z^{\frac{1}{2}} \frac{6x_2^2 + h}{2\sqrt{2}(x_1 - x_2)(x_3 - x_2)} \mathfrak{I}_2(z,t_2), \tag{10.28}$$

$$\xi^L_{(3),1}(z,t_2) = \xi^R_{(3),1}(z,t_2) = \pm i\pi^{-\frac{1}{2}} z^{\frac{1}{2}} \frac{6x_3^2 + h}{2\sqrt{2}(x_1 - x_3)(x_2 - x_3)} \mathfrak{I}_3(z,t_2), \tag{10.29}$$

have asymptotic expansions

$$\Psi_{21} e^{zu_2}\left(1 + O\left(\frac{1}{z}\right)\right), \quad \Psi_{31} e^{zu_3}\left(1 + O\left(\frac{1}{z}\right)\right),$$

respectively, both in $\Pi_{\rm left}$ and $\Pi_{\rm right}$. Thus, we can immediately say that the Stokes matrix computed at a point $(t_1,t_2,t_3)=(-\frac{1}{8}h^2,\varepsilon e^{i\phi},h)$ is of the form

$$S=\begin{pmatrix}1&0&0\\ *&1&0\\ *&0&1\end{pmatrix}.$$

Note that the arbitrariness of the orientations of the Lefschetz thimbles can be incorporated in the choice of the determinations of the entries of the Ψ matrix, and hence it will affect the monodromy data by the action of the group $(\mathbb{Z}/2\mathbb{Z})^3$.

After a careful analysis of the deformations of the Lefschetz thimbles, one finds that the solutions $\xi^L_{(1),1}(z,t_2)$, $\xi^R_{(1),1}(z,t_2)$ are respectively given by

$$\xi^R_{(1),1}(z,t_2)=\pm i\Psi_{11}\pi^{-\frac{1}{2}}z^{\frac{1}{2}}(6x_1^2+h)^{\frac{1}{2}}\left(\mathfrak{I}_1(z,t_2)+\mathfrak{I}_3(z,t_2)\right), \tag{10.30}$$
$$\xi^L_{(1),1}(z,t_2)=\pm i\Psi_{11}\pi^{-\frac{1}{2}}z^{\frac{1}{2}}(6x_1^2+h)^{\frac{1}{2}}\left(\mathfrak{I}_1(z,t_2)-\mathfrak{I}_2(z,t_2)\right), \tag{10.31}$$

having the asymptotic expansion

$$\Psi_{11}e^{zu_1}\left(1+O\left(\frac{1}{z}\right)\right)$$

in $\Pi_{\rm right}$ and $\Pi_{\rm left}$ respectively. This immediately allows one to compute the remaining entries of the Stokes matrix

$$S_{21}=\frac{\Psi_{11}(6x_1^2+h)^{\frac{1}{2}}}{\Psi_{21}(6x_2^2+h)^{\frac{1}{2}}}=\pm\frac{(6x_1^2+h)(x_3-x_2)}{(x_1-x_3)(6x_2^2+h)}\equiv\pm1,$$
$$S_{31}=\frac{\Psi_{11}(6x_1^2+h)^{\frac{1}{2}}}{\Psi_{31}(6x_3^2+h)^{\frac{1}{2}}}=\pm\frac{(6x_1^2+h)(x_2-x_3)}{(x_1-x_2)(6x_3^2+h)}\equiv\pm1.$$

This result is independent on the point $(t_1,t_2,t_3)=(-\frac{1}{8}h^2,\varepsilon e^{i\phi},h)$ of the chosen ℓ-chamber. It coincides with the Stokes matrix obtained at the coalescence point $(t_1,t_2,t_3)=(-\frac{1}{8}h^2,0,h)$, in complete accordance with our Theorem 9.1.

REMARK 10.4. It is interesting to note that the isomonodromy condition in this context is equivalent to the condition

$$\frac{f''(x_1)}{f''(x_2)}=\frac{x_1-x_3}{x_2-x_3},$$

a relation that the reader can easily show to be valid for any polynomial $f(x)$ of fourth degree with three non-degenerate critical points x_1,x_2,x_3.

Our Theorem 9.1 also states that as $t_2\to0$ the solutions (10.28), (10.29), (10.30), (10.31) must converge to the ones computed in the previous section at the coalescence point. We show this explicitly below. In order to do this, it suffices to set $t_2=0$ in the integral (10.25). With the change of variable

$x = 2^{-\frac{1}{4}} z^{-\frac{1}{4}} s^{\frac{1}{2}}$, we obtain

$$\begin{aligned}\mathfrak{I}_2(z,0) = \mathfrak{I}_3(z,0) &= 2^{-\frac{5}{4}} z^{-\frac{1}{4}} \int_L \exp\left\{ \frac{s^2}{2} + \left(\frac{h z^{\frac{1}{2}}}{\sqrt{2}} \right) s \right\} ds \\ &= 2^{-\frac{5}{4}} z^{-\frac{1}{4}} (2\pi)^{\frac{1}{2}} e^{-\frac{h^2 z}{8}} D_{-\frac{1}{2}} \left(\frac{h z^{\frac{1}{2}}}{\sqrt{2}} \right) \\ &= 2^{-\frac{3}{2}} e^{-\frac{h^2 z}{8}} h^{\frac{1}{2}} K_{\frac{1}{4}} \left(\frac{h^2 z}{8} \right) \\ &= \pi i \cdot 2^{-\frac{5}{2}} h^{\frac{1}{2}} e^{-\frac{h^2 z}{8}} e^{\frac{\pi i}{8}} H^{(1)}_{\frac{1}{4}} \left(\frac{i h^2 z}{8} \right).\end{aligned}$$

Here $D_\nu(z)$ is the Weber parabolic cylinder function of order ν, with integral representation ([**AS70**], page 688)

$$D_{-\frac{1}{2}}(z) = \pm \frac{e^{\frac{1}{2}z^2}}{(2\pi)^{\frac{1}{2}}} \int_L s^{-\frac{1}{2}} \exp\left(\frac{s^2}{2} + zs \right) ds, \quad \text{where} \begin{cases} (+) \text{ if } -\frac{3\pi}{2} + 2k\pi < \arg s < -\frac{\pi}{2} + 2k\pi, \\ (-) \text{ if } \frac{\pi}{2} + 2k\pi < \arg s < \frac{3\pi}{2} + 2k\pi, \end{cases}$$

the integration contour L being the one represented in Figure 10.10, together with the identities

$$D_{-\frac{1}{2}}(z) = \left(\frac{z}{2\pi} \right)^{\frac{1}{2}} K_{\frac{1}{4}} \left(\frac{1}{4} z^2 \right), \quad K_\nu(z) = \begin{cases} \frac{\pi i}{2} e^{\frac{\nu \pi i}{2}} H^{(1)}_\nu \left(z e^{\frac{\pi i}{2}} \right), \\ -\frac{\pi i}{2} e^{-\frac{\nu \pi i}{2}} H^{(2)}_\nu \left(z e^{-\frac{\pi i}{2}} \right). \end{cases}$$

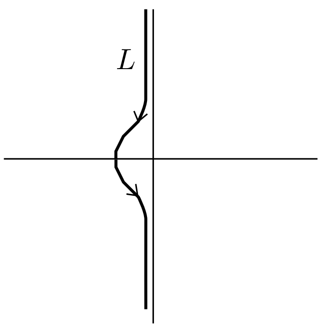


FIGURE 10.10. Integration contour L used in the integral representation of the Weber parabolic cylinder functions.

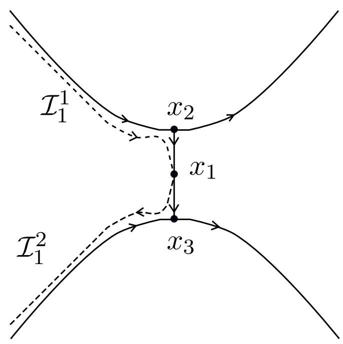


FIGURE 10.11. For $t_2 = 0$, we can decompose the integration cycle $\mathcal{I}_1$ into two pieces, $\mathcal{I}_1^1, \mathcal{I}_1^2$ used to define the functions $\mathfrak{I}_1^1$ and $\mathfrak{I}_1^2$. The continuous lines represent the Lefschetz thimbles through the critical points x_i's.

It follows that

$$\begin{aligned}\xi^L_{(2),1}(z,0)=\xi^R_{(2),1}(z,0)&=\pm i\pi^{-\frac{1}{2}}z^{\frac{1}{2}}\frac{6x_2^2+h}{2\sqrt{2}(x_1-x_2)(x_3-x_2)}\mathfrak{I}_2(z,0)\\&=\pm\frac{i\sqrt{\pi}}{8}h^{\frac{1}{2}}e^{\frac{5i\pi}{8}}z^{\frac{1}{2}}e^{-\frac{h^2z}{8}}H^{(1)}_{\frac{1}{4}}\left(\frac{ih^2z}{8}\right),\end{aligned}$$

which coincides (up to an irrelevant sign) with the solution computed in the previous section at the coalescence point. The computations for $\xi^L_{(3),1}(z,0)=\xi^R_{(3),1}(z,0)$ are identical.

The computations for $\xi^R_{(1),1}$ and $\xi^L_{(1),1}$ are a bit more laborious. First of all let us observe that the integral

$$g(z):=\int_0^\infty\exp\left(-\frac{t^2}{2}-zt\right)t^{-\frac{1}{2}}dt$$

is convergent for all $z\in\mathbb{C}$, defining an entire function[4]. Moreover we have

$$g(z)=\sqrt{\pi}e^{\frac{z^2}{4}}D_{-\frac{1}{2}}(z)=2^{-\frac{1}{2}}e^{\frac{z^2}{4}}z^{\frac{1}{2}}K_{\frac{1}{4}}\left(\frac{z^2}{4}\right).$$

With a change of variable $t=e^{-i\theta}\tau$ that rotates the half line $\mathbb{R}_+$ by θ, we find the following identity

$$g(z)=e^{-i\frac{\theta}{2}}\int_{e^{i\theta}\mathbb{R}_+}\exp\left(-e^{-2i\theta}\frac{\tau^2}{2}-e^{-i\theta}z\tau\right)\tau^{-\frac{1}{2}}d\tau. \tag{10.32}$$

For $t_2=0$ the integral $\mathfrak{I}_1(z,0)$ splits into two pieces:

$$\mathfrak{I}_1(z,0)=\mathfrak{I}_1^1(z)+\mathfrak{I}_1^2(z),\quad \mathfrak{I}_1^i(z):=\int_{\mathcal{I}_1^i}\exp(z(x^4+hx^2))dx,\quad i=1,2$$

where the paths $\mathcal{I}_1^i$ are as in Figure 10.11. Setting $x=2^{-\frac{1}{4}}z^{-\frac{1}{4}}s^{\frac{1}{2}}$, the image of the paths $\mathcal{I}_1^i$ are in two different sheets of the Riemann surface with local coordinate s. Keeping track of this, and of the orientations of the modified paths, using formula (10.32) for $\theta=\frac{3\pi i}{2},\frac{5\pi i}{2}$ and a small deformation of the paths of integration, we find that

$$\begin{aligned}\mathfrak{I}_1^1(z)&=2^{-\frac{5}{4}}z^{-\frac{1}{4}}\left(-\int_{e^{\frac{3\pi i}{2}}\mathbb{R}_+}\exp\left\{\frac{s^2}{2}+\frac{hz^{\frac{1}{2}}}{\sqrt{2}}s\right\}s^{-\frac{1}{2}}ds\right)\\&=-2^{-\frac{5}{4}}z^{-\frac{1}{4}}e^{\frac{3\pi i}{4}}g\left(e^{\frac{\pi i}{2}}\frac{hz^{\frac{1}{2}}}{\sqrt{2}}\right)\\&=-2^{-\frac{5}{4}}z^{-\frac{1}{4}}e^{\frac{3\pi i}{4}}\cdot 2^{-\frac{1}{2}}e^{-\frac{h^2z}{8}}\left(e^{\frac{\pi i}{2}}\frac{hz^{\frac{1}{2}}}{\sqrt{2}}\right)^{\frac{1}{2}}K_{\frac{1}{4}}\left(e^{\pi i}\frac{h^2z}{8}\right)=\frac{1}{4}e^{-\frac{h^2z}{8}}h^{\frac{1}{2}}K_{\frac{1}{4}}\left(e^{\pi i}\frac{h^2z}{8}\right),\end{aligned}$$

[4]This is in accordance with the expression of g in terms of the modified Bessel function K, which gives

$$g(e^{\pm\pi i}z)=2^{-\frac{1}{2}}e^{\frac{z^2}{4}}e^{\pm\frac{\pi i}{2}}z^{\frac{1}{2}}K_{\frac{1}{4}}\left(e^{\pm 2\pi i}\frac{z^2}{4}\right).$$

From the symmetry $K_{\frac{1}{4}}(e^{4\pi i}z)=-K_{\frac{1}{4}}(z)$ we deduce that $g(e^{-\pi i}z)=g(e^{\pi i}z)$.

and

$$\begin{aligned}\mathfrak{I}_1^2(z) &= 2^{-\frac{5}{4}}z^{-\frac{1}{4}}\left(\int_{e^{\frac{5\pi i}{2}}\mathbb{R}_+}\exp\left\{\frac{s^2}{2}+\frac{hz^{\frac{1}{2}}}{\sqrt{2}}s\right\}s^{-\frac{1}{2}}ds\right)\\ &= 2^{-\frac{5}{4}}z^{-\frac{1}{4}}e^{\frac{5\pi i}{4}}f\left(e^{-\frac{\pi i}{2}}\frac{hz^{\frac{1}{2}}}{\sqrt{2}}\right)\\ &= 2^{-\frac{5}{4}}z^{-\frac{1}{4}}e^{\frac{5\pi i}{4}}\cdot 2^{-\frac{1}{2}}e^{-\frac{h^2z}{8}}\left(e^{-\frac{\pi i}{2}}\frac{hz^{\frac{1}{2}}}{\sqrt{2}}\right)^{\frac{1}{2}}K_{\frac{1}{4}}\left(e^{-\pi i}\frac{h^2z}{8}\right) = -\frac{1}{4}e^{-\frac{h^2z}{8}}h^{\frac{1}{2}}K_{\frac{1}{4}}\left(e^{-\pi i}\frac{h^2z}{8}\right).\end{aligned}$$

Thus, in the limit $t_2 = 0$ we find that

$$\begin{aligned}\xi^R_{(1),1}(z,0) &= \pm i\Psi_{11}\pi^{-\frac{1}{2}}z^{\frac{1}{2}}(6x_1^2+h)^{\frac{1}{2}}\left(\mathfrak{I}_1^1(z)+\mathfrak{I}_1^2(z)+\mathfrak{I}_3(z,0)\right)\\ &= \pm i2^{-\frac{5}{2}}\pi^{-\frac{1}{2}}z^{\frac{1}{2}}e^{-\frac{h^2z}{8}}h^{\frac{1}{2}}\left\{K_{\frac{1}{4}}\left(e^{i\pi}\frac{h^2z}{8}\right)-K_{\frac{1}{4}}\left(e^{-i\pi}\frac{h^2z}{8}\right)+2^{\frac{1}{2}}K_{\frac{1}{4}}\left(\frac{h^2z}{8}\right)\right\}\\ &= \pm\pi^{\frac{1}{2}}z^{\frac{1}{2}}2^{-\frac{7}{2}}e^{-\frac{h^2z}{8}}h^{\frac{1}{2}}e^{-\frac{\pi i}{8}}\left\{H^{(2)}_{\frac{1}{4}}\left(e^{\frac{i\pi}{2}}\frac{h^2z}{8}\right)+e^{\frac{\pi i}{4}}H^{(1)}_{\frac{1}{4}}\left(e^{-\frac{i\pi}{2}}\frac{h^2z}{8}\right)+2^{\frac{1}{2}}H^{(2)}_{\frac{1}{4}}\left(e^{-\frac{i\pi}{2}}\frac{h^2z}{8}\right)\right\}\\ &= \pm\pi^{\frac{1}{2}}2^{-\frac{3}{2}}z^{\frac{1}{2}}e^{-\frac{h^2z}{8}}h^{\frac{1}{2}}e^{-\frac{\pi i}{8}}H^{(2)}_{\frac{1}{4}}\left(e^{\frac{i\pi}{2}}\frac{h^2z}{8}\right),\end{aligned}$$

which is exactly (modulo irrelevant signs) the solution at the coalescence point as computed in the previous section. We leave as an exercise for the industrious reader to show that all the other solutions $\xi^{R/L}_{(i),j}(z)$ converge to the ones computed at the coalescence point.

10.3. Reformulation of results for PVI$_\mu$ transcendents

Our result can be reinterpreted as an alternative and simpler approach w.r.t. Jimbo's procedures ([**Jim82**], [**DM00**] and see also [**Kan06**] and [**Guz06**]) for the computations of monodromy data of holomorphic branches of PVI$_\mu$ transcendents. Let us briefly recall how the WDVV problem for $n = 3$ is equivalent to Painlevé equations (see [**Dub96**],[**Dub99b**]): from equation (1.7), and the skew-symmetry of $V(u)$, one obtains that $\sum_i \partial_i V = \sum_i u_i\partial_i V = 0$, which imply the following functional form for V

$$V(u_1,u_2,u_3)\equiv V(t),\quad t:=\frac{u_2-u_1}{u_3-u_1},\quad V(t)=\begin{pmatrix}0 & \Omega_2(t) & -\Omega_3(t)\\ -\Omega_2(t) & 0 & \Omega_1(t)\\ \Omega_3(t) & -\Omega_1(t) & 0\end{pmatrix}.$$

Because of this functional form, for points with (u_1,u_2,u_3) pairwise distinct, i.e. $t\notin\{0,1,\infty\}$, the equation (1.5) can be rewritten as

$$\frac{dY}{dz}=\left(\begin{pmatrix}0&0&0\\0&t&0\\0&0&1\end{pmatrix}+\frac{V(t)}{z}\right)Y,$$

and compatibility conditions (1.7) read

$$\frac{d\Omega_1}{dt}=\frac{1}{t}\Omega_2\Omega_3,\quad \frac{d\Omega_2}{dt}=\frac{1}{1-t}\Omega_1\Omega_3,\quad \frac{d\Omega_3}{dt}=\frac{1}{t(1-t)}\Omega_1\Omega_2.$$

The function $V(t)$ can thus be expressed in terms of transcendents $y(t)$ satisfying the following Painlevé VI equation, called for brevity PVI$_\mu$,

$$\frac{d^2y}{dt^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right]\left(\frac{dy}{dt}\right)^2-\left[\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right]\frac{dy}{dt}+\frac{1}{2}\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[(2\mu-1)^2+\frac{t(t-1)}{(y-t)^2}\right],$$

where $\mu\in\mathbb{C}$ is a parameter such that $\{-\mu,0,\mu\}$ is the spectrum of $V(t)$. In particular, as shown in [**Guz01**], one has the explicit relations

$$\Omega_1(t)=i\frac{(y(t)-1)^{\frac{1}{2}}(y(t)-t)^{\frac{1}{2}}}{t^{\frac{1}{2}}}\left[\frac{A(t)}{(y(t)-1)(y(t)-t)}+\mu\right],\quad \Omega_2(t)=i\frac{y(t)^{\frac{1}{2}}(y(t)-t)^{\frac{1}{2}}}{(1-t)^{\frac{1}{2}}}\left[\frac{A(t)}{y(t)(y(t)-t)}+\mu\right],$$

$$\Omega_3(t)=-\frac{y(t)^{\frac{1}{2}}(y(t)-1)^{\frac{1}{2}}}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}\left[\frac{A(t)}{y(t)(y(t)-1)+\mu}\right],\quad A(t):=\frac{1}{2}\left[\frac{dy}{dt}t(t-1)-y(t)(y(t)-1)\right].$$

The "Painlevé transcendent" corresponding to the A_3-Frobenius manifold is the following algebraic solution of PVI$_{\frac{1}{4}}$ obtained in [**DM00**] (there is a missprint in $t(s)$ of [**DM00**]):

$$y(s)=\frac{(1-s)^2(1+3s)(9s^2-5)^2}{(1+s)(243s^6+1539s^4-207s^2+25)},\quad t(s)=\frac{(1-s)^3(1+3s)}{(1+s)^3(1-3s)}. \tag{10.33}$$

As it is shown in [**DM00**], the Jimbo's monodromy data of the Jimbo-Miwa-Ueno isomonodromic Fuchsian system associated with algebraic solutions of PVI$_\mu$ are $\mathrm{tr}(M_iM_j)=2-S_{ij}^2$, $1\leq i<j\leq 3$, where S is the Stokes matrix (in upper triangular form) of the corresponding Frobenius manifold. S is well known [**Dub96**], and $S+S^T$ is the Coxeter matrix of the reflection group A_3. Moreover, Jimbo's isomonodromic method [**Jim82**], as applied in [**DM00**] (see also [**Kan06**], [**Guz06**] for holomorphic solutions), provides $\mathrm{tr}(M_iM_j)$. The computations of this Chapter show that the application of Theorem 8.2 and Theorem 9.1 represents an alternative, and probably simpler, way for obtaining S.

CHAPTER 11

Quantum cohomology of the Grassmannian $\mathbb{G}(2,4)$ and its Monodromy Data

ABSTRACT. In this Chapter we consider the problem of computing the monodromy data of the (small) quantum cohomology of the complex Grassmannian $\mathbb{G}(2,4)$. This is the simplest case among all complex Grassmannians in which the coalescence phenomenon described in Chapter 4 manifests. After reducing the problem to the study of a generalized hypergeometric equation, an asymptotic analysis of its solutions in Mellin-Barnes form is carried on. This allows us to partially reconstruct both left and right solutions defining the Stokes matrices. By computing the first dominant terms of the asymptotic expansion of the topological-enumerative solution for $\mathbb{G}(2,4)$, we show how an application of constraints of Theorem 2.11 allows us to complete the computation of the monodromy data. It is shown that both the Stokes matrix and the Central connection matrix have a geometrical meaning in terms of objects of an explicit mutation of the Kapranov exceptional collection in $\mathcal{D}^b(\mathbb{G}(2,4))$.

We explicitly compute the monodromy data for the Frobenius manifold known as Quantum cohomology of the Grassmannian $\mathbb{G}(2,4)$. This manifold has a locus of coalescent semisimple points, known as small quantum cohomology, where Theorem 9.1 can be applied. This explicit computation seems to be missing from the literature. It is important to remark that the result, obtained by *analytic methods* and in completely *explicit way*, sheds new light on a conjecture of B. Dubrovin (formulated by B. Dubrovin in [**Dub98**], and then refined in [**Dub13**]), and on the strictly related Γ-conjectures of S. Galkin, V. Golyshev and H. Iritani ([**GGI16**, **GI15**]), in the case the quantum cohomology of the Grassmannian $\mathbb{G}(2,4)$ (Theorem 11.2 below, or Theorem 1.8). The importance of Theorem 9.1 is now clear.

11.1. Small Quantum Cohomology of $\mathbb{G}(2,4)$

11.1.1. Generalities and proof of its Semisimplicity. For simplicity, let us use the notation $\mathbb{G} := \mathbb{G}(2,4)$. From the general theory of Schubert Calculus exposed in previous chapter, it is known that $H^\bullet(\mathbb{G};\mathbb{C})$ is a complex vector space of dimension 6, and a basis is given by *Schubert classes*:

$$\sigma_0 := 1,\ \sigma_1,\ \sigma_2,\ \sigma_{1,1},\ \sigma_{2,1},\ \sigma_{2,2}$$

where σ_λ is a generator of $H^{2|\lambda|}(\mathbb{G};\mathbb{C})$. By posing

$$v_1 := \sigma_0,\ v_2 := \sigma_1,\ v_3 := \sigma_2,\ v_4 := \sigma_{1,1},\ v_5 := \sigma_{2,1},\ v_6 := \sigma_{2,2},$$

we will denote by t^i the coordinates with respect to v_i. The coordinates in the small quantum cohomology are

$$t = (0, t^2, 0, ..., 0).$$

By Pieri-Bertram and Giambelli formulas one finds that the matrix of the Poincaré pairing

$$\eta(\alpha,\beta) := \int_{\mathbb{G}} \alpha \wedge \beta$$

with respect to the basis above, is given by

$$\eta = \begin{pmatrix} 0&0&0&0&0&c\\ 0&0&0&0&c&0\\ 0&0&c&0&0&0\\ 0&0&0&c&0&0\\ 0&c&0&0&0&0\\ c&0&0&0&0&0 \end{pmatrix}, \quad c := \int_{G(2,\mathbb{C}^4)} \sigma_{2,2}.$$

Using Pieri-Bertram formula we deduce that the multiplication matrix of the operator of multiplication by $\lambda\sigma_1 + \mu\sigma_{1,1}$ is

$$\begin{pmatrix} 0&0&\mu q&0&\lambda q&0\\ \lambda&0&0&0&\mu q&\lambda q\\ 0&\lambda&0&0&0&\mu q\\ \mu&\lambda&0&0&0&0\\ 0&\mu&\lambda&\lambda&0&0\\ 0&0&0&\mu&\lambda&0 \end{pmatrix}, \qquad q := e^{t^2}. \tag{11.1}$$

The discriminant of the characteristic polynomial of this matrix is

$$16777216\ \lambda^4\mu^2q^8(\lambda^4+\mu^4q)^6$$

and so, if $\lambda \neq 0, \mu \neq 0$ and $\lambda^4 + q\mu^4 \neq 0$, its eigenvalues are pairwise distinct. This is a sufficient condition to state that the quantum cohomology of $G(2,\mathbb{C}^4)$ is semisimple.
Notice that the value at the point p of coordinates $(0,t^2,0,\dots,0)$ of the Euler field of quantum cohomology $QH^\bullet(G(2,\mathbb{C}^4))$ is [1] given by the first Chern class $c_1(\mathbb{G}) = 4\sigma_1$:

$$E|_p = 4\frac{\partial}{\partial t^2} \equiv 4\sigma_1.$$

The matrix $\mathcal{U}$ of multiplication by E at the point p is given by posing $\lambda = 4$, $\mu = 0$ in (11.1):

$$\mathcal{U}(0,t^2,0,\dots,0) \equiv 4\ \mathcal{C}_2(0,t^2,0,\dots,0) = \begin{pmatrix} 0&0&0&0&4q&0\\ 4&0&0&0&0&4q\\ 0&4&0&0&0&0\\ 0&4&0&0&0&0\\ 0&0&4&4&0&0\\ 0&0&0&0&4&0 \end{pmatrix}.$$

The characteristic polynomial is $p(z) = z^6 - 1024qz^2$, so that 0 is an eigenvalues with multiplicity 2. Therefore, the semisimple points with coordinates $(0,t^2,0,\dots,0)$ are *semisimple coalescence points in the bifurcation set.*

11.1.2. Idempotents at the points $(0,t^2,0,\dots,0)$. The multiplication by $\sigma_1+\sigma_{1,1}$ has pairwise distinct eigenvalues, at least at points for which $t^2 \neq i\pi(2k+1)$. Putting $\lambda = \mu = 1$ in (11.1), we deduce that the characteristic polynomial of this operator is

$$p(z) = (q+z^2)(-4q+q^2-8qz-2qz^2+z^4).$$

So the six eigenvalues are

$$iq^{\frac12}, \quad -iq^{\frac12},$$

$$\varepsilon_1 := -i\sqrt{2}q^{\frac14} - q^{\frac12}, \quad \varepsilon_2 := i\sqrt{2}q^{\frac14} - q^{\frac12}, \quad \varepsilon_3 := -\sqrt{2}q^{\frac14} + q^{\frac12}, \quad \varepsilon_4 := \sqrt{2}q^{\frac14} + q^{\frac12},$$

[1]We identify $T_pH^\bullet(\mathbb{G})$ with $H^\bullet(\mathbb{G})$ in the canonical way.

and the corresponding eigenvectors are

$$\begin{aligned}
&\pi_1 := -q - iq^{\frac{1}{2}}\sigma_2 + iq^{\frac{1}{2}}\sigma_{1,1} + \sigma_{2,2}, \qquad \pi_2 := -q + iq^{\frac{1}{2}}\sigma_2 - iq^{\frac{1}{2}}\sigma_{1,1} + \sigma_{2,2},\\
&\pi_{2+i} :=(q^2 + q\varepsilon_i^2) + (-q^2 + 2q\varepsilon_i + q\varepsilon_i^2)\sigma_1 + (2q + 2q\varepsilon_i)\sigma_2 + (2q + 2q\varepsilon_i)\sigma_{1,1}\\
&\qquad + (-2q - q\varepsilon_i + \varepsilon_i^3)\sigma_{2,1} + (q + \varepsilon_i^2)\sigma_{2,2}.
\end{aligned}$$

Then,

$$\pi_i \cdot \pi_j = 0 \ \text{ if } i \neq j, \quad \pi_i^2 = \lambda_i \pi_i \ \text{ where } \lambda_i > 0;$$

as a consequence, these vectors are orthogonal, since $\eta(\pi_i, \pi_j) = \eta(\pi_i \cdot \pi_j, 1) = \eta(0, 1) = 0$. Introducing the normalized eigenvectors

$$f_i := \frac{\pi_i}{\eta(\pi_i, \pi_i)^{\frac{1}{2}}}$$

we obtain an orthonormal frame of idempotent vectors, for any choice of the sign of the square roots. Let us now introduce a matrix $\Psi = (\psi_{ij})$ such that

$$\frac{\partial}{\partial t_\alpha} = \sum_i \psi_{i\alpha} f_i, \qquad \alpha = 1, 2, ..., n.$$

Note that necessarily we have

$$\Psi^T \Psi = \eta, \quad \psi_{i1} = \frac{\eta(\pi_i, 1)}{\eta(\pi_i, \pi_i)^{\frac{1}{2}}}.$$

After some computations, we obtain

$$\Psi = \frac{c^{\frac{1}{2}}}{2} \begin{pmatrix}
-iq^{-\frac{1}{2}} & 0 & -1 & 1 & 0 & iq^{\frac{1}{2}}\\
-iq^{-\frac{1}{2}} & 0 & 1 & -1 & 0 & iq^{\frac{1}{2}}\\
\frac{1}{\sqrt{2}q^{\frac{1}{2}}} & -\frac{i}{q^{\frac{1}{4}}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & iq^{\frac{1}{4}} & \frac{q^{\frac{1}{2}}}{\sqrt{2}}\\
\frac{1}{\sqrt{2}q^{\frac{1}{2}}} & \frac{i}{q^{\frac{1}{4}}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -iq^{\frac{1}{4}} & \frac{q^{\frac{1}{2}}}{\sqrt{2}}\\
\frac{1}{\sqrt{2}q^{\frac{1}{2}}} & -\frac{1}{q^{\frac{1}{4}}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -q^{\frac{1}{4}} & \frac{q^{\frac{1}{2}}}{\sqrt{2}}\\
\frac{1}{\sqrt{2}q^{\frac{1}{2}}} & \frac{1}{q^{\frac{1}{4}}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & q^{\frac{1}{4}} & \frac{q^{\frac{1}{2}}}{\sqrt{2}}
\end{pmatrix}.$$

This matrix diagonalizes $\mathcal{U}$ as follows

$$U := \Psi \mathcal{U} \Psi^{-1} = (\Psi^T)^{-1} \widehat{\mathcal{U}} \Psi^T =$$

$$= \begin{pmatrix}
u_1 & & & & & \\
& u_2 & & & & \\
& & u_3 & & & \\
& & & u_4 & & \\
& & & & u_5 & \\
& & & & & u_6
\end{pmatrix} = 4\sqrt{2}q^{\frac{1}{4}} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -i & 0 & 0 & 0\\
0 & 0 & 0 & i & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The eigenvalues u_i stand for $u_i(0, t^2, .., 0)$. Note that

$$u_i(0, t^2, 0, ..., 0) = q^{\frac{1}{4}} u_i(0, 0, ..., 0) = e^{\frac{t^2}{4}} u_i(0, 0, ..., 0). \tag{11.2}$$

11.1.3. Differential system expressing the Flatness of the Deformed Connection. The matrix μ is given by

$$\mu = \mathrm{diag}\Big(-2,-1,0,0,1,2\Big), \qquad \text{with eigenvalues } \mu_\alpha = \frac{\deg(\partial/\partial_\alpha)-4}{2}, \quad 1 \le \alpha \le 6. \tag{11.3}$$

Consider the system (2.2), rewritten as follows:

$$\partial_z \xi = (\widehat{\mathcal{U}} - \frac{1}{z}\mu)\ \xi \tag{11.4}$$

$$\partial_2 \xi = z\ \widehat{\mathcal{C}_2}\ \xi \tag{11.5}$$

where ξ is a column vector, whose components are $\xi_i = \partial_i \tilde{t}(t,z)$ (derivatives of a deformed flat coordinate), and

$$\widehat{\mathcal{U}} := \eta\, \mathcal{U}\ \eta^{-1} = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 4q & 0 & 0 & 0 & 0 & 4 \\ 0 & 4q & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widehat{\mathcal{C}_2} \equiv \frac{1}{4}\,\widehat{\mathcal{U}}.$$

Introducing a new function ϕ defined by

$$\phi(t,z) := \frac{\xi_1(t,z)}{z^2}$$

the first equation of the system becomes a single scalar partial differential equation

$$z^4\partial_z^5\phi + 10z^3\partial_z^4\phi + 25z^2\partial_z^3\phi + 15z\partial_z^2\phi + (1-1024qz^4)\partial_z\phi - 2048qz^3\phi = 0 \tag{11.6}$$

and the solution can be reconstructed from

$$\begin{aligned} \xi_1 &= z^2\phi \\ \xi_2 &= \frac{1}{4}z^2\partial_z\phi \\ \xi_3 &= \frac{1}{32}(z\partial_z\phi + z^2\partial_z^2\phi) + h \\ \xi_4 &= \frac{1}{32}(z\partial_z\phi + z^2\partial_z^2\phi) - h \\ \xi_5 &= \frac{1}{128}(\partial_z\phi + 3z\partial_z^2\phi + z^2\partial_z^3\phi) \\ \xi_6 &= \frac{1}{512}(-512qz^2\phi + \frac{1}{z}\partial_z\phi + 7\partial_z^2\phi + 6z\partial_z^3\phi + z^2\partial_z^4\phi) \end{aligned} \tag{11.7}$$

with h constant.

Remark 11.1. The third and the fourth equations follow from the fact that

$$\xi_3 + \xi_4 = \frac{1}{16}(z\partial_z\phi + z^2\partial_z^2\phi), \quad \partial_z(\xi_3 - \xi_4) = 0, \quad \partial_2(\xi_3 - \xi_4) = 0$$

so that $\xi_3 - \xi_4 = 2h$ constant.

From system (11.5) it follows that

$$\partial_2\phi = \frac{z}{4}\partial_z\phi$$

which implies the following functional form:

$$\phi(t_2, z) = \Phi\left(zq^{\frac{1}{4}}\right).$$

As a consequence, our problem (11.6) reduces to the solution of a single scalar ordinary differential equation for a function $\Phi(w)$, $w = zq^{\frac{1}{4}}$:

$$w^4\Phi^{(5)} + 10w^3\Phi^{(4)} + 25w^2\Phi^{(3)} + 15w\Phi'' + (1-1024w^4)\Phi' - 2048w^3\Phi = 0.$$

Multiplying by $w \in \mathbb{C}^*$, we can rewrite this equation in a more compact form:

$$\Theta^5\Phi - 1024w^4\Theta\Phi - 2048w^4\Phi = 0 \tag{11.8}$$

where Θ is the Euler's differential operator $w\frac{d}{dw}$. Moreover, defining $\varpi := 4w^4$, and writing $\Phi(w) = \tilde{\Phi}(\varpi)$, we can rewrite the equation in the form

$$\Theta_\varpi^5\tilde{\Phi} - \varpi\Theta_\varpi\tilde{\Phi} - \frac{1}{2}\varpi\tilde{\Phi} = 0 \qquad \Theta_\varpi := \varpi\frac{d}{d\varpi} = \frac{1}{4}\Theta_w.$$

11.1.4. Expected Asymptotic Expansions. Let Ξ be a fundamental matrix solution of system (11.4), and let Y be defined by

$$\Xi = \eta\Psi^{-1}Y. \tag{11.9}$$

Then, Y is a fundamental solution of system (2.26). The asymptotic theory for such Y's has been explained in Section 2.2.1, and Theorem 2.9 applies. To the formal solution (2.27), there corresponds a formal matrix solution

$$\Xi_{\text{formal}} = \eta\Psi^{-1}G(z)^{-1}e^{zU}.$$

To the fundamental solutions $Y_{\text{left/right}}$, there correspond solutions $\Xi_{\text{left/right}}$. For fixed t^2, then $\Xi_{\text{left/right}}(t^2, z) \equiv \Xi_{\text{left/right}}(e^{\frac{t^2}{4}}z)$ has the following asymptotic expansion for $z \to \infty$

$$\Xi_{\text{left/right}}(t^2, z) = \eta\Psi^{-1}\left(1 + O\left(\frac{1}{z}\right)\right)e^{zU} =$$

$$= \frac{c^{\frac{1}{2}}}{2}\left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} -\frac{ie^{zu_1}}{q^{\frac{1}{2}}} & -\frac{ie^{zu_2}}{q^{\frac{1}{2}}} & \frac{e^{zu_3}}{\sqrt{2}q^{\frac{1}{2}}} & \frac{e^{zu_4}}{\sqrt{2}q^{\frac{1}{2}}} & \frac{e^{zu_5}}{\sqrt{2}q^{\frac{1}{2}}} & \frac{e^{zu_6}}{\sqrt{2}q^{\frac{1}{2}}} \\ 0 & 0 & -\frac{ie^{zu_3}}{q^{\frac{1}{4}}} & \frac{ie^{zu_4}}{q^{\frac{1}{4}}} & -\frac{e^{zu_5}}{q^{\frac{1}{4}}} & \frac{e^{zu_6}}{q^{\frac{1}{4}}} \\ -e^{zu_1} & e^{zu_2} & -\frac{e^{zu_3}}{\sqrt{2}} & -\frac{e^{zu_4}}{\sqrt{2}} & \frac{e^{zu_5}}{\sqrt{2}} & \frac{e^{zu_6}}{\sqrt{2}} \\ e^{zu_1} & -e^{zu_2} & -\frac{e^{zu_3}}{\sqrt{2}} & -\frac{e^{zu_4}}{\sqrt{2}} & \frac{e^{zu_5}}{\sqrt{2}} & \frac{e^{zu_6}}{\sqrt{2}} \\ 0 & 0 & ie^{zu_3}q^{\frac{1}{4}} & -ie^{zu_4}q^{\frac{1}{4}} & -e^{zu_5}q^{\frac{1}{4}} & e^{zu_6}q^{\frac{1}{4}} \\ ie^{zu_1}q^{\frac{1}{2}} & ie^{zu_2}q^{\frac{1}{2}} & \frac{e^{zu_3}q^{\frac{1}{2}}}{\sqrt{2}} & \frac{e^{zu_4}q^{\frac{1}{2}}}{\sqrt{2}} & \frac{e^{zu_5}q^{\frac{1}{2}}}{\sqrt{2}} & \frac{e^{zu_6}q^{\frac{1}{2}}}{\sqrt{2}} \end{pmatrix}. \tag{11.10}$$

The first row above gives the asymptotics of $\phi(z, t^2)$. The correct value of h in (11.7) must be determined in order to match with the asymptotics of the third and fourth rows of (11.10). We find

$$h = -\frac{c^{\frac{1}{2}}}{2}, \quad \text{for the first column,} \tag{11.11}$$

$$h = \frac{c^{\frac{1}{2}}}{2}, \quad \text{for the second column,} \tag{11.12}$$

$$h = 0, \quad \text{for the remaining columns.} \tag{11.13}$$

The above result is determined as follows. For ϕ corresponding to the first two columns we respectively have

$$\phi(z,t^2) = -\frac{c^{\frac{1}{2}}}{2}\frac{ie^{zu_1}}{z^2q^{\frac{1}{2}}}\left(1 + O\left(\frac{1}{z}\right)\right) \quad \text{or} \quad \phi(z,t^2) = -\frac{c^{\frac{1}{2}}}{2}\frac{ie^{zu_2}}{z^2q^{\frac{1}{2}}}\left(1 + O\left(\frac{1}{z}\right)\right).$$

Since $u_1 = u_2 = 0$, the above is

$$\phi(z, t^2) = -\frac{c^{\frac{1}{2}}}{2}\frac{i}{z^2 q^{\frac{1}{2}}}\left(1 + O\left(\frac{1}{z}\right)\right).$$

Then

$$\frac{1}{32}(z\partial_z\phi + z^2\partial_z^2\phi) = O\left(\frac{1}{z}\right).$$

Comparing with the matrix elements $(3,1)$, $(4,1)$ and $(3,2)$, $(4,2)$ respectively, we obtain (11.11) and (11.12). For the remaining columns we proceed in the same way and find (11.13).

11.2. Solutions of the Differential Equation

11.2.1. Generalized Hypergeometric Equations. The equation (11.8) is an example of generalized hypergeometric differential equation, i.e. an equation of the form

$$\left(\prod_{j=1}^{q}(\Theta - \mu_j) + (-1)^h z \prod_{j=1}^{p}(\Theta - \nu_j + 1)\right)\Phi(z) = 0,$$

where $\Theta := z\frac{d}{dz}$ is the Euler operator, and μ_i, ν_i are complex parameters. This kind of equations was studied thoroughly by C.S. Meijer, who introduced in this context the class of G-functions. The problem reduces to a finite difference equation of order 1:

$$\prod_{j=1}^{q}(s + \mu_j)\tau(s) + (-1)^{h+p-q}\prod_{j=1}^{p}(s + \nu_j)\tau(s+1) = 0,$$

for the Mellin transform $\tau(s) := \mathfrak{M}(\Phi)(s) = \int_0^\infty \Phi(t)t^{s-1}dt$. Using the well known property of Γ function $z\Gamma(z) = \Gamma(z+1)$, it is easily seen that a function of the form

$$\tau(s) = \frac{\Gamma(s + \underline{\mu})}{\Gamma(s + \underline{\nu})} e^{\pi i \lambda s}\psi(s), \tag{11.14}$$

where $\psi(s)$ is a rational function of $e^{2\pi i s}$, $\lambda \equiv h + p - q + 1 \bmod(2)$ and $\Gamma(s + \underline{a})$ stands for

$$\prod_{j=1}^{|a|}\Gamma(s + a_j),$$

is a solution of the finite difference equation. So we expect that, if it is possible to apply *Mellin Inversion Theorem*, the functions

$$\Phi(z) := \mathfrak{M}^{-1}(\tau)(z) = \frac{1}{2\pi i}\int_\Lambda \frac{\Gamma(s + \underline{\mu})}{\Gamma(s + \underline{\nu})} e^{\pi i \lambda s}\psi(s) z^{-s} ds,$$

with Λ appropriate integration path, are solutions of the generalized hypergeometric equation. Essentially this is the generic form of a Meijer G-function. Note that, by the reflection property of Γ function $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we have that $\Gamma(\lambda + s)\Gamma(1 - \lambda - s)e^{\pm i\pi s}$ is a rational function of $e^{2\pi i s}$, so that we can move factors from denominator to numerator (or viceversa) in (11.14).

11.2.2. Solutions of $\Theta^5\Phi - 1024w^4\Theta\Phi - 2048w^4\Phi = 0$ and their asymptotics. We will apply the general methods exposed above to our equation

$$\Theta_{\varpi}^5\tilde{\Phi} - \varpi\Theta_{\varpi}\tilde{\Phi} - \frac{1}{2}\varpi\tilde{\Phi} = 0.$$

Applying the Mellin transform, we obtain the finite difference equation

$$s^5\tilde{\tau}(s) = \left(s + \frac{1}{2}\right)\tilde{\tau}(s+1). \tag{11.15}$$

Solutions of this equation are of the form

$$\tilde{\tau}(s) = \frac{\Gamma(s)^5}{\Gamma\left(s+\frac{1}{2}\right)}\psi(s), \quad \psi(s) = \psi(s+1).$$

So we expect that solutions of (11.8) are of the form

$$\Phi(w) = \frac{1}{2\pi i}\int_{\Lambda}\frac{\Gamma(s)^5}{\Gamma\left(s+\frac{1}{2}\right)}\psi(s)4^{-s}w^{-4s}ds,$$

for suitable chosen paths of integration Λ. Actually, we have the following

LEMMA 11.1. *The following functions are solutions of the generalized hypergeometric equation* (11.8)*:*

- *the function*
$$\Phi_1(w) := \frac{1}{2\pi i}\int_{\Lambda_1}\frac{\Gamma(s)^5}{\Gamma\left(s+\frac{1}{2}\right)}4^{-s}w^{-4s}ds,$$
defined for $-\frac{\pi}{2} < \arg w < \frac{\pi}{2}$, and where Λ_1 is any line in the complex plane from the point $c - i\infty$ to $c + i\infty$ for any $0 < c$;
- *the function*
$$\Phi_2(w) := \frac{1}{2\pi i}\int_{\Lambda_2}\Gamma(s)^5\Gamma\left(\frac{1}{2} - s\right)e^{i\pi s}4^{-s}w^{-4s}ds,$$
defined for $-\frac{\pi}{2} < \arg w < \pi$, and where Λ_2 is any line in the complex plane from the point $c - i\infty$ to $c + i\infty$ for any $0 < c < \frac{1}{2}$.

Before giving the proof of this Lemma, we recall the following well-known useful results

THEOREM 11.1 (Stirling). *The following estimate holds*

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{|s|}\right)$$

for $s \to \infty$ and $|\arg s| < \pi$, and where log *stands for the principal determination of the complex logarithm.*

COROLLARY 11.1. *For $|t| \to +\infty$ we have*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}t}\left(1 + O\left(\frac{1}{|t|}\right)\right),$$

uniformly on any strip of the complex plane $\sigma_1 \le \sigma \le \sigma_2$.

PROOF OF THE LEMMA 11.1. First of all let us prove that the functions Φ_1, Φ_2 are well defined on the regions above. Let us start with Φ_1. Denoting by $\mathcal{I}_1$ the integrand in Φ_1, and $s = c + it$, in virtue of Corollary 11.1 we have that

$$|\mathcal{I}_1| \sim (2\pi)^2 |t|^{5(c-\frac{1}{2})} e^{-\frac{5\pi}{2}|t|} |t|^{-c} e^{\frac{\pi}{2}|t|} e^{-c\log 4} e^{-4c\log|w| + 4t\arg w}.$$

The dominant part is

$$e^{-\frac{5\pi}{2}|t|} e^{\frac{\pi}{2}|t|} e^{4t\arg w}.$$

In order to have $|\mathcal{I}_1| \to 0$ for $t \to +\infty$ we must impose

$$-\frac{5\pi}{2} + \frac{\pi}{2} + 4\arg w < 0, \quad \text{i.e.} \ \arg w < \frac{\pi}{2};$$

analogously, for $t \to -\infty$ we have to impose

$$\frac{5\pi}{2} - \frac{\pi}{2} + 4\arg w > 0, \quad \text{i.e.} \ \arg w > -\frac{\pi}{2}.$$

Let us consider now the case of Φ_2. From Corollary 11.1 we deduce that

$$|\mathcal{I}_2| \sim (2\pi)^3 |t|^{5(c-\frac{1}{2})} e^{-\frac{5\pi}{2}|t|} |-t|^{-c} e^{-\frac{\pi}{2}|-t|} e^{-\pi t} e^{-c\log 4} e^{-4c\log|w| + 4t\arg w},$$

and now the dominant part is

$$e^{-\frac{5\pi}{2}|t|} e^{-\frac{\pi}{2}|-t|} e^{-\pi t} e^{4t\arg w}.$$

In order to have $|\mathcal{I}_2| \to 0$ for $t \to \pm\infty$, we find

$$-\frac{5\pi}{2} - \frac{\pi}{2} - \pi + 4\arg w < 0, \quad \text{i.e.} \ \arg w < \pi,$$

$$\frac{5\pi}{2} + \frac{\pi}{2} - \pi + 4\arg w > 0, \quad \text{i.e.} \ \arg w < -\frac{\pi}{2}.$$

Let us now prove that Φ_1 and Φ_2 are effectively solutions of equation (11.8). We have that

$$\begin{aligned}\Theta^5 \Phi_1(w) &= \frac{4^5}{2\pi i} \int_{\Lambda_1} -s^5 \frac{\Gamma(s)^5}{\Gamma\left(s+\frac{1}{2}\right)} 4^{-s} w^{-4s} ds \\ &= \frac{4^5}{2\pi i} \int_{\Lambda_1} -\left(s + \frac{1}{2}\right) \frac{\Gamma(s+1)^5}{\Gamma\left(s+\frac{3}{2}\right)} 4^{-s} w^{-4s} ds\end{aligned}$$

because of the identity (11.15). Changing variable $t := s + 1$, and consequently shifting the line of integration Λ_1 to $\Lambda_1 + 1$, we have

$$\begin{aligned}\Theta^5 \Phi_1(w) &= \frac{4^5}{2\pi i} \int_{\Lambda_1+1} -\left(t - \frac{1}{2}\right) \frac{\Gamma(t)^5}{\Gamma\left(t+\frac{1}{2}\right)} 4^{-t} \cdot 4 w^{-4(t-1)} dt \\ &= \frac{4^5}{2\pi i} \int_{\Lambda_1+1} (-4t) \frac{\Gamma(t)^5}{\Gamma\left(t+\frac{1}{2}\right)} 4^{-t} w^{-4(t-1)} dt + \frac{2 \cdot 4^5}{2\pi i} \int_{\Lambda_1+1} \frac{\Gamma(t)^5}{\Gamma\left(t+\frac{1}{2}\right)} 4^{-t} w^{-4(t-1)} dt.\end{aligned}$$

Note that in the region between Λ_1 and $\Lambda_1 + 1$ the two last integrands have no poles; so $\int_{\Lambda_1+1} = \int_{\Lambda_1}$ by Cauchy Theorem. This shows that

$$\Theta^5 \Phi_1 = 4^5 w^4 \Theta \Phi_1 + 2 \cdot 4^5 w^4 \Phi_1.$$

Analogously we have

$$\begin{aligned}\Theta^5\Phi_2 &= \frac{4^5}{2\pi i}\int_{\Lambda_2} -s^5\Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}w^{-4s}ds\\ &= \frac{4^5}{2\pi i}\int_{\Lambda_2} -\left(s+\frac{1}{2}\right)\Gamma(s+1)^5\Gamma\left(-\frac{1}{2}-s\right)e^{i\pi(s+1)}4^{-s}w^{-4s}ds\end{aligned}$$

where the second identity follows from equation (11.15). Note that the integrand function is holomorphic at $s=-\frac{1}{2}$: indeed we have

$$\lim_{s\to-\frac{1}{2}}\left(s+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}-s\right) = -1.$$

So in the strip of the complex plane $-1<\operatorname{Re} s<\frac{1}{2}$ there are no poles, and by Cauchy Theorem, we can change path of integration by shifting Λ_2 to Λ_2-1:

$$\Theta^5\Phi_2 = \frac{4^5}{2\pi i}\int_{\Lambda_2-1} -s^5\Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}4^{-s}w^{-4s}ds.$$

Posing now $t=s+1$, we can rewrite

$$\Theta^5\Phi_2 = \frac{4^5}{2\pi i}\int_{\Lambda_2} -\left(t-\frac{1}{2}\right)\Gamma(t)^5\Gamma\left(\frac{1}{2}-t\right)e^{i\pi t}4^{-(t-1)}w^{-4(t-1)}dt = 4^5w^4\Theta\Phi_2 + 2\cdot 4^5w^4\Phi_2.$$

This shows that effectively Φ_1 and Φ_2 are solutions. □

Note that solutions Φ_1 and Φ_2 are $\mathbb{C}$-linearly independent, since their Mellin transforms are. However we have the following identities

LEMMA 11.2. *By analytic continuation of the functions Φ_1 and Φ_2, we have*

$$\Phi_2(we^{i\frac{\pi}{2}}) = 2\pi\Phi_1(w) - \Phi_2(w) \tag{11.16}$$

$$\Phi_2(we^{-i\frac{\pi}{2}}) = 2\pi\Phi_1(we^{-i\frac{\pi}{2}}) - \Phi_2(w) \tag{11.17}$$

$$\Phi_2(we^{i\frac{\pi}{2}}) = 2\pi\Phi_1(w) + \Phi_2(we^{-i\frac{\pi}{2}}) - 2\pi\Phi_1(we^{-i\frac{\pi}{2}}) \tag{11.18}$$

PROOF. We have that

$$\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right) = \frac{\pi}{\sin\left(\pi\left(\frac{1}{2}+s\right)\right)} = \frac{2\pi e^{\pm i\pi s}}{e^{\pm 2i\pi s}+1}$$

for a coherent choice of the sign. So

$$e^{\pm 2i\pi s} = \frac{2\pi e^{\pm i\pi s}}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)} - 1.$$

First let us choose the one with $(-)$: we find that

$$\begin{aligned}\Phi_2(we^{i\frac{\pi}{2}}) &= \frac{1}{2\pi i}\int_{\Lambda_2}\Gamma(s)^5\Gamma\left(\frac{1}{2}-s\right)e^{i\pi s}\left(\frac{2\pi e^{-i\pi s}}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)}-1\right)4^{-s}w^{-4s}ds\\ &= 2\pi\Phi_1(w) - \Phi_2(w),\end{aligned}$$

which is the first identity. The second one can be deduce analogously using the formula with $(+)$ sign. Finally the third identity is the difference of (11.16) and (11.17). □

Let us now study the asymptotic behavior of these functions. By Stirling's formula we have that

$$\Phi_1(w) = \frac{(2\pi)^2}{2\pi i}\int_{\Lambda_1} e^{\phi(s)}ds,$$

where

$$\phi(s) = -5s + 5\left(s-\frac{1}{2}\right)\log s + s + \frac{1}{2} - s\log\left(s+\frac{1}{2}\right) - s\log 4 - 4s\log w + O\left(\frac{1}{|s|}\right)$$

for $s \to \infty$ and where log stands for the principal determination of logarithm. Let us find stationary points of $\phi(s)$ for large values of $|s|, |w|$. The derivative ϕ' is

$$\phi'(s) = -4 + 5\log s + \frac{10s-5}{2s} - \log\left(s+\frac{1}{2}\right) - \frac{s}{s+\frac{1}{2}} - \log 4 - 4\log w + O\left(\frac{1}{|s|}\right).$$

For $|s|$ large enough, we have

$$\frac{10s-5}{2s} \sim 5 - \frac{5}{2s}, \qquad \frac{s}{s+\frac{1}{2}} \sim 1 - \frac{1}{2s}, \qquad \log\left(s+\frac{1}{2}\right) = \log s + \log\left(1+\frac{1}{2s}\right) \sim \log s + \frac{1}{2s}.$$

Substituting these identities in ϕ', we find that the critical point $\bar{s}(w)$ in functions of w (for $|w|$ large)

$$\bar{s}(w) = \sqrt{2}w + \frac{5}{8} + O\left(\frac{1}{|w|}\right). \tag{11.19}$$

Note that for $-\frac{\pi}{2} < \arg w < \frac{\pi}{2}$, the point $\bar{s}(w)$ is in the half-plane Re $s > 0$, region in which there are no poles of the integrand functions in Φ_1. So we can shift the line Λ_1 in order that it passes through $\bar{s}$. In this way we obtain

$$\Phi_1(w) = \frac{(2\pi)^2}{2\pi i}e^{\phi(\bar{s})}\int_{\Lambda_1} e^{\phi(s)-\phi(\bar{s})}ds \;\sim\; \frac{(2\pi)^2}{2\pi i}e^{\phi(\bar{s})}\int_{\Lambda_1} e^{\frac{\phi''(\bar{s})}{2}(s-\bar{s})^2}ds.$$

The computation of this Gaussian integral shows that

$$\Phi_1(w) \sim \frac{(2\pi)^2}{2\pi}e^{\phi(\bar{s})}\frac{\sqrt{2\pi}}{\sqrt{\phi''(\bar{s})}} = (2\pi)^{\frac{3}{2}}\frac{e^{\phi(\bar{s})}}{\sqrt{\phi''(\bar{s})}},$$

where Re $\sqrt{\phi''(\bar{s})} > 0$. An explicit series expansion shows that

$$\phi(\bar{s}(w)) \sim -4\sqrt{2}w - \frac{5}{2}\log w - \frac{5}{8}\log 4 + O\left(\frac{1}{|w|}\right),$$

whereas

$$\phi''(\bar{s}(w)) \sim \frac{2\sqrt{2}}{w} + O\left(\frac{1}{|w|^3}\right)$$

and from this we deduce that

$$\Phi_1(w) \sim (2\pi)^{\frac{3}{2}}\frac{e^{-4\sqrt{2}w}}{4w^2}\left(1+O\left(\frac{1}{w}\right)\right).$$

Let us now focus on $\Phi_2(w)$. From Theorem 11.1 we deduce that

$$\Gamma(-s) = e^{-(s+\frac{1}{2})\log s}e^{-i\pi s}e^{s}(-i\sqrt{2\pi})\left(1+O\left(\frac{1}{|s|}\right)\right)$$

for $s \to \infty$ and $s \notin \mathbb{R}_+$. So,

$$\Phi_2(w) = \frac{(2\pi)^3}{2\pi i}\int_{\Lambda_2} e^{\phi(s)}ds,$$

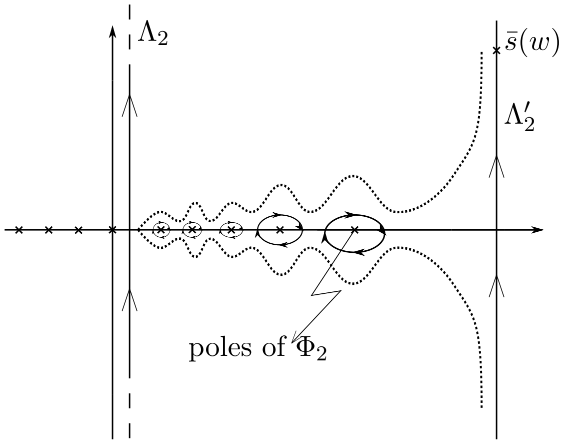


FIGURE 11.1. Deformation of path Λ_2

where

$$\phi(s) = 5\left(s - \frac{1}{2}\right)\log s - 5s - s\log\left(s - \frac{1}{2}\right) + s - \frac{1}{2} - s\log 4 - 4s\log w + O\left(\frac{1}{w}\right),$$

for $w \to \infty$. By computations analogous to those of the previous case, we find that ϕ has a critical point at

$$\bar{s}(w) = \sqrt{2}w + \frac{5}{4\sqrt{2}} + O\left(\frac{1}{w}\right)$$

for large values of $|w|$. Note explicitly that for $-\frac{\pi}{2} < \arg w < \frac{\pi}{2}$ this critical point is in the half-plane $\mathrm{Re}\ s > 0$.

By modifying the path of integration as in Figure 11.1, in order that it passes through the critical point, by Cauchy Theorem we have

$$\Phi_2(w) = \frac{1}{2\pi i}\int_{\Lambda_2'} \mathcal{I}_2(s)ds - \sum_{p\in P} \operatorname*{res}_{s=p} \mathcal{I}_2(s),$$

where P stands for the set of poles in the region between Λ_2 and Λ_2'. For the first summand we have an asymptotic behavior like before (Gaussian integral)

$$\int_{\Lambda_2'} \mathcal{I}_2(s)ds \sim \alpha\frac{e^{-4\sqrt{2}w}}{w^2}$$

with α constant. For the second summand, on the contrary, we have for $n \in \mathbb{N}$

$$\begin{aligned}\operatorname*{res}_{s=n+\frac{1}{2}} \mathcal{I}_2(s) &= \frac{(-1)^{n+1}}{n!}\Gamma\left(n + \frac{1}{2}\right)^5 e^{i\pi\left(n+\frac{1}{2}\right)} 4^{-n-\frac{1}{2}} w^{-4n-2} \\ &= -\frac{i}{n!}\left(\frac{(2n-1)!!}{2^n}\pi^{\frac{1}{2}}\right)^5 4^{-n-\frac{1}{2}} w^{-4n-2}.\end{aligned}$$

So

$$\sum_{p\in P} \operatorname*{res}_{s=p} \mathcal{I}_2(s) = -\frac{i\pi^{\frac{5}{2}}}{2w^2} - \frac{i\pi^{\frac{5}{2}}}{256w^6} + O\left(\frac{1}{w^{10}}\right).$$

In conclusion,

$$\Phi_2(w) \sim \frac{i\pi^{\frac{5}{2}}}{2w^2}\left(1+O\left(\frac{1}{w}\right)\right)$$

for $-\frac{\pi}{2} < \arg w < \frac{\pi}{2}$. Let us now use the identity (11.16) in the following form:

$$\Phi_2(w) = 2\pi\Phi_1(we^{-i\frac{\pi}{2}}) - \Phi_2(we^{-i\frac{\pi}{2}}), \qquad -\frac{\pi}{2} < \arg(we^{-i\frac{\pi}{2}}) < \frac{\pi}{2}.$$

It implies that

$$\Phi_2(w) \sim \frac{i\pi^{\frac{5}{2}}}{2w^2}\left(1+O\left(\frac{1}{w}\right)\right) \quad \text{on the whole sector } -\frac{\pi}{2} < \arg w < \pi.$$

Let us summarize our results (this will be later improved by Lemma 11.4):

LEMMA 11.3. *We have the following asymptotic expansions for Φ_1 and Φ_2:*

$$\Phi_1(w) = (2\pi)^{\frac{3}{2}}\frac{e^{-4\sqrt{2}w}}{4w^2}\left(1+O\left(\frac{1}{w}\right)\right) \quad \textit{in the domain } -\frac{\pi}{2} < \arg w < \frac{\pi}{2},$$

$$\Phi_2(w) = \frac{i\pi^{\frac{5}{2}}}{2w^2}\left(1+O\left(\frac{1}{w}\right)\right) \quad \textit{in the domain } -\frac{\pi}{2} < \arg w < \pi.$$

11.3. Computation of Monodromy Data

11.3.1. Solution at the Origin and computation of $\widetilde{\mathcal{C}}_0(\mathbb{G})$. Monodromy data at the origin $z=0$ are determined by the action of the first Chern class $c_1(\mathbb{G}) = 4\sigma_1$ on the classical cohomology ring. So,

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{pmatrix}. \tag{11.20}$$

By Theorem 2.2 and Theorem 2.4, there exists a fundamental matrix solution (2.12)

$$Y(z) = \Phi(t^2, z)z^\mu z^R,$$

for some appropriate converging power series $\Phi(t^2, z) = \mathbb{1} + O(z)$ such that

$$\Phi^T(t^2, -z)\ \eta\ \Phi(t^2, z) = \eta.$$

Thus, a fundamental matrix for our problem is given by

$$\Xi_0(z) = \eta\ \Phi(t^2, z)\ z^\mu z^R\ = \Phi^T(t^2, -z)^{-1}\eta\ z^\mu z^R.$$

By applying the iterative procedure in [Dub99b] for the proof of Theorem 2.2, at $t^2 = 0$ one finds the following fundamental solution

$$\Xi_0(0, z) = S(0, z)\eta z^\mu z^R, \tag{11.21}$$

$$S(0,z) = \begin{pmatrix} 2z^4+1 & 0 & 0 & 0 & 0 & 0 \\ 2z^3 & 1-4z^4 & 0 & 0 & 0 & 0 \\ z^2 & -z^3 & 1 & 0 & 0 & 0 \\ z^2 & -z^3 & 0 & 1 & 0 & 0 \\ z & 0 & -z^3 & -z^3 & 4z^4+1 & 0 \\ z^4 & z & -z^2 & -z^2 & 2z^3 & 1-2z^4 \end{pmatrix} + O(z^5).$$

REMARK 11.2. The solution (11.21) satisfies the condition

$$z^{-\mu}\left(\eta^{-1}S(0,z)\eta\right)z^{\mu} \text{ is holomorphic near } z=0,$$

$$z^{-\mu}\left(\eta^{-1}S(0,z)\eta\right)z^{\mu}=\begin{pmatrix}1-2z^4 & 2z^4 & -z^4 & -z^4 & z^4 & z^8\\ 0 & 4z^4+1 & -z^4 & -z^4 & 0 & z^4\\ 0 & 0 & 1 & 0 & -z^4 & z^4\\ 0 & 0 & 0 & 1 & -z^4 & z^4\\ 0 & 0 & 0 & 0 & 1-4z^4 & 2z^4\\ 0 & 0 & 0 & 0 & 0 & 2z^4+1\end{pmatrix}+O(z^9).$$

This means that $(\eta^{-1}S(0,z)\eta)\,z^{\mu}z^{R}$ coincides with the topological solution $Z_{\rm top}(0,z)$.

Notice that the leading term of the solution Ξ_0 in (11.21) is exactly

$$\eta\ z^{\mu}z^{R}=c\begin{pmatrix}\frac{64}{3}z^2\log^4(z) & \frac{64}{3}z^2\log^3(z) & 8z^2\log^2(z) & 8z^2\log^2(z) & 4z^2\log(z) & z^2\\ \frac{64}{3}z\log^3(z) & 16z\log^2(z) & 4z\log(z) & 4z\log(z) & z & 0\\ 8\log^2(z) & 4\log(z) & 1 & 0 & 0 & 0\\ 8\log^2(z) & 4\log(z) & 0 & 1 & 0 & 0\\ \frac{4\log(z)}{z} & \frac{1}{z} & 0 & 0 & 0 & 0\\ \frac{1}{z^2} & 0 & 0 & 0 & 0 & 0\end{pmatrix}.$$

From the first row, we deduce that near $z=0$ any solution of the equation (11.8), i.e.

$$\Theta^5\Phi-1024z^4\Theta\Phi-2048z^4\Phi=0$$

is of the form

$$\Phi(z)=\sum_{n\geq 0}z^n\left(a_n+b_n\log z+c_n\log^2 z+d_n\log^3 z+e_n\log^4 z\right),\tag{11.22}$$

where a_0,b_0,c_0,d_0,e_0 are arbitrary constants, and successive coefficients can be obtained recursively.

PROPOSITION 11.1. *Let R be as in (11.20) of R. Then, $\widetilde{\mathcal{C}}_0(\mu,R)$ is the algebraic abelian group of complex dimension 3 given by*

$$\widetilde{\mathcal{C}}_0(\mu,R)=\left\{\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ \alpha_1 & 1 & 0 & 0 & 0 & 0\\ \alpha_2 & \alpha_1 & 1 & 0 & 0 & 0\\ \alpha_3 & \alpha_1 & 0 & 1 & 0 & 0\\ \alpha_4 & \alpha_2+\alpha_3 & \alpha_1 & \alpha_1 & 1 & 0\\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & 1\end{pmatrix}:\alpha_i\in\mathbb{C}\ s.t.\ \begin{cases}\alpha_1^2-\alpha_2-\alpha_3=0\\ \alpha_2^2+\alpha_3^2-2\alpha_1\alpha_4+2\alpha_5=0\end{cases}\right\}.$$

In particular, if $F(t)\in\mathbb{C}[\![t]\!]$ is a formal power series of the form $F(t)=1+F_1t+F_2t^2+\dots$, then the matrix (computed w.r.t. the chosen Schubert basis $\sigma_0,\sigma_1,\sigma_2,\sigma_{1,1},\sigma_{2,1},\sigma_{2,2}$) representing the endomorphism

$$\lambda_F\cup(-)\colon H^\bullet(\mathbb{G};\mathbb{C})\to H^\bullet(\mathbb{G};\mathbb{C}),$$

where $\lambda_F\in H^\bullet(\mathbb{G};\mathbb{C})$ is such that

$$\widehat{F}(T\mathbb{G})\cup\lambda_F=\widehat{F}(T^*\mathbb{G}),$$

*is an element of $\widetilde{\mathcal{C}}_0(\mu,R)$. Here $\widehat{F}(V)$ denotes the Hirzebruch multiplicative characteristic class of the vector bundle $V\to\mathbb{G}$ associated to the formal power series $F(t)$ (see [**Hir78**]).*

Proof. The equations defining the group $\widetilde{\mathcal{C}}_0(\mu, R)$ are obtained by direct computation from the requirement that $P(z) := z^\mu z^R \cdot C \cdot z^{-R} z^{-\mu}$ is a polynomial of the form $P(z) = \mathbb{1} + A_1 z + A_2 z^2 + \dots$, together with the orthogonality condition $P(-z)^T \eta P(z) = \eta$. Notice that the polynomial for the generic matrix of the form above is equal to

$$P(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ z\alpha_1 & 1 & 0 & 0 & 0 & 0 \\ z^2\alpha_2 & z\alpha_1 & 1 & 0 & 0 & 0 \\ z^2\alpha_3 & z\alpha_1 & 0 & 1 & 0 & 0 \\ z^3\alpha_4 & z^2(\alpha_2+\alpha_3) & z\alpha_1 & z\alpha_1 & 1 & 0 \\ z^4\alpha_5 & z^3\alpha_4 & z^2\alpha_3 & z^2\alpha_2 & z\alpha_1 & 1 \end{pmatrix}.$$

We leave as an exercise to show that such a matrix group is abelian. Let $\delta_1, \dots, \delta_6$ be the Chern roots of $T\mathbb{G}$. Then, for some complex constants $a_{i,j} \in \mathbb{C}$, we have

$$\widehat{F}(T\mathbb{G}) := \prod_{j=1}^{6} F(\delta_j) = 1 + a_1\sigma_1 + a_2\sigma_2 + a_{1,1}\sigma_{1,1} + a_{2,1}\sigma_{2,1} + a_{2,2}\sigma_{2,2}, \tag{11.23}$$

$$\widehat{F}(T^*\mathbb{G}) := \prod_{j=1}^{6} F(-\delta_j) = 1 - a_1\sigma_1 + a_2\sigma_2 + a_{1,1}\sigma_{1,1} - a_{2,1}\sigma_{2,1} + a_{2,2}\sigma_{2,2}. \tag{11.24}$$

Thus, if

$$\lambda_F = 1 + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_{1,1} + x_4\sigma_{2,1} + x_5\sigma_{2,2},$$

from the condition $\widehat{F}(T\mathbb{G}) \cup \lambda_F = \widehat{F}(T^*\mathbb{G})$ we obtain the constraints

$$\begin{cases} x_1 = -2a_1, \\ x_2 = 2a_1^2, \\ x_3 = 2a_1^2, \\ x_4 = 2a_1(a_2 + a_{1,1}) - 4a_1^3 - 2a_{2,1}, \\ x_5 = 4a_1a_{2,1} - 4a_1^2(a_2 + a_{1,1}) + 4a_1^4. \end{cases}$$

From this it is immediately seen that $x_1^2 - x_2 - x_3 = 0$ and $x_2^2 + x_3^2 - 2x_1x_4 + 2x_5 = 0$. ☐

11.3.2. Stokes rays and computation of Ξ_{left}, Ξ_{right}. According to Theorem 9.1, monodromy data of $QH^\bullet(\mathbb{G})$ can be computed starting from a point $(0, t^2, 0, ..., 0)$ of the small quantum cohomology. Moreover, thanks to the Isomonodromy Theorems, it suffices to do the computation at $t^2 = 0$, i.e. $q = 1$, where the canonical coordinates (11.2) are

$$u_1 = u_2 = 0, \quad u_3 = -4i\sqrt{2}, \quad u_4 = 4i\sqrt{2}, \quad u_5 = -4\sqrt{2}, \quad u_6 = 4\sqrt{2}.$$

The Stokes rays (2.29) are easy seen to be

$$\begin{aligned} R_{13} = R_{23} &= \{-\rho \colon \rho \geq 0\}, \\ R_{14} = R_{24} = R_{34} &= \{\rho \colon \rho \geq 0\}, \\ R_{15} = R_{25} &= \{-i\rho \colon \rho \geq 0\}, \\ R_{16} = R_{26} = R_{56} &= \{i\rho \colon \rho \geq 0\}, \end{aligned}$$

$$R_{35} = \left\{\rho e^{-i\frac{\pi}{4}} : \rho \geq 0\right\}, \quad R_{36} = \left\{\rho e^{i\frac{\pi}{4}} : \rho \geq 0\right\},$$
$$R_{45} = \left\{-\rho e^{i\frac{\pi}{4}} : \rho \geq 0\right\}, \quad R_{46} = \left\{-\rho e^{-i\frac{\pi}{4}} : \rho \geq 0\right\}, \quad R_{ji} = -R_{ij}.$$

We fix the admissible line ℓ

$$\ell := \left\{ \rho e^{i\frac{\pi}{6}} : \rho \in \mathbb{R} \right\},$$

so that the sectors for the asymptotic expansion, containing $\Pi_{\text{left/right}}$ and extending up to the nearest Stokes rays are

$$\mathcal{S}_{\text{right}} = \{z : \ -\pi < \arg z < \pi/4\} \qquad \mathcal{S}_{\text{left}} = \{z : \ -0 < \arg z < \pi + \pi/4\}.$$

For such a choice of the line, according to Theorem 2.10, the structure of the Stokes matrix is

$$S = \begin{pmatrix} 1 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & 0 & * \\ * & * & * & 1 & 0 & * \\ * & * & * & * & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.25}$$

We use the following notation for fundamental matrices

$$\Xi_{\text{right}} = \begin{pmatrix} \xi^R_{(1),1} & \xi^R_{(2),1} & \xi^R_{(3),1} & \xi^R_{(4),1} & \xi^R_{(5),1} & \xi^R_{(6),1} \\ \\ \xi^R_{(1),2} & \xi^R_{(2),2} & \xi^R_{(3),2} & \xi^R_{(4),2} & \xi^R_{(5),1} & \xi^R_{(6),2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi^R_{(1),6} & \xi^R_{(2),6} & \xi^R_{(3),6} & \xi^R_{(4),6} & \xi^R_{(5),6} & \xi^R_{(6),6} \end{pmatrix}, \quad \Xi_{\text{left}} = \begin{pmatrix} \xi^L_{(1),1} & \xi^L_{(2),1} & \xi^L_{(3),1} & \xi^L_{(4),1} & \xi^L_{(5),1} & \xi^L_{(6),1} \\ \\ \xi^L_{(1),2} & \xi^L_{(2),2} & \xi^L_{(3),2} & \xi^L_{(4),2} & \xi^L_{(5),1} & \xi^L_{(6),2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi^L_{(1),6} & \xi^L_{(2),6} & \xi^L_{(3),6} & \xi^L_{(4),6} & \xi^L_{(5),6} & \xi^L_{(6),6} \end{pmatrix}$$

Note, in particular, that (11.25) implies that the fifth columns of Ξ_{right} and Ξ_{left} coincide. Then $\xi^L_{(5),1}$ is the analytical continuation of $\xi^R_{(5),1}$ on $\mathcal{S}_{\text{left}}$. Moreover, the exponential e^{zu_5} dominates all others e^{zu_j}'s in the sector between the rays R_{45} and R_{46}, i.e. for $-\pi - \pi/4 < \arg z < -\pi + \pi/4$. This implies that the asymptotics

$$\xi^L_{(5),1} = \xi^R_{(5),1} = \frac{c^{\frac{1}{2}}}{2\sqrt{2}} e^{zu_5} \left(1 + O\left(\frac{1}{z}\right)\right),$$

is valid in the whole sector $-\pi - \pi/4 < \arg z < \pi + \pi/4$. By lemma 11.3,

$$\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}} z^2 \Phi_1(z) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}} e^{zu_5} \left(1 + O\left(\frac{1}{z}\right)\right), \qquad \text{for } -\frac{\pi}{2} < \arg z < \frac{\pi}{2}.$$

Since the exponential e^{zu_5} is dominated by all others exponentials e^{zu_j} in the region between R_{35} and R_{36}, namely for $-\pi/4 < \arg z < \pi/4$, we conclude necessarily that

$$\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}} z^2 \Phi_1(z) = \xi^{L/R}_{(5),1}(z).$$

This determines the 5-th column of Ξ_{right} and Ξ_{left} in terms of Φ_1, using equations (11.7),(11.13). We also obtain an improvement of Lemma 11.3:

LEMMA 11.4. *Φ_1 and Φ_2 have the following asymptotic behaviour*

$$\Phi_1(w) = (2\pi)^{\frac{3}{2}} \frac{e^{-4\sqrt{2}w}}{4w^2} \left(1 + O\left(\frac{1}{w}\right)\right) \qquad \textit{in the domain } -\pi - \frac{\pi}{4} < \arg w < \pi + \frac{\pi}{4}$$

$$\Phi_2(w) = \frac{i\pi^{\frac{5}{2}}}{2w^2} \left(1 + O\left(\frac{1}{w}\right)\right) \qquad \textit{in the domain } -\frac{\pi}{2} < \arg w < \pi.$$

We are ready to determine the other columns of $\Xi_{\text{left/right}}$. By Lemma 11.4,

$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\frac{\pi}{2}}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_3}\left(1+O\left(\frac{1}{z}\right)\right), \quad \text{for } -2\pi+\frac{\pi}{4}<\arg z<\frac{3\pi}{4}, \tag{11.26}$$

$$\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\pi}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_6}\left(1+O\left(\frac{1}{z}\right)\right), \quad \text{for } -2\pi-\frac{\pi}{4}<\arg z<\frac{\pi}{4}. \tag{11.27}$$

We consider first (11.26). Being solutions of a differential equation, the following holds:

$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\frac{\pi}{2}}) = \text{ linear combination of the } \xi^R_{(1),i}, \quad 1\le i\le 6.$$

On the other hand, e^{zu_3} is dominated by all other e^{zu_i}'s in the sector $-\pi+\pi/4<\arg z<-\pi/2$ between R_{45} and R_{35}. This requires that the linear combination necessarily reduces to

$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\frac{\pi}{2}}) = \xi^R_{(3),1}$$

Now we consider (11.27). As above, since e^{zu_6} is dominated by all the other e^{zu_i}'s in the sector $-5\pi/4<\arg z<-3\pi/4$ between R_{46} and R_{45}, we conclude that

$$\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\pi}) = \xi^R_{(6),1}.$$

Analogously we find that

$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{-i\frac{\pi}{2}}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_4}\left(1+O\left(\frac{1}{z}\right)\right) \quad \text{for } -\frac{3\pi}{4}<\arg z<\pi+\frac{3\pi}{4},$$

$$\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{-i\pi}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_6}\left(1+O\left(\frac{1}{z}\right)\right) \quad \text{on } -\frac{\pi}{4}<\arg z<2\pi+\frac{\pi}{4}.$$

By dominance considerations as above, we conclude that

$$\xi^L_{(4),1} = -\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{-i\frac{\pi}{2}}), \quad \xi^L_{(6),1} = \frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{-i\pi}).$$

The above results reconstruct (using identities (11.7),(11.13)) three columns of matrices Ξ_{right} and Ξ_{left} respectively. As far as the first two columns are concerned, we invoke again Lemma 11.4 for Φ_2, which yileds

$$\frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}}z^2\Phi_2(ze^{i\frac{\pi}{2}}) = -\frac{ic^{\frac{1}{2}}}{2}\left(1+O\left(\frac{1}{z}\right)\right) \quad \text{on } -\pi<\arg z<\frac{\pi}{2},$$

$$\frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}}z^2\Phi_2(ze^{-i\frac{\pi}{2}}) = -\frac{ic^{\frac{1}{2}}}{2}\left(1+O\left(\frac{1}{z}\right)\right) \quad \text{on } 0<\arg z<\frac{3\pi}{2}.$$

Exactly as before, dominance relations of the exponentials e^{zu_i} yield

$$\frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}}z^2\Phi_2(ze^{i\frac{\pi}{2}}) = \xi^R_{(1),1} = \xi^R_{(2),1}, \qquad \frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}}z^2\Phi_2(ze^{-i\frac{\pi}{2}}) = \xi^L_{(1),1} = \xi^L_{(2),1},$$

Using (11.7),(11.11),(11.12), the first two columns are contructed. Summarizing, we have determined the following columns in terms of Φ_1 and Φ_2.

$$\Xi_{\text{right}} = \begin{pmatrix} \xi^R_{(1),1} & \xi^R_{(2),1} & \xi^R_{(3),1} & \text{unknown} & \xi^R_{(5)} & \xi^R_{(6),1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$\Xi_{\rm left}=\begin{pmatrix}\xi^L_{(1),1}&\xi^L_{(2),1}&\text{unknown}&\xi^L_{(4),1}&\xi^L_{(5),1}&\xi^L_{(6),1}\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}.$$

In Section 11.3.3 we show that the above partial information and the constraint (2) in Theorem 2.11 are enough to determine the Stokes and central connection matrices simultaneously. Since constraint (2) holds only in case S and C are related to Frobenius manifolds, we sketch below – for completeness sake – the general method to obtain the missing columns of $\Xi_{\rm left/right}$ and S, in a pure context of asymptotic analysis of differential equations.

We observe that

$$-\frac{c^{\frac12}}{2\pi^{\frac32}}z^2\Phi_1(ze^{-i\frac{\pi}{2}})=\frac{c^{\frac12}}{2\sqrt2}e^{u_4z}(1+O(1/z)),\qquad\text{for }-\pi+\frac{\pi}{4}<\arg z<\pi+\frac{3\pi}{4}.$$

The sub-sector $-\pi<\arg z<-3\pi/4$ of $\mathcal{S}_{\rm right}$ is not covered by the sector where the above asymptotic behaviour holds. On the sub-sector, the dominance relation $|e^{zu_4}|<|e^{zu_5}|$ holds. Thus,

$$\xi^R_{(4),1}=-\frac{c^{\frac12}}{2\pi^{\frac32}}z^2\Phi_1(ze^{-i\frac{\pi}{2}})+v\xi^R_{(5),1},\tag{11.28}$$

for some complex number $v\in\mathbb{C}$, to be determined. Analogously, we observe that

$$-\frac{c^{\frac12}}{2\pi^{\frac32}}z^2\Phi_1(ze^{-i\frac{3\pi}{2}})=\frac{c^{\frac12}}{2\sqrt2}e^{u_4z}(1+O(1/z)),\qquad\text{for }-2\pi-\frac{3\pi}{4}<\arg z<-\frac{\pi}{4}.$$

The sub-sector $-\pi/4<\arg z<\pi/4$ of $\mathcal{S}_{\rm right}$ is not covered by the sector where the asymptotic behaviour holds. Now, the following dominance relations hold: $|e^{zu_4}|<|e^{zu_i}|$, for $i=1,2,3,6$, in $0<\arg z<\pi/4$; for $i=6$ in $-\pi/4<\arg z<0$. Thus

$$\xi^R_{(4),1}=-\frac{c^{\frac12}}{2\pi^{\frac32}}z^2\Phi_1(ze^{i\frac{3\pi}{2}})+\gamma_1\xi^R_{(1),1}+\gamma_3\xi^R_{(3),1}+\gamma_6\xi^R_{(6),1}\tag{11.29}$$

for some complex number $\gamma_1,\gamma_3,\gamma_6\in\mathbb{C}$, to be determined[2]. The above (11.28) and (11.29) become a 6-terms linear relation between functions $\Phi_2(ze^{i\frac{k\pi}{2}})$, as follows

$$-\Phi_1(ze^{-i\frac{\pi}{2}})+v\Phi_1(z)=-\Phi_1(ze^{i\frac{3\pi}{2}})+\frac{\gamma_1}{\pi}\Phi_2(ze^{i\frac{\pi}{2}})-\gamma_3\Phi_1(ze^{i\frac{\pi}{2}})+\gamma_6\Phi_1(ze^{i\pi}),$$

$$\Phi_1(z)=\frac{1}{2\pi}\left[\Phi_2(z)+\Phi_2(ze^{i\frac{\pi}{2}})\right].$$

At this step, some further information is need. The equation $\Theta^5\Phi-1024z^4\Theta\Phi-2048z^4\Phi=0$ admits the symmetry $z\mapsto ze^{i\frac{\pi}{2}}$. This means that if Φ is a solution of the equation then also $\Phi(ze^{i\frac{\pi}{2}})$ is. Such a symmetry defines a linear map on the vector space of solutions of the equation defined in a neighborhood of $z=0$. Because of this symmetry, the form (11.22) can be refined as

$$\Phi(z)=\sum_{n\geq0}z^{4n}\left(a_n+b_n\log z+c_n\log^2 z+d_n\log^3 z+e_n\log^4 z\right),\tag{11.30}$$

where a_0,b_0,c_0,d_0,e_0 are arbitrary constants, and successive coefficients can be obtained recursively. In the basis of solutions of the form (11.30) with $(a_0,b_0,c_0,d_0,e_0)=(1,0,...,0)$, $(0,1,0,...,0)$ and so on, the matrix of the operator

$$(A\Phi)(z):=\Phi(ze^{i\frac{\pi}{2}})$$

is of triangular form with 1's on the diagonal. Hence, by Cayley-Hamilton Theorem we deduce that

$$(A-\mathbb{1})^5=0,$$

[2]There is no need to include a term $+\gamma_2\xi^R_{(2),1}$ in the linear combination, since $\xi^R_{(1),1}=\xi^R_{(2),1}$.

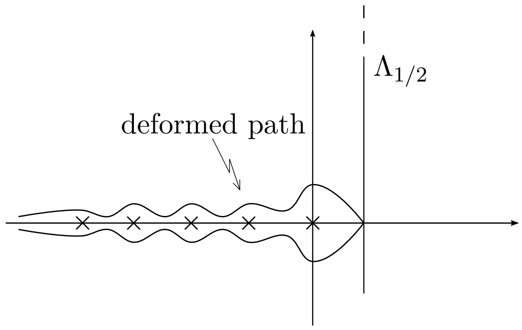


FIGURE 11.2. Deformation of the path $\Lambda_{1/2}$, in order to apply residue theorem. Poles are represented.

i.e.

$$A^5 - 5A^4 + 10A^3 - 10A^2 + 5A - \mathbb{1} = 0.$$

This proves the following

LEMMA 11.5. *The solutions of the equation* $\Theta^5\Phi - 1024z^4\Theta\Phi - 2048z^4\Phi = 0$ *satisfy the relation*

$$\Phi(ze^{i\frac{5\pi}{2}}) - 5\Phi(ze^{2\pi i}) + 10\Phi(ze^{i\frac{3\pi}{2}}) - 10\Phi(ze^{i\pi}) + 5\Phi(ze^{i\frac{\pi}{2}}) - \Phi(z) = 0. \tag{11.31}$$

The relation (11.31) applied to Φ_2 determines v, γ_1, γ_3, γ_6. For example, $v = 6$. This determines $\xi^R_{(4),1}$ through formula (11.28). The fourth column of Ξ_{right} is then constructed with formula (11.7) applied to $\xi^R_{(4),1}$ (with $h = 0$). The value $v = 6$ will be determined again in Section 11.3.3 making use of the constraint (2) of Theorem 2.11.

Proceeding in the same way, we also determine $\xi^L_{(3),1}$. One observes that

$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\frac{3\pi}{2}}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_3}(1 + O(1/z)), \qquad \text{for } \frac{\pi}{4} < \arg z < \frac{3\pi}{2} + 2\pi,$$
$$-\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{i\frac{\pi}{2}}) = \frac{c^{\frac{1}{2}}}{2\sqrt{2}}e^{zu_3}(1 + O(1/z)), \qquad \text{for } -2\pi - \frac{\pi}{2} < \arg z < \frac{3\pi}{4}.$$

The first asymptotic relation does not hold in the sub-sector $-\pi/4 < \arg z < \pi/4$ of $\mathcal{S}_{\text{left}}$, The second, does not in $3\pi/4 < \arg z < 5\pi/4$. Then, the dominance relations in these sub-sectors generate an 6-terms linear relation with unknown coefficients. The coefficients are determined by (11.31).

Once $\Xi_{\text{left/right}}$ has been determined, S can be computed by direct comparison of the two fundamental matrices (formula (11.31) need to be used at some point of the comparison). The final result is the Stokes matrix S of formula (11.33) below with $v = 6$.

11.3.3. Computation of Stokes and Central Connection Matrices, using constrain (2) of Theorem 2.11. We start from formula (11.28):

$$\xi^R_{(4),1} = -\frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\Phi_1(ze^{-i\frac{\pi}{2}}) + v\xi^R_{(5),1} \equiv \frac{c^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}}z^2\left(-\Phi_1(ze^{-i\frac{\pi}{2}}) + v\Phi_1(z)\right).$$

We show that the constraint (2) of Theorem 2.11 suffices to determine v and reconstruct both the Stokes and the central connection matrices, as follows.

The definition of the central connection matrix C and the transformation (11.9) imply that

$$\Xi_{\text{right}} = \Xi_0 C.$$

The matric C can be obtained by comparing the leading behaviours of Ξ_{right} and Ξ_0 near $z = 0$. The leading behaviour of Ξ_0 in (11.21) is $\eta z^\mu z^R$. In order to find the behaviour of Ξ_{right}, we need to compute the behaviour of Φ_1 and Φ_2 near $z = 0$. To this end, we consider the integral representations in Lemma 11.1, and deform both paths Λ_1 and Λ_2 to the left, as shown in Figure 11.2. By residue theorem, we obtain a representations of Φ_1 and Φ_2 as a series of residues at the poles $s = 0, -1, -2\dots$. Then, by the reconstruction dictated by equations (11.7),(11.11),(11.12),(11.13), for each entry of the matrix Ξ_{right} we obtain an expansion in z and $\log z$, converging for small $|z|$.

For example, let us compute the first and second columns of the matrix C: by deformation of the path Λ_2 we obtain that for small z the following series expansions hold:

$$\begin{aligned}
\xi^R_{(1),1} = \xi^R_{(2),1} &= \frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}} z^2 \Phi_2(z e^{i\frac{\pi}{2}}) \\
&= \frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}} z^2 \sum_{n=0}^{\infty} \operatorname*{res}_{s=-n} \left(\Gamma(s)^5 \Gamma\left(\frac{1}{2} - s\right) e^{-i\pi s} 4^{-s} z^{-4s} \right) \\
&= \alpha_1 z^2 \log^4 z + \alpha_2 z^2 \log^3 z + \alpha_3 z^2 \log^2 z + \alpha_4 z^2 \log z + \alpha_5 z^2 + O(z^4),
\end{aligned}$$

where α_i can be explicitly computed. By comparison with the first row of $\eta z^\mu z^R$ we determine the entries

$$\begin{aligned}
C_{11} = C_{12} &= \frac{3}{64c}\alpha_1, & C_{21} = C_{22} &= \frac{3}{64c}\alpha_2, \\
C_{51} = C_{52} &= \frac{1}{4c}\alpha_4, & C_{61} = C_{62} &= \frac{1}{c}\alpha_5.
\end{aligned}$$

For the other entries we have to consider expansions of $\xi^R_{(1),3}, \xi^R_{(2),3}, \xi^R_{(1),4}, \xi^R_{(2),4}$. For example,

$$\begin{aligned}
\xi^R_{(1),3} = \xi^R_{(2),4} &= \frac{c^{\frac{1}{2}}}{\pi^{\frac{5}{2}}} \cdot \frac{1}{32} (z \Phi_2'(z e^{i\frac{\pi}{2}}) + z^2 \Phi_2''(z e^{i\frac{\pi}{2}})) - \frac{c^{\frac{1}{2}}}{2} \\
&= -\frac{c^{\frac{1}{2}}}{2} + \frac{c^{\frac{1}{2}}}{2\pi^{\frac{5}{2}}} \sum_{n=0}^{\infty} \operatorname*{res}_{s=-n} \left(\Gamma(s)^5 \Gamma\left(\frac{1}{2} - s\right) e^{-i\pi s} 4^{-s} s^2 z^{-4s} \right) \\
&= \beta_1 \log^2 z + \beta_2 \log z + \beta_3 + O(z^4),
\end{aligned}$$

where β_i can be explicitly computed. So, by comparison of the third row of $g z^\mu z^R$ we obtain

$$C_{31} = C_{42} = \frac{\beta_3}{c}.$$

Analogously one obtains $C_{32} = C_{41}$. Note that the other entries C_{ij}, with $j = 3, 4, 5, 6$, are uniquely determined only by the expansion of $\xi^R_{(j),i}$ because of (11.13). The result of the explicit computation of C is reported in Appendix A. Note that only the fifth column of C is expressed in terms of the constant v. This v will now be determined.

Since S and C are associated to a Frobenius manifold, the constrain (2) of Theorem 2.11 holds:

$$S = C^{-1} e^{-\pi i R} e^{-\pi i \mu} \eta^{-1} (C^T)^{-1}. \tag{11.32}$$

Substituting C of Appendix A, with undetermined v, in the constraint above, we obtain the Stokes matrix

$$S = \begin{pmatrix} 1 & 0 & 4 & 0 & 0 & 4 \\ 0 & 1 & 4 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ -4 & -4 & -16 & 1 & 6-v & -6 \\ 4(v-1) & 4(v-1) & 16v-26 & -v & (v-6)v+1 & 6v-16 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.33}$$

By a direct comparison with the expected matrix form (11.25), which dictates that $S_{45} = 0$ and $S_{55} = 1$, we conclude that necessarily

$$v = 6.$$

In this way we have completely determined both the Stokes and central connection matrices as well as the fundamental matrix Ξ_{right}. See also (11.39) below.

11.3.4. Monodromy data and Exceptional collections in $\mathcal{D}^b(\mathbb{G})$ and Γ-conjecture. The monodromy data R and C computed above can be read as characteristic classes of objects of an exceptional collection in $\mathcal{D}^b(\mathbb{G})$, as it has been conjectured by B. Dubrovin ([**Dub98**]), though the formulation for the central connection matrix was not well understood. Following [**KKP**] where the role of the $\widehat{\Gamma}$-classes (characteristic classes obtained by the Hirzebruch's procedure starting from the series expansion of the functions $\Gamma(1 \pm t)$ near $t = 0$) was pointed out, we claim that the central connection matrix (for canonical coordinates in triangular/lexicographical order) can be identified with the matrix of the $\mathbb{C}$-linear morhisms

$$\mathfrak{X}^{\pm}_{\mathbb{G}} \colon K_0(\mathbb{G}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^{\bullet}(\mathbb{G};\mathbb{C})$$
$$E \mapsto \frac{1}{(2\pi)^2 c^{\frac{1}{2}}} \widehat{\Gamma}^{\pm}(\mathbb{G}) \cup \mathrm{Ch}(E)$$

$$\widehat{\Gamma}^{\pm}(\mathbb{G}) := \prod_j \Gamma(1 \pm \delta_j) \quad \text{where } \delta_j\text{'s are the Chern roots of } T\mathbb{G},$$

$$\mathrm{Ch}(V) := \sum_k e^{2\pi i x_k} \qquad x_k\text{'s are the Chern roots of a vector bundle } V,$$

expressed w.r.t.

- an exceptional basis $(\varepsilon_i)_i$ of $K_0(\mathbb{G}) \otimes_{\mathbb{Z}} \mathbb{C}$, i.e. satisfying $\chi(\varepsilon_i, \varepsilon_i) = 1$, and the Grothendieck-Euler-Poincaré orthogonality conditions $\chi(\varepsilon_i, \varepsilon_j) = 0$ for $i > j$, obtained by projection of a full exceptional collection $(E_i)_i$ in $\mathcal{D}^b(\mathbb{G})$;
- a basis in $H^{\bullet}(\mathbb{G};\mathbb{C})$ related to $(\sigma_0, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2})$ (the Schubert basis we have fixed) by a (η, μ)-orthogonal-parabolic G endomorphism (as described in Section 2.1.2) which commutes with the operator of classical $\cup$-multiplication $c_1(\mathbb{G}) \cup - \colon H^{\bullet}(\mathbb{G};\mathbb{C}) \to H^{\bullet}(\mathbb{G};\mathbb{C})$.

By application of the constraint (11.32) and the Grothendieck-Hirzebruch-Riemann-Roch Theorem, one can prove that the Stokes matrix (in triangular/lexicographical order) is equal to the inverse of the Gram matrix:

$$(S^{-1})_{ij} = \chi(\varepsilon_i, \varepsilon_j).$$

See [**CDG17a**] for a rigorous proof.

Remark 11.3. As exposed in Theorem 1.2 in the Introduction and in Section 2.3, some natural transformations are allowed, such as

- the left action of the group $\widetilde{\mathcal{C}}_0(\mu, R)$:

$$\text{no anction on } S, \qquad C \longmapsto G\ C, \tag{11.34}$$

where $G \in \widetilde{\mathcal{C}}_0(\mu, R)$ and has the form prescribed by Proposition 11.1;
- the right action of the group $(\mathbb{Z}/2\mathbb{Z})^{\times 6}$:

$$S \longmapsto \mathcal{I}\ S\ \mathcal{I}, \qquad C \longmapsto C\ \mathcal{I}, \tag{11.35}$$

where $\mathcal{I}$ is a diagonal matrix of 1's and -1's;
- the right action of the braid group $\mathcal{B}_6$:

$$S \longmapsto A^\beta\ S\ (A^\beta)^T, \qquad C \longmapsto C\ (A^\beta)^{-1}, \tag{11.36}$$

as in formulae (2.40) and (2.41).

The actions above naturally manifest respectively on the space $H^\bullet(\mathbb{G};\mathbb{C})$, on the set of full exceptional collections in the category $\mathcal{D}^b(\mathbb{G})$, and/or on the set of exceptional bases of the complexified Grothendieck group $K_0(\mathbb{G}) \otimes_{\mathbb{Z}} \mathbb{C}$. More precisely,

- $\widetilde{\mathcal{C}}_0(\mu, R)$ acts on $H^\bullet(\mathbb{G};\mathbb{C})$ as (η,μ)-orthogonal-parabolic endomorphisms commuting with the classical $\cup$-product by the first Chern class $c_1(\mathbb{G})$;
- the action of the shift functor $[1]\colon \mathcal{D}^b(\mathbb{G}) \to \mathcal{D}^b(\mathbb{G})$ on the objects of a full exceptional collection projects as an action of $(\mathbb{Z}/2\mathbb{Z})^{\times 6}$ on $K_0(\mathbb{G}) \otimes_{\mathbb{Z}} \mathbb{C}$ by changing of sings of the elements of the corresponding exceptional basis;
- the braid group $\mathcal{B}_6$ acts on the set of exceptional collections (and the corresponding exceptional bases) as follows: the generator $\beta_{i,i+1}$ $(1 \leq i \leq 5)$ transforms the collection $(E_1, \dots, E_{i-1}, E_i, E_{i+1}, E_{i+2} \dots, E_6)$ into $(E_1, \dots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \dots, E_6)$, where the object $L_{E_i}E_{i+1}$ is defined, up to unique isomorphism, by the distinguished triangle

$$L_{E_i}E_{i+1}[-1] \to \mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \to E_{i+1} \to L_{E_i}E_{i+1}.$$

Notice that our definition of braid mutations of exceptional objects differs by the one given, for example, in [GK04] by a shift: this difference is important in order to obtain the coincidence of the braid group action on the matrix representing the morphism $\mathfrak{X}^{\pm}_{\mathbb{G}}$ with the action on the central connection matrix.

Remark 11.4. The conjecture we are discussing was also formulated in [GGI16] contemporarily to [Dub13] for any Fano manifold X. In [GGI16] the authors seem to stress the relevance of the class $\widehat{\Gamma}^+(X)$, while in [Dub13] of $\widehat{\Gamma}^-(X)$. As we will show below, $\widehat{\Gamma}^+(X)$ and $\widehat{\Gamma}^-(X)$ can be interchanged by the action (11.34) of the group $\widetilde{\mathcal{C}}_0(\mu, R)$. For more details, see Section 14.5.

We now show that the monodromy data computed in the previous Section are of the form above for an exceptional collection in the same orbit of the Kapranov collection, under the action of the braid group. The Kapranov exceptional collection for $\mathbb{G}$ is formed by vector bundles $\mathbb{S}^\lambda(\mathcal{S}^*)$ ($\mathcal{S}$ is the tautological bundle), where $\mathbb{S}^\lambda$ denotes the Schur functor corresponding to the Young diagram λ [3]. In the general case of $\mathbb{G}(r,k)$, the graded Chern character of these bundles is given by

$$\mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = s_\lambda(e^{2\pi i x_1}, \dots, e^{2\pi i x_r}) := \frac{\det(e^{2\pi i x_i(\lambda_j + r - j)})_{1\leq i,j\leq r}}{\prod_{i<j}(e^{2\pi i x_i} - e^{2\pi i x_j})}$$

[3]The reader can find the definition of Schur functors as endo-functors of the category of vector spaces in [FH91]. The definition easily extends to the category of vector bundles.

i.e. the Schur polynomial calculated in the Chern roots $x_1, \dots, x_r$ of $\mathcal{S}^*$. In our case we obtain the following classes: posing $a := e^{2\pi i x_1}$ and $b := e^{2\pi i x_2}$ with $x_1 + x_2 = \sigma_1$ and $x_1 x_2 = \sigma_{1,1}$ we have that

$$\begin{aligned}
\text{for } \lambda = 0 \quad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = 1,\\
\text{for } \lambda = \ydiagram{1}, \qquad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = a + b,\\
\text{for } \lambda = \ydiagram{2}, \qquad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = (a+b)^2 - ab,\\
\text{for } \lambda = \ydiagram{1,1}, \qquad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = ab,\\
\text{for } \lambda = \ydiagram{2,1}, \qquad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = (a+b)ab,\\
\text{for } \lambda = \ydiagram{2,2}, \qquad & \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right) = a^2 b^2.
\end{aligned}$$

Observing that

$$\begin{aligned}
ab &= 1 + 2\pi i \sigma_1 - 2\pi^2(\sigma_2 + \sigma_1) - \frac{8}{3} i \pi^3 \sigma_{2,1} + \frac{4}{3}\pi^4 \sigma_{2,2},\\
a + b &= 2 + 2\pi i \sigma_1 - 2\pi^2 \sigma_2 + 2\pi^2 \sigma_{1,1} + \frac{4}{3} i \pi^3 \sigma_{2,1},
\end{aligned}$$

after some computations one obtains all graded Chern characters. Recalling the value of the $\widehat{\Gamma}^{\mp}$-class

$$\begin{aligned}
\widehat{\Gamma}^{\mp}(\mathbb{G}) = 1 \pm 4\gamma\sigma_1 + \frac{1}{6}\left(48\gamma^2 + \pi^2\right)(\sigma_{1,1} + \sigma_2) \pm \frac{4}{3}(16\gamma^3 + \gamma\pi^2 - \zeta(3))\sigma_{2,1}\\
+ \frac{1}{36}\left(768\gamma^4 + 96\gamma^2\pi^2 - \pi^4 - 192\gamma\zeta(3)\right)\sigma_{2,2}
\end{aligned}$$

we can explicitly compute all the classes

$$\frac{1}{4\pi^2 c^{\frac{1}{2}}}\left(\widehat{\Gamma}^{\mp}(\mathbb{G}) \wedge \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right)\right).$$

We denote by $C^{\mp}_{\mathrm{Kap}}$ the matrix obtained in this way: in appendix A the reader can find the entries of the matrix C^{-}_{Kap}.

The Stokes matrix can be put in triangular form by a suitable permutation of $(u_1, ..., u_6)$, to which a permutation matrix P is associated, according to the transformations (2.38). There are two permutations which yield PSP^{-1} in triangular form, namely

$$\tau_1 \colon (u_1, u_2, u_3, u_4, u_5, u_6) \mapsto (u_1', u_2', u_3', u_4', u_5', u_6') := (u_5, u_4, u_2, u_1, u_3, u_6), \tag{11.37}$$

$$\tau_2 \colon (u_1, u_2, u_3, u_4, u_5, u_6) \mapsto (u_1', u_2', u_3', u_4', u_5', u_6') := (u_5, u_4, u_1, u_2, u_3, u_6). \tag{11.38}$$

In both cases, the Stokes matrix S in (11.33), with $v = 6$, becomes

$$S \longmapsto PSP^{-1} = \begin{pmatrix} 1 & -6 & 20 & 20 & 70 & 20 \\ 0 & 1 & -4 & -4 & -16 & -6 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.39}$$

The matrix C in Appendix A, with $v = 6$, becomes

$$C \mapsto CP^{-1} \tag{11.40}$$

A direct computation proves the following:

THEOREM 11.2. *Consider the monodromy data of the quantum cohomology of the Grassmannian* $\mathbb{G}$ *at* $0 \in QH^\bullet(\mathbb{G})$, *as computed in Section 11.3.3 with respect to an admissible line*[4] $\ell = \ell(\phi)$ *of slope* $0 < \phi < \frac{\pi}{4}$ *and w.r.t. the basis of solutions (11.21). These are the matrix* S *in formula (11.33) and the matrix* C *in Appendix A, with* $v = 6$. *Arrange* S *in triangular form as in (11.39), with* P *associated to the one of the permutations* τ_1 *or* τ_2 *above, and transform* C *as in (11.40). The data so obtained are related to the Kapranov exceptional collection by a finite sequence of natural transformations (11.34), (11.35), (11.36). More precisely, the following sequence transforms* CP^{-1} *into* C^-_{Kap}*:*

- *(1) the change of sings in the normalised idempotents vector fields, determined by the action (11.35) of the diagonal matrix* $\mathcal{I} := \operatorname{diag}(1,-1,-1,1,-1,1)$ *(if we start from the cell where* τ_1 *is lexicographical), or* $\mathcal{I} := \operatorname{diag}(1,-1,1,-1,-1,1)$ *(if we start from the cell where* τ_2 *is lexicographical),*
- *(2) change of solution at the origin through the action (11.34), with* G *equal to*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2i\pi & 1 & 0 & 0 & 0 & 0 \\ -2\pi^2 & 2i\pi & 1 & 0 & 0 & 0 \\ -2\pi^2 & 2i\pi & 0 & 1 & 0 & 0 \\ -\frac{1}{3}\left(8i\pi^3\right) & -4\pi^2 & 2i\pi & 2i\pi & 1 & 0 \\ \frac{4\pi^4}{3} & -\frac{1}{3}\left(8i\pi^3\right) & -2\pi^2 & -2\pi^2 & 2i\pi & 1 \end{pmatrix} \in \widetilde{\mathcal{C}}_0(\mu, R),$$

- *(3) the action (11.36) with either the braid* $\beta_{12}\beta_{56}\beta_{45}\beta_{23}\beta_{34}$ *(if we start from the cell where* τ_1 *is lexicographical), or the braid* $\beta_{34}\beta_{12}\beta_{56}\beta_{45}\beta_{23}\beta_{34}$ *(if we start from the cell where* τ_2 *is lexicographical).*

Moreover, CP^{-1} *in* (11.40) *is transformed into* C^+_{Kap} *if, after the sequence of transformations (1),(2),(3) above, the following transformation is further applied:*

- *(4) the action (11.34), with matrix* G *equal to*

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -8\gamma & 1 & 0 & 0 & 0 & 0 \\ 32\gamma^2 & -8\gamma & 1 & 0 & 0 & 0 \\ 32\gamma^2 & -8\gamma & 0 & 1 & 0 & 0 \\ \frac{8}{3}\left(\zeta(3) - 64\gamma^3\right) & 64\gamma^2 & -8\gamma & -8\gamma & 1 & 0 \\ \frac{64}{3}\left(16\gamma^4 - \gamma\zeta(3)\right) & \frac{8}{3}\left(\zeta(3) - 64\gamma^3\right) & 32\gamma^2 & 32\gamma^2 & -8\gamma & 1 \end{pmatrix} \in \widetilde{\mathcal{C}}_0(\mu, R).$$

The connection matrix obtained from CP^{-1} *in* (11.40) *after the sequence (1),(2),(3) or (1),(2),(3),(4) will be denoted* C_{final}.

Let S_{final} *denote the Stokes matrix obtained from* PSP^{-1} *in* (11.39) *by either the sequence (1),(2),(3) or (1)(2)(3)(4) (recall that steps (2) and (4) do not act on* S*). Then,* $(S_{\text{final}})^{-1}$ *coincides with the Gram matrix*

$$G_{\text{Kap}} = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.41}$$

Thus, the data originally computed correspond to the exceptional block collections obtained from the Kapranov block collection by the action of the braid(s) $(\beta_{34})\beta_{12}\beta_{56}\beta_{45}\beta_{23}(\beta_{34})$ *(the action of* β_{34} *acting just as a permutation of the third and fourth elements of the block).*

REMARK 11.5. In both cases $C_{\text{final}} = C^{+}_{\text{Kap}}$ and $C_{\text{final}} = C^{-}_{\text{Kap}}$, the realtion (11.32) holds between C_{final} and S_{final}.

11.3.5. Reconstruction of Monodromy Data along the Small Quantum locus. In this section we reconstruct the monodromy data at all other points of the small quantum cohomology of $\mathbb{G}$, by applying the procedure described in Section 9.1.1, and already illustrated in Section 10.

We identify the small quantum choomology with the set of point $(0, t^2, 0, ..., 0)$. These points can be represented in the real plane $(\operatorname{Re} t^2, \operatorname{Im} t^2)$. At a point $(0, t^2, 0, ..., 0)$, the canonical coordinates are (11.2), so that the Stokes rays are

$$R_{ij}(t^2) = e^{\overline{t^2}/4} R_{ij}(0) \equiv e^{-i \operatorname{Im} t^2/4} R_{ij}(0),$$

where $R_{jj}(0)$ are the rays R_{ij} of Section 11.3.2. Let ℓ be a line of slope $\varphi \in]0, \pi/4[$, admissible for $t = 0$, i.e. for the the Stokes rays $R_{ij}(0)$. Then, whenever $\operatorname{Im} t^2 \in \pi \cdot \mathbb{Z} - 4\phi$, at least a pair of rays $R_{ij}(t^2)$ and $R_{ji}(t^2)$ lie along the line ℓ, for some (i, j). This means that the small quantum cohomology of $\mathbb{G}$ is split into the following horizontal bands of the $(\operatorname{Re} t^2, \operatorname{Im} t^2)$-plane:

$$\mathcal{H}_k := \{t^2 : \; k\pi - 4\phi < \operatorname{Im} t^2 < (k+1)\pi - 4\phi\}, \quad k \in \mathbb{Z}.$$

If t^2 varies along a curve connecting two neighbouring bands, at least a pair of opposite rays $R_{ij}(t^2)$ and $R_{ji}(t^2)$ cross ℓ in correspondence with t^2 crossing the border between the bands.

A point $(0, t^2, 0, ..., 0)$, such that t^2 is interior to a band, is a semisimple coalescence point, where Theorem 9.1 applies. The polydisc $\mathcal{U}_{\epsilon_1}(u(0, t^2, ..., 0))$ is split into two ℓ-cells. Each cell correspons, through the coordinate map $p \mapsto u(p)$, to the closure of an open connected subset of an ℓ-chamber of $QH^\bullet(\mathbb{G})$, as explained in Section 9.1.1. Therefore, each band $\mathcal{H}_k$ precisely belongs to the boundary of two ℓ-chambers corresponding to the two cells, while each line $\operatorname{Im} t^2 = k\pi - 4\phi$ between two bands $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$ belongs to the intersection of the boundaries of four neighbouring chambers of $QH^\bullet(\mathbb{G})$. As explained in Section 9.1.1, the monodromy data computed via Theorem 9.1 in $\mathcal{U}_{\epsilon_1}(u(0, t^2, ..., 0))$ are the data of the two chambers shearing the boundary $\mathcal{H}_k$. In particular, as a necessary consequence of Theorem 9.1, these data are the data at each point of $\mathcal{H}_k$. This means that *the monodromy data are constant in each band* $\mathcal{H}_k$.

In order to compute the monodromy data in each chamber of $QH^\bullet(\mathbb{G})$ is sufficies to apply the procedure of Section 9.1.1 starting from the data C, S computed at $t = 0$ in Section 11.3.3. Preliminary, by a permutation P, we have obtain upper triangular PSP^{-1} and the corresponding CP^{-1} in (11.39) and (11.40), which are the monodromy data in the cell of $\mathcal{U}_{\epsilon_1}(u'(0, 0, ..., 0))$ where $u'_1(0, 0, ..., 0), ..., u'_n(0, 0, ..., 0)$ are in lexicographical order as in (11.37) or (11.38). Thus, they are the data of the band $\mathcal{H}_0$. Then, the braid group actions (2.40) and (2.41) can be applied. In particular, we have computed the action of those braids which allow to pass from the chamber (with lexicographical order) whose boundary contains $\mathcal{H}_0$, to the chambers whose boundary contains $\mathcal{H}_k$, for $k = 1, 2, ..., 8$. The values of S and C so obtained are, as explained above, the constant monodromy data for $\mathcal{H}_0$, $\mathcal{H}_1$, ..., $\mathcal{H}_8$. They are reported in Table 11.1. From the table, we can read the monodromy data for the

[4]The computations have been done for $\phi = \pi/6$, but nothing changes if $0 < \phi < \frac{\pi}{4}$, since the sectors where the asymptotic behaviours are studied always are the same $\mathcal{S}_{\text{left/right}}$.

whole small quantum cohomology, since for any $k \in \mathbb{Z}$, the data for $\mathcal{H}_{k+8}$ are the same as for $\mathcal{H}_k$, as will be clear from the explanation below.

In order to determine the braid connecting neighbouring $\mathcal{H}_k$'s, it suffices to consider a fixed configuration of distinct $u_1(0,t^2,...,0),...,u_1(0,t^2,...,0)$ in lexicographical order, corresponding to a fixed $t=(0,t^2,0,..,0)$ slightly away from $t=0$. The corresponding rays $R_{ij}(t^2)$ are fixed. Then, we let ℓ rotate and keep track of the rays which are crossed by ℓ. Indeed, the motion of the point $(0,t^2,0,\dots,0)$ by increasing $\mathrm{Im}(t^2)$ determines a uniform clockwise rotation of the Stokes rays, whose effect is the same of a counter-clockwise rotation of the admissible line (by increasing its slope ϕ) and the consequent gliding of the ℓ-horizontal bands towards $\mathrm{Im}(t^2)\to-\infty$. The result is resumed in Figure 11.3. Note that each time ℓ crosses a ray, the coordinates u_i's must be relabelled in lexicographical order. As it appears in Figure 11.3, the passage from $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$ is obtained by an alternate compositions of the braids

$$\omega_1 := \beta_{12}\beta_{56}, \qquad \omega_2 := \beta_{23}\beta_{45}\beta_{34}\beta_{23}\beta_{45}.$$

Coherently with Lemma 2.5, after a complete mutation of the admissible line ℓ, the braid acting on the monodromy data is $(\omega_1\omega_2)^4 = (\beta_{12}\beta_{23}\beta_{34}\beta_{45}\beta_{56})^6$, the generator of the center of the braid group $\mathcal{B}_6$. This corresponds to the cyclical repetition of the same Stokes matrix in $\mathcal{H}_k$ and $\mathcal{H}_{k+8}$ (while C is shifted to $M_0^{-1}C$).

Remark 11.6. There is a remarkable symmetry between the above cyclical repetition and the fact that exceptional collections are organised in algebraic structures called *helices*, introduced in [**Gor88**] [**GR87**], and extensively developed in [**Gor90**] [**Gor94a**] [**GK04**]. This will be thoroughly explained in Part 4 (and in a forthcoming paper [**CDG17a**]).

Table 11.1: List of all possible Stokes matrices in bands decomposing the small quantum cohomology of $\mathbb{G}$: the computation is done at a point $(0,t^2,0,\dots,0)$ w.r.t. a line ℓ of slope $\phi\in]0,\pi/4[$, admissible for $t=0$. The starting matrix S_{lex} in $\mathcal{H}_0$ is PSP^{-1} of formula (11.39), with signs changed by (11.35) with $\mathcal{I}=\mathrm{diag}(-1,1,1,-1,1,-1)$. The braid acting on the monodromy data are $\omega_1:=\beta_{12}\beta_{56}$ and $\omega_2:=\beta_{23}\beta_{45}\beta_{34}\beta_{23}\beta_{45}$.

Band $\mathcal{H}_k$	S_{lex}	Braid
$0<\mathrm{Im}(t^2)+4\phi<\pi$	$\begin{pmatrix} 1 & 6 & -20 & 20 & -70 & 20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	id

Continued on next page

Table 11.1 – *Continued from previous page*

Band $\mathcal{H}_k$	S_{lex}	Braid
$\pi < \text{Im}(t^2) + 4\phi < 2\pi$	$\begin{pmatrix} 1 & -6 & -4 & 4 & 6 & 20 \\ 0 & 1 & 4 & -4 & -16 & -70 \\ 0 & 0 & 1 & 0 & -4 & -20 \\ 0 & 0 & 0 & 1 & 4 & 20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	ω_1
$2\pi < \text{Im}(t^2) + 4\phi < 3\pi$	$\begin{pmatrix} 1 & 6 & 20 & -20 & -70 & 20 \\ 0 & 1 & 4 & -4 & -16 & 6 \\ 0 & 0 & 1 & 0 & -4 & 4 \\ 0 & 0 & 0 & 1 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2$
$3\pi < \text{Im}(t^2) + 4\phi < 4\pi$	$\begin{pmatrix} 1 & -6 & 4 & -4 & 6 & 20 \\ 0 & 1 & -4 & 4 & -16 & -70 \\ 0 & 0 & 1 & 0 & 4 & 20 \\ 0 & 0 & 0 & 1 & -4 & -20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1$
$4\pi < \text{Im}(t^2) + 4\phi < 5\pi$	$\begin{pmatrix} 1 & 6 & -20 & 20 & -70 & 20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1\omega_2$
$5\pi < \text{Im}(t^2) + 4\phi < 6\pi$	$\begin{pmatrix} 1 & -6 & -4 & 4 & 6 & 20 \\ 0 & 1 & 4 & -4 & -16 & -70 \\ 0 & 0 & 1 & 0 & -4 & -20 \\ 0 & 0 & 0 & 1 & 4 & 20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1\omega_2\omega_1$

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Table 11.1 – *Continued from previous page*

Band $\mathcal{H}_k$	S_{lex}	Braid
$6\pi < \operatorname{Im}(t^2) + 4\phi < 7\pi$	$\begin{pmatrix} 1 & 6 & 20 & -20 & -70 & 20 \\ 0 & 1 & 4 & -4 & -16 & 6 \\ 0 & 0 & 1 & 0 & -4 & 4 \\ 0 & 0 & 0 & 1 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1\omega_2\omega_1\omega_2$
$7\pi < \operatorname{Im}(t^2) + 4\phi < 8\pi$	$\begin{pmatrix} 1 & -6 & 4 & -4 & 6 & 20 \\ 0 & 1 & -4 & 4 & -16 & -70 \\ 0 & 0 & 1 & 0 & 4 & 20 \\ 0 & 0 & 0 & 1 & -4 & -20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1\omega_2\omega_1\omega_2\omega_1$
$8\pi < \operatorname{Im}(t^2) + 4\phi < 9\pi$	$\begin{pmatrix} 1 & 6 & -20 & 20 & -70 & 20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\omega_1\omega_2\omega_1\omega_2\omega_1\omega_2\omega_1\omega_2$

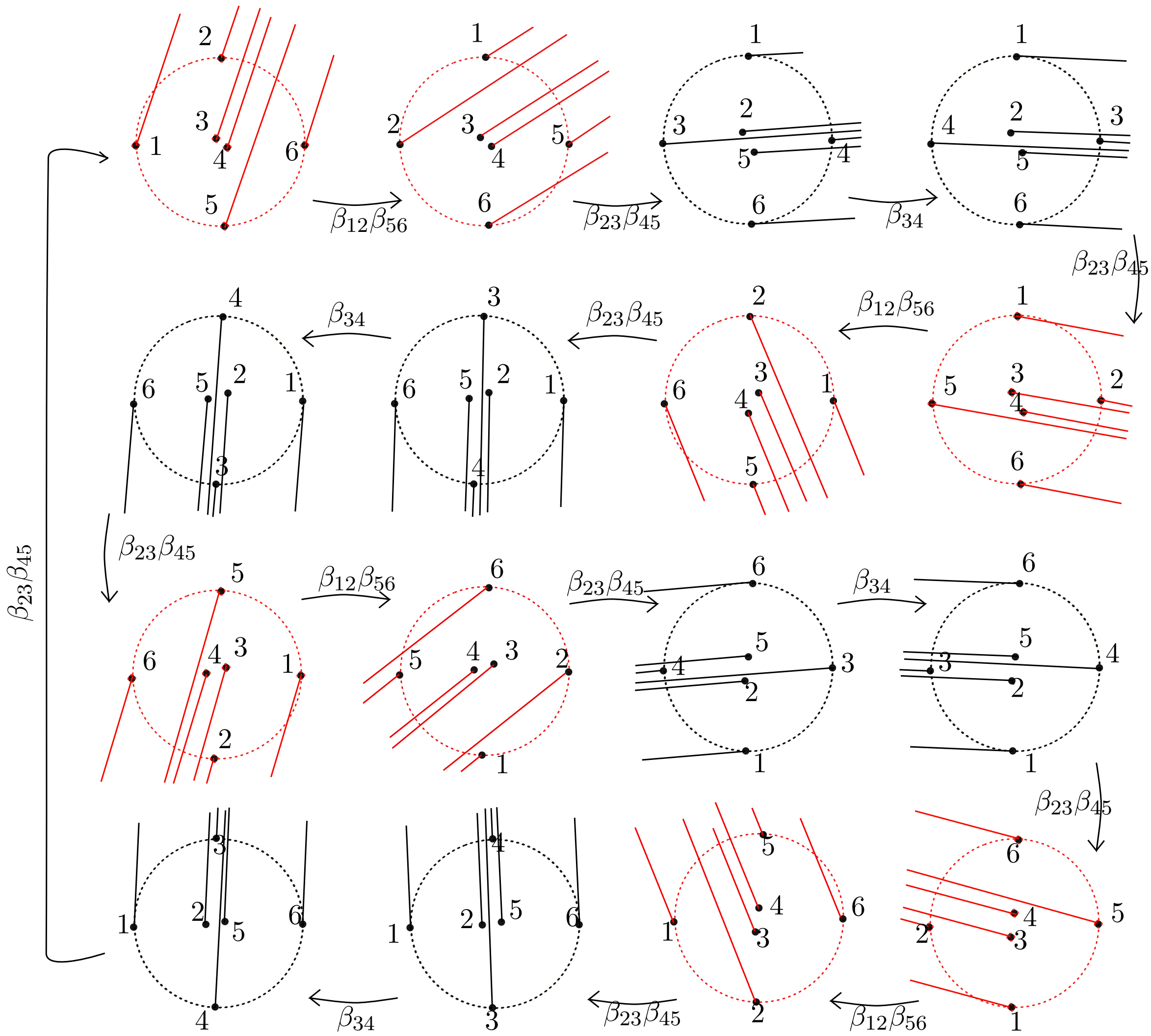

FIGURE 11.3. The picture, to be read in boustrophedon order, shows the braids corresponding to the passage from one band $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$. Starting from the configuration of the canonical coordinates at $0 \in QH^\bullet(\mathbb{G})$, we slightly split the coalescence as described in the first red picture in the first line. The numbers represent the lexicographical order of the canonical coordinates w.r.t. the admissible line. Letting the admissible line ℓ continuously rotate by increasing its slope, we determine all elementary braids acting in the mutation up to the next red configuration. By coalescence of the points u_3, u_4 in a red picture we obtain a configuration of canonical coordinates realized in the locus of small quantum cohomology. Thus we deduce that successive bands of the small quantum cohomology are related by alternate compositions of the braids $\omega_1 := \beta_{12}\beta_{56}$ and $\omega_2 := \beta_{23}\beta_{45}\beta_{34}\beta_{23}\beta_{45}$.

Part 4

Helix Structures in Quantum Cohomology of Fano Manifolds

περὶ τῶν ἀφανέων,
περὶ τῶν θνητῶν
σαφήνειαν μὲν θεοὶ ἔχοντι,
ὡς δὲ ἀνθρώποις
τεκμαίρεσθαι καὶ τὰ ἑξῆς.

Ἀλκμαίων, *Fr. B1 DK*

CHAPTER 12

Helix Theory in Triangulated Categories

ABSTRACT. In this Chapter we review the general theory of Helices in triangular categories as developed by the Moscow School of Algebraic Geometry (see [**Rud90**], [**GK04**]). We recall the notions of *exceptional objects*, *exceptional collections* in a $\mathbb{K}$-linear triangulated category $\mathscr{D}$, and we define their mutations under the action of the braid group. Then, we introduced the strictly related notion of *semiorthogonal decomposition* and we study the properties of *admissibility* of full triangulated subcategories as well as of *saturatedness* of $\mathscr{D}$. The problem of the existence of Serre functors is also discussed. We finally introduce the notions of *dual* exceptional collections and of *helix* generated by an exceptional collection.

12.1. Notations and preliminaries

Let $\mathbb{K}$ be a field[1]. We denote by $\mathrm{GrVect}_{\mathbb{K}}^{<\infty}$ the category of finite dimensional $\mathbb{Z}$-graded vector spaces[2]: it is a triangulated category, the shift being defined by

$$\mathrm{Gr}^p(V^\bullet[k]) := \mathrm{Gr}^{p+k}(V^\bullet), \quad p, k \in \mathbb{Z},$$

and we also have operations of *tensor product* and *dualization* with the usual gradations

$$\mathrm{Gr}^p(V^\bullet \otimes W^\bullet) := \bigoplus_{i+j=p} \mathrm{Gr}^i(V^\bullet) \otimes \mathrm{Gr}^j(W^\bullet), \qquad \mathrm{Gr}^p\left((V^\bullet)^\vee\right) := \left(\mathrm{Gr}^{-p}(V^\bullet)\right)^\vee.$$

The category $\mathrm{GrVect}_{\mathbb{K}}^{<\infty}$ is equivalent to the bounded derived category of finite dimensional $\mathbb{K}$-vector spaces, denoted by $\mathcal{D}^b(\mathbb{K})$: the equivalence is realized by the functors

$$\Phi\colon \mathrm{GrVect}_{\mathbb{K}}^{<\infty} \to \mathcal{D}^b(\mathbb{K})\colon V^\bullet \mapsto \bigoplus_{i\in\mathbb{Z}} (\mathrm{Gr}^i V^\bullet)[-i], \quad \text{with zero differentials},$$

$$H^\bullet\colon \mathcal{D}^b(\mathbb{K}) \to \mathrm{GrVect}_{\mathbb{K}}^{<\infty}\colon F^\bullet \mapsto H^\bullet(F^\bullet).$$

Let $\mathscr{D}$ be a triangulated category. We will assume that $\mathscr{D}$ is a $\mathbb{K}$-*linear category of finite type* (or Hom-*finite*), i.e. that

$$\mathrm{Hom}^\bullet(X, Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(X, Y)$$

is a finite dimensional graded $\mathbb{K}$-vector space for all $X, Y \in \mathrm{Ob}(\mathscr{D})$, and where we posed $\mathrm{Hom}^i(X, Y) := \mathrm{Hom}(X, Y[i])$ for any $i \in \mathbb{Z}$. Sometimes, it will be useful to consider the category $\mathscr{D}$ to be $\mathscr{D}^b(\mathbb{K})$-enriched, by identifying the graded vector spaces $\mathrm{Hom}^\bullet(X, Y)$ with the associated complex through the equivalence Φ above.

[1] Here we work on a general ground field $\mathbb{K}$, but Starting from Section 13.5 we will specialize to the case $\mathbb{K} = \mathbb{C}$.
[2] In what follows we will denote the p-th degree of $V^\bullet$ by $\mathrm{Gr}^p(V^\bullet)$ or V^p.

DEFINITION 12.1. Let $V^\bullet$ be a f.d. graded $\mathbb{K}$-vector space and X be an object in a $\mathbb{K}$-linear triangulated category $\mathscr{D}$. We define the tensor product $V^\bullet \otimes X$, an object of $\mathscr{D}$, as a solution of a universal problem, by requiring

$$\mathrm{Hom}^\bullet(Y, V^\bullet \otimes X) = V^\bullet \otimes \mathrm{Hom}^\bullet(Y, X) \quad \forall Y \in \mathrm{Ob}(\mathscr{D}).$$

Such a universal problem admits a solution: the tensor product can be constructed as

$$V^\bullet \otimes X := \bigoplus_i V^i \otimes X[-i],$$

where

$$V^i \otimes X[-i] := \underbrace{X[-i] \oplus \cdots \oplus X[-i]}_{\dim_{\mathbb{K}} V^i \text{ times}}.$$

REMARK 12.1. We can define the analogous operation of tensor product $-\otimes-\colon \mathcal{D}^b(\mathbb{K}) \times \mathscr{D} \to \mathscr{D}$ by compisition with the cohomology functor in the first entry:

$$\mathcal{D}^b(\mathbb{K}) \times \mathscr{D} \xrightarrow{H^\bullet \times \mathbb{1}_{\mathscr{D}}} \mathrm{GrVect}_{\mathbb{K}}^{<\infty} \times \mathscr{D} \xrightarrow{-\otimes-} \mathscr{D}.$$

In this way, the object $F^\bullet \otimes X$ depends only on the quasi-isomorphism class of $F^\bullet$.

LEMMA 12.1. *If $V^\bullet \in \mathrm{Ob}(\mathrm{GrVect}_{\mathbb{K}}^{<\infty})$, $X \in \mathrm{Ob}(\mathscr{D})$ and if $j, k \in \mathbb{Z}$, then*

$$V^\bullet[j] \otimes X[k] = (V^\bullet \otimes X)[j+k], \quad (V^\bullet[j])^\vee = (V^\bullet)^\vee[-j].$$

PROOF. For the first equality it is easy to see that the r.h.s. solves the universal problem which defines the l.h.s.. The second equality it is trivially deduced by a direct comparison of the gradings. ☐

DEFINITION 12.2. If $\mathscr{D}$ and $\mathscr{E}$ are two $\mathbb{K}$-linear triangulated categories, a covariant exact functor $F\colon \mathscr{D} \to \mathscr{E}$ is called *linear* if

$$F(V^\bullet \otimes X) = V^\bullet \otimes F(X)$$

for any graded vector space $V^\bullet$ and any object X. Analogously, a contravariant functor $F\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{E}$ is *linear* if it satisfies

$$F(V^\bullet \otimes X) = (V^\bullet)^\vee \otimes F(X)$$

for any graded vector space $V^\bullet$ and any object X.

So, in particular, the bifunctor $\mathrm{Hom}^\bullet(-,-)\colon \mathscr{D} \times \mathscr{D}^{\mathrm{op}} \to \mathrm{GrVect}_{\mathbb{K}}^{<\infty}$ is bilinear:

$$\mathrm{Hom}^\bullet(W^\bullet \otimes X, V^\bullet \otimes Y) = (W^\bullet)^\vee \otimes V^\bullet \otimes \mathrm{Hom}^\bullet(X, Y)$$

for any $X, Y \in \mathrm{Ob}(\mathscr{D})$ and any graded vector spaces $V^\bullet$ and $W^\bullet$. Thus, for any $X, Y \in \mathrm{Ob}(\mathscr{D})$, we have the identifications

$$\mathrm{End}(\mathrm{Hom}^\bullet(X,Y)) = \mathrm{Hom}^\bullet(\mathrm{Hom}^\bullet(X,Y) \otimes X, Y) = \mathrm{Hom}^\bullet(X, (\mathrm{Hom}^\bullet)^\vee(X,Y) \otimes Y).$$

Hence, the identity morphism $\mathrm{id}\colon \mathrm{Hom}^\bullet(X,Y) \to \mathrm{Hom}^\bullet(X,Y)$ induces two canonical morphisms

$$j^*(X,Y)\colon \mathrm{Hom}^\bullet(X,Y) \otimes X \to Y,$$
$$j_*(X,Y)\colon X \to (\mathrm{Hom}^\bullet)^\vee(X,Y) \otimes Y.$$

PROPOSITION 12.1. *Let $E \in \mathrm{Ob}(\mathscr{D})$ be a generic object. Let us define the functors*

$$\Phi_E\colon \mathcal{D}^b(\mathbb{K}) \to \mathscr{D}\colon \ V^\bullet \mapsto V^\bullet \otimes E,$$
$$\Phi_E^*\colon \mathscr{D} \to \mathcal{D}^b(\mathbb{K})\colon \ X \mapsto (\mathrm{Hom}^\bullet)^\vee(X, E),$$
$$\Phi_E^!\colon \mathscr{D} \to \mathcal{D}^b(\mathbb{K})\colon \ X \mapsto \mathrm{Hom}^\bullet(E, X).$$

We have the the adjunctions $\Phi_E^* \dashv \Phi_E \dashv \Phi_E^!$.

PROOF. This is a simple check of the definition of adjoint functors. Notice that the the unity of the adjunction $\Phi_E^* \dashv \Phi_E$ and counity of the adjunction $\Phi_E \dashv \Phi_E^!$ are given by the morphisms $j_*(-, E)$ and $j^*(E, -)$ respectively. □

DEFINITION 12.3 (Generated triangulated subcategory). If $\Omega \subseteq \mathrm{Ob}(\mathscr{D})$, we denote by $\langle \Omega \rangle$ the smallest full triangulated subcategory of $\mathscr{D}$ containing all objects of Ω.

DEFINITION 12.4. If $\mathcal{A}, \mathcal{B} \subseteq \mathrm{Ob}(\mathscr{D})$ we define the set

$$\mathcal{A} * \mathcal{B} := \{X \in \mathrm{Ob}(\mathscr{D})\colon A \to X \to B \to A[1], \text{ for some } A \in \mathcal{A},\ B \in \mathcal{B}\}\,.$$

Notice by the octahedral axiom (TR4) that the operation $*$ is associative.

The subcategory $\langle \Omega \rangle$ is obtained by taking the closure with respect to shifts and cones. More precisely, we have the following

PROPOSITION 12.2. *Let $\Omega \subseteq \mathrm{Ob}(\mathscr{D})$, and let us define*

$$\Omega_1 := \{X[n]\colon X \in \Omega, n \in \mathbb{Z}\}\,, \quad \Omega_r := \underbrace{\Omega_1 * \cdots * \Omega_1}_{r \text{ times}}\,.$$

Then

$$\langle \Omega \rangle \equiv \bigcup_{r \in \mathbb{N}^*} \Omega_r.$$

12.2. Exceptional Objects and Mutations

Let $\mathscr{D}$ be a $\mathbb{K}$-linear triangulated category.

DEFINITION 12.5 (Exceptional Object, Pair and Collection). An object $E \in \mathrm{Ob}(\mathscr{D})$ is called *exceptional* if $\mathrm{Hom}^\bullet(E, E)$ is a 1-dimensional $\mathbb{K}$-algebra generated by the identity morphism.
An ordered pair (E_1, E_2) of exceptional objects of $\mathscr{D}$ is called *exceptional* or *semiorthogonal* if

$$\mathrm{Hom}^\bullet(E_2, E_1) = 0.$$

More in general, an ordered collection $(E_1, E_2, \dots, E_k)$ of exceptional objects of $\mathscr{D}$ is called *exceptional* or *semiorthogonal* if

$$\mathrm{Hom}^\bullet(E_j, E_i) = 0 \quad \text{whenever } i < j.$$

An exceptional collection is said to be *full* if it generates $\mathscr{D}$, i.e. any full triangulated subcategory containing all objects E_i is equivalent to $\mathscr{D}$ via the inclusion functor.

PROPOSITION 12.3 ([Bon89]). *Let $E \in \mathrm{Ob}(\mathscr{D})$ be a generic object. Then E is exceptional if and only of the functor*

$$\Phi_E\colon \mathcal{D}^b(\mathbb{K}) \to \mathscr{D}\colon\ V^\bullet \mapsto V^\bullet \otimes E$$

is fully faithful. In particular, the category $\langle E \rangle \equiv \mathrm{Im}\, \Phi_E$ is equivalent to the category $\mathcal{D}^b(\mathbb{K})$.

PROOF. Using the notations of Proposition 12.1, Φ_E is fully faithful if and only if the natural transformation $\Phi_E^! \Phi_E \Leftarrow \mathbb{1}_{\mathcal{D}^b(\mathbb{K})}$ is a natural isomorphism. This holds if and only if $\mathrm{Hom}^\bullet(E, E) \cong \mathbb{K}$. □

REMARK 12.2. Given an exceptional collection in $\mathscr{D}$, there are several operations generating other such collections. Indeed, the group $\mathrm{Aut}(\mathscr{D})$ of isomorphism classes of auto-equivalences of the category $\mathscr{D}$ acts on the set of exceptional collections: the element $\Psi \in \mathrm{Aut}(\mathscr{D})$ acts in the obvious way, by associating to the exceptional collection $\mathfrak{E} := (E_1, \dots, E_n)$ the collection $\Psi\mathfrak{E} := (\Psi E_1, \dots, \Psi E_n)$. Analogously, the additive group $\mathbb{Z}^n$ acts on the sets of exceptional collection of length n by shifts:

if $\mathfrak{E} := (E_1, \dots, E_n)$ is an exceptional collection, then also $\mathfrak{E}[\mathbf{k}] := (E_1[k_1], E_2[k_2], \dots, E_n[k_n])$ is exceptional for any $(k_1, \dots, k_n) \in \mathbb{Z}^n$. The actions of both $\mathrm{Aut}(\mathscr{D})$ and $\mathbb{Z}^n$ preserve the fullness of an exceptional collection.

In what follows, we are going to define another nontrivial action of the *braid group* $\mathcal{B}_n$ on the set of (full) exceptional collections of length n.

Definition 12.6 (Orthogonal complements). Let $\mathscr{A}$ be a full triangulated subcategory of $\mathscr{D}$. We introduce the two full triangulated subcategories ${}^{\perp}\mathscr{A}$ and $\mathscr{A}^{\perp}$ defined by

$$
{}^{\perp}\mathscr{A} := \{X \in \mathrm{Ob}(\mathscr{D})\colon\ \mathrm{Hom}(X, A) = 0 \text{ for all } A \in \mathrm{Ob}(\mathscr{A})\},
$$

$$
\mathscr{A}^{\perp} := \{X \in \mathrm{Ob}(\mathscr{D})\colon\ \mathrm{Hom}(A, X) = 0 \text{ for all } A \in \mathrm{Ob}(\mathscr{A})\}.
$$

These subcategories are called respectively *left* and *right orthogonals* to $\mathscr{A}$ in $\mathscr{D}$.

Remark 12.3. It is easy to see that, if $\mathscr{A} = \langle E \rangle$ is the smallest triangulated subcategory containing an object $E \in \mathrm{Ob}(\mathscr{D})$, the following characterization of the orthogonal complements ${}^{\perp}\langle E \rangle$ and $\langle E \rangle^{\perp}$ holds:

$$
{}^{\perp}\langle E \rangle \equiv {}^{\perp}E, \quad \text{where } {}^{\perp}E := \{X \in \mathrm{Ob}(\mathscr{D})\colon\ \mathrm{Hom}^\bullet(X, E) = 0\},
$$

$$
\langle E \rangle^{\perp} \equiv E^{\perp}, \quad \text{where } E^{\perp} := \{X \in \mathrm{Ob}(\mathscr{D})\colon\ \mathrm{Hom}^\bullet(E, X) = 0\}.
$$

Definition 12.7 (Mutations of objects). Let $E \in \mathrm{Ob}(\mathscr{D})$ be an exceptional object. For any $X \in \mathrm{Ob}(\mathscr{D})$ we can define two new objects

$$
\mathbb{L}_E X \in \mathrm{Ob}(E^{\perp}), \quad \mathbb{R}_E X \in \mathrm{Ob}({}^{\perp}E)
$$

called respectively *left* and *right mutations* of X with respect to E. These two objects are defined as the cones

$$
\mathbb{L}_E(X) := \mathrm{Cone}\left(\Phi_E \Phi_E^!(X) \to X\right), \quad \mathbb{R}_E(X) := \mathrm{Cone}\left(X \to \Phi_E \Phi_E^*(X)\right)[-1],
$$

where the functors $\Phi_E, \Phi_E^*, \Phi_E^!$ are the ones introduced in Proposition 12.1. We thus have the distinguished triangles

$$
\mathbb{L}_E X[-1] \longrightarrow \mathrm{Hom}^\bullet(E, X) \otimes E \xrightarrow{j^*} X \longrightarrow \mathbb{L}_E X \tag{12.1}
$$

$$
\mathbb{R}_E X \longrightarrow X \xrightarrow{j_*} (\mathrm{Hom}^\bullet)^\vee(X, E) \otimes E \longrightarrow \mathbb{R}_E X[1] \tag{12.2}
$$

extending the canonical morphisms $j^*(E, X)$ and $j_*(X, E)$.

By applying the functor $\mathrm{Hom}^\bullet(E, -)$ to (12.1), and the functor $\mathrm{Hom}^\bullet(-, E)$ to (12.2), and using the fact that E is exceptional, we obtain the *orthogonality relations*

$$
\mathrm{Hom}^\bullet(E, \mathbb{L}_E X) = 0, \quad \mathrm{Hom}^\bullet(\mathbb{R}_E X, E) = 0. \tag{12.3}
$$

Remark 12.4. Our definitions of $\mathbb{L}_E X$ and $\mathbb{R}_E X$ differ from the original ones given in [GK04] by a shift operator. In particular,

- what here we denote $\mathbb{L}_E X$, in [GK04] is $\mathbb{L}_E X[1]$,
- and our $\mathbb{R}_E X$ in [GK04] is $\mathbb{R}_E X[-1]$.

The reason of such a choice will be explained later. In what follows we will reformulate some results of [GK04], adapted to our definition, and we leave to the reader the easy and small modifications of some proofs.

In general, the third term in a distinguished triangle is not *canonically* defined by the other two terms. In this case, however, the objects $\mathbb{L}_E X$ and $\mathbb{R}_E X$ are *unique up to unique* isomorphism because of the orthogonality relations. Indeed, let assume that we have the two following distinguished triangles[3]:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(E,X)\otimes E \arrow[r, "j^*"] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & S \arrow[r] \arrow[d, dotted, "h"] & (\mathrm{Hom}^\bullet(E,X)\otimes E)\,[1] \arrow[d, equal] \\
\mathrm{Hom}^\bullet(E,X)\otimes E \arrow[r, "j^*"] & X \arrow[r, "\alpha"] & \mathbb{L}_E X \arrow[r] & (\mathrm{Hom}^\bullet(E,X)\otimes E)\,[1]
\end{tikzcd}$$

Then, by axiom TR3 of triangulated category there exists a morphism $h\colon S \to \mathbb{L}_E X$, which necessarily is an isomorphism (the other two vertical maps being the identities). We want to show that h is unique. Let us apply the functor $\mathrm{Hom}^\bullet(-,\mathbb{L}_E X)$ to the first line of the diagram: recalling that the shift functor $T = (-)[1]$ is an auto-equivalence, and using the orthogonality relations (12.3) we obtain

$$\begin{aligned}
\mathrm{Hom}^\bullet\left((\mathrm{Hom}^\bullet(E,X)\otimes E)\,[1], \mathbb{L}_E X\right) &= \mathrm{Hom}^\bullet\left(\mathrm{Hom}^\bullet(E,X)\otimes E, \mathbb{L}_E X[-1]\right) \\
&= (\mathrm{Hom}^\bullet)^\vee(E,X)\otimes \mathrm{Hom}^\bullet(E,\mathbb{L}_E X[-1]) \\
&= 0.
\end{aligned}$$

So we get the long exact sequence

$$\dots \longrightarrow 0 \longrightarrow \mathrm{Hom}^\bullet(S,\mathbb{L}_E X) \longrightarrow \mathrm{Hom}^\bullet(X,\mathbb{L}_E X) \longrightarrow \dots$$

Since $\alpha \circ j^* = 0$, there exists $h \in \mathrm{Hom}^\bullet(S,\mathbb{L}_E X)$ as above, which must be unique by injectivity. Analogously one shows that $\mathbb{R}_E X$ is unique up to unique isomorphism. Notice, in particular, that $\mathbb{L}_E E = \mathbb{R}_E E = 0$.

Moreover, as a consequence of axiom TR1, we necessarily have that

$$\mathbb{L}_E X = X \quad \text{for all } X \in E^\perp, \tag{12.4}$$

$$\mathbb{R}_E X = X \quad \text{for all } X \in {}^\perp E, \tag{12.5}$$

which means that the operations $\mathbb{L}_E, \mathbb{R}_E$ are *projections* onto the subcategories $E^\perp$ and ${}^\perp E$. Some other useful properties of these projections are summarized in the following

PROPOSITION 12.4 ([GK04]). *Let $\mathscr{D}$ be a $\mathbb{K}$-linear triangulated category, and E an exceptional object.*

(1) *For any object $X \in \mathrm{Ob}(\mathscr{D})$ and any pair of integer $k,\ell \in \mathbb{Z}$ we have*

$$\mathbb{L}_{E[k]}\,(X[\ell]) = (\mathbb{L}_E X)\,[\ell], \quad \mathbb{R}_{E[k]}\,(X[\ell]) = (\mathbb{R}_E X)\,[\ell].$$

(2) *If $E' \in \mathrm{Ob}({}^\perp E), E'' \in \mathrm{Ob}(E^\perp)$ and $X \in \mathrm{Ob}(\mathscr{D})$, the following bifunctor isomorphisms hold*

$$\mathrm{Hom}^\bullet(E',X) = \mathrm{Hom}^\bullet(E',\mathbb{R}_E X) = \mathrm{Hom}^\bullet(E',\mathbb{L}_E X) = \mathrm{Hom}(\mathbb{L}_E E',\mathbb{L}_E X), \tag{12.6}$$

$$\mathrm{Hom}^\bullet(X,E'') = \mathrm{Hom}^\bullet(\mathbb{L}_E X,E'') = \mathrm{Hom}^\bullet(\mathbb{R}_E X,E'') = \mathrm{Hom}(\mathbb{R}_E X,\mathbb{R}_E E''). \tag{12.7}$$

(3) *The functors*

$$\mathscr{D} \xrightarrow{X \mapsto \mathbb{R}_E X} {}^\perp E$$

$$\mathscr{D} \xrightarrow{X \mapsto \mathbb{L}_E X} E^\perp$$

[3]Here we use the axiom TR2 of triangulated category.

are respectively the right adjoint functor to the inclusion ${}^{\perp}E \hookrightarrow \mathscr{D}$ *, and the left adjoint functor to the inclusion* $E^{\perp} \hookrightarrow \mathscr{D}$ *.*

(4) *The following identities hold*

$$\mathbb{L}_E \circ \mathbb{R}_E = \mathbb{L}_E, \quad \mathbb{R}_E \circ \mathbb{L}_E = \mathbb{R}_E.$$

(5) *The restrictions*

$$\mathbb{L}_E|_{{}^{\perp}E} \colon {}^{\perp}E \to E^{\perp} \quad \textit{and} \quad \mathbb{R}_E|_{E^{\perp}} \colon E^{\perp} \to {}^{\perp}E$$

are functors inverse to each other, establishing an isomorphism between these two subcategories.

(6) *The following isomorphism holds functorially on* X *and* E

$$(\mathrm{Hom}^{\bullet})^{\vee}(\mathbb{L}_E X[-1], E) = \mathrm{Hom}^{\bullet}(E, \mathbb{R}_E X).$$

Proof. Let us prove (1). Applying Lemma 12.1 we have that

$$\mathrm{Hom}^{\bullet}(E[k], X[\ell]) \otimes E[k] = (\mathrm{Hom}^{\bullet}(E, X) \otimes E)\,[\ell],$$

and that

$$(\mathrm{Hom}^{\bullet})^{\vee}(X[\ell], E[k]) \otimes E[k] = ((\mathrm{Hom}^{\bullet})^{\vee}(X, E) \otimes E)\,[\ell].$$

Moreover, it is easily seen that the following diagrams are commutative:

$$\begin{tikzcd} X[\ell] \arrow[r, "j_*(X[\ell],E[k])"] \arrow[dr, "j_*(X,E)[\ell]"'] & (\mathrm{Hom}^{\bullet})^{\vee}(X[\ell], E[k]) \otimes E[k] \arrow[d, equal] \\ & ((\mathrm{Hom}^{\bullet})^{\vee}(X, E) \otimes E)\,[\ell] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}^{\bullet}(E[k], X[\ell]) \otimes E[k] \arrow[r, "j^*(E[k],X[\ell])"] \arrow[d, equal] & X[\ell] \\ (\mathrm{Hom}(E, X) \otimes X)[\ell] \arrow[ur, "j^*(E,X)[\ell]"'] & \end{tikzcd}$$

Thus, applying the shift functor $(-)[\ell]$ to (12.1), (12.2) we obtain

$$\mathbb{L}_E X[\ell-1] \longrightarrow (\mathrm{Hom}^{\bullet}(E, X) \otimes E)[\ell] \xrightarrow{(-1)^{\ell} j^*[\ell]} X[\ell] \longrightarrow \mathbb{L}_E X[\ell],$$

$$\mathbb{R}_E X[\ell] \longrightarrow X[\ell] \xrightarrow{(-1)^{\ell} j_*[\ell]} ((\mathrm{Hom}^{\bullet})^{\vee}(X, E) \otimes E)[\ell] \longrightarrow \mathbb{R}_E X[\ell+1].$$

Recalling now that, by the axiom TR1 of triangulated category, one can change the sign of any two morphisms in a distinguished triangle, we conclude. For points (2),(3),(4),(5),(6) we refer to [GK04], where the reader can find and easily adapt the proofs by keeping track of the difference of shiftings in the definition of left and right mutations. □

Definition 12.8. If (E_1, E_2) is an exceptional pair, we define its *left* and *right mutations* to be the pairs

$$\mathbb{L}(E_1, E_2) := (\mathbb{L}_{E_1} E_2, E_1) \quad \text{and} \quad \mathbb{R}(E_1, E_2) := (E_2, \mathbb{R}_{E_2} E_1)$$

respectively.

Proposition 12.5. *The pairs* $\mathbb{L}(E_1, E_2), \mathbb{R}(E_1, E_2)$ *are exceptional. Moreover,* $\mathbb{L}, \mathbb{R}$ *act on the set of exceptional pairs as inverse transformations:*

$$\mathbb{L} \circ \mathbb{R}(E_1, E_2) = (E_1, E_2) \quad \textit{and} \quad \mathbb{R} \circ \mathbb{L}(E_1, E_2) = (E_1, E_2).$$

PROOF. The first statement follows from the orthogonality relations (12.3). From relations (12.4),(12.5) we have that

$$\mathbb{R}_{E_1}E_2 = E_2 \quad \text{and} \quad \mathbb{L}_{E_2}E_1 = E_1,$$

so that, by Proposition (12.4), we deduce

$$(\mathrm{Hom}^\vee)^\bullet(\mathbb{L}_{E_1}E_2[-1], E_1) = \mathrm{Hom}^\bullet(E_1, \mathbb{R}_{E_1}E_2) = \mathrm{Hom}^\bullet(E_1, E_2),$$
$$\mathrm{Hom}^\bullet(E_2, \mathbb{R}_{E_2}E_1[1]) = (\mathrm{Hom}^\vee)^\bullet(\mathbb{L}_{E_2}E_1, E_1) = (\mathrm{Hom}^\vee)^\bullet(E_1, E_2).$$

It follows that the triangles

$$\mathbb{L}_{E_1}E_2[-1] \longrightarrow \mathrm{Hom}^\bullet(E_1, E_2) \otimes E_1 \longrightarrow E_2 \longrightarrow \mathbb{L}_{E_1}E_2$$

$$\mathbb{L}_{E_1}E_2[-1] \longrightarrow (\mathrm{Hom}^\vee)^\bullet(\mathbb{L}_{E_1}E_2[-1], E_1) \otimes E_1 \longrightarrow \mathbb{R}_{E_1}\mathbb{L}_{E_1}E_2 \longrightarrow \mathbb{L}_{E_1}E_2$$

can be canonically identified, i.e. $\mathbb{R} \circ \mathbb{L}(E_1, E_2) = (E_1, E_2)$. Similarly the other identity follows. □

For a more general exceptional sequence, we give the following

DEFINITION 12.9. Let $(E_0, \dots, E_k)$ be an exceptional collection in $\mathscr{D}$. For $1 \leq i \leq k$ we define

$$\mathbb{L}_i(E_0, \dots, E_k) := (E_0, \dots, \mathbb{L}_{E_{i-1}}E_i, E_{i-1}, \dots, E_k),$$
$$\mathbb{R}_i(E_0, \dots, E_k) := (E_0, \dots, E_i, \mathbb{R}_{E_i}E_{i-1}, \dots, E_k).$$

PROPOSITION 12.6 ([BK89]). *The mutations preserve the exceptionality and satisfy*

$$\mathbb{L}_i\mathbb{R}_i = \mathbb{R}_i\mathbb{L}_i = \mathrm{Id} \tag{12.8}$$
$$\mathbb{L}_i\mathbb{L}_j = \mathbb{L}_j\mathbb{L}_i \quad \textit{for} \quad |i-j| > 1 \tag{12.9}$$
$$\mathbb{L}_{i+1}\mathbb{L}_i\mathbb{L}_{i+1} = \mathbb{L}_i\mathbb{L}_{i+1}\mathbb{L}_i \quad \textit{for} \quad 1 < i < k. \tag{12.10}$$

So the braid group $\mathcal{B}_{k+1}$ acts by the mutations on exceptional objects of length $(k+1)$.

PROOF. The fact that exceptionality is preserved, and the first two relations (12.8),(12.9) follow from the previous results. The only non-trivial relation is (12.10). Let (A, B, C) be an exceptional triple. We have to show the commutativity of the diagram

$$\begin{tikzcd}
 & (A,B,C) \arrow[dl, "\mathbb{L}_1"'] \arrow[dr, "\mathbb{L}_2"] & \\
(\mathbb{L}_A B, A, C) \arrow[d, "\mathbb{L}_2"'] & & (A, \mathbb{L}_B C, B) \arrow[d, "\mathbb{L}_1"] \\
(\mathbb{L}_A B, \mathbb{L}_A C, A) \arrow[dr, "\mathbb{L}_1"'] & & (\mathbb{L}_A\mathbb{L}_B C, A, B) \arrow[dl, "\mathbb{L}_2"] \\
 & (\mathbb{L}_{\mathbb{L}_A B}\mathbb{L}_A C, \mathbb{L}_A B, A) &
\end{tikzcd}$$

So we have to prove that

$$\mathbb{L}_{\mathbb{L}_A B}\mathbb{L}_A C = \mathbb{L}_A\mathbb{L}_B C.$$

Applying the exact linear functor $\mathbb{L}_A|_{\perp A}$ to the canonical triangle

$$\mathbb{L}_B C[-1] \to \mathrm{Hom}^\bullet(B, C) \otimes B \to C \to \mathbb{L}_B C,$$

and recalling that $\mathrm{Hom}^\bullet(B, C) = \mathrm{Hom}^\bullet(\mathbb{L}_A B, \mathbb{L}_A C)$ by Proposition (12.4) we find the triangle

$$\mathbb{L}_A \mathbb{L}_B C[-1] \to \mathrm{Hom}^\bullet(\mathbb{L}_A B, \mathbb{L}_A C) \otimes \mathbb{L}_A B \to \mathbb{L}_A C \to \mathbb{L}_A \mathbb{L}_B C.$$

□

REMARK 12.5. In the previous exposition, we have followed the main references on the subject and we have defined a *left* action of the braid group on the set of eceptional collections in a $\mathbb{K}$-linear triangulated category $\mathscr{D}$. In what follows, in order to establish a prefect correspondence between Helix theory and the theory of local monodromy invariants for quantum cohomologies of Fano manifolds, it will be convenient to consider the braid group $\mathcal{B}_{n+1}$ as acting on the *right* on the set of exceptional collections of length $n+1$: if we denote by $\beta_{i,i+1}$ with $1 \le i \le n$ the generators of the braid group, satisfying the relations

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1}, \quad |i-j| > 1,$$

$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2},$$

and if $\mathfrak{E} = (E_0, \dots, E_n)$ is an exceptional collection, we will define

$$\mathfrak{E}^{\beta_{i,i+1}} := \mathbb{L}_i \mathfrak{E}.$$

We will denote by $\sigma_{i,i+1}$ the inverse of the braid $\beta_{i,i+1}$.

12.3. Semiorthogonal decompositions, admissible subcategories, and mutations functors

Let $\mathscr{D}$ be a $\mathbb{K}$-linear triangulated category. In this section we introduce some definitions generalizing the ones of (full) exceptional collection and of left/right mutations w.r.t. them.

DEFINITION 12.10 ([**BK89**, **BO95**, **BO02**]). A sequence $\mathscr{A}_1, \dots, \mathscr{A}_n$ of full triangulated subcategories of $\mathscr{D}$ is said to be *semiorthogonal* if

$$\mathscr{A}_i \subseteq \mathscr{A}_j^{\perp} \quad \text{for all } i < j.$$

A semiorthogonal sequence $\mathscr{A}_1, \dots, \mathscr{A}_n$ is said to define a *semiorthogonal decoposition* of $\mathscr{D}$ if one of the following equivalent conditions holds:

(1) $\mathscr{D}$ is generated by the $\mathscr{A}_i$, i.e. $\mathscr{D} = \langle \mathscr{A}_i \rangle_{i=1}^n$;
(2) for any $X \in \mathrm{Ob}(\mathscr{D})$ there exists a chain of morphisms

$$\begin{tikzcd}
0 = X_n \arrow[r] & X_{n-1} \arrow[r] \arrow[dl] & \dots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] \arrow[dl] & X_0 = X \arrow[dl] \\
A_n \arrow[u, dotted] & & & A_2 \arrow[u, dotted] & A_1 \arrow[u, dotted] &
\end{tikzcd}$$

with $A_i \in \mathrm{Ob}(\mathscr{A}_i)$.

The equivalence of (1), (2) immediately follows from Proposition 12.2. The chain of morphisms of point (2) is usually called a *filtration* of the object X.

DEFINITION 12.11 (Filtrations and Postnikov systems). Given an object $X \in \mathrm{Ob}(\mathscr{D})$, we call *filtration* of X, the datum of a set of objects $\{X_i\}_{i=1}^m$, and a chain of morphisms

$$0 = X_m \to X_{m-1} \to \cdots \to X_2 \to X_1 \to X_0 = X.$$

These morphisms induce on the cones a family of arrows in the opposite direction, which fit into the diagram

$$\begin{tikzcd}
0 = X_n \arrow[rr] & & X_{n-1} \arrow[rr] \arrow[dl] & & \dots \arrow[rr] \arrow[dl] & & X_2 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & X_0 = X \arrow[dl] \\
& L_n \arrow[ul, dotted] & & L_{n-1} \arrow[ll, dotted, "\delta_{n-1}"] \arrow[ul, dotted] & & \dots \arrow[ll, dotted] & & L_2 \arrow[ll, dotted, "\delta_2"] \arrow[ul, dotted] & & L_1 \arrow[ll, dotted, "\delta_1"] \arrow[ul, dotted]
\end{tikzcd}$$

where $\delta_{i+1} \circ \delta_i = 0$, the bottom triangles are commutative, and the top triangles are distinguished (dashed arrows have degree 1). This diagram is called *(right) Postnikov system* (or *(right) Postnikov tower*), and the object X is called the *canonical convolution* of the Postnikov system.

The following is a simple but fundamental fact

PROPOSITION 12.7. *The Postnikov system induced by a semiorthogonal decomposition is functorial, i.e. given $X, X' \in \mathrm{Ob}(\mathscr{D})$ and a morphism $f\colon X \to X'$ there exists a* unique *prolongation to their Postnikov systems:*

$$\begin{tikzcd}
& A_n \arrow[dl, dotted] \arrow[dd] & & & & A_2 \arrow[dl, dotted] \arrow[dd] & & A_1 \arrow[dl, dotted] \arrow[dd] & \\
0 \arrow[rr] \arrow[dd] & & X_{n-1} \arrow[ul] \arrow[dd] & \dots & X_2 \arrow[rr] \arrow[dd] & & X_1 \arrow[ul] \arrow[rr] \arrow[dd] & & X \arrow[ul] \arrow[dd, "f"] \\
& A'_n \arrow[dl, dotted] & & & & A'_2 \arrow[dl, dotted] & & A'_1 \arrow[dl, dotted] & \\
0 \arrow[rr] & & X'_{n-1} \arrow[ul] & \dots & X'_2 \arrow[rr] & & X'_1 \arrow[ul] \arrow[rr] & & X' \arrow[ul]
\end{tikzcd}$$

In particular, the Postnikov system of an object X is unique up to a unique isomorphism.

PROOF. Since $X_1 \in \langle \mathscr{A}_2, \dots, \mathscr{A}_n \rangle$, we have that $\mathrm{Hom}^\bullet(X_1, A'_1) = 0$: thus we have the isomorphisms $\mathrm{Hom}(X_1, X'_1) \cong \mathrm{Hom}(X_1, X')$ and $\mathrm{Hom}(A_1, A'_1) \cong \mathrm{Hom}(X, A'_1)$. Consequently there exists a unique morphism of distinguished triangles from $X_1 \to X \to A_1 \to X_1[1]$ to $X'_1 \to X' \to A'_1 \to X'_1[1]$ which fits into the diagram. By induction one concludes. $\square$

REMARK 12.6. Let $\Phi\colon \mathscr{D} \to \mathscr{A}$ be a covariant cohomological functor with values in an abelian category $\mathscr{A}$, and set $\Phi^q(X) := \Phi(X[q])$ for any object $X \in \mathrm{Ob}(\mathscr{D})$, $q \in \mathbb{Z}$. Given an object $X \in \mathrm{Ob}(\mathscr{D})$, there exists a spectral sequence converging to $\Phi^\bullet(X)$. Let us realize X as the canonical convolution of a Postnikov system, as in Definition 12.11, and delete the $X_0 = X$ term:

$$\begin{tikzcd}
0 = X_n \arrow[rr] & & X_{n-1} \arrow[rr] \arrow[dl] & & \dots \arrow[rr] \arrow[dl] & & X_2 \arrow[rr] & & X_1 \arrow[dl] & \\
& L_n \arrow[ul, dotted] & & L_{n-1} \arrow[ll, dotted, "\delta_{n-1}"] \arrow[ul, dotted] & & \dots \arrow[ll, dotted] & & L_2 \arrow[ll, dotted, "\delta_2"] \arrow[ul, dotted] & & L_1 \arrow[ll, dotted, "\delta_1"] \arrow[ul, dotted]
\end{tikzcd}$$

By applying the functor Φ to this diagram we obtain a bigraded exact couple (D, E, i, j, k)

$$\begin{tikzcd}
D_1^{\bullet,\bullet} \arrow[rr, "i"] & & D_1^{\bullet,\bullet} \arrow[dl, "j"] \\
& E_1^{\bullet,\bullet} \arrow[ul, "k"] &
\end{tikzcd}$$

where $E_1^{p,q} := \Phi^q(L_{p+1})$, $D_1^{p,q} := \Phi^q(X_p)$, and the morphism i, j, k have degree $(-1,1), (0,0), (1,0)$ respectively. We thus obtain a spectral sequence $(E_1^{p,q}, d_1 := kj)$ which ca be shown to converge to $\Phi^{p+q}(X)$. For further details see [GM03], Ex. III.7.3c and Ex. IV.2.2a.

Let us now introduce the strictly related notion of admissibility of a subcategory.

DEFINITION 12.12 ([Bon89, BK89]). A full triangulated subcategory $\mathscr{A}$ of $\mathscr{D}$ is called

- *left admissible* if the inclusion functor $i\colon \mathscr{A} \to \mathscr{D}$ admits a left adjoint functor $i^*\colon \mathscr{D} \to \mathscr{A}$;
- *right admissible* if the inclusion functor $i\colon \mathscr{A} \to \mathscr{D}$ admits a right adjoint functor $i^!\colon \mathscr{D} \to \mathscr{A}$;
- *admissible* if it is both left and right admissible.

LEMMA 12.2 ([Bon89]). *Let $\mathscr{A}, \mathscr{B}$ be two full triangulated subcategories of $\mathscr{D}$.*

(1) *Let $\mathscr{D} = \langle \mathscr{A}, \mathscr{B} \rangle$ be a semiorthogonal decomposition of $\mathscr{D}$. Then $\mathscr{A}$ is left admissible and $\mathscr{B}$ is right admissible.*
(2) *Conversely, if $\mathscr{A}$ is left admissible and $\mathscr{B}$ is right admissible, then $\langle \mathscr{A}, {}^\perp\mathscr{A} \rangle$ and $\langle \mathscr{B}^\perp, \mathscr{B} \rangle$ are semiorthogonal decompositions of $\mathscr{D}$.*

PROOF. For any object $X \in \mathrm{Ob}(\mathscr{D})$ there exists a distinguished triangle

$$B \to X \to A \to B[1],$$

with $A \in \mathrm{Ob}(\mathscr{A})$ and $B \in \mathrm{Ob}(\mathscr{B})$. By Proposition 12.7, such a distinguished triangle is unique up to unique isomorphism. So, for point (1), the associations

$$i^*_{\mathscr{A}}(X) := A, \quad i^!_{\mathscr{B}}(X) = B,$$

are well defined and are respectively left/right adjoint functors to the inlcusions $i_{\mathscr{A}}, i_{\mathscr{B}}$. For point (2), given any object $X \in \mathrm{Ob}(\mathscr{D})$, by the properties of adjoint functors we have two morphisms

$$X \to i_{\mathscr{A}} i^*_{\mathscr{A}}(X), \quad i_{\mathscr{B}} i^!_{\mathscr{B}}(X) \to X.$$

By completing them to a distinguished triangle, and using the semiorthogonality condition, it is easily seen that the completing objects are respectively in ${}^\perp\mathscr{A}$ and $\mathscr{B}^\perp$. □

COROLLARY 12.1 ([Bon89]). *If $\mathscr{A}_1, \dots, \mathscr{A}_n$ is a semiorthogonal sequence of full triangulated subcategories of $\mathscr{D}$ such that*

- *$\mathscr{A}_1, \dots, \mathscr{A}_k$ are left admissible,*
- *$\mathscr{A}_{k+1}, \dots, \mathscr{A}_n$ are right admissible,*

then

$$\langle \mathscr{A}_1, \dots, \mathscr{A}_k, {}^\perp\langle \mathscr{A}_1, \dots, \mathscr{A}_k \rangle \cap \langle \mathscr{A}_{k+1}, \dots, \mathscr{A}_n \rangle^\perp, \mathscr{A}_{k+1}, \dots, \mathscr{A}_n \rangle$$

is a semiorthogonal decompositon.

COROLLARY 12.2. *If $(\mathscr{A}_1, \dots, \mathscr{A}_n)$ is a semiorthogonal sequence of full admissible triangulated subcategories of $\mathscr{D}$, then the following are equivalent*

(1) *$\mathscr{D} = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$ is a semiorthogonal decomposition,*
(2) *$\bigcap_{j=1}^n \mathscr{A}_j^\perp = 0$,*
(3) *$\bigcap_{j=1}^n {}^\perp\mathscr{A}_j = 0$.*

DEFINITION 12.13 (Mutations functors). Let $\mathscr{A}$ be a full triangulated subcategory of $\mathscr{D}$.

- Let us assume that $\mathscr{A}$ is *left admissible*. Then, we define a functor $\mathbb{R}_{\mathscr{A}}\colon \mathscr{D} \to \mathscr{D}$, called *right mutation functor w.r.t.* $\mathscr{A}$ as follows: for any $X \in \mathrm{Ob}(\mathscr{D})$ we define
$$\mathbb{R}_{\mathscr{A}}(X) := \mathrm{Cone}\,(X \to ii^*(X))\,[-1],$$
where $i\colon \mathscr{A} \to \mathscr{D}$ is the inclusion functor and i^* is its left adjoint.
- Let us assume that $\mathscr{A}$ is *right admissible*. Then, we define a functor $\mathbb{L}_{\mathscr{A}}\colon \mathscr{D} \to \mathscr{D}$, called *left mutation functor w.r.t.* $\mathscr{A}$ as follows: for any $X \in \mathrm{Ob}(\mathscr{D})$ we define
$$\mathbb{L}_{\mathscr{A}}(X) := \mathrm{Cone}\left(ii^!(X) \to X\right),$$
where $i\colon \mathscr{A} \to \mathscr{D}$ is the inclusion functor and $i^!$ is its right adjoint.

Proposition 12.8. *Let $(E_1, \ldots, E_k)$ be an exceptional collection in $\mathscr{D}$. Then the subcategory $\langle E_1, \ldots, E_k\rangle$ is admissible, and moreover*
$$\mathbb{R}_{\langle E_1,\ldots,E_k\rangle} = \mathbb{R}_{E_k} \circ \mathbb{R}_{E_{k-1}} \circ \cdots \circ \mathbb{R}_{E_1},$$
$$\mathbb{L}_{\langle E_1,\ldots,E_k\rangle} = \mathbb{L}_{E_1} \circ \mathbb{L}_{E_2} \circ \cdots \circ \mathbb{L}_{E_k}.$$
In particular, the r.h.s. depend only on $\langle E_1, \ldots, E_k\rangle$, and not on the exceptional collection $(E_1, \ldots, E_k)$.

Proof. Let us proceed by induction on the length k of the exceptional collection. If $k = 1$, then the statement is obvious for the results of the previous Section. Let us assume that it is true for all exceptional collections of length $k-1$. Then, we have two distinguished triangles
$$\mathbb{R}_{\langle E_1,\ldots E_{k-1}\rangle}X \to X \to F, \quad \text{with } F \in \mathrm{Ob}\langle E_1, \ldots, E_{k-1}\rangle,$$
$$\mathbb{R}_{E_k}\mathbb{R}_{\langle E_1,\ldots E_{k-1}\rangle}X \to \mathbb{R}_{\langle E_1,\ldots E_{k-1}\rangle}X \to F', \quad \text{with } F' \in \mathrm{Ob}\langle E_k\rangle.$$
We can fit these triangles into a bigger diagram which, by the octahedral axiom TR4, has exact column and rows:
$$\begin{tikzcd}
\mathbb{R}_{E_k}\mathbb{R}_{\langle E_1,\ldots E_{k-1}\rangle}X \arrow[r] \arrow[d] & X \arrow[r, dashed] \arrow[d, equal] & F'' \arrow[d, dashed] \\
\mathbb{R}_{\langle E_1,\ldots E_{k-1}\rangle}X \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, dashed] & F \arrow[d, dashed] \\
F' \arrow[r, dashed] & 0 \arrow[r, dashed] & F'[1]
\end{tikzcd}$$
Here the upper-left square is commutative. Focusing on the right column, we have that $F \in \mathrm{Ob}\langle E_1, \ldots, E_{k-1}\rangle$, $F' \in \mathrm{Ob}\langle E_k\rangle$, and consequently $F'' \in \mathrm{Ob}\langle E_1, \ldots, E_k\rangle$, being the subcategory triangulated, and thus closed by taking cones. The association $X \mapsto F''$ define a left adjoint for the inclusion $\langle E_1, \ldots, E_k\rangle \to \mathscr{D}$, and $\langle E_1, \ldots, E_k\rangle$ is left admissible. A similar argument shows that $\langle E_1, \ldots, E_k\rangle$ is right admissible. $\square$

Proposition 12.9. *Let $\mathscr{A}$ be an admissible full triangulated subcategory of $\mathscr{D}$.*

(1) *Both functors $\mathbb{L}_{\mathscr{A}}, \mathbb{R}_{\mathscr{A}}$ are vanishing if restricted to $\mathscr{A}$.*
(2) *For any $X \in \mathrm{Ob}(\mathscr{D})$ we have that $\mathbb{L}_{\mathscr{A}}(X) \in \mathrm{Ob}(\mathscr{A}^{\perp})$ and $\mathbb{R}_{\mathscr{A}}(X) \in \mathrm{Ob}({}^{\perp}\mathscr{A})$.*

(3) *The restricted functors* $\mathbb{L}_{\mathscr{A}}|_{{}^\perp\mathscr{A}}, \mathbb{R}_{\mathscr{A}}|_{\mathscr{A}^\perp}$ *induce mutually inverse equivalences* ${}^\perp\mathscr{A} \to \mathscr{A}^\perp$ *and* $\mathscr{A}^\perp \to {}^\perp\mathscr{A}$*, respectively.*

(4) *If* $\Psi \in \mathrm{Aut}(\mathscr{D})$ *is an auto-equivalence of* $\mathscr{D}$*, then*

$$\Psi \circ \mathbb{L}_{\mathscr{A}} = \mathbb{L}_{\Psi(\mathscr{A})} \circ \Psi, \quad \Psi \circ \mathbb{R}_{\mathscr{A}} = \mathbb{R}_{\Psi(\mathscr{A})} \circ \Psi.$$

12.4. Saturatedness and Serre Functors

Definition 12.14 ([**BK89**]). A triangulated $\mathbb{K}$-linear category $\mathscr{D}$ is *saturated* if and only if any (covariant/contravariant) cohomological functor of finite type, i.e. any functor

$$F\colon \mathscr{D} \to \mathrm{Vect}_{\mathbb{K}}^{<\infty}, \quad F\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{K}}^{<\infty}$$

such that

- F takes distinguished triangles into exact sequences,
- $\sum_{i\in\mathbb{Z}} \dim_{\mathbb{K}} F(A[i]) < \infty$ for any object $A \in \mathrm{Ob}(\mathscr{D})$,

is representable. This is equivalent to the requirement that any exact functor $\Phi\colon \mathscr{D} \to \mathcal{D}^b(\mathbb{K})$ is representable, the category $\mathscr{D}$ being $\mathcal{D}^b(\mathbb{K})$-enriched by seeing $\mathrm{Hom}^\bullet(X, Y)$ as a complex with trivial differentials.

The following results describes how the properties of admissimility and saturatedness interact with one other.

Proposition 12.10. *Let* $\mathscr{D}$ *be a* $\mathbb{K}$*-linear triangulated category.*

(1) *If* $\mathscr{D}$ *is saturated, and* $\mathscr{A} \subseteq \mathscr{D}$ *is left (or right) admissible, then* $\mathscr{A}$ *is saturated.*

(2) *If* $\mathscr{A}$ *is a saturated category, imbedded in* $\mathscr{D}$ *as a full triangulated subcategory, then* $\mathscr{A}$ *is admissible.*

(3) *If* $\mathscr{D} = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$ *is a semiorthogonal decomposition, and* $\mathscr{D}$ *is saturated, then each* $\mathscr{A}_i$ *is admissible.*

Proof. For a proof of the points (1) and (2), see [**Bon89**], [**BK89**] and [**Kuz07**]. For the point (3) let us proceed by induction on the length of the semiorthogonal decomposition. Since $\mathscr{D} = \langle \mathscr{A}_n^\perp, \mathscr{A}_n \rangle$, by point Lemma 12.2 it follows that $\mathscr{A}_n$ is right admissible, and $\mathscr{A}_n^\perp$ left admissible. Hence $\mathscr{A}_n$ and $\mathscr{A}_n^\perp$ are saturated (by (1)), and admissible (by (2)). A simple inductive argument completes the proof. □

Proposition 12.11. *Let* $\mathscr{D}$ *be a* $\mathbb{K}$*-linear triangular category.*

(1) *Let* $\mathscr{A}$ *be an admissible subcategory of* $\mathscr{D}$*. Suppose that both* $\mathscr{A}$ *and* $\mathscr{A}^\perp$ *are saturated. Then also* $\mathscr{D}$ *is saturated.*

(2) *Let* $\mathscr{D} = \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$ *be a given semiorthogonal decomposition. Then* $\mathscr{D}$ *is saturated if and only if each* $\mathscr{A}_i$ *is saturated.*

Proof. For a proof of point (1) see [**BK89**]. For point (2), let us suppose that $\mathscr{D}$ is saturated. Then by (3) and (1) of Proposition 12.10, we have that each $\mathscr{A}_i$ is saturated. Vice versa, an inductive argument completes the proof, using (1). □

Definition 12.15 ([BK89]). A *Serre functor* in a $\mathbb{K}$-linear category of finite type $\mathscr{D}$ is a $\mathbb{K}$-linear auto-equivalence $\kappa\colon \mathscr{D} \to \mathscr{D}$ such that there exist bi-functorial isomorphisms of $\mathbb{K}$-vector spaces

$$\eta_{A,B}\colon \operatorname{Hom}(A,B) \xrightarrow{\cong} \operatorname{Hom}(B,\kappa(A))^\vee$$

for any two objects $A, B \in \operatorname{Ob}(\mathscr{D})$.

If $\mathscr{D}$ is $\mathbb{K}$-linear triangulated category for which exists a Serre functor, then it is automatically compatible with the triangulated structure.

Proposition 12.12 ([BK89]). *Any Serre functor on a triangulated $\mathbb{K}$-linear category is exact, i.e.*

- *it commutes with shift operators,*
- *it takes distinguished triangles into distinguished triangles.*

Moreover, we have that

(1) *the category $\mathscr{D}$ has a Serre functor if and only if all functors* $\operatorname{Hom}(X,-)^\vee, \operatorname{Hom}(-,X)^\vee$, *for any object $X \in \operatorname{Ob}(\mathscr{D})$, are representable.*
(2) *Any two Serre functors κ_1 and κ_2 are connected by a canonical functor isomorphism.*

Because of the previous Proposition, it is clear that any Hom-finite saturated category admits a Serre functor, since the functors $\operatorname{Hom}(X,-)^\vee, \operatorname{Hom}(-,X)^\vee$ are cohomological of finite type, for any object $X \in \operatorname{Ob}(\mathscr{D})$.

Proposition 12.13 ([BK89]). *Let $\mathscr{D}$ be a triangulated* Hom*-finite $\mathbb{K}$-linear category admitting a full exceptional collection $\mathfrak{E} = (E_0, \dots, E_n)$. Then $\mathscr{D}$ is saturated, and hence it has a Serre functor.*

Proof. By Proposition 12.11, we already know that $\mathscr{D}$ is saturated if and only if each subcategory generated by an E_i is saturated. It is easily seen that the category generated by an exceptional object E is saturated: an exact functor $\Phi\colon \mathscr{D} \to \mathcal{D}^b(\mathbb{K})$ is represented by $\Phi(E)^\vee$ if Φ is covariant, $\Phi(E)$ otherwise. □

12.5. Dual Exceptional Collections and Helices

In the following subsections we will always suppose that the $\mathbb{K}$-linear triangulated category $\mathscr{D}$ admit a full exceptional collection $\mathfrak{E} := (E_0, E_1, \dots, E_n)$.

12.5.0.1. *Left and Right Dual Exceptional Collections.* Starting from the full exceptional collection $\mathfrak{E} := (E_0, E_1, \dots, E_n)$, we can define other two collections

$${}^\vee\mathfrak{E} := ({}^\vee E_0, {}^\vee E_1, \dots, {}^\vee E_n), \quad \mathfrak{E}^\vee := (E_0^\vee, E_1^\vee, \dots, E_n^\vee),$$

called respectively *left* and *right dual collections*, defined by iterated mutations

$${}^\vee E_k := \mathbb{R}_{E_n}\mathbb{R}_{E_{n-1}} \dots \mathbb{R}_{E_{n-k+1}} E_{n-k},$$

$$E_k^\vee := \mathbb{L}_{E_0}\mathbb{L}_{E_1} \dots \mathbb{L}_{E_{n-k-1}} E_{n-k}$$

for $k = 0, 1, \dots, n$. Adopting the conventions of Remark 12.5, we can define the dual exceptional collections through the action of the braids

$$\mathfrak{E}^\vee \equiv \mathfrak{E}^\beta, \quad \beta := (\beta_{n,n+1}, \beta_{n-1,n} \dots, \beta_{12})(\beta_{n,n+1}, \beta_{n-1,n} \dots, \beta_{23}) \dots \beta_{n,n+1},$$

$$^\vee\mathfrak{E} \equiv \mathfrak{E}^{\beta'}, \quad \beta' := (\sigma_{12}\sigma_{23} \dots \sigma_{n,n+1})(\sigma_{12}\sigma_{23} \dots \sigma_{n-1,n}) \dots \sigma_{12}.$$

Notice that we have

$$\mathrm{Hom}^\bullet(E_h, E_k^\vee) = \begin{cases} 0 & \text{if } h = 0, \dots, n-k-1, \quad \text{by definition of left mutation} \\ \mathbb{K} & \text{if } h = n-k, \\ 0 & \text{if } h = n-k+1, \dots, n, \quad \text{by iteration of (12.6)} \end{cases} \tag{12.11}$$

$$\mathrm{Hom}^\bullet(^\vee E_k, E_h) = \begin{cases} 0 & \text{if } h = 0, \dots, n-k-1, \quad \text{by iteration of (12.7)} \\ \mathbb{K} & \text{if } h = n-k, \\ 0 & \text{if } h = n-k+1, \dots, n, \quad \text{by definition of right mutation} \end{cases} \tag{12.12}$$

where the graded vector space $\mathbb{K}$ is concentrated in degree 0. In other words, we have that

- $\mathrm{Hom}^\alpha(E_h, E_k^\vee)$ vanish except $\alpha = 0$ and $h = n-k$ (in which case is $\mathbb{K}$),
- $\mathrm{Hom}^\alpha(^\vee E_k, E_h)$ vanish except $\alpha = 0$ and $h = n-k$ (in which case is $\mathbb{K}$).

These orthogonality relations actually define the left and right dual collections uniquely up to *unique* isomorphisms: this is a consequence of Yoneda Lemma, as the following results shows.

Proposition 12.14 ([GK04]). *If $\mathscr{D}$ is a $\mathbb{K}$-linear triangulated category generated by the exceptional collection $(E_0, \dots, E_n)$, then for any $k = 0, 1, \dots, n$ we have that:*

- *the object $^\vee E_k$ represents the covariant functor*
$$X \mapsto \mathrm{Hom}^\bullet(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X);$$
- *the object $E_k^\vee$ represents the contravariant functor*
$$X \mapsto (\mathrm{Hom}^\bullet)^\vee(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X).$$

In particular, for any object $X \in \mathrm{Ob}(\mathscr{D})$, we get the functorial isomorphisms
$$\mathrm{Hom}^\bullet(^\vee E_k, X) = (\mathrm{Hom}^\bullet)^\vee(X, E_k^\vee).$$

Proof. Observing that $\mathbb{L}_{E_0} \dots \mathbb{L}_{E_n} X = 0$ for any object X, since it is an object of the subcategory $\langle E_0, \dots, E_n \rangle^\perp = \mathscr{D}^\perp = 0$, and applying the functors $\mathrm{Hom}^\bullet(^\vee E_k, -)$ and $\mathrm{Hom}^\bullet(-, E_k^\vee)$ to the triangle

$$\mathrm{Hom}^\bullet(E_h, \mathbb{L}_{E_{h+1}} \dots \mathbb{L}_{E_n} X) \otimes E_h \to \mathbb{L}_{E_{h+1}} \dots \mathbb{L}_{E_n} X \to \mathbb{L}_{E_h} \dots \mathbb{L}_{E_n} X$$

starting from $h = 0$ up to $h = n-k-1$, we iteratively obtain

$$\mathrm{Hom}^\bullet(^\vee E_k, \mathbb{L}_{E_h} \dots \mathbb{L}_{E_n} X) = \mathrm{Hom}^\bullet(\mathbb{L}_{E_h} \dots \mathbb{L}_{E_n} X, E_k^\vee) = 0$$

for any $h \in \{0, \dots, n-k\}$. So, at the step $h = n-k$, applying $\mathrm{Hom}^\bullet({}^\vee E_k, -)$, we get

$$\begin{aligned}\mathrm{Hom}^\bullet({}^\vee E_k, X) &= \mathrm{Hom}^\bullet({}^\vee E_k, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X) \qquad \text{(by iteration of (12.6))}\\ &= \mathrm{Hom}^\bullet(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X) \otimes \mathrm{Hom}^\bullet({}^\vee E_k, E_{n-k})\\ &= \mathrm{Hom}^\bullet(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X)\end{aligned}$$

because of orthogonality relations (12.12). Analogously, applying $\mathrm{Hom}^\bullet(-, E_k^\vee)$ to the same triangle, we get

$$\begin{aligned}\mathrm{Hom}^\bullet(X, E_k^\vee) &= \mathrm{Hom}^\bullet(\mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X, E_k^\vee) \qquad \text{(by iteration of (12.7))}\\ &= (\mathrm{Hom}^\bullet)^\vee(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X) \otimes \mathrm{Hom}^\bullet(E_{n-k}, E_k^\vee)\\ &= (\mathrm{Hom}^\bullet)^\vee(E_{n-k}, \mathbb{L}_{E_{n-k+1}} \dots \mathbb{L}_{E_n} X)\end{aligned}$$

because of orthogonality relations (12.11). □

12.5.0.2. *Helices.* Following [GK04], we introduce the

Definition 12.16 (Helix). If $(E_0, \dots, E_n)$ is a full exceptional collection, we call *helix* the infinite collection $(E_i)_{i \in \mathbb{Z}}$ defined by the iterated mutations

$$E_{i+n+1} = \mathbb{R}_{E_{i+n}} \dots \mathbb{R}_{E_{i+1}} E_i, \quad E_{i-n-1} = \mathbb{L}_{E_{i-n}} \dots \mathbb{L}_{E_{i-1}} E_i, \quad i \in \mathbb{Z}.$$

Such a helix is said to be *of period* $n+1$, and any family of $n+1$ consequent objects $(E_i, E_{i+1}, \dots, E_{i+n})$ is called *helix foundation*. The braid group $\mathcal{B}_{n+1}$ acts on the set of helices of period $n+1$: the mutations functors $\mathbb{L}_i, \mathbb{R}_i$ act on the helix by replacing all the pairs

$$(E_{i-1+k(n+1)}, E_{i+k(n+1)}), \quad \text{with } k \in \mathbb{Z}$$

with their left/right mutations. In this way, the mutation of a helix is still a helix.

Proposition 12.15 ([GK04]). *Let $(E_{i-n-1}, \dots E_{i-1})$ and $(E_i, \dots, E_{i+n})$ be two consequent foundations of an helix in a $\mathbb{K}$-ilnear triangulated category $\mathscr{D}$. For any object $X \in \mathrm{Ob}(\mathscr{D})$ the following functorial isomorphisms holds:*

$$\mathrm{Hom}^\bullet(E_i, X) = (\mathrm{Hom}^\bullet)^\vee(X, E_{i-n-1}). \tag{12.13}$$

In particular, we deduce the periodicity condition

$$\mathrm{Hom}^\bullet(E_i, E_j) = \mathrm{Hom}^\bullet(E_{i-n-1}, E_{j-n-1}). \tag{12.14}$$

Proof. Notice that, if we consider two consequent foundations of a helix of period $n+1$, i.e.

$$E_{i-n-1}, \dots, E_{i-1}, \quad E_i, \dots, E_{i+n},$$

the collection $(F_0, \dots, F_n)$, defined by the relations

$$F_k := \mathbb{L}_{E_i} \dots \mathbb{L}_{E_{i+n-k-1}} E_{i+n-k} = \mathbb{R}_{E_{i-1}} \dots \mathbb{R}_{E_{i-k}} E_{i-k-1},$$

is at the same time right dual collection of $(E_i, \dots, E_{i+n})$ and left dual collection of $(E_{i-n-1}, \dots, E_{i-1})$. Thus, by Proposition 12.14, we deduce that for any object $X \in \mathrm{Ob}(\mathscr{D})$ it holds the functorial isomorphism

$$\mathrm{Hom}^\bullet(E_i, X) = (\mathrm{Hom}^\bullet)^\vee(X, E_{i-n-1}).$$

Applying it for $X = E_j$, we obtain

$$\mathrm{Hom}^\bullet(E_i, E_j) = (\mathrm{Hom}^\bullet)^\vee(E_j, E_{i-n-1});$$

analogously we have

$$\mathrm{Hom}^\bullet(E_j, X) = (\mathrm{Hom}^\bullet)^\vee(X, E_{j-n-1}),$$

and dualizing we get

$$(\mathrm{Hom}^\bullet)^\vee(E_j, X) = \mathrm{Hom}^\bullet(X, E_{j-n-1}).$$

If we take $X = E_{i-n-1}$ in the last isomorphism, we finally obtain the periodicity condition

$$\mathrm{Hom}^\bullet(E_i, E_j) = \mathrm{Hom}^\bullet(E_{i-n-1}, E_{j-n-1}).$$

□

We already know, by Proposition 12.13, that the category $\mathscr{D}$ admits a Serre functor $\kappa\colon \mathscr{D} \to \mathscr{D}$ (unique up to a canonical isomorphism): from the result above we deduce that if $\mathfrak{E} := (E_0, \dots, E_n)$ is an exceptional collection, then

$$\kappa(E_i) = E_{i-n-1}, \quad i \in \mathbb{Z},$$

for any exceptional object of the helix generated by $\mathfrak{E}$. Remarkably, the knowledge of the action of κ on such an helix is enough to reconstruct its action on the whole category $\mathscr{D}$.

Corollary 12.3 ([GK04]). *The action on the set of full exceptional collections $\mathfrak{E}$'s (of length $n+1$) of the central element of the braid group $\mathcal{B}_{n+1}$*

$$\beta^2 = (\beta_{n,n+1}\beta_{n-1,n}\dots\beta_{23}\beta_{12})^{n+1}, \quad \beta := (\beta_{n,n+1}, \beta_{n-1,n}\dots, \beta_{12})(\beta_{n,n+1}, \beta_{n-1,n}\dots, \beta_{23})\dots\beta_{n,n+1},$$

can be extended to a Serre functor κ of the category $\mathscr{D}$.

Proof. We have to show that given an object $X \in \mathrm{Ob}(\mathscr{D})$, the image $\kappa(X)$ is uniquely determined by the images of the objects of an exceptional collection $(E_0, \dots, E_n)$. Let us consider X as the canonical convolution of the Postnikov system whose associated complex is

$$0 \to V_0^\bullet \otimes E_0 \to V_1^\bullet \otimes E_1 \to \dots \to V_n^\bullet \otimes E_n \to 0, \quad V_k^\bullet := (\mathrm{Hom}^\bullet)^\vee(\mathbb{R}_{E_{k-1}}\dots\mathbb{R}_{E_0}X, E_k).$$

The differentials are given by an element of

$$\bigoplus_p \mathrm{Hom}^1(V_p^\bullet \otimes E_p, V_{p+1}^\bullet \otimes E_{p+1}) \cong \bigoplus_{p,\alpha} \mathrm{Hom}^{-\alpha}(V_p^\bullet, V_{p+1}^\bullet) \otimes \mathrm{Hom}^{\alpha+1}(E_p, E_{p+1}).$$

By (12.14), the same element defines differentials of the complex

$$0 \to V_0^\bullet \otimes E_{-n-1} \to V_1^\bullet \otimes E_{-n} \to \dots \to V_n^\bullet \otimes E_{-1} \to 0,$$

whose canonical convolution defines an object $\kappa(X)$. in order to show that $\mathrm{Hom}^\bullet(Y, X) \cong (\mathrm{Hom}^\bullet)^\vee(\kappa(X), Y)$, we use the procedure described in Remark 12.6 for both the linear covariant cohomological functors $\mathrm{Hom}^\bullet(Y, -)$ and $(\mathrm{Hom}^\bullet)^\vee(\kappa(-), Y)$. The first one is computed through the spectral sequence whose first sheet is

$$E_1^{p,q} = \mathrm{Hom}^q(Y, V_p^\bullet \otimes E_p) = \bigoplus_\alpha V_p^{-\alpha} \otimes \mathrm{Hom}^{q+\alpha}(Y, E_p),$$

and for the second one we have

$$E_1^{p,q} = \bigoplus_\alpha V_p^{-\alpha} \otimes \left(\mathrm{Hom}^{-q-\alpha}(E_{p-n-1}, Y)\right)^\vee.$$

By (12.13) one concludes. □

12.5.0.3. *m-Blocks.* Following B.V. Karpov and D.Y. Nogin ([**KN98**, **Kar90**]) we introduce the definition of *m-blocks* a particular class of exceptional collections.

DEFINITION 12.17 ([**KN98**]). If $\mathscr{D}$ is a triangulated $\mathbb{K}$-linear category of finite type, and if $\mathfrak{E} = (E_1, \ldots, E_k)$ is an exceptional collection, we will say that $\mathfrak{E}$ is a *block* if

$$\mathrm{Hom}^\bullet(E_i, E_j) = 0 \quad \text{whenever } i \neq j.$$

More in general, an m-block is an exceptional collection

$$(\mathfrak{E}_1, \ldots, \mathfrak{E}_m) = (E_{11}, \ldots, E_{1\alpha_1}, E_{21}, \ldots, E_{2\alpha_2}, \ldots, E_{m1}, \ldots, E_{m\alpha_m})$$

such that all subcollections $\mathfrak{E}_j = (E_{j1}, \ldots, E_{j\alpha_j})$ are blocks. We will call

- *type* of the m-block $\mathfrak{E}$ the m-tuple $(\alpha_1, \ldots, \alpha_m)$,
- *structure* of the m-block $\mathfrak{E}$ the set $\{\alpha_1, \ldots, \alpha_m\}$.

The number α_i will be called the *length* of the block $\mathfrak{E}_i$, and analogously $\sum_i \alpha_i$ the *lenght* of the m-block $\mathfrak{E}$.

A close notion of *levelled exceptional collection* has been introduced by L. Hille in [**Hil95**]. The following result is an immediate consequence of the vanishing condition defining an m-block.

PROPOSITION 12.16. *Let $\mathfrak{E}$ be an m-block exceptional collection of type $(\alpha_1, \ldots, \alpha_m)$. The left/right mutations of two objects in a same block $\mathfrak{E}_j$ act just as permutations and shifts.*

CHAPTER 13

Non-symmetric orthogonal geometry of Mukai lattices

Abstract. In this Chapter we focus on unimodular Mukai lattice structures: introduced and studied by A.L. Gorodentsev ([**Gor94b, Gor94a**]), they consist in the datum of a free abelian group V endowed with a non-necessarily symmetric bilinear non-degenerate form $\langle\cdot,\cdot\rangle$. Particular attention is given to the case of *exceptional Mukai lattices*. i.e. those admitting an exceptional basis, an important example being furnished by the Grothendieck group $K_0(\mathscr{D})$ of a $\mathbb{K}$-linear triangulated category $\mathscr{D}$ admitting a full exceptional collection. The mutations of exceptional bases under the action of the braid group, the canonical operator and the isometry group $\mathrm{Isom}(V,\langle\cdot,\cdot\rangle)$ are introduced and described. Furthemore, the complete isometric classification of Mukai spaces is outlined (Theorem 13.1 and Theorem 13.2). We also consider the geometrical case of the Grothendieck group of a smooth projective variety X admitting a full exceptional collection in $\mathcal{D}^b(X)$: it is shown that the existence of such a collection implies a *motivic decomposition* of X, and hence strong constraints are deduced on its geometry and topology. Finally, results on the isometric classification on the Grothendieck groups $K_0(X)\otimes_{\mathbb{Z}}\mathbb{C}$ with non-degenerate Euler-Poincaré form are presented.

13.1. Grothendieck Group and Mukai Lattices

Let $\mathscr{D}$ be a (small) triangulated category. Let us denote by $[\mathscr{D}]$ the set of isomorphism classes of objects of $\mathscr{D}$.

Definition 13.1. The *Grothendieck group* $K_0(\mathscr{D})$ is the group defined as the quotient of the free abelian group on $[\mathscr{D}]$ by the following *Euler relations*:

$$[B]=[A]+[C],$$

whenever there is a triangle in $\mathscr{D}$

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

This group is the solution of the following universal problem: *to find an abelian group X and a function $[-]\colon[\mathscr{D}]\to X$ such that, given a function $\varphi\colon[\mathscr{D}]\to G$, with values in an abelian group G, and preserving the Euler relations, there exists a unique group homomorphism $\overline{\varphi}\colon X\to G$ making the following diagram commutative*

$$\begin{tikzcd} {[\mathscr{D}]} \arrow[rr, "\varphi"] \arrow[dr, "{[-]}"'] & & G \\ & X \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

A triangulated functor $F\colon\mathscr{D}\to\mathscr{D}'$ induces a group homomorphism between $K_0(\mathscr{D})$ and $K_0(\mathscr{D}')$, by sending $[E]$ to $[F(E)]$. If $\mathscr{D}$ is $\mathbb{K}$-linear, we can naturally define the so called *Euler-Poincaré pairing*

$$\chi(E,F):=\sum_i(-1)^i\dim_{\mathbb{K}}\mathrm{Hom}^i(E,F),$$

for any objects $E,F\in\mathrm{Ob}\,\mathscr{D}$.

Remark 13.1. Let us write some useful identities valid in a Grothendieck group. First of all, note that $[0] = 0$. Moreover, from the distinguished triangle $A \to A \oplus B \to B \to A[1]$ we have
$$[A \oplus B] = [A] + [B],$$
whereas from $A \to 0 \to A[1] \to A[1]$ we deduce that $[A[1]] = -[A]$.

Lemma 13.1. *If $E \in \operatorname{Ob} \mathscr{D}$ is an exceptional object, then for any object $X \in \operatorname{Ob} \mathscr{D}$ the following identities hold in the Grothendieck group $K_0(\mathscr{D})$:*
$$[\mathbb{L}_E X] = [X] - \chi(E, X) \cdot [E],$$
$$[\mathbb{R}_E X] = [X] - \chi(X, E) \cdot [E].$$

Proof. From the distinguished triangle defining $\mathbb{L}_E X$ we have
$$\begin{aligned}
[\mathbb{L}_E X] &= [X] - [\operatorname{Hom}^\bullet(E, X) \otimes E] \\
&= [X] - \left[\bigoplus_i E[-i]^{\oplus \dim_{\mathbb{K}} \operatorname{Hom}^i(E,X)}\right] \\
&= [X] - \left(\sum_i (-1)^i \dim_{\mathbb{K}} \operatorname{Hom}^i(E, X)\right) [E].
\end{aligned}$$
Analogously, we have
$$\begin{aligned}
[\mathbb{R}_E X] &= [X] - [(\operatorname{Hom}^\bullet)^\vee(X, E) \otimes E] \\
&= [X] - \left[\bigoplus_i E[-i]^{\oplus \dim_{\mathbb{K}} (\operatorname{Hom}^{-i}(X,E))^\vee}\right] \\
&= [X] - \left(\sum_i (-1)^i \dim_{\mathbb{K}} \operatorname{Hom}^{-i}(X, E)\right) [E].
\end{aligned}$$
□

Let us assume that $\mathscr{D}$ admits a full exceptional collection $(E_0, \ldots, E_n)$: it then follows that $K_0(\mathscr{D})$ is freely generated by $([E_0], \ldots, [E_n])$. In this case $(K_0(\mathscr{D}), \chi(\cdot, \cdot))$ admits a structure of *exceptional unimodular Mukai lattice*:

Definition 13.2 (Mukai Lattice). A *unimodular Mukai lattice* is a finitely generated free $\mathbb{Z}$-module V endowed with a unimodular bilinear (not necessarily symmetric) form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Z}$. An element $e \in V$ will be said to be *exceptional* if $\langle e, e \rangle = 1$. A $\mathbb{Z}$-basis $\varepsilon := (e_0, \ldots, e_n)$ of the Mukai lattice is called *exceptional* if
$$\langle e_i, e_i \rangle = 1 \quad \text{for all } i, \quad \langle e_j, e_i \rangle = 0 \quad \text{for } j > i.$$
In other words, the Gram matrix must be of the upper triangular form
$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & * & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$
A Mukai lattice is called *exceptional* if it admits an exceptional basis.

It is thus clear that the projection on $K_0(\mathscr{D})$ of a full exceptional collection in $\mathscr{D}$ is an exceptional basis.

Definition 13.3 (Mutations of exceptional bases). Let $(V, \langle\cdot,\cdot\rangle)$ be an exceptional Mukai lattice of rank $n+1$. If $\varepsilon := (e_0,\dots,e_n)$ is an exceptional basis we define for any $1 \leq i \leq n$

$$\mathbb{L}_i\varepsilon := (e_0,\dots,\mathbb{L}_{e_{i-1}}e_i, e_{i-1},\dots,e_n), \quad \mathbb{L}_{e_{i-1}}e_i := e_i - \langle e_{i-1}, e_i\rangle \cdot e_{i-1},$$

$$\mathbb{R}_i\varepsilon := (e_0,\dots,e_i,\mathbb{R}_{e_i}e_{i-1},\dots,e_n), \quad \mathbb{R}_{e_i}e_{i-1} := e_{i-1} - \langle e_{i-1}, e_i\rangle \cdot e_i.$$

In particular, we still get exceptional basis, called *left* and *right mutations* of ε. It is easy to see that this defines an action[1] of the braid group $\mathcal{B}_n$ on the set of exceptional bases of V.

Note that, accordingly to Lemma 13.1, the projection on the Grothendieck group of the mutation of a full exceptional collection $(E_0,\dots,E_n)$ coincides with the corresponding mutation of the exceptional basis $([E_0],\dots,[E_n])$.

Definition 13.4 (Left and right dual exceptional bases). Given an exceptional basis $\varepsilon = (e_0,\dots,e_n)$ of an exceptional Mukai lattice $(V, \langle\cdot,\cdot\rangle)$, we define two other exceptional bases

$$^\vee\varepsilon = (^\vee e_0,\dots,{}^\vee e_n) \quad \text{and} \quad \varepsilon^\vee = (e_0^\vee,\dots,e_n^\vee)$$

called respectively *left* and *right dual exceptional bases* defined through the action of the braids

$$\varepsilon^\vee := \varepsilon^\beta, \quad \beta := (\beta_{n,n+1}, \beta_{n-1,n}\dots,\beta_{12})(\beta_{n,n+1},\beta_{n-1,n}\dots,\beta_{23})\dots\beta_{n,n+1},$$

$$^\vee\varepsilon := \varepsilon^{\beta'}, \quad \beta' := (\sigma_{12}\sigma_{23}\dots\sigma_{n,n+1})(\sigma_{12}\sigma_{23}\dots\sigma_{n-1,n})\dots\sigma_{12}.$$

Notice, in particular that we have the following orthogonality relations

$$\langle e_h, e_k^\vee\rangle = \delta_{h,n-k}, \quad \langle ^\vee e_k, e_h\rangle = \delta_{h,n-k}. \tag{13.1}$$

Proposition 13.1. *If $\varepsilon = (e_0,\dots,e_n)$ is an exceptional basis of $(V,\langle\cdot,\cdot\rangle)$, and G is the Gram matrix of $\langle\cdot,\cdot\rangle$ with respect to ε, i.e.*

$$G_{hk} := \langle e_h, e_k\rangle \quad 0 \leq h,k \leq n,$$

then the Gram matrix

- *with respect to the exceptional basis $\mathbb{L}_i\varepsilon$ is given by $H^i \cdot G \cdot H^i$, where*

$$H^i = \begin{pmatrix} 1 &&&&&&\\ & \ddots &&&&&\\ && 1 &&&&\\ &&& -G_{i-1,i} & 1 &&\\ &&& 1 & 0 &&\\ &&&&& 1 &\\ &&&&&& \ddots &\\ &&&&&&& 1 \end{pmatrix},$$

the entry $-G_{i-1,i}$ being in the place $(i-1,i-1)$;

[1] In what follows we will use the same conventions and notations of Remark 12.5 for the action of the braid group on the set of exceptional bases of a Mukai lattice.

- *with respect to the exceptional basis $\mathbb{R}_i\varepsilon$ is given by $K^i \cdot G \cdot K^i$, where*

$$K^i = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & -G_{i-1,i} & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix},$$

the entry $-G_{i-1,i}$ being in the place (i,i);
- *with respect to both the right and left dual exceptional basis $\varepsilon^\vee$ and ${}^\vee\varepsilon$ is given by*

$$J \cdot G^{-T} \cdot J,$$

where J is the anti-diagonal matrix

$$J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Proof. Lemma 13.1 implies that

$$(\mathbb{L}_i\varepsilon)_k = \sum_a (H^i)^a_k e_a, \quad (\mathbb{R}_i\varepsilon)_k = \sum_a (K^i)^a_k e_a,$$

from which the first two points immediately follow. If we define a matrix X such that

$$e_k^\vee = \sum_a X^h_a e_a,$$

then the orthogonality relations (13.1) imply that

$$G \cdot X = J,$$

so that the Gram matrix w.r.t. the basis $\varepsilon^\vee$ is given by

$$(G^{-1} \cdot J)^T \cdot G \cdot (G^{-1} \cdot J) = J \cdot G^{-T} \cdot J.$$

The computations for the basis ${}^\vee\varepsilon$ are identical, and are left as an exercise for the reader. □

Definition 13.5 (Left and right correlations). Given a Mukai lattice $(V, \langle\cdot,\cdot\rangle)$ there are two well defined correlations between V and its dual $V^* := \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, called respectively *emph* and *right correlations*:

$$\lambda\colon V \to V^*\colon x \mapsto \langle x, \cdot\rangle,$$

$$\rho\colon V \to V^*\colon x \mapsto \langle \cdot, x\rangle.$$

Because of the unimodularity of the pairing $\langle\cdot,\cdot\rangle$, both left and right correlations λ, ρ define isomorphisms of abelian groups.

13.2. Isometries and canonical operator

In the previous section, we have seen that in any triangulated category $\mathscr{D}$ also the group $\mathrm{Aut}(\mathscr{D})$ of isomorphism classes of auto-equivalence acts on the set of full exceptional collections. This action projects onto the Grothendieck group $K_0(\mathscr{D})$ through the actions of *isometries* preserving the Euler-Poincaré form, and hence acting on the set of exceptional bases.

Definition 13.6 (Isometries). Given two Mukai lattices $(V_1, \langle\cdot,\cdot\rangle_1), (V_2, \langle\cdot,\cdot\rangle_2)$, any $\mathbb{Z}$-linear map $\phi\colon V_1 \to V_2$ such that

$$\langle x, y\rangle_1 = \langle \phi(x), \phi(y)\rangle_2, \quad x, y \in V_1$$

is called *isometry* for the Mukai structures. If ϕ is invertible, then we will say that the Mukai structures $(V_1, \langle\cdot,\cdot\rangle_1), (V_2, \langle\cdot,\cdot\rangle_2)$ are isometrically isomorphic.

The set of all $\mathbb{Z}$-linear isometric automorphisms $\phi\colon V \to V$ of a Mukai lattice $(V, \langle\cdot,\cdot\rangle)$ is denoted by $\mathrm{Isom}_{\mathbb{Z}}(V, \langle\cdot,\cdot\rangle)$, or simply $\mathrm{Isom}_{\mathbb{Z}}$ if no confusion arises.

Since Serre functors are prototypical and important auto-equivalences in $\mathbb{K}$-linear triangulated categories, their projections on the Grothendieck group play a particularly important role.

Definition 13.7 (Canonical operator). Given a Mukai lattice $(V, \langle\cdot,\cdot\rangle)$, we call canonical operator the unique $\mathbb{Z}$-linear operator $\kappa\colon V \to V$ satisfying the property

$$\langle x, y\rangle = \langle y, \kappa(x)\rangle, \quad x, y \in V.$$

Although Serre functors do not always exist in $\mathbb{K}$-linear triangulated categories, at the level of Mukai structures this existence problem always admits a solution.

Proposition 13.2 ([Gor94a, Gor94b]). *Let $(V, \langle\cdot,\cdot\rangle)$ be a Mukai lattice.*

(1) *There exists a unique canonical operator $\kappa\colon V \to V$, and it is defined in terms of left and right canonical correlations as the composition*

$$\rho^{-1} \circ \lambda\colon V \to V.$$

(2) *Given any basis $(e_1, \dots, e_n)$ (not necessarily exceptional) of V, w.r.t. which the Gram matrix of the pairing $\langle\cdot,\cdot\rangle$ is G, then the matrix associated to the canonical operator κ is given by*

$$\kappa = G^{-1} \cdot G^T.$$

Proof. An exercise for the reader. □

13.3. Adjoint operators and canonical algebra

Let us consider a Mukai lattice $(V, \langle\cdot,\cdot\rangle)$.

Definition 13.8 (Left and right adjoint operators). Let $\phi \in \mathrm{End}_{\mathbb{Z}}(V)$. We define two new operators ${}^\vee\phi$ and $\phi^\vee$ called respectively *left* and *right adjoint to* ϕ through the following identities:

$$\langle {}^\vee\phi(x), y\rangle = \langle x, \phi(y)\rangle,$$
$$\langle \phi(x), y\rangle = \langle x, \phi^\vee(y)\rangle,$$

for any $x, y \in V$. Fixed a (non-necessarily exceptional) basis $(e_0, \dots, e_n)$ of V, in terms of matricial representation we have

$${}^\vee\phi = G^{-T} \cdot \phi^T \cdot G^T, \quad \phi^\vee = G^{-1} \cdot \phi^T \cdot G.$$

Because of the non-symmetry of the pairing $\langle\cdot,\cdot\rangle$, in general one has ${}^\vee\phi \neq \phi^\vee$.

DEFINITION 13.9 (Canonical algebra). An endomorphism $\phi \in \mathrm{End}_{\mathbb{Z}}(V)$ is called *reflexive* if ${}^{\vee}\phi = \phi^{\vee}$. The subalgebra $\mathcal{A} \subseteq \mathrm{End}_{\mathbb{Z}}(V)$ of all reflexive operators of V is called *canonical algebra*.

The proofs of the following Proposition is straightforward, and is left as an exercise for the reader.

PROPOSITION 13.3 ([Gor94a, Gor94b]). *Let $\phi \in \mathrm{End}_{\mathbb{Z}}(V)$. The following conditions are equivalent:*

(1) ${}^{\vee}\phi = \phi^{\vee}$,
(2) $\phi = \phi^{\vee\vee}$,
(3) $\phi =^{\vee\vee} \phi$,
(4) $\phi\kappa = \kappa\phi$.

Hence, the canonical algebra $\mathcal{A}$ coincides with the center of the canonical operator $\mathcal{Z}(\kappa)$.

PROPOSITION 13.4 ([Gor94a, Gor94b]). *The following sets are contained in the canonical algebra $\mathcal{A}$:*

(1) $\mathcal{A}^+ := \{\phi \in \mathrm{End}_{\mathbb{Z}}(V) \colon {}^{\vee}\phi = \phi^{\vee} = \phi\}$, *whose elements are called* self-adjoint operators*;*
(2) $\mathcal{A}^- := \{\phi \in \mathrm{End}_{\mathbb{Z}}(V) \colon {}^{\vee}\phi = \phi^{\vee} = -\phi\}$, *whose elements are called* anti-self-adjoint operators*;*
(3) $\mathrm{Isom}(V, \langle\cdot,\cdot\rangle) \equiv \{\phi \in \mathrm{Aut}_{\mathbb{Z}}(V) \colon {}^{\vee}\phi = \phi^{\vee} = \phi^{-1}\}$.

Given any field $\mathbb{K}$, we can extend scalars fro $\mathbb{Z}$ to $\mathbb{K}$, by considering the vector space $V \otimes_{\mathbb{Z}} \mathbb{K}$ endowed with the non-symmetric bilinear form $\langle\cdot,\cdot\rangle$, extended by $\mathbb{K}$-bilinearity. All previous definitions (and notations) can be trivially adapted to this extension of scalars.

PROPOSITION 13.5 ([Gor94a, Gor94b]). *If $\mathbb{K}$ is a field of characteristic not equal to 2, then the following direct sum of $\mathbb{K}$-vector spaces holds*

$$\mathcal{A}_{\mathbb{K}} = \mathcal{A}^+_{\mathbb{K}} \oplus \mathcal{A}^-_{\mathbb{K}}.$$

PROPOSITION 13.6 ([Gor94a, Gor94b]). *The Lie algebra of the complex Lie group $\mathrm{Isom}_{\mathbb{C}}$ is equal to $\mathcal{A}^-_{\mathbb{C}}$.*

13.4. Isometric classification of Mukai structures

The following Proposition underlines the importance of the canonical operator for the isometric classification of Mukai structures.

PROPOSITION 13.7 ([Gor94a, Gor94b, Gor16]). *Let V be a free $\mathbb{Z}$-module of finite rank, and let $\langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2$ two non-symmetric unimodular bilinear forms defining two Mukai lattice structures on V.*

(1) *The two Mukai structures share the same canonical operator if and only if there exists an invertible operator $\psi \in \mathcal{A}^+_{\langle\cdot,\cdot\rangle_1} \cap \mathcal{A}^+_{\langle\cdot,\cdot\rangle_2}$ and such that*
$$\langle x, y\rangle_1 = \langle x, \psi(y)\rangle_2, \quad x, y \in V.$$
(2) *If $\mathbb{K}$ is an algebraically closed field of characteristic zero, the two Mukai vector spaces $(V \otimes_{\mathbb{Z}} \mathbb{K}, \langle\cdot,\cdot\rangle_1)$ and $(V \otimes_{\mathbb{Z}} \mathbb{K}, \langle\cdot,\cdot\rangle_2)$ are isometrically isomorphic if and only if there exists an isomorphism $\phi \in \mathrm{Aut}_{\mathbb{K}}(V \otimes_{\mathbb{Z}} \mathbb{K})$ such that*
$$\phi \circ \kappa_1 = \kappa_2 \circ \phi.$$

PROOF. If λ_1, λ_2 and ρ_1, ρ_2 denote respectively the left and right correlations for the two Mukai structures on V, then the operator $\psi := \rho_2^{-1}\rho_1 = \lambda_2^{-1}\lambda_1$ satisfies all properties of point (1). Indeed, since $\kappa = \rho_1^{-1}\lambda_1 = \rho_2^{-1}\lambda_2$, we have that

$$\psi\kappa = (\rho_2^{-1}\rho_1)(\rho_1^{-1}\lambda_1) = \rho_2^{-1}\lambda_1 = (\rho_2^{-1}\lambda_2)(\lambda_2^{-1}\lambda_1) = \kappa\psi,$$
$$\langle x, \psi y\rangle_2 = [\rho_1(y)](x) = \langle x, y\rangle_1, \quad x, y \in V.$$

For point (2), if the two structures are isometric then the existence of an isomorphism ϕ intertwining the canonical operators is clear. Hence, let us suppose to have two different Mukai structures on the same vector space $V \otimes_{\mathbb{Z}} \mathbb{K}$ sharing the same canonical operator. By point (1), we deduce the existence of a self-dual isomorhism $\psi \in \mathrm{Aut}_{\mathbb{K}}(V \otimes_{\mathbb{Z}} \mathbb{K})$ such that

$$\langle x, y\rangle_1 = \langle x, \psi(y)\rangle_2, \quad x, y \in V \otimes_{\mathbb{Z}} \mathbb{K}.$$

The field $\mathbb{K}$ being algebraically closed, a polynomial $p \in \mathbb{K}[X]$ can be constructed in such a way that the operator $\alpha := p(\psi)$ satisfies $\alpha^2 = \psi$ (see Lemma 16.2 of [Gor16]). Such an operator α is self-adjoint, since

$$\alpha^\vee = p(\psi)^\vee = p(\psi^\vee) = p(\psi) = \alpha,$$

and it clearly satisfies the condition $\langle \alpha(x), \alpha(y)\rangle_2 = \langle x, y\rangle_1$. ☐

In particular, Proposition 13.7 implies that a non-degenerate non-symmetric bilinear form over algebraically closed field of characteristic zero is uniquely determined by Jordan normal form of its canonical operator.

DEFINITION 13.10. Given a Mukai lattice $(V, \langle\cdot,\cdot\rangle)$, and an algebraically closed field $\mathbb{K}$ of characteristic zero, the Mukai space $(V \otimes_{\mathbb{Z}} \mathbb{K}, \langle\cdot,\cdot\rangle)$ will be called *decomposable*, if there exist two subspaces U, V such that

(1) $V \otimes_{\mathbb{Z}} \mathbb{K} = U \oplus V$,
(2) the restrictions $\langle\cdot,\cdot\rangle$ to U and V are nondegenerate,
(3) U and V are *bi-orthogonal*, namely $\langle u, v\rangle = \langle v, u\rangle = 0$ for all $u \in U$ and $v \in V$.

The space will be called *indecomposable* if it is not decomposable.

The following result gives a complete classification of all indecomposable Mukai structures over an algebraically closed field $\mathbb{K}$ of characteristic zero.

THEOREM 13.1 ([Gor94a, Gor94b, Gor16]). *Let $(V, \langle\cdot,\cdot\rangle)$ be a Mukai lattice, and let $\mathbb{K}$ be an algebraically closed field of characteristic zero. If $(V \otimes_{\mathbb{Z}} \mathbb{K}, \langle\cdot,\cdot\rangle)$ is indecomposable, then it is isometrically isomorphic to one of the following Mukai spaces.*

(1) Space of type U_n: *consider the coordinate space $\mathbb{K}^n$ endowed with the non-degenerate bilinear form whose Gram matrix w.r.t. the standard basis is*

$$G = \begin{pmatrix} & & & & & 1 \\ & & & & -1 & 1 \\ & & & 1 & -1 & \\ & & \iddots & \iddots & & \\ & (-1)^{n-2} & (-1)^{n-3} & & & \\ (-1)^{n-1} & (-1)^{n-2} & & & & \end{pmatrix}.$$

In this case, by Proposition 13.2, we have that the canonical operator is of the form

$$\kappa = G^{-1} \cdot G^T = (-1)^{n-1}\mathbb{1} + M, \quad M^{n-1} \neq 0, \quad M^n = 0,$$

and its Jordan form is

$$J_n((-1)^{n-1}) := \begin{pmatrix} (-1)^{n-1} & 1 & & & \\ & (-1)^{n-1} & 1 & & \\ & & \ddots & \ddots & \\ & & & (-1)^{n-1} & 1 \\ & & & & (-1)^{n-1} \end{pmatrix}.$$

(2) Space of type $W_n(\lambda)$ with $\lambda \neq (-1)^{n-1}$: *consider the coordinate space $\mathbb{K}^{2n}$ endowed with the non-degenerate bilinear form whose Gram matrix w.r.t. the standard basis is*

$$G = \begin{pmatrix} 0 & \mathbb{1}_n \\ J_n(\lambda) & 0 \end{pmatrix}, \quad J_n(\lambda) := \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

In this case, by Proposition 13.2 the canonical operator is of the form

$$\kappa = G^{-1} \cdot G^T = \begin{pmatrix} J_n(\lambda)^{-1} & 0 \\ 0 & J_n(\lambda)^T \end{pmatrix},$$

and its Jordan form consists of two $n \times n$ blocks with nonzero inverse eigenvalues.

The two type of space $U_n, W_n(\lambda)$ with $\lambda \neq (-1)^{n-1}$ are not isometrically isomorphic.

Together with Theorem 13.1, the following result gives a complete classification of all Mukai spaces over algebraically closed field of characteristic zero.

Theorem 13.2 ([**Mal63**] Chapter VI-VII, [**Gor94a**, **Gor94b**]). *Let $(V, \langle \cdot, \cdot \rangle)$ is a vector space over an algebraically closed field $\mathbb{K}$ of characteristic zero endowed with a non-degenerate bilinear form. If $f \in \mathrm{End}(V)$ is an isometry, then V splits as a bi-orthogonal direct sum of subspaces V_λ, where*

(1) *for $\lambda = \pm 1$, V_λ is the root space*

$$V_\lambda := \bigoplus_{n \in \mathbb{N}} \ker (f - \lambda \mathbb{1})^n,$$

and the restriction of $\langle \cdot, \cdot \rangle$ on V_λ is non-degenerate;

(2) *for $\lambda \neq \pm 1$, the space V_λ is the sum of isotropic root subspaces*

$$\left(\bigoplus_{n \in \mathbb{N}} \ker (f - \lambda \mathbb{1})^n \right) \oplus \left(\bigoplus_{n \in \mathbb{N}} \ker \left(f - \lambda^{-1} \mathbb{1}\right)^n \right),$$

and the restriction of $\langle \cdot, \cdot \rangle$ on V_λ defines a non-degenerate pairing between these two subspaces.

13.5. Geometric case: the derived category $\mathcal{D}^b(X)$

In previous sections, we have treated the general case of a $\mathbb{K}$-linear triangulated category $\mathscr{D}$. Now we consider the case of the bounded derived category of coherent sheaves on a projective complex variety X. In order to work with a Hom-finite derived category, we assume that X is *smooth*: in this way each object is a *perfect complex*, i.e. locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank on X.

The condition of existence of a full exceptional collection in $\mathcal{D}^b(X)$ impose strict conditions on the topology and the geometry of X. The key property is a result of *motivic decomposition* for the rational Chow motive of X.

DEFINITION 13.11. Let X be a smooth projective variety over $\mathbb{C}$. We will say that the rational Chow motive of X, denote by $\mathfrak{h}(X)_{\mathbb{Q}} \in \mathrm{CHM}(\mathbb{C})_{\mathbb{Q}}$, is *discrete* (or of *Lefschetz type*) if it is polynomial in the Lefschetz motive $\mathbb{L}$, i.e. if it admits a decomposition as a direct sum

$$\mathfrak{h}(X)_{\mathbb{Q}} \cong \bigoplus_{i=1}^{n} \mathbb{L}^{\otimes a_i}, \quad a_i \in \{0, \dots, \dim_{\mathbb{C}} X\},$$

where by convention $\mathbb{L}^0 := \mathfrak{h}(\mathrm{Spec}(\mathbb{C}))_{\mathbb{Q}}$. The integer n will be called *length* of the motive $\mathfrak{h}(X)_{\mathbb{Q}}$.

THEOREM 13.3 ([**GO13**], [**MT15a**], [**BB12**]). *If X is a smooth projective variety over $\mathbb{C}$ with an exceptional collection in $\mathcal{D}^b(X)$, then the rational Chow motive $\mathfrak{h}(X)_{\mathbb{Q}}$ is discrete.*

There exist many proofs in literature of this fact, all differing in techniques. In [**GO13**] the statement was proved using K-motives. In [**MT15a**] (see also [**MT15b**]) a more general statement was proved (assuming that X is a smooth proper Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{K})$, for a perfect field $\mathbb{K}$) using the connection between Chow and non-commutative motives discovered by M. Kontsevich (see [**Tab13**]). In the case of smooth projective varieties, we can deduce Theorem 13.3 from the following result, essentially due to S. Kimura.

THEOREM 13.4 ([**Kim09**]). *Let X be a smooth projective variety over $\mathbb{C}$. The following conditions are equivalent:*

(1) *the Grothendieck group $K_0(X)_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space;*
(2) *the rational Chow motive of X is* discrete.

Furthermore, if these conditions hold true, the length of $\mathfrak{h}(X)_{\mathbb{Q}}$ coincides with $\dim_{\mathbb{Q}} K_0(X)_{\mathbb{Q}}$.

PROOF. The proof of the fact that (1) imples (2) follows from the main result of [**Kim09**]. Conversely, if

$$\mathfrak{h}(X)_{\mathbb{Q}} \cong \bigoplus_{i=1}^{n} \mathbb{L}^{\otimes a_i}, \quad a_i \in \{0, \dots, \dim_{\mathbb{C}} X\},$$

using the properties of the Lefschetz motive $\mathbb{L}$ we deduce that

$$\begin{aligned} CH^r(X)_{\mathbb{Q}} &\cong \mathrm{Hom}_{\mathrm{CHM}(\mathbb{C})_{\mathbb{Q}}}(\mathbb{L}^{\otimes r}, \mathfrak{h}(X)_{\mathbb{Q}}) \\ &\cong \bigoplus_{i=1}^{n} \mathrm{Hom}_{\mathrm{CHM}(\mathbb{C})_{\mathbb{Q}}}(\mathbb{L}^{\otimes r}, \mathbb{L}^{\otimes a_i}) \cong \mathbb{Q}^{N(r)}, \end{aligned}$$

where $N(r) := \mathrm{card}\,\{i \colon a_i = r\}$. Since the Chern character $\mathrm{ch}\colon K_0(X)_{\mathbb{Q}} \to CH^{\bullet}(X)_{\mathbb{Q}}$ is an isomorphism (not preserving the gradation), we conclude that $K_0(X)_{\mathbb{Q}}$ is finite dimensional. $\square$

COROLLARY 13.1 ([MT15a], [GKMS13]). *Let X be a smooth complex projective variety over $\mathbb{C}$ such that $K_0(X)$ is free of finite rank. Then:*

(1) *X is of Hodge-Tate type, i.e. $h^{p,q}(X) = 0$ if $p \neq q$.*
(2) *The cycle maps $c_X^r : CH^r(X)_{\mathbb{Q}} \to H^{2r}(X, \mathbb{Q})$ are isomorphisms.*
(3) *The forgetful morphism $K_0(X)_{\mathbb{Q}} \to K_0^{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}}$ is an isomorphism.*
(4) *$\mathrm{Pic}(X)$ is free of finite rank and $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism.*
(5) *$H_1(X, \mathbb{Z}) = 0$.*

PROOF. Since the Grothendieck group $K_0(X)$ is free and of finite rank, the conditions (1), (2) of Thereom 13.4 hold true. By the universal property of the Chow motives, any Weil cohomological functor $H^\bullet$ with values in $\mathrm{GrVect}_{\mathbb{Q}}^{<\infty}$ factorizes through $\mathrm{CHM}(\mathbb{C})_{\mathbb{Q}}$: hence, using the same notation of Theorem 13.4 and its proof, we have

$$H^\bullet(X) \cong \bigoplus_{i=1}^{n} H^\bullet(\mathbb{L})^{\otimes a_i},$$

and consequently

$$H^{2r}(X) \cong \mathbb{Q}^{N(r)}, \quad \text{since } H^{2i}(\mathbb{L}) \cong \begin{cases} \mathbb{Q}, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

By taking the Hodge realization of the rational Chow motive $\mathfrak{h}(X)_{\mathbb{Q}}$, we deduce point (1). Point (2) follows from the fact that $\mathrm{im}(c_X^r) \subseteq H^{r,r}(X)$ and that $CH^r(X)_{\mathbb{Q}}$ and $H^{2r}(X; \mathbb{Q})$ have the same dimension $N(r)$. Statement (3) follows from the commutative diagram

$$\begin{tikzcd} K_0(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[d] & CH^\bullet(X)_{\mathbb{Q}} \arrow[d] \\ K_0^{\mathrm{top}}(X^{\mathrm{an}})_{\mathbb{Q}} \arrow[r, "\mathrm{ch}^{\mathrm{top}}"] & H^\bullet_{dR}(X^{\mathrm{an}}; \mathbb{Q}) \end{tikzcd}$$

and from (2). Here the vertical arrows denote the natural forgetful morphisms, ch denotes the Chern character as defined in [Ful98], and $\mathrm{ch}^{\mathrm{top}}$ denotes the topological version of the Chern character. As shown in [GKMS13], the freeness of $K_0(X)$ implies that $\mathrm{Pic}(X)$ is free. From the exponential long exact sequence we have that

$$H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^1(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X),$$

and by point (1) we have that $h^{0,1}(X) = h^{0,2}(X) = 0$. Hence the first Chern class map is an isomoprhism. It follows that $\mathrm{Pic}(X)$ is of finite rank. For the last statement, by the Universal Coefficient Theorem we have a (non-canonical) isomorphism

$$H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\beta_2(X)} \oplus H_1(X, \mathbb{Z})^{\mathrm{tors}},$$

and by (4) we deduce that $H_1(X, \mathbb{Z})$ is a finitely generated torsion-free abelian group, and hence free. By point (1) we have that $h^{1,0}(X) = 0$. $\square$

COROLLARY 13.2 ([MT15a], [Kuz16]). *If X is a smooth projective variety over $\mathbb{C}$ such that $\mathcal{D}^b(X)$ admits a full exceptional collection, then X is of Hodge-Tate type, and the length of the collection is equal to $\sum_p h^{p,p}(X)$.*

REMARK 13.2. The proof of Corollary 13.2 which appears in [Kuz16] is not based on motivic decomposition techniques, rather on an *additivity property of Hochschild homology* ([Kuz09]). If X admits a semiorthogonal decomposition $\langle \mathscr{A}_i \rangle_{i=1}^n$ of $\mathcal{D}^b(X)$, then Hochschild homology admits the decomposition

$$HH_\bullet(X) \cong \bigoplus_{i=1}^n HH_\bullet(\mathscr{A}_i). \tag{13.2}$$

Using the following properties of Hochschild homology

- $HH_\bullet(\mathrm{Spec}(\mathbb{K})) \cong \mathbb{K}$ (Example 1.17 of [Kuz16]),
- and Hochschild-Kostant-Rosenberg Theorem

$$HH_\bullet(X) \cong \bigoplus_{q-p=k} H^q(X, \Omega_X^p),$$

by Proposition 12.3 and the decomposition (13.2) one easily concludes.

The following result shows that, in the isometric classification of Mukai structure, for all varieties X the vector spaces $(K_0(\mathbb{X})\otimes_{\mathbb{C}}, \chi(\cdot,\cdot))$ correspond just to one of the possible cases, namely bi-orthogonal sums of U_n-type spaces. It is based on the *dévissage property* of coherent sheaves on X (see e.g. [Sha13], Section II.6.3.3 and [CG10], Section 5.9).

PROPOSITION 13.8 ([CG10]). *Let X be a smooth projective variety over $\mathbb{C}$, and let E be a rank d vector bundle on X. The endomorphism*

$$\varphi_E \colon K_0(X)_{\mathbb{C}} \to K_0(X)_{\mathbb{C}} \colon [\mathscr{F}] \mapsto [\mathscr{F} \otimes (E - d \cdot \mathcal{O}_X)]$$

is nilpotent. In particular,

$$\varphi_E^{\dim_{\mathbb{C}} X+1} = 0.$$

COROLLARY 13.3. *Let X be a smooth projective variety over $\mathbb{C}$ for which the Euler-Poincaré product $\chi(\cdot,\cdot)$ is non-degenerate on $K_0(X)_{\mathbb{C}}$. The canonical morphism $\kappa \colon K_0(X)_{\mathbb{C}} \to K_0(X)_{\mathbb{C}}$ is of the form*

$$\kappa = (-1)^{\dim_{\mathbb{C}} X} \mathbb{1} + M, \quad \text{with } M \text{ nilpotent.}$$

Hence, the Mukai vector space $(K_0(X)_{\mathbb{C}}, \chi)$ is isomorphic to a direct sums of irreducible Mukai spaces of type U_n. In particular, a necessary condition for the irreducibility of $(K_0(X)_{\mathbb{C}}, \chi)$ is that

$$\sum_{j=0}^{\dim_{\mathbb{C}} X} \beta_j(X) \equiv \dim_{\mathbb{C}} X + 1 \pmod 2.$$

PROOF. The canonical morphism κ is defined by $\kappa([\mathscr{F}]) = (-1)^{\dim_{\mathbb{C}} X}[\mathscr{F} \otimes \omega_X]$. By Proposition 13.8, the morphism $\kappa + (-1)^{\dim_{\mathbb{C}} X+1}\phi_{\omega_X}$ is nilpotent. The last two statements follows from the classification of indecomposable Mukai spaces, Theorem 13.1, and from Theorem 13.2. $\square$

Hence, in the isometric classification, the case of Projective Spaces is of particular importance as the following results show.

THEOREM 13.5 ([**Gor94a**, **Gor94b**]). *The Mukai spaces $(K_0(\mathbb{P}^{k-1})_{\mathbb{C}}, \chi)$ are indecomposable of type U_k. Their isometry group*

$$\mathrm{Isom}_{\mathbb{C}}(K_0(\mathbb{P}^{k-1}_{\mathbb{C}})_{\mathbb{C}}, \chi)$$

has two connected components. The identity component is a unipotent abelian algebraic group of dimension[2] $[\frac{k}{2}]$.

Notice indeed that the necessary condition of Corollary 13.3 is satisfied in the case of complex Projective Spaces. From Corollary 13.3 and Theorem 13.5, we deduce the following result.

COROLLARY 13.4. *Let X be a smooth projective variety. The identity component of the isometry group*

$$\mathrm{Isom}_{\mathbb{C}}(K_0(X)_{\mathbb{C}}, \chi)_0$$

is unipotent and abelian.

[2] Here $[x]$ denotes the integer part of $x \in \mathbb{R}$.

CHAPTER 14

The Main Conjecture

Abstract. In this Chapter we review the original (incomplete) version of a conjecture formulated by B. Dubrovin ([**Dub98**]) stating the equivalence of the condition of semisimplicity of the quantum cohomology of a Fano variety X with the condition of existence of a full exceptional collection in $\mathcal{D}^b(X)$. This conjecture would also prescribe the monodromy data (S, C) in geometric terms w.r.t. the objects of the exceptional collection. After reviewing the results available in literature partially confirming the conjecture, we formulate a refined and complete version of the conjecture (Conjecture 14.2), including a prescription also for the central connection matrix C. We also explain how heuristically the conjecture should follows from M. Kontsevich's proposal of Homological Mirror Symmetry.

In the occasion of the 1998 ICM in Berlin, B. Dubrovin formulated a conjecture connecting two apparently different and unrelated aspects of the geometry of Fano varieties, namely their enumerative geometry (quantum cohomology) and their derived category of coherent sheaves.

14.1. Original version of the Conjecture and known results

Let us recall the original statement of the Conjecture formulated by B. Dubrovin.

Conjecture 14.1 ([**Dub98**]). *Let X be a Fano variety.*

(1) *The quantum cohomology $QH^\bullet(X)$ is semisimple if and only if the category $\mathcal{D}^b(X)$ admits a full exceptional collection $(E_1, \dots, E_n)$.*

(2) *The Stokes matrix S, computed w.r.t. a fixed oriented line ℓ admissible for the system and in lexicographical order, is equal to the Gram matrix of the Grothendieck-Euler-Poincaré product w.r.t. a full exceptional collection in $\mathcal{D}^b(X)$,*
$$S_{ij} := \chi(E_i, E_j).$$

(3) *The central connection matrix, connecting the solution $Y^{(0)}_{\text{right}}$ of Theorem 2.9 with the topological-enumerative solution $Y_{\text{top}} := \Psi \cdot Z_{\text{top}}$ of Proposition 3.1, is of the form*
$$C = C' \cdot C'',$$
where the columns of C'' are the components of the Chern characters $\mathrm{ch}(E_i)$, and the matrix C' represents an endomorphism of $H^\bullet(X;\mathbb{C})$ commuting with $c_1(X) \cup (-)\colon H^\bullet(X;\mathbb{C}) \to H^\bullet(X;\mathbb{C})$.

Let us summarize the main results obtained, which is some specific cases partially confirm the validity of Conjecture 14.1:

(1) In [**Guz99**] D. Guzzetti proved point (2) of Conjecture 14.1 for projective spaces (see Remark 14.1 for a precisation). He started a detailed analysis of the action of the braid group on the

set of monodromy data. Nevertheless, his results were not enough to explicitly determine the exceptional collections arising from these data.

(2) The results of D. Guzzetti have been recovered by S. Tanabé in [**Tan04**], who showed how to calculate the Stokes matrices of quantum cohomologies of projective spaces in terms of a certain hypergeometric group. Furthermore, in [**CMvdP15**], J. A. Cruz Morales and M. Van der Put showed another method to obtain the same results of Guzzetti for projective spaces, and also for the case of weighted projective spaces, using multisummation techniques.

(3) In [**Ued05b**] K. Ueda extended the results of D. Guzzetti to all complex grassmannians. His proof relies on a conjecture of K. Hori and C. Vafa ([**HV00**]), rigorously proved by A. Bertram, I. Ciocan-Fontanine, and B. Kim ([**BCFK05**]), relating quantum cohomology of grassmannians with quantum cohomology of projective spaces. The analysis of the action of the braid group is not treated. Note that in [**Ued05b**] the delicate *phenomenon of coalescence* for the isomonodromic system is neither discussed nor recognized: a priori, the monodromy data at points of small quantum cohomology of almost all grassmannians would not be well defined, and do not define local invariants. A rigorous analysis of this point has been developed in [**CDG17b**], and adapted to the geometry of Frobenius manifolds in [**CDG17c**].

(4) In [**Ued05a**] K. Ueda proved point (2) of Conjecture 14.1 for cubic surfaces, using a toric degeneration of the surfaces and A. Givental's mirror results ([**Giv98b**]).

(5) Out of the 106 deformation classes of smooth Fano threefolds (see [**Isk77, Isk78**], [**MM86, MM03**]), only 59 satisfy the condition of vanishing odd cohomology, necessary for the semisimplicity of the quantum cohomology. In [**Cio04, Cio05**], G. Ciolli proved the validity of point (1) of Conjecture 14.1 for 36 out of these 59 families.

(6) A. Bayer proved in [**Bay04**] that the family of varieties for which point (1) of Conjecture 14.1 holds true is closed under blow-ups at any number of points. Furthermore, Bayer also suggested to drop any reference to the condition of being Fano in the statement of Conjecture 14.1. No explicit result is available in the non-Fano case for points (2)-(3) of Conjecture 14.1.

(7) The results of Y. Kawamata [**Kaw06, Kaw13, Kaw16**] confirm the validity of point (1) of Conjecture 14.1 for projective toric manifolds.

(8) In [**Gol09**] V. Golyshev proved the validity of point (2) of Conjecture 14.1 for minimal Fano three-folds, i.e. with minimal cohomology

$$H^{2k+1}(X;\mathbb{Z}) \cong 0, \quad H^{2k}(X;\mathbb{Z}) \cong \mathbb{Z}.$$

(9) In [**GMS15**], S. Galkin, A. Mellit and M. Smirnov proved the validity of point (1) of Conjecture 14.1 for the symplectic isotropic grassmannian $IG(2,6)$. The importance of this result is due to the fact that it underlines the need of considering the whole big quantum cohomology for the formulation of the conjecture, the small quantum locus being contained in the caustic. This result has been generalized for all symplectic isotropic grassmannian $IG(n,2n)$: on the one hand it is known that these grassmannians admit full exceptional collections ([**Kuz08**], [**Sam07**]), on the other hand it has been proved by N. Perrin that their (big) quantum cohomology is generically semisimple (see [**Per14**]). See also [**CMMPS17**].

14.2. Gamma classes, graded Chern character, and morphisms $Д_X^{\pm}$

Let X be a smooth projective variety of (complex) dimension d with odd-vanishing cohomology, V be a complex vector bundle of rank r on X, and let $\delta_1, \dots, \delta_r$ be the Chern roots of the bundle V, so that

$$c_k(V) = \sigma_k(\delta_1, \dots, \delta_r), \quad 1 \le k \le r$$

where σ_k is the k-th elementary symmetric polynomial. Starting from the Taylor series expansion of the functions $\Gamma(1 \pm z)$ near $z = 0$, namely

$$\Gamma(1 \pm z) = \exp\left\{\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} (\mp z)^n\right\},$$

and applying the Hirzebruch's construction of characteristic classes, we can define two characteristic classes $\widehat{\Gamma}^{\pm}(V) \in H^\bullet(X, \mathbb{C})$ by

$$\widehat{\Gamma}^{\pm}(V) := \prod_{j=1}^{r} \Gamma(1 \pm \delta_j).$$

In particular we will denote by $\widehat{\Gamma}^{\pm}(X)$ the characteristic class $\widehat{\Gamma}^{\pm}(TX)$.

For any object $E \in \operatorname{Ob} \mathcal{D}^b(X)$ we define a *graded* version of (the Grothendieck's definition of) the Chern character: being X smooth, the object E is isomorphic in $\mathcal{D}^b(X)$ to a (bounded) complex of locally free sheaves $F^\bullet$. We thus define

$$\operatorname{Ch}(E) := \sum_j (-1)^j \operatorname{Ch}(F^j),$$

where

$$\operatorname{Ch}(F^j) := \sum_h e^{2\pi i \alpha_h}, \quad \text{the } \alpha_h\text{'s being the Chern roots of } F^j.$$

The definition is well posed, since it can be easily shown to be independent of the bounded complex $F^\bullet$ of locally free sheaves.

Let us now define two morphisms $Д_X^{\pm} \colon K_0(X)_{\mathbb{C}} \to H^\bullet(X, \mathbb{C})$ given by

$$Д_X^{\pm}(E) := \frac{i^{\overline{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}^{\pm}(X) \cup \exp(\pm \pi i c_1(X)) \cup \operatorname{Ch}(E),$$

where $\overline{d} \in \{0, 1\}$ is the residue class $d \bmod(2)$.

14.3. Refined statement of the Conjecture

We propose the following refinement of Conjecture 14.1:

CONJECTURE 14.2. *Let X be a smooth Fano variety of Hodge-Tate type.*

(1) *The quantum cohomology $QH^\bullet(X)$ is semisimple if and only if there exists a full exceptional collection in the derived category of coherent sheaves $\mathcal{D}^b(X)$.*

(2) *If $QH^\bullet(X)$ is semisimple, then for any oriented line ℓ (of slope $\phi \in [0; 2\pi[$) in the complex plane there is a correspondence between ℓ-chambers and founded helices, i.e. helices with a marked foundation, in the derived category $\mathcal{D}^b(X)$.*

(3) *The monodromy data computed in a ℓ-chamber Ω_ℓ, in lexicographical order, are related to the following geometric data of the corresponding exceptional collection $\mathfrak{E}_\ell = (E_1, \dots, E_n)$ (the marked foundation):*

   (a) *the Stokes matrix is equal to the* inverse *of the Gram matrix of the Grothendieck-Poincaré-Euler product on $K(X)_{\mathbb{C}}$, computed w.r.t. the exceptional basis $([E_i])_{i=1}^n$*

$$S_{ij}^{-1} = \chi(E_i, E_j);$$

(b) *the Central Connection matrix $C \equiv C^{(0)}$, connecting the solution $Y^{(0)}_{\text{right}}$ of Theorem 2.9 with the topological-enumerative solution $Y_{\text{top}} = \Psi \cdot Z_{\text{top}}$ of Proposition 3.1, coincides with the matrix associated to the $\mathbb{C}$-linear morphism*

$$Д^-_X \colon K_0(X)_{\mathbb{C}} \to H^\bullet(X;\mathbb{C}) \colon E \mapsto \frac{i^{\bar{d}}}{(2\pi)^{\frac{d}{2}}} \widehat{\Gamma}^-_X \cup \exp(-\pi i c_1(X)) \cup \mathrm{Ch}(E),$$

where $d = \dim_{\mathbb{C}} X$, and $\bar{d}$ is the residue class $d \pmod 2$. The matrix is computed w.r.t. the exceptional basis $([E_i])_{i=1}^n$ and any pre-fixed basis $(T_\alpha)_\alpha$ in cohomology (see Section 3.1.1).

Remark 14.1. Let us remark some important points of this revised version of the Conjecture.

(1) Let X be a smooth projective variety with semisimple quantum cohomology. From the original Conjecture 14.1, in [**Dub98**] it was conjectured the existence of an atlas of $QH^\bullet(X)$ whose charts, denoted $Fr(S,C)$, are expected to be in one-to-one correspondence with exceptional collections in $\mathcal{D}^b(X)$. Point (2) of Conjecture 14.2 clarifies this point. In order to have such a correspondence, each of the charts discussed in [**Dub98**] should cover a single ℓ-chamber. The correspondence with exceptional collections is not one-to-one, since two foundations of a same helix, obtained one another by iterated applications of the Serre functor (or its inverse), are associated with monodromy data computed w.r.t. other solutions $Y^{(k)}_{\text{left/right}}$ of Theorem 2.9. In other words, the choice of the foundation of the helix corresponds to the choice of the branch of the logarithmic term $z^\mu z^{c_1(X)\cup(-)}$ of $Y_0(z)$.

(2) One of the main difference between the statement of Conjecture 14.1 and Conjecture 14.2 is the point concerning the Stokes matrix S. The identification of S with the inverse of the Gram matrix is forced by the Grothendieck-Hirzebruch-Riemann-Roch Theorem and the constraint of monodromy data, as it will be evident from Proposition 14.1 and Corollary 14.1. Since this point could be confusing for the reader, let us focus on the example of Projective Spaces. It is usually claimed that in [**Guz99**] it has been proved that the Stokes matrix of $QH^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}})$ coincide (up to mutations) to the Gram matrix associated with the Beilinson exceptional collections $\mathfrak{B} = (\mathcal{O}(i))_{i=0}^{k-1}$. What actually has been found is an explicit braid relating S with the *inverse* of the Gram matrix w.r.t. $\mathfrak{B}$. Then, using the identity of Proposition 13.1

$$[S^{-1}]^\beta = PS^TP, \quad \beta = \beta_{12}(\beta_{23}\beta_{12})(\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\dots\beta_{12}), \quad P^\alpha_\beta = \delta_{\alpha+\beta,1+k},$$

where S is any $k \times k$ Stokes matrix (see also [**Zas96**]), together with the numerical "coincidence"

$$PG^TP = G, \quad \text{for } G_{ab} := \chi(\mathcal{O}(a-1), \mathcal{O}(b-1)) = \binom{k-1+b-a}{b-a}, \quad 1 \le a,b \le k,$$

it was deduced that G and its inverse are in the same orbit under the action of the braid group. We do not know if it is valid for other/all smooth projective varieties X rather than $\mathbb{P}^{k-1}_{\mathbb{C}}$.

Proposition 14.1. *Let X be a smooth projective variety of complex dimension d.*

(1) *Let $E, F \in \operatorname{Ob}\mathcal{D}^b(X)$. Then*

$$\text{Д}^{\pm}_X(E) \cup \text{Д}^{\mp}_X(F) = \frac{(-1)^{\overline{d}}}{(2\pi)^d}\operatorname{Td}(X) \cup \operatorname{Ch}(E) \cup \operatorname{Ch}(F) \cup e^{-i\pi c_1(X)},$$

where $\operatorname{Td}(X) \in H^\bullet(X, \mathbb{C})$ is the graded Todd characteristic class

$$\operatorname{Td}(X) := \prod_{j=1}^d \frac{2\pi i \delta_j}{1 - e^{-2\pi i \delta_j}},$$

where $\delta_1, \ldots, \delta_d$ are the Chern roots of the tangent bundle TX.

(2) *Let us naturally identify the tangent bundle $TQH^\bullet(X)$ with $H^\bullet(X;\mathbb{C})$. Then for any $E \in \operatorname{Ob}\mathcal{D}^b(X)$ we have*

$$e^{-i\pi\mu}(\text{Д}^{\pm}_X(E)) = i^d \text{Д}^{\mp}_X(E^\vee),$$

where $\mu \in \operatorname{End}(H^\bullet(X;\mathbb{C}))$ is the grading operator defined in (2.1).

(3) *Given $E, F \in \operatorname{Ob}\mathcal{D}^b(X)$ the following identity holds true*

$$\int_X e^{-i\pi\mu}(\text{Д}^{\pm}_X(E)) \cup e^{i\pi c_1(X)} \cup \text{Д}^{\pm}_X(F) = \chi(E, F).$$

PROOF. From the well known relation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

we get

$$\Gamma(1+z)\Gamma(1-z) = \frac{2\pi i z}{1 - e^{-2\pi i z}} e^{-i\pi z}.$$

Thus,

$$\text{Д}^{+}_X(E) \cup \text{Д}^{-}_X(F) = \frac{(-1)^{\overline{d}}}{(2\pi)^d}\left(\prod_{j=1}^d \frac{2\pi i \delta_j}{1 - e^{-2\pi i \delta_j}} e^{-i\pi\delta_j}\right) \cup \operatorname{Ch}(E) \cup \operatorname{Ch}(F),$$

and we conclude the proof of (1) since $c_1(X) = \sum_j \delta_j$. For (2) notice that if $\phi \in H^\bullet(X, \mathbb{C})$, $\phi = \sum_p \phi_p$ with $\phi_p \in H^{2p}(X, \mathbb{C})$ then

$$e^{-i\pi\mu}(\phi) = i^d \sum_p (-1)^p \phi_p,$$

and one easily concludes. For the last point (3), we can apply (1),(2) and the Grothendieck-Hirzebruch-Riemann-Roch Theorem as follows

$$\begin{aligned}
&\int_X e^{-i\pi\mu}(\text{Д}^{\pm}_X(E)) \cup e^{i\pi c_1(X)} \cup \text{Д}^{\pm}_X(F) = \\
&\quad i^d \int_X \text{Д}^{\mp}_X(E^\vee) \cup e^{i\pi c_1(X)} \cup \text{Д}^{\pm}_X(F) = && \text{(by (2))}\\
&\quad \frac{(-1)^{\overline{d}}}{(2\pi)^d} i^d \int_X \operatorname{Td}(X) \cup \operatorname{Ch}(E^\vee) \cup \operatorname{Ch}(F) = && \text{(by (1))}\\
&\quad \frac{(-1)^{\overline{d}}}{(2\pi)^d} i^d (2\pi i)^d \chi(E, F) && \text{(by GHRR-Thm).}
\end{aligned}$$

□

COROLLARY 14.1. *Let X be a Fano smooth projective variety for which points* (3b) *of the Conjecture 14.2 holds true. Then also point* (3a) *holds true.*

PROOF. The Stokes and central connection matrices must satisfy the constraint, which is equivalent to

$$(e^{-i\pi\mu}C)^T\eta e^{i\pi R}C = S^{-1},$$

with $R = c_1(X) \cup (-)$. By point (3) of Proposition 14.1 we conclude. □

THEOREM 14.1. *Let X be a smooth Fano variety of Hodge-Tate type for which Conjecture 14.2 holds true. Then, all admissible operations on the monodromy data have a geometrical counterpart in the derived category $\mathcal{D}^b(X)$, as summarized in Table 14.1 at the end of this Chapter. In particular, we have the following:*

(1) *Mutations of the monodromy data (S, C) correspond to mutations of the exceptional basis.*
(2) *The monodromy data computed w.r.t. the others solutions $Y^{(k)}_{\text{left/right}}$, i.e. $(S, C^{(k)})$, are associated, as in points* (3a)-(3b) *of Conjecture 14.2, with different foundations of the helix, related to the marked one by an iterated application of the Serre functor $(\omega_X \otimes -)[\dim_{\mathbb{C}} X] : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$.*
(3) *The group $\widetilde{\mathcal{C}}_0(X)$ is isomorphic to a subgroup of the identity component of the isometry group* $\mathrm{Isom}_{\mathbb{C}}(K_0(X)_{\mathbb{C}}, \chi)$*: more precisely, the morphism*

$$\widetilde{\mathcal{C}}_0(X) \to \mathrm{Isom}_{\mathbb{C}}(K_0(X)_{\mathbb{C}}, \chi)_0 \colon A \mapsto \left(Д^-_X\right)^{-1} \circ A \circ Д^-_X \tag{14.1}$$

defines a monomorphism. In particular, $\widetilde{\mathcal{C}}_0(X)$ is abelian.
(4) *The monodromy matrix $M_0 := e^{2\pi i\mu}e^{2\pi i R}$, with $R = c_1(X) \cup (-)$, has spectrum contained in $\{-1, 1\}$.*

PROOF. Claim (1) immediately follows from the definition of the action of the braid group on the monodromy data, and on the exceptional bases (Proposition 13.1). For claim (2), recall that the pairs of monodromy data $(S, C^{(k)})$ are related one another by a power of the generator of the center of the braid group. Hence one concludes by Corollary 12.3. Point (3) follows from the identification of S with the inverse of the Gram matrix, from constraint and from properties of elements of $\widetilde{\mathcal{C}}_0(X)$: indeed, if $A \in \widetilde{\mathcal{C}}_0(X)$, then

$$\begin{aligned}(C^{-1}AC)^T S^{-1}(C^{-1}AC) &= C^T A^T C^{-T} S^{-1} C^{-1} AC \\ &= C^T A^T \eta e^{\pi i\mu} e^{\pi i R} AC \\ &= C^T A^T \eta e^{\pi i\mu} A e^{\pi i R} C \\ &= C^T \eta e^{\pi i\mu} e^{\pi i R} C \\ &= S^{-1}.\end{aligned}$$

Moreover, since $\widetilde{\mathcal{C}}_0(X)$ is unipotent (and since we are working in characteristic zero) it is connected ([**DG70**], Prop. IV.2.4.1). From Corollary 13.4, we deduce that $\widetilde{\mathcal{C}}_0(X)$ is abelian. The last statement follows from Lemma 2.5 and from Corollary 13.3 and point (2): indeed the inverse matrix $M_0^{-1} \in \widetilde{\mathcal{C}}_0(X)$ corresponds to the canonical operator $\kappa \in \mathrm{Isom}_{\mathbb{C}}(\chi)_0$. □

Proposition 14.2. *The class of Fano variety for which Conjecture 14.2 holds true is closed under finite products.*

Proof. Let X, Y be Fano varieties for which Conjecture 14.2 holds true, an let us define the canonical projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

If X, Y have semisimple quantum cohomology, then also the tensor product of Frobenius manifolds $QH^\bullet(X\times Y) = QH^\bullet(X) \otimes QH^\bullet(Y)$ is semisimple. Furthermore, if $(E_0, \dots, E_n)$ and $(F_0, \dots, F_m)$ are full exceptional collections in $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$, respectively, then the collection $(E_i \boxtimes F_j)_{(i,j)}$, indexed by all pairs (i,j), is a full exceptional collection for $\mathcal{D}^b(X\times Y)$ (see e.g. [**Kuz11**]). Here we set

$$E \boxtimes F := \pi_1^* E \otimes \pi_2^* F.$$

The order of the objects is intended to be the lexicographical one on the pairs (i,j). Using the identities

$$\begin{aligned} \widehat{\Gamma}^{\pm}_{X\times Y} &= \pi_1^* \widehat{\Gamma}^{\pm}_X \cup \pi_2^* \widehat{\Gamma}^{\pm}_Y, \\ \mathrm{Ch}\,(E_i \boxtimes F_j) &= \pi_1^* \mathrm{Ch}(E_i) \cup \pi_2^* \mathrm{Ch}(F_j), \\ c_1(X\times Y) &= \pi_1^* c_1(X) + \pi_2^* c_1(Y), \\ \dim(X\times Y) &= \dim X + \dim Y, \end{aligned}$$

and recalling that if M, M' are two semisimple Frobenius manifolds we have that

$$S_{M\otimes M'} = S_M \otimes S_{M'}, \quad C_{M\otimes M'} = C_M \otimes C_{M'}$$

(see [**Dub99b**], Lemma 4.10), we easily conclude. □

14.4. Relations with Kontsevich's Homological Mirror Symmetry

The validity of the Conjecture 14.2, at least of its points (1) and (3a), can be heuristically deduced from M. Kontsevich's proposal of *Homological Mirror Symmetry* ([**Kon95, Kon98**]). More precisely, Conjecture 14.2 establish an explicit relationship between the two different geometrical aspects of a same Fano manifold X, the *symplectic* one (the A-side) and the *complex* one (the B-side), which can be connected through the study of a object *mirror dual* to X.

Although Mirror Symmetry phenomena were originally studied in the case of Calabi-Yau varieties, several mirror conjectural correspondences have been generalized also to the Fano setting by the works of A. Givental ([**Giv95, Giv97, Giv98b**]), M. Kontsevich ([**Kon98**]), K. Hori and C. Vafa ([**HV00**]). If X is a Fano manifold satisfying the semisimplicity condition of Conjecture 14.2, its mirror dual is conjectured to be a pair (V, f) (the *Landau-Ginzburg model*), where

- V is a non-compact Kähler manifold (with symplectic form ω),
- and $f\colon V \to \mathbb{C}$ is a holomorphic function which defines a Lefschetz fibration, i.e. f admits only isolated non-degenerate critical points $\{p_1, \dots, p_n\}$, with only A_1-type singularities (i.e. Morse-type), and whose fibers are symplectic submanifolds of V.

To such an object, one can associate two different categories, codifying respectively symplectic and complex geometrical properties of the the pair (V, f). Let us briefly recall their constructions. The symplectic geometry, also called *A-side* or *Landau-Ginzburg A-model*, is described by a Fukaya-type $\mathcal{A}_\infty$-category, originally introduced by M. Kontsevich and later by K. Hori, and whose explicit and rigorous construction has been formalized by P. Seidel ([**Sei01b, Sei01a, Sei02**]). On the fibration $f\colon V \to \mathbb{C}$ one can consider a symplectic transport, by considering as horizontal spaces the symplectic orthogonal complement of vertical subspaces, i.e.

$$\mathcal{H}_p := (\ker df_p)^{\perp\omega}, \quad p \in V.$$

For a fixed regular value $z_0 \in \mathbb{C}$, by choosing n paths γ_i connecting z_0 with the critical values[1] $z_i := f(p_i)$ for $i = 1, \dots, n$, so that one can symplectically transport along the arc γ_i the vanishing cycles at p_i. In this way one obtains a Lagrangian disc $D_i \subseteq V$ fibered above γ_i (such a disc is called the *Lefschetz thimble* over γ_i), and whose boundary is a Lagrangian sphere L_i in the fiber $f^{-1}(z_0)$. Assuming genericity conditions, in particular that all the paths intersect each other only at z_0 and that all Lagrangian spheres intersect transversally in $f^{-1}(z_0)$, one can introduce the so called *directed Fukaya category of* $(f, \{\gamma_i\})$.

Definition 14.1 ([**Sei01b, Sei01a**]). The directed Fukaya category $\mathcal{F}uk(V, f, \{\gamma_i\})$ is defined as the $\mathcal{A}_\infty$-category whose objects are the Lagrangian spheres $L_1, \dots, L_n$ and whose morphisms are given by

$$\operatorname{Hom}(L_i, L_j) := \begin{cases} CF^\bullet(L_i, L_j; \mathbb{C}) \cong \mathbb{C}^{|L_i \cap L_j|}, & \text{if } i < j, \\ \mathbb{C} \cdot \operatorname{Id}, & \text{if } i = j, \\ 0, & \text{if } i > j, \end{cases}$$

where the Floer cochain complex $CF^\bullet(L_i, L_j; \mathbb{C})$ with complex coefficients, the differential m_1, the composition m_2 and all other higher degree products m_k's are defined in terms of Floer Lagrangian (co)homology in the fiber $f^{-1}(z_0)$.

The directed Fukaya category is unique up to quasi-isomorphism, and the derived category $\mathcal{D}\mathcal{F}uk(V, f)$ only depends on $f\colon V \to \mathbb{C}$ ([**Sei01b**], Corollary 6.5). Furthermore, the objects $(L_1, \dots, L_n)$ define a full exceptional collection of $\mathcal{D}\mathcal{F}uk(V, f)$, and different choices of paths $\{\gamma_i\}$ (actually inside a same *Hurwtiz equivalence* class) reflect on different choices of full exceptional collections, related one another by operations called *mutations* (not totally coinciding with the ones discussed in Section 12). For more details the reader can consult the cited references.

The second category associated to the pair (V, f), encoding its complex geometrical aspects, is the so called *triangulated category of singularities* defined by D. Orlov ([**Orl04, Orl09**]). If Y is an algebraic variety over $\mathbb{C}$, in what follows we denote by $\mathfrak{Perf}(Y)$ the full triangulated subcategory of $\mathcal{D}^b(X)$ formed by perfect complex, i.e. objects locally isomorphic to a bounded complex of coherent sheaves of finite type: in particular, if Y is smooth, then $\mathfrak{Perf}(Y) \equiv \mathcal{D}^b(Y)$.

Definition 14.2 ([**Orl04, Orl09**]). We define the *triangulated category of singularities of* (V, f) as the disjoint union

$$\mathcal{D}_{\text{sing}}(V, f) := \coprod_{z \in \mathbb{C}} \mathcal{D}_{\text{sing}}(V_z), \quad V_z := f^{-1}(z),$$

where we introduced the quotient category,

$$\mathcal{D}_{\text{sing}}(V_z) := \mathcal{D}^b(V_z)/\mathfrak{Perf}(V_z).$$

[1] We assume that the critical values, and the paths are numbered in clockwise order around the regular value z_0.

Such a quotient is defined by localizing the category $\mathcal{D}^b(V_z)$ w.r.t. the class of morphisms s embedding into an exact triangle

$$X \xrightarrow{s} Y \longrightarrow Z \longrightarrow X[1], \quad \text{with } Z \in \mathrm{Ob}(\mathfrak{Perf}(V_z)).$$

In particular, note that $\mathcal{D}_{\mathrm{sing}}(V_z)$ is non-trivial only at the critical values $z_1, \ldots, z_n$ of f.

The crucial point in our discussion is the following homological formulation of Mirror Symmetry in the Fano case.

Conjecture 14.3 (Homological Mirror Symmetry, [**Kon98**]). *Let X be a Fano variety. There exist equivalences of triangulated categories as follows:*

$$\begin{array}{ccc} A\text{-Model} & & B\text{-Model} \\ & & \\ \mathcal{F}uk(X) & & \mathcal{D}^b(X) \\ & \times & \\ \mathcal{D}\mathcal{F}uk(V,f) & & \mathcal{D}_{sing}(V,f) \end{array}$$

It is believed that the net of equivalences described above could be recast in terms of isomorphy of Frobenius manifolds structures, associated with X and (V, f), respectively. More precisely, we have that

- the Frobenius manifold related to the symplectic geometry of X (the A-side) is the quantum cohomology $QH^\bullet(X)$;
- the Frobenius manifold associated with (V, f), and encoding information about its complex geometrical aspects (the B-side), is the Frobenius manifold structure defined on the space of miniversal unfodings of f. The general construction is well-defined thanks to the works of A. Douai and C. Sabbah [**DS03, DS04, Sab08**], C. Hertling [**Her02, Her03**] and also of S. Barannikov's construction of Frobenius structures arising from semi-infinite variations of Hodge structures ([**Bar00**]). These efforts can be seen as a generalization of the construction of K. Saito [**Sai83**], who considered the case of germs of functions defined on $\mathbb{C}^n$.

The A-model and B-model of the Landau-Ginzburg mirror (V, f) are conjectured to *numerically* related in the following way:

Conjecture 14.4. *The Stokes matrix of the B-model Frobenius manifold associated to (V, f) equals the Gram matrix of the Grothendieck-Euler-Poincaré product $\chi(\cdot, \cdot)$ product on $\mathcal{D}\mathcal{F}uk(V, f)$.*

Putting together Conjecture 14.4 and Conjectures 14.3, it is clear that points (1) and (3.a) should (heuristically) follow.

14.5. Galkin-Golyshev-Iritani Gamma Conjectures and its relationship with Conjecture 14.2

Some months before the beginning of the research project of this Thesis, two papers by S. Galkin, V. Golyshev and H. Iritani appeared ([**GGI16, GI15**]). In *loc. cit.*, the authors proposed two conjectures, called Γ-conjectures, describing the exponential asymptotic of flat sections for the extended deformed connection $\widehat{\nabla}$ on the (semisimple) quantum cohomology of Fano varieties. Although they focused attention on flat *vector fields*, rather than flat *differentials* defining deformed flat coordinates, Galkin, Golyshev and Iritani claimed that from Γ-conjectures it should follows a refinement of the third part of the original Dubrovin's Conjecture 14.1: it is claimed, that the columns of the central connection matrix of the semisimple quantum cohomology of a Fano variety X, computed w.r.t. the topological solution of Section 3.3, should be equal to the components of the form

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\widehat{\Gamma}^{+}_{X}\cup \mathrm{Ch}(E_i),\quad d:=\dim_{\mathbb{C}} X,$$

for some exceptional collection $(E_1,\dots,E_n)$. Some months before the paper [**GGI16**] was issued, B. Dubrovin formulated an *ansatz* for the same connection matrix: in his formulation, the class $\widehat{\Gamma}^{+}_{X}$ is replaced by the class $\widehat{\Gamma}^{-}_{X}$ ([**Dub13**]). The computation of this Part of the Thesis show that although both of these formulations are *admissible*, they do not correspond to the topological-enumerative choice of a solution in Levelt normal form at $z=0$, and actually the action of two different elements of $\widetilde{\mathcal{C}}_0(X):=\widetilde{\mathcal{C}}_0(\mu,R)$, with $R=c_1(X)\cup(-)$, is necessary. The exact form of the central connection matrix computed w.r.t. the topological-enumerative solution is the one prescribed by Conjecture 14.2.

TABLE 14.1.

Frobenius Manifolds $QH^\bullet(X)$	Derived category $\mathcal{D}^b(X)$	Grothendieck group $K_0(X)_\mathbb{C}$
Stokes matrix S		inverse of the Gram matrix $G_{ij} := \chi(E_i, E_j)$ for an exceptional basis $([E_i])_i$
Central connection matrix C		matrix associated with the morphism $Д^-_X \colon K_0(X)_\mathbb{C} \to H^\bullet(X;\mathbb{C})$
action of the braid group $\mathcal{B}_n$	action of $\mathcal{B}_n$ on the set of exceptional collections	action of $\mathcal{B}_n$ on exceptional bases
action of the group $(\mathbb{Z}/2\mathbb{Z})^{\times n}$	shifts of exceptional collections	projected shifts, i.e. change of signs, of exceptional bases
action of the group $\widetilde{\mathcal{C}}_0(X)$	action of a subgroup of autoequivalences $\mathrm{Aut}(\mathcal{D}^b(X))$	action of a subgroup of the identity component of $\mathrm{Isom}_\mathbb{C}(K_0(X)_\mathbb{C}, \chi)$
complete ccw rotation of the line ℓ, action of the generator of the center $Z(\mathcal{B}_n)$, action of the matrix $M_0 \in \widetilde{\mathcal{C}}_0(X)$, $M_0 := \exp(2\pi i\mu)\exp(2\pi iR)$	action of the Serre functor $(\omega_X \otimes -)[\dim_\mathbb{C} X]$ on the set of exceptional collections	action of the canonical operator $\kappa \colon K_0(X)_\mathbb{C} \to K_0(X)_\mathbb{C}$ on the set of exceptional bases

CHAPTER 15

Proof of the Main Conjecture for Projective Spaces

ABSTRACT. In this Chapter we prove the validity of Conjecture 14.2 for all complex Projective Spaces $\mathbb{P}^{k-1}_{\mathbb{C}}$. After computing the topological-enumerative solution for the system of deformed flat coordinates, we show that the group $\widetilde{\mathcal{C}}_0(\mathbb{P}^{k-1}_{\mathbb{C}})$, which describes the ambiguity in the choice of a solutions in Levelt normal form at $z = 0$, is isomorphic to the identity component of the isometry group $\mathrm{Isom}_{\mathbb{C}}(K_0(\mathbb{P}^{k-1}_{\mathbb{C}}), \chi(\cdot,\cdot))_0$. Hence, we compute the central connection matrix at the point $0 \in QH^\bullet(\mathbb{P}^{k-1}_{\mathbb{C}})$ w.r.t a line ℓ of slope $0 < \phi < \frac{\pi}{k}$. By completing the braid analysis developed by D. Guzzetti in [Guz99], we recognize in the computed monodromy data the geometric information, as prescribed by the Conjecture 14.2, associated with an explicit mutations of the Beilinson exceptional collection (see Theorem 15.2 and Corollary 15.2). After studying in detail the trace of the ℓ-chamber decomposition along the small quantum locus, a property of *quasi-periodicity* of Stokes matrix is shown (Theorem 15.5). From this property, we deduce that the only Projective Space for which the monodromy data are the ones associated with the Beilinson collection (modulo suitable choices of branches of the Ψ-matrix, choice of the line ℓ, etc.) are $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{P}^2_{\mathbb{C}}$ (Corollary 15.3). For all other Projective Spaces the data corresponding to the Beilinson exceptional collection can be computed in chambers of the big quantum cohomology.

15.1. Notations and preliminaries

In what follows

- the symbol $\mathbb{P}$ will stand for $\mathbb{P}^{k-1}_{\mathbb{C}}$, $k \geq 2$;
- we denote σ the generator of the 2-nd cohomology group $H^2(\mathbb{P};\mathbb{C})$, so that

$$H^\bullet(\mathbb{P};\mathbb{C}) \cong \frac{\mathbb{C}[\sigma]}{(\sigma^k)}.$$

We also assume that σ is normalized so that

$$\int_{\mathbb{P}} \sigma^{k-1} = 1.$$

The flat coordinates $t^1, \dots, t^k$ for the quantum cohomology of $\mathbb{P}$ are the coordinates w.r.t. the homogeneous basis

$$(1, \sigma, \sigma^2, \dots, \sigma^{k-1}),$$

the matrix of the Poincaré metric being constant

$$\eta_{\alpha\beta} = \eta\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right) = \delta_{\alpha+\beta,k+1}.$$

Notice that the unity vector field is $e = \frac{\partial}{\partial t^1}$, and the Euler vector field is

$$E = \sum_{\alpha \neq 2} (1 - q_\alpha) t^\alpha \frac{\partial}{\partial t^\alpha} + k\frac{\partial}{\partial t^2}, \quad q_h = h - 1 \quad \text{for } h = 1, \dots, k.$$

If ζ is a column vector whose components are the components of the gradient of a deformed flat coordinate, w.r.t the frame $\left(\frac{\partial}{\partial t_i}\right)_i = (\sigma^i)_i$, then it must satisfies the system

$$\begin{cases} \partial_\alpha \zeta = z\mathcal{C}_\alpha \zeta, \\ \partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta. \end{cases}$$

If we restrict to the locus of *small quantum cohomology*, i.e. to the points $(0, t^2, 0, \dots, 0)$, the system above reduces to the two equations

$$\partial_2 \zeta = z\mathcal{C}_2 \zeta, \tag{15.1}$$

$$\partial_z \zeta = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta, \tag{15.2}$$

where at the point $(0, t^2, 0, \dots, 0)$

$$\mathcal{U} := \begin{pmatrix} 0 & & & & kq \\ k & 0 & & & \\ & k & 0 & & \\ & & \ddots & \ddots & \\ & & & k & 0 \end{pmatrix}, \quad q := e^{t^2}, \quad \mathcal{C}_2 = \frac{1}{k}\mathcal{U}, \quad \mu = \operatorname{diag}\left(-\frac{k-1}{2}, -\frac{k-3}{2}, \dots, \frac{k-3}{2}, \frac{k-1}{2}\right).$$

We study the monodromy phenomenon of the second differential equation of the system:

$$\frac{d\zeta}{dz} = \left(\mathcal{U} + \frac{1}{z}\mu\right)\zeta. \tag{15.3}$$

The eigenvalues of the matrix $\mathcal{U}(0, t^2, 0, \dots, 0)$ are

$$u_h = ke^{\frac{2\pi i(h-1)}{k}} q^{\frac{1}{k}} \quad h = 1, \dots, k,$$

and let us compute the corresponding eigenvectors $x_1, \dots, x_k$: the equations for $x_h = (x_h^1, \dots, x_h^k)$ read

$$kx_h^\ell = u_h x_h^{\ell+1}, \quad \ell = 1, \dots, k-1,$$
$$kqx_h^k = u_h x_h^1.$$

By choosing $x_h^k = e^{\frac{i\pi(h-1)}{k}}$, we get all the entries

$$x_h^\ell = \left(\frac{u_h}{k}\right)^{k-\ell} x_h^k = q^{\frac{k-\ell}{k}} e^{(1-2\ell)i\pi\frac{(h-1)}{k}} \quad h, \ell = 1, \dots, k.$$

Since the norm of the eigenvector x_h is

$$\eta(x_h, x_h) = kq^{\frac{k-1}{k}},$$

we find (choosing signs of square roots) the orthogonal vectors $f_1, \dots, f_k$

$$f_h^\ell = k^{-\frac{1}{2}} q^{\frac{k+1-2\ell}{2k}} e^{(1-2\ell)i\pi\frac{(h-1)}{k}} \quad h, \ell = 1, \dots, k.$$

Thus the matrix Ψ is given by

$$\Psi = \left(\begin{array}{c|c|c|c} f_1 & f_2 & \dots & f_k \end{array}\right)^{-1}.$$

Instead of working with the differential equation (15.3) we consider the gauge equivalent system of differential equations in $\xi(z,t^2) := \eta\cdot\zeta(z,t^2)$, a column vector whose components are the ones of the differential of a deformed flat coordinate:

$$\partial_2\xi = z\mathcal{C}_2^T\xi, \tag{15.4}$$

$$\partial_z\xi = \left(\mathcal{U}^T - \frac{1}{z}\mu\right)\xi. \tag{15.5}$$

A simple computation shows that with the following substitution

$$\xi_\alpha(z,t^2) = \frac{1}{k^{\alpha-1}} z^{\frac{k-1}{2}-\alpha+1}\vartheta^{\alpha-1}\Phi(z,t^2),$$

for any $\alpha = 1,2,\dots,k$ and where $\vartheta := z\frac{d}{dz}$, the system (15.4)-(15.5) is equivalent to the equations

$$\vartheta^k\Phi - (kz)^k q\Phi = 0,$$
$$\partial_2^k\Phi - z^k q\Phi = 0.$$

The compatibility of these equations implies the following functional dependence of Φ on (z,t^2):

$$\Phi(t^2,z) = \Phi(q^{\frac{1}{k}}z).$$

Thus, the study of the system (15.5), restricted to the point $t^2=0$, is equivalent to the study of the generalized hypergeometric equation

$$\vartheta^k\Phi(z) - (kz)^k\Phi(z) = 0, \tag{15.6}$$

where $\vartheta := z\frac{d}{dz}$. Given a solution Φ of (15.6), the corresponding solution of equation (15.5) is given by

$$\xi = \begin{pmatrix} z^{\frac{k-1}{2}}\Phi(z) \\ \vdots \\ \frac{1}{k^{\alpha-1}}z^{\frac{k-1}{2}-\alpha+1}\vartheta^{\alpha-1}\Phi(z) \\ \vdots \\ \frac{1}{k^{k-1}}z^{\frac{1-k}{2}}\vartheta^{k-1}\Phi(z)\end{pmatrix}.$$

15.2. Computation of the Topological-Enumerative Solution

In this section, we use the characterization of the topological-enumerative solution described in Proposition 3.1.

Lemma 15.1. *The formal series* $\Phi(z) \in \mathbb{C}[\sigma,\log z][\![z]\!]$

$$\Phi(z) = e^{k\sigma\log(z)}\sum_{n=0}^{\infty} f(n)z^{kn}, \quad f(n)\in\mathbb{C}[\sigma]/(\sigma^k)$$

satisfies equation

$$\vartheta^k\Phi(z) - (kz)^k\Phi(z) = 0,$$

if and only if the coefficients $f(n)$ *satisfy the following difference equation*

$$(\sigma+n)^k f(n) = f(n-1), \quad n\geq 1.$$

PROOF. Observe that

$$\begin{aligned}\vartheta\Phi(z) &= z e^{k\sigma\log(z)}\left(\frac{k\sigma}{z}\sum_{n=0}^{\infty} f(n)z^{kn} + \sum_{n=0}^{\infty} f(n)knz^{kn-1}\right)\\ &= k e^{k\sigma\log(z)}\sum_{n=0}^{\infty}(\sigma+n)f(n)z^{kn}.\end{aligned}$$

By an inductive argument, one easily can shows that

$$\vartheta^{\alpha}\Phi(z) = k^{\alpha}e^{k\sigma\log(z)}\sum_{n=0}^{\infty}(\sigma+n)^{\alpha}f(n)z^{kn}.$$

So, using the fact that $\sigma^k = 0$, we have that

$$\vartheta^k\Phi(z) = (kz)^k\Phi(z),$$

if and only if

$$(\sigma+n)^k f(n) = f(n-1) \quad \text{for all } n \geq 1.$$

□

PROPOSITION 15.1. (1) *For any fixed value $f(0) \in H^{\bullet}(\mathbb{P};\mathbb{C})$, the corresponding formal solution of (15.6) is given by*

$$\Phi(z) = f(0)\cdot\sum_{p=0}^{k-1}\left(\sum_{l=0}^{p}\frac{(k\log z)^{p-l}}{(p-l)!}a_l(z)\right)\sigma^p,$$

where, for $0 \leq l \leq k-1$, we have introduced the notation

$$a_l(z) := \sum_{n=0}^{\infty}\alpha_{n,l}z^{kn}, \quad \alpha_{0,l} := \delta_{0,l}, \quad \alpha_{n,l} := \sum_{\substack{h_1+\cdots+h_n=l\\ 0\leq h_i\leq k-1}}\left(\prod_{j=1}^{n}\frac{(-1)^{h_j}}{j^{k+h_j}}\binom{k-1+h_j}{h_j}\right). \tag{15.7}$$

Representing $\Phi(z) = \sum_{i=1}^{k}\Phi_i(z)\sigma^{k-i}$, we deduce that each component

$$\Phi_i(z) := f(0)\cdot\sum_{l=0}^{k-i}\frac{(k\log z)^{k-i-l}}{(k-i-l)!}a_l(z)$$

is a solution of (15.6).

(2) *Another representation of the solution is given by the formula*

$$\Phi(z) = f(0)e^{k\sigma\log z}\sum_{n=0}^{\infty}\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k}e^{\pm k\pi i n}z^{kn}$$

for any choice of the sign $(\pm)$.

(3) *Moreover, if $f(0) = 1$, the fundamental solution Ξ of (15.5), given by*

$$\Xi_0(z) = \begin{pmatrix} z^{\frac{k-1}{2}}\Phi_1(z) & \dots & z^{\frac{k-1}{2}}\Phi_k(z)\\ \vdots & & \vdots\\ \frac{1}{k^{\alpha-1}}z^{\frac{k-1}{2}-\alpha+1}\vartheta^{\alpha-1}\Phi_1(z) & \dots & \frac{1}{k^{\alpha-1}}z^{\frac{k-1}{2}-\alpha+1}\vartheta^{\alpha-1}\Phi_k(z)\\ \vdots & & \vdots\\ \frac{1}{k^{k-1}}z^{\frac{1-k}{2}}\vartheta^{k-1}\Phi_1(z) & \dots & \frac{1}{k^{k-1}}z^{\frac{1-k}{2}}\vartheta^{k-1}\Phi_k(z)\end{pmatrix}, \tag{15.8}$$

is of the form

$$\Xi(z)=\eta\Theta_{\rm top}(z)z^{\mu}z^{c_1(\mathbb{P})\cup(-)},\quad \Theta_{\rm top}(z)^{\alpha}_{\gamma}=\delta^{\alpha}_{\gamma}+\sum_{n=0}^{\infty}\sum_{\lambda}\sum_{\beta\in{\rm Eff}(\mathbb{P})\setminus\{0\}}\langle\tau_n\sigma^{\gamma},\sigma^{\lambda}\rangle^{\mathbb{P}}_{0,2,\beta}\eta^{\lambda\alpha}z^{n+1},$$

$$\textit{with}\quad \langle\tau_n\sigma^{\gamma},\sigma^{\lambda}\rangle^{\mathbb{P}}_{0,2,\beta}:=\int_{[\overline{\mathcal{M}}_{0,2}(X,\beta)]^{\rm vir}}\psi_1^n\cup{\rm ev}_1^*(\sigma^{\gamma})\cup{\rm ev}_2^*(\sigma^{\lambda}).$$

Proof. From the identity

$$(1+\sigma)^{-1}=1-\sigma+\sigma^2-\cdots+(-1)^{k-1}\sigma^{k-1},$$

one easily shows that if $n\geq 1$, then

$$(n+\sigma)^{-k}=n^{-k}\left(1+\frac{\sigma}{n}\right)^{-k}=\sum_{h=0}^{k-1}\frac{(-1)^h}{n^{k+h}}\binom{k-1+h}{h}\sigma^h.$$

As a consequence, we have that

$$f(n)=\sum_{l=0}^{k-1}f(0)\sigma^l\alpha_{n,l},$$

where the numbers $\alpha_{n,l}\in\mathbb{Q}$ are defined as in (15.7). It follows that

$$\begin{aligned}\Phi(z)&=f(0)e^{k\sigma\log z}\sum_{n=0}^{\infty}\sum_{l=0}^{k-1}f(0)\sigma^l\alpha_{n,l}z^{kn}\\&=f(0)\left(\sum_{m=0}^{k-1}\frac{(k\log z)^m}{m!}\sigma^m\right)\cdot\left(\sum_{n=0}^{\infty}\sum_{l=0}^{k-1}\sigma^l\alpha_{n,l}z^{kn}\right)\\&=f(0)\left(\sum_{m=0}^{k-1}\frac{(k\log z)^m}{m!}\sigma^m\right)\left(\sum_{l=0}^{k-1}a_l(z)\sigma^l\right)\\&=f(0)\cdot\sum_{p=0}^{k-1}\left(\sum_{l=0}^{p}\frac{(k\log z)^{p-l}}{(p-l)!}a_l(z)\right)\sigma^p.\end{aligned}$$

This proves point (1). For the second point, observe that also the functions

$$f_{\pm}(n):=\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k}e^{\pm k\pi i n}$$

satisfy the relation

$$(\sigma+n)^kf_{\pm}(n)=f_{\pm}(n-1).$$

For the last claim, if we write the solution Ξ_0 in the form

$$\Xi_0(z)=z^{-\mu}A(z)\eta z^R, R\equiv c_1(\mathbb{P})\cup(-)\colon H^\bullet(\mathbb{P};\mathbb{C})\to H^\bullet(\mathbb{P};\mathbb{C}),$$

by Proposition 3.2 it is sufficient to prove that $A(z)$ is holomorphic in $z=0$ and $A(0)=\mathbb{1}$. From the identity

$$\Phi(z)=z^{k\sigma}\sum_{n=0}^{\infty}f(n)z^{kn},$$

we obtain for $1 \leq \alpha \leq k$ the relation

$$\vartheta^{\alpha-1}\Phi(z) = z^{k\sigma}\left\{(k\sigma)^{\alpha-1} + \sum_{p=0}^{\alpha-2}\binom{\alpha-1}{p}k^{\alpha-1}\sigma^p\sum_{n=0}^{\infty}f(n)n^{\alpha-1-p}z^{kn}\right\},$$

and by definition of $A(z)$ we have the identity

$$\sum_{j=1}^{k}A(z)_j^{\alpha}\sigma^{j-1} = \frac{1}{k^{\alpha-1}}\left\{(k\sigma)^{\alpha-1} + \sum_{p=0}^{\alpha-2}\binom{\alpha-1}{p}k^{\alpha-1}\sigma^p\sum_{n=0}^{\infty}f(n)n^{\alpha-1-p}z^{kn}\right\}.$$

This shows that $A(z)$ is holomorphic in $z = 0$, and furthermore that $A(0) = \mathbb{1}$. □

15.3. Computation of the group $\widetilde{\mathcal{C}}_0(\mathbb{P})$

Let us introduce the $k \times k$ matrices J_i, $i \geq 0$, defined by

$$(J_i)_{ab} := \delta_{i,a-b}.$$

THEOREM 15.1. *The group $\widetilde{\mathcal{C}}_0(\mathbb{P})$ is an abelian unipotent algebraic group of dimension $[\frac{k}{2}]$. In particular, the exponential map defines an isomorphism*

$$\widetilde{\mathcal{C}}_0(\mathbb{P}) \cong \underbrace{\mathbb{C}\oplus\cdots\oplus\mathbb{C}}_{[\frac{k}{2}]\ \textit{copies}}.$$

With respect to the basis $(1, \sigma, \dots, \sigma^{k-1})$ of $H^\bullet(\mathbb{P};\mathbb{C})$, the group $\widetilde{\mathcal{C}}_0(\mathbb{P})$ is described as follows

$$\widetilde{\mathcal{C}}_0(\mathbb{P}) = \left\{C \in GL(k,\mathbb{C})\colon C = \sum_{i=0}^{k-1}\alpha_i J_i,\quad \alpha_0 = 1,\quad 2\alpha_{2n} + \sum_{\substack{i+j=2n\\1\leq i,j}}(-1)^i\alpha_i\alpha_j = 0,\quad 2 \leq 2n \leq k-1\right\}.$$

PROOF. If $C \in \widetilde{\mathcal{C}}_0(\mathbb{P})$, in order to have that $P(z) := z^\mu z^R C z^{-R} z^{-\mu}$ is polynomial in z, where R is the operator of classical multiplication by the first Chern class $c_1(\mathbb{P})$, the matrix C must be of the form

$$C = \sum_{i=0}^{k-1}\alpha_i J_i, \quad \alpha_0 = 1.$$

We have that $C \in \tilde{\mathcal{C}}_0(\mathbb{P})$ if and only if

$$\left(\sum_{i=0}^{k-1}(-1)^i\alpha_i z^i J_i^T\right)\eta\left(\sum_{i=0}^{k-1}\alpha_i z^i J_i\right) = \eta.$$

The l.h.s is equal to

$$\eta + \sum_{i=1}^{k-1}\alpha_i z^i \eta J_i + \sum_{i=1}^{k-1}(-1)^i\alpha_i z^i J_i^T\eta + \sum_{h=2}^{2k-2}\left(\sum_{\substack{i+j=h\\1\leq i,j\leq k-1}}(-1)^i\alpha_i\alpha_j J_i^T\eta J_j\right)z^h,$$

and using the relations

$$\eta J_i = J_i^T \eta, \tag{15.9}$$
$$(J_i J_j)_{ab} = \delta_{i+j,a-b} = (J_{i+j})_{ab}, \tag{15.10}$$
$$J_h = 0 \quad \text{if } h \geq k, \tag{15.11}$$

we obtain the equation

$$\sum_{\substack{1\leq i\leq k-1\\ i \text{ even}}} 2\alpha_i z^i \eta J_i + \sum_{h=2}^{k-1}\left(\sum_{\substack{i+j=h\\ 1\leq i,j\leq k-1}} (-1)^i \alpha_i\alpha_j\right) z^h \eta J_h = 0.$$

So, we have the following constraints on the constants α_i's:

$$\begin{aligned} 2\alpha_2 - \alpha_1^2 &= 0,\\ 2\alpha_4 - 2\alpha_1\alpha_3 + \alpha_2^2 &= 0,\\ 2\alpha_6 - 2\alpha_1\alpha_5 + 2\alpha_2\alpha_4 - \alpha_3^2 &= 0,\\ &\dots\\ 2\alpha_{2n} + \sum_{\substack{i+j=2n\\ 1\leq i,j}} (-1)^i\alpha_i\alpha_j &= 0, \quad 2\leq 2n\leq k-1. \end{aligned}$$

The Lie algebra of the group is

$$\widetilde{\mathfrak{g}}_0(\mathbb{P}) = \left\{ C \in \mathfrak{gl}(k,\mathbb{C}) \colon C = \sum_{i=0}^{k-1} \alpha_i J_i, \quad \alpha_{\text{even}} = 0 \right\},$$

which is abelian by (15.10), coherently with Theorem 14.1. In characteristic zero the structure of unipotent abelian group is well-known: in particular, the exponential map defines an isomorphism of groups (see [**DG70**], Ch. IV.2.4 Proposition 4.1). □

The following result immediately follows from Theorem 14.1 and Theorem 13.5.

Corollary 15.1. *The groups $\widetilde{\mathcal{C}}_0(\mathbb{P})$ and the identity component* $\mathrm{Isom}_{\mathbb{C}}(K_0(\mathbb{P})_{\mathbb{C}}, \chi)_0$ *are isomorphic.*

Remarkably, notice that the equations obtained above for the group $\widetilde{\mathcal{C}}_0(\mathbb{P})$ essentially coincide with those obtained by A.L. Gorodentsev for $\mathrm{Isom}_{\mathbb{C}}(K_0(\mathbb{P})_{\mathbb{C}}, \chi)_0$ in [**Gor94a, Gor94b**].

15.4. Computation of the Central Connection Matrix

Using the labeling of the canonical coordinates $u_1, \dots, u_n$ introduced in the section 15.1, we introduce the corresponding Stokes' rays:

$$R_{rs} := \{ z = -i\rho(\overline{u_r} - \overline{u_s}), \quad \rho > 0 \}.$$

At a generic point of the small quantum cohomology $(0, t^2, 0, \dots, 0)$, we have

$$
\begin{aligned}
-i(\overline{u_r} - \overline{u_s}) &= -ikq^{-\frac{1}{k}}\left(e^{-\frac{2\pi i(r-1)}{k}} - e^{-\frac{2\pi i(s-1)}{k}}\right)\\
&= 2kq^{-\frac{1}{k}} \sin\left(\frac{\pi}{k}(s-r)\right) \exp\left(i\left[\frac{2\pi}{k} - \frac{\pi}{k}(r+s)\right]\right).
\end{aligned}
$$

So if $r < s$ the Stokes' rays at a generic point $(0, t^2, 0, \dots, 0)$ are

$$
\begin{gathered}
R_{rs} = \left\{z\colon z = \rho \exp\left(i\left[\frac{2\pi}{k} - \frac{\pi}{k}(r+s) - \frac{\tau}{k}\right]\right)\right\}, \quad \tau := \Im(t^2), \\
R_{sr} = -R_{rs}.
\end{gathered}
\tag{15.12}
$$

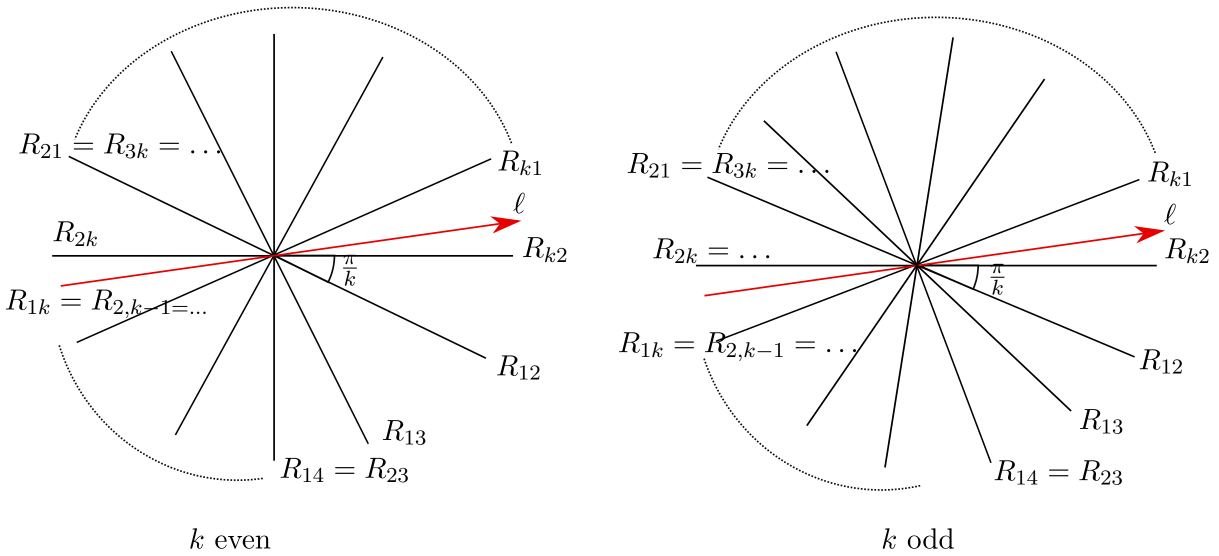

FIGURE 15.1. Configuration of Stokes rays for k odd and k even.

Since we want compute the central connection matrix at $t^2 = 0$ we have to fix an admissible line: following [**Guz99**] we choose a line ℓ with slope $0 < \phi < \frac{\pi}{k}$.

PROPOSITION 15.2 ([**Guz99**]). *Let*

$$
g(z) := \begin{cases} \frac{1}{(2\pi)^{\frac{k+1}{2}}} \int_\Lambda \Gamma(-s)^k z^{ks} ds, & k \text{ even} \\[2ex] \frac{1}{(2\pi)^{\frac{k+1}{2}} i} \int_\Lambda \Gamma(-s)^k e^{-i\pi s} z^{ks} ds, & k \text{ odd} \end{cases}
$$

where Λ is a straight line going from $-c - i\infty$ to $-c + i\infty$, $c > 0$. Fix a line ℓ with slope $0 < \epsilon < \frac{\pi}{k}$. Then, for k even, the fundamental solution Ξ_R, having asymptotic expansion

$$
\Xi = \eta \Psi^{-1} e^{zU}\left(\mathbb{1} + O\left(\frac{1}{z}\right)\right) \quad \text{on } \Pi_R,
$$

is reconstructed from the solutions of (15.6)

$$\Phi_R(z)^T := \begin{pmatrix} (-1)^{\frac{k}{2}}\left(g(ze^{-i\pi}) - \binom{k}{1}g\left(ze^{-i\pi+i\frac{2\pi}{k}}\right) + \cdots - \binom{k}{k-1}g\left(ze^{i(\pi-\frac{2\pi}{k})}\right)\right) \\ \vdots \\ g\left(ze^{-\frac{4\pi i}{k}}\right) - \binom{k}{1}g\left(ze^{-\frac{2\pi i}{k}}\right) + \binom{k}{2}g(z) - \binom{k}{3}g\left(ze^{\frac{2\pi i}{k}}\right) \\ -g\left(ze^{-\frac{2\pi i}{k}}\right) + \binom{k}{1}g(z) \\ g(z) \\ -g\left(ze^{\frac{2\pi i}{k}}\right) \\ g\left(ze^{\frac{4\pi i}{k}}\right) \\ \vdots \\ (-1)^{\frac{k}{2}-1}g\left(ze^{\frac{2\pi i}{k}(\frac{k}{2}-1)}\right) \end{pmatrix}$$

and the entry corresponding to $g(z)$ *is the* $n(k) := \left(\frac{k}{2}+1\right)$*-th one. We can write in a more compact form this vector as follows:*

$$\Phi_R^\alpha(z) = \begin{cases} (-1)^{\frac{k}{2}+1-\alpha}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}g\left(ze^{-i\pi+(\alpha+h-1)\frac{2\pi i}{k}}\right) & \text{if } 1 \le \alpha \le \frac{k}{2}, \\ (-1)^{\alpha-\frac{k}{2}-1}g\left(ze^{\frac{2\pi i}{k}(\alpha-\frac{k}{2}-1)}\right) & \text{if } \frac{k}{2}+1 \le \alpha \le k. \end{cases}$$

For k *odd we have*

$$\Phi_R(z)^T := \begin{pmatrix} (-1)^{\frac{k-1}{2}}\left(g\left(ze^{-\frac{2\pi i}{k}(\frac{k-1}{2})}\right) - \binom{k}{1}g\left(ze^{-\frac{2\pi i}{k}(\frac{k-3}{2})}\right) + \cdots + \binom{k}{k-1}g\left(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}\right)\right) \\ \vdots \\ g\left(ze^{-\frac{4\pi i}{k}}\right) - \binom{k}{1}g\left(ze^{-\frac{2\pi i}{k}}\right) + \binom{k}{2}g(z) - \binom{k}{3}g\left(ze^{\frac{2\pi i}{k}}\right) + \binom{k}{4}g\left(ze^{\frac{4\pi i}{k}}\right) \\ -g\left(ze^{-\frac{2\pi i}{k}}\right) + \binom{k}{1}g(z) - \binom{k}{2}g\left(ze^{\frac{2\pi i}{k}}\right) \\ g(z) \\ -g\left(ze^{\frac{2\pi i}{k}}\right) \\ g\left(ze^{\frac{4\pi i}{k}}\right) \\ \vdots \\ (-1)^{\frac{k-1}{2}}g\left(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}\right) \end{pmatrix}$$

and the entry corresponding to $g(z)$ *is the* $n(k) := \frac{k+1}{2}$*-th one. We can write in a more compact form*

$$\Phi_R^\alpha(z) = \begin{cases} (-1)^{\frac{k+1}{2}-\alpha}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}g\left(ze^{\frac{2\pi i}{k}(\alpha-\frac{k+1}{2}+h)}\right) & \text{if } 1 \le \alpha \le \frac{k+1}{2}, \\ (-1)^{\alpha-\frac{k+1}{2}}g\left(ze^{\frac{2\pi i}{k}(\alpha-\frac{k+1}{2})}\right) & \text{if } \frac{k+1}{2}+1 \le \alpha \le k. \end{cases}$$

Now we compute the entries of the central connection matrix. We will denote by $\Phi_{\text{top}}(z)$ the solution of Proposition 15.1 corresponding to the choice $f(0) = 1$. The computations will be done in cases, depending on the parity of k.

CASE k EVEN: If $1 \leq \alpha \leq \frac{k}{2}$, we have that

$$\begin{aligned}
\Phi_R^\alpha(z) &= -\frac{2\pi i(-1)^{\frac{k}{2}+1-\alpha}}{(2\pi)^{\frac{k+1}{2}}}\sum_{h=0}^{k+1-2\alpha}\sum_{n=0}^{\infty}(-1)^h\binom{k}{h}\operatorname*{res}_{s=n}\left(\Gamma(-s)^k z^{ks}e^{(\alpha+h-\frac{k}{2}-1)2\pi i s}\right)ds\\
&= \frac{i(-1)^{\frac{k}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{h=0}^{k+1-2\alpha}\sum_{n=0}^{\infty}(-1)^h\binom{k}{h}\operatorname*{res}_{w=0}\left(\Gamma(-w-n)^k z^{k(w+n)}e^{(\alpha+h-\frac{k}{2}-1)2\pi i(w+n)}\right)dw\\
&= \frac{i(-1)^{\frac{k}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{h=0}^{k+1-2\alpha}\sum_{n=0}^{\infty}(-1)^h\binom{k}{h}\int_{\mathbb{P}}\left(\frac{\Gamma(-\sigma-n)^k z^{k(\sigma+n)}}{\Gamma(-\sigma)^k}\Gamma(1-\sigma)^k e^{(\alpha+h-\frac{k}{2}-1)2\pi i\sigma}\right)\\
&= \frac{i(-1)^{\frac{k}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\int_{\mathbb{P}}\left\{\left(\Phi_{\mathrm{top}}(z)\cup e^{-k\pi i\sigma}\right)\cup\widehat{\Gamma}^-(\mathbb{P})\cup\mathrm{Ch}(\mathcal{O}(\alpha+h-1))\right\}.
\end{aligned}$$

If $\frac{k}{2}+1 \leq \alpha \leq k$

$$\begin{aligned}
\Phi_R^\alpha(z) &= -\frac{2\pi i(-1)^{\alpha-\frac{k}{2}-1}}{(2\pi)^{\frac{k+1}{2}}}\sum_{n=0}^{\infty}\operatorname*{res}_{s=n}\left(\Gamma(-s)^k z^{ks}e^{(\alpha-\frac{k}{2}-1)2\pi i s}\right)ds\\
&= \frac{i(-1)^{\alpha-\frac{k}{2}}}{(2\pi)^{\frac{k-1}{2}}}\sum_{n=0}^{\infty}\operatorname*{res}_{w=0}\left(\Gamma(-w-n)^k z^{k(w+n)}e^{(\alpha-\frac{k}{2}-1)2\pi i(w+n)}\right)dw\\
&= \frac{i(-1)^{\alpha-\frac{k}{2}}}{(2\pi)^{\frac{k-1}{2}}}\sum_{n=0}^{\infty}\int_{\mathbb{P}}\left(\frac{\Gamma(-\sigma-n)^k z^{k(\sigma+n)}}{\Gamma(-\sigma)^k}\Gamma(1-\sigma)^k e^{(\alpha-\frac{k}{2}-1)2\pi i\sigma}\right)\\
&= \frac{i(-1)^{\alpha-\frac{k}{2}}}{(2\pi)^{\frac{k-1}{2}}}\int_{\mathbb{P}}\left\{\left(\Phi_{\mathrm{top}}(z)\cup e^{-k\pi i\sigma}\right)\cup\widehat{\Gamma}^-(\mathbb{P})\cup\mathrm{Ch}(\mathcal{O}(\alpha-1))\right\}.
\end{aligned}$$

CASE k ODD: If $1 \leq \alpha \leq \frac{k+1}{2}$ we have

$$\begin{aligned}
\Phi_R^\alpha(z) &= -\frac{2\pi i(-1)^{\frac{k+1}{2}-\alpha}}{(2\pi)^{\frac{k+1}{2}}i}\sum_{n=0}^{\infty}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\operatorname*{res}_{s=n}\left(\Gamma(-s)^k e^{-i\pi s}z^{ks}e^{2\pi i s\left(\alpha-\frac{k+1}{2}+h\right)}\right)ds\\
&= \frac{(-1)^{\frac{k-1}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{n=0}^{\infty}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\operatorname*{res}_{w=0}\left(\Gamma(-w-n)^k e^{-i\pi(w+n)}z^{k(w+n)}e^{2\pi i(w+n)\left(\alpha-\frac{k+1}{2}+h\right)}\right)dw\\
&= \frac{(-1)^{\frac{k-1}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{n=0}^{\infty}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\int_{\mathbb{P}}\left(-\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k}\Gamma(1-\sigma)^k e^{-i\pi(\sigma+n)}z^{k(\sigma+n)}e^{2\pi i(\sigma+n)\left(\alpha-\frac{k+1}{2}+h\right)}\right)\\
&= \frac{(-1)^{\frac{k+1}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{n=0}^{\infty}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\int_{\mathbb{P}}\left(\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k}\Gamma(1-\sigma)^k e^{-ki\pi(\sigma+n)}z^{k(\sigma+n)}e^{2\pi i\sigma(\alpha+h-1)}\right)\\
&= \frac{(-1)^{\frac{k+1}{2}-\alpha}}{(2\pi)^{\frac{k-1}{2}}}\sum_{h=0}^{k+1-2\alpha}(-1)^h\binom{k}{h}\int_{\mathbb{P}}\left\{\left(\Phi_{\mathrm{top}}(z)\cup e^{-ki\pi\sigma}\right)\cup\widehat{\Gamma}^-(\mathbb{P})\cup\mathrm{Ch}(\mathcal{O}(\alpha+h-1))\right\}.
\end{aligned}$$

If $\frac{k+1}{2}+1 \leq \alpha \leq k$ we have

$$
\begin{aligned}
\Phi_R^\alpha(z) &= -\frac{2\pi i(-1)^{\alpha-\frac{k+1}{2}}}{(2\pi)^{\frac{k+1}{2}} i} \sum_{n=0}^{\infty} \operatorname*{res}_{s=n} \left(\Gamma(-s)^k e^{-i\pi s} z^{ks} e^{2\pi i s\left(\alpha-\frac{k+1}{2}\right)}\right) ds \\
&= \frac{(-1)^{\alpha-\frac{k-1}{2}}}{(2\pi)^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} \operatorname*{res}_{w=0} \left(\Gamma(-w-n)^k e^{-i\pi(w+n)} z^{k(w+n)} e^{2\pi i(w+n)\left(\alpha-\frac{k+1}{2}\right)}\right) dw \\
&= \frac{(-1)^{\alpha-\frac{k-1}{2}}}{(2\pi)^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} \int_{\mathbb{P}} \left(-\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k} e^{-k(\sigma+n)\pi i} z^{k(\sigma+n)} \Gamma(1-\sigma)^k e^{2\pi i(\sigma+n)(\alpha-1)}\right) \\
&= \frac{(-1)^{\alpha-\frac{k+1}{2}}}{(2\pi)^{\frac{k-1}{2}}} \sum_{n=0}^{\infty} \int_{\mathbb{P}} \left(\frac{\Gamma(-\sigma-n)^k}{\Gamma(-\sigma)^k} e^{-kn\pi i} z^{k(\sigma+n)} e^{-ki\pi\sigma} \Gamma(1-\sigma)^k e^{2\pi i\sigma(\alpha-1)}\right) \\
&= \frac{(-1)^{\alpha-\frac{k+1}{2}}}{(2\pi)^{\frac{k-1}{2}}} \int_{\mathbb{P}} \left\{\left(\Phi_{\mathrm{top}}(z) \cup e^{-ki\pi\sigma}\right) \cup \widehat{\Gamma}^-(\mathbb{P}) \cup \mathrm{Ch}(\mathcal{O}(\alpha-1))\right\}.
\end{aligned}
$$

The form $\Phi_{\mathrm{top}}(z) \cup e^{-k\pi i\sigma}$ corresponds to the choice of another fundamental basis at the origin $\widetilde{\Xi}_0$, related to (15.8) by a right multiplication of a matrix:

$$
\widetilde{\Xi}_0(z) = \Xi_0(z) \begin{pmatrix}
1 & & & & & \\
-k\pi i & 1 & & & & \\
-\frac{k^2\pi^2}{2} & -k\pi i & 1 & & & \\
\vdots & & & \ddots & & \\
\frac{(-k\pi i)^m}{m!} & \dots & \dots & \dots & 1 & \\
\vdots & & & & & \ddots \\
\frac{(-k\pi i)^{k-1}}{(k-1)!} & \dots & \dots & \dots & \dots & \dots \quad 1
\end{pmatrix}.
$$

We claim that such a matrix is an element of the group $\widetilde{\mathcal{C}}_0(\mathbb{P})$ (see the previous section). Indeed if

$$
\alpha_m := \frac{(-k\pi i)^m}{m!}
$$

then, for $2 \leq 2n \leq k-1$, we have that

$$
\begin{aligned}
& 2\alpha_{2n} + \sum_{\substack{i+j=2n \\ 1\leq i,j}} (-1)^i \alpha_i \alpha_j \\
&= \frac{(-k\pi i)^{2n}}{(2n)!} \left(2 + \sum_{j=1}^{2n-1} (-1)^j \binom{2n}{j}\right) \\
&= 0.
\end{aligned}
$$

15.5. Reduction to Beilinson Form

Let us recall that the canonical coordinates can always be reordered so that the corresponding Stokes matrix is upper triangular (*lexicographical order* w.r.t the line ℓ). For the case of quantum cohomology of projective spaces, and for the choice of an admissible line ℓ with slope $0 < \epsilon < \frac{\pi}{k}$, such an order is the one described in picture.

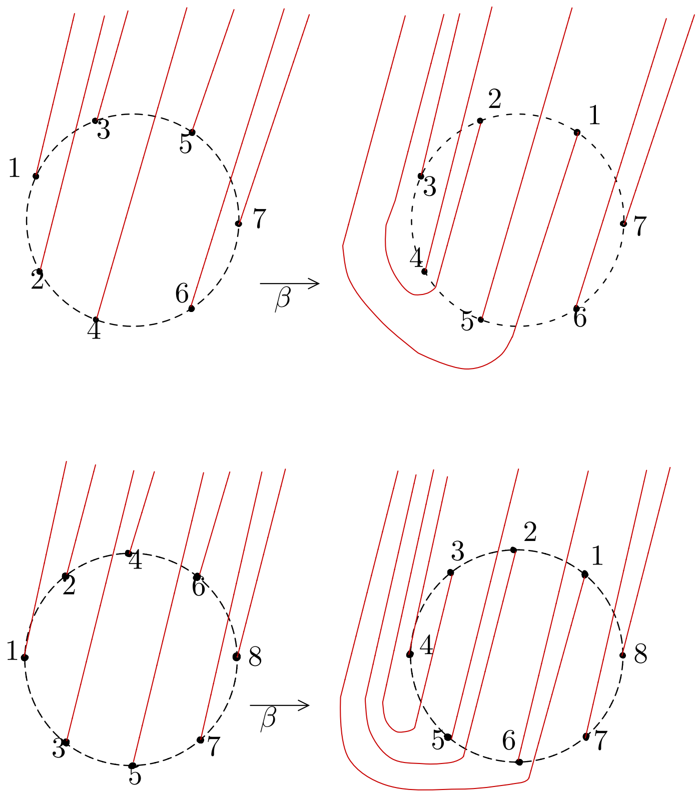

FIGURE 15.2. Action of the braid β found by D. Guzzetti: in the figure above we draw the case $k = 7$, below the case $k = 8$.

The matrices P associated to this permutations are

- for k even

$$P = \begin{pmatrix} & & & & & & 1 & 0 & & & & & \\ & & & & & 1 & 0 & 0 & & & & & \\ & & & & 0 & 0 & 0 & 1 & 0 & & & & \\ & & & & 1 & 0 & 0 & 0 & 0 & & & & \\ & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & 1 & & & & & & & & & & 0 & \\ 0 & 0 & & & & & & & & & & 0 & 1 \\ 1 & 0 & & & & & & & & & & 0 & 0 \end{pmatrix}$$

where the 1 on the first row in on the $\frac{k}{2} + 1$-th column;

- for k odd

$$P = \begin{pmatrix}
 & & & & & 0 & 1 & & & & & & \\
 & & & & & 0 & 0 & 1 & & & & & \\
 & & & & 0 & 1 & 0 & 0 & 0 & & & & \\
 & & & & 0 & 0 & 0 & 0 & 1 & & & & \\
 & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 1 & & & & & & & & & & 0 & \\
0 & 0 & & & & & & & & & & 0 & 1 \\
1 & 0 & & & & & & & & & & 0 & 0
\end{pmatrix}$$

where the 1 on the first row in os the $\frac{k+1}{2}$-th column.

After such a renumeration of $u_1, \dots, u_n$, as a consequence of the computations of the preceding section, the central connection matrix is, for k even

$$C_{\mathrm{lex}} = \frac{i}{(2\pi)^{\frac{k-1}{2}}} \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ \pm\widehat{\Gamma}^0 & \mp\widehat{\Gamma}^1 & \pm\widehat{\Gamma}^2 & \dots & \mp\widehat{\Gamma}^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \cdot A_k,$$

where:

- $\widehat{\Gamma}^j$ is a column vector whose components are the components of the characteristic classes

$$\widehat{\Gamma}^-(\mathbb{P}) \cup \mathrm{Ch}(\mathcal{O}(j));$$

- the sign $(+)$ is chosen if $\frac{k}{2} - 1$ is even, $(-)$ if $\frac{k}{2} - 1$ is odd;

- the matrix A_k is the $k \times k$ matrix

$$
A_k := \left(\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} \\
 & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} \\
 & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} \\
 & & & & & \dots & * & \vdots & * & \vdots & * \\
0 & 0 & 0 & 1 & 0 & & * & \vdots & * & \vdots & * \\
0 & 1 & 0 & \binom{k}{1} & 0 & \dots & * & \vdots & * & \vdots & * \\
1 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \dots & * & \vdots & * & \vdots & * \\
0 & 0 & 1 & \binom{k}{3} & 0 & & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \dots & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & & * & \vdots & * & \vdots & * \\
 & & & & & & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & \binom{k}{k-3} \\
 & & & & & & 0 & 1 & \binom{k}{k-3} & 0 & \binom{k}{k-2} \\
 & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-1}
\end{array}\right),
$$

where the 1 of the first column is on the $(\frac{k}{2}+1)$-th row.

Analogously, if k is odd the central connection matrix in lexicographical order is

$$
C_{\text{lex}} = \frac{1}{(2\pi)^{\frac{k-1}{2}}} \begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ \pm\widehat{\Gamma}^0 & \mp\widehat{\Gamma}^1 & \pm\widehat{\Gamma}^2 & \dots & \pm\widehat{\Gamma}^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \cdot A_k,
$$

where:

- $\widehat{\Gamma}^j$ is as before;
- the sign $(+)$ is chosen if $\frac{k-1}{2}$ is even, $(-)$ if $\frac{k-1}{2}$ is odd;

- the matrix A_k is the $k \times k$ matrix

$$A_k := \left(\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 1 \\
 & & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} \\
 & & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} \\
 & & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} \\
 & & & & & & \dots & * & \vdots & * & \vdots & * \\
0 & 0 & 0 & 0 & 1 & 0 & & * & \vdots & * & \vdots & * \\
0 & 0 & 1 & 0 & \binom{k}{1} & 0 & \dots & * & \vdots & * & \vdots & * \\
1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \dots & * & \vdots & * & \vdots & * \\
0 & 1 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & & * & \vdots & * & \vdots & * \\
0 & 0 & 0 & 1 & \binom{k}{4} & 0 & & * & \vdots & * & \vdots & * \\
0 & 0 & 0 & 0 & 0 & 1 & & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \dots & * & \vdots & * & \vdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & * & \vdots & * & \vdots & * \\
 & & & & & & & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & \binom{k}{k-3} \\
 & & & & & & & 0 & 1 & \binom{k}{k-3} & 0 & \binom{k}{k-2} \\
 & & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-1}
\end{array}\right),$$

  where the 1 of the first column is in the $\frac{k+1}{2}$-th row.

Proposition 15.3 ([**Guz99**]). *The action of the braid*

$$\beta := (\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots$$

$$\dots\beta_{\frac{k}{2}-2,\frac{k}{2}-1}(\beta_{k-3,k-2}\beta_{k-4,k-3}\dots\beta_{12})$$

for k even, and

$$\beta := (\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots$$

$$\dots(\beta_{\frac{k-3}{2},\frac{k-1}{2}}\beta_{\frac{k-5}{2},\frac{k-3}{2}})(\beta_{k-3,k-2}\beta_{k-4,k-3}\dots\beta_{12})$$

for k odd, is represented by the multiplication of the matrix

$$
A^{\beta}(S) := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} & 0 & 0 \\
 & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & 0 \\
 & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & 0 \\
 & & & & & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 1 & 0 & \binom{k}{1} & 0 & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
1 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 1 & \binom{k}{3} & 0 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
 & & & & & & \binom{k}{k-7} & 0 & \binom{k}{k-6} & 0 & \binom{k}{k-5} & 0 & 0 \\
 & & & & & & 0 & 1 & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-3} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

for k even, and

$$
A^\beta(S) := \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} & 0 & 0 \\
 & & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & 0 \\
 & & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & 0 \\
 & & & & & & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 1 & 0 & \binom{k}{1} & 0 & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 1 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 0 & 1 & \binom{k}{4} & 0 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \cdots & * & \vdots & * & \vdots & * & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & * & \vdots & * & \vdots & * & \vdots & \vdots \\
 & & & & & & & \binom{k}{k-7} & 0 & \binom{k}{k-6} & 0 & \binom{k}{k-5} & 0 & 0 \\
 & & & & & & & 0 & 1 & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-3} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

for k odd. Under the action of this braid, the Stokes matrix becomes

$$
S^\beta = \begin{pmatrix}
1 & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} & \binom{k}{4} & \cdots & -\binom{k}{k-1} \\
 & 1 & \binom{k}{1} & \binom{k}{2} & \binom{k}{3} & \cdots & -\binom{k}{k-2} \\
 & & 1 & \binom{k}{1} & \binom{k}{2} & \cdots & -\binom{k}{k-3} \\
 & & & 1 & \binom{k}{1} & \cdots & -\binom{k}{k-4} \\
 & & & & \ddots & & \vdots \\
 & & & & & & 1
\end{pmatrix}.
$$

Observe that, in both cases k even/odd, we obtain that

$$A_k(A^\beta(S))^{-1} = \begin{pmatrix} 0 & \dots & & & & & 1 \\ 1 & 0 & \dots & & & & \binom{k}{k-1} \\ & 1 & 0 & \dots & & & \binom{k}{k-2} \\ & & 1 & 0 & \dots & & \binom{k}{k-3} \\ & & & \ddots & & & \vdots \\ & & & & \ddots & & \vdots \\ & & & & & 1 & \binom{k}{1} \end{pmatrix}. \tag{15.13}$$

Indeed, observe that

$$A_k = \left(\begin{array}{ccc|cc} 0 & \dots & 0 & 0 & 1 \\ \hline & & & 0 & * \\ & X & & \vdots & \vdots \\ & & & 0 & * \\ \hline 0 & \dots & 0 & 1 & * \end{array}\right) \quad \text{and} \quad A^\beta(S) = \left(\begin{array}{c|cc} X & & \\ \hline & 1 & \\ & & 1 \end{array}\right).$$

The matrix (15.13) is the matrix corresponding to the braid

$$\beta' := \beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12},$$

that is

$$A^{\beta'}(S^\beta).$$

This is easily seen from the fact that

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & x_1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & x_2 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 0 & 1 & & \\ & & & 1 & 0 & x_1 & & \\ & & & 0 & 1 & x_2 & & \\ & & & & & & 1 & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

and that $A^{\beta_1\beta_2}(S)=A^{\beta_2}(S^{\beta_1})A^{\beta_1}(S)$. As a consequence, we have that

$$A^{\beta'}(S^{\beta})=\begin{pmatrix}0&\dots&&&&&1\\1&0&\dots&&&&*\\&1&0&\dots&&&*\\&&1&0&\dots&&*\\&&&\ddots&&&\vdots\\&&&&\ddots&&*\\&&&&&1&*\end{pmatrix}$$

and the entries $*$ are exactly those of the k-th column of $A^{\beta_{i,i+1}}(S^{\beta})$, from the top to the bottom, namely

$$-S^{\beta}_{1,k}=\binom{k}{k-1}$$

$$-S^{\beta}_{2,k}=\binom{k}{k-2}$$

$$\dots$$

$$-S^{\beta}_{k-1,k}=\binom{k}{1}.$$

We have thus obtained the following

THEOREM 15.2. *Consider the central connection matrix for the quantum cohomology of $\mathbb{P}^{k-1}_{\mathbb{C}}$, computated w.r.t the solution (15.8) and the solutions of Proposition (15.2) and set it in the lexicographical form $C_{\rm lex}$. Modulo the action of the group $\widetilde{\mathcal{C}_0}\left(\mathbb{P}^{k-1}_{\mathbb{C}}\right)$, and the action of the braid*

$$\begin{aligned}\beta\beta':=&(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots\\&\dots\beta_{\frac{k}{2}-2,\frac{k}{2}-1}(\beta_{k-3,k-2}\beta_{k-4,k-3}\dots\beta_{12})(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})\end{aligned}$$

for k even, and

$$\begin{aligned}\beta\beta':=&(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots\\&\dots(\beta_{\frac{k-3}{2},\frac{k-1}{2}}\beta_{\frac{k-5}{2},\frac{k-3}{2}})(\beta_{k-3,k-2}\beta_{k-4,k-3}\dots\beta_{12})(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})\end{aligned}$$

for k odd, the central connection matrix is

$$C_{lex}=\frac{i}{(2\pi)^{\frac{k-1}{2}}}\begin{pmatrix}\vdots&\vdots&\vdots&&\vdots\\\pm\widehat{\Gamma}^0&\mp\widehat{\Gamma}^1&\pm\widehat{\Gamma}^2&\dots&\mp\widehat{\Gamma}^{k-1}\\\vdots&\vdots&\vdots&&\vdots\end{pmatrix},$$

for k even, and

$$C_{lex}=\frac{1}{(2\pi)^{\frac{k-1}{2}}}\begin{pmatrix}\vdots&\vdots&\vdots&&\vdots\\\pm\widehat{\Gamma}^0&\mp\widehat{\Gamma}^1&\pm\widehat{\Gamma}^2&\dots&\mp\widehat{\Gamma}^{k-1}\\\vdots&\vdots&\vdots&&\vdots\end{pmatrix},$$

for k odd. Here

- $\widehat{\Gamma}^j$ *is a column vector whose components are the components of the characteristic classes*
$$\widehat{\Gamma}^-(\mathbb{P}) \cup \mathrm{Ch}(\mathcal{O}(j));$$
- *if k is even, the sign $(+)$ is chosen if $\frac{k}{2}-1$ is even, $(-)$ if $\frac{k}{2}-1$ is odd;*
- *if k is odd, the sign $(+)$ is chosen if $\frac{k-1}{2}$ is even, $(-)$ if $\frac{k-1}{2}$ is odd.*

The corresponding Stokes matrix (using the relation (2.36)) is in the canonical form
$$s_{ij} = \binom{k}{i-j}, \quad i<j.$$
After the conjugation by
$$(-1)^{\frac{k}{2}-1}\operatorname{diag}(1,-1,1,-1,\dots,1,-1)$$
if k is even, or by
$$(-1)^{\frac{k-1}{2}}\operatorname{diag}(1,-1,1,-1,\dots,1,-1,1)$$
if k is odd, the central connection matrix is in the canonical form
$$C_{lex} = \frac{i}{(2\pi)^{\frac{k-1}{2}}}\begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ \widehat{\Gamma}^0 & \widehat{\Gamma}^1 & \widehat{\Gamma}^2 & \dots & \widehat{\Gamma}^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$
for k even, and
$$C_{lex} = \frac{1}{(2\pi)^{\frac{k-1}{2}}}\begin{pmatrix} \vdots & \vdots & \vdots & & \vdots \\ \widehat{\Gamma}^0 & \widehat{\Gamma}^1 & \widehat{\Gamma}^2 & \dots & \widehat{\Gamma}^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix},$$
for k odd. The corresponding Stokes matrix is in the form
$$s_{ij} = (-1)^{j-i}\binom{k}{i-j}, \quad i<j.$$

REMARK 15.1. Notice that, the braid β found by D. Guzzetti takes the canonical coordinates in cyclic counterclockwise order (see Figure 15.2). If we further act with the braid β' above, then the canonical coordinates dispose in cyclic counterclockwise order starting from 1.

15.6. Mutations of the Exceptional Collections

LEMMA 15.2. *The computed braid can be rewritten as the product*
$$\beta\beta' = \beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})$$
for k even,
$$\beta\beta' = (\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})$$
for k odd.

PROOF. Consider the case k even. The only thing that we have to prove is that the braid

$$(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots\beta_{\frac{k}{2}-2,\frac{k}{2}-1} \tag{15.14}$$

is equal to

$$\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-5,k-4}\dots\beta_{12}).$$

Note that the braid above ends with the product

$$\dots\beta_{\frac{k}{2}-2,\frac{k}{2}-1}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\beta_{\frac{k}{2}-2,\frac{k}{2}-1}.$$

By Yang-Baxter equations this product is equal to

$$\dots\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\beta_{\frac{k}{2}-2,\frac{k}{2}-1}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}.$$

Because of commutation relations, we can shift the first term on the left till we find

$$\dots\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\beta_{\frac{k}{2}-4,\frac{k}{2}-3}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\dots,$$

which is equal to

$$\dots\beta_{\frac{k}{2}-4,\frac{k}{2}-3}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\beta_{\frac{k}{2}-4,\frac{k}{2}-3}\dots.$$

Again, starting from the first term, we can shift it on the left (until commutation law allows), then use Yang-Baxter relations. Continuing this procedure, at the end we have eliminated the last term of (15.14), and we obtain a new first term:

$$\beta_{12}(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots$$
$$\dots(\beta_{\frac{k}{2}-1,\frac{k}{2}}\beta_{\frac{k}{2}-2,\frac{k}{2}-1}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}).$$

Now we continue the procedure of elimination of the last braid: we start from its first term, i.e. $\beta_{\frac{k}{2}-1,\frac{k}{2}}$, we shift it on the left, use Yang- Baxter relations, and so on, till we find

$$\beta_{12}\beta_{34}(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots$$
$$\dots(\beta_{\frac{k}{2}-2,\frac{k}{2}-1}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}).$$

Applying again the same procedure, before for $\beta_{\frac{k}{2}-2,\frac{k}{2}-1}$, and after for $\beta_{\frac{k}{2}-3,\frac{k}{2}-2}$, we have eliminated the last braid and we obtain

$$\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{k-5,k-4}\beta_{k-6,k-5}\dots\beta_{12})(\beta_{k-6,k-5}\beta_{k-7,k-6}\dots\beta_{23})(\beta_{k-7,k-6}\dots\beta_{34})\dots$$
$$\dots(\beta_{\frac{k}{2},\frac{k}{2}+1}\beta_{\frac{k}{2}-1,\frac{k}{2}}\beta_{\frac{k}{2}-2,\frac{k}{2}-1}\beta_{\frac{k}{2}-3,\frac{k}{2}-2}\beta_{\frac{k}{2}-4,\frac{k}{2}-3}).$$

Iterating the same procedure, one obtains the braid

$$\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12}).$$

The case k odd is analogous, and we left the details to the reader. □

EXAMPLE 15.1. Consider for example $k = 12$. we have that

$$\beta\beta' = (\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34})\beta_{45}\cdot$$
$$\cdot(\beta_{9,10},\dots\beta_{12})(\beta_{11,12}\dots\beta_{12}).$$

We have to rearrange the first 4 braids. Let us apply the procedure described abobe:

$$
\begin{aligned}
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34})\beta_{45} =\\
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{34}\beta_{45})\beta_{34} =\\
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})\beta_{34}(\beta_{56}\beta_{45}\beta_{34}) =\\
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{23}\beta_{34})\beta_{23}(\beta_{56}\beta_{45}\beta_{34}) =\\
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\beta_{23}(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34}) =\\
&(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{12}\beta_{23})\beta_{12}(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34}) =\\
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34}).
\end{aligned}
$$

Now we continue by eliminating the last braid, starting from its first term:

$$
\begin{aligned}
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{56}\beta_{45}\beta_{34}) =\\
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{56}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{45}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{45}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}(\beta_{78}\beta_{67}\beta_{56}\beta_{34}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})(\beta_{45}\beta_{34}) =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{45}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{34}\beta_{45}\beta_{34}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{34}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{23}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23})\beta_{34} =\\
&\beta_{12}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{23}\beta_{34})\beta_{23} =\\
&\beta_{12}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\beta_{23}(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{12}\beta_{23})\beta_{12}(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}).
\end{aligned}
$$

As a final step, we have to eliminate the final braid, always starting from its first term:

$$
\begin{aligned}
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{78}\beta_{67}\beta_{56}\beta_{67}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{78}\beta_{56}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{56}\beta_{34}\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}(\beta_{78}\beta_{67}\beta_{45}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}) =
\end{aligned}
$$

$$
\begin{aligned}
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{45}\beta_{23}\beta_{12})(\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{34}\beta_{23}) =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{34}\beta_{12})\beta_{23} =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{23}\beta_{34}\beta_{23}\beta_{12})\beta_{23} =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\beta_{23} =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\beta_{23} =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})\beta_{56}\beta_{45}\beta_{34}\beta_{23}(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{12}\beta_{23})\beta_{12} =\\
&\beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})(\beta_{78}\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12}).
\end{aligned}
$$

In what follows we will denote by $\mathcal{T}$ the tangent sheaf of $\mathbb{P}$, by Ω the cotangent sheaf, and we will use the shorthands

$$\bigwedge^p \mathcal{T}(k) := \left(\bigwedge^p \mathcal{T}\right) \otimes \mathcal{O}(k), \quad \bigwedge^p \Omega(k) := \left(\bigwedge^p \Omega\right) \otimes \mathcal{O}(k).$$

The following formulae, due to R. Bott ([**Bot57**], [**OSS11**], [**DG88**]), will be useful:

$$\dim_{\mathbb{C}} H^q\left(\mathbb{P}^n_{\mathbb{C}}, \bigwedge^p \mathcal{T}(k)\right) = \begin{cases} \binom{k+n+p+1}{p}\binom{k+n}{n-p}, & q=0, \quad k>-p-1,\\ 1, & q=n-p, \quad k=-n-1,\\ \binom{-k-p-1}{-k-n-1}\binom{-k-n-2}{p}, & q=n, \quad k<-n-p-1,\\ 0, & \text{otherwise,}\end{cases} \tag{15.15}$$

$$\dim_{\mathbb{C}} H^q\left(\mathbb{P}^n_{\mathbb{C}}, \bigwedge^p \Omega(k)\right) = \begin{cases} \binom{k+n-p}{k}\binom{k-1}{p}, & q=0, \quad 0\le p\le n, \quad k>p,\\ 1, & k=0, \quad 0\le q=p\le n,\\ \binom{-k+p}{-k}\binom{-k-1}{n-p}, & q=n, \quad 0\le p\le n, \quad k<p-n,\\ 0, & \text{otherwise.}\end{cases} \tag{15.16}$$

Consider Beilinson's exceptional collection $\mathfrak{B} := (\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(k-1))$ in $\mathcal{D}^b(\mathbb{P})$, with $\mathbb{P} = \mathbb{P}(V)$ ($\dim_{\mathbb{C}} V = k$), and the well known Euler exact sequence, together with its exterior powers

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & V\otimes\mathcal{O}(1) & \longrightarrow & \mathcal{T} & \longrightarrow & 0,\\
0 & \longrightarrow & \mathcal{T} & \longrightarrow & \bigwedge^2 V\otimes\mathcal{O}(2) & \longrightarrow & \bigwedge^2\mathcal{T} & \longrightarrow & 0,\\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \bigwedge^{h-1}\mathcal{T} & \longrightarrow & \bigwedge^h V\otimes\mathcal{O}(h) & \longrightarrow & \bigwedge^h\mathcal{T} & \longrightarrow & 0,\\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \bigwedge^{k-2}\mathcal{T} & \longrightarrow & \bigwedge^{k-1} V\otimes\mathcal{O}(k-1) & \longrightarrow & \mathcal{O}(k) & \longrightarrow & 0.
\end{array} \tag{15.17}
$$

By Bott formulae (15.15)-(15.16), we deduce that both $\mathrm{Hom}^\bullet(\mathcal{O}(h), \bigwedge^h \mathcal{T})$ and $\mathrm{Hom}^\bullet\left(\bigwedge^{h-1}\mathcal{T}, \mathcal{O}(h)\right)$ are concentrated in degree 0 and they have the same dimension $\binom{k}{h}$. Hence, the short exact sequences (15.17), together with the identifications

$$\bigwedge^h V = \mathrm{Hom}^\bullet\left(\mathcal{O}(h), \bigwedge^h \mathcal{T}\right) = (\mathrm{Hom}^\bullet)^\vee\left(\bigwedge^{h-1}\mathcal{T}, \mathcal{O}(h)\right),$$

allow us to explicitly compute successive right mutations of the sheaf $\mathcal{O}$: namely, for $0 < h \leq k-1$, we have

$$\mathbb{R}_{\langle \mathcal{O}(1)\dots\mathcal{O}(h)\rangle}\mathcal{O} = \left(\bigwedge^h \mathcal{T}\right)[-h].$$

Being the sheaf $\mathcal{O}(j)$ locally free, the functor $\mathcal{O}(j)\otimes(-)$ preserves the short exact sequences (15.17); moreover, observing that

$$\mathrm{Hom}(\mathcal{O}(l), \mathcal{O}(m)) \cong \mathrm{Hom}(\mathcal{O}(l+n), \mathcal{O}(m+n))$$

for all $l, m, n \in \mathbb{Z}$, we deduce that for $j < h \leq k-1$

$$\mathbb{R}_{\langle \mathcal{O}(j+1),\dots,\mathcal{O}(h)\rangle}\mathcal{O}(j) = \left(\bigwedge^{h-j}\mathcal{T}(j)\right)[j-h].$$

Corollary 15.2. *The central connection and the Stokes matrices of the quantum cohomology of $\mathbb{P}^{k-1}_{\mathbb{C}}$, computed at $0 \in QH^\bullet(\mathbb{P})$ and with respect to a line ℓ with slope $0 < \epsilon < \frac{\pi}{k}$, corresponds (modulo action of $(\mathbb{Z}/2\mathbb{Z})^k$) to the exceptional collections*

$$\left(\mathcal{O}\left(\frac{k}{2}\right), \bigwedge^1 \mathcal{T}\left(\frac{k}{2}-1\right), \mathcal{O}\left(\frac{k}{2}+1\right), \bigwedge^3 \mathcal{T}\left(\frac{k}{2}-2\right), \dots, \mathcal{O}(k-1), \bigwedge^{k-1}\mathcal{T}\right)$$

for k even, and

$$\left(\mathcal{O}\left(\frac{k-1}{2}\right), \mathcal{O}\left(\frac{k+1}{2}\right), \bigwedge^2 \mathcal{T}\left(\frac{k-3}{2}\right), \mathcal{O}\left(\frac{k+3}{2}\right), \bigwedge^4 \mathcal{T}\left(\frac{k-5}{2}\right), \dots, \mathcal{O}(k-1), \bigwedge^{k-1}\mathcal{T}\right)$$

for k odd.

Proof. Denoting by σ_{ij} the inverse braid β_{ij}^{-1}, from Theorem (15.2) and from Lemma (15.2), we have that the monodromy data computed at 0 with respect to the line ℓ correspond to the exceptional collection

$$\mathfrak{B}^\sigma, \quad \sigma := (\sigma_{12}\sigma_{23}\dots\sigma_{k-1,k})\dots(\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{45}\sigma_{56})(\sigma_{12}\sigma_{23}\sigma_{34})\sigma_{12}$$

for k even, and to

$$\mathfrak{B}^\sigma, \quad \sigma := (\sigma_{12}\sigma_{23}\dots\sigma_{k-1,k})\dots(\sigma_{12}\sigma_{23}\sigma_{34}\sigma_{45})(\sigma_{12}\sigma_{23})$$

for k odd. Using the previous observations, one obtains the collections above. $\square$

15.7. Reconstruction of the monodromy data along the small quantum cohomology, and some results on the big quantum cohomology

From the partial knowledge of Corollary 15.2 we are able now to determine the monodromy data at *any* point of the small quantum cohomology w.r.t. *any* line ℓ, together with the corresponding full exceptional collections. Notice that if we fix a line ℓ of slope ϕ, the small quantum cohomology is decomposed in open regions, namely the traces of the ℓ-chambers. Accordingly to equation (15.12), these

	$S_{\rm lex}$	Exceptional Collection	Braid
$0 < 3\phi + \Im(t^1) < \pi$	$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$(\mathcal{O}(1), \mathcal{O}(2), \bigwedge^2 \mathcal{T})$	id
$\pi < 3\phi + \Im(t^1) < 2\pi$	$\begin{pmatrix} 1 & -3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$\left(\mathcal{O}(1), \bigwedge^1 \mathcal{T}, \mathcal{O}(2)\right)$	$\omega_{1,3}$
$2\pi < 3\phi + \Im(t^1) < 3\pi$	$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$	$\omega_{1,3}\omega_{2,3}$
$3\pi < 3\phi + \Im(t^1) < 4\pi$	$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$(\mathcal{O}, \Omega(2), \mathcal{O}(1))$	$\omega_{1,3}\omega_{2,3}\omega_{1,3}$
$4\pi < 3\phi + \Im(t^1) < 5\pi$	$\begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$\left(\bigwedge^2 \Omega(2), \mathcal{O}, \mathcal{O}(1)\right)$	$(\omega_{1,3}\omega_{2,3})^2$
$5\pi < 3\phi + \Im(t^1) < 6\pi$	$\begin{pmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$\left(\bigwedge^2 \Omega(2), \Omega(1), \mathcal{O}\right)$	$(\omega_{1,3}\omega_{2,3})^2\omega_{1,3}$
$6\pi < 3\phi + \Im(t^1) < 7\pi$	$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$\left(\bigwedge^2 \Omega(1), \bigwedge^2 \Omega(2), \mathcal{O}\right)$	$(\omega_{1,3}\omega_{2,3})^3$

TABLE 15.1. In this table we represent all possible Stokes matrices along the small quantum cohomology of $\mathbb{P}^2_{\mathbb{C}}$, in ℓ-lexicographical order for a line ℓ of slope ϕ. We also write the corresponding (modulo shifts) exceptional collections associated with the monodrodmy data. Notice that the Beilinson exceptional collection $\mathfrak{B}$ appears along the small quantum locus: it is obtained from the one of Corollary 15.2 by applying the braids $\omega_{1,3}\omega_{2,3}$.

regions are unbounded horizontal strips in the complex plane $(0, t^2, 0, \dots, 0) \in \mathbb{C}$, whose boundaries are the lines

$$\tau \in \pi \cdot \mathbb{Z} - k\phi, \quad \text{where } \tau = \Im(t^2). \tag{15.18}$$

For the points of this lines, ℓ is not admissible, and by the Isomonodromy Theorem 2.12 the monodromy data are constant in each horizontal strip[1]. The data in different ℓ-chambers are related by a braid: let us explain this in more details.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two ℓ-chambers, pick two points $p_1 \in \mathcal{C}_1$ and $p_2 \in \mathcal{C}_2$ and consider a piece of straight line connecting them: the monodromy data, as functions on this closed interval, are discontinuous at points corresponding to the intersections with the lines (15.18). This is due to the fact that some Stokes rays cross the line ℓ: the precise order[2] of these crossings give us the braids acting on the data.

[1]For simplicity we will call also these strips ℓ-chambers, though they are the intersection of proper ℓ-chambers with the locus of small quantum cohomology.

[2]Recall that the Stokes rays must be labelled w.r.t. the lexicographical order in any ℓ-chambers.

Up to now we have fixed a line ℓ and considered the data in these "static" ℓ-chambers. If we now let vary the line ℓ, say by increasing its slope ϕ, then the ℓ-chambers *glide* over the small quantum cohomology, according to equation (15.18). Consider a point $p \in \mathcal{C}_1(\ell)$, and let vary the line ℓ by increasing its slope: the ℓ-chamber $\mathcal{C}_1(\ell)$ glides towards $\Im(t^2) \to -\infty$, so that, at the end of the transformation of ℓ, p belongs to another chamber, say $\mathcal{C}_2(\ell') = \mathcal{C}_2(\ell) - c$ for some positive constant c. The crosses with the (static) Stokes rays during the rotation of ℓ lead to the same braids obtained from the point of view described in the previous paragraph (see also Section 2.3). In conclusion, if we know the data at a point of the small quantum cohomology w.r.t. some line ℓ, then we can reconstruct the data at any point w.r.t. any line.

Starting from $0 \in QH^\bullet(\mathbb{P})$ with a line ℓ of slope $0 < \phi < \frac{\pi}{k}$, we let increase ϕ, so that the line ℓ rotates counter-clockwise. From the geometry of the disposition of the Stokes rays, it is easily seen that the *first* crossing of Stokes rays is described as follows[3]:

- if $k \geq 2$ is *even*, the line ℓ firstly cross $\frac{k}{2}$ Stokes rays (which coincide) and the corresponding braid is
$$\omega_{1,k} := \prod_{\substack{i=2\\ i \text{ even}}}^{k} \beta_{i-1,i};$$
- if $k \geq 3$ is *odd*, then the line ℓ firstly cross $\frac{k-1}{2}$ Stokes rays (which coincide) and the corresponding braid is
$$\omega_{1,k} := \prod_{\substack{i=3\\ i \text{ odd}}}^{k} \beta_{i-1,i}.$$

The *second* crossing of the Stokes rays is instead:

- if $k \geq 2$ is *even*, the line ℓ secondly cross $\frac{k}{2} - 1$ Stokes rays (which coincide) and the corresponding braid is
$$\omega_{2,k} := \prod_{\substack{i=3\\ i \text{ odd}}}^{k-1} \beta_{i-1,i};$$
- if $k \geq 3$ is *odd*, then the line ℓ firstly cross $\frac{k-1}{2}$ Stokes rays (which coincide) and the corresponding braid is
$$\omega_{2,k} := \prod_{\substack{i=2\\ i \text{ even}}}^{k-1} \beta_{i-1,i}.$$

Furthermore, using symmetries of regular polygons (see Figure 15.3), it is easy to see that the braids corresponding to subsequent crossings are *alternatively* $\omega_{1,k}$ and $\omega_{2,k}$: in this ways, if we let rotate counterclockwise the line ℓ, and we have N crossings in total, the resulting acting braid is the composition
$$\omega_{1,k}\omega_{2,k}\omega_{1,k}\omega_{2,k}\dots$$
with N braids ω's in total. Notice that after a complete rotation of ℓ, the resulting braid is
$$(\omega_{1,k}\omega_{2,k})^k,$$
which, accordingly to Lemma 2.5, is easily seen to be the be the central element $(\beta_{12}, \dots, \beta_{k-1,k})^k$, using the braid Yang-Baxter relations.

[3]Products $\prod_{i=a}^{b}(\dots)$ with $a > b$ must be set equal to the identity 1.

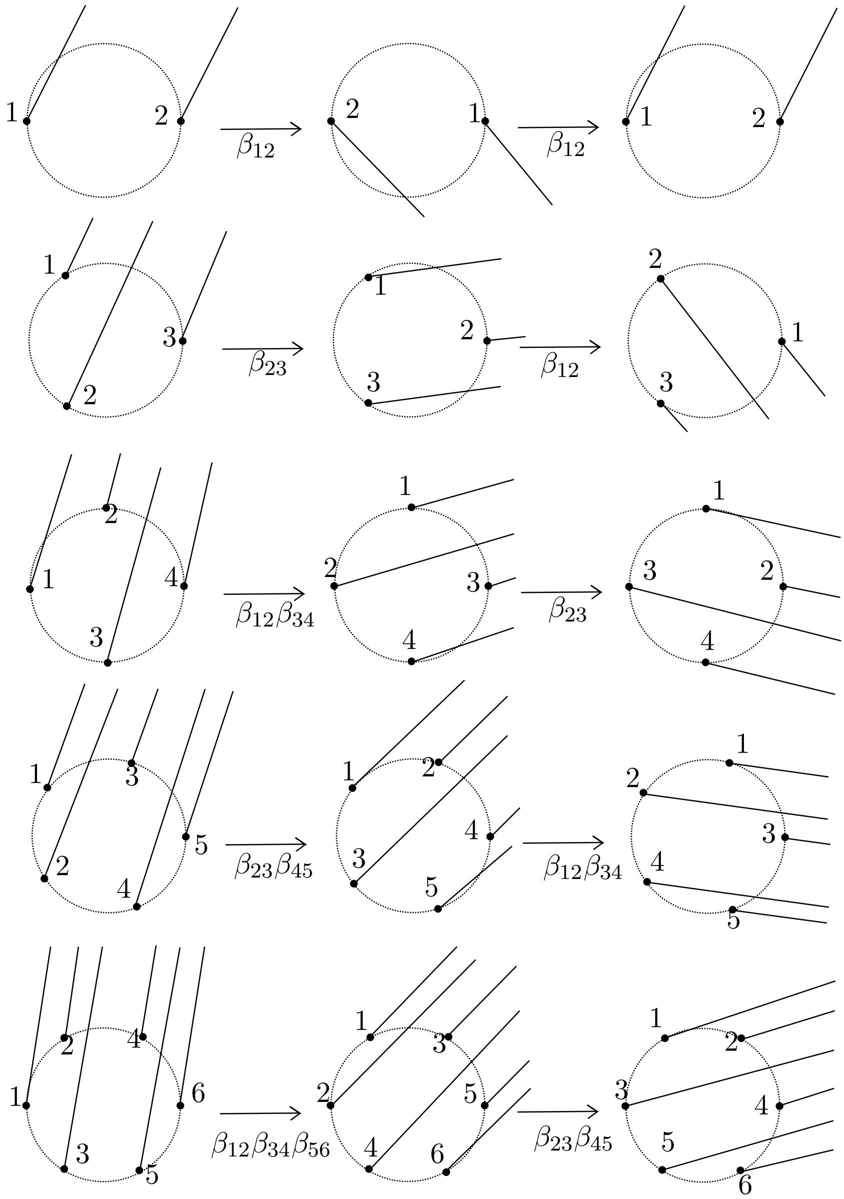

FIGURE 15.3. Here we represent the action of the braids $\omega_{1,k}, \omega_{2,k}$ for $2 \leq k \leq 6$. On the left column the reader can find the canonical coordinates in ℓ-lexicographical order for ℓ of slope $\phi \in]0; \frac{\pi}{k}[$. In the central column we represent the action of the braid $\omega_{1,k}$, whereas in the right column the consecutive action of the braid $\omega_{2,k}$.

THEOREM 15.3. *The braids of Lemma 15.2, i.e.*

$$\beta\beta' = \beta_{12}(\beta_{34}\beta_{23}\beta_{12})(\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})$$

for k even,

$$\beta\beta' = (\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}\beta_{12})\dots(\beta_{k-1,k}\beta_{k-2,k-1}\dots\beta_{12})$$

for k odd, which take the monodromy data computed at $0 \in QH^\bullet(\mathbb{P}_{\mathbb{C}}^{k-1})$ (w.r.t. a line of slope $0 < \phi < \frac{\pi}{k}$) to the data corresponding to the Beilinson's exceptional collection, are of the form

$$\omega_{1,k}\omega_{2,k}\omega_{1,k}\omega_{2,k}\dots$$

if and only if $k = 2$ or $k = 3$. Thus, they do not correspond to analytic continuation along paths in the small quantum cohomology for $k \geq 4$.

Proof. For $k = 2, 3$ the braids $\beta\beta'$ are

$$\omega_{1,2} = \beta_{12} \quad \text{and} \quad \omega_{1,3}\omega_{2,3} = \beta_{23}\beta_{12}$$

respectively (see also Table 15.1). So, let us suppose that $k \geq 4$ and that $\beta\beta'$ can be expressed as a product

$$\omega_{1,k}\omega_{2,k}\omega_{1,k}\omega_{2,k}\dots. \tag{15.19}$$

Let us start from the following observation: if a generic braid can be represented as a product of positive powers of elementary braids $\beta_{i,i+1}$, then any other of its factorizations in positive powers of elementary braids must consist of the same numbers of factors (this follows immediately from the relations defining the braid group $\mathcal{B}_n$). Thus, the product (15.19) should be a product of

$$\left(\frac{k}{2}\right)^2 \text{ factors for } k \text{ even}, \quad \frac{k^2-1}{4} \text{ factors for } k \text{ odd}.$$

We firstly consider the case k *even*: we are supposing the existence of a number $n \in \mathbb{N}^*$ such that the product (15.19) contains n times the braid $\omega_{1,k}$ and n or $n-1$ times the braid $\omega_{2,k}$. So, we must have

$$n\frac{k}{2} + m\left(\frac{k}{2} - 1\right) = \left(\frac{k}{2}\right)^2$$

for some $n \in \mathbb{N}^*$ and $m \in \{n-1, n\}$, so that

$$k = (n+m) \pm \frac{1}{2}(4(n+m)^2 - 16m)^{\frac{1}{2}}. \tag{15.20}$$

As a necessary condition we have that

$$4(n+m)^2 - 16m, \quad \text{with } m \in \{n-1, n\}$$

must be the square of some integer. Since

- for $m = n$ the number $16(n^2 - n)$ is a perfect square only for $n = 1$,
- for $m = n-1$ the number $16(n-1)^2 + 4$ is a perfect square only for $n = 1$,

accordingly to (15.20) the only possible value of k is $k = 2$. Analogously, for the case $k \geq 3$ and *odd*, if we suppose that it exists a number $n \in \mathbb{N}^*$ such that the product (15.19) contains n times the braid $\omega_{1,k}$ and n or $n-1$ times the braid $\omega_{2,k}$, we necessarily must have

$$n\frac{k-1}{2} + m\frac{k-1}{2} = \frac{k^2-1}{4}, \quad \text{with } m \in \{n-1, n\}$$

$$\Longrightarrow n + m = \frac{k+1}{2}, \quad \text{with } m \in \{n-1, n\}.$$

Thus, for any odd number $k \geq 3$, we have found a composition of n times $\omega_{1,k}$ and n or $n-1$ times $\omega_{2,k}$ whose length equals the length of $\beta\beta'$. In particular, we have that

- if $k = 4n-1$ then $\omega_{1,k}$ and $\omega_{2,k}$ appear the same number $n = m$ of times;
- if $k = 4n-3$ then $\omega_{1,k}$ appears n times and $\omega_{2,k}$ appears $m = n-1$ times.

Notice in particular that for $k = 3$ we are in the first case, accordingly with what said at the beginning of the proof. We want now to show that $k = 3$ is the only case in which the braid we have found is actually $\beta\beta'$.

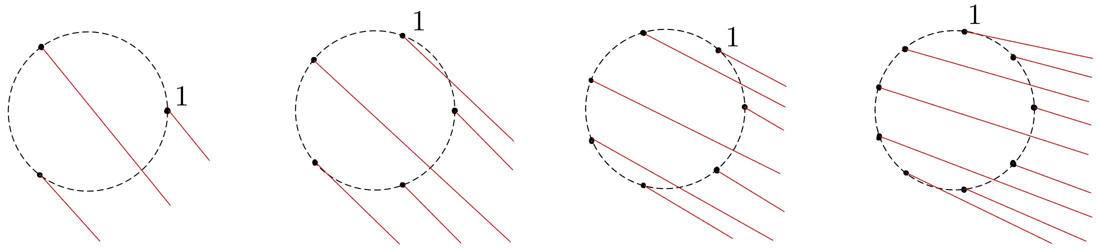

FIGURE 15.4. Configuration of the canonical coordinates, for k odd ($k = 3, 5, 7, 9$) after the action of the candidate braid $\omega_{1,k}\omega_{2,k}\dots$. Notice that the final arrangement of the canonical coordinates is $(\dots, \underset{n\text{-th}}{1}, \dots)$.

For this, notice that the braid $\beta\beta'$ takes the canonical coordinates in a ordered cyclic disposition $(u_1, u_2, \dots, u_k)$ starting from 1 and going counter-clockwise along the regular k-agon formed by the canonical coordinates (Remark 15.1): we will denote this arrangement by the k-tuple $(1, 2, 3, \dots, k)$. Instead, the product of ω's we have found takes the canonical coordinates in another disposition (Figure 15.4): for example, the canonical coordinate u_1 is not taken in first position but in the n-th in both cases $k = 4n - 1$ or $k = 4n - 3$: the corresponding k-tuple is of the form

$$(\dots, \underset{n\text{-th}}{1}, \dots).$$

Again we find that the only admissible case is $n = 1$, and so $k = 3$. This completes the proof. □

15.8. Symmetries and Quasi-Periodicity of Stokes matrices along the small quantum locus

In this section we describe a curious property of *quasi-periodicity* of the Stokes matrices S computed at a point of the small quantum cohomology of $\mathbb{P}$ w.r.t. all possible admissible lines ℓ. Because of the discussion at the beginning of the previous section, we can do the computation at any point, say, to fix ideas, at $0 \in QH^\bullet(\mathbb{P})$.

For this let us introduce a new labeling of Stokes rays which is useful for describing the *Stokes factors* in which the matrix S factorizes. Let us fix an admissible line ℓ in $\mathbb{C}$ and choose an admissible direction τ in the universal cover $\mathcal{R}$ which projects onto ℓ_+. We label the Stokes rays in $\mathcal{R}$ as follows: the rays are labelled in counter-clockwise order (i.e. increasing the value of the argument) starting from the first one in Π_{right} which will be R_0. In this way

$$R_0, \dots, R_{k-1} \subseteq \Pi_{\text{right}},$$

$$R_k, \dots, R_{2k-1} \subseteq \Pi_{\text{left}}.$$

The labeling is then extended to all integers, increasing the index in counter-clockwise direction, so to obtain a whole family $\{R_i\}_{i\in\mathbb{Z}}$. For the choice of ℓ with slope $0 < \phi < \frac{\pi}{k}$ we have that

$$\text{the ray } R_0 \text{ projects onto } R_{1k},$$

$$\text{the ray } R_1 \text{ projects onto } R_{1,k-1},$$

$$\dots$$

where we use not the lexicographical labeling but the original one (see equation (15.12)). If we denote by $\mathscr{S}_j$ the sector in $\mathcal{R}$ bounded by R_{j-1} and R_{j+k}, then $\mathscr{S}_j$ has angular width of $\pi+\frac{\pi}{k}$ and consequently there exists a unique genuine solution Ξ_j of the system (15.5) with the required asymptotic expansions on $\mathscr{S}_j$. We define the Stokes factors to be the connection matrices K_j such that

$$\Xi_{j+1} = \Xi_j K_j, \quad j \in \mathbb{Z}.$$

In this way we have that

$$S = K_0 K_1 \dots K_{k-2} K_{k-1}.$$

Moreover, notice that the first row of $\Xi_j(z)$ is equal to $z^{\frac{k-1}{2}}\Phi(z)$ so that

$$\Phi_{j+1} = \Phi_j K_j.$$

Notice that if

$$F(z) = \left(\frac{1}{\sqrt{k}}\frac{1}{z^{\frac{k-1}{2}}}\exp(kz), \frac{1}{\sqrt{k}}\frac{e^{\frac{i\pi}{k}}}{z^{\frac{k-1}{2}}}\exp(ke^{\frac{2\pi i}{k}}z), \dots, \frac{1}{\sqrt{k}}\frac{e^{\frac{i\pi}{k}(k-1)}}{z^{\frac{k-1}{2}}}\exp(ke^{\frac{2\pi i}{k}(k-1)}z)\right)$$

is the row vector whose entries are the first term of the asymptotic expansions of an actual solutions $\Phi(z)$ of the generalized hypergeometric equation, it is easily seen that

$$F(ze^{\frac{2\pi i}{k}}) = F(z)T_F, \quad T_F = \begin{pmatrix} 0 & & & & \dots & 1 \\ -1 & 0 & & & & \\ & -1 & 0 & & & \\ & & -1 & & & \\ & & & \ddots & \ddots & \vdots \\ & & & & -1 & 0 \end{pmatrix}.$$

As a consequence, if $\Phi_m(z)$ is the unique genuine solution of the hypergeometric equation such that

$$\Phi_m(z) \sim F(z) \quad z \to \infty \quad z \in \mathscr{S}_m,$$

then

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}}) \sim F(ze^{\frac{2\pi i}{k}}) = F(z)T_F \quad z \in \mathscr{S}_m,$$

so that

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}})T_F^{-1} \sim F(z) \quad z \in \mathscr{S}_m.$$

By unicity, this implies that

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}})T_F^{-1} = \Phi_m(z) \quad z \in \mathcal{R}.$$

We deduce from this identity the following properties of the Stokes factors

Lemma 15.3. *For any $m, p \in \mathbb{Z}$ the following identity holds*

$$K_{m+2p} = T_F^{-p} K_m T_F^p.$$

Proof. We have from the definitions of the K_i's that

$$\begin{aligned} \Phi_{m+1}(z) = \Phi_m(z)K_m &= \Phi_{m+2}(ze^{\frac{2\pi i}{k}})T_F^{-1}K_m \\ &= \Phi_{m+3}(ze^{\frac{2\pi i}{k}})K_{m+2}^{-1}T_F^{-1}K_m \\ &= \Phi_{m+1}(z)T_F K_{m+2}^{-1}T_F^{-1}K_m. \end{aligned}$$

Hence, $K_{m+2} = T_F^{-1}K_m T_F^1$. A simple inductive argument completes the proof. □

From this one can deduce the following

THEOREM 15.4 ([Guz99]). *Let ℓ be an admissible line, and let us enumerate the rays as described above, and introduce the corresponding Stokes factors K_i's. The Stokes matrix of the system (15.5), and equivalently of the hypergeometric equation (15.6), for $k>3$, is given by*

$$S=\begin{cases}(K_0K_1T_F^{-1})^{\frac{k}{2}}T_F^{\frac{k}{2}}\equiv T_F^{\frac{k}{2}}(T_F^{-1}K_{k-2}K_{k-1})^{\frac{k}{2}}, & k \text{ even}\\ \\ (K_0K_1T_F^{-1})^{\frac{k-1}{2}}K_0T_F^{\frac{k-1}{2}}\equiv T_F^{\frac{k-1}{2}}K_{k-1}(T_F^{-1}K_{k-2}K_{k-1})^{\frac{k-1}{2}}, & k \text{ odd.}\end{cases}$$

Moreover, the two Stokes factors K_{k-2} and K_{k-1} are given by:

- *for k **even** we have*

$$(K_{k-2})_{2,1}=-\binom{k}{1},\quad (K_{k-2})_{j,j}=1 \text{ for } j=1,\dots,k$$
$$(K_{k-2})_{j,k-j+3}=\binom{k}{2j-3} \text{ for } j=3,\dots,\frac{k}{2}+1$$
$$(K_{k-1})_{j,j}=1 \text{ for } j=1,\dots,k,\quad (K_{k-1})_{j,k-j+2}=\binom{k}{2(j-1)} \text{ for } j=2,\dots,\frac{k}{2}$$

  and all other entries of K_{k-2},K_{k-1} are zero.
- *for k **odd** we have*

$$(K_{k-2})_{2,1}=-\binom{k}{1},\quad (K_{k-2})_{j,j}=1 \text{ for } j=1,\dots,k$$
$$(K_{k-2})_{j,k-j+3}=\binom{k}{2j-3} \text{ for } j=3,\dots,\frac{k+1}{2}$$
$$(K_{k-1})_{j,j}=1 \text{ for } j=1,\dots,k,\quad (K_{k-1})_{j,k-j+2}=\binom{k}{2(j-1)} \text{ for } j=2,\dots,\frac{k+1}{2}$$

  and all other entries of K_{k-2},K_{k-1} are zero.

With this results, we can now resume the symmetries and quasi-periodicity relations of the Stokes matrices

THEOREM 15.5. *Let p be a point of the small quantum cohomology of $\mathbb{P}_{\mathbb{C}}^{k-1}$, let $\ell(\phi)$ be an admissible line of slope $\phi\in\mathbb{R}$ at p, and denote by $S(p,\ell)^{lex}$ (or $S(p,\phi)^{lex}$) the Stokes matrix computed at p, w.r.t. the line ℓ and in ℓ-lexicographical order.*

(1) *The Stokes matrix has the following functional form*

$$S(t\sigma,\phi)^{\mathrm{lex}}=S(\mathrm{Im}(t)+k\phi),\quad t\in\mathbb{C}.$$

(2) *The Stokes matrix satisfies the* quasi-periodicity *condition*

$$S(p,\phi)^{\mathrm{lex}}\sim S\left(p,\phi+\frac{2\pi i}{k}\right)^{\mathrm{lex}},$$

where $A\sim B$ means that the matrices A,B are in the same $(\mathbb{Z}/2\mathbb{Z})^k$-orbit w.r.t. the action of Theorem 1.2. Moreover

$$S(p,\phi)^{\mathrm{lex}}=S\left(p,\phi+2\pi i\right)^{\mathrm{lex}}.$$

(3) *The entries*

$$S(p,\phi)^{\mathrm{lex}}_{i,i+1} \text{ and } S\left(p,\phi+\frac{\pi i}{k}\right)^{\mathrm{lex}}_{i,i+1}$$

differ for some signs for all $p \in H^2(\mathbb{P}^{k-1}_{\mathbb{C}};\mathbb{C})$, $\phi \in \mathbb{R}$ *and for any* $i = 1, \dots k-1$. *In particular, the* $(k-1)$*-tuple*

$$\left(\left|S(p,\phi)^{\mathrm{lex}}_{1,2}\right|, \left|S(p,\phi)^{\mathrm{lex}}_{2,3}\right|, \dots, \left|S(p,\phi)^{\mathrm{lex}}_{k-1,k}\right|\right)$$

does not depend on p *and* ϕ. *In particular, it is equal to*

$$\left(\binom{k}{1}, \dots, \binom{k}{k-1}\right).$$

Proof. The first point of the statement follows from the discussion at the beginning of Section 15.7. For proving the rest of Theorem, we can consider just the origin $0 \in QH^{\bullet}(\mathbb{P})$ as point at which we compute the monodromy data: for brevity, we will just omit the index $p = 0$ from the Stokes matrix in the following formlulae. So, if we have fixed an admissible line ℓ, and if $S(\ell)^{\mathrm{lex}}$ is the Stokes matrix in lexicographical form, then the matrix $S(e^{\frac{2\pi i}{k}}\ell)^{\mathrm{lex}}$ is nothing else than

$$\left(S(\ell)^{\mathrm{lex}}\right)^{\omega_1\omega_2} \quad \text{or} \quad \left(S(\ell)^{\mathrm{lex}}\right)^{\omega_2\omega_1}.$$

Notice that if we label the Stokes rays as in the above discussion, then the Stokes matrix (not in the upper triangular form) is given by

$$S(\ell) = K_0K_1 \dots K_{k-2}K_{k-1},$$

and for getting it in the lexicographical form we can just act by conjugation $PS(\ell)P^{-1}$, for a *unique* permutation matrix corresponding to the ℓ-lexicographical order. Consequently, we have that

$$S(e^{\frac{2\pi i}{k}}\ell) = K_2K_3 \dots K_kK_{k+1},$$

and accordingly to the previous Theorem we can deduce that

$$K_k = T_F^{-\frac{k}{2}}K_0T_F^{\frac{k}{2}}, \quad K_{k+1} = T_F^{-\frac{k}{2}}K_1T_F^{\frac{k}{2}}.$$

From now on, we will restrict to the case k *even*: the case k *odd* being analogous, we leave to the reader the easy and necessary adjustments of the proof. Under this assumption we have

$$\begin{aligned}
S(e^{\frac{2\pi i}{k}}\ell) &= T_F^{\frac{k}{2}}\left(T_F^{-1}K_kK_{k+1}\right)^{\frac{k}{2}} \\
&= T_F^{\frac{k}{2}}\left(T_F^{-1}T_F^{-\frac{k}{2}}K_0K_1T_F^{\frac{k}{2}}\right)^{\frac{k}{2}} \\
&= T_F^{-1}(K_0K_1T_F^{-1})^{\frac{k}{2}}T_F^{\frac{k}{2}+1} \\
&= T_F^{-1}S(\ell)T_F.
\end{aligned}$$

If we want to put the matrix $S(e^{\frac{2\pi i}{k}}\ell)$ in lexicographical form, we have to conjugate it by a suitable permutation matrix, say Q (corresponding to the lexicographical order w.r.t. the rotated line $e^{\frac{2\pi i}{k}}\ell$):

$$\begin{aligned}
S(e^{\frac{2\pi i}{k}}\ell)^{\mathrm{lex}} &= Q \cdot S(e^{\frac{2\pi i}{k}}\ell) \cdot Q^{-1} \\
&= (QT_F^{-1}) \cdot S(\ell) \cdot (QT_F^{-1})^{-1}.
\end{aligned}$$

From the uniqueness of the permutation P above, we deduce that P and QT_F^{-1} must have the same "shape", namely:

$$P_{ij} = 0 \Longleftrightarrow (QT_F^{-1})_{ij} = 0,$$

and the other entries may differ by a sign. In other words we have that $QT_F^{-1} = \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1) \cdot P$ for some choice of the signs (in particular, there will be $k-1$ times entries (-1)'s and just one entry $(+1)$, as in the matrix T_F^{-1}). This proves the first statement.
For the second statement, it is sufficient to prove it just for the choice of ℓ with slope $0 < \phi < \frac{\pi}{k}$. From the explicit expressions for the Stokes factors K_{k-2} and K_{k-1} of the previous Theorem, after some computations, one finds that the entries in the first upper-diagonals of the matrix $S(\ell)^{\text{lex}}$ are

$$\begin{pmatrix}
1 & -\binom{k}{1} & -\binom{k}{1}\binom{k}{k-2} + \binom{k}{k-1} & \binom{k}{1}^2 - \binom{k}{2} & \dots & & \dots & \\
 & 1 & \binom{k}{k-2} & -\binom{k}{1} & \dots & & \dots & \\
 & & 1 & -\binom{k}{3} & \dots & & \dots & \\
 & & & \ddots & \ddots & & & \\
 & & & & 1 & -\binom{k}{k-3} & -\binom{k}{k-3}\binom{k}{2} + \binom{k}{k-1} & \binom{k}{k-3}\binom{k}{1} - \binom{k}{k-2} \\
 & & & & & 1 & \binom{k}{2} & -\binom{k}{1} \\
 & & & & & & 1 & -\binom{k}{k-1} \\
 & & & & & & & 1
\end{pmatrix},$$

i.e. along the diagonals we have the general form

$$\begin{pmatrix}
\ddots & \ddots & & & & \\
 & 1 & \textcolor{red}{-\binom{k}{2n-1}} & -\binom{k}{2n-1}\binom{k}{k-2n} + \binom{k}{k-1} & \textcolor{blue}{\binom{k}{2n-1}\binom{k}{1} - \binom{k}{2n}} & \dots \\
 & & 1 & \binom{k}{k-2n} & \textcolor{red}{-\binom{k}{1}} & \dots \\
 & & & 1 & -\binom{k}{2n+1} & \dots \\
 & & & & 1 & \ddots
\end{pmatrix}$$

for $n = 1, \dots, \frac{k}{2} - 1$. Since the Stokes matrix $S(e^{\frac{\pi i}{k}}\ell)^{\text{lex}}$ is equal to $(S(\ell)^{\text{lex}})^{\omega_1} = A^{\omega_1} \cdot S(\ell)^{\text{lex}} \cdot A^{\omega_1}$, where

$$A^{\omega_1} = \begin{pmatrix}
0 & 1 & & & & & \\
1 & \binom{k}{1} & & & & & \\
 & & 0 & 1 & & & \\
 & & 1 & \binom{k}{3} & & & \\
 & & & & \ddots & & \\
 & & & & & 0 & 1 \\
 & & & & & 1 & \binom{k}{k-1}
\end{pmatrix},$$

we find that

$$\left(S(e^{\frac{\pi i}{k}}\ell)^{\text{lex}}\right)_{2i+1,2i+2}=\binom{k}{2i+1}\qquad i=0,\dots,\frac{k}{2}-1$$

and

$$\begin{aligned}\left(S(e^{\frac{\pi i}{k}}\ell)^{\text{lex}}\right)_{2i,2i+1}&=\left(S(\ell)^{\text{lex}}\right)_{2i-1,2i+2}-\left(S(\ell)^{\text{lex}}\right)_{2i-1,2i-1}\cdot\left(S(\ell)^{\text{lex}}\right)_{2i,2i+2}\\&=\binom{k}{2n-1}\binom{k}{1}-\binom{k}{2n}+\binom{k}{2n-1}\binom{k}{1}\\&=-\binom{k}{2n}\end{aligned}$$

for $i=1,\dots,\frac{k}{2}-1$. This completes the proof. □

Corollary 15.3. *The Beilinson exceptional collection $\mathfrak{B}$ corresponds to the monodromy data computed at some point of the small quantum cohomology of $\mathbb{P}^{k-1}_{\mathbb{C}}$ if and only if $k=2,3$.*

Proof. Note that the inverse of the Gram matrix of the Grothendieck-Euler-Poincaré product, which would coincide the Stokes matrix, has the following entries on the upper diagonal:

$$(-k,-k,\dots,-k,-k).$$

□

Remark 15.2. Note that the Corollary above cannot be deduced from Theorem 15.3. The reason is that *a priori* the subgroup of $\mathcal{B}_k$ of braids fixing up to shifts the Beilinson exceptional collection $\mathfrak{B}$

$$\left\{\beta\in\mathcal{B}_k\colon\mathfrak{B}^\beta\equiv\mathfrak{B}[\mathbf{m}]\right\},\quad\mathbf{m}:=(m_1,\dots,m_k)\in\mathbb{Z}^k,$$

could be non-trivial. In general, it is still an open problem to study transitiveness and freeness of the braid group action on the set of exceptional collections. See [GK04] fur further details.

CHAPTER 16

Proof of the Main Conjecture for Grassmannians

ABSTRACT. In this Chapter, using the Quantum Satake identification described in Chapter 4, we prove the validity of Conjecture 14.2 by using the results of the previous Chapter. In particular, we show that the monodromy data computed at the points of the small quantum cohomology, w.r.t. an oriented line ℓ in the complex plane, are the prescribed geometric data associated with an exceptional collection which can be mutated into the Kapranov exceptional collection twisted by a line bundle (Theorem 16.1).

In what follows we will adopt the same notations of Chapter 4. In particular,

- r, k will be natural numbers such that $1 \leq r < k$.
- We will denote by $\mathbb{P}$ the complex projective space $\mathbb{P}^{k-1}_{\mathbb{C}}$;
- $\mathbb{G}$ will be the complex Grassmannian $\mathbb{G}(r,k)$ of r-planes in $\mathbb{C}^k$;
- $\sigma \in H^2(\mathbb{P};\mathbb{C})$ will be the generator of the cohomology of $\mathbb{P}$, normalized so that

$$\int_{\mathbb{P}} \sigma^{k-1} = 1.$$

- σ_λ will denote the λ-th Schubert class of $\mathbb{G}$, identified with

$$\sigma_{\lambda_1+r-1} \wedge \cdots \wedge \sigma_{\lambda_r} \in \bigwedge^r H^\bullet(\mathbb{P};\mathbb{C}),$$

  through the identification $(j \circ \vartheta)$.

16.1. Computation of the fundamental systems of solutions and monodromy data

In all this Chapter, if V denotes a complex vector space and $\phi \in \mathrm{End}_{\mathbb{C}}(V)$, we denote by $\wedge^r\phi \in \mathrm{End}_{\mathbb{C}}(\bigwedge^r V)$ its r-exterior power: if a basis $(v_1, \dots, v_n)$ of V is fixed, and if A denotes the matrix associated with ϕ, then the matrix $\wedge^r A$ associated with $\wedge^r\phi$ is the one obtained by taking the $r \times r$ minors of A (also called r-th *compound* matrix of A, see [**Gan60**]). The entries are disposed according to a pre-fixed ordering of the induced basis $(v_{i_1} \wedge \cdots \wedge v_{i_r})_{1 \leq i_1 < \cdots < i_r \leq n}$ of $\bigwedge^r V$. Notice that a natural ordering for the induced basis is the lexicographical one.

In practice, the space V will be intended to be the classical cohomology space $H^\bullet(\mathbb{P};\mathbb{C})$, and its r-th exterior power will be identified with the classical cohomology space $H^\bullet(\mathbb{G}(r,k),\mathbb{C})$ through the identification $(j \circ \vartheta)$ described in Chapter 4. In particular, any ordering of the normalized idempotents vector fields (and consequently of the canonical coordinates) for $\mathbb{P}$ induced a natural lexicographical[1] order for the normalized idempotents (and of canonical coordinates) for $\mathbb{G}$.

[1] *Caveat lector:* do not confuse this lexicographical order with the one induced by the choice of an admissible line ℓ.

PROPOSITION 16.1. *Let $Z^{\mathbb{P}}(z,t^2)$ be a solution of the system of differential equations* (15.1)-(15.2), *i.e.*

$$\frac{\partial}{\partial t^2} Z^{\mathbb{P}}(z,t^2) = z\mathcal{C}_2^{\mathbb{P}}(t^2) Z^{\mathbb{P}}(z,t^2), \quad \mathcal{C}_2^{\mathbb{P}}(t^2) := (\sigma)\circ^{\mathbb{P}}_{t^2\sigma}, \tag{16.1}$$

$$\frac{\partial}{\partial z} Z^{\mathbb{P}}(z,t^2) = \left(\mathcal{U}^{\mathbb{P}}(t^2) + \frac{1}{z}\mu^{\mathbb{P}}\right) Z^{\mathbb{P}}(z,t^2). \tag{16.2}$$

Then, the r-exterior power

$$Z^{\mathbb{G}}(z,t^2) := \bigwedge^r \left(Z^{\mathbb{P}}(z,t^2+(r-1)\pi i)\right) \tag{16.3}$$

defines a solution of the system corresponding to the Grassmannian $\mathbb{G}$, namely

$$\frac{\partial}{\partial t^2} Z^{\mathbb{G}}(z,t^2) = z\mathcal{C}_2^{\mathbb{G}}(t^2) Z^{\mathbb{G}}(z,t^2), \quad \mathcal{C}_2^{\mathbb{G}}(t^2) := (\sigma_1)\circ^{\mathbb{G}}_{t^2\sigma_1}, \tag{16.4}$$

$$\frac{\partial}{\partial z} Z^{\mathbb{G}}(z,t^2) = \left(\mathcal{U}^{\mathbb{G}}(t^2) + \frac{1}{z}\mu^{\mathbb{G}}\right) Z^{\mathbb{G}}(z,t^2). \tag{16.5}$$

Furthermore, if $Z^{\mathbb{P}}(z,t^2)$ is in Levelt normal form at $z=0$, then also (16.3) *is in Levelt normal form at $z=0$.*

PROOF. Let us notice that

$$\frac{\partial}{\partial t^2}(Z^{\mathbb{G}})^A_B\Big|_{(z,t^2)} = \sum_{a=1}^r \sum_{\ell=1}^k \det \begin{pmatrix} (Z^{\mathbb{P}})^{\alpha_1}_{\beta_1} & \dots & (Z^{\mathbb{P}})^{\alpha_1}_{\beta_r} \\ \vdots & & \vdots \\ X^{\alpha_a}_{\ell}(Z^{\mathbb{P}})^{\ell}_{\beta_1} & \dots & X^{\alpha_a}_{\ell}(Z^{\mathbb{P}})^{\ell}_{\beta_r} \\ \vdots & & \vdots \\ (Z^{\mathbb{P}})^{\alpha_r}_{\beta_1} & \dots & (Z^{\mathbb{P}})^{\alpha_r}_{\beta_r} \end{pmatrix}\Bigg|_{(z,t^2+\pi i(r-1))}, \quad X(z,t^2) = z\mathcal{C}_2^{\mathbb{P}}.$$

Using the results of Corollary 4.4, the r.h.s. is easily seen to be equal to

$$\left(z\mathcal{C}_2^{\mathbb{G}}(t^2)Z^{\mathbb{G}}(z,t^2)\right)^A_B.$$

Analogously, we have that

$$\frac{\partial}{\partial z}(Z^{\mathbb{G}})^A_B\Big|_{(z,t^2)} = \sum_{a=1}^r \sum_{\ell=1}^k \det \begin{pmatrix} (Z^{\mathbb{P}})^{\alpha_1}_{\beta_1} & \dots & (Z^{\mathbb{P}})^{\alpha_1}_{\beta_r} \\ \vdots & & \vdots \\ W^{\alpha_a}_{\ell}(Z^{\mathbb{P}})^{\ell}_{\beta_1} & \dots & W^{\alpha_a}_{\ell}(Z^{\mathbb{P}})^{\ell}_{\beta_r} \\ \vdots & & \vdots \\ (Z^{\mathbb{P}})^{\alpha_r}_{\beta_1} & \dots & (Z^{\mathbb{P}})^{\alpha_r}_{\beta_r} \end{pmatrix}\Bigg|_{(z,t^2+\pi i(r-1))},$$

where we set $W(z,t^2) = \left(\mathcal{U}^{\mathbb{P}}(t^2) + \frac{1}{z}\mu^{\mathbb{P}}\right)$. Using Proposition 4.2 and Corollary 4.4, one identifies the r.h.s. with

$$\left[\left(\mathcal{U}^{\mathbb{G}}(t^2) + \frac{1}{z}\mu^{\mathbb{G}}\right)\cdot Z^{\mathbb{G}}(z,t^2)\right]^A_B.$$

For the last statement, notice that if

$$Z^{\mathbb{P}}(z,t^2) = \Phi(z,t^2)z^{\mu^{\mathbb{P}}}z^{c_1(\mathbb{P})\cup(-)}, \quad \Phi(-z,t^2)^T\eta^{\mathbb{P}}\Phi(z,t^2) = \eta^{\mathbb{P}},$$

then using the generalized Cauchy-Binet identity for the minors of a product, and invoking Corollary 4.2, Corollary 4.3 and Proposition 4.2, one obtains that

$$Z^{\mathbb{G}}(z,t^2)=\widetilde{\Phi}(z,t)z^{\mu^{\mathbb{G}}}z^{c_1(\mathbb{G})\cup(-)},\quad \widetilde{\Phi}(-z,t^2)^T\eta^{\mathbb{G}}\widetilde{\Phi}(z,t^2)=\eta^{\mathbb{G}},\quad \widetilde{\Phi}(z,t^2)=\bigwedge^r\Phi(z,t^2+\pi i(r-1)).$$

This concludes the proof. □

Corollary 16.1. *Let $Z^{\mathbb{P}}_{\rm top}(z,t^2)$ be the restriction to the small quantum locus of the topological-enumerative solution of $\mathbb{P}$. Then, the topological-enumerative solution of $\mathbb{G}$, restrictedd to the small quantum cohomology is given by*

$$Z^{\mathbb{G}}_{\rm top}(z,t^2)=\left(\bigwedge^r Z^{\mathbb{P}}_{\rm top}(z,t^2+\pi i(r-1))\right)\cdot e^{-\pi i(r-1)\sigma_1\cup(-)}.$$

Proof. According to Proposition 3.1 and Proposition 3.2, we have

$$Z^{\mathbb{P}}_{\rm top}(z,t^2)=\Theta^{\mathbb{P}}_{\rm top}(z,t^2)z^{\mu^{\mathbb{P}}}z^{c_1(\mathbb{P})\cup(-)},$$

and $\Theta^{\mathbb{P}}_{\rm top}$ is characterized by the fact that

$$z^{-\mu^{\mathbb{P}}}\Theta^{\mathbb{P}}_{\rm top}(z,t^2)z^{\mu^{\mathbb{P}}}=\exp(t^2\sigma\cup(-))+\sum_{i=1}^{\infty}A_iz^i,\quad A_i\in\mathfrak{gl}(k,\mathbb{C}).$$

Hence, from Proposition 16.1, we deduce that

$$\left(\bigwedge^r Z^{\mathbb{P}}_{\rm top}(z,t^2+\pi i(r-1))\right)=H(z,t^2)z^{\mu^{\mathbb{G}}}z^{c_1(\mathbb{G})\cup(-)},$$

where

$$z^{-\mu^{\mathbb{G}}}H(z,t^2)z^{\mu^{\mathbb{G}}}=\exp((t^2+\pi i(r-1))\sigma_1\cup(-))+\sum_{i=1}^{\infty}A_i'z^i,\quad A_i'\in\mathfrak{gl}\left(\binom{k}{r},\mathbb{C}\right),$$

by Proposition 4.2 and Corollary 4.3. Using Proposition 3.2, we conclude. □

Let us consider a fixed determination $\Psi^{\mathbb{P}}(t^2)$ of the Ψ-matrix for $\mathbb{P}$ along points $t^2\sigma\in H^2(\mathbb{P};\mathbb{C})$ of the small quantum cohomology. By point (3) of Corollary 4.4, a determination of the Ψ-matrix for the Grassmannian $\mathbb{G}$ is given by the r-exterior power

$$\Psi^{\mathbb{G}}(t^2):=i^{\binom{r}{2}}\bigwedge^r\Psi^{\mathbb{P}}(t^2+\pi i(r-1)),\quad t^2\sigma_1\in H^2(\mathbb{G};\mathbb{C}).\tag{16.6}$$

If we set $Y^{\mathbb{P}/\mathbb{G}}:=\Psi^{\mathbb{P}/\mathbb{G}}\cdot Z^{\mathbb{P}/\mathbb{G}}$, we can consider the corresponding systems of differential equations (2.24). The following results establish the relationship between the solutions of these differential systems, and their Stokes phenomena.

Proposition 16.2. *Let ℓ be an oriented line in the complex plane, with slope $\phi\in[0;2\pi[$, admissible at both points*

$$p:=t^2\sigma_1\in H^2(\mathbb{G};\mathbb{C}),\quad \hat{p}:=(t^2+\pi i(r-1))\sigma\in H^2(\mathbb{P};\mathbb{C}).$$

Let us denote by $Y^{\mathbb{P}/\mathbb{G}}_{\rm formal}(z,u)$ the formal solutions of the differential systems (2.24) associated to the quantum cohomology of $\mathbb{P}$ and $\mathbb{G}$, respectively. If $Y^{(k),\mathbb{P}/\mathbb{G}}_{\rm left/right}(z,u)$ denote the solutions of these systems, uniquely characterized by the asymptotic expansion

$$Y^{(k),\mathbb{P}/\mathbb{G}}_{\rm left/right}(z,u)\sim Y^{\mathbb{P}/\mathbb{G}}_{\rm formal}(z,u),\quad |z|\to\infty,\quad z\in e^{2\pi ik}\Pi_{\rm left/right}(\phi),$$

uniformly in u, then we have the following identifications:

$$Y^{\mathbb{G}}_{\text{formal}}(z,u(p)) = \bigwedge^r Y^{\mathbb{P}}_{\text{formal}}(z,u(\hat{p})),$$
$$Y^{\mathbb{G}}_{\text{left/right}}(z,u(p)) = \bigwedge^r Y^{\mathbb{P}}_{\text{left/right}}(z,u(\hat{p})).$$

PROOF. The claim immediately follows from identity (16.6), Proposition 16.1, and from the simple observation that

$$\bigwedge^r \left[\exp(zU^{\mathbb{P}}(\hat{p}))\left(\mathbb{1}+\sum_{h\geq 1}\frac{1}{z^h}A_h\right)\right] = \exp(zU^{\mathbb{G}}(p))\left(\mathbb{1}+\sum_{h\geq 1}\frac{1}{z^h}A'_h\right).$$

□

COROLLARY 16.2. *If ℓ is an oriented line in the complex plane, with slope $\phi\in[0;2\pi[$, admissible at both points*

$$p := t^2\sigma_1 \in H^2(\mathbb{G};\mathbb{C}),\quad \hat{p} := (t^2+\pi i(r-1))\sigma \in H^2(\mathbb{P};\mathbb{C}),$$

and if

$$S^{\mathbb{P}}(\hat{p};\ell),\ C^{(k),\mathbb{P}}(\hat{p};\ell),\quad S^{\mathbb{G}}(p;\ell),\ C^{(k),\mathbb{G}}(p;\ell)$$

denote the Stokes and Central connection matrices of $\mathbb{P}$, and $\mathbb{G}$ respectively, computed at a point $\hat{p}$, and p respectively, w.r.t. the oriented line ℓ, then the following identities hold true:

$$S^{\mathbb{G}}(p;\ell) = \bigwedge^r S^{\mathbb{P}}(\hat{p};\ell),\quad C^{(k),\mathbb{G}}(p;\ell) = i^{-\binom{r}{2}}\left(\bigwedge^r C^{(k),\mathbb{P}}(\hat{p};\ell)\right)\cdot e^{\pi i(r-1)\sigma_1\cup(-)}.$$

In particular, if the canonical coordinates $(u_i)_{i=1}^k$ of $\mathbb{P}$ are in ℓ-lexicographical order, then the induced lexicographical order of canonical coordinates $(u_{i_1}+\dots+u_{i_r})_{1\leq i_!<\dots<i_r\leq k}$ of $\mathbb{G}$ is a ℓ-triangular order in the sense of Definition 2.17.

16.2. Reduction to (twisted) Kapranov Form

PROPOSITION 16.3. *Let $(V,\langle\cdot,\cdot\rangle)$ be a Mukai lattice of rank k, and define a Mukai structure on the free $\mathbb{Z}$-module $\wedge^r V$ by setting*

$$\langle\alpha_I,\alpha_J\rangle^{\wedge r} := \det\left(\langle\alpha_{i_h},\alpha_{j_\ell}\rangle\right)_{1\leq h,\ell\leq r},$$

where $\alpha_I := \alpha_{i_1}\wedge\dots\wedge\alpha_{i_r}$, with $1\leq i_1<\dots<i_r\leq k$, and analogously α_J denote two decomposable elements. If $(\varepsilon_i)_i$ and $(\tilde{\varepsilon}_i)_i$ are two exceptional bases of V related by the action of a braid in $\mathcal{B}_k$, then the exceptional bases $(\varepsilon_I)_I$ and $(\tilde{\varepsilon}_I)_I$ of $\wedge^r V$, obtained by lexicographical ordering, are in the same orbit through the action of braids in $\mathcal{B}_{\binom{k}{r}}$ and $(\mathbb{Z}/2\mathbb{Z})^{\times\binom{k}{r}}$.

PROOF. It is clearly enough to prove the statement for two exceptional bases of V related by the action of an elementary braid. Let us suppose, for example, that the exceptional bases of V

$$(\varepsilon_1,\dots,\varepsilon_i,\varepsilon_{i+1},\dots,\varepsilon_k),\quad (\varepsilon_1,\dots,\varepsilon_{i+1},\tilde{\varepsilon}_i,\varepsilon_{i+2},\dots,\varepsilon_k),\quad \tilde{\varepsilon}_i := \mathbb{R}_{\varepsilon_{i+1}}\varepsilon_i,$$

are related by the action of the elementary braid $\sigma_{i,i+1}$. Let us now consider the exceptional bases of $\wedge^r V$ obtained by lexicographical ordering. The elements of the second basis can be classified in three different types:

(1) those of the form ε_J with $\varepsilon_{j_h} \notin \{\varepsilon_{i+1}, \tilde{\varepsilon}_i\}$ for all $h = 1, \dots, r$,
(2) those of the form $(\dots \wedge \varepsilon_{i+1} \wedge \tilde{\varepsilon}_i \wedge \dots)$,
(3) and those of the form
$$\left(\bigwedge_{a=1}^{\ell-1} \varepsilon_{j_a}\right) \wedge \tilde{\varepsilon}_i \wedge \left(\bigwedge_{a=\ell+1}^{r} \varepsilon_{j_a}\right), \tag{16.7}$$
for some ℓ.

Using the definition $\mathbb{R}_{\varepsilon_{i+1}}\varepsilon_i := \varepsilon_i - \langle\varepsilon_i, \varepsilon_{i+1}\rangle\varepsilon_{i+1}$, it is evident that for the elements of the class (2) the following identity holds:
$$\dots \wedge \varepsilon_{i+1} \wedge \tilde{\varepsilon}_i \wedge \dots = \dots \wedge \varepsilon_{i+1} \wedge \varepsilon_i \wedge \dots = -(\dots \wedge \varepsilon_i \wedge \varepsilon_{i+1} \wedge \dots).$$
Consequently, they are the opposites of elements of the first exceptional basis. For the elements of the class (3), notice that all the elements between the *first* one of the type (16.7), and the corresponding one obtained by replacing $\tilde{\varepsilon}_i$ with ε_{i+1}, are of the form
$$\left(\bigwedge_{a=1}^{\ell-1} \varepsilon_{h_a}\right) \wedge \varepsilon_{i+1} \wedge \left(\bigwedge_{a=\ell+1}^{r} \varepsilon_{h_a}\right),$$
with $j_a = h_a$ for $a \in \{1, \dots, \ell-1\} \cup \{\ell+1, \dots, n\}$, and $j_{n+1} < h_{n+1}$, for some $n \in \{\ell+1, \dots, r\}$. The scalar product of these elements with the first element (16.7) is given by the determinant
$$\det\left(\begin{array}{c|c} D_1 & D_2 \\ \hline 0 & D_3 \end{array}\right) = 0,$$
since the matrices D_1, D_3 are upper triangular, $\operatorname{diag}(D_1) = (1, \dots, 1, \langle\varepsilon_{i+1}, \tilde{\varepsilon}_i\rangle)$ and D_3 has at least a zero element on the diagonal (at least $(D_3)_{n+1,n+1} = 0$). Hence, we can successively mutate the first element (16.7) on the left, till we obtain the following configuration of exceptional basis:
$$\left(\dots, \underbrace{\left(\bigwedge_{a=1}^{\ell-1} \varepsilon_{j_a}\right) \wedge \varepsilon_{i+1} \wedge \left(\bigwedge_{a=\ell+1}^{r} \varepsilon_{j_a}\right)}_{A_{i+1}}, \underbrace{\left(\bigwedge_{a=1}^{\ell-1} \varepsilon_{j_a}\right) \wedge \tilde{\varepsilon}_i \wedge \left(\bigwedge_{a=\ell+1}^{r} \varepsilon_{j_a}\right)}_{\tilde{A}_i}, \dots\right).$$
At this point, notice that
$$\tilde{A}_i = A_{i+1} - \langle A_i, A_{i+1}\rangle^{\wedge r} A_{i+1}, \quad A_i := \left(\bigwedge_{a=1}^{\ell-1} \varepsilon_{j_a}\right) \wedge \varepsilon_i \wedge \left(\bigwedge_{a=\ell+1}^{r} \varepsilon_{j_a}\right),$$
since $\langle A_i, A_{i+1}\rangle^{\wedge r} = \langle\varepsilon_i, \varepsilon_{i+1}\rangle$. The procedure continues and iterates with the new first term of the type (16.7). At the end of the procedure, one obtaines a factor decomposition of the braids taking the second exceptional basis into the first one (modulo signs for elements of the class (2)). Notice that elements of the class (1) do not mutate. □

Example 16.1. An example will clarify the procedure. Let us consider the case $(r, k) = (3, 6)$ and let $(\varepsilon_1, \dots, \varepsilon_6)$ be an exceptional basis of V. Through the action of the braid σ_{23} we obtain a new exceptional collection
$$(\varepsilon_1, \varepsilon_3, \tilde{\varepsilon}_2, \varepsilon_4, \varepsilon_5, \varepsilon_6).$$

By lexicographical ordering, from the first basis we obtain the exceptional basis

$$\varepsilon_{123},\ \varepsilon_{124},\ \varepsilon_{125},\ \varepsilon_{126},\ \varepsilon_{134},\ \varepsilon_{135},\ \varepsilon_{136},\ \varepsilon_{145},\ \varepsilon_{146},\ \varepsilon_{156},\ \varepsilon_{234},\ \varepsilon_{235},\ \varepsilon_{236},\ \varepsilon_{245},\ \varepsilon_{246},\ \varepsilon_{256},\ \varepsilon_{345},\ \varepsilon_{346},\ \varepsilon_{356},\ \varepsilon_{456}. \tag{16.8}$$

Analogously, from the second basis we obtain the exceptional one

$$\varepsilon_{13\tilde{2}},\ \varepsilon_{134},\ \varepsilon_{135},\ \varepsilon_{136},\ \varepsilon_{1\tilde{2}4},\ \varepsilon_{1\tilde{2}5},\ \varepsilon_{1\tilde{2}6},\ \varepsilon_{145},\ \varepsilon_{146},\ \varepsilon_{156},\ \varepsilon_{3\tilde{2}4},\ \varepsilon_{3\tilde{2}5},\ \varepsilon_{3\tilde{2}6},\ \varepsilon_{345},\ \varepsilon_{346},\ \varepsilon_{356},\ \varepsilon_{\tilde{2}45},\ \varepsilon_{\tilde{2}46},\ \varepsilon_{\tilde{2}56},\ \varepsilon_{456}. \tag{16.9}$$

We want to determine the transformation which transform (16.9) into (16.8). In red we have colored elements of the class (2), in blue the elements of the class (3). Black elements are in class (1). Notice that red elements are just the opposite of the corresponding elements in (16.8) obtained by the exchange $(3 \to 2, \tilde{2} \to 3)$. Let us now start with the first blue element, i.e. $\varepsilon_{1\tilde{2}4}$: we have that

$$\langle \varepsilon_{135}, \varepsilon_{1\tilde{2}4} \rangle = 0, \quad \langle \varepsilon_{136}, \varepsilon_{1\tilde{2}4} \rangle = 0.$$

Hence, by acting on (16.9) with the braid $\beta_{45}\beta_{34}$, we obtain

$$-\varepsilon_{123},\ \varepsilon_{134},\ \varepsilon_{1\tilde{2}4},\ \varepsilon_{135},\ \varepsilon_{136}, \dots.$$

Acting now with the braid β_{23}, we obtain

$$-\varepsilon_{123},\ \varepsilon_{124},\ \varepsilon_{134},\ \varepsilon_{135},\ \varepsilon_{136}, \dots.$$

We can continue with the next blue element, i.e. $\varepsilon_{1\tilde{2}5}$, till we obtain the sequence

$$-\varepsilon_{123},\ \varepsilon_{124},\ \varepsilon_{125},\ \varepsilon_{134},\ \varepsilon_{135},\ \varepsilon_{136},\ \varepsilon_{1\tilde{2}6}, \dots.$$

By iterating the mutation procedure of the next blue elements, we arrive at the exceptional basis (16.8) (modulo signs of the red elements).

The following computation already appears in the papers [**GGI16, GI15**], although only for the class $\widehat{\Gamma}^{+}_{\mathbb{G}}$.

Lemma 16.1. *The following identity holds true:*

$$(j \circ \vartheta)\left[\widehat{\Gamma}^{\pm}_{\mathbb{G}} \cup \mathrm{Ch}(\mathbb{S}^{\mu}\mathcal{S}^{\vee})\right] = (2\pi i)^{-\binom{r}{2}} e^{-\pi i(r-1)\sigma_1} \bigwedge_{h=1}^{r} \widehat{\Gamma}^{\pm}_{\mathbb{P}} \cup \mathrm{Ch}(\mathcal{O}(\mu_h + r - h)).$$

Proof. As in Section 4.2.1, denote by $x_1, \dots, x_r$ the Chern roots of the bundle $\mathcal{S}^{\vee}$ on $\mathbb{G}$. Starting from the generalized Euler sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{G}}^{\oplus k} \to \mathcal{Q} \to 0,$$

and applying to it the tensor product $\mathcal{S}^{\vee} \otimes -$, in the Grothendieck group $K_0(\mathbb{G})$ we obtain the identity

$$[T\mathbb{G}] = [\mathcal{S}^{\vee} \otimes \mathcal{Q}] = k[\mathcal{S}^{\vee}] - [\mathcal{S}^{\vee} \otimes \mathcal{S}].$$

Hence, by the multiplicative property of the $\widehat{\Gamma}^{\pm}$-classes, we obtain

$$\widehat{\Gamma}^{\pm}_{\mathbb{G}} = \prod_{i,h=1}^{r} \frac{\Gamma(1 \pm x_i)^k}{\Gamma(1 \pm x_i \mp x_h)}.$$

Notice that

$$\begin{aligned}\prod_{i,h=1}^{r}\Gamma(1\pm x_i\mp x_h)&=\prod_{i<h}\Gamma(1\pm x_i\mp x_h)\Gamma(1\mp x_i\pm x_h)\\&=\prod_{i<h}\frac{2\pi i(x_i-x_h)}{e^{\pi i(x_i-x_h)}-e^{\pi i(x_h-x_i)}}\\&=(2\pi i)^{\binom{r}{2}}\prod_{i<h}(x_i-x_h)\prod_{i<h}\frac{e^{\pi i(x_i+x_h)}}{e^{2\pi i x_i}-e^{2\pi i x_h}}\\&=(2\pi i)^{\binom{r}{2}}\left(\prod_{i<h}\frac{x_i-x_h}{e^{2\pi i x_i}-e^{2\pi i x_h}}\right)e^{(r-1)\pi i\sigma_1},\end{aligned}$$

where for the last equality we used the fact that $\{x_i+x_h\}_{i<h}$ are the Chern roots of $\bigwedge^2\mathcal{S}^\vee$, so that

$$\begin{aligned}\prod_{i<h}e^{\pi i(x_i+x_h)}&=\exp\left(\pi i\sum_{i<h}x_i+x_h\right)\\&=\exp\left(\pi i c_1\left(\bigwedge^2\mathcal{S}^\vee\right)\right)\\&=\exp\left(\pi i(r-1)c_1(\mathcal{S}^\vee)\right).\end{aligned}$$

We have thus obtained the formula

$$\widehat{\Gamma}^{\pm}_{\mathbb{G}}=(2\pi i)^{-\binom{r}{2}}e^{-\pi i(r-1)\sigma_1}\prod_{i<h}\frac{e^{2\pi i x_i}-e^{2\pi i x_h}}{x_i-x_h}\prod_{i=1}^{r}\Gamma(1\pm x_i)^k. \tag{16.10}$$

At this point, if we recall that the Chern character defines a morphism of rings, from the definition of Schur polynomials, we obtain the identity

$$\mathrm{Ch}(\mathbb{S}^\mu\mathcal{S}^\vee)=\frac{\det(e^{2\pi i x_i(\mu_h+r-h)})_{i,h}}{\prod_{i<h}e^{2\pi i x_i}-e^{2\pi i x_h}}. \tag{16.11}$$

The claim follows from equations (16.10) and (16.11). ∎

THEOREM 16.1. *The central connection matrix, in lexicographical order, of $QH^\bullet(\mathbb{G})$, computed at $t=0$ w.r.t. an admissible oriented line ℓ is the matrix associated to the morphism $Д^-_{\mathbb{G}}\colon K_0(\mathbb{G})\otimes\mathbb{C}\to H^\bullet(\mathbb{G};\mathbb{C})$ w.r.t. an exceptional basis of the Grothendieck group $K_0(\mathbb{G})$, related by suitable mutations and elements of $\mathbb{Z}^{\binom{k}{r}}$ to the twisted Kapranov basis*

$$\left([\mathbb{S}^\mu\mathcal{S}^\vee\otimes\mathscr{L}]\right)_\mu,\quad \mathscr{L}:=\det\left(\bigwedge^2\mathcal{S}^\vee\right).$$

In particular, the Conjecture 14.2 holds true.

PROOF. If C is the matrix associated with the morphism $Д^-_{\mathbb{P}}$ w.r.t.

- the Beilinson basis $([\mathcal{O}],\dots,[\mathcal{O}(k-1)])$ of $K_0(\mathbb{P})\otimes\mathbb{C}$,
- the basis $(1,\sigma,\dots,\sigma^{k-1})$ of $H^\bullet(\mathbb{P};\mathbb{C})$,

then by Corollary 4.3 and Lemma 16.1 it follows that the matrix

$$i^{-\binom{r}{2}}\left(\bigwedge^r C\right)e^{\pi i(r-1)\sigma_1\cup(-)} \tag{16.12}$$

is the matrix associated[2] with $Д_{\mathbb{G}}^{-}$ w.r.t.

- the twisted Kapranov basis $([\mathbb{S}^{\mu}\mathcal{S}^{\vee}\otimes\mathscr{L}])_{\mu}$,
- the induced Schubert basis $(\sigma_{\mu})_{\mu}$.

The line bundle $\mathscr{L}$ is uniquely determined by its first Chern class $c_1(\mathscr{L}) = (r-1)\sigma_1$, by point (4) of Corollary 13.1 (or even because $\mathbb{G}$ is Fano). Thus, by Corollary 14.1, it follows that the association

$$\left(\bigwedge^r K_0(\mathbb{P}), \wedge^r\chi^{\mathbb{P}}\right) \to (K_0(\mathbb{G}), \chi^{\mathbb{G}})\colon \bigwedge_{h=1}^{r}[\mathcal{O}(\mu_h + r - h)] \mapsto [\mathbb{S}^{\mu}\mathcal{S}^{\vee}\otimes\mathscr{L}],$$

defines an isomorphism of Mukai lattices. By Proposition 16.3, the claim follows. □

16.3. Symmetries and Quasi-Periodicity of the Stokes matrices along the small quantum locus

We conclude this Chapter with the following result, concerning the symmetries and quasi-periodicity properties of the Stokes matrix S of $QH^{\bullet}(\mathbb{G})$ computed at points of the small quantum cohomology. It is an immediate consequence of the analogous properties of the Stokes matrix for $QH^{\bullet}(\mathbb{P})$ and of Corollary 16.2.

THEOREM 16.2. *The Stokes matrix $S_{\mathbb{G}(r,k)}(p,\phi)$, computed at a point $p \ni H^2(\mathbb{G};\mathbb{C})$ w.r.t. an admissible line ℓ of slope $\phi\in\mathbb{R}$ and in ℓ-lexicographical order, satisfies the following conditions:*

(1) *it has the following functional form*
$$S_{\mathbb{G}(r,k)}(t\sigma_1,\phi) = S(\mathrm{Im}t + k\phi);$$
(2) *it is* quasi-periodic *along the small quantum locus, in the sense that*
$$S_{\mathbb{G}(r,k)}(p,\phi) \sim S_{\mathbb{G}(r,k)}\left(p, \phi + \frac{2\pi i}{k}\right),$$
where $A\sim B$ means that the matrices A and B are in the same orbit under the action of $(\mathbb{Z}/2\mathbb{Z})^{\binom{k}{r}}$. Moreover, we have that
$$S_{\mathbb{G}(r,k)}(p,\phi) = S_{\mathbb{G}(r,k)}\left(p,\phi+2\pi i\right);$$
(3) *the upper-diagonal entries*
$$S_{\mathbb{G}(r,k)}(p,\phi)_{i,i+1},\quad S_{\mathbb{G}(r,k)}\left(p,\phi+\frac{\pi i}{k}\right)_{i,i+1}$$
differ for some signs, and we have that
$$|S_{\mathbb{G}(r,k)}(p,\phi)_{i,i+1}| \in \left\{\binom{k}{1},\dots,\binom{k}{k-1}\right\}\cup\{0\}.$$

From this Theorem, Corollary 15.3, Proposition 14.1 and from Lemma 16.1, we finally deduce the following result.

[2] Note that the numerical factor $i^{\bar{d}}$ in (16.12) is exactly with $d = r(k-r)$, since $r \equiv r^2 \pmod 2$.

COROLLARY 16.3. *The Kapranov exceptional collection* $(\mathbb{S}^{\lambda}\mathcal{S}^{\vee})_{\lambda}$, *twisted by a suitable line bundle, is associated with the monodromy data of* $\mathbb{G}(r,k)$ *at points of the small quantum locus if and only if* $(r,k) = (1,2),(1,3),(2,3)$. *In this cases, the line bundle is trivial, and the Kapranov collection coincides with the Beilinson one*[3].

[26]Notice that $\mathbb{G}(2,3) \cong \mathbb{P}((\mathbb{C}^3)^{\vee}) \cong \mathbb{P}^2_{\mathbb{C}}$ by duality.

Appendices

Non senza fatiga si giunge al fine

Girolamo Frescobaldi,
Toccata IX

A

Examples of Cell Decomposition

Example A.1. Let

$$\Lambda(t)=\operatorname{diag}(u_1(t),u_2(t),u_3(t)):=\operatorname{diag}(0,t,1).$$

In this example, the coalescence locus in a neighbourhood of $t=0$ is $\{0\}$, while the global coalescence locus in $\mathbb{C}$ is $\{0,1\}$. At $t=0$ we have

$$\arg(u_1(0)-u_3(0))=\arg(0-1),\qquad \arg(u_3(0)-u_1(0))=\arg(1-0).$$

We *choose* $\widehat{\arg}(1)=0$, $\widehat{\arg}(-1)=\pi$. This implies that an admissible direction η such that $\eta-2\pi<\widehat{\arg}(u_i(0)-u_j(0))<\eta$ must satisfy

$$\eta-2\pi<0<\eta,\qquad \eta-2\pi<\pi<\eta \qquad\Longrightarrow\qquad \pi<\eta<2\pi.$$

Therefore $\tau=3\pi/2-\eta$ satisfies

$$-\frac{\pi}{2}<\tau<\frac{\pi}{2}.$$

– At $t\neq 0$: $u_1(t)=u_1(0)$ and $u_3(t)=u_3(0)$, and

$$\arg(u_1(t)-u_2(t))=\arg(-t),\qquad \arg(u_2(t)-u_1(t))=\arg(t),$$
$$\arg(u_3(t)-u_2(t))=\arg(1-t),\qquad \arg(u_2(t)-u_3(t))=\arg(t-1).$$

We *impose:*

$$\eta-2\pi<\widehat{\arg}(-t)<\eta,\qquad \eta-2\pi<\widehat{\arg}(t)<\eta,$$
$$\Downarrow$$
$$\eta-2\pi<\widehat{\arg}(t)<\eta-\pi \ \text{ out } \ \eta-\pi<\widehat{\arg}(t)<\eta.$$

The above gives the 2 cells of $\mathcal{U}_{\epsilon_0}(0)$ for $\epsilon_0<1$.

$$c(-):=\{t\in\mathcal{U}_{\epsilon_0}(0)\mid \eta-2\pi<\arg(t)<\eta-\pi\},\qquad c(+):=\{t\in\mathcal{U}_{\epsilon_0}(0)\mid \eta-\pi<\arg(t)<\eta\}.$$

Since $u(t)$ is globally defined (and $t=1$ is another coalescence point), one can globally divide the t-plane into cells. Accordingly, we also impose the condition

$$\eta-2\pi<\widehat{\arg}(1-t)<\eta,\qquad \eta-2\pi<\widehat{\arg}(t-1)<\eta,$$
$$\Downarrow$$
$$\eta-2\pi<\widehat{\arg}(t-1)<\eta-\pi \ \text{ out } \ \eta-\pi<\widehat{\arg}(t-1)<\eta.$$

Therefore, the t plane is globally partitioned into 3 cells by the above relation, as in figure A.1.

Example A.2. Let

$$\Lambda(t)=\ \operatorname{diag}\Big(u_1(t),u_2(t),u_3(t),u_4(t),u_5(t)\Big):=\ \operatorname{diag}\Big(0,\ t,\ te^{i\frac{\pi}{2}},\ te^{i\pi},\ te^{i\frac{3\pi}{2}}\Big).$$

The coalescence locus is $t=0$. The admissible direction η can be chosen arbitrarily, because $\Lambda(0)=0$ has no Stokes rays. Once η is fixed, we impose $\eta-2\pi<\widehat{\arg}(u_i(t)-u_j(t))<\eta$. Thus, for $0\le l,k\le 3$:

$$\eta-2\pi<\widehat{\arg}(te^{i\frac{\pi}{2}k})<\eta,\quad \eta-2\pi<\widehat{\arg}(-te^{i\frac{\pi}{2}k})<\eta,\quad \eta-2\pi<\widehat{\arg}\left(t(e^{i\frac{\pi}{2}l}-e^{i\frac{\pi}{2}k})\right)<\eta.$$

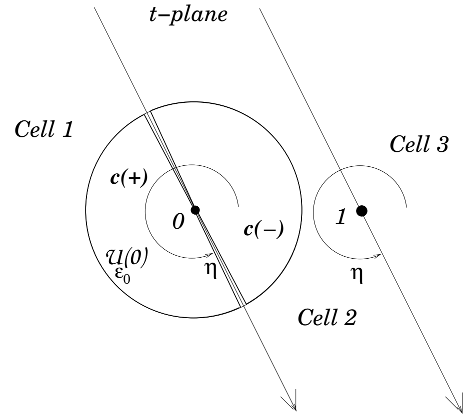

FIGURE A.1. Cell partition (Cell 1, Cell 2, Cell 3) of the t-sheet $\eta-2\pi < \arg(t) < \eta$ and $\eta-2\pi < \arg(t-1) < \eta$. The neighbourhood $\mathcal{U}_{\epsilon_0}(0)$ (the disk) splits into two cells $c(+)$ and $c(-)$.

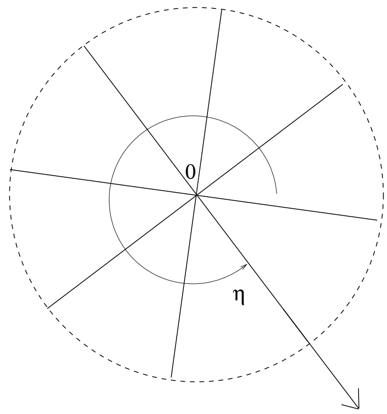

FIGURE A.2. The cells of $\mathcal{U}_{\epsilon_0}(0)$ of Example A.2.

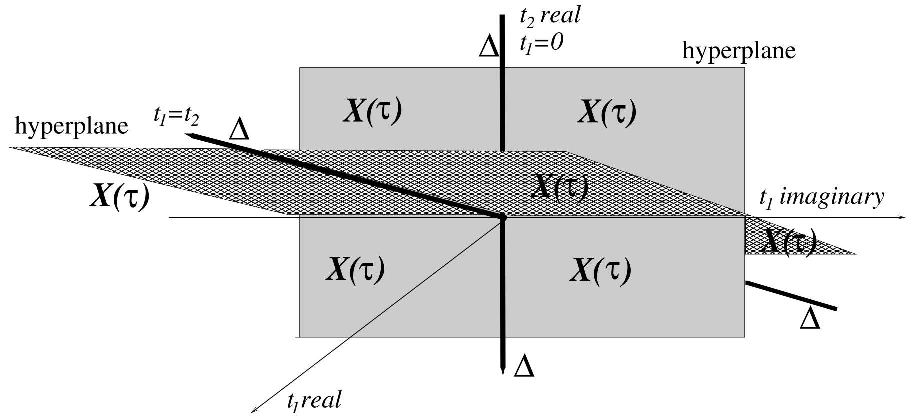

FIGURE A.3. Example A.3, with $\eta = 3\pi/2$. The horizontal plane is $t_1 \in \mathbb{C}$. The vertical axis is $t_2 \in \mathbb{R}$. The thick lines $t_1 = t_2$ (real) and $t_1 = 0$ (t_2 real) are the projection of $\Delta_{\mathbb{C}^2}$. The planes (minus Δ) are the projection of the crossing locus $X(\tau)$. The full planes (which include the thick lines) are the projection of $W(\tau)$. They disconnect $\{t \in \mathbb{C}^2 \mid t_2 \in \mathbb{R}\}$.

The first two constraints imply

$$\eta - 2\pi - \frac{\pi}{2}k < \arg t < \eta - \pi - \frac{\pi}{2}k, \qquad \text{or} \qquad \eta - \pi - \frac{\pi}{2}k < \arg t < \eta - \frac{\pi}{2}k.$$

By prosthaphaeresis formulas we have $e^{i\frac{\pi}{2}l} - e^{i\frac{\pi}{2}k} = 2i \sin\frac{\pi}{4}(l-k)\ e^{i\frac{\pi}{4}(l+k)}$. Therefore, the third constraint gives

$$\eta - 2\pi - \frac{\pi}{4}(l+k) < \arg t < \eta - \pi - \frac{\pi}{4}(l+k), \qquad \text{or} \qquad \eta - \pi - \frac{\pi}{4}(l+k) < \arg t < \eta - \frac{\pi}{4}(l+k).$$

It turns out that the cell-partition of $\mathcal{U}_{\epsilon_0}(0)$ is into 8 slices of angular width $\pi/4$, with angles determined by η. See figure A.2.

Example A.3. We consider $t = (t_1, t_2) \in \mathbb{C}^2$ and $\Lambda(t) = \mathrm{diag}(0, t_1, t_2)$. The coalescence locus can be studied globally on $\mathbb{C}^2$:

$$\Delta_{\mathbb{C}^2} = \{t \in \mathbb{C}^2 \mid t_1 = t_2\} \cup \{t \in \mathbb{C}^2 \mid t_1 = 0\} \cup \{t \in \mathbb{C}^2 \mid t_2 = 0\}.$$

This is the union of complex lines (complex dimension $= 1$) of complex co-dimension $= 1$. In particular, $t = 0$ is the point of maximal coalescence. $\Lambda(0) = 0$ has has no Stokes rays, thus we choose η freely. The cell-partition for a chosen η is given (see previous examples) by:

$$\eta - 2\pi < \arg(t_i) < \eta - \pi, \qquad \text{or} \qquad \eta - \pi < \arg(t_i) < \eta, \qquad i = 1, 2,$$

and

$$\eta - 2\pi < \arg(t_1 - t_2) < \eta - \pi, \qquad \text{or} \qquad \eta - \pi < \arg(t_1 - t_2) < \eta, \qquad i = 1, 2.$$

In figure A.3 we represent the projection of $\mathbb{C}^2$ onto the subspace $\{t \in \mathbb{C}^2 \mid t_2 \in \mathbb{R}\}$, for the choice $\eta = 3\pi/2$. The two thick lines

$$t_1 = t_2 \text{ real}, \qquad t_1 = 0 \text{ with } t_2 \text{ real},$$

are the projection of $\Delta_{\mathbb{C}^2}$. The following planes, without the thick lines,

$$\Big\{t \mid \arg(t_1 - t_2) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \bmod 2\pi\Big\} \cup \Big\{t \mid \arg(t_1) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \bmod 2\pi\Big\}$$

are the projection of the crossing locus $X(\tau)$. The planes, including the thick lines, are the projections of $W(\tau)$.

B

Central Connection matrix of $\mathbb{G}(2,4)$

We report the explicit values for the columns of the central connection matrix $C=(C_{ij})$, computed in Section 11.3.3, where v is indicated. The correct value is $v=6$ (v was first introduced in (11.28)).

$$C_{i1}=\begin{pmatrix} \frac{1}{2\sqrt{c}\pi^2} \\ \frac{4\gamma+i\pi}{2\sqrt{c}\pi^2} \\ \frac{48\gamma^2+24i\gamma\pi-5\pi^2}{12\sqrt{c}\pi^2} \\ \frac{48\gamma^2+24i\gamma\pi+7\pi^2}{12\sqrt{c}\pi^2} \\ \frac{64\gamma^3+48i\gamma^2\pi+4\gamma\pi^2+3i\pi^3-4\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4+768i\gamma^3\pi+96\gamma^2\pi^2+144i\gamma\pi^3-\pi^4-48(4\gamma+i\pi)\zeta(3)}{72\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i2}=\begin{pmatrix} \frac{1}{2\sqrt{c}\pi^2} \\ \frac{4\gamma+i\pi}{2\sqrt{c}\pi^2} \\ \frac{48\gamma^2+24i\gamma\pi+7\pi^2}{12\sqrt{c}\pi^2} \\ \frac{48\gamma^2+24i\gamma\pi-5\pi^2}{12\sqrt{c}\pi^2} \\ \frac{64\gamma^3+48i\gamma^2\pi+4\gamma\pi^2+3i\pi^3-4\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4+768i\gamma^3\pi+96\gamma^2\pi^2+144i\gamma\pi^3-\pi^4-48(4\gamma+i\pi)\zeta(3)}{72\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i3}=\begin{pmatrix} -\frac{1}{4\sqrt{c}\pi^2} \\ \frac{-2\gamma-i\pi}{2\sqrt{c}\pi^2} \\ \frac{-48\gamma^2-48i\gamma\pi+11\pi^2}{24\sqrt{c}\pi^2} \\ \frac{-48\gamma^2-48i\gamma\pi+11\pi^2}{24\sqrt{c}\pi^2} \\ \frac{2\zeta(3)-(2\gamma+i\pi)(4\gamma+i\pi)(4\gamma+3i\pi)}{6\sqrt{c}\pi^2} \\ \frac{-768\gamma^4-1536i\gamma^3\pi+1056\gamma^2\pi^2-23\pi^4+96i\pi\zeta(3)+96\gamma\left(3i\pi^3+2\zeta(3)\right)}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i4}(v)=\begin{pmatrix} \frac{v-1}{4\sqrt{c}\pi^2} \\ \frac{2\gamma(v-1)+i\pi}{2\sqrt{c}\pi^2} \\ \frac{48\gamma^2(v-1)+48i\gamma\pi+(v+11)\pi^2}{24\sqrt{c}\pi^2} \\ \frac{48\gamma^2(v-1)+48i\gamma\pi+(v+11)\pi^2}{24\sqrt{c}\pi^2} \\ \frac{32\gamma^3(v-1)+48i\gamma^2\pi+2\gamma(v+11)\pi^2-3i\pi^3-2(v-1)\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4(v-1)+1536i\gamma^3\pi+96\gamma^2(v+11)\pi^2-(v+23)\pi^4-96i\pi\zeta(3)+96\gamma\left(-3i\pi^3-2(v-1)\zeta(3)\right)}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i4}(6) = \begin{pmatrix} \frac{5}{4\sqrt{c}\pi^2} \\ \frac{10\gamma+i\pi}{2\sqrt{c}\pi^2} \\ \frac{240\gamma^2+48i\gamma\pi+17\pi^2}{24\sqrt{c}\pi^2} \\ \frac{240\gamma^2+48i\gamma\pi+17\pi^2}{24\sqrt{c}\pi^2} \\ \frac{160\gamma^3+48i\gamma^2\pi+34\gamma\pi^2-3i\pi^3-10\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{3840\gamma^4+1536i\gamma^3\pi+1632\gamma^2\pi^2-288i\gamma\pi^3-29\pi^4-960\gamma\zeta(3)-96i\pi\zeta(3)}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i5} = \begin{pmatrix} \frac{1}{4\sqrt{c}\pi^2} \\ \frac{\gamma}{\sqrt{c}\pi^2} \\ \frac{48\gamma^2+\pi^2}{24\sqrt{c}\pi^2} \\ \frac{48\gamma^2+\pi^2}{24\sqrt{c}\pi^2} \\ \frac{-\zeta(3)+16\gamma^3+\gamma\pi^2}{3\sqrt{c}\pi^2} \\ -\frac{192\gamma\zeta(3)-768\gamma^4+\pi^4-96\gamma^2\pi^2}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$C_{i6} = \begin{pmatrix} \frac{1}{4\sqrt{c}\pi^2} \\ \frac{\gamma+i\pi}{\sqrt{c}\pi^2} \\ \frac{48\gamma^2+96i\gamma\pi-47\pi^2}{24\sqrt{c}\pi^2} \\ \frac{48\gamma^2+96i\gamma\pi-47\pi^2}{24\sqrt{c}\pi^2} \\ \frac{(\gamma+i\pi)(4\gamma+3i\pi)(4\gamma+5i\pi)-\zeta(3)}{3\sqrt{c}\pi^2} \\ \frac{768\gamma^4+3072i\gamma^3\pi-4512\gamma^2\pi^2-2880i\gamma\pi^3+671\pi^4-192(\gamma+i\pi)\zeta(3)}{144\sqrt{c}\pi^2} \end{pmatrix}.$$

We report now the entries of the matrix C^-_{Kap} whose columns are given by the components of the characteristic classes

$$\frac{1}{4\pi c^{\frac{1}{2}}}\widehat{\Gamma}^-(\mathbb{G}) \cup \mathrm{Ch}\left(\mathbb{S}^\lambda(\mathcal{S}^*)\right);$$

the order of the column is given by $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, $\lambda = (1,1)$, $\lambda = (2,1)$ and $\lambda = (2,2)$.

$$\left(C^-_{\text{Kap}}\right)_0 = \begin{pmatrix} \frac{1}{4\sqrt{c}\pi^2} \\ \frac{\gamma}{\sqrt{c}\pi^2} \\ \frac{\frac{1}{24}+\frac{2\gamma^2}{\pi^2}}{\sqrt{c}} \\ \frac{\frac{1}{24}+\frac{2\gamma^2}{\pi^2}}{\sqrt{c}} \\ \frac{-\zeta(3)+16\gamma^3+\gamma\pi^2}{3\sqrt{c}\pi^2} \\ -\frac{192\gamma\zeta(3)-768\gamma^4+\pi^4-96\gamma^2\pi^2}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$\left(C^{-}_{\mathrm{Kap}}\right)_{\ydiagram{1}} = \begin{pmatrix} \frac{1}{2\sqrt{c}\pi^2} \\ \frac{4\gamma+i\pi}{2\sqrt{c}\pi^2} \\ \frac{\frac{2\gamma(2\gamma+i\pi)}{\pi^2}-\frac{5}{12}}{\sqrt{c}} \\ \frac{\frac{2\gamma(2\gamma+i\pi)}{\pi^2}+\frac{7}{12}}{\sqrt{c}} \\ \frac{64\gamma^3+48i\gamma^2\pi+4\gamma\pi^2+3i\pi^3-4\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4+768i\gamma^3\pi+96\gamma^2\pi^2+144i\gamma\pi^3-\pi^4-48(4\gamma+i\pi)\zeta(3)}{72\sqrt{c}\pi^2} \end{pmatrix},$$

$$\left(C^{-}_{\mathrm{Kap}}\right)_{\ydiagram{2}} = \begin{pmatrix} \frac{3}{4\sqrt{c}\pi^2} \\ \frac{3(2\gamma+i\pi)}{2\sqrt{c}\pi^2} \\ \frac{\frac{6\gamma(\gamma+i\pi)}{\pi^2}-\frac{19}{8}}{\sqrt{c}} \\ \frac{\frac{6\gamma(\gamma+i\pi)}{\pi^2}+\frac{13}{8}}{\sqrt{c}} \\ \frac{32\gamma^3+48i\gamma^2\pi-6\gamma\pi^2+5i\pi^3-2\zeta(3)}{2\sqrt{c}\pi^2} \\ \frac{-6\gamma^2+\frac{16\gamma^4}{\pi^2}+\frac{32i\gamma^3}{\pi}+10i\gamma\pi+\frac{7\pi^2}{48}-\frac{2(2\gamma+i\pi)\zeta(3)}{\pi^2}}{\sqrt{c}} \end{pmatrix},$$

$$\left(C^{-}_{\mathrm{Kap}}\right)_{\ydiagram{1,1}} = \begin{pmatrix} \frac{1}{4\sqrt{c}\pi^2} \\ \frac{2\gamma+i\pi}{2\sqrt{c}\pi^2} \\ \frac{\frac{2\gamma(\gamma+i\pi)}{\pi^2}-\frac{11}{24}}{\sqrt{c}} \\ \frac{\frac{2\gamma(\gamma+i\pi)}{\pi^2}-\frac{11}{24}}{\sqrt{c}} \\ \frac{(2\gamma+i\pi)(4\gamma+i\pi)(4\gamma+3i\pi)-2\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4+1536i\gamma^3\pi-1056\gamma^2\pi^2-288i\gamma\pi^3+23\pi^4-96(2\gamma+i\pi)\zeta(3)}{144\sqrt{c}\pi^2} \end{pmatrix},$$

$$\left(C^{-}_{\mathrm{Kap}}\right)_{\ydiagram{2,1}} = \begin{pmatrix} \frac{1}{2\sqrt{c}\pi^2} \\ \frac{4\gamma+3i\pi}{2\sqrt{c}\pi^2} \\ \frac{\frac{2\gamma(2\gamma+3i\pi)}{\pi^2}-\frac{29}{12}}{\sqrt{c}} \\ \frac{\frac{2\gamma(2\gamma+3i\pi)}{\pi^2}-\frac{17}{12}}{\sqrt{c}} \\ \frac{(4\gamma+i\pi)(4\gamma+3i\pi)(4\gamma+5i\pi)-4\zeta(3)}{6\sqrt{c}\pi^2} \\ \frac{768\gamma^4+2304i\gamma^3\pi-2208\gamma^2\pi^2-720i\gamma\pi^3+47\pi^4-48(4\gamma+3i\pi)\zeta(3)}{72\sqrt{c}\pi^2} \end{pmatrix},$$

$$\left(C^{-}_{\mathrm{Kap}}\right)_{\ydiagram{2,2}} = \begin{pmatrix} \frac{1}{4\sqrt{c}\pi^2} \\ \frac{\gamma+i\pi}{\sqrt{c}\pi^2} \\ \frac{\frac{2\gamma(\gamma+2i\pi)}{\pi^2}-\frac{47}{24}}{\sqrt{c}} \\ \frac{\frac{2\gamma(\gamma+2i\pi)}{\pi^2}-\frac{47}{24}}{\sqrt{c}} \\ \frac{(\gamma+i\pi)(4\gamma+3i\pi)(4\gamma+5i\pi)-\zeta(3)}{3\sqrt{c}\pi^2} \\ \frac{768\gamma^4+3072i\gamma^3\pi-4512\gamma^2\pi^2-2880i\gamma\pi^3+671\pi^4-192(\gamma+i\pi)\zeta(3)}{144\sqrt{c}\pi^2} \end{pmatrix}.$$

By application of the constraint

$$S = \left(C^-_{\mathrm{Kap}}\right)^{-1} e^{-\pi i R} e^{-\pi i \mu} \eta^{-1} \left((C^-_{\mathrm{Kap}})^T\right)^{-1},$$

we find

$$S_{\mathrm{Kap}} = \begin{pmatrix} 1 & -4 & 6 & 10 & -20 & 20 \\ 0 & 1 & -4 & -4 & 16 & -20 \\ 0 & 0 & 1 & 0 & -4 & 6 \\ 0 & 0 & 0 & 1 & -4 & 10 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S^{-1}_{\mathrm{Kap}} = \begin{pmatrix} 1 & 4 & 10 & 6 & 20 & 20 \\ 0 & 1 & 4 & 4 & 16 & 20 \\ 0 & 0 & 1 & 0 & 4 & 10 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, S^{-1}_{Kap} coincides with the Gram matrix $(\chi(E_i, E_j))_{i,j}$ of the Kapranov exceptional collection.

C

Tabulation of Stokes matrices for $\mathbb{G}(r,k)$ for small k

In this appendix we tabulate all the Stokes matrices computed along the small quantum cohomology of Grassmannians $\mathbb{G}(r,k)$ for $k \leq 5$, w.r.t. an oriented line of slope $\phi \in \mathbb{R}$, and for a suitable choice of the branch of the Ψ-matrix. From this tables, the quasi-periodicity properties proved in Section 15.8 and Section 16.3 are evident. The matrices are obtained in the following way: the matrix S for $\mathbb{P}^{k-1}_{\mathbb{C}}$ with $0 < \mathrm{Im}(t) + k\phi < \pi$ is the one computed by D. Guzzetti in [**Guz99**]. The other Stokes matrices of $\mathbb{P}^{k-1}_{\mathbb{C}}$ are obtained through an action of the braids $\omega_{1,k}, \omega_{2,k}$ described in Section 15.7. The Stokes matrices for $\mathbb{G}(r,k)$ are obtained by applying Corollary 16.2. Colors keep track of the shifts of the quantum Satake identification: a matrix in the r-th column is the r-th exterior power of the matrix in the first column and of the same color.

TABLE C.1. Case $k = 2$

	$\mathbb{P}^1_{\mathbb{C}}$
$0 < \mathrm{Im}(t) + 2\phi < \pi$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
$\pi < \mathrm{Im}(t + 2\phi) < 2\pi$	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

TABLE C.2. Case $k = 3$

	$\mathbb{P}^2_{\mathbb{C}}$	$\mathbb{G}(2,3)$
$0 < \mathrm{Im}(t) + 3\phi < \pi$	$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$
$\pi < \mathrm{Im}(t) + 3\phi < 2\pi$	$\begin{pmatrix} 1 & -3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$
$2\pi < \mathrm{Im}(t) + 3\phi < 3\pi$	$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$
$3\pi < \mathrm{Im}(t) + 3\phi < 4\pi$	$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$
$4\pi < \mathrm{Im}(t) + 3\phi < 5\pi$	$\begin{pmatrix} 1 & -3 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$
$5\pi < \mathrm{Im}(t) + 3\phi < 6\pi$	$\begin{pmatrix} 1 & -3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

TABLE C.3. Case $k = 4$

$\mathrm{Im}(t) + 4\phi$	$\mathbb{P}^3_{\mathbb{C}}$	$\mathbb{G}(2,4)$	$\mathbb{G}(3,4)$
$]0;\pi[$	$\begin{pmatrix} 1 & -4 & -20 & 10 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6 & -20 & -20 & -70 & 20 \\ 0 & 1 & 4 & 4 & 16 & -6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 & 20 & -10 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]\pi;2\pi[$	$\begin{pmatrix} 1 & 4 & -4 & -10 \\ 0 & 1 & -6 & -20 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 & 4 & -4 & -6 & -20 \\ 0 & 1 & 4 & -4 & -16 & -70 \\ 0 & 0 & 1 & 0 & -4 & -20 \\ 0 & 0 & 0 & 1 & 4 & 20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -4 & -4 & -10 \\ 0 & 1 & 6 & 20 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]2\pi;3\pi[$	$\begin{pmatrix} 1 & -4 & -20 & -10 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 & -20 & 20 & -70 & -20 \\ 0 & 1 & -4 & 4 & -16 & -6 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 0 & 1 & -4 & -4 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 & -20 & -10 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]3\pi;4\pi[$	$\begin{pmatrix} 1 & 4 & 4 & -10 \\ 0 & 1 & 6 & -20 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6 & -4 & -4 & -6 & 20 \\ 0 & 1 & 4 & 4 & 16 & -70 \\ 0 & 0 & 1 & 0 & 4 & -20 \\ 0 & 0 & 0 & 1 & 4 & -20 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -4 & -4 & 10 \\ 0 & 1 & 6 & -20 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]4\pi;5\pi[$	$\begin{pmatrix} 1 & 4 & -20 & -10 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 & -20 & -20 & 70 & 20 \\ 0 & 1 & -4 & -4 & 16 & 6 \\ 0 & 0 & 1 & 0 & -4 & -4 \\ 0 & 0 & 0 & 1 & -4 & -4 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 & -20 & 10 \\ 0 & 1 & -6 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]5\pi;6\pi[$	$\begin{pmatrix} 1 & -4 & -4 & 10 \\ 0 & 1 & 6 & -20 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6 & -4 & 4 & 6 & -20 \\ 0 & 1 & 4 & -4 & -16 & 70 \\ 0 & 0 & 1 & 0 & -4 & 20 \\ 0 & 0 & 0 & 1 & 4 & -20 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -4 & 4 & 10 \\ 0 & 1 & -6 & -20 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]6\pi;7\pi[$	$\begin{pmatrix} 1 & -4 & 20 & 10 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6 & 20 & -20 & 70 & -20 \\ 0 & 1 & -4 & 4 & -16 & 6 \\ 0 & 0 & 1 & 0 & 4 & -4 \\ 0 & 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -4 & -20 & 10 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$]7\pi;8\pi[$	$\begin{pmatrix} 1 & 4 & -4 & 10 \\ 0 & 1 & -6 & 20 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6 & -4 & -4 & 6 & 20 \\ 0 & 1 & -4 & -4 & 16 & 70 \\ 0 & 0 & 1 & 0 & -4 & -20 \\ 0 & 0 & 0 & 1 & -4 & -20 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 & -4 & -10 \\ 0 & 1 & -6 & -20 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\mathrm{Im}(t)+5\phi$	$\mathbb{P}^4_{\mathbb{C}}$	$\mathbb{G}(2,5)$
$]0;\pi[$	$\begin{pmatrix} 1 & 5 & -5 & -40 & 15 \\ 0 & 1 & -10 & -95 & 40 \\ 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 & -5 & -15 & -5 & 10 & 40 & 75 & 325 & -50 \\ 0 & 1 & -10 & -45 & -5 & 50 & 225 & 435 & 1990 & -325 \\ 0 & 0 & 1 & 5 & 0 & -5 & -25 & -45 & -225 & 40 \\ 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & -45 & 15 \\ 0 & 0 & 0 & 0 & 1 & -10 & -45 & -95 & -435 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 10 & 50 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]\pi;2\pi[$	$\begin{pmatrix} 1 & -5 & -45 & 15 & 35 \\ 0 & 1 & 10 & -5 & -15 \\ 0 & 0 & 1 & -10 & -45 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -10 & -95 & -40 & -45 & -435 & -185 & 75 & 50 & 175 \\ 0 & 1 & 10 & 5 & 5 & 50 & 25 & -10 & -10 & -50 \\ 0 & 0 & 1 & 5 & 0 & 5 & 25 & -5 & -25 & -185 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & -5 & -40 \\ 0 & 0 & 0 & 0 & 1 & 10 & 5 & -5 & -10 & -75 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & -10 & -50 & -435 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]2\pi;3\pi[$	$\begin{pmatrix} 1 & 5 & -5 & -40 & -15 \\ 0 & 1 & -10 & -95 & -40 \\ 0 & 0 & 1 & 10 & 5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 & 5 & -15 & -5 & -10 & 40 & -75 & 325 & 50 \\ 0 & 1 & 10 & -45 & -5 & -50 & 225 & -435 & 1990 & 325 \\ 0 & 0 & 1 & -5 & 0 & -5 & 25 & -45 & 225 & 40 \\ 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & -45 & -15 \\ 0 & 0 & 0 & 0 & 1 & 10 & -45 & 95 & -435 & -75 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & 10 & -50 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]3\pi;4\pi[$	$\begin{pmatrix} 1 & -5 & -45 & -15 & 35 \\ 0 & 1 & 10 & 5 & -15 \\ 0 & 0 & 1 & 10 & -45 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 & -95 & -40 & 45 & -435 & -185 & -75 & -50 & 175 \\ 0 & 1 & -10 & -5 & 5 & -50 & -25 & -10 & -10 & 50 \\ 0 & 0 & 1 & 5 & 0 & 5 & 25 & 5 & 25 & -185 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & 5 & -40 \\ 0 & 0 & 0 & 0 & 1 & -10 & -5 & -5 & -10 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 10 & 50 & -435 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]4\pi;5\pi[$	$\begin{pmatrix} 1 & 5 & 5 & -40 & -15 \\ 0 & 1 & 10 & -95 & -40 \\ 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -10 & -5 & 15 & -5 & -10 & 40 & 75 & -325 & -50 \\ 0 & 1 & 10 & -45 & 5 & 50 & -225 & -435 & 1990 & 325 \\ 0 & 0 & 1 & -5 & 0 & 5 & -25 & -45 & 225 & 40 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & -45 & -15 \\ 0 & 0 & 0 & 0 & 1 & 10 & -45 & -95 & 435 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & -10 & 50 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE C.4. Case $k=5$ (first part)

$\mathrm{Im}(t)+5\phi$	$\mathbb{G}(3,5)$	$\mathbb{G}(4,5)$
$]0;\pi[$	$\begin{pmatrix} 1 & 10 & 5 & -5 & -10 & -75 & -15 & -40 & -325 & 50 \\ 0 & 1 & 5 & -10 & -50 & -435 & -45 & -225 & -1990 & 325 \\ 0 & 0 & 1 & 0 & -10 & -95 & 0 & -45 & -435 & 75 \\ 0 & 0 & 0 & 1 & 5 & 45 & 5 & 25 & 225 & -40 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & 5 & 50 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & 45 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & -5 & -40 & 15 \\ 0 & 1 & 10 & 95 & -40 \\ 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]\pi;2\pi[$	$\begin{pmatrix} 1 & 10 & -45 & 95 & -435 & -75 & -40 & 185 & 50 & 175 \\ 0 & 1 & -5 & 10 & -50 & -10 & -5 & 25 & 10 & 50 \\ 0 & 0 & 1 & 0 & 10 & 5 & 0 & -5 & -10 & -75 \\ 0 & 0 & 0 & 1 & -5 & -5 & -5 & 25 & 25 & 185 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & -5 & -50 & -435 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 & -40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & -45 & -15 & -35 \\ 0 & 1 & -10 & -5 & -15 \\ 0 & 0 & 1 & 10 & 45 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]2\pi;3\pi[$	$\begin{pmatrix} 1 & -10 & -5 & -5 & -10 & 75 & -15 & -40 & 325 & 50 \\ 0 & 1 & 5 & 10 & 50 & -435 & 45 & 225 & -1990 & -325 \\ 0 & 0 & 1 & 0 & 10 & -95 & 0 & 45 & -435 & -75 \\ 0 & 0 & 0 & 1 & 5 & -45 & 5 & 25 & -225 & -40 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 & 5 & -50 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & -5 & 40 & 15 \\ 0 & 1 & 10 & -95 & -40 \\ 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]3\pi;4\pi[$	$\begin{pmatrix} 1 & 10 & -45 & -95 & 435 & 75 & -40 & 185 & 50 & -175 \\ 0 & 1 & -5 & -10 & 50 & 10 & -5 & 25 & 10 & -50 \\ 0 & 0 & 1 & 0 & -10 & -5 & 0 & -5 & -10 & 75 \\ 0 & 0 & 0 & 1 & -5 & -5 & 5 & -25 & -25 & 185 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & 5 & 50 & -435 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 & 40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & -45 & -15 & 35 \\ 0 & 1 & -10 & -5 & 15 \\ 0 & 0 & 1 & 10 & -45 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]4\pi;5\pi[$	$\begin{pmatrix} 1 & -10 & -5 & -5 & -10 & 75 & 15 & 40 & -325 & -50 \\ 0 & 1 & 5 & 10 & 50 & -435 & -45 & -225 & 1990 & 325 \\ 0 & 0 & 1 & 0 & 10 & -95 & 0 & -45 & 435 & 75 \\ 0 & 0 & 0 & 1 & 5 & -45 & -5 & -25 & 225 & 40 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 & -5 & 50 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & -5 & 40 & -15 \\ 0 & 1 & 10 & -95 & 40 \\ 0 & 0 & 1 & -10 & 5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE C.5. Case $k = 5$ (second part)

$\mathrm{Im}(t)+5\phi$	$\mathbb{P}^4_{\mathbb{C}}$	$\mathbb{G}(2,5)$
$]5\pi;6\pi[$	$\begin{pmatrix} 1 & 5 & -45 & -15 & 35 \\ 0 & 1 & -10 & -5 & 15 \\ 0 & 0 & 1 & 10 & -45 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 & -95 & -40 & -45 & 435 & 185 & 75 & 50 & -175 \\ 0 & 1 & -10 & -5 & -5 & 50 & 25 & 10 & 10 & -50 \\ 0 & 0 & 1 & 5 & 0 & -5 & -25 & -5 & -25 & 185 \\ 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & -5 & 40 \\ 0 & 0 & 0 & 0 & 1 & -10 & -5 & -5 & -10 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 10 & 50 & -435 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]6\pi;7\pi[$	$\begin{pmatrix} 1 & -5 & -5 & 40 & 15 \\ 0 & 1 & 10 & -95 & -40 \\ 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -10 & -5 & 15 & 5 & 10 & -40 & -75 & 325 & 50 \\ 0 & 1 & 10 & -45 & -5 & -50 & 225 & 435 & -1990 & -325 \\ 0 & 0 & 1 & -5 & 0 & -5 & 25 & 45 & -225 & -40 \\ 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & 45 & 15 \\ 0 & 0 & 0 & 0 & 1 & 10 & -45 & -95 & 435 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & -10 & 50 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]7\pi;8\pi[$	$\begin{pmatrix} 1 & -5 & 45 & 15 & -35 \\ 0 & 1 & -10 & -5 & 15 \\ 0 & 0 & 1 & 10 & -45 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -10 & 95 & 40 & -45 & 435 & 185 & -75 & -50 & 175 \\ 0 & 1 & -10 & -5 & 5 & -50 & -25 & 10 & 10 & -50 \\ 0 & 0 & 1 & 5 & 0 & 5 & 25 & -5 & -25 & 185 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & -5 & 40 \\ 0 & 0 & 0 & 0 & 1 & -10 & -5 & 5 & 10 & -75 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & -10 & -50 & 435 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 & 95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]8\pi;9\pi[$	$\begin{pmatrix} 1 & 5 & -5 & 40 & 15 \\ 0 & 1 & -10 & 95 & 40 \\ 0 & 0 & 1 & -10 & -5 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 10 & -5 & 15 & -5 & 10 & -40 & 75 & -325 & 50 \\ 0 & 1 & -10 & 45 & -5 & 50 & -225 & 435 & -1990 & 325 \\ 0 & 0 & 1 & -5 & 0 & -5 & 25 & -45 & 225 & -40 \\ 0 & 0 & 0 & 1 & 0 & 0 & -5 & 0 & -45 & 15 \\ 0 & 0 & 0 & 0 & 1 & -10 & 45 & -95 & 435 & -75 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & 10 & -50 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]9\pi;10\pi[$	$\begin{pmatrix} 1 & -5 & -45 & 15 & -35 \\ 0 & 1 & 10 & -5 & 15 \\ 0 & 0 & 1 & -10 & 45 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -10 & -95 & 40 & -45 & -435 & 185 & 75 & -50 & -175 \\ 0 & 1 & 10 & -5 & 5 & 50 & -25 & -10 & 10 & 50 \\ 0 & 0 & 1 & -5 & 0 & 5 & -25 & -5 & 25 & 185 \\ 0 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & -5 & -40 \\ 0 & 0 & 0 & 0 & 1 & 10 & -5 & -5 & 10 & 75 \\ 0 & 0 & 0 & 0 & 0 & 1 & -5 & -10 & 50 & 435 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE C.6. Case $k=5$ (third part)

$\operatorname{Im}(t)+5\phi$	$\mathbb{G}(3,5)$	$\mathbb{G}(4,5)$
$]5\pi;6\pi[$	$\begin{pmatrix} 1 & 10 & -45 & -95 & 435 & 75 & 40 & -185 & -50 & 175 \\ 0 & 1 & -5 & -10 & 50 & 10 & 5 & -25 & -10 & 50 \\ 0 & 0 & 1 & 0 & -10 & -5 & 0 & 5 & 10 & -75 \\ 0 & 0 & 0 & 1 & -5 & -5 & -5 & 25 & 25 & -185 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & -5 & -50 & 435 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & 45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -5 & 40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & -45 & 15 & 35 \\ 0 & 1 & -10 & 5 & 15 \\ 0 & 0 & 1 & -10 & -45 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]6\pi;7\pi[$	$\begin{pmatrix} 1 & -10 & -5 & 5 & 10 & -75 & 15 & 40 & -325 & 50 \\ 0 & 1 & 5 & -10 & -50 & 435 & -45 & -225 & 1990 & -325 \\ 0 & 0 & 1 & 0 & -10 & 95 & 0 & -45 & 435 & -75 \\ 0 & 0 & 0 & 1 & 5 & -45 & 5 & 25 & -225 & 40 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 & 5 & -50 & 10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -45 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & 5 & 40 & -15 \\ 0 & 1 & -10 & -95 & 40 \\ 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]7\pi;8\pi[$	$\begin{pmatrix} 1 & -10 & 45 & -95 & 435 & -75 & 40 & -185 & 50 & 175 \\ 0 & 1 & -5 & 10 & -50 & 10 & -5 & 25 & -10 & -50 \\ 0 & 0 & 1 & 0 & 10 & -5 & 0 & -5 & 10 & 75 \\ 0 & 0 & 0 & 1 & -5 & 5 & -5 & 25 & -25 & -185 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 & -5 & 50 & 435 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & 5 & 40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -5 & -45 & 15 & 35 \\ 0 & 1 & 10 & -5 & -15 \\ 0 & 0 & 1 & -10 & -45 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]8\pi;9\pi[$	$\begin{pmatrix} 1 & 10 & -5 & -5 & 10 & 75 & -15 & 40 & 325 & -50 \\ 0 & 1 & -5 & -10 & 50 & 435 & -45 & 225 & 1990 & -325 \\ 0 & 0 & 1 & 0 & -10 & -95 & 0 & -45 & -435 & 75 \\ 0 & 0 & 0 & 1 & -5 & -45 & 5 & -25 & -225 & 40 \\ 0 & 0 & 0 & 0 & 1 & 10 & 0 & 5 & 50 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -5 & -45 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & -5 & -40 & 15 \\ 0 & 1 & -10 & -95 & 40 \\ 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$
$]9\pi;10\pi[$	$\begin{pmatrix} 1 & -10 & -45 & -95 & -435 & 75 & 40 & 185 & -50 & -175 \\ 0 & 1 & 5 & 10 & 50 & -10 & -5 & -25 & 10 & 50 \\ 0 & 0 & 1 & 0 & 10 & -5 & 0 & -5 & 10 & 75 \\ 0 & 0 & 0 & 1 & 5 & -5 & -5 & -25 & 25 & 185 \\ 0 & 0 & 0 & 0 & 1 & -10 & 0 & -5 & 50 & 435 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -5 & -45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 & -5 & -40 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & -95 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 5 & 45 & -15 & -35 \\ 0 & 1 & 10 & -5 & -15 \\ 0 & 0 & 1 & -10 & -45 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

TABLE C.7. Case $k=5$ (fourth part)

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