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Semichiral fields on S^2 and generalized Kähler geometry

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Abstract: We study a class of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theories, given by semichiral multiplets coupled to the usual vector multiplet. In the UV, these theories are traditional gauge theories deformed by a gauged Wess-Zumino term. In the IR, they give rise to nonlinear sigma models on noncompact generalized Kähler manifolds, which contain a three-form field H and whose metric is not Kähler. We place these theories on S^2 and compute their partition function exactly with localization techniques. We find that the contribution of instantons to the partition function that we define is insensitive to the deformation, and discuss our results from the point of view of the generalized Kähler target space.

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Contents

1 Introduction

Starting with the work of Pestun [1], there has been substantial application of localization techniques [2, 3] to supersymmetric field theories in various dimensions, leading to the exact computation of Euclidean path-integrals on various manifolds and backgrounds. This new wave of exact results has affected two-dimensional physics [4–20], in particular in the study of two-dimensional $\mathcal{N} = (2,2)$ gauge theories [4, 5]. One of the most interesting applications in two dimensions is to gauged linear sigma models (GLSMs) with chiral multiplets, realizing nonlinear sigma models (NLSMs) on Kähler manifolds in the infrared (IR) [21]. In this case, the exact computation of field theory observables leads to interesting quantities associated to the corresponding Kähler target spaces. For instance, in (untwisted) A-type localization the S^2 partition function computes the exact Kähler potential on the quantum Kähler moduli space of Calabi-Yau manifolds [6, 22], while in B-type localization it computes the exact Kähler potential on the complex structure moduli space [13]. Among other applications, this has been used to compute Gromov-Witten invariants [22, 23] and the Seiberg-Witten prepotential in novel ways [24]. The extension of localization techniques to theories involving twisted chiral multiplets has shed new light on mirror symmetry, showing that the S^2 partition function for the Landau-Ginzburg models proposed by Hori and Vafa [25, 26] reproduces the partition function of the corresponding mirror Abelian GLSMs [6].

In this paper we extend the localization techniques to more general $\mathcal{N} = (2,2)$ gauge theories by including *semichiral multiplets*. A salient feature of these GLSMs is that they include a gauged Wess-Zumino term and they realize NLSMs on generalized Kähler manifolds in the IR, rather than Kähler when only chiral (or only twisted chiral) multiplets are present. In fact, chiral, twisted chiral, and semichiral multiplets are all required (and sufficient) to describe the most general $\mathcal{N} = (2,2)$ NLSMs with torsion. The gauge theories we consider here are the S^2 versions of the GLSMs discussed in detail in [27].[1]

A generalized Kähler structure on a manifold $\mathcal{M}$ consists of the triplet $(g, J_\pm, H)$, where g is a Riemannian metric, $J_\pm$ are two integrable complex structures, and H is a closed three-form (that can be locally written as $H = db$), subject to some constraints. This is the most general target space for $\mathcal{N} = (2,2)$ NLSMs, containing Kähler geometry as the special case $H = 0$ and $J_+ = \pm J_-$. The complex structures are covariantly constant, $\nabla^\pm J_\pm = 0$, each with respect to a connection with torsion $\nabla^\pm = \nabla^0 \pm \frac{1}{2} g^{-1} db$, where ∇^0 is the Levi-Civita connection. The presence of torsion implies that the geometry is generically not Kähler: the forms $\omega_\pm = gJ_\pm$ are not closed. This structure was originally discovered in [35], where it was termed bi-Hermitian geometry. More recently, it has been reformulated as the analog of Kähler geometry in the context of generalized complex geometry [36, 37].[2]

Locally, on a generalized Kähler manifold one can always choose coordinates which are adapted to the decomposition of the tangent bundle

$$T_{\mathcal{M}} = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \mathrm{coIm}[J_+, J_-] \ ,$$

[1] For some previous work on GLSMs with semichiral multiplets in flat space see [28–33]; for a study of gauge theories with a gauged Wess-Zumino term with on-shell $\mathcal{N} = (2,2)$ supersymmetry see [34].

[2] For introductory lectures on generalized complex geometry and its relation to supersymmetry, see for instance [38, 39]. For a review of generalized Kähler geometry and general $\mathcal{N} = (2,2)$ NLSMs see [40].

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where the last factor is the co-image [41, 42]. We denote the corresponding coordinates by $\Phi \oplus \chi \oplus (\mathbb{X}_L, \mathbb{X}_R)$, respectively. In terms of the $\mathcal{N} = (2, 2)$ sigma model, these correspond to different matter multiplets: chiral, twisted chiral, left semichiral and right semichiral, respectively. From the perspective of generalized Kähler geometry, semichiral fields are as fundamental as chiral and twisted chiral: the latter parametrize directions along which the two complex structures commute, while the former parametrize directions along which they do not. As in Kähler geometry, the full geometric data is locally encoded in a single function: the generalized Kähler potential K [40]. Apart from obeying certain inequalities for the metric to be positive definite, $K = K(\Phi, \bar{\Phi}; \chi, \bar{\chi}; \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$ is otherwise an arbitrary real function of the coordinates on the manifold. As in the Kähler case, K serves as the action for the NLSM in superspace. The case of complex dimension three, and in particular the case of one pair of semichiral fields and one chiral field, is especially relevant to supergravity [43].

It is shown in [27] that GLSMs with semichiral fields coupled to the usual vector multiplet are continuous deformations of certain GLSMs with chiral fields only, which realize noncompact Calabi-Yau manifolds. The deformation preserves the R-symmetry at the quantum level, but deforms the geometric structure of the target from Kähler to generalized Kähler by introducing torsion. Here we place those gauge theories on S^2 (with the untwisted background of [4, 5]), and compute the exact partition function by supersymmetric localization "on the Coulomb branch". It turns out that gauge theories with semichiral fields do not admit enough real masses to lift all massless modes (which appear as non-compact directions in the IR NLSM), therefore their partition function is threatened by divergences. We propose a contour prescription to remove those divergences. We show that the parameters controlling the non-Kähler deformation enter in a $\mathcal{Q}$-exact term. As a consequence, the S^2 partition function should not depend on this deformation and should coincide with the partition function in the Kähler case. We verify this fact explicitly.

We will also discuss some consequences of our result for topological A/B-models on generalized Kähler manifolds. As shown in [44] the topological A-model localizes to generalized holomorphic maps,

$$(1 - iJ_+)\bar{\partial}X = 0 \, , \qquad (1 + iJ_-)\partial X = 0 \, , \tag{1.1}$$

where X are real coordinates on the target $\mathcal{M}$. For generic $J_\pm$, these equations are very restrictive and the only solutions are constant maps. In our models, the S^2 partition function defined with our contour prescription does receive instanton corrections — equal to the ones in the Calabi-Yau before deformation — and yet there are no real compact solutions to (1.1).[3] A possible resolution of this puzzle is that, in fact, our partition function receives contributions from complexified solutions to (1.1), which are less restricted and may well exist even for generic $J_\pm$. We leave this puzzle as an open question.

[3]We stress that our analysis of the generalized Kähler structure is carried out only in the UV of the NLSM, which is enough to capture the instanton corrections. An important question is what the behavior of these theories is in the deep IR. The conditions for conformal invariance at the quantum level, and the relation to the generalized Calabi-Yau condition of Hitchin [45], is discussed in [43, 46, 47]. This, however, goes beyond the scope of this paper.

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The paper is organized as follows. In section 2 we review the field components, supersymmetry transformations, gauged supersymmetric actions for semichiral fields on $\mathbb{R}^2$ and discuss the NLSMs these theories realize. In section 3 we place these gauge theories on S^2. In section 4 we study the BPS configurations, localize the path-integral, and discuss issues of the integration contour and instanton contributions. In section 5 we review some aspects of topologically twisted NLSMs on generalized Kähler manifolds, and study the phenomenon of type-change in an example of the resolved conifold with a generalized Kähler structure. We conclude with a discussion in section 6.

2 Semichiral fields on $\mathbb{R}^2$

We begin by reviewing some basic aspects of $\mathcal{N} = (2,2)$ supersymmetry and defining our notation and conventions. We then discuss gauge theories for semichiral fields and their description as NLSMs.

The algebra of $\mathcal{N} = (2,2)$ superderivatives is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = \pm 2i\partial_{\pm\pm}\ , \tag{2.1}$$

where $\pm$ are spinor indices, $\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm$ are superderivatives and $\partial_{\pm\pm} = \frac{1}{2}(\partial_1 \mp i\partial_2)$ are space-time derivatives; the precise definitions are given in appendix A. The SUSY transformations are generated by

$$\delta = \bar{\epsilon}^+\mathbb{Q}_+ + \bar{\epsilon}^-\mathbb{Q}_- + \epsilon^+\bar{\mathbb{Q}}_+ + \epsilon^-\bar{\mathbb{Q}}_-\ , \tag{2.2}$$

where ϵ, $\bar{\epsilon}$ are anticommuting Dirac spinors while $\mathbb{Q}$, $\bar{\mathbb{Q}}$ are the supercharges satisfying $\{\mathbb{Q}_\pm, \bar{\mathbb{Q}}_\pm\} = \mp 2i\partial_{\pm\pm}$ and anticommuting with the spinor derivatives: $\{\mathbb{Q}_\pm, \mathbb{D}_\pm\} = 0$, etc.

2.1 Supermultiplets

The basic matter supermultiplets are chiral, twisted chiral and semichiral fields. In *Lorentzian* signature these fields are defined by the following set of constraints:

$$\begin{aligned}
\text{Chiral}: &\ \bar{\mathbb{D}}_+\Phi = 0\ , & \bar{\mathbb{D}}_-\Phi = 0\ , && \mathbb{D}_+\bar{\Phi} = 0\ , && \mathbb{D}_-\bar{\Phi} = 0\ ,\\
\text{Twisted Chiral}: &\ \bar{\mathbb{D}}_+\chi = 0\ , & \mathbb{D}_-\chi = 0\ , && \mathbb{D}_+\bar{\chi} = 0\ , && \bar{\mathbb{D}}_-\bar{\chi} = 0\ ,\\
\text{Left semichiral}: &\ \bar{\mathbb{D}}_+\mathbb{X}_L = 0\ , & && \mathbb{D}_+\bar{\mathbb{X}}_L = 0\ , && \\
\text{Right semichiral}: &\ & \bar{\mathbb{D}}_-\mathbb{X}_R = 0\ , && && \mathbb{D}_-\bar{\mathbb{X}}_R = 0\ .
\end{aligned} \tag{2.3}$$

In Lorentzian signature, complex conjugation acts on superderivatives as $\mathbb{D}_\pm^\dagger = \bar{\mathbb{D}}_\pm$ and on superfields as $\mathbb{X}^\dagger = \bar{\mathbb{X}}$. The SUSY constraints (2.3) are compatible with complex conjugation.

In *Euclidean* signature, however, the conjugation of superderivatives changes the helicity, namely $\mathbb{D}_\pm^\dagger = \bar{\mathbb{D}}_\mp$, and taking the complex conjugate of the constraints (2.3) may lead to additional constraints. In the case of a twisted chiral field χ, for instance, this implies that the field be constant. The well-known resolution is to complexify the multiplet and consider χ and $\bar{\chi}$ as independent fields. Although this problem does not arise for semichiral fields, we nonetheless choose to complexify them.[4] That is, we will consider $\mathbb{X}_L$ a left semichiral field and $\bar{\mathbb{X}}_L$ an independent left anti-semichiral field, and similarly for $\mathbb{X}_R$ and $\bar{\mathbb{X}}_R$. The SUSY constraints (and their Euclidean conjugates) read:

$$\begin{aligned} &\bar{\mathbb{D}}_+\mathbb{X}_L = 0\,, \quad \mathbb{D}_+\bar{\mathbb{X}}_L = 0\,, \quad \bar{\mathbb{D}}_-\mathbb{X}_R = 0\,, \quad \mathbb{D}_-\bar{\mathbb{X}}_R = 0\,, \\ &\mathbb{D}_-\mathbb{X}_L^\dagger = 0\,, \quad \bar{\mathbb{D}}_-\bar{\mathbb{X}}_L^\dagger = 0\,, \quad \mathbb{D}_+\mathbb{X}_R^\dagger = 0\,, \quad \bar{\mathbb{D}}_+\bar{\mathbb{X}}_R^\dagger = 0\,. \end{aligned} \tag{2.4}$$

The target space geometry of these models is the complexification of the target space geometry of the corresponding models defined in Lorentzian signature. See [48] for a discussion of these issues.

2.2 Components and supersymmetry transformations

Semichiral fields were originally introduced in [49]. Since they are less known than chiral and twisted chiral fields, we review some of their basic properties here. Each left or right semichiral multiplet consists of 3 complex scalars, 4 Weyl fermions, and one complex chiral vector. We denote these by

$$\begin{aligned} &\mathbb{X}_L: \quad (X_L, \psi_\pm^L, F_L, \bar{\chi}_-, M_{-+}, M_{--}, \bar{\eta}_-)\,, \qquad \bar{\mathbb{X}}_L: \quad (\bar{X}_L, \bar{\psi}_\pm^L, \bar{F}_L, \chi_-, \bar{M}_{-+}, \bar{M}_{--}, \eta_-)\,, \\ &\mathbb{X}_R: \quad (X_R, \psi_\pm^R, F_R, \bar{\chi}_+, M_{+-}, M_{++}, \bar{\eta}_+)\,, \qquad \bar{\mathbb{X}}_R: \quad (\bar{X}_R, \bar{\psi}_\pm^R, \bar{F}_R, \chi_+, \bar{M}_{+-}, \bar{M}_{++}, \eta_+)\,, \end{aligned} \tag{2.5}$$

where $\psi_\alpha, \chi_\alpha, \eta_\alpha$ are fermionic and $X, F, M_{\alpha\beta}$ are bosonic fields, all valued in the same representation $\mathfrak{R}$ of a gauge group G (see appendix A.3 for details). Compared to the field content of chiral and twisted chiral fields, semichiral fields have twice the number of bosonic and fermionic components.

To treat left and right semichiral fields in a unified way, we define a superfield $\mathbb{X}$ that satisfies at least one chiral constraint (either $\bar{\mathbb{D}}_+\mathbb{X} = 0$ or $\bar{\mathbb{D}}_-\mathbb{X} = 0$, or both), but we do not specify which one until the end of the calculation. Similarly, $\bar{\mathbb{X}}$ is an independent field satisfying at least one antichiral constraint (either $\mathbb{D}_+\bar{\mathbb{X}} = 0$ or $\mathbb{D}_-\bar{\mathbb{X}} = 0$, or both). We denote the field content by

$$\mathbb{X}: \quad (X, \psi_\alpha, F, \bar{\chi}_\alpha, M_{\alpha\beta}, \bar{\eta}_\alpha)\,, \qquad \text{with} \quad \alpha, \beta = \pm\,, \tag{2.6}$$

and similarly for $\bar{\mathbb{X}}$. By setting $\bar{\chi}_\alpha = M_{\alpha\beta} = \bar{\eta}_\alpha = 0$, the multiplet $\mathbb{X}$ describes a chiral multiplet $\Phi : (X, \psi_\alpha, F)$. By setting $\bar{\chi}_+ = M_{+-} = M_{++} = \bar{\eta}_+ = 0$ it describes a left semichiral multiplet, and by setting $\bar{\chi}_- = M_{-+} = M_{--} = \bar{\eta}_- = 0$ it describes a right semichiral field.[5]

We couple the multiplet $\mathbb{X}$ to the usual vector multiplet V. One could also consider couplings to other vector multiplets, but we do not do that here (see instead [27]). The

[4] One may choose not to do so. Then, a left semichiral field $\mathbb{Y}_L$ satisfies $\bar{\mathbb{D}}_+\mathbb{Y}_L = 0$ and its (Euclidean) Hermitian conjugate $\bar{\mathbb{Y}}_L$ satisfies $\mathbb{D}_-\bar{\mathbb{Y}}_L = 0$, and similarly for a right semichiral field. However, the target space geometry of these models is not well understood. Since ultimately we are interested in learning about the target space geometry of models in Lorentzian signature, we choose to complexify semichiral fields.

[5] With appropriate identifications, it can also describe a twisted chiral multiplet. However, since we are interested in minimally coupling $\mathbb{X}$ to the usual vector multiplet, we do not consider that case here.

field content of the usual vector multiplet is $(A_\mu, \sigma_1, \sigma_2, \lambda_\pm, D)$, where σ_1, σ_2 are real in Lorentzian but complex in Euclidean signature and we define

$$\sigma = i\sigma_1 - \sigma_2 \,, \qquad \bar{\sigma} = -i\sigma_1 - \sigma_2 \,. \tag{2.7}$$

The SUSY transformation rules for the multiplet $\mathbb{X}$, minimally coupled to the vector multiplet V, are derived in appendix A.3 and read:

$$\begin{aligned}
\delta\psi_\alpha &= \big([i\gamma^\mu D_\mu X + i\sigma_1 X + \sigma_2 X\gamma_3]\,\epsilon\big)_\alpha - \epsilon^\beta M_{\beta\alpha} + \bar{\epsilon}_\alpha F & \delta X &= \bar{\epsilon}\psi + \epsilon\bar{\chi} \\
\delta F &= [-i\sigma_1\psi - \sigma_2\psi\gamma_3 - i\lambda X - i(D_\mu\psi)\gamma^\mu + \bar{\eta}]\,\epsilon & \delta\bar{\chi}_\alpha &= \bar{\epsilon}^\beta M_{\alpha\beta} \\
\delta M_{\alpha\beta} &= -\bar{\eta}_\alpha\bar{\epsilon}_\beta - i\sigma_1\bar{\chi}_\alpha\epsilon_\beta - \sigma_2\bar{\chi}_\alpha\gamma_3\epsilon_\beta - i(D_\mu\bar{\chi}_\alpha)(\gamma^\mu\epsilon)_\beta \\
\delta\bar{\eta}_\alpha &= -i(\epsilon\lambda)\bar{\chi}_\alpha + i(\epsilon\gamma^\mu)^\beta D_\mu M_{\alpha\beta} - i\sigma_1\epsilon^\beta M_{\alpha\beta} - \sigma_2(\gamma_3\epsilon)^\beta M_{\alpha\beta} \,,
\end{aligned} \tag{2.8}$$

and similarly for the multiplet $\bar{\mathbb{X}}$:

$$\begin{aligned}
\delta\bar{\psi}_\alpha &= \big(\big[i\gamma^\mu D_\mu\bar{X} - i\sigma_1\bar{X} + \sigma_2\bar{X}\gamma_3\big]\,\bar{\epsilon}\big)_\alpha - \bar{\epsilon}^\beta\bar{M}_{\beta\alpha} + \epsilon_\alpha\bar{F} & \delta\bar{X} &= \epsilon\bar{\psi} + \bar{\epsilon}\chi \\
\delta\bar{F} &= \big[i\sigma_1\bar{\psi} - \sigma_2\bar{\psi}\gamma_3 - i\bar{\lambda}\bar{X} - i(D_\mu\bar{\psi})\gamma^\mu + \eta\big]\,\bar{\epsilon} & \delta\chi_\alpha &= \epsilon^\beta\bar{M}_{\alpha\beta} \\
\delta\bar{M}_{\alpha\beta} &= -\eta_\alpha\epsilon_\beta + i\sigma_1\chi_\alpha\bar{\epsilon}_\beta - \sigma_2\chi_\alpha\gamma_3\bar{\epsilon}_\beta - i(D_\mu\chi_\alpha)(\gamma^\mu\bar{\epsilon})_\beta \\
\delta\eta_\alpha &= -i(\bar{\epsilon}\bar{\lambda})\chi_\alpha + i(\bar{\epsilon}\gamma^\mu)^\beta D_\mu\bar{M}_{\alpha\beta} + i\sigma_1\bar{\epsilon}^\beta\bar{M}_{\alpha\beta} - \sigma_2(\gamma_3\bar{\epsilon})^\beta\bar{M}_{\alpha\beta} \,.
\end{aligned} \tag{2.9}$$

Here $D_\mu = \partial_\mu - iA_\mu$ is the gauge-covariant derivative. To keep the notation compact in what follows, it is convenient to introduce the operator

$$\mathcal{P}_{\alpha\beta} \equiv \begin{pmatrix} 2iD_{++} & \sigma \\ \bar{\sigma} & -2iD_{--} \end{pmatrix} .$$

2.3 Supersymmetric actions

The gauge-invariant kinetic action for semichiral fields in flat space is built out of terms of the form:

$$\begin{aligned}
\mathcal{L}_{\mathbb{X}}^{\mathbb{R}^2} &= \int d^4\theta\, \bar{\mathbb{X}}\,\mathbb{X} \\
&= D_\mu\bar{X}D^\mu X + \bar{X}\left(\sigma_1^2 + \sigma_2^2 + iD\right)X + \bar{F}F - \bar{M}_{\alpha\beta}M^{\beta\alpha} - \bar{X}\mathcal{P}_{\alpha\beta}M^{\alpha\beta} + \bar{M}^{\alpha\beta}\mathcal{P}_{\beta\alpha}X \\
&\quad - \bar{\psi}\left(i\gamma^\mu D_\mu - i\sigma_1 + \gamma_3\sigma_2\right)\psi + i\bar{\psi}\lambda X - i\bar{X}\bar{\lambda}\psi - \eta\psi - \bar{\psi}\bar{\eta} \\
&\quad + \bar{\chi}\left(i\gamma^\mu D_\mu - i\sigma_1 + \gamma_3\sigma_2\right)\chi + i\bar{X}\lambda\bar{\chi} - i\chi\bar{\lambda}X \,.
\end{aligned} \tag{2.10}$$

Setting some fields to zero, according to the discussion below (2.6), gives the corresponding action for chiral and semichiral fields. An important aspect of semichiral fields, however, is that in order to obtain standard quadratic kinetic terms after having integrated out the auxiliary fields, they must come in pairs $(\mathbb{X}_L, \mathbb{X}_R)$. Consider for instance a (neutral, for simplicity) left semichiral field with kinetic action

$$\int d^4\theta\, \bar{\mathbb{X}}_L\,\mathbb{X}_L \,.$$

It is easy to see that the equations of motion set $M_{-+} = \bar{M}_{-+} = \psi_+ = \bar{\psi}_+ = 0$ and give the first order equations

$$\partial_{++}X_L = \partial_{++}\bar{X}_L = \partial_{++}M_{--} = \partial_{++}\bar{M}_{--} = 0$$
$$\partial_{++}\psi_- = \partial_{++}\bar{\psi}_- = \partial_{++}\chi_- = \partial_{++}\bar{\chi}_- = 0\,,$$

which describe two left-moving bosonic and two left-moving fermionic modes. Although interesting, we leave the study of such Lagrangians for future work.

To obtain sigma models with standard kinetic terms, one must consider models with the same number of left and right semichiral fields and an appropriate coupling between them, either of the form $(\bar{\mathbb{X}}_L\mathbb{X}_R + c.c.)$ or $(\mathbb{X}_L\mathbb{X}_R + c.c.)$. In such models, integrating out the auxiliary fields leads to standard kinetic terms as well as (gauged) Wess-Zumino couplings. Note that depending on the type of the off-diagonal term one chooses, left and right semichiral fields must be either in the same or in conjugate representations of the gauge group. Thus, from now on we restrict ourselves to models containing pairs of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ either in a representation $(\mathfrak{R}, \mathfrak{R})$ or $(\mathfrak{R}, \overline{\mathfrak{R}})$ of the gauge group.

Same representation. Consider a pair of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ in representation $(\mathfrak{R}, \mathfrak{R})$. The most general gauge-invariant quadratic action follows from the Lagrangian

$$\mathcal{L}_{LR}^{\mathbb{R}^2} = -\int d^4\theta\, \left[\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \alpha\big(\bar{\mathbb{X}}_L\mathbb{X}_R + \bar{\mathbb{X}}_R\mathbb{X}_L\big)\right]\,, \tag{2.11}$$

where $\alpha > 1$ is a real parameter.[6] Before gauging, the action (2.11) describes flat space with a constant B-field controlled by α, which of course could be gauged away; the significance of the parameter α is that it determines a choice of complex structures $J_\pm$ on the space.[7] We note that due to the the off-diagonal term, $\mathbb{X}_L$ and $\mathbb{X}_R$ must have the same R-charge.

The general case with multiple pairs of semichiral fields is very similar. Under the assumption that the metric is positive definite, one can always diagonalize the kinetic term and reduce it to multiple decoupled copies of (2.11) by field redefinitions; see appendix B.1 for details.

To gain some intuition on GLSMs for semichiral fields, and their difference with standard GLSMs for chiral fields, we reduce the Lagrangian (2.11) to component fields and integrate out the auxiliary components in the semichiral multiplet. This gives

$$\mathcal{L}_{LR}^{\mathbb{R}^2} = \big(g_{mn} + b_{mn}\big)\, D_{++}X^m D_{--}X^n + \frac{\alpha}{2}\Big(\bar{X}\big(|\sigma|^2 + D\big)X + \bar{\tilde{X}}\big(|\sigma|^2 + D\big)\tilde{X}\Big) + \ldots\,, \tag{2.12}$$

[6] One may begin with the general Lagrangian

$$\mathcal{L} = \int d^4\theta\, \left[\beta\,\bar{\mathbb{X}}_L\mathbb{X}_L + \gamma\,\bar{\mathbb{X}}_R\mathbb{X}_R - \alpha\,\bar{\mathbb{X}}_L\mathbb{X}_R - \alpha^*\,\bar{\mathbb{X}}_R\mathbb{X}_L\right]$$

where β, γ are real parameters and α is complex. This describes flat space and requiring that the metric be positive requires $\beta, \gamma \neq 0$. Then, by rescaling the fields one can set $\beta = \pm 1$ and $\gamma = \pm 1$. By a further phase redefinition of the fields, α can be made real and non-negative. Finally, the requirement of a positive-definite metric implies $\beta = \gamma = -1$, $\alpha > 1$.

[7] See [27] and references therein for a more detailed discussion on this point.

where we have omitted fermionic kinetic terms and Yukawa couplings, and defined

$$g_{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \qquad b_{mn} = \sqrt{\alpha^2 - 1} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \tag{2.13}$$

where we denote $X^m = (X, \bar{X}, \tilde{X}, \bar{\tilde{X}})$ and we made the change of variables

$$X_L = \sqrt{\frac{\alpha}{4(\alpha+1)}}\, X + \sqrt{\frac{\alpha}{4(\alpha-1)}}\, \bar{\tilde{X}} , \qquad X_R = \sqrt{\frac{\alpha}{4(\alpha+1)}}\, X - \sqrt{\frac{\alpha}{4(\alpha-1)}}\, \bar{\tilde{X}} . \tag{2.14}$$

We note that after this field redefinition one may take the special limit $\alpha \to 1$, for which $b_{mn} \to 0$. In fact, in this limit the whole Lagrangian (2.12) coincides with the component Lagrangian of a standard GLSM for chiral multiplets (with lowest components X, $\tilde{X}$) in gauge representations $(\mathfrak{R}, \overline{\mathfrak{R}})$. Thus, the parameter α controls a deformation of such model which includes the addition of a gauged Wess-Zumino term. The deformation is not generic: it is such that by adding suitable auxiliary fields $\mathcal{N} = (2,2)$ SUSY can be realized off-shell in terms of semichiral fields.[8]

As we shall discuss in section 2.4, these models give rise to NLSMs on generalized Kähler manifolds with non-zero three-form H, controlled by the parameter α. In the limit $\alpha \to 1$, the H field vanishes and g becomes a Kähler metric, as expected.

Conjugate representations. For a pair of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ in conjugate representations $(\mathfrak{R}, \overline{\mathfrak{R}})$, the most general gauge-invariant quadratic Lagrangian is

$$\mathcal{L}^{\mathbb{R}^2}_{LR} = \int d^4\theta \left[\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \beta \big(\mathbb{X}_L \mathbb{X}_R + \bar{\mathbb{X}}_R \bar{\mathbb{X}}_L \big) \right] , \tag{2.15}$$

where $\beta > 1$ is a real parameter. In this model $\mathbb{X}_L$ and $\mathbb{X}_R$ have opposite R-charges. Once again, reducing the Lagrangian to components and integrating out the auxiliary fields, one observes that the theory is a deformation of a standard GLSM with chiral fields in representation $(\mathfrak{R}, \overline{\mathfrak{R}})$ by a gauged Wess-Zumino term controlled by the parameter β, which vanishes in the limit $\beta \to \infty$. In fact, the gauge theories (2.11) and (2.15) are related off-shell by a simple field redefinition corresponding to a change of coordinates on the target space and sending $\alpha \to 1$ corresponds to sending $\beta \to \infty$ (see appendix B.2). Thus, without loss of generality, we shall consider only one of these actions, choosing the most convenient one for a particular calculation. In the multiflavor case, by field redefinitions, one can always rewrite the Lagrangian as multiple decoupled copies of (2.15).

Finally, we comment that traditional superpotential terms for semichiral fields are not possible because they break supersymmetry. Fermionic superpotential terms (integrals over $d^3\theta$) are possible if a fermionic semichiral multiplet is present, but this is not the case here.

[8] For a discussion of more general gauge theories with gauged Wess-Zumino terms, but only *on-shell* $\mathcal{N} = (2,2)$ SUSY, see [34].

2.4 NLSM description

As mentioned above, one of the main motivations to study gauge theories with semichiral fields is that they realize NLSMs on generalized Kähler manifolds, as opposed to Kähler manifolds when only chiral (or only twisted chiral) fields are present. Let us briefly illustrate this point in the case $G = \mathrm{U}(1)$, although the discussion holds with arbitrary gauge group.

Consider N_F pairs of semichiral fields $(\mathbb{X}_L^i, \mathbb{X}_R^i)$, $i = 1, \ldots, N_F$, charged under a U(1) vector multiplet with charges $(Q_i, -Q_i)$. The Lagrangian for the GLSM is given by

$$\mathcal{L} = \mathcal{L}_{\mathrm{VM}} + \sum_{i=1}^{N_F} \int d^4\theta \left[\bar{\mathbb{X}}_L^i \mathbb{X}_L^i + \bar{\mathbb{X}}_R^i \mathbb{X}_R^i + \beta_i \big(\mathbb{X}_L^i \mathbb{X}_R^i + \bar{\mathbb{X}}_R^i \bar{\mathbb{X}}_L^i\big)\right] + \frac{i}{2} \int d^2\tilde{\theta}\, t\, \Sigma + c.c.\,, \tag{2.16}$$

with $\beta_i > 1$, where

$$\mathcal{L}_{\mathrm{VM}} = -\frac{1}{2e^2} \int d^4\theta\, \bar{\Sigma}\Sigma\,, \qquad \Sigma = \bar{\mathbb{D}}_+ \mathbb{D}_- V\,, \tag{2.17}$$

and $t = i\xi + \frac{\theta}{2\pi}$ is the complexified Fayet-Iliopoulos (FI) parameter. In addition, one may have any number of chiral multiplets charged under the same vector multiplet V with arbitrary charges. In the case that only semichiral fields are present, a better formulation of these theories in terms of a new vector multiplet is discussed in [27].

Just as in GLSMs with chiral fields [21], on the Higgs branch of the theory the vector multiplet becomes massive and at low energies (compared to the gauge coupling e) it can be integrated out. The effective theory is a NLSM on the space of vacua, which is determined by the vanishing of the scalar potential U, modulo the action of the gauge group. The full moduli space of the GLSMs at hand is analyzed in [27]. For the theory in (2.16), there is a Higgs branch given by[9]

$$\mathcal{M} = \left\{ \sum\nolimits_i Q_i \Big(\big|X_L^i\big|^2 - \big|X_R^i\big|^2 \Big) - \xi = 0 \right\} \Big/ \mathrm{U}(1)\,, \tag{2.18}$$

where U(1) acts by $X_L^i \to e^{i\alpha Q^i} X_L^i$, $X_R^i \to e^{-i\alpha Q^i} X_R^i$. The complex dimension of $\mathcal{M}$ is $2N_F - 1$. Topologically, $\mathcal{M}$ coincides with the Higgs branch of a GLSM with chiral (as opposed to semichiral) fields $(\Phi^i, \widetilde{\Phi}^i)$ with gauge charges $(Q_i, -Q_i)$. However, in the semichiral model the space is endowed with a family of generalized Kähler structures controlled by the parameters β_i. In the limit $\beta_i \to \infty$ the space becomes Kähler, but at finite β_i there are two independent complex structures $J_\pm$ and, due to the presence of semichiral fields, $[J_+, J_-] \neq 0$ at generic points on the manifold. We will give an explicit example in section 5.2.

One may also consider a model with gauge charges (Q_i, Q_i). As discussed above, this is completely equivalent to the model with charges $(Q_i, -Q_i)$. Thus, the moduli space of

[9]One way to see this is to compute the scalar potential explicitly by going down to components and working, say, in Wess-Zumino gauge. Alternatively, one may work in superspace, by writing $\mathbb{X}^{(0)\,i} = \mathbb{X}^i$ and $\bar{\mathbb{X}}^{(0)\,i} = \bar{\mathbb{X}}^i e^{-Q_i V}$ in (2.16) to introduce the vector multiplet explicitly (here we are following similar notation to that in [21]). Then, the lowest component of the equation of motion for V leads to the constraint below. Note, in particular, that due to the absence of e^V terms in the off-diagonal terms, the β_i do not enter in the constraint; they do however determine the geometric structure on $\mathcal{M}$.

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these models is always noncompact, for any choice of gauge charges. The generalization of this discussion to generic gauge groups G is straightforward,[10] at the classical level (the analysis of quantum effects would require a careful treatment, as in [50]).

It is well known that the moduli spaces (2.18) admit not only a Kähler, but in fact a Calabi-Yau structure (the condition $\sum_a Q_a = 0$ is automatically satisfied). It would be interesting to study whether they also admit a generalized Calabi-Yau structure [45]. This could answer the question of the behavior of the theories in the deep IR, which we do not address here.

Before we proceed we would like to make a comment on couplings to other vector multiplets. Since semichiral fields are less constrained than chiral or twisted chiral fields, they admit minimal couplings to various vector multiplets. In addition to the usual vector multiplet, they may couple minimally to the twisted vector multiplet, as well as to the Semichiral Vector Multiplet (SVM) and the Large Vector Multiplet (LVM) introduced in [29, 51]. The corresponding GLSMs and the structure of their moduli space is discussed in [27]. We do not study these models here, but we comment briefly on the coupling to the Abelian SVM in section 4.3.

3 Semichiral fields on S^2

The main goal of this section is to place the gauge theories (2.11) — or equivalently (2.15) — on the round sphere S^2 with no twist, i.e., on the supersymmetric background constructed in [4, 5] which preserves four supercharges. Neutral semichiral multiplets, as well as general supersymmetry in two dimensions, have been studied in [52]. We explicitly construct the supersymmetry variations for gauged semichiral multiplets and their action on S^2. As we now show, it turns out that the action is $\mathcal{Q}_A$-exact, therefore the partition function will not depend on the parameters α (or β) therein. We work in components, rather than in superspace.

3.1 Supersymmetry transformations

One way to determine the supersymmetry transformations on S^2 is by first constructing the $\mathcal{N} = (2, 2)$ superconformal transformations and then specializing to an SU(2|1) sub-algebra. The superconformal transformations can be deduced from the $\mathcal{N} = (2, 2)$ super-Poincaré transformations (2.8) by covariantizing them with respect to Weyl transformations, as we now review.

Let the scalar component X transform under Weyl transformations with a weight $\frac{q}{2}$, i.e., under an infinitesimal Weyl transformation $\delta X = -\frac{q}{2}\Omega\, X$. The supersymmetry transformations (2.8) are not covariant under Weyl transformations, but can be covariantized by adding suitable terms proportional to $\nabla_\pm \epsilon$, as explained in [5] and reviewed in appendix A.5. Following this procedure, we find that the Weyl-covariant transformations for

[10] This is certainly the case when there are also charged chiral fields in the model. If only semichiral fields are present, the generalization to the nonabelian case may be more subtle [27].

the superfields $\mathbb{X}$ and $\bar{\mathbb{X}}$ are:

$$\begin{aligned}
\delta\psi_\alpha &= \left(\left[i(D_\mu X)\gamma^\mu + i\sigma_1 X + \sigma_2 X\gamma_3 + i\frac{q}{2}X\overleftarrow{\nabla\!\!\!/}\right]\epsilon\right)_\alpha - \epsilon^\beta M_{\beta\alpha} + \bar\epsilon_\alpha F \qquad & \delta X &= \bar\epsilon\psi + \epsilon\bar\chi \\
\delta F &= \left[-i\sigma_1\psi - \sigma_2\psi\gamma_3 - i\lambda X - i(D_\mu\psi)\gamma^\mu - i\frac{q}{2}\psi\overleftarrow{\nabla\!\!\!/} + \bar\eta\right]\epsilon & \delta\bar\chi_\alpha &= \bar\epsilon^\beta M_{\alpha\beta} \\
\delta M_{\alpha\beta} &= -\bar\eta_\alpha\bar\epsilon_\beta - i\sigma_1\bar\chi_\alpha\epsilon_\beta - \sigma_2\bar\chi_\alpha(\gamma_3\epsilon)_\beta - i(D_\mu\bar\chi_\alpha)(\gamma^\mu\epsilon)_\beta \\
&\quad - i\frac{q+1}{2}\bar\chi_\alpha(\nabla\!\!\!/\epsilon)_\beta + \frac{i}{2}(\gamma_3\nabla\!\!\!/\epsilon)_\beta(\gamma_3\bar\chi)_\alpha \\
\delta\bar\eta_\alpha &= -i(\epsilon\lambda)\bar\chi_\alpha + i(\epsilon\gamma^\mu)^\beta D_\mu M_{\alpha\beta} - i\sigma_1\epsilon^\beta M_{\alpha\beta} - \sigma_2(\gamma_3\epsilon)^\beta M_{\alpha\beta} \\
&\quad + i\frac{q+1}{2}\left((\nabla_\mu\epsilon)\gamma^\mu\right)^\beta M_{\alpha\beta} + \frac{i}{2}(\nabla_\mu\epsilon\gamma_3\gamma^\mu)^\beta(\gamma_3)_\alpha{}^\rho M_{\rho\beta}\,,
\end{aligned}$$

and

$$\begin{aligned}
\delta\bar\psi_\alpha &= \left(\left[i(D_\mu\bar X)\gamma^\mu - i\sigma_1\bar X + \sigma_2\bar X\gamma_3 + i\frac{q}{2}\bar X\overleftarrow{\nabla\!\!\!/}\right]\bar\epsilon\right)_\alpha - \bar\epsilon^\beta\bar M_{\beta\alpha} + \epsilon_\alpha\bar F \qquad & \delta\bar X &= \epsilon\bar\psi + \bar\epsilon\chi \\
\delta\bar F &= \left[i\sigma_1\bar\psi - \sigma_2\bar\psi\gamma_3 - i\bar\lambda\bar X - i(D_\mu\bar\psi)\gamma^\mu - i\frac{q}{2}\bar\psi\overleftarrow{\nabla\!\!\!/} + \eta\right]\bar\epsilon & \delta\chi_\alpha &= \epsilon^\beta\bar M_{\alpha\beta} \\
\delta\bar M_{\alpha\beta} &= -\eta_\alpha\epsilon_\beta + i\sigma_1\chi_\alpha\bar\epsilon_\beta - \sigma_2\chi_\alpha(\gamma_3\bar\epsilon)_\beta - i(D_\mu\chi_\alpha)(\gamma^\mu\bar\epsilon)_\beta \\
&\quad - i\frac{q+1}{2}\chi_\alpha(\nabla\!\!\!/\bar\epsilon)_\beta + \frac{i}{2}(\gamma_3\nabla\!\!\!/\bar\epsilon)_\beta(\gamma_3\chi)_\alpha \\
\delta\eta_\alpha &= -i(\bar\epsilon\bar\lambda)\chi_\alpha + i(\bar\epsilon\gamma^\mu)^\beta D_\mu\bar M_{\alpha\beta} + i\sigma_1\bar\epsilon^\beta\bar M_{\alpha\beta} - \sigma_2(\gamma_3\bar\epsilon)^\beta\bar M_{\alpha\beta} \\
&\quad + i\frac{q+1}{2}\left((\nabla_\mu\bar\epsilon)\gamma^\mu\right)^\beta\bar M_{\alpha\beta} + \frac{i}{2}(\nabla_\mu\bar\epsilon\gamma_3\gamma^\mu)^\beta(\gamma_3)_\alpha{}^\rho\bar M_{\rho\beta}\,.
\end{aligned} \tag{3.1}$$

Here $D_\mu = \nabla_\mu - iA_\mu$ is the gauge-covariant derivative on S^2. Splitting $\delta = \delta_\epsilon + \delta_{\bar\epsilon}$ and imposing the Killing spinor equations $\nabla_\mu\epsilon = \gamma_\mu\check\epsilon$, $\nabla_\mu\bar\epsilon = \gamma_\mu\check{\bar\epsilon}$ for some spinors $\check\epsilon$, $\check{\bar\epsilon}$, one finds that the superconformal algebra is realized on semichiral fields as:

$$\begin{aligned}
[\delta_\epsilon,\,\delta_{\bar\epsilon}]X &= \xi^\mu\partial_\mu X + i\Lambda X + \frac{q}{2}\rho X + iq\alpha X\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\bar X &= \xi^\mu\partial_\mu\bar X - i\Lambda\bar X + \frac{q}{2}\rho\bar X - iq\alpha\bar X\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\psi &= \xi^\mu\partial_\mu\psi + i\Lambda\psi + \frac{q+1}{2}\rho\psi + i(q-1)\alpha\psi + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\psi + i\beta\gamma_3\psi\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\bar\psi &= \xi^\mu\partial_\mu\bar\psi - i\Lambda\bar\psi + \frac{q+1}{2}\rho\bar\psi - i(q-1)\alpha\bar\psi + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\psi - i\beta\gamma_3\bar\psi\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]F &= \xi^\mu\partial_\mu F + i\Lambda F + \frac{q+2}{2}\rho F + i(q-2)\alpha F\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\bar F &= \xi^\mu\partial_\mu\bar F - i\Lambda\bar F + \frac{q+2}{2}\rho\bar F - i(q-2)\alpha\bar F\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\bar\chi_\alpha &= \xi^\mu\partial_\mu\bar\chi_\alpha + i\Lambda\bar\chi_\alpha + \frac{q+1}{2}\rho\bar\chi_\alpha + i(q+1)\alpha\bar\chi_\alpha + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\chi_\alpha - i\beta(\gamma_3\bar\chi)_\alpha\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]\chi_\alpha &= \xi^\mu\partial_\mu\chi_\alpha - i\Lambda\chi_\alpha + \frac{q+1}{2}\rho\chi_\alpha - i(q+1)\alpha\chi_\alpha + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\chi_\alpha + i\beta(\gamma_3\chi)_\alpha\,, \\
[\delta_\epsilon,\,\delta_{\bar\epsilon}]M_{\alpha\beta} &= \xi^\mu\partial_\mu M_{\alpha\beta} + i\Lambda M_{\alpha\beta} + \frac{q+2}{2}\rho M_{\alpha\beta} + iq\alpha M_{\alpha\beta} + \frac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\rho M_{\rho\beta} \\
&\quad + \frac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\beta{}^\rho M_{\alpha\rho} - i\beta(\gamma_3)_\alpha{}^\rho M_{\rho\beta} + i\beta(\gamma_3)_\beta{}^\rho M_{\alpha\rho}\,,
\end{aligned}$$

$$
\begin{aligned}
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar M_{\alpha\beta} &= \xi^\mu \partial_\mu \bar M_{\alpha\beta} - i\Lambda \bar M_{\alpha\beta} + \frac{q+2}{2}\rho \bar M_{\alpha\beta} - iq\alpha \bar M_{\alpha\beta} + \frac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\alpha{}^\rho \bar M_{\rho\beta} \\
&\quad + \frac{1}{4}\Theta^{\mu\nu}(\gamma_{\mu\nu})_\beta{}^\rho \bar M_{\alpha\rho} + i\beta(\gamma_3)_\alpha{}^\rho \bar M_{\rho\beta} - i\beta(\gamma_3)_\beta{}^\rho \bar M_{\alpha\rho}\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\bar\eta_\alpha &= \xi^\mu \partial_\mu \bar\eta_\alpha + i\Lambda\bar\eta_\alpha + \frac{q+3}{2}\rho\bar\eta_\alpha + i(q-1)\alpha\bar\eta_\alpha + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\bar\eta_\alpha - i\beta(\gamma_3\bar\eta)_\alpha\,, \\
[\delta_\epsilon,\, \delta_{\bar\epsilon}]\eta_\alpha &= \xi^\mu \partial_\mu \eta_\alpha - i\Lambda\eta_\alpha + \frac{q+3}{2}\rho\eta_\alpha - i(q-1)\alpha\eta_\alpha + \frac{1}{4}\Theta^{\mu\nu}\gamma_{\mu\nu}\eta_\alpha + i\beta(\gamma_3\eta)_\alpha\,,
\end{aligned} \tag{3.2}
$$

where the parameters are given by

$$
\begin{aligned}
\xi_\mu &\equiv i(\bar\epsilon\gamma_\mu\epsilon)\,, & \Lambda &\equiv -\xi^\mu A_\mu + (\bar\epsilon\epsilon)\sigma_1 - i(\bar\epsilon\gamma_3\epsilon)\sigma_2\,, \\
\Theta^{\mu\nu} &\equiv D^{[\mu}\xi^{\nu]} + \xi^\rho\omega_\rho{}^{\mu\nu}\,, & \rho &\equiv \frac{i}{2}(D_\mu\bar\epsilon\gamma^\mu\epsilon + \bar\epsilon\gamma^\mu D_\mu\epsilon) = \frac{1}{2}D_\mu\xi^\mu\,, \\
\alpha &\equiv -\frac{1}{4}(D_\mu\bar\epsilon\gamma^\mu\epsilon - \bar\epsilon\gamma^\mu D_\mu\epsilon)\,, & \beta &\equiv \frac{1}{4}(D_\mu\bar\epsilon\gamma_3\gamma^\mu\epsilon - \bar\epsilon\gamma_3\gamma^\mu D_\mu\epsilon)\,,
\end{aligned}
$$

and $\omega_\rho{}^{\mu\nu}$ is the spin connection. Here ξ_μ parametrizes translations, Λ is a gauge parameter, ρ is a parameter for dilations, and α, β parametrize vector and axial R-symmetry transformations, respectively. From here one reads off the charges of the fields under these transformations, which are summarized in table 1. Note that the vector R-charge of the multiplet is twice its Weyl weight, as for chiral fields. All other commutators vanish, $[\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar\epsilon}, \delta_{\bar\epsilon}] = 0$, if one imposes the extra condition $\Box\epsilon = h\epsilon$, $\Box\bar\epsilon = h\bar\epsilon$ with the same function h.

	$\mathbb{D}_+$	$\mathbb{D}_-$	$\bar{\mathbb{D}}_+$	$\bar{\mathbb{D}}_-$	X	ψ	$\bar\chi$	F	$M_{\alpha\beta}$	$\bar\eta$
w	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{q}{2}$	$\frac{q+1}{2}$	$\frac{q+1}{2}$	$\frac{q+2}{2}$	$\frac{q+2}{2}$	$\frac{q+3}{2}$
q_V	-1	-1	1	1	q	$q-1$	$q+1$	$q-2$	q	$q-1$
q_A	1	-1	-1	1	0	1	-1	0	$-2\epsilon_{\alpha\beta}$	-1

Table 1. Weyl weight, vector and axial R-charge for the component fields of the semichiral multiplet.

There are four complex Killing spinors on S^2 satisfying $\nabla_\mu\epsilon = \pm\frac{i}{2r}\gamma_\mu\epsilon$. Restricting the transformations (3.1) to spinors $\epsilon, \bar\epsilon$ that satisfy

$$
\nabla_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon\,, \qquad \nabla_\mu\bar\epsilon = \frac{i}{2r}\gamma_\mu\bar\epsilon\,, \tag{3.3}
$$

the algebra (3.2) does not contain dilations nor axial R-rotations (i.e. $\rho = \beta = 0$). This is an SU(2|1) subalgebra of the superconformal algebra that we identify as the $\mathcal{N} = (2,2)$ SUSY on S^2 and denote it by SU$(2|1)_A$, as in [4, 5]. The transformation rules in (3.1) simplify to

$$\begin{aligned}
\delta\psi_\alpha &= \epsilon^\beta(\mathcal{P}_{\beta\alpha}X - M_{\beta\alpha}) + \bar{\epsilon}_\alpha F - \frac{q}{2r}X\epsilon_\alpha && \delta X = \bar{\epsilon}\psi + \epsilon\bar{\chi} \\
\delta F &= \epsilon^\alpha\mathcal{P}_{\alpha\beta}\psi^\beta - i(\epsilon\lambda)X + \epsilon\bar{\eta} + \frac{q}{2r}\epsilon\psi && \delta\bar{\chi}_\alpha = M_{\alpha\beta}\bar{\epsilon}^\beta \\
\delta M_{\alpha\beta} &= \epsilon^\gamma\mathcal{P}_{\gamma\beta}\bar{\chi}_\alpha - \bar{\eta}_\alpha\bar{\epsilon}_\beta + \frac{q-2}{2r}\bar{\chi}_\alpha\epsilon_\beta + \frac{2}{r}\bar{\chi}_{(\alpha}\epsilon_{\beta)} \\
\delta\bar{\eta}_\alpha &= \epsilon^\kappa\mathcal{P}_{\kappa\gamma}M_{\alpha\beta}C^{\gamma\beta} - i(\epsilon\lambda)\bar{\chi}_\alpha + \frac{q}{2r}M_{\alpha\beta}\epsilon^\beta + \frac{2}{r}M_{[\alpha\beta]}\epsilon^\beta\,,
\end{aligned}$$

and

$$\begin{aligned}
\delta\bar{\psi}_\alpha &= \bar{\epsilon}^\beta(\mathcal{P}_{\alpha\beta}\bar{X} - \bar{M}_{\beta\alpha}) + \epsilon_\alpha\bar{F} - \frac{q}{2r}\bar{X}\bar{\epsilon}_\alpha && \delta\bar{X} = \epsilon\bar{\psi} + \bar{\epsilon}\chi \\
\delta\bar{F} &= \bar{\epsilon}^\alpha\mathcal{P}_{\beta\alpha}\bar{\psi}^\beta - i(\bar{\epsilon}\bar{\lambda})\bar{X} + \bar{\epsilon}\eta + \frac{q}{2r}\bar{\epsilon}\bar{\psi} && \delta\chi_\alpha = \bar{M}_{\alpha\beta}\epsilon^\beta \\
\delta\bar{M}_{\alpha\beta} &= \bar{\epsilon}^\gamma\mathcal{P}_{\beta\gamma}\chi_\alpha - \eta_\alpha\epsilon_\beta + \frac{q-2}{2r}\chi_\alpha\bar{\epsilon}_\beta + \frac{2}{r}\chi_{(\alpha}\bar{\epsilon}_{\beta)} \\
\delta\eta_\alpha &= \bar{\epsilon}^\kappa\mathcal{P}_{\gamma\kappa}\bar{M}_{\alpha\beta}C^{\gamma\beta} - i(\bar{\epsilon}\bar{\lambda})\chi_\alpha + \frac{q}{2r}\bar{M}_{\alpha\beta}\bar{\epsilon}^\beta + \frac{2}{r}\bar{M}_{[\alpha\beta]}\bar{\epsilon}^\beta\,,
\end{aligned} \tag{3.4}$$

where $C^{\alpha\beta}$ is the antisymmetric tensor with $C^{+-} = 1$ and $[\alpha\beta]\,,\,(\alpha\beta)$ denotes (anti-) symmetrization of indices, respectively.[11] Another way to derive these transformation rules is by coupling the theory to background supergravity, along the lines of [53]. Using this method, the SUSY transformations (in the case of neutral fields) on more general Riemann surfaces are given in [52].

3.2 Supersymmetric actions

The flat-space action (2.10) is not invariant under the curved-space transformations (3.4). However, it is possible to add suitable $\frac{1}{r}$ and $\frac{1}{r^2}$ terms to obtain an invariant Lagrangian:

$$\begin{aligned}
\mathcal{L}^{S^2}_{\mathbb{X}} &= \mathcal{L}^{\mathbb{R}^2}_{\mathbb{X}} + \delta\mathcal{L}\,, \\
\delta\mathcal{L} &= \frac{iq}{r}\,\bar{X}\sigma_1 X + \frac{q(2-q)}{4r^2}\,\bar{X}X - \frac{q}{2r}\Big(\bar{\psi}\psi + \chi\bar{\chi} + \bar{X}C^{\alpha\beta}M_{\alpha\beta} + C^{\alpha\beta}\bar{M}_{\alpha\beta}X\Big)\,.
\end{aligned} \tag{3.5}$$

The first three terms in $\delta\mathcal{L}$ are the ones also needed in the case of a chiral field, while the last three terms are additional ones required for semichiral fields. The action is not only supersymmetric, it is also $\mathcal{Q}_A$-exact, namely:

$$\bar{\epsilon}\epsilon\int d^2x\,\mathcal{L}^{S^2}_{\mathbb{X}} = \delta_\epsilon\delta_{\bar{\epsilon}}\int d^2x\left(\bar{\psi}\psi + \chi\bar{\chi} - 2i\bar{X}\sigma_1 X + \frac{q-1}{r}\,\bar{X}X + \bar{X}C^{\alpha\beta}M_{\alpha\beta} + C^{\alpha\beta}\bar{M}_{\alpha\beta}X\right). \tag{3.6}$$

Thus, one can use $\mathcal{L}^{S^2}_{\mathbb{X}}$ itself for localization, which is an important simplification in evaluating the one-loop determinant using spherical harmonics.[12]

[11] Here we have written the transformations using explicit representations for the gamma matrices and properties of spinors (see appendix A.1). We find this convenient for calculations in the upcoming sections.

[12] Strictly speaking, one should use a $\mathcal{Q}_A$-exact action which is positive definite, so that one localizes to the zero-locus. $\mathcal{L}^{S^2}_{\mathbb{X}}$ has positive definite real part, provided that $0 \leq q \leq 2$.

Same representation. Let us consider first the case of a pair of semichiral fields in representation $(\mathfrak{R},\mathfrak{R})$ with the flat-space action (2.11). Since so far we treated $\bar{\mathbb{X}}$ and $\mathbb{X}$ as independent fields, we can use the result (3.5) for each individual term in (2.11) and the Lagrangian on S^2 is therefore given by

$$\begin{aligned}\mathcal{L}_{LR}^{S^2} = & D_\mu \bar{X}^i D^\mu X_i + \bar{X}^i \left(\sigma_1^2+\sigma_2^2+iD\right) X_i + \bar{F}^i F_i \\ & - \bar{M}^i_{\alpha\beta} M_i^{\beta\alpha} - \bar{X}^i \mathcal{P}_{\alpha\beta} M_i^{\alpha\beta} + \bar{M}^{\alpha\beta,i}\mathcal{P}_{\beta\alpha} X_i \\ & - i\bar{\psi}^i\gamma^\mu D_\mu \psi_i + \bar{\psi}^i \left(i\sigma_1 - \gamma_3\sigma_2\right)\psi_i + i\bar{\psi}^i\lambda X_i - i\bar{X}^i\bar{\lambda}\psi_i - \eta^i\psi_i - \bar{\psi}^i\bar{\eta}_i \\ & + i\bar{\chi}_i\gamma^\mu D_\mu\chi^i - \bar{\chi}_i\left(i\sigma_1-\gamma_3\sigma_2\right)\chi^i + i\bar{X}^i\lambda\bar{\chi}_i - i\chi^i\bar{\lambda}X_i \\ & + \frac{iq}{r}\bar{X}^i\sigma_1 X_i + \frac{q(2-q)}{4r^2}\bar{X}^i X_i - \frac{q}{2r}\left(\bar{\psi}^i\psi_i + \chi^i\bar{\chi}_i + \bar{X}^i C^{\alpha\beta} M_{i\,\alpha\beta} + C^{\alpha\beta}\bar{M}^i_{\alpha\beta}X_i\right),\end{aligned} \tag{3.7}$$

where the flavor indices $i=(L,R)$ are contracted with

$$\mathcal{M}_{\bar{\imath}j} = -\begin{pmatrix} 1 & \alpha \\ \alpha & 1\end{pmatrix}. \tag{3.8}$$

The action (3.7) is $\mathcal{Q}_A$-exact, being a sum of $\mathcal{Q}_A$-exact terms.

In the simple model with a single pair of semichiral fields and gauge group U(1), the R-charge q is unphysical, because it can be set to the canonical value $q=0$ by mixing the R-current with the gauge current (this is no longer true if we have multiple semichiral pairs charged under the same U(1)). However, we keep q for now and set it to zero only at the end of the calculation; this will reduce the number of BPS configurations to be taken into account in the localization.

Conjugate representations. Let us move to the case of a pair of semichiral fields in conjugate representations, whose flat-space Lagrangian includes off-diagonal terms of the form $\mathbb{X}_L\mathbb{X}_R$ appearing in (2.15):

$$\mathcal{L}_\beta^{\mathbb{R}^2} = \beta\int d^4\theta\left(\mathbb{X}_L\mathbb{X}_R + \bar{\mathbb{X}}_L\bar{\mathbb{X}}_R\right) = \beta\left(M^{\alpha\beta}M_{\alpha\beta} - \bar{\eta}\bar{\chi} + \bar{M}^{\alpha\beta}\bar{M}_{\alpha\beta} - \eta\chi\right). \tag{3.9}$$

This Lagrangian is actually invariant under the S^2 SUSY transformations (3.4), with no need for $\frac{1}{r}$ improving terms, i.e., $\mathcal{L}_\beta^{S^2} = \mathcal{L}_\beta^{\mathbb{R}^2}$. Furthermore, it is also $\mathcal{Q}_A$-exact, namely:

$$\begin{aligned}\bar{\epsilon}\epsilon\int d^2x\,\mathcal{L}_\beta^{S^2} = \delta_\epsilon\delta_{\bar{\epsilon}}\int d^2x\Big(& X_L M_{+-} - X_R M_{-+} + \bar{\chi}_-\psi_+^R + \psi_-^L\bar{\chi}_+ - \frac{1}{r}X_R X_L \\ & + \bar{X}_L\bar{M}_{+-} - \bar{X}_R\bar{M}_{-+} + \chi_-\bar{\psi}_+^R + \bar{\psi}_-^L\chi_+ - \frac{1}{r}\bar{X}_L\bar{X}_R\Big).\end{aligned} \tag{3.10}$$

Note that although $\frac{1}{r}$ terms appear inside the integral on the right-hand side, these are cancelled against $\frac{1}{r}$ terms coming from $\delta_\epsilon\delta_{\bar{\epsilon}}$. Summarizing, the generalization of (2.15) to S^2 is the sum of (3.7) (where flavor indices are contracted instead with $\mathcal{M}_{\bar{\imath}j} = \delta_{\bar{\imath}j}$) and (3.9).

Since a general Lagrangian for a number N_F of semichiral pairs can always be rewritten as N_F decouped copies of a single pair, the elements given above are sufficient to write the general action on S^2 for any number of semichiral multiplets.

4 Localization on the Coulomb branch

In this section, we compute the S^2 partition function of the gauge theories at hand by means of localization on the Coulomb-branch localization.

We wish to compute the path integral

$$Z_{S^2} = \int \mathcal{D}\varphi\, e^{-\mathcal{S}[\varphi]} ,$$

where φ are all fields in the theory, namely those in vector multiplets, semichiral multiplets, and possibly chiral multiplets. The action is given by

$$\mathcal{S} = \int d^2x \big(\mathcal{L}_{\text{VM}} + \mathcal{L}_{\text{FI}} + \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{semichiral}}\big) , \tag{4.1}$$

where each term is the appropriate Lagrangian on S^2 and $\mathcal{L}_{\text{FI}} = -i\xi D + \frac{i\theta}{2\pi} F_{12}$ is the standard FI term (which needs no curvature couplings on S^2). To perform the localization, we pick a Killing spinor ϵ (our choice is in (A.28), but all choices are equivalent up to rotations of S^2) and call $\mathcal{Q}_A$ the generator of supersymmetry variations along ϵ and $\bar{\epsilon} = \epsilon^c$, as in [4]. The supercharge $\mathcal{Q}_A$ generates an SU(1|1) superalgebra, that we will use for localization. Following the usual arguments [2, 3], the partition function localizes on the BPS configurations given by $\{\mathcal{Q}_A \cdot \text{fermions} = 0\}$, and it is given exactly by the one-loop determinant around such configurations. The contribution from vector and chiral multiplets was studied in [4, 5], that from twisted chiral multiplets in [6], and that from twisted vector multiplets in [13]. Here we study the contribution from semichiral multiplets.

We begin by studying the BPS equations. These follow from setting the variations of all fermions in (3.4) to zero and are analyzed in detail in appendix A.6. We show that for a generic value of $q \neq 0$, the only smooth solution is

$$X = \bar{X} = F = \bar{F} = M_{\alpha\beta} = \bar{M}_{\alpha\beta} = 0 . \tag{4.2}$$

Thus, like in the case of chiral multiplets, the BPS configuration for generic q is only the trivial one. The BPS configurations for the vector multiplet are given by [4, 5]

$$0 = F_{12} - \frac{\sigma_2}{r} = D + \frac{\sigma_1}{r} = D_\mu \sigma_1 = D_\mu \sigma_2 = [\sigma_1, \sigma_2] . \tag{4.3}$$

Flux quantization of F_{12} implies that, up to gauge transformations, $\sigma_2 = \frac{\mathfrak{m}}{2r}$ where $\mathfrak{m}$ is a co-weight (i.e. $\mathfrak{m}$ belongs to the Cartan subalgebra of the gauge group algebra and $\rho(\mathfrak{m}) \in \mathbb{Z}$ for any weight ρ of any representation $\mathfrak{R}$). Thus, the set of BPS configurations are parametrized by the continuous variable σ_1 and the discrete fluxes $\mathfrak{m}$.

As shown in the previous section, the kinetic actions for semichiral fields are $\mathcal{Q}_A$-exact. Thus, we can use the kinetic actions themselves as a deformation term for localization and we should compute the one-loop determinants arising from those actions. We now compute the determinant for semichiral fields in the same gauge representation, as well as in conjugate representations; the determinants coincide, as they should, since such theories are related by a simple change of variables.

4.1 One-loop determinants

Same representation. Consider the Lagrangian (3.7) and expand it at quadratic order around the BPS background. Let us look at bosonic fields first, using the basis

$$\mathcal{X} = \left(X^L, X^R, M^L_{-+}, M^L_{--}, M^R_{++}, M^R_{+-}, F^L, F^R\right)^{\mathsf{T}} .$$

The bosonic part of the quadratic action is given by $\bar{\mathcal{X}}\mathcal{O}_B\mathcal{X}$, where $\mathcal{O}_B$ is the 8×8 operator

$$\mathcal{O}_B = \begin{pmatrix} \mathcal{O}_X & \alpha\,\mathcal{O}_X & \frac{q}{2r}-\sigma & 2iD_{++} & -\alpha\,2iD_{--} & -\alpha\left(\frac{q}{2r}+\bar\sigma\right) & 0 & 0 \\ \alpha\,\mathcal{O}_X & \mathcal{O}_X & \alpha\left(\frac{q}{2r}-\sigma\right) & \alpha\,2iD_{++} & -2iD_{--} & -\left(\frac{q}{2r}+\bar\sigma\right) & 0 & 0 \\ \alpha\left(\sigma-\frac{q}{2r}\right) & -\frac{q}{2r}+\sigma & 0 & 0 & 0 & -1 & 0 & 0 \\ \alpha\,2iD_{--} & 2iD_{--} & 0 & \alpha & 0 & 0 & 0 & 0 \\ -2iD_{++} & -\alpha\,2iD_{++} & 0 & 0 & \alpha & 0 & 0 & 0 \\ \frac{q}{2r}+\bar\sigma & \alpha\left(\frac{q}{2r}+\bar\sigma\right) & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \end{pmatrix} \tag{4.4}$$

and

$$\mathcal{O}_X = -\Box + \sigma_1^2 + \sigma_2^2 + i\frac{(q-1)\sigma_1}{r} + \frac{q(2-q)}{4r^2} . \tag{4.5}$$

All appearances of σ_1, σ_2 in this matrix (and all matrices below) are to be understood as $\rho(\sigma_1)$, $\rho(\sigma_2)$, but we omit this to avoid cluttering the matrices; we will reinstate them in the expressions for the determinants below. The analysis of the eigenvalues of (4.4) contains different cases, depending on the angular momentum j on S^2. Assuming $\alpha \neq 0$, and putting all cases together, the determinant in the bosonic sector is given by (see appendix C):

$$\begin{aligned} \operatorname{Det}\mathcal{O}_B = \prod_{\rho\in\mathfrak{R}} \frac{\alpha^{|\rho(\mathfrak{m})|-1}}{\alpha^{|\rho(\mathfrak{m})|+1}} \prod_{j=\frac{|\rho(\mathfrak{m})|}{2}}^{\infty} \left[j^2 + \frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4} - \alpha^2\left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2 \right]^{2j+1} \times \\ \times \left[(j+1)^2 + \frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4} - \alpha^2\left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2 \right]^{2j+1} \left(\frac{(\alpha^2-1)^2}{r^4}\right)^{2j+1} . \end{aligned} \tag{4.6}$$

Now we turn to the fermionic determinant. In the basis

$$\Psi = \left(\psi^{L+}, \psi^{L-}, \psi^{R+}, \psi^{R-}, \bar\eta^{L+}, \bar\eta^{R-}, \bar\chi^{L+}, \bar\chi^{R-}\right)^{\mathsf{T}} ,$$

the quadratic action for fermions is $\bar{\Psi}\mathcal{O}_F\Psi$ where $\mathcal{O}_F$ reads

$$\begin{pmatrix} -\left(\frac{q}{2r}+\bar{\sigma}\right) & 2iD_{--} & -\alpha\left(\frac{q}{2r}+\bar{\sigma}\right) & 2i\alpha D_{--} & -1 & 0 & 0 & 0 \\ -2iD_{++} & \frac{q}{2r}-\sigma & -2i\alpha D_{++} & \alpha\left(\frac{q}{2r}-\sigma\right) & 0 & \alpha & 0 & 0 \\ -\alpha\left(\frac{q}{2r}+\bar{\sigma}\right) & 2i\alpha D_{--} & -\left(\frac{q}{2r}+\bar{\sigma}\right) & 2iD_{--} & -\alpha & 0 & 0 & 0 \\ -2i\alpha D_{++} & \alpha\left(\frac{q}{2r}-\sigma\right) & -2iD_{++} & \frac{q}{2r}-\sigma & 0 & 1 & 0 & 0 \\ -\alpha & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha\left(\frac{q}{2r}-\sigma\right) & -2iD_{--} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2iD_{++} & \alpha\left(\frac{q}{2r}+\bar{\sigma}\right) \end{pmatrix}.$$

An analysis of the eigenvalues of this operator gives the determinant

$$\begin{aligned} \text{Det}\,\mathcal{O}_F = & \prod_{\rho\in\mathfrak{R}} \left(\frac{(1-\alpha^2)\alpha^2}{r^2}\right)^{|\rho(\mathfrak{m})|} \left[\frac{\rho(\mathfrak{m})^2}{4} - \left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2\right]^{|\rho(\mathfrak{m})|} \times \\ & \times \prod_{j=\frac{|\rho(\mathfrak{m})|+1}{2}}^{\infty} \left(\frac{\alpha^2-1}{r^2}\right)^{4j+2} \left[\left(j+\frac{1}{2}\right)^2 + \frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4} - \alpha^2\left(\frac{q}{2}-ir\rho(\sigma_1)\right)^2\right]^{4j+2}. \end{aligned} \tag{4.7}$$

Bringing the bosonic and fermionic determinants together leads to many cancellations and the final result is

$$Z_{LR} = \frac{\text{Det}\,\mathcal{O}_F}{\text{Det}\,\mathcal{O}_B} = \prod_{\rho\in\mathfrak{R}} \frac{(-1)^{|\rho(\mathfrak{m})|}}{\frac{\rho(\mathfrak{m})^2}{4} - \left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2}. \tag{4.8}$$

Note that in this expression, the dependence on the parameter α has cancelled, as expected from the fact that this parameter appears in a $\mathcal{Q}_A$-exact term.[13]

This expression has simple poles at $\rho(\sigma_1) = -\frac{i}{2r}(q \pm \rho(\mathfrak{m}))$ for $\rho(\mathfrak{m}) \neq 0$, and a double pole at $\rho(\sigma_1) = -\frac{iq}{2r}$ for $\rho(\mathfrak{m}) = 0$. Using properties of the Γ-function, (4.8) can be written as

$$Z_{LR} = \prod_{\rho\in\mathfrak{R}} \frac{\Gamma\left(\frac{q}{2} - ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1-\frac{q}{2} + ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)} \cdot \frac{\Gamma\left(-\frac{q}{2} + ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1+\frac{q}{2} - ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)}. \tag{4.9}$$

The radius r can be reabsorbed into σ_1, making it into a dimensionless variable. In fact (4.9) coincides with the one-loop determinant for two chiral fields in conjugate representations of the gauge group, opposite R-charges, and no twisted mass parameter turned on. Each Γ-function in the numerator has an infinite tower of poles, most of which cancel against the poles of the denominator (such a cancellation does not occur for a pair of chiral multiplets with generic twisted masses and R-charges).

[13] In fact, here we have dropped overall factors of r^2 and $(\alpha^2 - 1)$, which can be reabsorbed in the path-integral measure.

Conjugate representations. Here we compute the one-loop determinant for semichiral fields in representation $(\mathfrak{R}, \overline{\mathfrak{R}})$. The flat-space Lagrangian is the semichiral dual to the one in the previous section (see appendix B.2). To place it on S^2 we take (3.7), where the indices are contracted appropriately, and the off-diagonal term (3.9). To second order in the fluctuations, the bosonic action is $\bar{\mathcal{X}}\widetilde{\mathcal{O}}_B\mathcal{X}$ where

$$\mathcal{X} = \left(X^L, \bar{X}^R, M^L_{-+}, M^L_{--}, \bar{M}^R_{++}, \bar{M}^R_{+-}, F^L, \bar{F}^R\right)^\mathsf{T}$$

and

$$\widetilde{\mathcal{O}}_B = \begin{pmatrix} \alpha_s\mathcal{O}_X^{(q)} & 0 & \alpha_s\left(\frac{q}{2r}-\sigma\right) & \alpha_s 2iD_{++} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{O}_X^{(-q)} & 0 & 0 & -2iD_{--} & \frac{q}{2r}-\sigma & 0 & 0 \\ 0 & \frac{q}{2r}+\bar{\sigma} & -\alpha & 0 & 0 & -1 & 0 & 0 \\ 0 & 2iD_{--} & 0 & \alpha & 0 & 0 & 0 & 0 \\ -\alpha_s 2iD_{++} & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ \alpha_s\left(\frac{q}{2r}+\bar{\sigma}\right) & 0 & -\alpha_s & 0 & 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

in which we defined $\alpha_s = \alpha^2 - 1$ and $\mathcal{O}_X^{(q)}$ is the operator in (4.5) using R-charge q. The second-order fermionic action is $\bar{\Psi}\widetilde{\mathcal{O}}_F\Psi$ with

$$\Psi = \left(\psi^{L+}, \psi^{L-}, \bar{\psi}^{R+}, \bar{\psi}^{R-}, \bar{\eta}^{L+}, \eta^{R-}, \bar{\chi}^{L+}, \chi^{R-}\right)^\mathsf{T}$$

and $\widetilde{\mathcal{O}}_F$ equals

$$\begin{pmatrix} -\alpha_s\left(\frac{q}{2r}+\bar{\sigma}\right) & \alpha_s 2iD_{--} & 0 & 0 & -\alpha_s & 0 & 0 & 0 \\ -\alpha_s 2iD_{++} & \alpha_s\left(\frac{q}{2r}-\sigma\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{q}{2r}-\sigma & 2iD_{--} & 0 & 0 & 0 & 0 \\ 0 & 0 & -2iD_{++} & -\left(\frac{q}{2r}+\bar{\sigma}\right) & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \alpha & 0 \\ 0 & \alpha_s & 0 & 0 & 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & -2iD_{--} \\ 0 & 0 & 0 & 0 & 0 & -\alpha & \alpha_s 2iD_{++} & 0 \end{pmatrix}.$$

Evaluating the determinants, we again find (4.9), as expected.

4.2 Integration contour and instanton corrections

The full partition function requires integration over the zero-mode σ_1, as well as summation over the flux sectors $\mathfrak{m}$ (which are co-weights of the gauge group):

$$Z_{LR}^{S^2} = \frac{1}{|\mathcal{W}|}\sum_{\mathfrak{m}}\int\frac{d\sigma_1}{2\pi}\, e^{-4\pi i\xi\,\mathrm{Tr}\,\sigma_1 - i\theta\,\mathrm{Tr}\,\mathfrak{m}}\, Z_{\text{gauge}}\, Z_{LR}\,, \tag{4.10}$$

where Z_{LR} is given in (4.9), $|\mathcal{W}|$ is the order of the Weyl group, and Z_{gauge} is the contribution from the vector multiplet given in [4, 5]. The integral is over some contour in the complex plane which needs to be specified, as we are going to discuss.

Let us make our point through a concrete example. Consider N_F pairs of semichiral fields coupled to a U(1) gauge field, each pair having charges $(1,-1)$ and R-charges $(q,-q)$. From now on we set the radius $r=1$ to avoid cluttering the formulæ. We can simply take the result (4.9) for each pair and obtain

$$Z_{LR}^{S^2}(\xi,\theta)=\sum_{\mathfrak{m}}\int\frac{d\sigma_1}{2\pi}e^{-4\pi i\xi\sigma_1-i\theta\mathfrak{m}}\left(\frac{\Gamma\left(\frac{q}{2}-i\sigma_1-\frac{\mathfrak{m}}{2}\right)}{\Gamma\left(1-\frac{q}{2}+i\sigma_1-\frac{\mathfrak{m}}{2}\right)}\cdot\frac{\Gamma\left(-\frac{q}{2}+i\sigma_1+\frac{\mathfrak{m}}{2}\right)}{\Gamma\left(1+\frac{q}{2}-i\sigma_1+\frac{\mathfrak{m}}{2}\right)}\right)^{N_F}.$$

We have not included any twisted mass. As we have discussed, the integrand coincides with that of N_F chiral fields of charge 1 and N_F chiral fields of charge -1. It has poles of order N_F at $\sigma_1=-\frac{i}{2}(q\pm\mathfrak{m})$ for $\mathfrak{m}\neq 0$, and a pole of order $2N_F$ at $\sigma_1=-\frac{i}{2}q$ for $\mathfrak{m}=0$. If the R-charge is set to $q=0$, in the $\mathfrak{m}=0$ sector one encounters a pole on the real line. Unless this pole is avoided by the integration contour, it leads to a divergence; such a divergence is physical, as it is a manifestation of the non-compactness of the target space. To avoid the divergence and extract some useful information, the contour prescription should be modified.

The way this is dealt with in the case of chiral fields is to introduce twisted masses. For instance, in the case $N_F=1$ of two chiral fields Φ, $\tilde{\Phi}$ there is a $\mathrm{U}(1)_g\times\mathrm{U}(1)_F$ symmetry: the first $\mathrm{U}(1)_g$ is gauged, while the second remains as a global flavor symmetry. One can turn on a twisted mass $\widetilde{M}$ associated to $\mathrm{U}(1)_F$ which splits the double pole for $\mathfrak{m}=0$ into two poles, one above the real axis and one below it. Integrating along the real line (i.e., going through the split poles for $\mathfrak{m}=0$) leads to a finite result, regulated by $\widetilde{M}$. From the point of view of the low-energy NLSM, a twisted mass corresponds to a quadratic potential on the target which effectively compactifies the model.

Unfortunately, in the semichiral case this resolution is not possible. As we have discussed, the only U(1) symmetry in these models is the one being gauged[14] — the would-be flavour symmetry $\mathrm{U}(1)_F$ is broken by the off-diagonal terms in (2.11) or (2.15). Thus, we are forced to choose a different contour prescription to avoid the singularity. One possible choice is to simply go around the double pole for $\mathfrak{m}=0$, either above or below it, avoiding the classical (divergent) contribution; this is the contour prescription we choose.

From the point of view of the large-volume NLSM, at least in the Abelian case, the prescription can be interpreted as follows. Going back to the model with chiral fields, suppose that $\xi<0$ which selects a particular large-volume limit. One can rewrite the integral along the real line as a sum of the residues at the poles in the upper half-plane. Each residue can be interpreted as the contribution from a different instanton sector to the NLSM path-integral [22, 23, 54], and in particular the residue at the pole collapsing with its partner (as $\widetilde{M}\to 0$) reproduces the classical and one-loop contribution, divergent in the limit. We choose, instead, a contour that goes above the collapsing pair of poles: this essentially misses the collapsing pole, and therefore it computes the NLSM path-integral

[14] As discussed in [27], there is a way to introduce additional mass parameters in these theories using the SVM. This is achieved by giving the chiral field strength (of R-charge 2) in the SVM a VEV, but since this field has a non-zero R-charge, this breaks A-type supersymmetry on the sphere.

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without the zero-instanton contribution. This makes sense because the space of NLSM field configurations is the disconnected sum of instanton sectors, therefore it is a well-defined operation to remove one of them from the path-integral. Had we considered the large-volume limit at $\xi > 0$, we would have taken the contour that goes below the collapsing pair of poles. Notice that the instanton corrections captured by this prescription are identical to the ones in the corresponding Kähler case.

Performing the contour integral in the simple Abelian model above, with $\xi < 0$, using Cauchy's integral formula and summing over $\mathfrak{m}$, one finds

$$
\begin{aligned}
Z^{\text{inst}}_{N_F}(\xi,\theta) = \sum_{j=0}^{N_F-1} \frac{(2N_F-2-j)!}{(N_F-1)!^2} \binom{N_F-1}{j} (-4\pi\xi)^j \times \\
\times \left[\mathrm{Li}_{2N_F-1-j}\left((-1)^{N_F} e^{i\theta+2\pi\xi}\right) + \mathrm{Li}_{2N_F-1-j}\left((-1)^{N_F} e^{-i\theta+2\pi\xi}\right)\right] \ , \quad (4.11)
\end{aligned}
$$

where we have set $q = 0$ after performing the integral.

A case of particular interest is $N_F = 2$ $\big($or $N_F = 1$, and in addition two chiral fields with gauge charges $(1,-1)\big)$, for which the target space has complex dimension three. As we discuss in section 5.2, this gauge theory describes a conifold with a generalized Kähler structure. The contribution due to instantons reads

$$
Z^{\text{inst}}_{\text{conifold}}(\xi,\theta) = 2\left(\mathrm{Li}_3(e^{2\pi\xi+i\theta}) + \mathrm{Li}_3(e^{2\pi\xi-i\theta})\right) - 4\pi\xi\left(\mathrm{Li}_2(e^{2\pi\xi+i\theta}) + \mathrm{Li}_2(e^{2\pi\xi-i\theta})\right) . \quad (4.12)
$$

We discuss instanton corrections from the point of view of the NLSM in section 5, but before that we make a comment on gauge theories with other vector multiplets.

4.3 Coupling to other vector multiplets

As mentioned earlier, a salient feature of semichiral fields is that they can couple minimally to various vector multiplets. So far we have studied the coupling to the usual vector multiplet, but they may also couple to the twisted vector and to the SVM. For the SVM one can define two gauge-invariant field strengths $(\mathbb{F}, \tilde{\mathbb{F}})$, which are chiral and twisted chiral, respectively (see appendix A.3 for a brief overview). In the Abelian case, a gauge-invariant action in flat space is:

$$
\begin{aligned}
\mathcal{L}_{\text{SVM},\, LR} = -\frac{1}{2e^2}\int d^4\theta \left(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}\right) + \left(i\int d^2\theta\, s\,\mathbb{F} + c.c.\right) + \left(i\int d^2\tilde{\theta}\, t\,\tilde{\mathbb{F}} + c.c.\right) \\
+ \sum_{i=1}^{N_F}\int d^4\theta \left[\bar{\mathbb{X}}^i_L e^{Q_i V_L}\mathbb{X}^i_L + \bar{\mathbb{X}}^i_R e^{Q_i V_R}\mathbb{X}^i_R + \beta_i\left(\mathbb{X}^i_L e^{-iQ_i\mathbb{V}}\mathbb{X}^i_R + c.c.\right)\right] . \quad (4.13)
\end{aligned}
$$

The first line is the kinetic action for the SVM and s, t are two complex FI parameters. The Higgs branch has complex dimension $2N_F - 2$ and for generic value of the parameters β_i the geometry is generalized Kähler [27]. For a special choice of the parameters the geometry becomes hyperkähler [32].

Now, we can promote s in (4.13) to a chiral field Φ:

$$
s \quad \to \quad \Phi \, .
$$

As discussed in [27], by doing so Φ acts as a Lagrange multiplier imposing $\mathbb{F} = 0$ and by choosing an appropriate SVM gauge, the action reduces to (2.16), namely the action of semichiral fields coupled to the usual vector multiplet. From the superspace point of view, this is a more natural way to formulate the gauge theories (2.16). For our purposes here, however, it is simpler to derive the couplings to curvature when working in the formulation (2.16). Note that the theory in (4.13) and the one with dynamical Φ only differ by a superpotential term, moreover the Lagrange multiplier must be neutral and with R-charge 0. It follows that the partition functions in $\mathcal{Q}_A$-localization for the two Abelian theories coincide.

5 Sigma models on generalized Kähler manifolds

The GLSMs discussed in this paper realize NLSMs on generalized Kähler manifolds. In this section we discuss instanton corrections from the point of view of the NLSM on the generalized Kähler space, and comment on possible interpretations for the GLSM partition function. We first review some known results on topologically twisted sigma models with Kähler and generalized Kähler target spaces, and then discuss some interesting issues that the localization computation raises for the models at hand.

5.1 Topologically twisted theories

The topological twist is a powerful method to obtain exact results about specific sectors in supersymmetric theories [2]. After a topological twist, a linear combination of the supercharges becomes a scalar charge $\mathcal{Q}_{\mathrm{BRST}}$, referred to as the BRST operator. In the case of nonlinear sigma models with a Kähler target space, the classical action of the NLSM can be written as the sum of a BRST-exact term and a topological term [3]. The path-integral localizes on fixed points of $\mathcal{Q}_{\mathrm{BRST}}$ and therefore the partition function is given by a sum over such configurations, with a weight given by the exponential of the topological term. In the A-model, the fixed points of $\mathcal{Q}_{\mathrm{BRST}}$ are holomorphic maps, while in the B-model they are constant maps.

The topologically twisted versions of sigma models with a generalized Kähler target space were first studied in [44] (see [55–57] for further discussions). A BRST operator can be constructed and its fixed points are given (in the case of an A-twist) by configurations satisfying

$$\frac{1}{2}\big(1 - iJ_+\big)\,\bar\partial X = 0\,, \qquad \frac{1}{2}\big(1 + iJ_-\big)\,\partial X = 0\,, \tag{5.1}$$

where X are *real* coordinates on the target space and target space indices have been omitted. These equations are a generalization of both the usual holomorphic maps of the ordinary A-model and the constant maps of the ordinary B-model.

As in the Kähler case, it is natural to expect that the classical action can be written as a sum of a BRST-exact term and a topological term. This was proven when $[J_+, J_-] = 0$, and conjectured to also hold when $[J_+, J_-] \neq 0$ [58].

Here we wish to focus on the space of solutions to (5.1). As discussed in [44], for generic $J_\pm$, and at a generic point on the manifold, these equations are more restrictive than the ordinary holomorphic map condition: in fact the only solutions are constant maps and non-trivial instanton solutions are not possible. However, if $J_+ = J_-$ then the two equations in (5.1) become complex conjugates to each other and they reduce to the standard holomorphic map condition, allowing for non-trivial instanton solutions. Thus, instanton corrections can only arise from (compact) submanifolds on which the pull-back of $\omega \equiv g(J_+ - J_-)$ is degenerate.[15]

We now discuss in more detail the generalized Kähler structure realized by the GLSMs discussed in a particular example.

5.2 Example: a generalized Kähler metric on the conifold

Consider a pair of semichiral fields and a pair of chiral fields $(\mathbb{X}_L, \mathbb{X}_R, \Phi_1, \Phi_2)$, charged under a U(1) vector multiplet with charges $(1, -1, 1, -1)$, respectively.[16] The kinetic Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_{\text{VM}} + \int d^4\theta \Big(\bar{\mathbb{X}}_L \mathbb{X}_L + \bar{\mathbb{X}}_R \mathbb{X}_R + \beta \big(\mathbb{X}_L \mathbb{X}_R + \bar{\mathbb{X}}_L \bar{\mathbb{X}}_R \big) + \bar{\Phi}_1 \Phi_1 + \bar{\Phi}_2 \Phi_2 \Big) + \frac{i}{2} \int d^2\tilde{\theta}\, t\, \Sigma + c.c.\,, \tag{5.2}$$

where $\mathcal{L}_{\text{VM}}$ is given in (2.17), $t = i\xi + \frac{\theta}{2\pi}$ and we take $\xi > 0$. As discussed in section 2.4 (and including the usual contribution from chiral fields), the space of vacua of this theory is given by solutions to

$$|X_L|^2 + |\phi_1|^2 - |X_R|^2 - |\phi_2|^2 = \xi\,, \tag{5.3}$$

modulo U(1) gauge transformations. As a quotient space, this is the description of the resolved conifold, a $\mathbb{C}^2$ bundle over $\mathbb{CP}^1$ [59]. For $\xi = 0$ the space has a conical singularity at the origin (the tip of the cone), but for finite ξ the singularity is blown-up to an S^2 of radius $|\xi|^{\frac{1}{2}}$. For $\xi > 0$, the S^2 at the tip is given by setting $X_R = \phi_2 = 0$. Gauge-invariant combinations are

$$\mathbb{X}'_L = \frac{\mathbb{X}_L}{\Phi_1}\,, \qquad \mathbb{X}'_R = \Phi_1 \mathbb{X}_R\,, \qquad \Phi = \Phi_1 \Phi_2\,. \tag{5.4}$$

These are good coordinates on the target in the patch $\Phi_1 \neq 0$.

It is well known that this space admits a Kähler metric (in fact, a Calabi-Yau metric) and can be realized by a GLSM with four chiral fields with charges $(1, 1, -1, -1)$ [60]. In the description (5.2) this corresponds to the limit $\beta \to \infty$.

The genus-zero partition function of the topological string on the conifold was computed in [61].[17] In addition to the classical contribution it contains nontrivial instanton corrections, which arise from multi-coverings of the worldsheet onto the S^2 at the tip. In supersymmetric localization these are captured by (4.12).

[15] Similarly, in the case of a B-twist instanton corrections arise only from compact submanifolds on which $g(J_+ + J_-)$ is degenerate.

[16] One may consider a GLSM involving semichiral fields only. This model is discussed in more detail in [27] but requires the introduction of a new vector multiplet and calculations are less straightforward. Nonetheless, the discussion below also applies to those models.

[17] For a review of the results for higher genus contributions and the interesting relation to 3d Chern-Simons theory, see [62, 63] and references therein.

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As mentioned in the introduction, the full geometric data is encoded in the NLSM's $\mathcal{N} = (2,2)$ superspace Lagrangian K — the generalized Kähler potential. This function can be easily computed in the UV by classically integrating out the vector multiplet V. In the limit $e \to \infty$, the equation of motion for the vector multiplet is simply a quadratic equation for e^V. Solving it and plugging e^V back into the action (5.2), one obtains the generalized Kähler potential

$$K_{\rm UV} = \sqrt{\xi^2 + 4\,r^2} - \xi\,\log\left(\frac{\xi + \sqrt{\xi^2+4\,r^2}}{1+|\mathbb{X}_L|^2}\right) + \beta\big(\mathbb{X}_L\mathbb{X}_R + \bar{\mathbb{X}}_L\bar{\mathbb{X}}_R\big)\,, \tag{5.5}$$

where we have written the potential in terms of the gauge-invariant coordinates (5.4) (and dropped the primes) and defined the radial coordinate $r^2 \equiv \big(1+|\mathbb{X}_L|^2\big)\big(|\mathbb{X}_R|^2 + |\Phi|^2\big)$. Both square roots in (5.5) should be taken as the positive root, since we have chosen a particular branch of the solution corresponding to a real e^V. The subscript UV reminds us that, although this is a description of the gauge theory at low energies with respect to the gauge coupling e, it is still at high energies from the point of view of the NLSM: such a NLSM is not conformal and will continue to flow towards the IR.

Using the formulæ in appendix D.1, one can compute the metric $g_{\rm UV}$ and the NS-NS three-form field $H_{\rm UV}$ from the generalized potential (5.5). Since we do not expect $g_{\rm UV}, H_{UV}$ to solve the supergravity equations,[18] we are not particularly interested in their explicit expression, apart from the fact that for finite β one finds $H_{UV} \neq 0$ and $J_+ \neq J_-$ and therefore the target in the UV is generalized Kähler. In the special limit $\beta \to \infty$, the H-field vanishes and one recovers the Kähler case, as expected. The full expression for $H_{\rm UV}$ is rather lengthy, but can be computed explicitly.

We now wish to address the issue of instanton corrections. Since these are independent of the RG scale, one may compute them in the UV NLSM, without knowledge of the IR metric and flux. In the Kähler case these arise from multi-coverings of the worldsheet onto the blown-up S^2, which in our coordinates corresponds to setting $X_R = \phi = 0$.

As discussed above, nontrivial solutions to (5.1) can arise only from submanifolds on which $(J_+ - J_-)$ is degenerate. We now investigate whether this model contains such submanifolds, of complex dimension one which may harbor nontrivial instanton corrections to Z_{S^2}. One possibility is the submanifold $X_R = X_L = 0$ which, however, in our model is non-compact and therefore cannot harbor nontrivial instanton corrections (moreover, this submanifold has nothing to do with the S^2 that hosts the instantons in the $\beta \to \infty$ limit). This seems to be at odds with the supersymmetric localization result (4.12).

Another possibility relates to the interesting phenomenon of type-change in generalized Kähler manifolds [36, 64, 65]. This occurs when $J_+ - J_-$ (or $J_+ + J_-$) develops a *new* zero eigenvalue on a certain locus, signaling that on that locus a pair of semichiral fields has turned into a pair of chiral (or twisted chiral) fields (see discussion in appendix D.2).

[18]These can be written as a single differential equation for K [47], which in the case of precisely one pair of semichiral fields and a chiral field was given in [43].

Let us investigate the possibility of type-change in the model (5.5). Using the formula (D.6), the eigenvalues of $\frac{1}{2}(J_+ - J_-)$ are $\{0, \pm i\lambda\}$ (each with multiplicity two) with

$$\lambda^2 = -\frac{\xi\Big(|X_L|^2|\phi|^2 - |X_R|^2\Big) - \Big(|X_R|^2 + \big(2+|X_L|^2\big)|\phi|^2\Big)\sqrt{\xi^2+4r^2}}{2r^2\beta^2\left(\sqrt{\xi^2+4r^2} + \frac{1}{\beta}\big(X_L X_R + \bar{X}_L\bar{X}_R\big)\right)} . \tag{5.6}$$

The only locus on which the numerator vanishes is $\phi = X_R = 0$. However in this case $r = 0$ and the denominator vanishes as well. Taking the limit one finds $\lambda^2 \to \frac{1}{\beta^2(1+|X_L|^2)}$. Thus, for finite β there are no points in the patch under consideration where $\lambda = 0$; a similar analysis in the other patch $\Phi_2 \neq 0$ leads to the conclusion that there are no type-change loci for $J_+ - J_-$. One can then check that ω is also non-degenerate since the metric is well defined at the tip.

This result is a bit surprising. On the one hand, the absence of compact submanifolds where $J_+ - J_-$ is degenerate seems to imply that the only finite-action solutions to (5.1) are constant maps, and therefore that the partition function does not receive instanton corrections. This is the expectation, for instance, in [44]. On the other hand, the partition function computed in (4.12) seems to represent instanton corrections (which are identical to those of the Kähler model in the $\beta \to \infty$ limit). Although we do not have a clear resolution of this puzzle, that we leave as an open question, we propose the following possibility: that the partition function, computed with our contour prescription, captures complexified solutions to the equations (5.1) for the fields in Euclidean signature. Another possibility that we cannot exclude, though, is that — because of the unavoidable divergences — the partition function computed in this paper is not a well-defined object.

6 Discussion

In this paper we have studied a class of two-dimensional GLSMs with a gauged Wess-Zumino term and off-shell $\mathcal{N} = (2,2)$ SUSY. These involve chiral and semichiral fields coupled to the usual vector multiplet, and are described at low energies by NLSMs on generalized Kähler manifolds, as opposed to the more standard case of Kähler manifolds when there is no gauged Wess-Zumino term. As we have shown, the GLSMs can be placed on the round S^2 (with the untwisted background of [4, 5]) while preserving all supercharges [52], and we have explicitly constructed their actions. We have also shown that the parameters controlling the gauged Wess-Zumino coupling enter in $\mathcal{Q}_A$-exact action terms. Thus, localization should be insensitive to the non-Kähler deformation, which we have verified explicitly. Unfortunately, these theories do not admit enough twisted masses to remove all massless modes, and their partition functions are inherently divergent. We have computed the partition functions by means of supersymmetric localization on the Coulomb branch, and proposed a contour prescription to remove the singularities.

In principle, the techniques described here provide a method for computing the partition function of NLSMs on certain generalized Kähler manifolds, for which currently no other method exists. However, as discussed, this approach raises some puzzles which we have not fully resolved. Although we discussed possible resolutions, this certainly deserves further study.

As a simple but interesting example illustrating our point we have considered a GLSM realizing a one-parameter family of generalized Kähler structures on the conifold. Although the generalized holomorphic equations (5.1) of the A/B-model do not admit real, compact, solutions in this case (apart from constant maps), the partition function as computed by localization does seem to exhibit instanton contributions. This raises the question of how to reconcile the two statements. We hope that our observation can thrust some progress in the study of these topological models.

We should mention that semichiral fields can be T-dualized to a chiral plus twisted chiral field [30, 66]; it would be very interesting if supersymmetric localization could shed light into aspects of generalized mirror symmetry, especially non-perturbative ones. Regarding possible extensions, it may be interesting to apply localization techniques to GLSMs realizing generalized Kähler manifolds without semichiral fields, which are constructed by coupling chiral and twisted chiral fields to the Large Vector Multiplet [29, 51]. These models can realize compact manifolds with an H-field as well as noncompact ones.

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A $\mathcal{N} = (2,2)$ supersymmetry

A.1 Conventions for spinors in Euclidean space

Here we give our conventions for spinors in Euclidean signature and some useful identities. We use anticommuting Dirac spinors, and contract them as

$$\epsilon\lambda \equiv \epsilon^{\alpha}\lambda_{\alpha} \,, \qquad \epsilon\gamma^{\mu}\lambda \equiv \epsilon^{\alpha}(\gamma^{\mu})_{\alpha}{}^{\beta}\lambda_{\beta} \,, \tag{A.1}$$

where the spinors have components labelled as $\lambda_\alpha = \binom{\lambda_+}{\lambda_-}$ (we take lower index to denote a "column vector") and the gamma matrices with the above index structure read

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \gamma^3 = -i\gamma^1\gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.2}$$

The spinor indices can be raised and lowered using the antisymmetric tensors $C^{\alpha\beta}$ and $C_{\alpha\beta}$, respectively, with $C^{+-} = C_{-+} = 1$. For instance:

$$\lambda^\alpha = C^{\alpha\beta}\lambda_\beta , \quad \lambda_\alpha = C_{\alpha\beta}\lambda^\beta \qquad \Rightarrow \qquad \lambda^+ = \lambda_- , \quad \lambda^- = -\lambda_+ . \tag{A.3}$$

A.2 Supersymmetry on $\mathbb{R}^2$

The algebra of $\mathcal{N} = (2, 2)$ spinor derivatives on flat space is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = \pm 2i\partial_{\pm\pm} \tag{A.4}$$

where $\partial_{\pm\pm} = \frac{1}{2}\big(\partial_1 \mp i\partial_2\big)$ are spacetime derivatives and $\Box = 2\{\partial_{++}, \partial_{--}\}$, while all other anticommutators vanish. They can be written in terms of spinor coordinates $\theta^\pm$, $\bar\theta^\pm$ as

$$\mathbb{D}_\pm = \frac{\partial}{\partial\theta^\pm} \pm i\bar\theta^\pm\partial_{\pm\pm} , \qquad \bar{\mathbb{D}}_\pm = \frac{\partial}{\partial\bar\theta^\pm} \pm i\theta^\pm\partial_{\pm\pm} . \tag{A.5}$$

We will use the notation

$$M\big| \equiv M\big|_{\theta^\pm=\bar\theta^\pm=0} \tag{A.6}$$

for the bottom component of a multiplet. The SUSY transformations are generated by

$$\delta = \bar\epsilon\,\mathbb{Q} + \epsilon\,\bar{\mathbb{Q}} = \bar\epsilon^+\mathbb{Q}_+ + \bar\epsilon^-\mathbb{Q}_- + \epsilon^+\bar{\mathbb{Q}}_+ + \epsilon^-\bar{\mathbb{Q}}_- . \tag{A.7}$$

We consider ϵ and $\bar\epsilon$ as two independent anticommuting Dirac spinors. The supercharges satisfy $\{\mathbb{Q}_\pm, \bar{\mathbb{Q}}_\pm\} = \mp 2i\partial_\pm$ and anticommute with the spinor derivatives. In terms of spinor coordinates:

$$\mathbb{Q}_\pm = \frac{\partial}{\partial\theta^\pm} \mp i\bar\theta^\pm\partial_{\pm\pm} , \qquad \bar{\mathbb{Q}}_\pm = \frac{\partial}{\partial\bar\theta^\pm} \mp i\theta^\pm\partial_{\pm\pm} . \tag{A.8}$$

A.3 $\mathcal{N} = (2, 2)$ supermultiplets

Vector multiplet. To formulate gauge theories, one introduces a vector multiplet in superspace. There are various such vector multiplets. The most standard one is the vector multiplet V, with gauge transformation $\delta_g V = i(\bar\Lambda - \Lambda)$, with Λ a chiral gauge parameter. To derive SUSY transformation rules and explicit actions in component form, it is most convenient to introduce gauge-covariant superderivatives $\nabla_\pm$, $\bar\nabla_\pm$. These can be constructed in different representations, according to the matter fields they are acting on. In chiral representation (where all objects transform with a chiral gauge parameter), they are given in terms of the usual superderivatives by

$$\nabla_\pm = e^{-V}\mathbb{D}_\pm e^V , \qquad \bar\nabla_\pm = \bar{\mathbb{D}}_\pm , \tag{A.9}$$

and they satisfy the algebra

$$\{\nabla_\pm, \bar{\nabla}_\pm\} = \pm 2i D_{\pm\pm} \ , \qquad \Sigma = \{\bar{\nabla}_+, \nabla_-\} \ , \tag{A.10}$$

where $D_{\pm\pm}$ denote the gauge-covariant spacetime derivatives while Σ is the field strength supermultiplet. In the Abelian case $\Sigma = \bar{\mathbb{D}}_+ \mathbb{D}_- V$.

The component fields of the vector multiplet Σ are defined by

$$\sigma = \Sigma| \ , \qquad i\lambda_+ = \nabla_+ \Sigma| \ , \qquad -i\lambda_- = \nabla_- \bar{\Sigma}| \ , \qquad -i\tilde{D} = \nabla_+ \bar{\nabla}_- \Sigma| \ , \tag{A.11}$$

$$\bar{\sigma} = \bar{\Sigma}| \ , \qquad i\bar{\lambda}_+ = \bar{\nabla}_+ \bar{\Sigma}| \ , \qquad -i\bar{\lambda}_- = \bar{\nabla}_- \Sigma| \ , \qquad -i\bar{\tilde{D}} = \bar{\nabla}_+ \nabla_- \bar{\Sigma}| \ , \tag{A.12}$$

where we used the complex notation

$$\sigma = i\sigma_1 - \sigma_2 \ , \qquad \bar{\sigma} = -i\sigma_1 - \sigma_2 \ , \qquad \tilde{D} = -iF_{12} + D \ , \qquad \bar{\tilde{D}} = -iF_{12} - D \ , \tag{A.13}$$

and $-iF_{12} = [D_1, D_2] = 2i[D_{--}, D_{++}]$.

One may similarly define gauge-covariant supercharges $\mathcal{Q}_\pm$, $\bar{\mathcal{Q}}_\pm$ such that the SUSY transformations are generated by

$$\delta = \bar{\epsilon}\mathcal{Q} + \epsilon\bar{\mathcal{Q}} = \bar{\epsilon}^+ \mathcal{Q}_+ + \bar{\epsilon}^- \mathcal{Q}_- + \epsilon^+ \bar{\mathcal{Q}}_+ + \epsilon^- \bar{\mathcal{Q}}_- \ . \tag{A.14}$$

Matter multiplets. To define the components of $\mathbb{X}$ and $\bar{\mathbb{X}}$ we use the gauge-covariant superderivatives:

$$\begin{aligned}
&X = \mathbb{X}\big| \ , && \psi_\pm = \nabla_\pm \mathbb{X}\big| \ , && F = \nabla_+ \nabla_- \mathbb{X}\big| \ , \\
&\bar{X} = \bar{\mathbb{X}}\big| \ , && \bar{\psi}_\pm = \bar{\nabla}_\pm \bar{\mathbb{X}}\big| \ , && \bar{F} = \bar{\nabla}_+ \bar{\nabla}_- \bar{\mathbb{X}}\big| \ , \\
&\bar{\chi}_\pm = \bar{\nabla}_\pm \mathbb{X}\big| \ , \quad M_{\mp\pm} = \nabla_\pm \bar{\nabla}_\mp \mathbb{X}\big| \ , && M_{\pm\pm} = \nabla_\pm \bar{\nabla}_\pm \mathbb{X}\big| \ , && \bar{\eta}_\pm = \nabla_+ \nabla_- \bar{\nabla}_\pm \mathbb{X}\big| \ , \\
&\chi_\pm = \nabla_\pm \bar{\mathbb{X}}\big| \ , \quad \bar{M}_{\mp\pm} = \bar{\nabla}_\pm \nabla_\mp \bar{\mathbb{X}}\big| \ , && \bar{M}_{\pm\pm} = \bar{\nabla}_\pm \nabla_\pm \bar{\mathbb{X}}\big| \ , && \eta_\pm = \bar{\nabla}_+ \bar{\nabla}_- \nabla_\pm \bar{\mathbb{X}}\big| \ .
\end{aligned} \tag{A.15}$$

The multiplet $\mathbb{X}$ can describe chiral, twisted chiral,[19] as well as left and right semichiral multiplets as follows:

$$\begin{aligned}
\text{Chiral}: &\quad \bar{\chi}_\pm = M_{\pm\pm} = M_{\pm\mp} = \bar{\eta}_\pm = 0 \\
\text{Twisted Chiral}: &\quad \psi_- = \bar{\chi}_+ = F = M_{+\pm} = \bar{\eta}_+ = 0, \ M_{--} = -2iD_{--}X, \ \bar{\eta}_- = -2iD_{--}\psi_+ \\
\text{Left Semichiral}: &\quad \bar{\chi}_+ = M_{+\pm} = \bar{\eta}_+ = 0 \\
\text{Right Semichiral}: &\quad \bar{\chi}_- = M_{-\pm} = \bar{\eta}_- = 0 \ .
\end{aligned}$$

The most convenient way to determine the SUSY transformations of the component fields, defined by expressions such as $\left[\nabla_\pm^n \bar{\nabla}_\pm^m \mathbb{X}\right]\big|$ above, is by using identities such as

$$\delta\left[\nabla_\pm^n \bar{\nabla}_\pm^m \mathbb{X}\right]\big| \equiv \left[\nabla_\pm^n \bar{\nabla}_\pm^m \delta \mathbb{X}\right]\big| = \left[(\bar{\epsilon}\nabla + \epsilon\bar{\nabla})\nabla_\pm^n \bar{\nabla}_\pm^m \mathbb{X}\right]\big| \ , \tag{A.16}$$

[19] A twisted chiral field cannot couple minimally to the vector multiplet. Thus, in the case of a twisted chiral field, these identifications and the SUSY transformations are valid only when the field is neutral. For instance, the twisted chiral field could be taken to be the Abelian vector multiplet Σ with the following identifications: $X = \sigma$, $\psi_+ = i\lambda_+$, $\bar{\chi}_- = -i\lambda_-$, $M_{-+} = -i\tilde{D}$.

where we have used the fact that $\nabla_\pm$'s and $\mathcal{Q}_\pm$'s anticommute and that $\nabla_\pm| = \mathcal{Q}_\pm|$, and similarly for the barred operators. From this we find:

$$\begin{aligned}
\delta X &= \bar\epsilon\psi + \epsilon\bar\chi\,,\\
\delta\psi_+ &= -\bar\epsilon^- F + \epsilon^+ 2iD_{++}X + \epsilon^-\bar\sigma X - \epsilon^+ M_{++} - \epsilon^- M_{-+}\,,\\
\delta\psi_- &= \bar\epsilon^+ F + \epsilon^+\sigma X - \epsilon^- 2iD_{--}X - \epsilon^+ M_{+-} - \epsilon^- M_{--}\,,\\
\delta F &= \epsilon^+ 2iD_{++}\psi_- + \epsilon^- 2iD_{--}\psi_+ - \epsilon^+\sigma\psi_+ + \epsilon^-\bar\sigma\psi_- - i(\epsilon^+\lambda_+ + \epsilon^-\lambda_-)X + \epsilon\bar\eta\,,\\
\delta\bar\chi_\pm &= \bar\epsilon_- M_{\pm+} - \bar\epsilon_+ M_{\pm-}\,,\\
\delta M_{\pm+} &= \bar\epsilon_+\bar\eta_\pm + \epsilon_- 2iD_{++}\bar\chi_\pm - \epsilon_+\bar\sigma\bar\chi_\pm\,,\\
\delta M_{\pm-} &= \bar\epsilon_-\bar\eta_\pm + \epsilon_+ 2iD_{--}\bar\chi_\pm + \epsilon_-\sigma\bar\chi_\pm\,,\\
\delta\bar\eta_+ &= -i(\epsilon\lambda)\bar\chi_+ + \epsilon_+(-2iD_{--}M_{++} - \bar\sigma M_{+-}) + \epsilon_-(2iD_{++}M_{+-} - \sigma M_{++})\,,\\
\delta\bar\eta_- &= -i(\epsilon\lambda)\bar\chi_- + \epsilon_-(2iD_{++}M_{--} - \sigma M_{-+}) + \epsilon_+(-2iD_{--}M_{-+} - \bar\sigma M_{--})\,,
\end{aligned} \tag{A.17}$$

as well as similar transformations for $\bar{\mathbb{X}}$. Using identities such as

$$\bar\epsilon_- 2iD_{++}\psi_- - \bar\epsilon_+ 2iD_{--}\psi_+ = i\bar\epsilon\gamma^\mu D_\mu\psi\,,$$

one may write these transformations in the form (2.8). For certain computations (such as proving invariance of the action on S^2) we find it convenient to keep the index notation and introduce the operator

$$\mathcal{P}_{\alpha\beta} \equiv \begin{pmatrix} 2iD_{++} & \sigma \\ \bar\sigma & -2iD_{--}\end{pmatrix}\,,$$

whose supersymmetric variation is $\delta\mathcal{P}_{\alpha\beta} = i\left(\epsilon_\alpha\bar\lambda_\beta + \bar\epsilon_\beta\lambda_\alpha\right)$. In this way, all transformation rules can be written in the compact form:

$$\begin{aligned}
\delta X &= \bar\epsilon\psi + \epsilon\bar\chi\,, & \delta\bar X &= \epsilon\bar\psi + \bar\epsilon\chi\,,\\
\delta\psi_\alpha &= \epsilon^\beta(\mathcal{P}_{\beta\alpha}X - M_{\beta\alpha}) + \bar\epsilon_\alpha F\,, & \delta\bar\psi_\alpha &= \bar\epsilon^\beta(\mathcal{P}_{\alpha\beta}\bar X - \bar M_{\beta\alpha}) + \epsilon_\alpha\bar F\,,\\
\delta F &= \epsilon^\alpha\mathcal{P}_{\alpha\beta}\psi^\beta - i(\epsilon\lambda)X + \epsilon\bar\eta\,, & \delta\bar F &= \bar\epsilon^\alpha\mathcal{P}_{\beta\alpha}\bar\psi^\beta - i(\bar\epsilon\bar\lambda)\bar X + \bar\epsilon\eta\,,\\
\delta\bar\chi_\alpha &= M_{\alpha\beta}\bar\epsilon^\beta\,, & \delta\chi_\alpha &= \bar M_{\alpha\beta}\epsilon^\beta\,,\\
\delta M_{\alpha\beta} &= \epsilon^\gamma\mathcal{P}_{\gamma\beta}\bar\chi_\alpha - \bar\eta_\alpha\bar\epsilon_\beta\,, & \delta\bar M_{\alpha\beta} &= \bar\epsilon^\gamma\mathcal{P}_{\beta\gamma}\chi_\alpha - \eta_\alpha\epsilon_\beta\,,\\
\delta\bar\eta_\alpha &= \epsilon^\kappa\mathcal{P}_{\kappa\gamma}M_{\alpha\beta}C^{\gamma\beta} - i(\epsilon\lambda)\bar\chi_\alpha\,, & \delta\eta_\alpha &= \bar\epsilon^\kappa\mathcal{P}_{\gamma\kappa}\bar M_{\alpha\beta}C^{\gamma\beta} - i(\bar\epsilon\bar\lambda)\chi_\alpha\,.
\end{aligned} \tag{A.18}$$

By setting the appropriate fields to zero, these become the SUSY transformations for chiral, twisted chiral and semichiral multiplets.

The component action is found by writing the spinorial measure as

$$\int d^4\theta\;\bar{\mathbb{X}}\mathbb{X} = \left[\nabla_+\nabla_-\bar\nabla_+\bar\nabla_-\left(\bar{\mathbb{X}}\mathbb{X}\right)\right]\Big|\,, \tag{A.19}$$

using that the rules for Grassmann integration and differentiation are the same, and that the integrand is gauge-invariant.

Semichiral vector multiplet. Here we give a brief review of the Semichiral Vector Multiplet (SVM for short), following [29, 51]. For simplicity we describe the Abelian case. The SVM gauges isometries that act only on semichiral fields and is defined by four vector multiplets $(V_L, V_R, \mathbb{V}, \tilde{\mathbb{V}})$, with gauge transformations

$$\delta V_L = i(\bar{\Lambda}_L - \Lambda_L) \,, \quad \delta V_R = i(\bar{\Lambda}_R - \Lambda_R) \,, \quad i\delta\mathbb{V} = i(\Lambda_L - \Lambda_R) \,, \quad i\delta\tilde{\mathbb{V}} = i(\Lambda_L - \bar{\Lambda}_R) \,, \tag{A.20}$$

where $\Lambda_{L,R}$ are semichiral fields. These vector multiplets are not independent, but satisfy

$$-\frac{1}{2}V' \equiv \mathrm{Re}\,\tilde{\mathbb{V}} = \mathrm{Re}\,\mathbb{V} \,, \qquad \mathrm{Im}\,(\tilde{\mathbb{V}} - \mathbb{V}) = V_R \,, \qquad \mathrm{Im}\,(\tilde{\mathbb{V}} + \mathbb{V}) = V_L \,. \tag{A.21}$$

Then,

$$\mathbb{V} = \frac{1}{2}\big(-V' + i(V_L - V_R)\big) \,, \qquad \tilde{\mathbb{V}} = \frac{1}{2}\big(-V' + i(V_L + V_R)\big) \,, \tag{A.22}$$

where V' must transform under gauge transformations as $\delta V' = (\Lambda_R + \bar{\Lambda}_R - \Lambda_L - \bar{\Lambda}_L)$. There are two field strengths which are invariant under the full gauge symmetry (A.20):

$$\mathbb{F} \equiv \bar{\mathbb{D}}_+\bar{\mathbb{D}}_-\mathbb{V} \qquad \text{and} \qquad \tilde{\mathbb{F}} \equiv \bar{\mathbb{D}}_+\mathbb{D}_-\tilde{\mathbb{V}} \,, \tag{A.23}$$

chiral and twisted chiral, respectively. The kinetic action for the SVM is given by

$$\mathcal{L}_{\mathrm{SVM}} = -\frac{1}{2e^2}\int d^4\theta\left(\bar{\tilde{\mathbb{F}}}\tilde{\mathbb{F}} - \bar{\mathbb{F}}\mathbb{F}\right) \,, \tag{A.24}$$

and the FI terms are given by

$$\mathcal{L}_{\mathrm{FI}} = \left(i\,t\int d^2\tilde{\theta}\,\tilde{\mathbb{F}} + c.c.\right) + \left(i\,s\int d^2\theta\,\mathbb{F} + c.c.\right) \,. \tag{A.25}$$

From the definitions (A.23), the FI terms can also be written as D-terms.

A.4 Supersymmetry on S^2

The metric on the round sphere of radius r is

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \,.$$

We consider Killing spinors on S^2 satisfying

$$D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon \,, \qquad D_\mu\bar{\epsilon} = \frac{i}{2r}\gamma_\mu\bar{\epsilon} \,. \tag{A.26}$$

The gauge-covariant derivatives on S^2 read

$$\begin{aligned} \mathcal{P}_{++} = iD_1 + D_2 &= \frac{i}{r}\left(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi\right) - \frac{is}{r}\frac{\cos\theta}{\sin\theta} \,, \\ \mathcal{P}_{--} = -iD_1 + D_2 &= -\frac{i}{r}\left(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\right) - \frac{is}{r}\frac{\cos\theta}{\sin\theta} \,, \end{aligned} \tag{A.27}$$

where $s = s_z - \frac{\rho(\mathfrak{m})}{2}$ is the effective spin. Using these derivatives, one can check that

$$\begin{pmatrix}\epsilon_+\\ \epsilon_-\end{pmatrix} = e^{i\frac{\varphi}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\ i\sin\frac{\theta}{2}\end{pmatrix} \,, \qquad \begin{pmatrix}\bar{\epsilon}_+\\ \bar{\epsilon}_-\end{pmatrix} = e^{-i\frac{\varphi}{2}}\begin{pmatrix}i\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} \tag{A.28}$$

satisfy the Killing spinor equations (A.26). We use the latter spinors ϵ, $\bar{\epsilon}$ to construct the localizing supercharge $\mathcal{Q}_A$.

A.5 Weyl covariance

The additional terms that are supplemented to the $\mathbb{R}^2$ transformations of fields of a given R-charge, are determined by the requirement that the SUSY transformations (A.18) are covariant under Weyl transformations. The metric transforms infinitesimally as $\delta g_{\mu\nu} = 2\Omega\, g_{\mu\nu}$, hence it follows that the spin connection transforms as

$$\delta\omega_\mu{}^{mn} = e_\mu{}^m e_\nu{}^n \partial^\nu \Omega\, . \tag{A.29}$$

For a field φ of spin s_z and weight w under a Weyl transformation, i.e., $\delta\varphi = -w\Omega\,\varphi$, we have:

$$\delta(\mathcal{P}_{\pm\pm}\varphi) = -w\Omega\,\mathcal{P}_{\pm\pm}\varphi - (w \pm s_z)(\mathcal{P}_{\pm\pm}\Omega)\varphi\, . \tag{A.30}$$

This uniquely determines the additional terms one must add to the flat-space transformation rules (in addition to replacing the derivatives by covariant derivatives) to determine the transformation rules on S^2. We assume that ϵ is a "positive" Killing spinor satisfying

$$D_\mu\epsilon = \frac{i}{2r}\gamma_\mu\epsilon \qquad \Rightarrow \qquad \mathcal{P}_{\pm\pm}\epsilon^\pm = -\frac{1}{r}\epsilon_\pm\, . \tag{A.31}$$

We also take the spinor $\bar\epsilon$ to satisfy the same equation. Then the additional terms follow by the replacement rule

$$\begin{aligned} \epsilon^\pm\mathcal{P}_{\pm\pm}\varphi \quad &\xrightarrow{\text{replace}} \quad \epsilon^\pm\mathcal{P}_{\pm\pm}\varphi \,+\, (-1)^F(w\pm s_z)\varphi\,\mathcal{P}_{\pm\pm}\epsilon^\pm \\ &\qquad = \epsilon^\pm\mathcal{P}_{\pm\pm}\varphi \,-\, (-1)^F(w\pm s_z)\frac{1}{r}\,\varphi\,\epsilon_\pm\, , \end{aligned} \tag{A.32}$$

where $F = 0$ if φ is a boson and $F = 1$ if it is a fermion, and we have used the Killing spinor equation (A.31).

We denote the scaling dimension of the lowest component $X = \mathbb{X}|$ by $\frac{q}{2}$. From the definitions (A.15), Weyl weights and R-charges of all the other component fields are determined and given in table 1. Using those charges and following the prescription (A.32), we find that the SUSY transformations on S^2 are given by (3.4).

A.6 BPS equations for the semichiral multiplet

Here we show that the BPS equations following from the transformation rules (3.4) have only one smooth solution for $q \neq 0$: $X = F = M = 0$. Let us first write down the BPS equations following from $\mathcal{Q}_A\psi_\pm = \mathcal{Q}_A\bar\psi_\pm = 0$. Using (A.28) we find

$$\begin{aligned} 0 &= -\sin\frac{\theta}{2}\left(2D_{++}X - ie^{-i\varphi}F + iM_{++}\right) - \cos\frac{\theta}{2}\left(\bar\sigma X + \frac{q}{2r}X - M_{-+}\right), \\ 0 &= +\cos\frac{\theta}{2}\left(2D_{--}X - ie^{-i\varphi}F - iM_{--}\right) + \sin\frac{\theta}{2}\left(\sigma X - \frac{q}{2r}X - M_{+-}\right), \\ 0 &= +\cos\frac{\theta}{2}\left(2D_{++}\bar X - ie^{i\phi}\bar F + i\bar M_{++}\right) + \sin\frac{\theta}{2}\left(\bar X\sigma - \frac{q}{2r}\bar X + \bar M_{-+}\right), \\ 0 &= -\sin\frac{\theta}{2}\left(2D_{--}\bar X - ie^{i\phi}\bar F - i\bar M_{--}\right) - \cos\frac{\theta}{2}\left(\bar X\bar\sigma + \frac{q}{2r}\bar X + \bar M_{+-}\right). \end{aligned} \tag{A.33}$$

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Recall that for a particular semichiral field, only some components of $M_{\alpha\beta}$ are non-zero in these equations. Note also that for $M_{\alpha\beta}=0$, these reduce to the BPS equations for a chiral field of R-charge q.

Let us now look at the equations following from other spinor variations. Take for instance $\mathbb{X}_R$ and $\bar{\mathbb{X}}_R$, i.e., $\mathcal{Q}_A\bar{\chi}_+ = \mathcal{Q}_A\chi_+ = \mathcal{Q}_A\bar{\eta}_+ = \mathcal{Q}_A\eta_+ = 0$:

$$0 = M_{++}\cos\frac{\theta}{2} - iM_{+-}\sin\frac{\theta}{2}\,, \tag{A.34}$$

$$0 = i\bar{M}_{++}\sin\frac{\theta}{2} - \bar{M}_{+-}\cos\frac{\theta}{2}\,, \tag{A.35}$$

$$0 = \cos\frac{\theta}{2}\left(2iD_{--}M_{++} + \left(\bar{\sigma}+\frac{q+2}{2r}\right)M_{+-}\right) + \sin\frac{\theta}{2}\left(2D_{++}M_{+-} + i\left(\sigma-\frac{q}{2r}\right)M_{++}\right), \tag{A.36}$$

$$0 = \sin\frac{\theta}{2}\left(2D_{--}\bar{M}_{++} - i\left(\sigma+\frac{q+2}{2r}\right)\bar{M}_{+-}\right) + \cos\frac{\theta}{2}\left(2iD_{++}\bar{M}_{+-} - \left(\bar{\sigma}-\frac{q}{2r}\right)\bar{M}_{++}\right). \tag{A.37}$$

Consider the equations (A.34) and (A.35) and their complex conjugates. Imposing the reality condition $X^\dagger = \bar{X}$, $\psi^\dagger = \bar{\psi}$, $M^\dagger = \bar{M}$, etc., immediately leads to

$$0 = M_{++}\cos\theta = M_{+-}\cos\theta \qquad \Rightarrow \qquad M_{++} = M_{+-} = 0\,, \tag{A.38}$$

and (A.36) and (A.37) are trivially satisfied.[20] The same analysis holds for left semichiral fields and thus one also concludes $M_{--} = M_{-+} = 0$. Finally, plugging $M_{\alpha\beta} = 0$ into $\mathcal{Q}_A\psi = \mathcal{Q}_A\bar{\psi} = 0$ reduces those equations to the BPS equations for a chiral field. As discussed in [4, 5], for generic q the only smooth solution is $X = F = M = 0$.

B GLSMs for semichiral fields

B.1 Kinetic action and positivity of the metric

Consider a theory with N_F pairs of semichiral fields $(\mathbb{X}_L^i, \mathbb{X}_R^i)$ with $i = 1, \ldots, N_F$, where the left and right partners transform in the same representation $\mathfrak{R}$ of the gauge and/or flavor group. Then the most general gauge-invariant quadratic kinetic action follows from the superspace Lagrangian

$$\mathcal{L} = \int d^4\theta\,\left(\beta_{ij}\,\bar{\mathbb{X}}_L^i\mathbb{X}_L^j + \gamma_{ij}\,\bar{\mathbb{X}}_R^i\mathbb{X}_R^j - \alpha_{ij}\,\bar{\mathbb{X}}_L^i\mathbb{X}_R^j - \alpha_{ij}^\dagger\,\bar{\mathbb{X}}_R^i\mathbb{X}_L^j\right)\,, \tag{B.1}$$

where β, γ are Hermitian matrices while α is a generic complex matrix. By field redefinitions one can set β and γ to be diagonal with entries $\pm1, 0$. Requiring the metric to be positive-definite after having integrated out the auxiliary fields, leads to the following two conditions:

$$(\alpha^\dagger)^{-1}\gamma\alpha^{-1} < 0 \qquad\qquad \text{and} \qquad\qquad \beta(\alpha^\dagger)^{-1}\gamma\alpha^{-1}\beta - \beta > 0\,. \tag{B.2}$$

[20] At the special point $\theta = \frac{\pi}{2}$ the fields M need not vanish, but we restrict ourselves to smooth field configurations.

These force $\beta = \gamma = -\mathbb{1}$ and by the singular value decomposition theorem we can, by further unitary field redefinitions, reduce α to a diagonal matrix with non-negative entries which now have to satisfy $\alpha_{ii} > 1$ as in the case $N_F = 1$. Thus, in the presence of multiple semichiral pairs, we can always choose a basis that diagonalizes the quadratic kinetic action.

B.2 Semichiral-semichiral duality

An interesting feature of semichiral fields is that a pair $(\mathbb{X}_L, \mathbb{X}_R)$ in a representation $(\mathfrak{R}, \mathfrak{R})$ of the gauge and/or flavor group, is "dual" to a pair in representation $(\mathfrak{R}, \overline{\mathfrak{R}})$ or $(\overline{\mathfrak{R}}, \mathfrak{R})$ [66]. Unlike T-duality, this is simply a change of coordinates which does not change the geometry (one may call this a coordinate duality). To see how this works, consider a pair of semichiral fields in representation $(\mathfrak{R}, \mathfrak{R})$ with an action of the form (2.11). The idea is to relax the condition of semichirality on $\mathbb{X}_R$, imposing it by a semichiral Lagrange multiplier $\tilde{\mathbb{X}}_R$, i.e.,

$$\mathcal{L} = -\int d^4\theta \left[\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\mathbb{X}}_R\mathbb{X}_R + \alpha\big(\bar{\mathbb{X}}_L\mathbb{X}_R + \bar{\mathbb{X}}_R\mathbb{X}_L\big) - \big(\tilde{\mathbb{X}}_R\mathbb{X}_R + \bar{\tilde{\mathbb{X}}}_R\bar{\mathbb{X}}_R\big)\right] . \tag{B.3}$$

This action is gauge-invariant provided $\tilde{\mathbb{X}}_R$ is in representation $\overline{\mathfrak{R}}$. Integrating out $\tilde{\mathbb{X}}_R$ simply imposes that $\mathbb{X}_R$ is right semichiral and leads to the original model. On the other hand, integrating out $\mathbb{X}_R$ leads to the change of coordinates $\tilde{\mathbb{X}}_R = \bar{\mathbb{X}}_R + \alpha\,\bar{\mathbb{X}}_L$ and the dual action reads

$$\mathcal{L} = \int d^4\theta \Big[(\alpha^2-1)\bar{\mathbb{X}}_L\mathbb{X}_L + \bar{\tilde{\mathbb{X}}}_R\tilde{\mathbb{X}}_R - \alpha\big(\tilde{\mathbb{X}}_R\mathbb{X}_L + \bar{\mathbb{X}}_L\bar{\tilde{\mathbb{X}}}_R\big)\Big] . \tag{B.4}$$

This is a GLSM for semichiral fields in representation $(\mathfrak{R}, \overline{\mathfrak{R}})$. After a rescaling of $\mathbb{X}_L$ to normalize the first term, we see that the relation between α in (2.11) and β in (2.15) is

$$\beta = \frac{\alpha}{\sqrt{\alpha^2-1}} . \tag{B.5}$$

Of course, one could similarly relax the semichirality condition on $\mathbb{X}_L$ instead, which would lead to a model with semichiral fields in representation $(\overline{\mathfrak{R}}, \mathfrak{R})$.

C One-loop determinants

We first evaluate the bosonic determinant. In addition to the lowest components X_L and X_R we also have the four fields $M_{\alpha\beta}$ and two auxiliary fields F_L, F_R, therefore the quadratic terms read $\bar{\mathcal{X}}\mathcal{O}_B\mathcal{X}$ with $\bar{\mathcal{X}} = (\bar{X}^L, \bar{X}^R, \bar{M}^R_{+-}, \bar{M}^R_{++}, \bar{M}^L_{--}, \bar{M}^L_{-+}, \bar{F}^L, \bar{F}^R)$ and $\mathcal{X} = (X^L, X^R, M^L_{-+}, M^L_{--}, M^R_{++}, M^R_{+-}, F^L, F^R)^\mathsf{T}$. The 8×8 matrix of the kinetic operator

$\mathcal{O}_B$ is then given by

$$\begin{pmatrix} \mathcal{O}_X & \alpha\,\mathcal{O}_X & \frac{q}{2r}-\sigma & 2iD_{++} & -\alpha\,2iD_{--} & -\alpha\left(\frac{q}{2r}+\bar\sigma\right) & 0 & 0 \\ \alpha\,\mathcal{O}_X & \mathcal{O}_X & \alpha\left(\frac{q}{2r}-\sigma\right) & \alpha\,2iD_{++} & -2iD_{--} & -\left(\frac{q}{2r}+\bar\sigma\right) & 0 & 0 \\ \alpha\left(-\frac{q}{2r}+\sigma\right) & -\frac{q}{2r}+\sigma & 0 & 0 & 0 & -1 & 0 & 0 \\ \alpha\,2iD_{--} & 2iD_{--} & 0 & \alpha & 0 & 0 & 0 & 0 \\ -2iD_{++} & -\alpha\,2iD_{++} & 0 & 0 & \alpha & 0 & 0 & 0 \\ \frac{q}{2r}+\bar\sigma & \alpha\left(\frac{q}{2r}+\bar\sigma\right) & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \end{pmatrix} \tag{C.1}$$

where $\mathcal{O}_X$ is given in (4.5). Expanding the fields in $\mathcal{X}$ in terms of the spherical harmonics $\left(Y^s_{j,j_3}, Y^s_{j,j_3}, Y^s_{j,j_3}, Y^{s-1}_{j,j_3}, Y^{s+1}_{j,j_3}, Y^s_{j,j_3}, Y^s_{j,j_3}, Y^s_{j,j_3}\right)$ and using

$$D_{\pm\pm}Y^s_{j,j_3} = \pm\frac{s_\pm}{2r}Y^{s\pm1}_{j,j_3} \qquad \text{with } s_\pm = \sqrt{j(j+1)-s(s\pm1)}\,, \tag{C.2}$$

we obtain for $j \geq \frac{|\rho(\mathfrak{m})|}{2}+1$:

$$\begin{aligned} \det\mathcal{O}_B = \frac{\left(\alpha^2-1\right)^2}{r^4}&\left[j^2+\frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4}-\alpha^2\left(\frac{q}{2}-ir\rho(\sigma_1)\right)^2\right]\times \\ &\times\left[(j+1)^2+\frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4}-\alpha^2\left(\frac{q}{2}-ir\rho(\sigma_1)\right)^2\right], \end{aligned} \tag{C.3}$$

with multiplicity $2j+1$. Here $\rho(\cdot)$ denotes a weight vector in the representation $\mathfrak{R}$. There are three more cases that need to be considered before we can write down the full determinant.

- For $j = \frac{|\rho(\mathfrak{m})|}{2} \geq \frac{1}{2}$, either Y^{s+1}_{j,j_3} or Y^{s-1}_{j,j_3} does not exist. For instance, for $\rho(\mathfrak{m}) \geq 1$, Y^{s-1}_{j,j_3} does not exist, then we can remove the fourth row/column of the matrix $\mathcal{O}_B$. Similiarly, for $\rho(\mathfrak{m}) \leq -1$, Y^{s+1}_{j,j_3} does not exist, then we can remove the fifth row/column of the matrix $\mathcal{O}_B$. In this case, the determinant becomes $\frac{1}{\alpha}\det\mathcal{O}_B$ at $j = \frac{|\rho(\mathfrak{m})|}{2}$.
- For $j = \rho(\mathfrak{m}) = 0$, only Y^0_{0,j_3} exists, and we remove both the fourth and the fifth row/column of the matrix $\mathcal{O}_B$. The determinant becomes $\frac{1}{\alpha^2}\det\mathcal{O}_B$ at $j = \rho(\mathfrak{m}) = 0$.
- For $j = \frac{|\rho(\mathfrak{m})|}{2} - 1 \geq 0$, only one of Y^{s+1}_{j,j_3} and Y^{s-1}_{j,j_3} exists. In both cases, the determinant is only α with multiplicity $|\rho(\mathfrak{m})|-1$.

Putting all these cases together, we obtain the full determinant in the bosonic sector (ignoring overall constants):

$$\begin{aligned} \mathrm{Det}\,\mathcal{O}_B = \prod_{\rho\in\mathfrak{R}}\frac{\alpha^{|\rho(\mathfrak{m})|-1}}{\alpha^{|\rho(\mathfrak{m})|+1}}\prod_{j=\frac{|\rho(\mathfrak{m})|}{2}}^{\infty}&\left[j^2+\frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4}-\alpha^2\left(\frac{q}{2}-ir\rho(\sigma_1)\right)^2\right]^{2j+1}\times \\ \times\left[(j+1)^2+\frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4}-\alpha^2\left(\frac{q}{2}-ir\rho(\sigma_1)\right)^2\right]^{2j+1}&\left(\frac{\left(\alpha^2-1\right)^2}{r^4}\right)^{2j+1}. \end{aligned} \tag{C.4}$$

Now let us work out the fermionic determinant. First we work out the determinant factor produced by the fields ψ and η. Proceeding as before, we write the action as $\bar{\Psi}\mathcal{O}_F\Psi$ with $\bar{\Psi} = \left(\bar{\psi}^{L-}, \bar{\psi}^{L+}, \bar{\psi}^{R-}, \bar{\psi}^{R+}, \eta^{R-}, \eta^{L+}, \chi^{R-}, \chi^{L+}\right)$, $\Psi = (\psi^{L+}, \psi^{L-}, \psi^{R+}, \psi^{R-}, \bar{\eta}^{L+}, \bar{\eta}^{R-}, \bar{\chi}^{L+}, \bar{\chi}^{R-})^{\mathsf{T}}$ and $\mathcal{O}_F$ is given by

$$\begin{pmatrix} -\left(\frac{q}{2r}+\bar{\sigma}\right) & 2iD_{--} & -\alpha\left(\frac{q}{2r}+\bar{\sigma}\right) & 2i\alpha D_{--} & -1 & 0 & 0 & 0 \\ -2iD_{++} & \frac{q}{2r}-\sigma & -2i\alpha D_{++} & \alpha\left(\frac{q}{2r}-\sigma\right) & 0 & \alpha & 0 & 0 \\ -\alpha\left(\frac{q}{2r}+\bar{\sigma}\right) & 2i\alpha D_{--} & -\left(\frac{q}{2r}+\bar{\sigma}\right) & 2iD_{--} & -\alpha & 0 & 0 & 0 \\ -2i\alpha D_{++} & \alpha\left(\frac{q}{2r}-\sigma\right) & -2iD_{++} & \frac{q}{2r}-\sigma & 0 & 1 & 0 & 0 \\ -\alpha & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha\left(\frac{q}{2r}-\sigma\right) & -2iD_{--} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2iD_{++} & \alpha\left(\frac{q}{2r}+\bar{\sigma}\right) \end{pmatrix}. \tag{C.5}$$

Expanding Ψ in the spherical harmonics $\left(Y^{s-\frac{1}{2}}_{j,j_3}, Y^{s+\frac{1}{2}}_{j,j_3}, Y^{s-\frac{1}{2}}_{j,j_3}, Y^{s+\frac{1}{2}}_{j,j_3}, Y^{s-\frac{1}{2}}_{j,j_3}, Y^{s+\frac{1}{2}}_{j,j_3}, Y^{s-\frac{1}{2}}_{j,j_3},\right.$ $\left.Y^{s+\frac{1}{2}}_{j,j_3}\right)$, we obtain for $j \geq \frac{|\rho(\mathfrak{m})|}{2} + \frac{1}{2}$:

$$\det \mathcal{O}_F = \frac{\left(\alpha^2-1\right)^2}{r^4}\left[\left(j+\frac{1}{2}\right)^2 + \frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4} - \alpha^2\left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2\right]^2, \tag{C.6}$$

with multiplicity $2j+1$. For $j = \frac{|\rho(\mathfrak{m})|}{2} - \frac{1}{2} \geq 0$ only one of the eigenfunctions exists. For instance, when $\rho(\mathfrak{m}) \geq 1$, only $Y^{s+\frac{1}{2}}_{j,j_3}$ exists, which leads to the following eigenvalue with multiplicity $|\rho(\mathfrak{m})|$:

$$-\frac{\left(\alpha^2-1\right)\alpha^2}{r^2}\left(\frac{|\rho(\mathfrak{m})|}{2} + \frac{q}{2} - ir\rho(\sigma_1)\right)\left(\frac{|\rho(\mathfrak{m})|}{2} - \frac{q}{2} + ir\rho(\sigma_1)\right).$$

For $\rho(\mathfrak{m}) \leq -1$ only $Y^{s-\frac{1}{2}}_{j,j_3}$ exists, and the eigenvalue is same as above. The case $\rho(\mathfrak{m}) = 0$ does not exist but formally we can still use the same expression. Putting all these cases together, we obtain the full determinant in the fermionic sector:

$$\begin{aligned} \mathrm{Det}\,\mathcal{O}_F = \prod_{\rho\in\mathfrak{R}} &\left(\frac{(1-\alpha^2)\alpha^2}{r^2}\right)^{|\rho(\mathfrak{m})|}\left[\frac{\rho(\mathfrak{m})^2}{4} - \left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2\right]^{|\rho(\mathfrak{m})|} \times \\ \times \prod_{j=\frac{|\rho(\mathfrak{m})|+1}{2}}^{\infty} &\left(\frac{\alpha^2-1}{r^2}\right)^{4j+2}\left[\left(j+\frac{1}{2}\right)^2 + \frac{(\alpha^2-1)\rho(\mathfrak{m})^2}{4} - \alpha^2\left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2\right]^{4j+2}. \end{aligned} \tag{C.7}$$

Putting these bosonic and fermionic determinants together leads to many cancellations and ignoring overall factors, the one-loop determinant reads

$$Z_{LR} = \frac{\mathrm{Det}\,\mathcal{O}_F}{\mathrm{Det}\,\mathcal{O}_B} = \prod_{\rho\in\mathfrak{R}} \frac{(-1)^{|\rho(\mathfrak{m})|}}{\frac{\rho(\mathfrak{m})^2}{4} - \left(\frac{q}{2} - ir\rho(\sigma_1)\right)^2}. \tag{C.8}$$

Now we show that (C.8) can be rewritten in a more recognizable form, as a one-loop determinant for chiral fields. For a pair of chiral fields with opposite R-charges and gauge charges one has the expression in (4.9) [4, 5]. Let us set $q = 0$ for now and then shift $r\rho(\sigma_1)$ to $r\rho(\sigma_1)+i\frac{q}{2}$ in the final result. Using the property of the Γ-function, $\Gamma(z+n) = (z)_n \cdot \Gamma(z)$, where $(z)_n \equiv \prod_{k=0}^{n-1}(z+k)$ is the Pochhammer symbol, we obtain for $\rho(\mathfrak{m}) \geq 1$ (the cases $\rho(\mathfrak{m}) = 0$ and $\rho(\mathfrak{m}) \leq -1$ are similar)

$$
\begin{aligned}
&\prod_{\rho\in\mathfrak{R}} \frac{\Gamma\left(-ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1+ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)} \frac{\Gamma\left(ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)}{\Gamma\left(1-ir\rho(\sigma_1) + \frac{\rho(\mathfrak{m})}{2}\right)} \\
&= \prod_{\rho\in\mathfrak{R}} \frac{\left(1+ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)_{\rho(\mathfrak{m})-1}}{\left(-ir\rho(\sigma_1) - \frac{\rho(\mathfrak{m})}{2}\right)_{\rho(\mathfrak{m})+1}} = \prod_{\rho\in\mathfrak{R}} \frac{(-1)^{\rho(\mathfrak{m})}}{\frac{\rho(\mathfrak{m})^2}{4} - (-ir\rho(\sigma_1))^2}\,,
\end{aligned}
$$

which coincides with (C.8) after the appropriate shift.

D Target space geometry

D.1 Metric, B-field and complex structures

For the reader's convenience, here we give some relevant formulæ to compute the metric, the B-field and the complex structures $J_\pm$ from the generalized Kähler potential K. For a comprehensive review and details see [40]. Defining $E = \frac{1}{2}(g + B)$ one has

$$
\begin{aligned}
E_{LL} &= C_{LL}K^{-1}_{LR}J_sK_{RL} & E_{cL} &= C_{cL}K^{-1}_{LR}J_sK_{RL} \\
E_{LR} &= J_sK_{LR}J_s + C_{LL}K^{-1}_{LR}C_{RR} & E_{cR} &= J_cK_{cR}J_s + C_{cL}K^{-1}_{LR}C_{RR} \\
E_{Lc} &= K_{Lc} + J_sK_{Lc}J_c + C_{LL}K^{-1}_{LR}C_{Rc} & E_{cc} &= K_{cc} + J_cK_{cc}J_c + C_{cL}K^{-1}_{LR}C_{Rc} \\
E_{Lt} &= -K_{Lt} - J_sK_{Lt}J_t + C_{LL}K^{-1}_{LR}A_{Rt} & E_{ct} &= -K_{ct} - J_cK_{ct}J_t + C_{cL}K^{-1}_{LR}A_{Rt} \\
E_{RL} &= -K_{RL}J_sK^{-1}_{LR}J_sK_{RL} & E_{tL} &= C_{tL}K^{-1}_{LR}J_sK_{RL} \\
E_{RR} &= -K_{RL}J_sK^{-1}_{LR}C_{RR} & E_{tR} &= J_tK_{tR}J_s + C_{tL}K^{-1}_{LR}C_{RR} \\
E_{Rc} &= K_{Rc} - K_{RL}J_sK^{-1}_{LR}C_{Rc} & E_{tc} &= K_{tc} + J_tK_{tc}J_c + C_{tL}K^{1}_{LR}C_{Rc} \\
E_{Rt} &= -K_{Rt} - K_{RL}J_sK^{-1}_{LR}A_{Rt} & E_{tt} &= -K_{tt} - J_tK_{tt}J_t + C_{tL}K^{-1}_{LR}A_{Rt}\,.
\end{aligned}
\tag{D.1}
$$

Here A and C are matrices defined as follows (with the two indices suppressed)

$$
A = \begin{pmatrix} 2iK & 0 \\ 0 & -2iK \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 2iK \\ -2iK & 0 \end{pmatrix}, \tag{D.2}
$$

where K itself is a matrix whose entries are second derivatives of the generalized potential with c, t, s denoting chiral, twisted chiral and semichiral directions, respectively. For instance

$$
K_{LR} \equiv \begin{pmatrix} \frac{\partial^2 K}{\partial X_L \partial X_R} & \frac{\partial^2 K}{\partial X_L \partial \bar{X}_R} \\ \frac{\partial^2 K}{\partial \bar{X}_L \partial X_R} & \frac{\partial^2 K}{\partial \bar{X}_L \partial \bar{X}_R} \end{pmatrix}, \tag{D.3}
$$

and similarly for $K_{Rc} = \frac{\partial^2 K}{\partial X_R \partial \phi}$, etc.

The complex structures read [40, 67]

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1}C_{LL} & K_{RL}^{-1}J_sK_{LR} & K_{RL}^{-1}C_{Lc} & K_{RL}^{-1}C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} \tag{D.4}$$

and

$$J_- = \begin{pmatrix} K_{LR}^{-1}J_sK_{RL} & K_{LR}^{-1}C_{RR} & K_{LR}^{-1}C_{Rc} & K_{LR}^{-1}A_{Rt} \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix}, \tag{D.5}$$

where $J_{c,t,s}$ are the canonical complex structures of the form $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and of the appropriate dimension.

D.2 Type-change loci

The *type* of a generalized Kähler structure is given by

$$(k_+, k_-) = \big(\dim_{\mathbb{C}} \ker(J_+ - J_-)\ ,\ \dim_{\mathbb{C}} \ker(J_+ + J_-)\big)\ .$$

In terms of $\mathcal{N} = (2,2)$ multiplets, k_+ and k_- simply count the number of chiral and twisted chiral fields, respectively; the number of semichiral fields is given by

$$\dim_{\mathbb{C}} \operatorname{coIm}[J_+, J_-] = d - k_+ - k_-\ ,$$

where coIm is the co-image, while d the complex dimension of the manifold. An important aspect of generalized complex geometry is that the type may jump discontinuously on so-called type-changing loci [68]. On these loci, a pair of semichiral fields $(\mathbb{X}_L, \mathbb{X}_R)$ becomes either a pair of chiral or a pair of twisted chiral fields.

To study this phenomenon one may compute the eigenvalues of $J_+ \pm J_-$. For concreteness, let us consider an arbitrary generalized potential K that depends on a single chiral field and a pair of semichiral fields. At generic points on the manifold the type is $(k_+, k_-) = (1, 0)$. From (D.4) and (D.5) one finds that the eigenvalues of $\frac{1}{2}(J_+ - J_-)$ are $\{0, \pm i\lambda\}$, each with multiplicity two, with

$$\lambda = \sqrt{\frac{|K_{\bar{l}r}|^2 - K_{r\bar{r}}K_{l\bar{l}}}{|K_{\bar{l}r}|^2 - |K_{lr}|^2}}\ . \tag{D.6}$$

The eigenvalues of $\frac{1}{2}(J_+ + J_-)$ are $\{\pm i, \pm i\tilde{\lambda}\}$, each with multiplicity two, with

$$\tilde{\lambda} = \sqrt{\frac{|K_{lr}|^2 - K_{r\bar{r}}K_{l\bar{l}}}{|K_{lr}|^2 - |K_{\bar{l}r}|^2}}\ . \tag{D.7}$$

On the locus $\lambda = 0$ the type jumps to $(k_+, k_-) = (3, 0)$, where the manifold is locally described by three chiral fields, and on the locus $\tilde{\lambda} = 0$ the type jumps to $(k_+, k_-) = (1, 2)$, where it is locally described by one chiral field and two twisted chiral fields. Whether these loci exist or not depends on the specific potential K. For a study of type-change in various WZW models see, for example, [69].

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