3-dimensional left-invariant sub-Lorentzian contact structures

3-DIMENSIONAL LEFT-INVARIANT SUB-LORENTZIAN CONTACT STRUCTURES

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Abstract. We provide a classification of ts-invariant sub-Lorentzian structures on 3 dimensional contact Lie groups. Our approach is based on invariants arising form the construction of a normal Cartan connection.

1. Introduction

1.1. **Sub-Lorentzian Geometry.** Let M be a smooth manifold. *A sub-Lorentzian structure* (or *a metric*) on M is, by definition, a pair (H, g) where H is a smooth bracket generating distribution on M and g is a smooth Lorentzian metric on H. A triple (M, H, g) where M is a manifold endowed with a sub-Lorentzian structure (H, g) is called *a sub-Lorentzian manifold*. Such a manifold is said to be contact if the distribution H is contact.

The theory of sub-Lorentzian manifolds is a subject presently in it's infancy but interest is growing, see [3, 4, 5] for outlines of the general theory and [12, 10, 7] where the main emphasis was on investigating the causal structure of such manifolds (geodesics, the structure of reachable sets, normal forms). There are also two papers [8, 9] where the authors started the investigation of the theory of invariants for sub-Lorentzian structures.

Sub-Lorentzian metrics which are simultaneously time and space oriented will be called ts-oriented (see section 2 for more details). The aim of this paper is to classify all left invariant ts-oriented sub-Lorentzian structures on 3-dimensional Lie groups. Two basic invariants which plays fundamental role in the study were introduced in [8]. These are: a $(1, 1)$-tensor $h = \left(\begin{smallmatrix} a & b \\ -b & -a \end{smallmatrix}\right)$ on H and a smooth function κ on M (see Section 4.4 for exact formulas). These invariants can be obtained from a variety of perspectives with varying degrees of complexity. For example in [9] a Riemannian approach is used while an approach using null lines should also be possible. In both cases, the fact that the underlying manifold is contact plays a significant role and so the techniques do not obviously generalize.

In this work we use the standard machinery of Cartan connections with the goal of giving a thorough explanation of the origins of the invariants via Cartan theory. The general approach proceeds as follows: to begin, we construct the so called first order geometric structure corresponding to the sub-Lorentzian structure (M, H, g) (see section 3.1 for the definitions). Specifically, we construct the symbol algebra corresponding to a canonical filtration of the tangent bundle generated by H. The symbol algebra then defines the weighted frame bundle. The first order geometric structure is an $SO_{1,1}$ reduction of the bundle of weighted frames. A frame on M is said to be adapted if it corresponds in a natural manner with a weighted frame and the bundle over M consisting of adapted frames admits a canonical Cartan connection. The curvature function of this connection generates the algebra of invariants for the corresponding sub-Lorentzian structure. In particular the invariants κ and h arise in a canonical decomposition of the curvature function for 3-dimensional contact sub-Lorentzian structure.

One can go further and obtain canonical frames using the $SO^{+}_{1,1}(\mathbb{R})$-equivariance of the structure function. This leads to the full classification of 3-dimensional left-invariant sub-Lorentzian contact structures. In the following theorem we use the notation of Šnobl and Winternitz [17] for 3-dimensional real Lie algebras. In particular, $L(3,1) = \mathfrak{h}_3$ is the Heisenberg Lie algebra; the Lie algebras $L(3,2,\eta)$, $L(3,4,\eta)$ and $L(3,3)$ are solvable with the special cases $L(3,2,-1) = \mathfrak{p}_{1,1}$ and $L(3,4,0) = \mathfrak{e}_2$ being the Poincaré and Euclidean Lie algebras respectively; $L(3,5)$ and $L(3,6)$ are $\mathfrak{sl}_2$ and $\mathfrak{su}_2$ respectively. Our main results is:

Theorem 1. *All 3-dimensional left-invariant sub-Lorentzian contact structures up to local ts-isometries are given by Table 1, where each connected simply connected Lie group is associated with it's Lie algebra. Each row in the table corresponds to a different structure. In particular*

(1) If $\det h \leq 0$ then the structure is completely determined by the invariants h, κ and τ if τ is defined.
(2) If $\det h > 0$ then every value of κ and h admits 3 locally non-isomorphic structures.

Table 1. 3D left-invariant contact sub-Lorentzian structures

h equivalent to	κ	τ, if defined	Lie Algebra
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\kappa = 0$	$-$	$L(3,1) = \mathfrak{h}_3$
	$\kappa \in \mathbb{R}^*$	$-$	$L(3,5) = \mathfrak{sl}_2$
$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$	$\kappa = 0$	$\tau = 2$	$L(3,3)$
	$\kappa = 0$	$\lvert\tau\rvert > 2$	$L\left(3, 2, \frac{-\tau-\sqrt{\tau^2-4}}{-\tau+\sqrt{\tau^2-4}}\right)$
	$\kappa = 0$	$\lvert\tau\rvert < 2$	$L\left(3, 4, \frac{\lvert\tau\rvert}{\sqrt{\tau^2-4}}\right)$
	$\kappa \in \mathbb{R}^*$	$-$	$L(3,5) = \mathfrak{sl}_2$
$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$	$\kappa = 0$	$\tau \in \mathbb{R}$	$L\left(3, 2, \frac{\tau-\sqrt{\tau^2+4}}{\tau+\sqrt{\tau^2+4}}\right)$
	$\kappa \in \mathbb{R}^*$	$-$	$L(3,5) = \mathfrak{sl}_2$
$\begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}, \chi \neq 0$	$\lvert\kappa\rvert < -\chi$	$-$	$L(3,6) = \mathfrak{su}_2$
	$\lvert\kappa\rvert > -\chi, \lvert\kappa\rvert \neq \chi$	$-$	$L(3,5) = \mathfrak{sl}_2$
	$\chi = \pm\kappa > 0$,	$-$	$L(3,2,-1) = \mathfrak{p}_{1,1}$
	$\chi = \pm\kappa < 0$,	$-$	$L(3,4,0) = \mathfrak{e}_2$
$\begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}, \chi \neq 0$	$\kappa = -7\chi$	$-$	$L(3,3)$
	$\kappa > -7\chi$	$-$	$L\left(3, 2, \frac{\sqrt{\lvert\chi-\kappa\rvert}-\sqrt{7\chi+\kappa}}{\sqrt{\lvert\chi-\kappa\rvert}+\sqrt{7\chi+\kappa}}\right)$
	$\kappa < -7\chi$,	$-$	$L\left(3, 4, \frac{\sqrt{\lvert\chi-\kappa\rvert}}{\sqrt{-7\chi-\kappa}}\right)$
$\begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}, \chi \neq 0$	$\kappa = 7\chi$	$-$	$L(3,3)$
	$\kappa < 7\chi$	$-$	$L\left(3, 2, \frac{\sqrt{\lvert\chi+\kappa\rvert}-\sqrt{7\chi-\kappa}}{\sqrt{\lvert\chi+\kappa\rvert}+\sqrt{7\chi-\kappa}}\right)$
	$\kappa > 7\chi$,	$-$	$L\left(3, 4, \frac{\sqrt{\lvert\chi+\kappa\rvert}}{\sqrt{-7\chi+\kappa}}\right)$
$\begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}, \chi \neq 0$	$\kappa \in \mathbb{R}$	$-$	$L(3,5) = \mathfrak{sl}_2$

We outline the main similarities and differences with the sub-Riemannian case. First of all, as a byproduct of our classification procedure, we see specific similarities with phenomena which arise in [1], namely that the affine group $A_1(\mathbb{R}) \oplus \mathbb{R}$ is locally ts-isometric to $SL_2(R)$. Secondly, the scalar invariants are similar to the sub-Riemannian case with the differences simply in sign. However, the tensor h represents a significant difference. In particular we have 4 cases to consider: $h = 0$; $\det h = 0, h \neq 0$; $\det h > 0$ and $\det h < 0$. Finally, the case $\det h = 0, h \neq 0$, $\kappa = 0$ has a non-discrete family of non-equivalent structures. We introduced the invariant τ, which is in fact the covariant derivative of h, to parametrise this family.

The paper is organized as follows. In section 2 we make precise some terminology concerning orientation and isometries as well as provide a motivating example comming from control theory. In section 3 we outline Cartan geometry and the construction of canonical Cartan connections. In section 4 we apply the Cartan methods to our specific case to produce the invariants of the sub-Lorentzian srtuctures of interest. In section 5 we combine the invariants with the classsification procedure of Šnobl and Winternitz [17] to prove our main theorem 1.

2. Sub-Lorentzian preliminaries

2.1. **Orientation.** Suppose that (M, H, g) is a sub-Lorentzian manifold and let $q \in M$. A vector $v \in H_q$ is said to be timelike if $g(v, v) < 0$, null if $g(v, v) = 0$ and $v \neq 0$, nonspacelike if $g(v, v) \leq 0$, $v \neq 0$, and finally spacelike if $g(v, v) > 0$ or $v = 0$. A vector field on M is called timelike (nonspacelike, null) if its values $X(q)$ have such a property for every $q \in M$. The definition of course implies that any timelike, nonspacelike or null vector field is horizontal in the sense that it is a section of the bundle $H \to M$.

If (H, g) is a sub-Lorentzian metric on M, then it can be proved that H admits a splitting $H = H^- \oplus H^+$ of sub-bunbdles such that H^- is of rank 1 and g is negative (positive) definite on H^- (H^+). Any splitting with the above-mentioned properties is called a causal decomposition for (H, g). We say that the metric (H, g) is time-orientable (resp. space-orientable) if the bundle $H^- \to M$ (resp. $H^+ \to M$) is orientable. Consequently, by a time (space) orientation we mean a given orientation of the bundle $H^- \to M$ $(H^+ \to M)$. Note that since a rank 1 bundle is orientable if and only if it is trivial, time orientability of (M, H, g) is equivalent to the existence of a timelike vector field on M. Thus it is sometimes more convenient to define a time orientation of (M, H, g) as a choice of a timelike vector field X on M.

Since causal decompositions are not unique we must make it precise when two pairs of bundles $H_1^+ \to M$, $H_2^+ \to M$ and $H_1^- \to M$, $H_2^- \to M$ have compatible orientations, where $H = H_1^- \oplus H_1^+$and $H = H_2^- \oplus H_2^+$ are causal decompositions. So we say that the given orientations of $H_1^- \to M$, $H_2^- \to M$ are compatible if $g(X_1, X_2) < 0$, where X_i is the section of $H_i^- \to M$ which agrees with the orientation of $H_i^- \to M$, $i = 1, 2$. On the other hand, $H_1^+ \to M$, $H_2^+ \to M$ have compatible orientations if the following condition is satisfied: for any point $q \in M$ there exists its neighbourhood U and linearly independent sections $X_{i,1}, ..., X_{i,k}$: $U \to H_i^+$, $rank\ H_i^+ = k$, such that $X_{i,1}, ..., X_{i,k}$ agrees with the orientation of $H_i^+ \to M$, $i = 1, 2$, and $\det \left(g(X_{1,i}, X_{1,j})\right)_{i,j=1,...,k} > 0$.

As mentioned earlier, Sub-Lorentzian metrics which are simultaneously time and space oriented will be called ts-oriented. If (M, H, g) is time-oriented by a vector field X then a nonspacelike $v \in H_q$ is said to be future directed if $g(v, X(q)) < 0$.

2.2. **Isometries.** Let (M, H, g) be a sub-Lorentzian manifold. A diffeomorphism $f : M \to M$ is called an isometry if (i) $d_q f(H_q) = H_{f(q)}$ for every $q \in M$, and (ii) $d_q f : H_q \to H_{f(q)}$ is a linear isometry for every q, that is to say if

$g(d_q f(v), d_q f(w)) = g(v,w)$ for all $v,w \in H_q$ and $q \in M$. Of course, all isometries of a given sub-Lorentzian manifold form a group. Let $f : M \to M$ be an isometry and let $H = H^- \oplus H^+$ be a causal decomposition. Then $H = H_1^- \oplus H_1^+$ is also a causal decomposition, where $H_1^\pm = df(H^\pm)$. Suppose now that (M,H,g) is time-oriented (resp. space-oriented). We say that f is a t-isometry (resp. s-isometry) if the fiber bundle map $df|_{H^-} : H^- \to H_1^-$ (resp. $df|_{H^+} : H^+ \to H_1^+$) is orientation preserving. In case (M,H,g) is ts-oriented, f is called a ts-isometry if it preserves both orientations. Clearly, any ts-isometry f is characterized by the condition $d_q f \in SO_{1,1}(k)$ for every $q \in M$.

2.3. **An application.** Sub-Lorentzian manifolds arise in control theory. Suppose that (M,H,g) is a time-oriented sub-Lorentzian manifold, $\operatorname{rank} H = k+1$. Let $X_0, X_1, \dots, X_k$ be an orthonormal basis for (H,g), defined on an open set $U \subset M$, where X_0 is a time orientation. An absolutely continious curve $\gamma\colon (a,b) \to M$ is said to be *nonspacelike future directed*, if $\dot\gamma(t) \in H_{\gamma(t)}$, $g(\dot\gamma(t), X_0(\gamma(t))) \le 0$ and $g(\dot\gamma(t), \dot\gamma(t)) \le 0$ for almost every t in (a,b). It can be proved (see [7]) that up to a reparameterization all nonspacelike future directed curves in U can be obtained as trajectories of the following affine control system on U:

$$\dot q = X_0 + \sum_{i=1}^{k} u_i X_i, \tag{1}$$

where the set of control parameters is equal to

$$\mathcal{C} = \left\{ u \in \mathbb{R}^k;\ \sum_{i=1}^{k} u_i^2 \le 1 \right\},$$

i.e. to the unit ball centered at 0. Note that (1) is not uniquely determined by the structure $(H|_U, g)$, it depends on the choice of an orthonormal frame $X_0, X_1, \dots, X_k$, where X_0 is a time orientation. However, any two such systems are equivalent in the sense that they have the same set of trajectories.

Affine control systems frequently arise in various fields of mathematics and physics (cf. ,[11], [14]), so it is worth noting that our results can be used to classify such systems in the case where M is a 3 dimensional Lie group, g is left-invariant and $k = 1$. For more general systems, Cartan geometry applies equally well, and so the outline given here can also be used in such cases.

3. Cartan Geometries Associated with Structures on Filtered Manifolds

3.1. **Geometric structures on filtered manifolds.** A filtered manifold is basically a manifold endowed with a filtration of the tangent bundle. A manifold with a bracket generating distribution endowed with a (pseudo) sub-Riemannian structure with constant symbol sits within the purview of the general theory of geometric structures on filtered manifolds. In this section we briefly describe some basic constructions from nilpotent differential geometry [13]. Our aim is to illustrate a general strategy for the construction of canonical Cartan connections related to geometries on filtered manifolds.

Definition 1. A filtration of the tangent bundle of a manifold M is a sequence $\{F^i\}_{i\in\mathbb{Z}}$ of sub-bundles of TM such that

i $F^0 = \{0\}$,
ii $F^{i+1} \subset F^i$,
iii $\cup_{i\in\mathbb{Z}} F^i = TM$,
iv $[\underline{F}^i, \underline{F}^j] \subset \underline{F}^{i+j}, \quad \forall i,j \in \mathbb{Z}$,

where $\underline{F}^i$ denotes the sheaf of germs of local sections of F^i. A *filtered manifold* is a smooth manifold M equipped with a filtration of the tangent bundle.

A filtration naturally arises from a smooth bracket generating distribution D. Set $\underline{F}^i = \{0\}$ for all $i \geq 0$ and let $\underline{F}^{-1}$ denote the sheaf of germs of local sections of D. For $i < 0$, we inductively define a sequence of sheaves by setting $\underline{F}^{i-1} = \underline{F}^i + [\underline{F}^i, \underline{F}^{-1}]$. The filtration is the sequence $\{F^i\}$ where each F^i is the union of stalks $F^i = \cup_{p\in M}\underline{F}^i|_p$.

If $\operatorname{gr}_{-i} T_pM = F_p^{-i}/F_p^{-i+1}$, then the graded tangent space is the vector space

$$\operatorname{gr} T_pM = \bigoplus_{i=1}^{k} \operatorname{gr}_{-i} T_pM.$$

If $X \in \Gamma(F^{-i})$ and $Y \in \Gamma(F^{-j})$ are local sections defined on a neighborhood of p, then we have

$$[X + \Gamma(F^{-i+1}), Y + \Gamma(F^{-j+1})] = [X, Y] + \Gamma(F^{-(i+j)+1})$$

on some neighborhood of p, and it follows that the Lie bracket of vector fields induces a well defined Lie bracket on $\operatorname{gr} T_pM$ thus defining a stratified nilpotent Lie algebra of step k. Lie algebra $\operatorname{gr} T_pM$ is often called *a symbol Lie algebra* of the filtration F in the point p. We say that a filtered manifold is *of type* $\mathfrak{m}$ if $\operatorname{gr} T_pM$ equipped with the induced Lie bracket is isomorphic with $\mathfrak{m}$ as a stratified nilpotent Lie algebra at every point p.

Definition 2. Let U be an open subset of a filtered manifold M of type $\mathfrak{m}$. *A local weighted frame* is a map $\Phi : U \times \mathfrak{m} \to \operatorname{gr} TM$ such that for each point $p \in U$ the map $\Phi_p : \mathfrak{m} \to \operatorname{gr} T_pM$ is a strata preserving Lie algebra isomorphism. The union of germs of local weighted frames over all points of M defines an $\operatorname{Aut}(\mathfrak{m})$-principle bundle which we refer to as the *weighted frame bundle* of the filtered manifold M of type $\mathfrak{m}$.

Definition 3. If M is filtered of type $\mathfrak{m}$ then a local frame $\{E_j^{-i}\}$ of TM is said to be *adapted* if there exists a weighted frame Φ such that $\Phi(p, e_j^{-i}) = E_j^{-i}(p) + F_p^{-i+1}$ where e_j^{-i} form a basis of $\mathfrak{m}_{-i}$. If $\{\omega_{-i}^j\} \subset T^*M$ is the set of dual elements to $\{E_j^{-i}\}$, then the map $e_j^{-i} \otimes \omega_{-i}^j \colon TM \to \mathfrak{m}$ is called *an adapted coframe.*

Definition 4. A *first order geometric structure* on a filtered manifold of type $\mathfrak{m}$ is a reduction of the weighted frame bundle to the subgroup $G_0 \subset \operatorname{Aut}(\mathfrak{m})$. We call a weighted frame which belongs to the reduced bundle *a weighted frame adapted to the structure.*

Consider a (pseudo) sub-Riemannian structure which is defined by a pseudo-Riemannian metric g on the distribution $\mathcal{H} \subset TM$. If the distribution is bracket generating then M is a filtered manifold and the metric g defines a metric (pseudo) sub-Riemannian symbol $(g, \mathfrak{m})$ in every point. If metric symbols are equivalent for all points we say that the corresponding structure has constant metric symbol. For such structures the weighted frame bundle can be reduced to the structure group $G_0 \subseteq SO(\mathfrak{m}_{-1})$ which gives us a canonically defined first order geometric structure on M.

3.2. **Cartan geometries as a generalization of homogeneous spaces.** A homogeneous space for a Lie group G is a manifold M on which the Lie group G acts from the right both transitively and effectively. If H is the stabilizer of an arbitrary point $p \in M$ then H is a closed Lie subgroup of G. The manifold M is diffeomorphic to the right coset space G/H under the action $(Ha)g = H(ag)$.

Consider the 3 natural actions which are defined for all Lie groups G:

- left multiplication $L_g(a) = ga$,
- right multiplication $R_g(a) = ag$,
- conjugation $C_g(a) = gag^{-1}$,

All maps above are diffeomorphisms and their differentials are denoted by L_{g*}, R_{g*} and Ad_g respectively. In particular, the tangent map

$$L_{g^{-1}*} \colon T_gG \to T_eG$$

is an isomorphisms of tangent spaces. The *Maurer-Cartan form* $\tilde{\omega} \colon TG \to \mathfrak{g}$ is defined pointwise by $\tilde{\omega}_g = L_{g^{-1}*}$ and satisfies the following 3 properties which will be crucial in the definition of Cartan connection:

(1) $\tilde{\omega}_g \colon T_gG \to \mathfrak{g}$ is an isomorphism;
(2) $R_g^*\tilde{\omega} = Ad_{g^{-1}}\tilde{\omega}$;
(3) for all left-invariant vector field X we have $\tilde{\omega}_g(X_g) = X_e$.

We introduce the commutator on forms with values in a Lie algebra $\mathfrak{g}$ defined by

$$[\alpha, \beta](X, Y) = [\alpha(X), \beta(Y)] + [\beta(X), \alpha(Y)].$$

In particular

$$[\alpha(X), \alpha(Y)] = \frac{1}{2}[\alpha, \alpha](X, Y).$$

If e_i is basis of $\mathfrak{g}$, $\alpha = \sum_i e_i \otimes \alpha_i$ and $\beta = \sum_i e_i \otimes \beta_i$, then $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \sum_{i,j} [e_i, e_j] \otimes \alpha_i \wedge \beta_j.$$

One of the key properties of the Maurer-Cartan form $\tilde{\omega}$ is that the following structure equation holds:

$$d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = 0.$$

To show that structure equation holds, it is sufficient to check it for left-invariant vector fields. Using Cartan's formula we obtain

$$\begin{aligned} d\tilde{\omega}(X, Y) &= X\tilde{\omega}(Y) - Y\tilde{\omega}(X) - \tilde{\omega}([X, Y]) \\ &= -\tilde{\omega}([X, Y]) = -[\tilde{\omega}(X), \tilde{\omega}(Y)] = -\frac{1}{2}[\tilde{\omega}, \tilde{\omega}](X, Y), \end{aligned}$$

where X and Y are left-invariant vector fields.

Cartan geometries generalize homogeneous spaces $G \to G/H$ simply by considering a general principal H-bundle $\mathcal{G} \to \mathcal{G}/H$ and prescribing an object on $\mathcal{G}$ analogous to the Maurer Cartan form on G, where properties analogous to (2) and (3) above are only required to hold with respect to H.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of the Lie groups G and H respectively.

Definition 5. A *Cartan geometry* of infinitesimal type $(\mathfrak{g}, \mathfrak{h})$ on a manifold M is a principal H-bundle $\mathcal{G}$ over M together with a form $\tilde{\omega} : T\mathcal{G} \to \mathfrak{g}$, called the *Cartan connection form*, having the following properties:

(1) $\tilde{\omega}_p \colon T_p\mathcal{G} \to \mathfrak{g}$ is an isomorphism;
(2) $R_h^*\tilde{\omega} = \mathrm{Ad}_{h^{-1}}\,\tilde{\omega}$ for all $h \in H$;
(3) $\tilde{\omega}(X^*) = X$ where X^* is a fundamental vector field corresponding to $X \in \mathfrak{h}$, i.e., $X^*f(p) = \frac{d}{dt}f(p\exp(tX))|_{t=0}$.

The Maurer-Cartan structure equation doesn't hold for general Cartan connections. The $\mathfrak{g}$-valued 2-form $\tilde{\Omega} \in \mathfrak{g} \otimes \bigwedge^2 T\mathcal{G}^*$ given by the formula

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}],$$

is called the *curvature form.* A fundamental property of the curvature is that $v_p \lrcorner \tilde{\Omega}_p = 0$ for all v_p belonging to the vertical sub-bundle $\mathcal{V} = \ker \pi_*$, where $\pi : \mathcal{G} \to M$ is the natural projection, see [16, p. 187].

On the manifold M the relevant object to study is the pull back of the Cartan connection by a section of the principle bundle $\mathcal{G}$.

Definition 6. Given an arbitrary section $s\colon M \to \mathcal{G}$, the *Cartan gauge* corresponding to s is the one form $\omega = s^*\tilde{\omega}$.

Consider a change of section $\bar{s} = sh$, where $h\colon M \to H$. Then the Cartan gauge changes in the following way:

$$\bar{\omega} = \bar{s}^*\tilde{\omega} = \operatorname{Ad}_{h^{-1}} \omega + h^*\omega_H = Ad_{h^{-1}}\omega + h^{-1}dh, \tag{2}$$

where ω_H is Maurer-Cartan form on H.

The pull-back of the curvature $\tilde{\Omega}$ on the principle H-bundle $\mathcal{G}$ is a two form $\Omega = s^*\tilde{\Omega}$ on the manifold M and is given by the following formula

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

If we change section $\bar{s} = sh$ the curvature on the manifold changes by the adjoint action of h^{-1}:

$$\bar{\Omega} = \bar{s}^*\tilde{\Omega} = \operatorname{Ad}_{h^{-1}} \Omega. \tag{3}$$

3.3. **Cartan connections associated with structures on filtered manifold.** The problem of equivalence between geometric structures on manifolds is typically solved by applying Cartan's method of equivalence to produce a Cartan connection and use its curvature as the natural invariant.

When the underlying manifolds are filtered, the target Lie algebra for a Cartan connection is given by the Tanaka prolongation of the pair $(\mathfrak{m}, \mathfrak{g}_0)$ where $\mathfrak{g}_0$ is a subalgebra of the strata preserving derivations of $\mathfrak{m}$ such that $G_0 = \exp(\mathfrak{g}_0)$, see [19].

Consider a graded nilpotent Lie algebra $\mathfrak{m} = \mathfrak{m}_{-k} \oplus \cdots \oplus \mathfrak{m}_{-1}$. Let $\mathfrak{g}_0$ be a subalgebra of the grading preserving derivations of $\mathfrak{m}$. The Tanaka prolongation of the pair $(\mathfrak{m}, \mathfrak{g}_0)$ is the graded Lie algebra $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ where $\mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m}_i$ for $-k \le i < 0$, $\mathfrak{g}_0(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{g}_0$ and for each $i > 0$, $\mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0)$ is inductively defined by

$$\mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0) = \Big\{\varphi \in \bigoplus_{p>0} \mathfrak{g}_{i-p}(\mathfrak{m}, \mathfrak{g}_0) \otimes \mathfrak{g}^*_{-p} \mid \varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)]\Big\}.$$

The pair $(\mathfrak{m}, \mathfrak{g}_0)$ is said to be of *finite type* if $\mathfrak{g}_i(\mathfrak{m}, \mathfrak{g}_0) = \{0\}$ for some i, otherwise it is of *infinite type* and $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is infinite dimensional.

Consider a first order geometric structure of type $\mathfrak{m}$ on the filtered manifold M. Let $\mathfrak{g} = \mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ and $\mathfrak{g}_+ = \oplus_{i>0}\mathfrak{g}_i$. We define a trivial H-principal bundle $\mathcal{G} = H \times M$ where

$$H = G_0 \times \exp(\mathfrak{g}_+).$$

With the given first order geometric structure we associate a family of adapted Cartan connections of type $(\mathfrak{g}, \mathfrak{h})$.

Definition 7. A Cartan connection $\tilde{\omega}\colon T\mathcal{G} \to \mathfrak{g}$ is called *adapted* if for an arbitrary section $s\colon M \to \mathcal{G}$, the corresponding Cartan gauge $\omega\colon TM \to \mathfrak{g}$, has the property that the $\mathfrak{m}$ valued part forms an adapted coframe.

To obtain invariants of the initial geometric structure we want to associate a unique adapted Cartan connection to the structure. The construction of the desired connection can be done using normalization of the structure function.

Definition 8. The *curvature function* $\tilde{k}\colon \mathcal{G} \to \mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})$ of a Cartan connection $\tilde{\omega}$ is defined by the formula

$$\tilde{k}(X, Y) = \tilde{\Omega}\left(\tilde{\omega}^{-1}(X), \tilde{\omega}^{-1}(Y)\right).$$

A built in property of the curvature function is that it is H-equivariant, i.e.,

$$R_h^*\tilde{k} = Ad_{h^{-1}}\tilde{k}.$$

The vector space $\mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g}) = \mathfrak{g} \otimes \bigwedge^2 \mathfrak{m}^*$ has a natural grading. Elements in the subspace $\mathfrak{g}_l \otimes \mathfrak{m}^*_{-i} \wedge \mathfrak{m}^*_{-j}$ are assigned weight $w = l + i + j$.

Definition 9. We call a Cartan connection and underlying Cartan geometry *regular* if $\tilde{k}$ takes values in $\mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})_+$, where $+$ subscript means a positive degree part of a space.

Definition 10. A subspace $N \subset \mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})_+$ is called a normal module if:

(1) N is an H module with respect to the adjoint action of H on $\mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})$;
(2) $\mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})_+ = N \oplus \partial\left(\mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g})_+\right)$, where ∂ is the Lie algebra differential.

The existence of unique Cartan connection associated with a geometric structure is a fundamental starting point in the study of equivalence problems for the given geometric structure. For example regular parabolic geometries (i.e. with semi-simple model group) admit a natural and uniform notion of normal Cartan connection [2]. The general result regarding existence of normal Cartan connections can be formulated as follows:

Theorem 2. [13, p. 92] *Consider a geometric structure with an infinitesimal model* $(\mathfrak{g}, \mathfrak{h})$ *on a filtered manifold. Then for every normal module*

$$N \subset \mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})_+$$

there exists a unique regular Cartan connection adapted to the structure such that the curvature function takes values in N*.*

In the next section we are going to construct various invariant objects associated with sub-Lorentzian structures on a contact 3-manifold M. The main ingredients are a normal module and the corresponding normal Cartan connection given by Theorem 2. A canonical pullback of the Cartan connection to M induces differential invariants for the structures at the level of M and allows us to construct canonical frames for sub-Lorentzian contact structures.

4. Invariants of 3-dimensional sub-Lorentzian contact structures

4.1. **First order geometric structures associated with 3-dimensional sub-Lorentzian contact structures.** The sub-Lorentzian contact structure is given by a contact distribution $\mathcal{H}$ and a sub-Lorentzian metric g which is defined on $\mathcal{H}$. Let X_1 and X_2 be an orthogonal frame of $\mathcal{H}$ in the following sense:

$$g(X_1, X_1) = -1,\; g(X_1, X_2) = 0,\; g(X_2, X_2) = 1.$$

We choose a contact form η so that $d\eta(X_1, X_2) = \eta([X_2, X_1]) = 1$ and denote the corresponding Reeb vector field by X_3, i.e., $X_3 \lrcorner d\eta = 0$ and $\eta(X_3) = -1$. The Lie brackets are then given by 6 structure functions according to the following relations:

$$[X_1, X_3] = c^1_{13}X_1 + c^2_{13}X_2$$
$$[X_2, X_3] = c^1_{23}X_1 + c^2_{23}X_2$$
$$[X_1, X_2] = c^1_{12}X_1 + c^2_{12}X_2 + X_3.$$

The coframe dual to the frame $\{X_1, X_2, X_3\}$ is an adapted coframe and denoted $\{\theta_1, \theta_2, \theta_3\}$. Using Cartan's formula we get following structure equations for the adapted coframe:

$$\begin{aligned} d\theta_1 &= c^1_{12}\theta_2 \wedge \theta_1 + c^1_{13}\theta_3 \wedge \theta_1 + c^1_{23}\theta_3 \wedge \theta_2 \\ d\theta_2 &= c^2_{12}\theta_2 \wedge \theta_1 + c^2_{13}\theta_3 \wedge \theta_1 + c^2_{23}\theta_3 \wedge \theta_2 \\ d\theta_3 &= \theta_2 \wedge \theta_1 \end{aligned}$$

Since $d^2\theta_3 = 0$ we immediately get that $c^1_{13} + c^2_{23} = 0$, hence letting $c^1_{13} = c$ and $c^2_{23} = -c$ we get

$$\begin{aligned} [X_1, X_3] &= cX_1 + c^2_{13}X_2 \\ [X_2, X_3] &= c^1_{23}X_1 - cX_2 \\ [X_1, X_2] &= c^1_{12}X_1 + c^2_{12}X_2 + X_3. \end{aligned} \tag{4}$$

and

$$\begin{aligned} d\theta_1 &= c^1_{12}\theta_2 \wedge \theta_1 + c\theta_3 \wedge \theta_1 + c^1_{23}\theta_3 \wedge \theta_2 \\ d\theta_2 &= c^2_{12}\theta_2 \wedge \theta_1 + c^2_{13}\theta_3 \wedge \theta_1 - c\theta_3 \wedge \theta_2 \\ d\theta_3 &= \theta_2 \wedge \theta_1. \end{aligned} \tag{5}$$

If $\mathcal{H}$ denotes the contact distribution and g denotes the sub-Lorentzian metric on $\mathcal{H}$, then the filtration of the tangent bundle is given by

$$F^0 = \{0\}, \quad F^{-1} = \mathcal{H}, \quad F^{-2} = \mathcal{H} + [\mathcal{H}, \mathcal{H}],$$

where $\mathrm{gr}_{-1} TM = \mathcal{H}$. The type in such cases is given by the Heisenberg algebra. In particular

$$\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1},$$

where $\mathfrak{m}_{-1} = \langle e_1, e_2 \rangle$, $\mathfrak{m}_{-2} = \langle e_3 \rangle$ and $[e_1, e_2] = e_3$.

An adapted to sub-Lorentzian 3-dimensional contacts structure weighted frame takes the form

$$\begin{aligned} \Phi(p, e_1) &= Y_1 \in H \\ \Phi(p, e_2) &= Y_2 \in H \\ \Phi(p, e_3) &= [Y_1, Y_2] \in TM/H \end{aligned}$$

where $g(Y_1, Y_1) = -1$, $g(Y_1, Y_2) = 0$, $g(Y_2, Y_2) = 1$. Since our interest is in *ts*-isometric equivalence, we consider the $SO^+_{1,1}(\mathbb{R})$-principle bundle of ts-oriented weighted frames. It follows that adapted to the structure frame is of the form

$$\begin{aligned} E_1(p) &= a_{12}(p)X_1 + a_{22}(p)X_2, \\ E_2(p) &= a_{12}(p)X_1 + a_{22}(p)X_2, \\ E_3(p) &= b_1(p)X_1 + b_2(p)X_2 + X_3, \end{aligned}$$

where $\left(\begin{smallmatrix} a_{11}(p) & a_{12}(p) \\ a_{21}(p) & a_{22}(p) \end{smallmatrix}\right) \in SO^+_{1,1}(\mathbb{R})$ and X_1, X_2, X_3 are as in (4).

For sub-Lorentzian structures on a contact three manifold, the pair $(\mathfrak{m}, \mathfrak{g}_0)$ consists of the Heisenberg algebra $\mathfrak{m} = \operatorname{span}\{e_1, e_2, e_3 = [e_1, e_2]\}$ and $\mathfrak{g}_0$ is spanned by $\{e_4\}$ with relations

$$[e_1, e_2] = e_3, \; [e_4, e_1] = e_2, \; [e_4, e_2] = e_1.$$

Lemma 1. *The Tanaka prolongation for this structure is* $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$.

Proof. Consider an arbitrary element $\varphi \in \mathfrak{g}_0 \otimes \mathfrak{g}_{-1}^* \oplus \mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}^*$ in the first prolongation. Then

$$0 = \varphi([e_1, e_3]) = [\varphi(e_1), e_3] + [e_1, \varphi(e_3)] = [e_1, \varphi(e_3)],$$
$$0 = \varphi([e_2, e_3]) = [\varphi(e_2), e_3] + [e_2, \varphi(e_3)] = [e_2, \varphi(e_3)].$$

Since $\varphi(e_3) \in \mathfrak{g}_{-1}$ it must be equal to 0. Let $\varphi(e_1) = a_1 e_4$ and $\varphi(e_2) = a_2 e_4$. Then the following equality shows that $\varphi = 0$:

$$0 = \varphi(e_3) = \varphi([e_1, e_2]) = [a_1 e_4, e_2] + [e_1, a_2 e_4] = a_1 e_1 - a_2 e_2.$$

□

4.2. **Normal Cartan geometry associated with 3-dimensional sub-Lorentzian contact structures.** As shown in Lemma 1, the infinitesimal flat model for sub-Lorentzian structures on contact 3–manifolds is given by the 4-dimensional graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

with basis $\{e_1, e_2, e_3, e_4\}$ satisfying the following relations:

$$[e_1, e_2] = e_3, \ [e_4, e_1] = e_2 \quad \text{and} \quad [e_4, e_2] = e_1,$$

where $\mathfrak{g}_{-2} = \operatorname{span}\{e_3\}$, $\mathfrak{g}_{-1} = \operatorname{span}\{e_1, e_2\}$ and $\mathfrak{g}_0 = \operatorname{span}\{e_4\}$.

A Cartan connection for sub-Lorentzian contact structure on a 3-dimensional manifold M is defined on a principle $SO^+_{1,1}(\mathbb{R})$-bundle $\mathcal{G}$. Since we are interested in local equivalence, we can assume $\mathcal{G}$ is the trivial bundle $SO^+_{1,1}(\mathbb{R}) \times U$ where U is an open subset of M.

Consider an arbitrary Cartan connection $\tilde{\omega} = \sum_{i=1} \tilde{\omega}_i e_i \colon T\mathcal{G} \to \mathfrak{g}$ for sub-Lorentzian structure on a 3-dimensional contact manifold M. The curvature of this connection is

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \sum_{l=1}^{4} \sum_{1 \le i < j \le 3} k^l_{ij} e_l \otimes \tilde{\omega}_i \wedge \tilde{\omega}_j \colon \ \wedge^2 T\mathcal{G} \to \mathfrak{g},$$

and the corresponding curvature function has the form

$$\tilde{k} = \sum_{l=1}^{4} \sum_{1 \le i < j \le 3} k^l_{ij} e_l \otimes e^*_i \wedge e^*_j \colon \mathcal{G} \to \operatorname{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g}).$$

Proposition 1. *For an arbitrary sub-Lorentzian structure there exists a unique Cartan connection $\tilde{\omega} = \sum_i \tilde{\omega}_i e_i \colon T\mathcal{G} \to \mathfrak{g}$ with the curvature function taking values in the following 6-dimensional $SO^+_{1,1}(\mathbb{R})$-module $\tilde{N}$:*

$$e_1 \otimes e^*_1 \wedge e^*_3 - e_2 \otimes e^*_2 \wedge e^*_3; \ e_1 \otimes e^*_2 \wedge e^*_3; \ e_2 \otimes e^*_1 \wedge e^*_3;$$
$$e_1 \otimes e^*_1 \wedge e^*_3 + e_2 \otimes e^*_2 \wedge e^*_3; \ e_4 \otimes e^*_1 \wedge e^*_3; \ e_4 \otimes e^*_2 \wedge e^*_3.$$

Proof. In order to use Theorem 2 we need to show that $\tilde{N}$ is an $SO^+_{1,1}(\mathbb{R})$-module and is complementary to the image of Lie algebra differential

$$\partial \colon \operatorname{Hom}(\mathfrak{g}_-, \mathfrak{g})_+ \to \operatorname{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g})_+.$$

The image of the differential ∂ is generated by the following 5 elements:

$$e_1 \otimes e^*_1 \wedge e^*_2 - e_3 \otimes e^*_2 \wedge e^*_3; \ e_2 \otimes e^*_1 \wedge e^*_2 + e_3 \otimes e^*_1 \wedge e^*_3;$$
$$e_1 \otimes e^*_1 \wedge e^*_2; \ e_2 \otimes e^*_1 \wedge e^*_2; \ e_4 \otimes e^*_1 \wedge e^*_2 - e_2 \otimes e^*_1 \wedge e^*_3 - e_1 \otimes e^*_2 \wedge e^*_3,$$

and doesn't intersect $\tilde{N}$. The space $\operatorname{Hom}(\wedge^2 \mathfrak{g}_-, \mathfrak{g})_+$ is 11-dimensional, therefor $\tilde{N}$ is complementary to $\operatorname{im} \partial$.

An element $h \in SO^+_{1,1}(\mathbb{R})$ acts naturally on 2-dimensional space $\langle e_1, e_2 \rangle$, acts by inverse transform h^{-1} on $\langle e^*_1, e^*_2 \rangle$ and trivially by identity on $\{e_3, e_4, e^*_3, e^*_4\}$. This defines and action of $SO^+_{1,1}(\mathbb{R})$ on $\mathrm{Hom}(\wedge^2\mathfrak{g}_-,\mathfrak{g})_+$ and we can see that $\tilde{N}$ is in fact an $SO^+_{1,1}(\mathbb{R})$-module. $\square$

Proposition 2. *Assume that the the curvature function of the Cartan connection belongs to the module $\tilde{N}$ defined in Proposition 1. Then the coefficient of*

$$e_1\otimes e_1^*\wedge e_3^*+e_2\otimes e_2^*\wedge e_3^*$$

is equal to zero.

Furthermore, the coefficients of $e_4\otimes e_1^*\wedge e_3^*$ and $e_4\otimes e_2^*\wedge e_3^*$ are linear combinations of the covariant derivatives of the coefficients of

$$e_1\otimes e_1^*\wedge e_3^*-e_2\otimes e_2^*\wedge e_3^*,\quad e_1\otimes e_2^*\wedge e_3^*\quad \textit{and}\quad e_2\otimes e_1^*\wedge e_3^*.$$

Proof. Let $\tilde{\Omega}=d\tilde{\omega}+\frac{1}{2}[\tilde{\omega},\tilde{\omega}]$ be the curvature of the normal Cartan connection $\tilde{\omega}$. The fact that curvature function $\tilde{k}(\cdot,\cdot)=\tilde{\Omega}\left(\tilde{\omega}^{-1}(\cdot),\tilde{\omega}^{-1}(\cdot)\right)$ belongs to $\tilde{N}$ is equivalent by definition to

$$\begin{aligned}\tilde{\Omega}=&k_1(e_1\otimes\tilde{\omega}_1\wedge\tilde{\omega}_3+e_2\otimes\tilde{\omega}_2\wedge\tilde{\omega}_3)+k_2(e_1\otimes\tilde{\omega}_1\wedge\tilde{\omega}_3-e_2\otimes\tilde{\omega}_2\wedge\tilde{\omega}_3)+\\ &k_3e_1\otimes\tilde{\omega}_2\wedge\tilde{\omega}_3+k_4e_2\otimes\tilde{\omega}_1\wedge\tilde{\omega}_3+k_5e_4\otimes\tilde{\omega}_1\wedge\tilde{\omega}_3+k_6e_4\otimes\tilde{\omega}_2\wedge\tilde{\omega}_3.\end{aligned}\tag{6}$$

The Bianchi identity [16, p. 193] states that

$$d\tilde{\Omega}+[\tilde{\omega},\tilde{\Omega}]=0.\tag{7}$$

By (6), $d\tilde{\Omega}$ has a trivial projection onto e_3. Moreover, using $\tilde{\omega}=\sum_i e_i\otimes\tilde{\omega}_i$ and directly calculating gives

$$[\tilde{\omega},\tilde{\Omega}]=2k_1e_3\otimes\tilde{\omega}_1\wedge\tilde{\omega}_2\wedge\tilde{\omega}_3\mod\langle e_1,e_2\rangle$$

and so $k_1=0$.

Consider now the projection of Bianchi identity on $\mathfrak{g}_{-1}=\langle e_1,e_2\rangle$. The $\mathfrak{g}_{-1}$-part of $[\tilde{\omega},\tilde{\Omega}]$ is given by

$$[e_1\tilde{\omega}_1+e_2\tilde{\omega}_2,k_5e_4\otimes\tilde{\omega}_1\wedge\tilde{\omega}_3+k_6e_4\otimes\tilde{\omega}_2\wedge\tilde{\omega}_3]=k_5e_1\tilde{\omega}_1\wedge\tilde{\omega}_2\wedge\tilde{\omega}_3-k_6e_2\tilde{\omega}_1\wedge\tilde{\omega}_2\wedge\tilde{\omega}_3.\tag{8}$$

The formula for $\mathfrak{g}_{-1}$-part of $d\tilde{\Omega}$ is also straightforward:

$$d\tilde{\Omega}=\Big((\tilde{X}_1k_3-\tilde{X}_2k_2)e_1-(\tilde{X}_2k_4+\tilde{X}_1k_2)e_2\Big)\tilde{\omega}_1\wedge\tilde{\omega}_2\wedge\tilde{\omega}_3\mod\tilde{\omega}_4,\tag{9}$$

where $\tilde{X}_i=\tilde{\omega}^{-1}(e_i)\in\Gamma(T\mathcal{G})$ are universal covariant differentiations defined by the normal Cartan connection, see [16, p. 194]. Comparing (8) and (9) we conclude that

$$k_5=\tilde{X}_2k_2-\tilde{X}_1k_3,\ k_6=\tilde{X}_2k_4+\tilde{X}_1k_2.$$

$\square$

We denote by N the "essential" part of $\tilde{N}$ which by Proposition 2 is the submodule generated by

$$\begin{aligned}K&=e_1\otimes e_1^*\wedge e_3^*-e_2\otimes e_2^*\wedge e_3^*,\\ X&=e_1\otimes e_2^*\wedge e_3^*,\\ Y&=e_2\otimes e_1^*\wedge e_3^*,\end{aligned}\tag{10}$$

Since $SO^+_{1,1}(\mathbb{R})$ acts on e_3^* by identity we see that actually $SO^+_{1,1}(\mathbb{R})$ acts on K, X and Y as on $e_1\otimes e_1^*-e_2\otimes e_2^*$, $e_1\otimes e_2^*$ and $e_2\otimes e_1^*$. The later is exactly the adjoint action of $SO^+_{1,1}(\mathbb{R})$ on $\mathfrak{sl}_2(\mathbb{R})$ given by $A\to TAT^{-1}$ where

$$K=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad\text{and}\quad T=\begin{pmatrix}\cosh(t)&\sinh(t)\\ \sinh(t)&\cosh(t)\end{pmatrix}.\tag{11}$$

4.3. **Local computation of normal Cartan connection.** Consider a principle $SO^+_{1,1}(\mathbb{R})$-bundle $\mathcal{G} = SO^+_{1,1}(\mathbb{R}) \times U$ where U is an open subset of M. Let $\tilde{\omega}\colon T\mathcal{G} \to \mathfrak{g}$ be an arbitrary Cartan connection adapted to the sub-Lorentzian structure $(M, \mathcal{H}, g)$. For an arbitrary section $s\colon U \to \mathcal{G}$ the corresponding Cartan gauge $\omega = s^*\tilde{\omega}\colon TU \to \mathfrak{g}$ adapted to the sub-Lorentzian structure has the form:

$$\omega = e_1 \otimes \omega_1 + e_2 \otimes \omega_2 + e_3 \otimes \omega_3 + e_4 \otimes \omega_4,$$

where

$$\begin{aligned}
\omega_1 &= \bar{a}_{11}\theta_1 + \bar{a}_{12}\theta_2 + \bar{\alpha}_1\theta_3,\\
\omega_2 &= \bar{a}_{21}\theta_1 + \bar{a}_{22}\theta_2 + \bar{\alpha}_2\theta_3,\\
\omega_3 &= \theta_3,\\
\omega_4 &= \bar{\beta}_1\theta_1 + \bar{\beta}_2\theta_2 + \bar{\beta}_3\theta_3
\end{aligned}$$

and the matrix $(\bar{a}_{ij})$ is an element of $SO^+_{1,1}(\mathbb{R})$. Changing the section by the right action of suitable element in $SO^+_{1,1}(\mathbb{R})$ according to formula 2 gives the following lemma.

Lemma 2. *There exists a unique section such that a Cartan gauge adapted to the sub-Lorentzian structure has the form:*

$$\begin{aligned}
\omega_1 &= \theta_1 + \alpha_1\theta_3,\\
\omega_2 &= \theta_2 + \alpha_2\theta_3,\\
\omega_3 &= \theta_3,\\
\omega_4 &= \beta_1\theta_1 + \beta_2\theta_2 + \beta_3\theta_3.
\end{aligned}$$

If $\omega = s^*\tilde{\omega}$ then $\Omega = s^*\tilde{\Omega}\colon \wedge^2 TM \to \mathfrak{g}$ is given by $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ which for our specific case takes the form:

$$\Omega = e_1 \otimes \Omega_1 + e_2 \otimes \Omega_2 + e_3 \otimes \Omega_3 + e_4 \otimes \Omega_4,$$

where

$$\begin{aligned}
\Omega_1 &= d\omega_1 - \omega_2 \wedge \omega_4,\\
\Omega_2 &= d\omega_2 - \omega_1 \wedge \omega_4,\\
\Omega_3 &= d\omega_3 - \omega_2 \wedge \omega_1,\\
\Omega_4 &= d\omega_4.
\end{aligned}$$

In accordance with Propositions 1 and 2, the normal Cartan connection satisfies the conditions

$$\begin{aligned}
&\Omega_3 = 0,\\
&\Omega_i = 0 \mod \omega_3,\ i = 1, 2, 4.
\end{aligned}$$

The first condition is :

$$\Omega_3 = d\omega_3 - \omega_2 \wedge \omega_1 = \alpha_1\omega_3 \wedge \omega_2 - \alpha_2\omega_3 \wedge \omega_1 = 0.$$

Therefore, $\alpha_1 = \alpha_2 = 0$. The second normalization condition is $\Omega_1 \mod \omega_3 = \Omega_2 \mod \omega_3 = 0$. This condition gives us:

$$\begin{aligned}
\Omega_1 &= d\omega_1 - \omega_2 \wedge \omega_4 = (\beta_1 - c^1_{12})\omega_1 \wedge \omega_2 = 0 \mod \omega_3,\\
\Omega_2 &= d\omega_2 - \omega_1 \wedge \omega_4 = (-\beta_2 - c^2_{12})\omega_1 \wedge \omega_2 = 0 \mod \omega_3,
\end{aligned}$$

From the formulas above we obtain $\beta_1 = c^1_{12}$ and $\beta_2 = -c^2_{12}$. The last normalization condition is $\Omega_4 \mod \omega_3 = 0$:

$$\Omega_4 = d\omega_4 = (-\beta_3 - X_2(\beta_1) + X_1(\beta_2) - \beta_1 c^1_{12} - \beta_2 c^2_{12})\omega_1 \wedge \omega_2 = 0 \mod \omega_3$$

We obtain that $\beta_3 = (c^2_{12})^2 - (c^1_{12})^2 - X_1(c^2_{12}) - X_2(c^1_{12})$.

To summarize, the coefficients of Ω have the form:

$$\Omega_1 = -c\omega_1 \wedge \omega_3 - (c^1_{23} + \beta_3)\omega_2 \wedge \omega_3, \tag{12}$$

$$\Omega_2 = -(c^2_{13} + \beta_3)\omega_1 \wedge \omega_3 + c\omega_2 \wedge \omega_3, \tag{13}$$

$$\Omega_3 = 0, \tag{14}$$

$$\Omega_4 = \big(X_1(\beta_3) - X_3(\beta_1) - \beta_1 c - \beta_2 c^2_{13}\big)\,\omega_1 \wedge \omega_3 \tag{15}$$

$$+ \big(X_2(\beta_3) - X_3(\beta_2) - \beta_1 c^1_{23} + \beta_2 c\big)\,\omega_2 \wedge \omega_3. \tag{16}$$

Where $\beta_1 = c^1_{12}$, $\beta_2 = -c^2_{12}$ and $\beta_3 = (c^2_{12})^2 - (c^1_{12})^2 - X_1(c^2_{12}) - X_2(c^1_{12})$.

4.4. **Invariants of sub-Lorentzian structure.** A normal Cartan connection is a special type of absolute parallelism. The problem of equivalence of absolute parallelisms is a classical subject and was studied in details for example in [18]. In particular, local invariants of an absolute parallelism are precisely its structure function and its consecutive covariant derivatives. A finite number of these invariants uniquely (up to local equivalence) determines the absolute parallelism.

Applied to a normal Cartan connection, this means that all local invariants of a given Cartan connection can be derived from its structure function and consecutive covariant derivatives. However, due to H-equivariance of the structure function $\tilde{k}\colon \mathcal{G} \to \mathrm{Hom}(\wedge^2\mathfrak{g}_-, \mathfrak{g})$, one can simplify k by introducing a canonical section $s\colon M \to \mathcal{G}$ and considering a canonical pullback $k = s^*\tilde{k}$. This allows us to obtain invariants generated by k that are defined on the manifold M instead of the principal bundle $\mathcal{G}$.

Let k_N be the part of the curvature function taking values in the module N generated by (10). According to Proposition 2, the entire curvature function k can be expressed through k_N using covariant differentiation. Therefore we need only focus our attention on $k_N : M \to \mathfrak{sl}_2(\mathbb{R})$.

Under the adjoint action, the $SO^+_{1,1}(\mathbb{R})$-module $\mathfrak{sl}_2(\mathbb{R})$ has 2 irreducible submodules. One is generated by the matrix

$$f_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{17}$$

and the other generated by the pair of matrices

$$f_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{18}$$

If we write $k_N = \kappa f_0 + a f_1 + b f_2$ then

$$\kappa = \beta_3 + \frac{c^1_{23} + c^2_{13}}{2} = (c^2_{12})^2 - (c^1_{12})^2 - X_1(c^2_{12}) - X_2(c^1_{12}) + \frac{c^1_{23} + c^2_{13}}{2},$$
$$a = c$$
$$b = (c^1_{23} - c^2_{13})/2$$

Under the change of section $\bar{s} = R_h(s)$, $h\colon M \to SO^+_{1,1}$ we get

$$\bar{k}_N = \bar{s}^*\tilde{k}_N = Ad_{h^{-1}}(s^*\tilde{k}_N) = Ad_{h^{-1}}(k_N).$$

Since $SO^+_{1,1}(\mathbb{R})$ acts on (17) as the identity, κ doesn't depend on the choice of section and is an invariant. We summarize this observation in the following proposition.

Proposition 3. *The following expression*

$$\kappa = \beta_3 + \frac{c^1_{23} + c^2_{13}}{2} = (c^2_{12})^2 - (c^1_{12})^2 - X_1(c^2_{12}) - X_2(c^1_{12}) + \frac{c^1_{23} + c^2_{13}}{2}. \tag{19}$$

is an invariant of time-space orientation preserving structure.

Consider now the submodule of $\mathfrak{sl}_2(\mathbb{R})$ generated by

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix}. \tag{20}$$

The corresponding part of k_N is

$$h = af_1 + bf_2 = \begin{pmatrix} c & \frac{c^1_{23} - c^2_{13}}{2} \\ \frac{c^2_{13} - c^1_{23}}{2} & -c \end{pmatrix} \tag{21}$$

and it depends on the choice of section $s\colon M \to \mathcal{G}$. To obtain an absolute invariant of the structure we factor the expression at (21) by the action of $SO^+_{1,1}(\mathbb{R})$.

Proposition 4. *For every 3-dimensional contact sub-Lorentzian ts-oriented manifold there exists a section s such that the invariant h has the following form at a given point:*

(1) If $h = 0$ then:

$$h \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm 1 \\ \mp 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & \pm 1 \\ \mp 1 & 1 \end{pmatrix} \right\}.$$

(2) If $\det h > 0$ then:

$$h = \begin{pmatrix} 0 & \chi \\ -\chi & 0 \end{pmatrix}.$$

(3) If $\det h < 0$ then:

$$h = \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}.$$

Moreover if $h \neq 0$ such a section is unique.

Proof. If $T = \left(\begin{smallmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{smallmatrix}\right) \in SO^+_{1,1}(\mathbb{R})$ then the adjoint action on h is given by

$$T^{-1} h\, T = \begin{pmatrix} \cosh(2t)a + \sinh(2t)b & \sinh(2t)a + \cosh(2t)b \\ -\sinh(2t)a - \cosh(2t)b & -\cosh(2t)a - \sinh(2t)b \end{pmatrix}. \tag{22}$$

If $\det h = b^2 - a^2 > 0$ then the equation

$$\cosh(2t)a + \sinh(2t)b = 0$$

has the solution $t = \frac{1}{4}\ln(\frac{b-a}{b+a})$ and h takes the form in item (2) with

$$\chi = \operatorname{sgn}(a+b)\sqrt{b^2 - a^2}. \tag{23}$$

Similarly, if $\det h < 0$ then the equation

$$\sinh(2t)a + \cosh(2t)b = 0$$

has the solution $t = \frac{1}{4}\ln(\frac{a-b}{b+a})$ and h takes the form in item (3) with

$$\chi = \operatorname{sgn}(a+b)\sqrt{a^2 - b^2}. \tag{24}$$

If $\det h = b^2 - a^2 = 0$ then

$$h = \begin{pmatrix} a & \pm a \\ \mp a & -a \end{pmatrix} \tag{25}$$

and the adjoint action of T is simply $T^{-1} h\, T = \exp(\pm 2t)h$, which depending on $\operatorname{sgn}(a)$, has exactly one of the forms in item (1). $\square$

Since $a = c$ and $b = (c^1_{23} - c^2_{13})/2$ we have the following Corollary.

Corollary 1. *Assume that* $\det h \neq 0$ *for sub-Lorentzian contact structure. Then the following expression is a local invariant of ts-oriented structures:*

$$\chi = \operatorname{sgn}\left(c + \frac{c^1_{23} - c^2_{13}}{2}\right)\sqrt{\left|\left(\frac{c^1_{23} - c^2_{13}}{2}\right)^2 - c^2\right|}.$$

Corollary 2. *If a sub-Lorentzian contact structure on a contact* 3 *manifold* M *satisfies* $h \neq 0$*, then Proposition 4 defines a unique normal frame* $\theta = s^*\omega_-$*.*

Proof. The change of section $s \to s\cdot h$, $h\colon M \to SO^+_{1,1}(\mathbb{R})$ for the Cartan connection is equivalent to the change $\theta \to \mathrm{Ad}_h(\theta)$ of associated frame $\theta = s^*\omega_-$ due to the formula (2). Therefor existence and uniqueness follows from Proposition 4. □

5. Classification of left-invariant sub-Lorentzian structures

5.1. **Classification of real 3-dimensional Lie algebras.** To begin we review the classification of real 3-dimensional Lie algebras following Šnobl and Winternitz [17]. If $\mathfrak{g}$ is a real 3-dimensional Lie algebra then we define $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and divide into the cases : $\dim \mathfrak{g}^{(1)} = 0, 1, 2, 3$. The cases $\dim \mathfrak{g}^{(1)} = 0$ is the abelian algebra and the case $\dim \mathfrak{g}^{(1)} = 1$ determines two classes, namely the Heisenberg algebra and the affine algebra. The cases $\dim \mathfrak{g}^{(1)} = 2$ give rise to significantly more classes corresponding to eigenvalues of $\mathrm{ad}_X|_{\mathfrak{g}^{(1)}}$, where $X \notin \mathfrak{g}^{(1)}$. Finally, in the case $\mathfrak{g}^{(1)} = 3$ there are only 2 non-isomorphic semi-simple Lie algebras, namely $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{su}_2(\mathbb{R})$

Theorem 3. *Any* 3*-dimensional Lie algebra is isomorphic to exactly one of the following algebras:*

$\dim \mathfrak{g}^{(1)} = 0$

$L(3,0) : [E_1, E_2] = 0,\ [E_1, E_3] = 0,\ [E_2, E_3] = 0,\ (\mathbb{R}^3)$

$\dim \mathfrak{g}^{(1)} = 1$

$L(3,1) : [E_1, E_2] = E_3,\ [E_1, E_3] = 0,\ [E_2, E_3] = 0,$ (*Heisenberg*)

$L(3,-1) : [E_1, E_2] = E_1,\ [E_1, E_3] = 0,\ [E_2, E_3] = 0,\ (A^+(\mathbb{R}) \oplus \mathbb{R})$

$\dim \mathfrak{g}^{(1)} = 2$

$L(3,2,\eta) : [E_1, E_2] = 0,\ [E_1, E_3] = E_1,\ [E_2, E_3] = \eta E_2,\ 0 < |\eta| \leq 1,$

$L(3,4,\eta) : [E_1, E_2] = 0,\ [E_1, E_3] = \eta E_1 - E_2,\ [E_2, E_3] = E_1 + \eta E_2,\ \eta \in [0, \infty),$

$L(3,3) : [E_1, E_2] = 0,\ [E_1, E_3] = E_1,\ [E_2, E_3] = E_1 + E_2,$

$\dim \mathfrak{g}^{(1)} = 3$

$L(3,5) : [E_1, E_2] = E_1,\ [E_1, E_3] = -2E_2,\ [E_2, E_3] = E_3,\ \mathfrak{sl}_2(\mathbb{R}),$

$L(3,6) : [E_1, E_2] = E_3,\ [E_1, E_3] = -E_2,\ [E_2, E_3] = E_1,\ \mathfrak{su}_2(\mathbb{R}).$

We review the proof of the classification theorem as it provides the procedures for putting a given algebra into its canonical form.

Proof. If $\dim \mathfrak{g}^{(1)} = 0$ then $\mathfrak{g}$ is the three dimensional abelian Lie algebra $L(3,0)$. If $\dim \mathfrak{g}^{(1)} = 1$ then there exists $Z \in \mathfrak{g}$ such that $\mathfrak{g}^{(1)} = \mathrm{span}\,\{Z\}$ and so it follows that for any basis of the form $\{X, Y, Z\}$ we have

$$[X, Y] = \alpha_1 Z, \quad [X, Z] = \alpha_2 Z, \quad [Y, Z] = \alpha_3 Z.$$

Direct calculation shows that $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] = \{0\}$ if and only if $\alpha_2 = \alpha_3 = 0$. Moreover, if $\alpha_2 = \alpha_3 = 0$ then $\mathfrak{g}$ is the Heisenberg algebra. Indeed, if $E_1 = X$, $E_2 = Y$ and $E_3 = \alpha_1 Z$ then

$$[E_1, E_2] = E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

If $\alpha_3 \neq 0$ then

$$E_1 = Z, \quad E_2 = -\frac{1}{\alpha_3} Y, \quad E_3 = \alpha_3 X - \alpha_2 Y + \alpha_1 Z$$

is a basis with bracket relations:

$$[E_1, E_2] = E_1, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0. \tag{26}$$

Similarly, if $\alpha_2 \neq 0$ then

$$E_1 = Z, \quad E_2 = -\frac{1}{\alpha_2} X, \quad E_3 = -\alpha_3 X + \alpha_2 Y - \alpha_1 Z$$

is a basis with the same bracket relations as in (26) above.

The Lie algebra defined by (26) is denoted $L(3,-1)$ and has the decomposition $L(3,-1) = L(2,1) \oplus L(1,0)$ where $L(2,1)$ is the subalgebra spanned by $\{E_1, E_2\}$ and $L(1,0) = \operatorname{span}\{E_3\}$.

Next we assume $\dim \mathfrak{g}^{(1)} = 2$. There are only two 2-dimensional Lie algebras. Therefor $\mathfrak{g}^{(1)}$ is abelian since $\mathfrak{g}^{(1)}$ doesn't contain semi-simple elements ($\mathfrak{g}$ is solvable). Furthermore, rank $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}} = 2$ for any nonzero $X \notin \mathfrak{g}^{(1)}$. We conclude that the map $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$ is an isomorphism $\mathfrak{g}^{(1)} \to \mathfrak{g}^{(1)}$ which doesn't depend on $X \notin \mathfrak{g}^{(1)}$.

There are three subcases to consider: $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$ diagonalises in $GL(\mathfrak{g}^{(1)})$, $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$ diagonalises in $GL(\mathfrak{g}^{(1)} \otimes_{\mathbb{R}} \mathbb{C})$, $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$ does not diagonalises. We get the following classes:

$$\begin{aligned}
L(3,2,\eta) &: [E_1,E_2] = 0,\ [E_1,E_3] = E_1,\ [E_2,E_3] = \eta E_2,\ 0 < |\eta| \le 1,\\
L(3,4,\eta) &: [E_1,E_2] = 0,\ [E_1,E_3] = \eta E_1 - E_2,\ [E_2,E_3] = E_1 + \eta E_2,\ \eta \in [0,\infty),\\
L(3,3) &: [E_1,E_2] = 0,\ [E_1,E_3] = E_1,\ [E_2,E_3] = E_1 + E_2.
\end{aligned}$$

Case: ad_X diagonalises in $GL(\mathfrak{g}^{(1)})$. Suppose the eigen values of $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$ are $\lambda_1, \lambda_2 \in \mathbb{R}$, $|\lambda_1| \ge |\lambda_2|$ with $V_1, V_2 \in \mathfrak{g}^{(1)}$ denoting corresponding linearly independent eigen vectors. Then the basis

$$E_1 = V_1, \quad E_2 = V_2, \quad E_3 = -\frac{1}{\lambda_1} X$$

satisfies the $L(3,2,\eta)$ bracket relations with $\eta = \frac{\lambda_2}{\lambda_1}$, $0 < |\eta| \le 1$.

Case: ad_X diagonalises in $GL(\mathfrak{g}^{(1)} \otimes_{\mathbb{R}} \mathbb{C})$. In this case the eigenvalues are a conjugate pair $(\lambda, \bar{\lambda})$ and there exists a nonzero $W = U + iV \in \mathfrak{g}^{(1)} \otimes_{\mathbb{R}} \mathbb{C}$ such that $[X, W] = \lambda W$. If $\Re\lambda/\Im\lambda \ge 0$ then the basis

$$E_1 = U, \quad E_2 = V, \quad E_3 = -\frac{1}{\Im\lambda} X$$

satisfies the $L(3,4,\eta)$ bracket relations with $\eta = \Re\lambda/\Im\lambda$. If $\Re\lambda/\Im\lambda < 0$ then

$$E_1 = U, \quad E_2 = -V, \quad E_3 = \frac{1}{\Im\lambda} X$$

satisfies the $L(3,4,\eta)$ bracket relations with $\eta = -\Re\lambda/\Im\lambda$. It is known that $L(3,4,s) \simeq L(3,4,t)$ if and only if $t = \pm s$.

Case: ad_X does not diagonalises. In this we consider the Jordan form of $\operatorname{ad}_X |_{\mathfrak{g}^{(1)}}$, in particular we can choose the basis $\{Y, Z\}$ so that ad_X is given by the matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \frac{1}{2} \operatorname{tr} \operatorname{ad}_X.$$

Then the basis

$$E_1 = \frac{1}{\lambda}Y, \quad E_2 = Z, \quad E_3 = -\frac{1}{\lambda}X$$

satisfies the $L(3,3)$ bracket relations.

In the case $\dim \mathfrak{g}^{(1)} = 3$ we use the fact that there are just 2 semi-simple real Lie algebras of dimension 3. One can distinguish $L(3,5)$ and $L(3,6)$ via the Killing form. Indeed $L(3,5) = \mathfrak{sl}_2(\mathbb{R})$ is a split real form of $\mathfrak{sl}_2(\mathbb{C})$ with sign-indefinite Killing form and $L(3,6) = \mathfrak{su}_2(\mathbb{R})$ is a compact real form with negative negative definite Killing form.

□

5.2. **Left-invariant sub-Lorentzian contact structures in dimension 3.** Now we are going to proof Theorem 1. If the sub-Lorentzian structure is a left-invariant structure on a Lie group then all the invariants are constant. In particular for the 3-dimensional sub-Lorentzian case

$$\kappa = -(c_{12}^1)^2 + (c_{12}^2)^2 + \frac{c_{23}^1 + c_{13}^2}{2}$$

and

$$h = \begin{pmatrix} c & \frac{c_{23}^1 - c_{13}^2}{2} \\ \frac{c_{13}^2 - c_{23}^1}{2} & -c \end{pmatrix}.$$

In order to obtain classification we consider canonical frames given by Proposition 4 and Corollary 2.

5.2.1. ***Case*** $h = 0$. This case needs special consideration since Proposition 4 and Corollary 2 doesn't provide canonical frame for this particular case.

We have $c = 0$ and $c_{13}^2 = c_{23}^1 = \gamma$, hence the Lie brackets are

$$\begin{aligned}
[X_1, X_3] &= \gamma X_2 \\
[X_2, X_3] &= \gamma X_1 \\
[X_1, X_2] &= c_{12}^1 X_1 + c_{12}^2 X_2 + X_3
\end{aligned}$$

and

$$\begin{aligned}
[X_1, [X_2, X_3]] + [[X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] &= [X_3, [X_1, X_2]] \\
&= -\gamma c_{12}^1 X_2 - \gamma c_{12}^2 X_1.
\end{aligned}$$

The Jacobi identity forces $\gamma = 0$ or $c_{12}^1 = c_{12}^2 = 0$, which lead to the following two possible Lie algebra structures:

$$A: \quad \begin{aligned} &[X_1, X_3] = 0 \\ &[X_2, X_3] = 0 \quad (\kappa = (c_{12}^2)^2 - (c_{12}^1)^2) \\ &[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 + X_3 \end{aligned} \qquad B: \quad \begin{aligned} &[X_1, X_3] = \gamma X_2 \\ &[X_2, X_3] = \gamma X_1 \quad (\kappa = \gamma) \\ &[X_1, X_2] = X_3. \end{aligned}$$

If $c_{12}^1 = c_{12}^2 = \gamma = 0$, then both algebras A and B are isomorphic to the Heisenberg algebra $L(3,1)$. If $\gamma \neq 0$, then $\gamma = \kappa$ and B is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) = L(3,5)$ via the isomorphism given by the following change of basis:

$$E_1 = X_1 + X_2, \quad E_2 = \frac{1}{\kappa}X_3 \quad E_3 = \frac{1}{\kappa}(X_1 - X_2).$$

Furthermore, the sub-Lorentzian metric is $-(1/2\kappa)K$ where K is the Killing form.

If $c_{12}^1 \neq 0$ or $c_{12}^2 \neq 0$, then the algebra A is isomorphic to $L(3,-1)$. Indeed, according to $c_{12}^1 \neq 0$ or $c_{12}^2 \neq 0$, the isomorphism is given by the corresponding change of basis:

$$\begin{aligned}
&1). \quad E_1 = X_1 + \frac{c^2_{12}}{c^1_{12}} X_2 + \frac{1}{c^1_{12}} X_3 && 2). \quad E_1 = \frac{c^1_{12}}{c^2_{12}} X_1 + X_2 + \frac{1}{c^2_{12}} X_3 \\
&\qquad\ E_2 = \frac{1}{c^1_{12}} X_2 && \qquad\ E_2 = -\frac{1}{c^2_{12}} X_1 \\
&\qquad\ E_3 = X_3 && \qquad\ E_3 = -X_3.
\end{aligned} \tag{27}$$

Theorem 4. *If κ is nonzero and identical for A and B, then the sub-Lorentzian structures are locally ts-isometric.*

Proof. To show this we exploit the fact that corresponding structures are constant curvature structures.

Definition 11. We say that sub-Lorentzian structure is a constant curvature structure if the curvature of the corresponding Cartan connection is constant on the whole principle bundle.

One can see that sub-Lorentzian structure is a constant curvature structure only if $SO^+_{1,1}(\mathbb{R})$ acts trivially (identically) on the curvature function. Otherwise curvature function wouldn't be constant along fibers of the principle bundle.

Any two constant curvature structures with the same curvature are isomorphic due to the following theorem.

Theorem 5. [15, Thm 14.18, p. 433] *Let θ and $\bar\theta$ be two coframes on m-dimensional manifolds M and $\bar M$, having the same constant structure functions. Then for any pair of points $p \in M$ and $q \in \bar M$, there exists a unique local diffeomorphism $\Phi : M \to \bar M$ such that $q = \Phi(p)$ and $\phi^*\bar\theta_i = \theta_i$ for $i = l, ..., m$.*

If follows that there exists a local diffeomorphism of the corresponding principle bundles which preserves fibers and maps one Cartan connection to the other. Since the action of G_0 is preserved, the projection of this diffeomorphism gives rise to the required isometry between the underlying manifolds.

Indeed, let $\pi_1 \colon \mathcal{G}_1 \to M_1$ and $\pi_2 \colon \mathcal{G}_2 \to M_2$ be principle bundles corresponding to sub-Lorentzian manifolds M_1 and M_2. Let $\tilde\omega_1 \colon T\mathcal{G}_1 \to \mathfrak{g}$ and $\tilde\omega_2 \colon T\mathcal{G}_1 \to \mathfrak{g}$ be Cartan connections induced by the sub-Lorentzian structure, and let $\varphi \colon \mathcal{G}_1 \to \mathcal{G}_2$ be the local diffeomorphism such that $\tilde\omega_1 = \varphi^*\tilde\omega_2$. Then for any section $s_1 \colon M_1 \to \mathcal{G}_1$ with image contained in the domain of φ, the diffeomorphism $\pi_2 \circ \varphi \circ s_1$ maps the frame adapted to sub-Lorentzian structure on M_1 to the frame adapted to sub-Lorentzian structure on M_2. Therefore this diffeomorphism automatically preserves the sub-Lorentzian structure.

The last step in the proof is to check that A and B are constant curvature structures with the same curvature function. Indeed, formulas on page 13 shows that curvature function for both cases is

$$k = \kappa(e_1 \otimes e_2^* \wedge e_3^* + e_2 \otimes e_1^* \wedge e_3^*)$$

and $SO^+_{1,1}(\mathbb{R})$ acts trivially on it.

□

5.2.2. ***Case*** $\det h = 0$, $h \neq 0$. In this case we have $c \in \{-1, 1\}$ and $c^1_{23} - c^2_{13} = \pm 2$, and the brackets are

$$\begin{aligned}
[X_1, X_3] &= cX_1 + c^2_{13} X_2 \\
[X_2, X_3] &= (c^2_{13} \pm 2) X_1 - cX_2 \\
[X_1, X_2] &= c^1_{12} X_1 + c^2_{12} X_2 + X_3.
\end{aligned}$$

The cases $c = 1$ and $c = -1$ can be obtained one from the other by a reversal of the time orientation or a reversal of the space orientation (but not both simultaneously). The underlying transformation is isometric but not ts-isometric. However the underlying group is unaffected and thus we only consider the case $c = 1$.

The Jacobi identity

$$\begin{aligned}[X_1,[X_2,X_3]] + [[X_2,[X_3,X_1]] + [X_3,[X_1,X_2]] &= [X_3,[X_1,X_2]] \\ &= -(c^1_{12} + c^2_{12}(c^2_{13} \pm 2))X_1 + (c^2_{12} - c^1_{12}c^2_{13})X_2.\end{aligned}$$

implies that we must also have the following equations:

$$c^1_{12}(1 + c^2_{13}(c^2_{13} \pm 2)) = 0, \quad \text{and} \quad c^2_{12} = c^1_{12}c^2_{13}.$$

There are three possible solutions:

(1) $\pm 2 = -2$, $c^2_{13} = 1$, $c^2_{12} = c^1_{12}$, $\kappa = 0$,
(2) $\pm 2 = 2$, $c^2_{13} = -1$, $c^2_{12} = -c^1_{12}$, $\kappa = 0$,
(3) $c^1_{12} = 0$, $c^2_{12} = 0$, $\kappa = (c^1_{23} + c^2_{13})/2$.

We see that solutions (1) and (2) give rise to two families of sub-Lorentzian structures which couldn't be distinguished by invariants κ and h. Therefor we introduce $\tau = c^1_{12}$ which is an additional invariant for these particular cases. One could check that for solutions (1) and (2) τ is a covariant derivative of h along X_1.

In solution (1) the brackets are

$$\begin{aligned}[X_1,X_3] &= X_1 + X_2 \\ [X_2,X_3] &= -(X_1 + X_2) \\ [X_1,X_2] &= \tau(X_1 + X_2) + X_3.\end{aligned} \tag{28}$$

which implies that $\mathfrak{g}^{(1)} = \operatorname{span}\{X_1 + X_2, X_3\}$. Furthermore we have that

$$\operatorname{ad}_{X_1}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} \tau & 1 \\ 1 & 0 \end{pmatrix}$$

relative to the basis $\{X_1 + X_2, X_3\}$. The characteristic polynomial of $\operatorname{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 - \tau t - 1$ and the eigenvalues are $(\tau \pm \sqrt{\tau^2 + 4})/2$. Following the classification procedure the algebra is $L(3, 2, \frac{\lambda_1}{\lambda_2}) = L(3, 2, \frac{\lambda_2}{\lambda_1})$ where λ_1 and λ_2 are the eigenvalues.

In solution (2) the brackets are

$$\begin{aligned}[X_1,X_3] &= X_1 - X_2 \\ [X_2,X_3] &= X_1 - X_2 \\ [X_1,X_2] &= \tau(X_1 - X_2) + X_3.\end{aligned} \tag{29}$$

which implies that $\mathfrak{g}^{(1)} = \operatorname{span}\{X_1 - X_2, X_3\}$. Furthermore we have that

$$\operatorname{ad}_{X_1}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} -\tau & 1 \\ -1 & 0 \end{pmatrix} \tag{30}$$

relative to the basis $\{X_1 - X_2, X_3\}$. The characteristic polynomial of $\operatorname{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 + \tau t + 1$ and the eigenvalues are $(-\tau \pm \sqrt{\tau^2 - 4})/2$. Following the classification procedure we get the following three possibilities:

(a) If $|\tau| = 2$ then the algebra is $L(3, 3)$ since (30) does not diagonalises,
(b) If $|\tau| > 2$ then the algebra is $L(3, 2, \frac{\lambda_1}{\lambda_2}) = L(3, 2, \frac{\lambda_2}{\lambda_1})$ where λ_1 and λ_2 are the eigenvalues,
(c) If $|\tau| < 2$ then the algebra is $L(3, 4, |\tau|/\sqrt{4 - \tau^2})$.

We remark that in solutions (1) and (2)(b) we do have distinct groups. Indeed suppose that

$$\frac{c^1_{12} - \sqrt{(c^1_{12})^2+4}}{c^1_{12} + \sqrt{(c^1_{12})^2+4}} = \left(\frac{-C^1_{12} - \sqrt{(C^1_{12})^2-4}}{-C^1_{12} + \sqrt{(C^1_{12})^2-4}}\right)^{\pm 1}$$

where c^k_{ij} denote the structure constant in solution (1) and C^k_{ij} denote the structure constant in solution (2)(b). Then it follows that $c^1_{12} = C^1_{12} = 0$ which contradicts $|C^1_{12}| > 2$.

In solution (3) the brackets are

$$\begin{aligned}
[X_1, X_3] &= X_1 + (\kappa \mp 1)X_2 \\
[X_2, X_3] &= (\kappa \pm 1)X_1 - X_2 \\
[X_1, X_2] &= X_3.
\end{aligned}$$

It follows that $\dim \mathfrak{g}^{(1)} < 3$ if and only if $\kappa = 0$ which reduces to particular cases of solutions (1) or (2). If $\dim \mathfrak{g}^{(1)} = 3$ the Killing form is

$$K = (2\kappa \mp 2)(x_1)^2 - 4x_1x_2 - (2\kappa \pm 2)(x_1)^2 + 2\kappa^2(x_3)^2.$$

and so $\kappa \neq 0$ implies $\mathfrak{g} \simeq L(3,5)$.

5.2.3. ***Case*** $\det h > 0$. In this case we have $c = 0$ and $c^2_{13} - c^1_{23} = 2\chi$, hence the brackets are

$$\begin{aligned}
[X_1, X_3] &= c^2_{13}X_2 \\
[X_2, X_3] &= (c^2_{13} - 2\chi)X_1 \\
[X_1, X_2] &= c^1_{12}X_1 + c^2_{12}X_2 + X_3
\end{aligned}$$

and

$$\begin{aligned}
[X_1,[X_2,X_3]] + [[X_2,[X_3,X_1]] + [X_3,[X_1,X_2]] &= [X_3,[X_1,X_2]] \\
&= -c^1_{12}c^2_{13}X_2 - c^2_{12}(c^2_{13} - 2\chi)X_1.
\end{aligned}$$

The Jacobi identity implies that we also have the following equations:

$$c^1_{12}c^2_{13} = 0 \quad c^2_{12}(c^2_{13} - 2\chi) = 0.$$

There are three possible solutions:

(1) $c^1_{12} = 0$, $c^2_{12} = 0$, $(\kappa = c^2_{13} - \chi)$
(2) $c^1_{12} = 0$, $c^2_{13} - 2\chi = 0$, $(\kappa = (c^2_{12})^2 + \chi)$
(3) $c^2_{13} = 0$, $c^2_{12} = 0$, $(\kappa = -(c^1_{12})^2 - \chi)$.

In solution (1) the brackets are

$$\begin{aligned}
[X_1, X_3] &= (\kappa + \chi)X_2 \\
[X_2, X_3] &= (\kappa - \chi)X_1 \\
[X_1, X_2] &= X_3.
\end{aligned}$$

We note that $\dim \mathfrak{g}^{(1)} = 3$ if and only if $\kappa^2 - \chi^2 \neq 0$. The Killing form is

$$K = 2(\kappa + \chi)(x_1)^2 - 2(\kappa - \chi)(x_2)^2 + 2(\kappa^2 - \chi^2)(x_3)^2.$$

Hence we conclude that if $\kappa + \chi < 0$ and $\kappa - \chi > 0$ then K is negative definite and the algebra is $L(3,6)$ otherwise if $\kappa^2 - \chi^2 \neq 0$ the algebra is $L(3,5)$.

If $\chi = \kappa$ then $\mathfrak{g}^{(1)} = \operatorname{span}\{X_2, X_3\}$. Furthermore we have that

$$\operatorname{ad}_{X_1}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} 0 & 2\kappa \\ 1 & 0 \end{pmatrix}$$

relative to the basis $\{X_2, X_3\}$. The characteristic polynomial of $\mathrm{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 - 2\kappa$. If $\kappa > 0$ then the eigenvalues are $\pm\sqrt{2\kappa}$ and the classification procedure implies that the algebra is $L(3,2,-1)$. If $\kappa < 0$ then the eigenvalues are $\pm i\sqrt{-2\kappa}$ and the classification procedure implies that the algebra is $L(3,4,0)$.

If $\chi = -\kappa$ then $\mathfrak{g}^{(1)} = \mathrm{span}\{X_1, X_3\}$. Furthermore we have that

$$\mathrm{ad}_{X_2}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} 0 & 2\kappa \\ -1 & 0 \end{pmatrix}$$

relative to the basis $\{X_1, X_3\}$. The characteristic polynomial of $\mathrm{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 + 2\kappa$. If $\kappa > 0$ then the eigenvalues are $\pm i\sqrt{2\kappa}$ and the classification procedure implies that the algebra is $L(3,4,0)$. If $\kappa < 0$ then the eigenvalues are $\pm\sqrt{-2\kappa}$ and the classification procedure implies that the algebra is $L(3,2,-1)$.

In solution (2) the brackets are

$$\begin{aligned} [X_1, X_3] &= 2\chi X_2 \\ [X_2, X_3] &= 0 \\ [X_1, X_2] &= c_{12}^2 X_2 + X_3, \quad ((c_{12}^2)^2 = \kappa - \chi). \end{aligned}$$

and we see that $\mathfrak{g}^{(1)} = \mathrm{span}\{X_2, X_3\}$. Furthermore, we also have that

$$\mathrm{ad}_{X_1}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} c_{12}^2 & 2\chi \\ 1 & 0 \end{pmatrix} \tag{31}$$

relative to the basis $\{X_2, X_3\}$. The characteristic polynomial of $\mathrm{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 - c_{12}^2 t - 2\chi$ and the eigenvalues are $(c_{12}^2 \pm \sqrt{(c_{12}^2)^2 + 8\chi})/2$. Following the classification procedure we get the following three possibilities:

(a) If $(c_{12}^2)^2 = -8\chi$ then the algebra is $L(3,3)$ since (31) does not diagonalises.
(b) If $(c_{12}^2)^2 > -8\chi$ then the algebra is $L(3,2,\frac{\lambda_1}{\lambda_2}) = L(3,2,\frac{\lambda_2}{\lambda_1})$ where λ_i are the eigenvalues.
(c) If $(c_{12}^2)^2 < -8\chi$ then the algebra is $L(3,4,c_{12}^2/\sqrt{-8\chi - (c_{12}^2)^2})$.

In solution (3) the brackets are

$$\begin{aligned} [X_1, X_3] &= 0 \\ [X_2, X_3] &= -2\chi X_1 \\ [X_1, X_2] &= c_{12}^1 X_1 + X_3, \quad ((c_{12}^1)^2 = -\kappa - \chi). \end{aligned}$$

and we see that $\mathfrak{g}^{(1)} = \mathrm{span}\{X_1, X_3\}$. Furthermore, we also have that

$$\mathrm{ad}_{X_2}|_{\mathfrak{g}^{(1)}} = \begin{pmatrix} -c_{12}^1 & -2\chi \\ -1 & 0 \end{pmatrix} \tag{32}$$

relative to the basis $\{X_1, X_3\}$. The characteristic polynomial of $\mathrm{ad}_{X_1}|_{\mathfrak{g}^{(1)}}$ is $t^2 + c_{12}^1 t - 2\chi$ and the eigenvalues are $(-c_{12}^1 \pm \sqrt{(c_{12}^1)^2 + 8\chi})/2$. Following the classification procedure we get the following three possibilities:

(a) If $(c_{12}^1)^2 = -8\chi$ then the algebra is $L(3,3)$ since (32) does not diagonalises.
(b) If $(c_{12}^1)^2 > -8\chi$ then the algebra is $L(3,2,\frac{\lambda_1}{\lambda_2}) = L(3,2,\frac{\lambda_2}{\lambda_1})$ where λ_i are the eigenvalues.
(c) If $(c_{12}^1)^2 < -8\chi$ then the algebra is $L(3,4,c_{12}^1/\sqrt{-(c_{12}^1)^2 - 8\chi})$.

We remark that the solutions (2)(b) and (3)(b) are distinct except when $c_{12}^1 = \pm c_{12}^2$ and solutions (2)(c) and (3)(c) are distinct except when $c_{12}^1 = \pm c_{12}^2$. In fact case (3) can be obtained from cases (2) by multiplying the metric by -1 (i.e., timelike becomes spacelike and vice versa).

5.2.4. ***Case*** $\det h < 0$. In this case we have $c = \chi$ and $c^1_{23} = c^2_{13}$, hence the brackets are

$$\begin{aligned}
[X_1, X_3] &= \chi X_1 + c^2_{13} X_2 \\
[X_2, X_3] &= c^2_{13} X_1 - \chi X_2 \\
[X_1, X_2] &= c^1_{12} X_1 + c^2_{12} X_2 + X_3
\end{aligned}$$

and

$$\begin{aligned}
[X_1, [X_2, X_3]] + [[X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] &= [X_3, [X_1, X_2]] \\
&= -(c^1_{12}\chi + c^2_{12} c^2_{13}) X_1 + (c^2_{12}\chi - c^1_{12} c^2_{13}) X_2.
\end{aligned}$$

The Jacobi identity implies that we also have the following equations:

$$c^1_{12}\chi + c^2_{12} c^2_{13} = 0 \quad \text{and} \quad c^2_{12}\chi - c^1_{12} c^2_{13} = 0.$$

It follows that $c^1_{12} = 0$, $c^2_{12} = 0$ and $\kappa = c^2_{13}$, moreover the brackets are

$$\begin{aligned}
[X_1, X_3] &= \chi X_1 + \kappa X_2, \\
[X_2, X_3] &= \kappa X_1 - \chi X_2, \\
[X_1, X_2] &= X_3,
\end{aligned}$$

implying $\dim \mathfrak{g}^{(1)} = 3$. The Killing form is

$$K = 2\kappa \left((x_1)^2 - (x_2)^2\right) - 4\chi x_1 x_2 + 2(\kappa^2 + \chi^2)(x_3)^2.$$

It is sign-indefinite hence the algebra is $L(3, 5)$.

The classification is complete.

References

[1] A. Agrachev, D. Barilari, *Sub-Riemannian structures on 3D Lie groups*, Journal of Dynamical and Control Systems, Vol. 18, 2012

[2] A. Čap and J. Slovák. *Parabolic geometries. I. Background and general theory.* Mathematical Surveys and Monographs, 154. American Mathematical Society, Providence, 2009.

[3] M. Grochowski, *Geodesics in the sub-Lorentzian Geometry*, Bull. Polish Acad.M. 50 (2002), no. 2, 161–178.

[4] M. Grochowski, Properties of Reachable Sets in the sub-Lorentzian Geometry, J. Geom. Phys. 59 (2009), no. 7, 885900.

[5] M. Grochowski, *Some remarks on the global sub-Lorentzian geometry*, Anal.Math.Phys. (2013) 3:295309. Sci. Math. 50 (2002), no. 2, 161–178.

[6] M. Grochowski, *Normal Forms of Germs of Contact sub-Lorentzian Structures on $\mathbb{R}^3$. Differentiability of the sub-Lorentzian Distance Function*, J. Dynam. Control Systems 9 (2003), no. 4, 531-547.

[7] M. Grochowski, *Reachable Sets For Contact sub-Lorentzian Metrics on $\mathbb{R}^3$. Application to control Affine Systems with the Scalar Input*, J. Math. Sci. (N. Y.) 177 (2011), no. 3, 383-394.

[8] M. Grochowski, B. Warhurst *Invariants and infinitesimal transformations for contact sub-Lorentzian structures on 3-dimensional manifolds* SIGMA 11, 2015.

[9] M. Grochowski, W. Kryński, *Invariants of contact sub-pseudo-Riemannian structures and Einstein-Weyl geometry*, submitted (2015), arXiv:1503.06868v1

[10] E. Grong, A. Vasilev, *Sub-Riemannian and sub-Lorentzian geometry on $SU(1,1)$ and on its universal cover*, J. Geom. Mech. 3 (2) (2011) 225260.

[11] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics, 52. Cambridge University Press, Cambridge, 1997

[12] A. Korolko, I. Markina, *Nonholonomic Lorentzian geometry on some H -type groups*, J. Geom. Anal. 19 (4) (2009).

[13] T. Morimoto. Geometric structures on filtered manifolds. *Hokkaido Math. J.*, 22:263–347, 1993.

[14] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, 91. American Mathematical Society, Providence, RI, 2002.

[15] P. Olver. *Equivalence, invariants, and symmetry*, Cambridge University Press, Cambridge, 1995.

[16] R.W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program.* Graduate texts in mathematics, 166. Springer-Verlag, New York, 1997.

[17] L. Šnobl and P. Winternitz *Classification and Identification of Lie Algebras*. CRM Monograph Series, Volume 33, 2014.

[18] S. Sternberg. *Lectures on Differential Geometry*. Graduate texts in mathematics, 166. Springer-Verlag, New York, 1997.

[19] N. Tanaka. *On differential systems, graded Lie algebras and pseudogroups*, J. Math. Kyoto Univ., 10:1–82, 1970.