Published for SISSA by Springer

Received: *May 27, 2010*
Accepted: *June 13, 2010*
Published: *June 28, 2010*

The Hilbert series of the one instanton moduli space

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Abstract: The moduli space of k G-instantons on $\mathbb{R}^4$ for a classical gauge group G is known to be given by the Higgs branch of a supersymmetric gauge theory that lives on Dp branes probing D$(p+4)$ branes in Type II theories. For $p = 3$, these $(3 + 1)$ dimensional gauge theories have $\mathcal{N} = 2$ supersymmetry and can be represented by quiver diagrams. The F and D term equations coincide with the ADHM construction. The Hilbert series of the moduli spaces of one instanton for classical gauge groups is easy to compute and turns out to take a particularly simple form which is previously unknown. This allows for a G invariant character expansion and hence easily generalisable for exceptional gauge groups, where an ADHM construction is not known. The conjectures for exceptional groups are further checked using some new techniques like sewing relations in Hilbert Series. This is applied to Argyres-Seiberg dualities.

Keywords: Supersymmetry and Duality, Supersymmetric gauge theory, Duality in Gauge Field Theories, Solitons Monopoles and Instantons

ArXiv ePrint: 1005.3026

doi:10.1007/JHEP06(2010)100

Contents

1 Introduction

Yang-Mills Instantons [1] have attracted great interest from both physicists and mathematicians since their discovery in 1975. They have served as a powerful tool in studying a number of physical and mathematical problems, ranging from the Yang-Mills vacuum structure (e.g., [2–4]) to the classification of four-manifolds [5].

A method for constructing a self-dual Yang-Mills instanton solution on $\mathbb{R}^4$ is due to Atiyah, Drinfeld, Hitchin and Manin (ADHM) [6] in 1978. The ADHM construction is known for the classical gauge groups, SU(N), SO(N) and Sp(N) (see, e.g., [7–12] for explicit constructions); there is no known such construction, however, for the exceptional groups. The space of all solutions to the self-dual Yang-Mills equation modulo gauge transformations, in a given winding sector k and gauge group G is said to be the *moduli space of k G-instantons* on $\mathbb{R}^4$. In 1994-1996, Douglas and Witten [25–28] discovered that the ADHM construction can be realised in string theory. In particular, the moduli space of instantons on $\mathbb{R}^4$ is identical to the Higgs branch of supersymmetric gauge theories on a system of Dp-D$(p+4)$ branes (see, e.g., [13] for a review).[1] These theories are quiver gauge theories with 8 supercharges ($\mathcal{N} = 2$ supersymmetry in $(3 + 1)$ dimensions for $p = 3$). In section 3 of this paper, we present the $\mathcal{N} = 2$ quiver diagram of each theory as well as provide a prescription for writing down the corresponding $\mathcal{N} = 1$ quiver diagram and the superpotential. The Hilbert series of the one instanton moduli space is easily computed using the ADHM construction for classical gauge groups and is written in a form that provides a natural conjectured generalization for exceptional gauge groups (even though the ADHM construction does not exist for the latter).

In addition to the ADHM construction, there exists an alternative description of the moduli space of instantons for simply laced (A, D and E) groups via three dimensional mirror symmetry [14]. This symmetry exchanges the Coulomb branch and the Higgs branch, and therefore maps the Coulomb branch of the ADE quiver gauge theories to moduli spaces of instantons. On the contrary to Higgs branch, one expects the Coulomb branch to receive many non-perturbative quantum corrections. As argued in [14], quantum effects correct the Coulomb branch to be the moduli space of one ADE-instanton, with the point at the origin corresponding to an instanton of zero size.[2] Nevertheless, due to such quantum corrections, this description of the instanton moduli space is not useful for exact computations using Hilbert series.

In the last section of this paper, exceptional groups are considered, and checks that the Hilbert Series above predicts the correct dimension of the moduli space. In the case of E_n it is known [15, 16] that $\mathcal{N} = 2$ CFTs realise the moduli space of one E_n instanton. We use Argyres-Seiberg S-dualities in $\mathcal{N} = 2$ supersymmetric gauge theories [17–23] to match the Hilbert series of the theories on both sides of the duality, providing a consistency check.

2 Hilbert series for one-instanton moduli spaces on $\mathbb{C}^2$

We are interested in computing the partition function that counts holomorphic functions (Hilbert series) on the moduli space of k G-instantons on $\mathbb{C}^2$, were G is a gauge group of finite rank r. It is well known that this moduli space has quaternionic dimension kh_G

[1] The Higgs branch of D3 branes near E_n type 7 branes is the moduli space of E_n instantons. Since there is no known Lagrangian for this class of theories, it is not clear how to compute the ADHM analog.

[2] The Coulomb branch of the gauge theory with quiver diagram G (where G is A, D or E) and all ranks multiplied by k is $kh_G - 1$ quaternionic dimensional [14], where h_G is the dual coxeter number of G. This precisely agrees with the fact that the coherent component (eliminating the translation on $\mathbb{R}^4$) of the one G-instanton moduli space is $h_G - 1$ quaternionic dimensional.

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where h_G is the dual Coxeter number of the gauge group G. The present paper will focus on the case of a single instanton moduli space. The moduli space is reducible into a trivial $\mathbb{C}^2$ component, physically corresponding to the position of the instanton in $\mathbb{C}^2$, and the remaining irreducible component of quaternionic dimension $h_G - 1$. Henceforth, we shall call this component the *coherent component* or the *irreducible component*. The Hilbert series for the coherent component takes the form

$$g^{\mathrm{Irr}}_G(t; x_1, \ldots, x_r) = \sum_{k=0}^{\infty} \chi[R_G(k)] t^{2k} \,, \tag{2.1}$$

where $R_G(k)$ is a series of representations of G and $\chi[R]$ is the character of the representation R.[3] The fugacities x_i (with $i = 1, \ldots, r$) are conjugate to the charges of each holomorphic function under the Cartan subalgebra of G. The moduli space of instantons is a non-compact hyperKähler space, and so there are infinitely many holomorphic functions which are graded by degrees d. Setting $x_1 = \cdots = x_r = 1$, we obtain the (finite) number of holomorphic functions of degree d.

The main result of this paper is the following:

The representation $R_G(k)$ is the irreducible representation Adj^k ,

where Adj^k denotes the irreducible representation whose Dynkin labels are $\theta_k = k\theta$, with θ the highest root of G.[4] By convention $R_G(0)$ is the trivial, one-dimensional, representation (this corresponds to the space being connected), and $R_G(1)$ is the adjoint representation.

In the case of classical gauge groups A_n, B_n, C_n, D_n it is possible to directly verify the above statement by explicit counting of the chiral operators on the Higgs branch of a certain $\mathcal{N} = 2$ supersymmetric gauge theory with a one dimensional Coulomb branch and a $A_n, \ldots, D_n$ global symmetry. The specific gauge theory can be derived in string theory by a simple system of Dp branes which probe a background of D$(p + 4)$ branes in Type II theories. The moduli space of k G-instantons on $\mathbb{C}^2$ is identified with the Higgs branch of the gauge theory living on the k Dp branes. The gauge group G, which is interpreted as a global symmetry on the world volume of the Dp branes, lives on the D$(p + 4)$ branes and can be chosen to be any of the classical gauge groups by an appropriate choice of a background with or without an orientifold plane. The gauge theory living on the Dp branes is a simple quiver gauge theory and is discussed in detail in section 3. The F and D term equations for the Higgs branch of these theories coincides with the ADHM construction of the moduli space of instantons for classical gauge groups. Unfortunately, such a simple construction is not available for exceptional groups and other methods need to be applied. It is therefore not possible to explicitly compute the Hilbert series for exceptional groups and the main statement of this paper is a conjecture for these cases. This conjecture is subject to a collection of tests which are presented in section 5.

[3] In this paper, we represent an irreducible representation of a group G by its Dynkin labels (which is also the highest weight of such a representation) $[a_1, \ldots, a_r]$, where $r = \mathrm{rank}\ G$. Since a representation is determined by its character, we slightly abuse terminology by referring to a character by the corresponding representation.

[4] For the A_n series $\theta_k = [k, 0, \ldots, 0, k]$, for the B_n and D_n series $\theta_k = [0, k, 0, \ldots, 0]$, for the C_n series $\theta_k = [2k, 0, \ldots, 0]$, for E_6 $\theta_k = [0, k, 0, 0, 0, 0]$, for G_2 $\theta_k = [0, k]$, for all other exceptional groups $\theta_k = [k, 0, \ldots, 0]$.

An example of D_4. An explicit counting of chiral operators in the well known $\mathcal{N}=2$ supersymmetric gauge theory of SU(2) with 4 flavours (see section 3.3.1 for details), gives the Hilbert series for the coherent component of the one $D_4 = \mathrm{SO}(8)$ instanton moduli space (omitting the trivial component $\mathbb{C}^2$):

$$g^{\mathrm{Irr}}_{D_4}(t; x_1, x_2, x_3, x_4) = \sum_{k=0}^{\infty} [0, k, 0, 0]_{D_4} t^{2k}, \tag{2.2}$$

Setting these fugacities y_i to 1, we get the unrefined Hilbert series:

$$\begin{aligned} g^{\mathrm{Irr}}_{D_4}(t) &= \sum_{k=0}^{\infty} dim[0, k, 0, 0]_{D_4} t^{2k} \\ &= \frac{(1+t^2)(1+17t^2+48t^4+17t^6+t^8)}{(1-t^2)^{10}} \\ &= 1 + 28t^2 + 300t^4 + \cdots \, . \end{aligned} \tag{2.3}$$

An explicit expression for the dimension of each such representation is given by

$$\dim \ [0, k, 0, 0]_{D_4} = \frac{(k+1)(k+2)^3(k+3)^3(k+4)(2k+5)}{4320} \, . \tag{2.4}$$

Notice that summing the series we get a closed formula with a pole of order 10 at $t = 1$. This means that the space is 10-complex dimensional, and is in agreement with the fact that the non-trivial component of the one-instanton moduli space for D_4 has quaternionic dimension 5 (the dual Coxeter number $h_{D_4} = 6$).

In general, summing up the unrefined Hilbert series for any group G gives rational functions of the form

$$g^{\mathrm{Irr}}_G(t) = \frac{P_G(t^2)}{(1-t^2)^{2h-2}} \, , \tag{2.5}$$

where $P_G(x)$ is a palindromic polinomial of degree $h_G - 1$.

A dimension formula for Adj^k. Formula (2.4) can be generalised to any classical and exceptional group. Defining

$$G_{a,b}(h, k) = \frac{\binom{(1+a)h/2-b-1+k}{k}}{\binom{(1-a)h/2+b-1+k}{k}} \, , \tag{2.6}$$

the dimension of the Adj^k representation is given by

$$\dim \ Adj^k = G_{1,1}(h, k) G_{a,b}(h, k) G_{1-a,1-b}(h, k) \frac{2k+h-1}{h-1} \, . \tag{2.7}$$

where (a, b, h) are given in table 2.[5]

[5]Formula (2.7) generalises the Proposition 1.1 of [24]

$$\dim Adj^k = \frac{3c+2k+5}{3c+5} \frac{\binom{k+2c+3}{k}\binom{k+5c/2+3}{k}\binom{k+3c+4}{k}}{\binom{k+c/2+1}{k}\binom{k+c+1}{k}} \, ,$$

which gives the results for A_1, A_2, G_2, D_4, F_4, E_6, E_7 and E_8 if we use $c = \frac{1}{3}h_G - 2$.

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Lie group	Dynkin label of Adj^k	Dual coxeter number	(a,b)	$\mathcal{N}=2$ gauge theory
$A_n = \mathrm{SU}(n+1)$	$[k,0,\ldots,0,k]$	$n+1$	$(1,1)$	Quiver diagram 6
$B_{n\geq 3} = \mathrm{SO}(2n+1)$	$[0,k,0,\ldots,0]$	$2n-1$	$(1,2)$	Quiver diagram 8
$C_{n\geq 2} = \mathrm{Sp}(2n)$	$[2k,0,\ldots,0]$	$n+1$	$(1,1/2)$	Quiver diagram 10
$D_{n\geq 4} = \mathrm{SO}(2n)$	$[0,k,0,\ldots,0]$	$2n-2$	$(1,2)$	Quiver diagram 8
E_6	$[0,k,0,0,0,0]$	12	$(1/3,0)$	3 M5s on 3-punctured sphere
$E_{7,8}$	$[k,0,\ldots,0]$	$18,30$	$(1/3,0)$	4, 6 M5s on 3-punctured sphere
F_4	$[k,0,0,0]$	9	$(1/3,0)$	
G_2	$[0,k]$	4	$(1/3,0)$	

Table 1. Useful information on classical and exceptional groups. The last column indicates the $\mathcal{N}=2$ gauge theories, for which the Higgs branch is identified with the corresponding moduli space of instantons on $\mathbb{R}^4$.

3 Gauge theories on Dp-D$(p+4)$ brane systems

The moduli space of instantons is known to be the Higgs branch of certain supersymmetric gauge theories [26–28]. For classical gauge groups there is an explicit construction, while for exceptional gauge groups there is a puzzle on how to explicitly write it down. Below we recall the string theory embedding of the gauge theories for classical gauge groups as worldvolume theories of Dp branes in backgrounds of D$(p+4)$ branes and summarize the gauge theory data for these theories. It is perhaps convenient to take $p=3$, so that the worldvolume theories have $\mathcal{N}=2$ supersymmetry in $(3+1)$ dimensions. The presence of these branes breaks space-time into $\mathbb{R}^{1,3}\times\mathbb{C}^2\times\mathbb{C}$. There is a U(2) symmetry that acts on the $\mathbb{C}^2$ and acts as an R symmetry on the different supermultiplets in the theory. This symmetry is used below to distinguish some of the gauge invariant operators.

The gauge theory on the D3 branes is most conveniently written in terms of $\mathcal{N}=2$ quiver diagrams but for the purpose of computing the Hilbert series, it is more convenient to work using an $\mathcal{N}=1$ notation. Section 3.1 summarizes the basic rules of translating an $\mathcal{N}=2$ quiver diagram to an $\mathcal{N}=1$ quiver diagram with a superpotential.

3.1 Quiver diagrams

To write down a Lagrangian for a gauge theory with $\mathcal{N}=2$ supersymmetry it is enough to specify the gauge group, transforming in a vector multiplet, and the matter fields, transforming in hyper multiplets. This can be simply summarized by a quiver with 2 objects - nodes and lines but nevertheless has a two-fold ambiguity on how to assign the objects. A traditional mathematical approach, first introduced to the string theory literature in [29], is to assign nodes to vector multiplets and lines to hyper multiplets. This is the so called quiver diagram used below. The more physically inspired approach [30], is to assign lines to

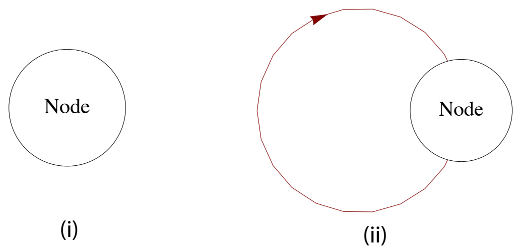

Figure 1. A node in the $\mathcal{N} = 2$ quiver diagram (labelled (i)) becomes a node with an adjoint chiral multiplet in the $\mathcal{N} = 1$ quiver diagram (labelled (ii)).

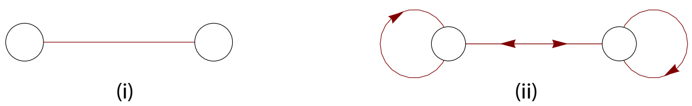

Figure 2. A line in the $\mathcal{N} = 2$ quiver diagram (labelled (i)) becomes a bi-directional line in the $\mathcal{N} = 1$ quiver diagram (labelled (ii)).

vector multiplets and nodes to hyper multiplets. This notation turns out to be more useful when the hyper multiplets carry more than two charges. On the other hand, to write down the Lagrangian for a gauge theory with $\mathcal{N} = 1$ supersymmetry the data which is needed consists of 3 objects: the gauge group, the matter fields, and the interaction terms written in the form of a superpotential. This can be summarized by an oriented quiver, namely it has arrows which are absent in the $\mathcal{N} = 2$ quiver, and is supplemented by a superpotential W. A simple dictionary exists between the two formulations. It goes as follows:

- A node in the $\mathcal{N} = 2$ quiver diagram becomes a node with an adjoint chiral multiplet in the $\mathcal{N} = 1$ quiver diagram. This adjoint chiral multiplet comes from the $\mathcal{N} = 2$ vector multiplet which decomposes as a $\mathcal{N} = 1$ vector multiplet and a $\mathcal{N} = 1$ chiral multiplet. The map is shown in figure 1.

- A line in the $\mathcal{N} = 2$ quiver diagram becomes a bi-directional line in the $\mathcal{N} = 1$ quiver diagram. This is shown in figure 2.

- The superpotential is given by the sum of contributions from all lines in the $\mathcal{N} = 2$ quiver diagram. Each line stretched between two nodes in the $\mathcal{N} = 2$ quiver diagram contributes two cubic superpotential terms. Let the two nodes be labeled by 1 and 2. Associated with each node, there is an adjoint field denoted respectively by Φ_1 and Φ_2. A line connecting between two nodes contains two $\mathcal{N} = 1$ bi-fundamental chiral multi-

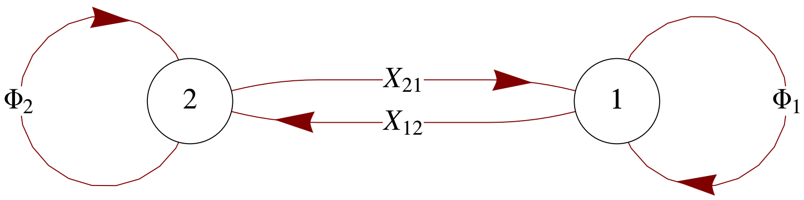

Figure 3. An $\mathcal{N} = 1$ quiver diagram with the superpotential : $X_{21} \cdot \Phi_1 \cdot X_{12} - X_{12} \cdot \Phi_2 \cdot X_{21}$.

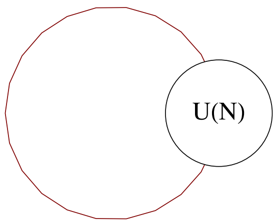

Figure 4. The $\mathcal{N} = 2$ quiver diagram for the $\mathcal{N} = 4$ SYM theory with gauge group U(N). The loop around the U(N) gauge group denotes an adjoint hypermultiplet.

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plets X_{12} and X_{21}. (The $\mathcal{N} = 1$ quiver diagram is drawn in figure 3.) The corresponding superpotential term is written as an adjoint valued mass term for the X fields:

$$X_{21} \cdot \Phi_1 \cdot X_{12} - X_{12} \cdot \Phi_2 \cdot X_{21} \,, \tag{3.1}$$

This notation means as follows. Denote the rank of nodes 1 and 2 by r_1 and r_2 respectively. then $\Phi_1, \Phi_2, X_{12}, X_{21}$ can be chosen to be $r_1 \times r_1, r_2 \times r_2, r_1 \times r_2, r_2 \times r_1$ matrices, respectively. The $\cdot$ corresponds to matrix multiplication and an impiicit trace is assumed. Note that this is a schematic notation which does not specify the index contraction whose details depend on the gauge and flavour groups. As a special case, a line from one node to itself would naturally produce a commutator.

As an example, we give the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ quiver diagrams for the U(N) $\mathcal{N} = 4$ super Yang-Mills (SYM) respectively in figure 4 and figure 5.

3.2 k SU(N) instantons on $\mathbb{C}^2$

With this quiver notation it is now very simple to write down the gauge theory living on the world volume of k D3 branes in the background of N D7 branes. In fact, the brane system very naturally forms a quiver and we can just write down a dictionary between the branes and the objects in the quiver. We will write down the theory using $\mathcal{N} = 2$ quivers and then translate it to $\mathcal{N} = 1$ quivers. First, the gauge theory on k D3 branes is the well known $\mathcal{N} = 4$ supersymmetric theory with gauge group U(k) depicted in figure 4. The D7 branes are heavier and therefore give rise to a global U(N) symmetry on the worldvolume of the D3 branes. As discussed below, the global U(1) of U(N) may be absorbed into the local U(1)

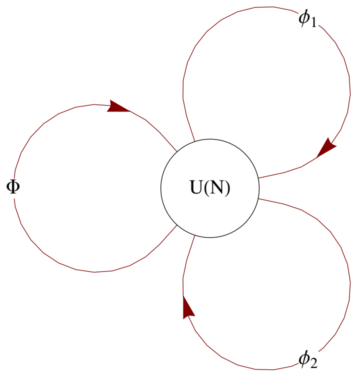

Figure 5. The $\mathcal{N} = 1$ quiver diagram of the $\mathcal{N} = 4$ SYM theory. The adjoint field Φ comes from the $\mathcal{N} = 2$ vector multiplet, whereas the adjoint fields ϕ_1, ϕ_2 come from the $\mathcal{N} = 2$ adjoint hypermultiplet. The superpotential is $W = \mathrm{Tr}(\phi_1 \cdot \Phi \cdot \phi_2 - \phi_2 \cdot \Phi \cdot \phi_1) = \mathrm{Tr}\,(\Phi \cdot [\phi_1, \phi_2])$.

of U(k); therefore global SU(N) symmetry is represented by a square node with index N. Finally strings stretched between the D3 branes and the D7 branes are represented by a line connecting the circular node to the square node. The resulting quiver is depicted in figure 6.

It is now straightforward to apply the rules of section 3.1 to write down the $\mathcal{N} = 1$ quiver diagram which is depicted in figure 7 and its corresponding superpotential. To write down the superpotential we need explicit notation for the quiver fields and the line between the circular node and the square node corresponds to two chiral fields denoted by Q and $\widetilde{Q}$. Putting this together, W takes the form

$$\begin{aligned} W &= X_{21} \cdot \Phi \cdot X_{12} + \left(\phi^{(1)} \cdot \Phi \cdot \phi^{(2)} - \phi^{(2)} \cdot \Phi \cdot \phi^{(1)}\right) \\ &= X_{21} \cdot \Phi \cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)} \cdot \Phi \cdot \phi^{(\beta)} \; . \end{aligned} \tag{3.2}$$

Note that the rules for writing the quiver imply the existence of another term coming from the adjoint in the vector multiplet of the D7 branes. This term corresponds to an adjoint U(N) valued mass term for the bifundamental fields X_{12}, X_{21}. In this paper we will not treat this mass term and set it to 0, even though it is interesting to consider the effects of such a term. The adjoint fields are parametrizing the position of the D3 branes in $\mathbb{C}^2$. Since there is a natural $\mathrm{U}(2)_g = \mathrm{SU}(2)_g \times \mathrm{U}(1)_g$ symmetry that acts on $\mathbb{C}^2$, the fields ϕ_1 and ϕ_2 transform as a doublet of $\mathrm{SU}(2)_g$ symmetry and with charge 1 under $\mathrm{U}(1)_g$. The superpotential should therefore be invariant under $\mathrm{SU}(2)_g$ and carry charge 2 under $\mathrm{U}(1)_g$.

We list the charges and the representations under which the fields transform in table 2.

From table 2, it can be seen that the U(1) of U(N) can be absorbed into the local U(1) (e.g. by means of redefining the fugacity z/q). From the brane perspective, the vector multiplet of the local U(1) contains a scalar which parametrises the position of the D3-brane in the directions transverse to the D7 branes. One can set the origin of these directions

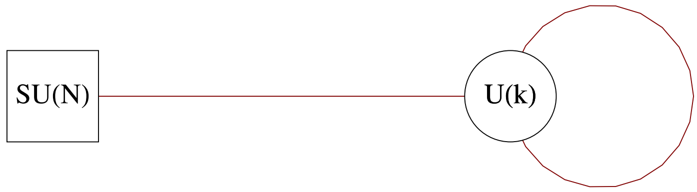

Figure 6. The $\mathcal{N}=2$ quiver diagram for k SU(N) instantons on $\mathbb{C}^2$. The circular node represents the U(k) gauge symmetry and the square node represents the SU(N) flavour symmetry. The line connecting the SU(N) and U(k) groups denotes kN bi-fundamental hypermultiplets, and the loop around the U(k) group denotes the adjoint hypermultiplet.

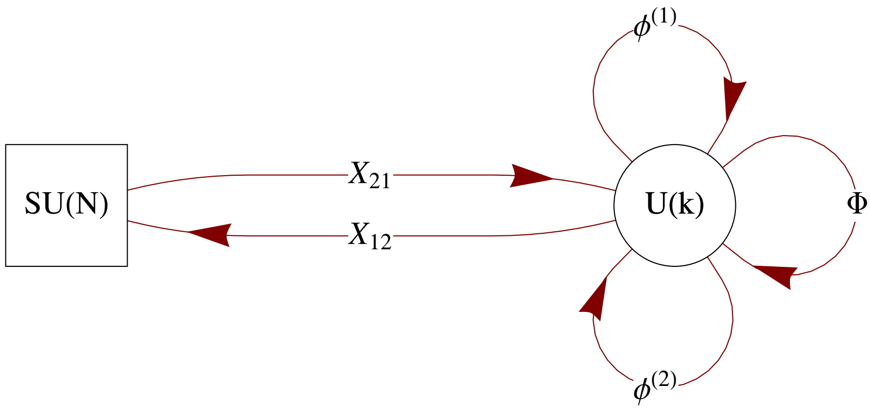

Figure 7. Flower quiver; The $\mathcal{N}=1$ quiver diagram for k SU(N) instantons on $\mathbb{C}^2$ with the corresponding superpotential, $W = X_{21}\cdot\Phi\cdot X_{12} + \epsilon_{\alpha\beta}\phi^{(\alpha)}\cdot\Phi\cdot\phi^{(\beta)}$.

Field	U(k)		U(N)		SU(2)$_g$	U(1)$_g$
	SU(k)	U(1)	SU(N)	U(1)	global	global
Fugacity:	$z_1,\ldots,z_{k-1}$	z	$x_1,\ldots,x_{N-1}$	q	x	t
Φ	$[1,0,\ldots,0,1]$	0	$[0,\ldots,0]$	0	[0]	0
$\phi^{(1)},\phi^{(2)}$	$[1,0,\ldots,0,1]$	0	$[0,\ldots,0]$	0	[1]	1
X_{12}	$[1,0,\ldots,0]$	1	$[0,\ldots,0,1]$	-1	[0]	1
X_{21}	$[0,0,\ldots,0,1]$	-1	$[1,0\ldots,0]$	1	[0]	1
Tr Φ	[0,...,0]	0	[0,...,0]	0	[0]	0
Tr $\phi^{(1)}$, Tr $\phi^{(2)}$	[0,...,0]	0	[0,...,0]	0	[1]	1

Table 2. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table. The U(2)$_g$ global symmetry acts on $\phi^{(1)}$ and $\phi^{(2)}$. It is the symmetry group of $\mathbb{C}^2$, the trivial component in the moduli space.

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to be at the CoM of the D7-branes and thereby eliminate the corresponding background U(1) vector multiplet.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N}=2$ quiver diagram, the line connecting the SU(N) and U(k) groups denotes kN hypermultiplets, and the loop around the U(k) group denotes k^2 hypermultiplets. Hence, we have in total $kN+k^2$ quarternionic degrees of freedom. On a generic point on the Higgs branch, the gauge group U(k) is completely broken and hence there are k^2 broken generators. As a result of the Higgs mechanism, the vector multiplet gains k^2 degrees of freedom and becomes massive. Hence, the $(kN+k^2)-k^2 = kN$ quarternionic degrees of freedom are left massless. Thus, the Higgs branch is kN quaternionic dimensional or $2kN$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}_{k,N} = 2kN = 2kh \ . \tag{3.3}$$

This agrees with the dual coxeter number of SU(N) which is $h_{\text{SU}(N)} = N$.

From the brane perspective, the VEV of the scalar Φ correspond to the position of the D3-branes along the directions transverse to the D7-branes. On the Higgs branch, the gauge fields become massive freezing the whole vector multiplet and hence $\langle\Phi\rangle = 0$, setting the D3 branes to lie within the D7 branes and possibly form bound states. The hypermultiplets acquire non-zero VEVs at a generic point on the Higgs branch that parametrize all possible bound states of D3 and D7 branes. From the point of view of the D7 brane gauge theory, the D3 branes are interpreted as instantons and hence, the moduli space of classical instantons on $\mathbb{C}^2$ is identified with the Higgs branch of the quiver theory [26].

3.2.1 One SU(N) instanton: $k=1$

The gauge theory for 1 SU(N) instanton on $\mathbb{C}^2$ is particularly simple and lives on the world volume of 1 D3 brane, $k=1$. The gauge group is U(1) and the adjoints Φ, ϕ_1, ϕ_2 are simply complex numbers, and hence the second term of (3.2) vanishes,

$$W = X_{21}\cdot\Phi\cdot X_{12}. \tag{3.4}$$

The Higgs branch. On the Higgs branch, $\Phi = 0$ and $X_{12}\cdot X_{21} = 0$. The space of F-term solutions (which we will call the F-flat space and denote by $\mathcal{F}^\flat$) is obviously a complete intersection. Using (3.3) the dimension of the moduli space is $2N$. On the other hand there are $2N$ bifundamental fields X_{12}, X_{21} and 2 ϕ's which are subject to 1 relation. This gives an F-flat moduli space which is $2N+1$ dimensional and after imposing the D-term equations we get a $2N$ dimensional moduli space, as expected. The F-flat Hilbert series can be written down according to table 3 as[6]

$$\begin{aligned} g^{\mathcal{F}^\flat}_{k=1,N}(t,x_1,\ldots,x_{N-1},x,q,z) = (1-t^2)\text{PE}\Big[&[1]_{\text{SU}(2)_g}t + [1,0,\ldots,0]_{\text{SU}(N)}\frac{tz}{q} \\ &+[0,0,\ldots,0,1]_{\text{SU}(N)}\frac{tq}{z}\Big] \ . \end{aligned} \tag{3.5}$$

[6]The *plethystic exponential (PE)* of a multi-variable function $g(t_1,\ldots,t_n)$ that vanishes at the origin, $g(0,\ldots,0)=0$, is defined to be $\text{PE}[g(t_1,\ldots,t_n)] := \exp\left(\sum_{r=1}^\infty \frac{g(t_1^r,\ldots,t_n^r)}{r}\right)$. The reader is referred to [31–39] for more details.

Note that the first term in the square bracket corresponds to $\phi^{(1)}$ and $\phi^{(2)}$, the second term corresponds to X_{12} and the third term correspond to X_{21}, and the factor in front of the PE corresponds to the relation.

Notice from (3.5) that the U(1) of U(N) can in fact be absorbed into the local U(1). This can be seen by redefining the fugacity for the local U(1) as

$$w = \frac{z}{q}\,, \tag{3.6}$$

and rewrite

$$\begin{aligned} g^{\mathcal{F}^\flat}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x, w) \;=&\; (1-t^2)\mathrm{PE}\Big[[1]_{\mathrm{SU}(2)_g} t + [1, 0, \ldots, 0]_{\mathrm{SU}(N)} tw \\ &+[0, 0, \ldots, 0, 1]_{\mathrm{SU}(N)} \frac{t}{w}\Big]\,. \end{aligned} \tag{3.7}$$

The right hand side can explicitly be written as a rational function:

$$\begin{aligned} &(1-t^2)\times \frac{1}{(1-tx)(1-\frac{t}{x})} \times \frac{1}{(1-twx_1)\left(1-\frac{tw}{x_{N-1}}\right)\prod_{k=2}^{N-1}(1-tw\frac{x_k}{x_{k-1}})} \\ &\times \frac{1}{\left(1-\frac{t}{w}\frac{1}{x_1}\right)(1-\frac{t}{w}x_{N-1})\prod_{k=2}^{N-1}(1-\frac{t}{w}\frac{x_{k-1}}{x_k})}\,. \end{aligned} \tag{3.8}$$

The Hilbert series. Now we project (3.8) onto the gauge invariant subrepresentation by performing an integration over the U(1) gauge group.[7] The Hilbert series of the Higgs branch is therefore given by

$$g^{\mathrm{Higgs}}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x) = \frac{1}{2\pi i}\oint_{|w|=1} \frac{\mathrm{d}w}{w} g^{\mathcal{F}^\flat}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x, w)\,. \tag{3.9}$$

Using the residue theorem on (3.8), where the poles are located at[8]

$$w = t\frac{1}{x_1},\; t\frac{x_1}{x_2},\; \ldots,\; t\frac{x_{N-2}}{x_{N-1}},\; tx_{N-1}\,, \tag{3.10}$$

we can write the Hilbert series in terms of representations as

$$g^{\mathrm{Higgs}}_{k=1,N}(t, x_1, \ldots, x_{N-1}, x) = \frac{1}{(1-tx)\left(1-\frac{t}{x}\right)} \sum_{k=0}^{\infty} [k, 0, \ldots, 0, k]_{\mathrm{SU}(N)} t^{2k}. \tag{3.11}$$

The factor $\frac{1}{(1-tx)(1-\frac{t}{x})}$ indicates the Hilbert series for the complex plane $\mathbb{C}^2$, whose symmetry is $\mathrm{U}(2)_g$ (with the fugacities t, x). This space $\mathbb{C}^2$ is parametrised by $\phi^{(1)}$ and $\phi^{(2)}$ and corresponds to the position of the D3-brane inside the D7-branes. The second factor corresponds to the coherent component of the one SU(N) instanton moduli space. Unrefining by setting $x_1 = \cdots = x_{N-1} = x = 1$, we obtain

$$g^{\mathrm{Higgs}}_{k=1,N}(t, 1, \ldots, 1) = \frac{1}{(1-t)^2} \times \frac{\sum_{k=0}^{N-1} \binom{N-1}{k}^2 t^{2k}}{(1-t^2)^{2(N-1)}}\,. \tag{3.12}$$

[7] This is called the *Molien-Weyl integral formula* (see, e.g., [35–39]).

[8] Note that $|t| < 1$ and only poles located inside the unit circle $|w| = 1$ are included.

The order of the pole $t = 1$ is $2N$, and hence the dimension of the Higgs branch is $2N$, in accordance with (3.3). Note that (3.12) can also be derived directly from (3.9) as follows. Setting $x_1 = \cdots = x_{N-1} = x = 1$ in (3.9), we obtain

$$g^{\rm Higgs}_{k=1,N}(t,1,\ldots,1) = \frac{(1-t^2)}{(1-t)^2}\frac{1}{2\pi i}\oint_{|w|=1}\frac{{\rm d}w}{w}\frac{1}{(1-tw)^N(1-\frac{t}{w})^N}\ . \tag{3.13}$$

The contribution to the integral comes from the pole at $w = t$, which is of order N. Using the residue theorem, we find that

$$g^{\rm Higgs}_{k=1,N}(t,1,\ldots,1) = \frac{(1-t^2)}{(1-t)^2}\times\frac{1}{(N-1)!}\frac{{\rm d}^{N-1}}{{\rm d}w^{N-1}}\left[\frac{w^{N-1}}{(1-tw)^N}\right]_{w=t} \tag{3.14}$$

Using Leibniz's rule for differentiation, we thus arrive at (3.12).

The plethystic logarithm can be written as

$$\begin{aligned}{\rm PL}[g^{\rm Higgs}_{k=1,N}(t,x_1,\ldots,x_{N-1},x)] = {} & [1]_{{\rm SU}(2)_g}t + [1,0,\ldots,0,1]_{{\rm SU}(N)}t^2 - ([0,1,0,\ldots,0,1,0] + \\ & + [1,0,\ldots,0,1] + [0,\ldots,0])_{{\rm SU}(N)}\, t^4 + \cdots\ . \end{aligned}\tag{3.15}$$

Hence, the generators are ${\rm Tr}\,\phi^{(1)}$, ${\rm Tr}\,\phi^{(2)}$ at order t and the adjoints [1,0,…,0,1] of SU(N) at the order t^2. The basic relations transform in the SU(N) representation $[0,1,0,\ldots,0,1,0]+[1,0,\ldots,0,1]+[0,\ldots,0]$.

3.3 k SO(N) instantons on $\mathbb{C}^2$

As pointed out in [28], the moduli space of k SO(N) instantons can be realised on a system of k D3-branes with N half D7-branes on top of an O7$^-$ orientifold plane. (If the number of branes is odd, the combination of half D7 brane stuck on the O7$^-$ plane form an orientifold plane which is called $\widetilde{O7}^-$ plane.) The brane picture is similar to the one described in the previous subsection and therefore the quiver looks the same. We only need to figure out the action of the orientifold plane on the different objects in the quiver. All together, there are 4 objects in figure 6.

- The gauge group on the D7 branes is projected to SO(N). This is a global symmetry for the gauge theory on the D3 branes. $\mathcal{N} = 2$ supersymmetry restricts the gauge theory on the D3 branes to be Sp(k). Hence,
- The gauge group on the D3 branes is projected down to Sp(k).
- The bi-fundamental fields become bi-fundamentals of SO(N) $\times$ Sp(k).
- The loop around the U(k) gauge group undergoes a $\mathbb{Z}_2$ projection which leaves two options - the second rank symmetric or antisymmetric representation of Sp(k). To find which one, we notice that only the anti-symmetric representation is reducible into a singlet plus the rest. Since the center of mass of the instanton is physically decoupled from the rest of the moduli space, we conclude that the projection is to the antisymmetric representation.

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Figure 8. The $\mathcal{N}=2$ quiver diagram for k SO(N) instantons on $\mathbb{C}^2$. The circular node represents the Sp(k) gauge symmetry and the square node represents the SO(N) flavour symmetry. The line connecting the SO(N) and Sp(k) groups denotes $2kN$ half-hypermultiplets, and the loop around the Sp(k) gauge group denotes a hypermultiplet transforming in the (reducible) second rank antisymmetric tensor.

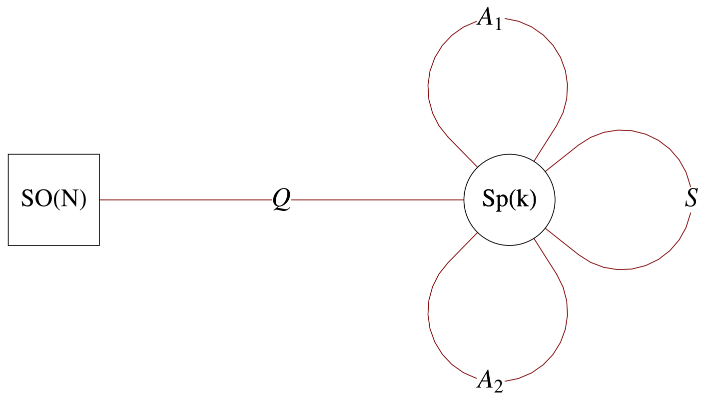

Figure 9. The $\mathcal{N}=1$ quiver diagram for k SO(N) instantons on $\mathbb{C}^2$. The chiral multiplet transforming in the second rank symmetric tensor (adjoint field) of Sp(k) is denoted by S and the second rank antisymmetric tensors are denoted by A_1, A_2. The superpotential is given by $W = Q \cdot S \cdot Q + \epsilon_{\alpha\beta} A_\alpha \cdot S \cdot A_\beta$.

The resulting $\mathcal{N}=2$ quiver diagram is depicted in figure 8.

Using the rules of section 3.1 it is easy to find the $\mathcal{N}=1$ quiver diagram given in figure 9 and the superpotential,

$$\begin{aligned} W &= Q \cdot S \cdot Q + (A_1 \cdot S \cdot A_2 - A_2 \cdot S \cdot A_1) \\ &= Q \cdot S \cdot Q + \epsilon_{\alpha\beta} A_\alpha \cdot S \cdot A_\beta \,, \end{aligned} \tag{3.16}$$

where we have suppressed the contractions over the gauge indices by the tensor J^{ab} (an invariant tensor of Sp(k)) and the contractions over the flavour indices by δ_{ij} (an invariant tensor of SO(N)). The epsilon tensor $\epsilon_{\alpha\beta}$ in the second line is an invariant tensor of the global SU(2) symmetry which interchanges A_1 and A_2. The mass term for Q coming from the adjoint of SO(N) is set to 0.

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Field	Sp(k)	SO(N)	SU(2)$_g$	U(1)$_g$
Fugacity:	$z_1, \ldots, z_k$	$x_1, \ldots, x_{\lfloor N/2 \rfloor}$	x	t
S	$[2, 0, \ldots, 0]$	$[0, \ldots, 0]$	$[0]$	0
A_1, A_2	$[0, 1, 0, \ldots, 0] + [0, \ldots, 0]$	$[0, \ldots, 0]$	$[1]$	1
Q	$[1, 0, \ldots, 0]$	$[1, 0, \ldots, 0]$	$[0]$	1

Table 3. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N} = 2$ quiver diagram, the lines connecting the SO(N) and Sp(k) groups denotes $2kN$ half-hypermultiplets (equivalently, kN hypermultiplets), and the loop around the Sp(k) group gives $k(2k-1)$ hypermultiplets. Hence, we have in total $kN+k(2k-1)$ quarternionic degrees of freedom. On the Higgs branch, Sp(k) is completely broken and hence there are $k(2k+1)$ broken generators. As a result of the Higgs mechanism, the vector multiplet gains $k(2k+1)$ degrees of freedom and becomes massive. Hence, the $kN + k(2k-1) - k(2k+1) = k(N-2)$ degrees of freedom are left massless. Thus, the Higgs branch is $k(N-2)$ quaternionic dimensional or $2k(N-2)$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\text{Higgs}}_{k,N} = 2k(N-2) = 2k h_{\text{SO}(N)} \ . \tag{3.17}$$

Note that $h_{\text{SO}(N)} = N - 2$ is the dual coxeter number of the SO(N) group.

The charges and the representations under which the fields transform are given in table 3 [45].

3.3.1 One SO(N) instanton on $\mathbb{C}^2$: $k = 1$

In the special case $k = 1$, the gauge group is Sp(1) = SU(2) and the superpotential (3.16) becomes

$$W_{k=1} = \epsilon^{ab}\epsilon^{cd} Q^i_a S_{bc} Q^i_d \ . \tag{3.18}$$

The Higgs branch. The Higgs branch is given by the F-term conditions: $S = 0$ and $Q^i_a Q^i_b + Q^i_b Q^i_a = 0$, and the D-term condition. The Hilbert series of the F-flat moduli space is

$$\begin{aligned} g^{\mathcal{F}^\flat}(t, z, x_1, \ldots, x_{\lfloor N/2 \rfloor}, x) = {}& (1-t^2)\left(1 - \frac{t^2}{z^2}\right)(1-t^2z^2)\,\text{PE}\Big[[1]_{\text{SU}(2)_g} t \\ & +[1,0,\ldots,0]_{\text{SO}(N)} t\left(z + \frac{1}{z}\right)\Big] \ . \end{aligned} \tag{3.19}$$

We note that the relation transforms in the representation [2] of Sp(1) and that the F-flat moduli space is a complete intersection of dimension $2 + 2N - 3 = 2N - 1$. Noting that the characters of the fundamental representations of $B_n = \text{SO}(2n+1)$ and $D_n = \text{SO}(2n)$

respectively are

$$
\begin{aligned}
[1,0,\ldots,0]_{B_n}(x_a) &= 1+\sum_{a=1}^{n}\left(x_a+\frac{1}{x_a}\right) , \\
[1,0,\ldots,0]_{D_n}(x_a) &= \sum_{a=1}^{n}\left(x_a+\frac{1}{x_a}\right) ,
\end{aligned} \tag{3.20}
$$

we can write down (3.19) as a rational functional function

$$
\begin{aligned}
g^{\mathcal{F}^\flat}(t,z,x_1,\ldots,x_n,x)_{B_n,D_n} &= \frac{(1-t^2)}{(1-tx)(1-t/x)} \times \\
&\quad \times \frac{\left(1-\frac{t^2}{z^2}\right)(1-t^2z^2)}{(1-t)^\delta \prod_{a=1}^{n}(1-tzx_a)(1-\frac{tz}{x_a})(1-\frac{t}{z}x_a)(1-\frac{t}{zx_a})} ,
\end{aligned} \tag{3.21}
$$

where $\delta=1$ for B_n and $\delta=0$ for D_n.

Performing the Molien-Weyl integral over the gauge group Sp(1), we obtain the Higgs branch Hilbert series as

$$
\begin{aligned}
g^{\mathrm{Higgs}}(t,x_1,\ldots,x_n,x)_{B_n,D_n} &= \frac{1}{2\pi i}\oint_{|z|=1}\mathrm{d}z\left(\frac{1-z^2}{z}\right) g^{\mathcal{F}^\flat}(t,z,x_1,\ldots,x_n,x)_{B_n,D_n} \\
&= \frac{1}{(1-tx)(1-t/x)}\times\sum_{k=0}^{\infty}[0,k,0,\ldots,0]_{B_n,D_n}t^{2k} ,
\end{aligned} \tag{3.22}
$$

where the contributions to the integral come from the poles:

$$
z = tx_1,\ldots,tx_n,\frac{t}{x_1},\ldots,\frac{t}{x_n} . \tag{3.23}
$$

The factor $\frac{1}{(1-tx)(1-t/x)}$ is the Hilbert series for $\mathbb{C}^2$ (whose symmetry is $\mathrm{U}(2)_g$) and is parametrised by the singlets in A_1, A_2; this corresponds to the position of the D3-brane inside the D7-branes. The second factor corresponds to the coherent component of the one SO(N) instanton moduli space.

Example: $N=8$. The expression (3.19) can be written as a rational function:

$$
\frac{(1-t^2)}{(1-tx)(1-t/x)}\times\frac{\left(1-\frac{t^2}{z^2}\right)(1-t^2z^2)}{\prod_{a=1}^{4}(1-tzx_a)(1-\frac{tz}{x_a})(1-\frac{tx_a}{z})(1-\frac{t}{zx_a})} . \tag{3.24}
$$

The poles which contribute to the Molien-Weyl integral (3.22) are

$$
z = tx_1,\ldots,tx_4,\frac{t}{x_1},\ldots,\frac{t}{x_4} . \tag{3.25}
$$

The integral (3.22) gives

$$
g^{\mathrm{Higgs}}(t,x_1,\ldots,x_4,x) = \frac{1}{(1-tx)(1-t/x)}\times\sum_{k=0}^{\infty}[0,k,0,0]_{\mathrm{SO}(8)}t^{2k} . \tag{3.26}
$$

Unrefining by setting $x_1 = \cdots = x_4 = x = 1$, we obtain

$$g^{\rm Higgs}(t,1,1,1,1,1) = \frac{1}{(1-t)^2} \times \frac{\left(1+t^2\right)\left(1+17t^2+48t^4+17t^6+t^8\right)}{\left(1-t^2\right)^{10}} \,. \tag{3.27}$$

Observe that the pole at $t = 1$ is of order 12, and so the Higgs branch is indeed 12 dimensional, in agreement with (3.17). The plethystic logarithm is

$$\begin{aligned} \mathrm{PL}\left[g^{\rm Higgs}(t,x_1,x_2,x_3,x_4,x)\right] &= [1]_{\mathrm{SU}(2)_g} t + [0,1,0,0]_{\mathrm{SO}(8)} t^2 - ([2,0,0,0]+[0,0,2,0] \\ &\quad +[0,0,0,2]+[0,0,0,0])_{\mathrm{SO}(8)} t^4 + \cdots \,, \end{aligned} \tag{3.28}$$

indicating that the relations are invariant under the triality of SO(8).

3.4 k Sp(N) instantons on $\mathbb{C}^2$

As pointed out in [26], the moduli space of k Sp(N) instantons can be realised on a system of k D3-branes with N D7-branes on top of an O7$^+$ orientifold plane. As a result, the gauge group is projected to SO(k),[9] and the scalar in the vector multiplet becomes an antisymmetric tensor, denoted by A_{ab} (where the SO(k) gauge indices take values $a, b = 1, \ldots, k$). The adjoint hypermultiplet becomes a symmetric tensor, as it is the reducible second rank tensor of SO(k), and is denoted by two chiral multiplets S_1 and S_2. Since representations of the SO(k) group are real, the flavour symmetry is Sp(N) and we have $2kN$ half-hypermultiplets. We denote the complex scalar in each half-hypermultiplet as Q^i_a (where the Sp(N) flavour indices take values $i, j = 1, \ldots, 2N$).

The $\mathcal{N} = 2$ and $\mathcal{N} = 1$ quiver diagrams are given respectively in figure 10 and figure 11. The $\mathcal{N} = 1$ superpotential is

$$\begin{aligned} W &= Q \cdot A \cdot Q + (S_1 \cdot A \cdot S_2 - S_2 \cdot A \cdot S_1) \\ &= Q \cdot A \cdot Q + \epsilon_{\alpha\beta} S_\alpha \cdot A \cdot S_\beta \,, \end{aligned} \tag{3.29}$$

where we have suppressed the contractions over the flavour indices by the tensor J_{ij} (an invariant tensor of Sp(N)) and the contractions over the gauge indices by δ^{ab} (an invariant tensor of SO(k)). The epsilon tensor $\epsilon_{\alpha\beta}$ in the second line is an invariant tensor of the global SU(2) symmetry which interchanges S_1 and S_2. The mass term transforming in the adjoint of Sp(N) is set to 0.

Let us compute the quaternionic dimension of the Higgs branch. From the $\mathcal{N} = 2$ quiver diagram, the lines connecting the Sp(N) and $O(k)$ groups denotes $2kN$ half-hypermultiplets (equivalently, kN hypermultiplets), and the loop around the $O(k)$ group gives $\frac{1}{2}k(k+1)$ hypermultiplets. Hence, we have in total $kN + \frac{1}{2}k(k+1)$ degrees of freedom. On the Higgs branch, we assume that $O(k)$ is completely broken and hence there are $\frac{1}{2}k(k-1)$ broken generators. As a result of the Higgs mechanism, the vector multiplet gains $\frac{1}{2}k(k-1)$ degrees of freedom and becomes massive. Hence, the $\left[kN + \frac{1}{2}k(k+1)\right] -$

[9] For $k = 1$ we take the convention that SO(1) is $\mathbb{Z}_2$. For higher values of k, the computations in this paper do not distinguish between a gauge group $O(k)$ and a gauge group SO(k) and hence this $\mathbb{Z}_2$ ambiguity is ignored.

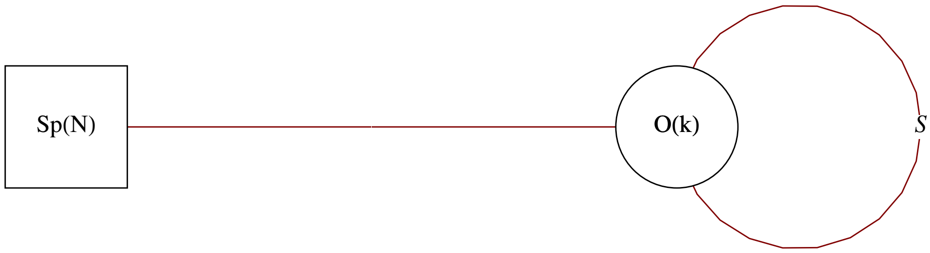

Figure 10. The $\mathcal{N}=2$ quiver diagram for k $\mathrm{Sp}(N)$ instantons on $\mathbb{C}^2$. The circular node represents the $O(k)$ gauge symmetry and the square node represents the $\mathrm{Sp}(N)$ flavour symmetry. The line connecting the $\mathrm{Sp}(N)$ and $O(k)$ groups denotes $2kN$ half-hypermultiplets, and the loop around the $O(k)$ group denotes the second rank (reducible) symmetric tensor.

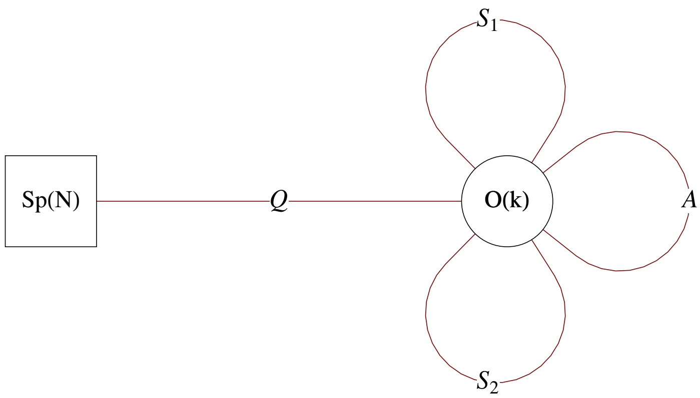

Figure 11. The $\mathcal{N}=1$ quiver (flower) diagram for k $\mathrm{Sp}(N)$ instantons on $\mathbb{C}^2$, with A being an antisymmetric tensor (adjoint field) and S_1, S_2 being symmetric tensors of $\mathrm{Sp}(k)$. The superpotential is $W = Q \cdot A \cdot Q + \epsilon_{\alpha\beta} S_\alpha \cdot A \cdot S_\beta$.

$\frac{1}{2}k(k-1) = k(N+1)$ degrees of freedom are left massless. Thus, the Higgs branch is $k(N+1)$ quaternionic dimensional or $2k(N+1)$ complex dimensional:

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{Higgs}}_{k,N} = 2k(N+1) = 2k h_{\mathrm{Sp}(N)}\,, \tag{3.30}$$

where $h_{\mathrm{Sp}(N)} = N+1$ is the dual coxeter number of the $\mathrm{Sp}(N)$ gauge group.

We list the charges and the representations under which the fields transform in table 4.

3.4.1 One $\mathrm{Sp}(N)$ instanton on $\mathbb{C}^2$: $k=1$

For $k=1$, the gauge group becomes $O(1) \cong \mathbb{Z}_2$. Recall that we have $2N$ hypermultiplets Q^i and two gauge singlets S_1 and S_2. It is then easy to see that the moduli space in this case is

$$\mathcal{M}^{\mathrm{Higgs}}_{k=1,N} = \mathbb{C}^{2N}/\mathbb{Z}_2 \times \mathbb{C}^2\,, \tag{3.31}$$

Field	SO(k)	Sp(N)	SU$(2)_g$ global	U(1) global
Fugacity:	$z_1,\dots,z_k$	$x_1,\dots,x_{\lfloor N/2 \rfloor}$	x	t
A	$[0,1,\dots,0]$	$[0,\dots,0]$	$[0]$	0
S_1, S_2	$[2,0,\dots,0]+[0,\dots,0]$	$[0,\dots,0]$	$[1]$	1
Q	$[1,0,\dots,0]$	$[1,0,\dots,0]$	$[0]$	1

Table 4. The charges and the representations under which various fields transform. The fugacites of each field are assigned according to this table.

where the factor $\mathbb{C}^2$ is parametrised by S_1 and S_2, the $\mathbb{C}^{2N}$ is parametrised by Q^i, and the orbifold action $\mathbb{Z}_2$ is -1 on each coordinate of $\mathbb{C}^{2N}$. Observe that $\mathcal{M}^{\text{Higgs}}_{k=1,N}$ is $2(N+1)$ complex dimensional, in accordance with (3.30). Physically, the $\mathbb{C}^2$ corresponds to the position (4 real coordinates) of the instanton. The coherent component of the one Sp(N) instanton moduli space is therefore $\mathbb{C}^{2N}/\mathbb{Z}_2$.

One can see the last statement clearly from the Hilbert series. The Hilbert series of $\mathbb{C}^{2N}/\mathbb{Z}_2$ is given by the discrete Molien formula (see, e.g., [31–34]):

$$\begin{aligned} g(t,x_1,\dots,x_N;\mathbb{C}^{2N}/\mathbb{Z}_2) &= \frac{1}{2}\left(\mathrm{PE}\left[[1,0,\dots,0]_{\mathrm{Sp}(N)}t\right]+\mathrm{PE}\left[[1,0,\dots,0]_{\mathrm{Sp}(N)}(-t)\right]\right) \\ &= \sum_{k=0}^{\infty}[2k,0,\dots,0]t^{2k}\,, \end{aligned} \tag{3.32}$$

where the plethystic exponential can be written explicitly as

$$\mathrm{PE}\left[[1,0,\dots,0]_{\mathrm{Sp}(N)}t\right] = \frac{1}{\prod_{a=1}^{N}(1-tx_a)(1-t/x_a)} = \sum_{n=0}^{\infty}[n,0,\dots,0]_{\mathrm{Sp}(N)}\,t^n,$$

and the $\mathbb{Z}_2$ acts on t by projecting to even powers. The last equality of (3.32) follows from the fact that the plethystic exponential generates symmetrisation. This is indeed the Hilbert series for the coherent component of the one Sp(N) instanton moduli space. The choice of x_a in this formula is not the natural choice of weights in the representation but rather a linear combination of weights which is convenient for writing this particular formula.

4 $\mathcal{N}=2$ supersymmetric SU(N_c) gauge theory with N_f flavours

This section deals with the computation of the Hilbert series for the Higgs branch of the $\mathcal{N}=2$ SU(N_c) supersymmetric gauge theory with N_f flavours. It serves as a preparation for the discussion in section section 5, were the results will be used in checking Argyres-Seiberg duality. The global symmetry of this theory is U(N_f) = U$(1)_B\times$SU(N_f) and since it plays a crucial role on the Higgs branch this theory will sometimes be called the U(N_f) theory. The special case of $N_c=2$ and $N_f=4$ is discussed in section 3.3.1 and is revisited below. The $\mathcal{N}=2$ quiver diagram for this theory is depicted in figure 12.

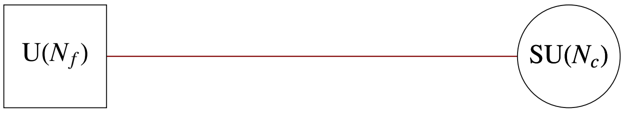

Figure 12. $\mathcal{N}=2$ quiver diagram for SU(N_c) gauge theory with N_f flavours.

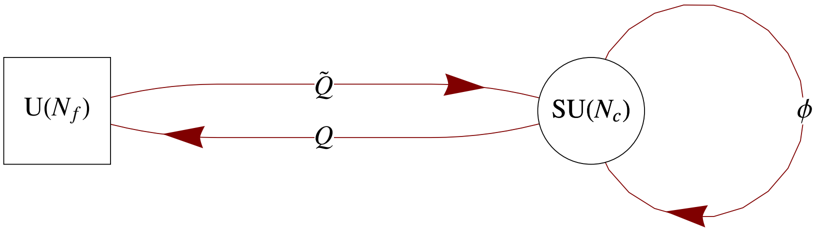

Figure 13. $\mathcal{N}=1$ quiver diagram for SU(N_c) gauge theory with N_f flavours. The superpotential is $W=\widetilde{Q}\cdot\phi\cdot Q$.

The $\mathcal{N}=1$ quiver diagram is depicted in figure 13 and the superpotential after setting the masses to 0 is given by

$$W=\widetilde{Q}\cdot\phi\cdot Q, \tag{4.1}$$

giving the F-term equations on the Higgs branch, $\phi=0$ and $Q\widetilde{Q}=0$, where the last equation has only N_c^2-1 equations and not N_c^2. The trace meson $\widetilde{Q}\cdot Q$ need not vanish.

The Higgs branch of this theory has a Hilbert series which is easy to write down as an integral over the Haar measure of SU(N_c). The reason for this lies partly with supersymmetry and partly with the simplicity of the gauge and matter content. We first argue that the F-flat moduli space is a complete intersection. Since the quaternionic dimension of the Higgs branch is $N_cN_f-(N_c^2-1)$, the complex dimension of the F-flat moduli space is expected to be N_c^2-1 higher than this one. Adding these together, we get that the complex dimension of the F-flat moduli space is $2N_cN_f-(N_c^2-1)$. On the other hand, these are precisely the number of degrees of freedom. There are $2N_cN_f$ complex variables which are subject to N_c^2-1 equations on the Higgs branch. We therefore conclude that the F-flat moduli space is a complete intersection and its Hilbert series can be written as a ratio of two plethystic exponentials,

$$g^{\mathcal{F}^\flat}_{N_c,N_f}=\frac{\mathrm{PE}\left[[1,0,\dots,0]_{\mathrm{SU}(N_c)}[0,\dots,0,1]_{\mathrm{SU}(N_f)}t_1+[0,\dots,0,1]_{\mathrm{SU}(N_c)}[1,0,\dots,0]_{\mathrm{SU}(N_f)}t_2\right]}{\mathrm{PE}\left[[1,0,\dots,0,1]_{\mathrm{SU}(N_c)}t^2\right]}, \tag{4.2}$$

where $t_1=tb$ and $t_2=t/b$ are respectively the global U(1) fugacities for Q and $\widetilde{Q}$ and b is the fugacity for the baryonic symmetry $\mathrm{U}(1)_B$. The Higgs branch is given by integrating this Hilbert series using the SU(N_c) Haar measue,

$$g^{\mathrm{Higgs}}_{N_c,N_f}=\int d\mu_{\mathrm{SU}(N_c)}g^{\mathcal{F}^\flat}_{N_c,N_f}\ . \tag{4.3}$$

4.1 The case of $N_c = 3$ and $N_f = 6$

In this subsection, we focus on the $\mathcal{N} = 2$ supersymmetric SU(3) gauge theory with 6 flavours.

From (4.2), the F-flat Hilbert series after setting all U(6) fugacities to 1 can be written as

$$g^{\mathcal{F}^\flat}_{N_c=3,N_f=6} = \frac{(1-t^2)^2\left(1-\frac{t^2 z_1}{z_2^2}\right)\left(1-\frac{t^2}{z_1 z_2}\right)\left(1-\frac{t^2 z_1^2}{z_2}\right)\left(1-\frac{t^2 z_2}{z_1^2}\right)(1-t^2 z_1 z_2)\left(1-\frac{t^2 z_2^2}{z_1}\right)}{(1-tz_1)^6 (1-tz_2)^6 \left(1-\frac{t}{z_1}\right)^6 \left(1-\frac{t}{z_2}\right)^6 \left(1-\frac{tz_1}{z_2}\right)^6 \left(1-\frac{tz_2}{z_1}\right)^6}, \tag{4.4}$$

where z_1 and z_2 are the SU(3) fugacities. The Haar measure for SU(3) is

$$\int d\mu_{\mathrm{SU}(3)} = \frac{1}{(2\pi i)^2} \oint_{|z_1|=1} \frac{dz_1}{z_1} \oint_{|z_2|=1} \frac{dz_2}{z_2} (1-z_1 z_2)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right), \tag{4.5}$$

After integrating over z_1 and z_2, we obtain the Hilbert series:[10]

$$g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t) = \frac{P(t)}{(1-t)^{20}(1+t)^{16}(1+t+t^2)^{10}}, \tag{4.6}$$

where the numerator $P(t)$ is a palindromic polynomial of degree 36:

$$\begin{aligned} P(t) = {} & 1 + 6t + 41t^2 + 206t^3 + 900t^4 + 3326t^5 + 10846t^6 + 31100t^7 + 79677t^8 + \\ & + 183232t^9 + 381347t^{10} + 720592t^{11} + 1242416t^{12} + 1959850t^{13} + \\ & + 2837034t^{14} + 3774494t^{15} + 4624009t^{16} + 5220406t^{17} + 5435982t^{18} \\ & + \cdots \text{ (palindrome) } \ldots + t^{36}. \end{aligned} \tag{4.7}$$

Note that the space is $20 = 2(3 \cdot 6 - 8)$ complex-dimensional, as expected. The first few orders of the power expansion of (4.6) reads

$$g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t) = 1 + 36t^2 + 40t^3 + 630t^4 + 1120t^5 + \cdots. \tag{4.8}$$

The plethystic logarithm is

$$PL\left[g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t)\right] = 36t^2 + 40t^3 - 36t^4 - 320t^5 - 435t^6 + \cdots. \tag{4.9}$$

The fully refined Hilbert series. In fact, one can obtain the fully refined Hilbert series directly from (4.2) and (4.3). The result can be written as a power series

$$\begin{aligned} & g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t_1,t_2;x_1,x_2,x_3,x_4,x_5) = \\ & = \frac{1}{1-t_1t_2} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty} [n_1,n_2,n_3+n_4,n_2,n_1]_{\mathrm{SU}(6)} {t_1}^{n_1+2n_2+3n_3} {t_2}^{n_1+2n_2+3n_4}. \end{aligned} \tag{4.10}$$

[10] In using the residue theorem, the non-trivial contributions to the first integral over z_1 come from the poles $z_1 = t,\ tz_2$, and the non-trivial contributions to the second integral over z_2 come from the poles $z_2 = t,\ t^2$.

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where $x_1, \ldots, x_5$ are the SU(6) fugacities.

The plethystic logarithm of (4.10) is

$$\begin{aligned} \mathrm{PL}\left[g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t_1,t_2;x_1,x_2,x_3,x_4,x_5)\right] &= ([0,0,0,0,0]+[1,0,0,0,1])t_1t_2 + \\ +[0,0,1,0,0](t_1^3+t_2^3) - ([0,0,0,0,0]&+[1,0,0,0,1])t_1^2t_2^2+\cdots\,, \end{aligned} \tag{4.11}$$

where the gauge invariant operators in the representation $[0,0,0,0,0]+[1,0,0,0,1]$ of SU(6) can be identified as *mesons* (see (4.17)) and the operators in the representation $[0,0,1,0,0]$ of SU(6) can be identified as *baryons* and *antibaryons* (see (4.18)).

4.2 Generalisation to the case $N_f = 2N_c$

The formula (4.16) can be generalised to the case $N_f = 2N_c$. Let us first consider the simplest case of: $N_f = 2N_c = 4$, discussed in section 3.3.1.

The $N_c = 2$ and $N_f = 4$ case. From (3.26), the Hilbert series of the coherent component of the Higgs branch is

$$g^{\mathrm{Higgs}}_{N_c=2,N_f=4}(t;x_1,x_2,x_3,x_4) = \sum_{k=0}^{\infty}[0,k,0,0]_{\mathrm{SO}(8)}t^{2k}\,, \tag{4.12}$$

The branching rule of the representation $[0,k,0,0]$ of SO(8) to the subgroup $\mathrm{SU}(4)\times\mathrm{U}(1)_B$ is given by

$$[0,k,0,0]_{\mathrm{SO}(8)} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}[n_1,n_2+n_3,n_1]_{\mathrm{SU}(4)}b^{2n_2-2n_3}\delta(k-n_1-n_2-n_3-n_4)\,, \tag{4.13}$$

or equivalently the decomposition identity

$$\sum_{k=0}^{\infty}[0,k,0,0]_{\mathrm{SO}(8)}t^{2k} = \frac{1}{1-t^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]_{\mathrm{SU}(4)}b^{2n_2-2n_3}t^{2n_1+2n_2+2n_3}\,, \tag{4.14}$$

where b is the fugacity of $\mathrm{U}(1)_B$. Substituting (4.13) into (4.12), we obtain

$$\begin{aligned} g^{\mathrm{Higgs}}_{N_c=2,N_f=4}(t;x_1,x_2,x_3;b) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]t^{2n_1+2n_2+2n_3+2n_4}b^{2n_2-2n_3} \\ &= \frac{1}{1-t_1t_2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}[n_1,n_2+n_3,n_1]t_1^{n_1+2n_2}t_2^{n_1+2n_3}, \end{aligned} \tag{4.15}$$

where in the last line we take $t_1 = tb$ and $t_2 = tb^{-1}$.

Generalisation. From (4.10) and (4.15), we conjecture that the Hilbert series for the Higgs branch of the $\mathrm{SU}(N_c)$ gauge theory with $N_f = 2N_c$ flavours can be written in terms of SU($2N_c$) representations as

$$
\begin{aligned}
& g^{\text{Higgs}}_{N_f=2N_c}(t_1,t_2;x_1,\ldots,x_{2N_c-1}) = \\
& = \frac{1}{1-t_1t_2}\sum_{n_1=0}^{\infty} \\
& \cdots \sum_{n_{N_c+1}=0}^{\infty} [n_1,n_2,\ldots,n_{N_c-1},n_{N_c}+n_{N_c+1},n_{N_c-1},\ldots,n_2,n_1] t_1^{d+N_cn_{N_c}} t_2^{d+N_cn_{N_c+1}} \,,
\end{aligned} \tag{4.16}
$$

where $d = \sum_{k=1}^{N_c-1} kn_k$. This formula can be checked by plugging in the dimensions of the representations, one finds that the Higgs branch is $2(N_c^2+1)$ complex dimensional, as expected. Note the similarity between (4.16) and the Hilbert series of $\mathcal{N}=1$ SQCD (see (5.1) of [39]); however, they are not identical — the moduli space of $\mathcal{N}=1$ SQCD with $N_f \geq N_c$ is $2N_cN_f-(N_c^2-1)$ complex dimensional, whereas the moduli space of the $\mathcal{N}=2$ gauge theory is $2N_cN_f-2(N_c^2-1)$ complex dimensional.

The plethystic logarithm of (4.16) indicates that:

- At the order t_1t_2, there are gauge invariants transforming in the representation $[0,\ldots,0]+[1,0,\ldots,0,1]$ of SU(N_f) and carrying U$(1)_B$ charge 0 These operators are *mesons*:

$$
M^i_j = Q^i_a \widetilde{Q}^a_j \,, \tag{4.17}
$$

  where $a=1,\ldots,N_c$ and $i,j=1,\ldots,N_f$.

- At the order $t_1^{N_c}$ and $t_2^{N_c}$, there are gauge invariants transforming in the representation $[0,\ldots,0,1,0,\ldots0]$ of SU(N_f) and carrying U$(1)_B$ charges N_c and $-N_c$. These operators are respectively *baryons* and *antibaryons*:

$$
\begin{aligned}
B^{i_1,\ldots,i_{N_c}} &= \epsilon^{a_1\ldots a_{N_c}} Q^{i_1}_{a_1}\ldots Q^{i_{N_c}}_{a_{N_c}} \,, \\
\widetilde{B}_{i_1,\ldots,i_{N_c}} &= \epsilon_{a_1\ldots a_{N_c}} \widetilde{Q}^{a_1}_{i_1}\ldots \widetilde{Q}^{a_{N_c}}_{i_{N_c}} \,.
\end{aligned} \tag{4.18}
$$

These generators are indeed identical to those of the $\mathcal{N}=1$ SQCD. Hence, they satisfy the relations given by (3.11) and (3.12) of [39]:

$$
\begin{aligned}
(*B)\widetilde{B} &= *(M^{N_c}) \,, \\
M\cdot *B &= M\cdot *\widetilde{B} = 0 \,.
\end{aligned} \tag{4.19}
$$

where $(*B)_{i_{N_c+1}\ldots i_{N_f}} = \frac{1}{N_c!}\epsilon_{i_1\ldots i_{N_f}} B^{i_1\ldots i_{N_c}}$ and a '$\cdot$' denotes a contraction of an upper with a lower flavour index. In addition, the F-terms impose further relations. These are given by (2.23) and (2.24) of [40]:

$$
\begin{aligned}
M'\cdot B = \widetilde{B}\cdot M' &= 0 \,, \\
M\cdot M' &= 0 \,,
\end{aligned} \tag{4.20}
$$

where

$$
(M')^i_j = M^i_j - \frac{1}{N_c}(\operatorname{Tr} M)\delta^i_j \,. \tag{4.21}
$$

5 Exceptional groups and Argyres-Seiberg dualities

In this section, we consider the Hilbert series of a single G instanton on $\mathbb{R}^4$ where G is one of the 5 exceptional groups. It is shown that the conjecture is consistent with the dimension of the instanton moduli space, by explicitly summing the unrefined Hilbert series. In the cases of E_6 and E_7, we also check that the proposed Hilbert Series are consistent with Argyres-Seiberg dualities found in [17–23]. Only for the case of E_6, we are able to carry out a full all-order check. In the case of E_7, we just match the lower dimension BPS operators. Notice that the check for BPS operators of scaling dimension 2 is equivalent to the check that the symmetries on both sides of the duality are the same. This is because BPS operators of scaling dimension 2 are in the same super multiplet of the flavour currents.

Notation. In this section, when there is no ambiguity, we denote special unitary (SU) groups in the quiver diagrams by their ranks. Each U(1) global symmetry is associated with a hypermultiplet and hence each solid line connecting two nodes represents a U(1) global symmetry. The dashed lines are not associated with bi-fundamental hypermultiplets and do not correspond to U(1) global symmetries. Square nodes with an index 1 do not count as a U(1) global symmetry.

5.1 E_6

The Hilbert series of one E_6-instanton on $\mathbb{R}^4$ is given by (2.1):

$$g^{\mathrm{Irr}}_{E_6}(t; x_1, \ldots, x_6) = \sum_{k=0}^{\infty} [0, k, 0, 0, 0, 0] t^{2k}. \tag{5.1}$$

By setting the E_6 fugacities to 1, this equation can be resumed and written in the form of (2.5):

$$g^{\mathrm{Irr}}_{E_6}(t; 1, \ldots, 1) = \frac{P_{E_6}(t)}{(1-t^2)^{22}}, \tag{5.2}$$

where

$$\begin{aligned} P_{E_6}(t) = &(1+t^2)(1+55t^2+890t^4+5886t^6+17929t^8+26060t^{10}+ \\ &+ \ldots \text{ (palindrome) } \ldots + t^{20}) . \end{aligned} \tag{5.3}$$

This confirms that the complex dimension of the moduli space is $2h_{E_6} - 2 = 22$, where $h_{E_6} = 12$ is the dual Coxeter number of E_6.

5.1.1 Duality between the $6 - \bullet - 2 - 1$ quiver theory and the SU(3) gauge theory with 6 flavours

As discussed in [20], the strongly interacting SCFT with E_6 flavour symmetry can be realised as 3 M5-branes wrapping a sphere with 3 punctures. These punctures are of the maximal type, each one is associated to SU(3) global symmetry. The global symmetry $\mathrm{SU}(3)^3$ enhances to E_6. This theory is also known as the T_3 theory [15, 16, 20, 21] and is denoted by the left picture of figure 14. There is no known Lagrangian description for this theory.

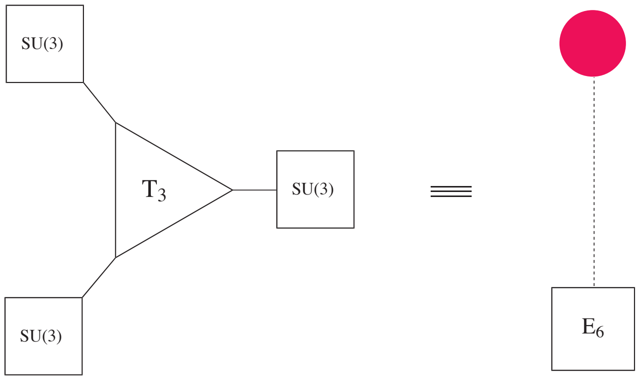

Figure 14. *Left:* The E_6 theory arising from 3 M5-branes wrapping a sphere with 3 maximal punctures, each is associated to SU(3) global symmetry. The $\mathrm{SU}(3)^3$ symmetry enhances to E_6. *Right:* The quiver diagram representing the E_6 theory. The red blob denotes a theory with an unknown Lagrangian description. The E_6 global symmetry is indicated in the square node.

The E_6 theory is denoted by a 'quiver diagram' which is analogous to those in previous sections. This is given in the right picture of figure 14. The red blob denotes a theory with an unknown Lagrangian. The E_6 global symmetry is indicated in the square node. Below it is demonstrated that even though the Lagrangian is not known, it is still possible to make statements about the spectrum of operators for this theory.

The E_6 theory can be used to construct a quiver gauge theory called the $6 - \bullet - 2 - 1$ theory, depicted in figure 18. This theory is proposed by Argyres and Seiberg [17] to be dual to an SU(3) gauge theory with 6 flavours, whose quiver diagram is shown in figure 16. The appearance of the tail in figure 15 seems to be a generic feature of these dualities and follows from the splitting of branes when ending on the same brane - see figure 20 of [30].

Let us summarise a construction of the $6-\bullet-2-1$ quiver theory. The global symmetry E_6 can be decomposed into the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(6)$. The SU(2) symmetry is gauged and is coupled to the $2-1$ tail, as depicted in figure 15. The resulting theory is the *the* $6 - \bullet - 2 - 1$ *quiver theory.* The U(1) global symmetry is associated with the solid line in the quiver diagram. The global symmetry is thus $\mathrm{SU}(6) \times \mathrm{U}(1) \cong \mathrm{U}(6)$.

Note that a necessary condition for two theories to be dual is that they have the same global symmetry. Indeed, both of the $6 - \bullet - 2 - 1$ quiver theory and the SU(3) gauge theory with 6 flavours have the same global symmetry U(6), even though these symmetries arise from different sources in each case.

A branching rule for E_6 to $\mathrm{SU}(2) \times \mathrm{SU}(6)$. To proceed, we first decompose the E_6 representations into representations of $\mathrm{SU}(2) \times \mathrm{SU}(6)$. For this it is useful to introduce the fugacity map. The fugacities $u_1, u_2, \ldots, u_6$ of E_6 can be mapped to the fugacities x of SU(2) and $y_1, \ldots, y_5$ of SU(6) as follows:

$$u_1 = x y_5, \quad u_2 = y_1 y_5, \quad u_3 = y_5^2, \quad u_4 = y_2 y_5^2, \quad u_5 = y_3 y_5, \quad u_6 = y_4 \,. \tag{5.4}$$

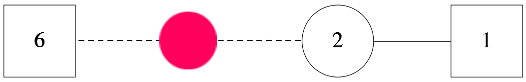

Figure 15. *The* $6 - \bullet - 2 - 1$ *quiver theory:* From the E_6 theory, the global symmetry E_6 is decomposed into the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(6)$. The SU(2) symmetry is gauged and is coupled to the $2 - 1$ tail. The U(1) global symmetry is associated with the solid line in the quiver diagram. The flavour symmetry is $\mathrm{SU}(6) \times \mathrm{U}(1)$.

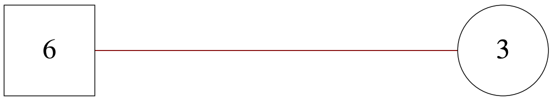

Figure 16. The SU(3) gauge theory with 6 flavours. This theory is conjectured by Argyres-Seiberg to be dual to the $6 - \bullet - 2 - 1$ quiver theory.

Using this map, one can decompose the character of an E_6 representation into the characters of $\mathrm{SU}(2) \times \mathrm{SU}(6)$ representations. For example, if we denote a representation of $\mathrm{SU}(2) \times \mathrm{SU}(6)$ of highest weight m for SU(2) and highest weights $n_1, \ldots, n_5$ for SU(6) by $[m; n_1, \ldots, n_5]$, then one finds that

$$
\begin{aligned}
[0,1,0,0,0,0]_{E_6} &= [0;1,0,0,0,1] + [1;0,0,1,0,0] + [2;0,0,0,0,0], \\
[0,2,0,0,0,0]_{E_6} &= [0;0,0,0,0,0] + [0;0,1,0,1,0] + [0;2,0,0,0,2] + \\
&\quad +[1;0,0,1,0,0] + [1;1,0,1,0,1] + [2;1,0,0,0,1] + \\
&\quad +[2;0,0,2,0,0] + [3;0,0,1,0,0] + [4;0,0,0,0,0] \; .
\end{aligned} \tag{5.5}
$$

These equalities can be checked by matching the characters of the representations on both sides. The general formula for the decompositions of Adj^k for any k is given in (5.11).

The decompositions (5.5) can be written in terms of dimensions as

$$
\begin{aligned}
78 &\to (1,35) \oplus (2,20) \oplus (3,1) \\
2430 &\to (1,1) \oplus (1,189) \oplus (1,405) \oplus (2,20) \oplus (2,540) \oplus (3,35) \oplus \\
&\quad\; (3,175) \oplus (4,20) \oplus (5,1) \; .
\end{aligned} \tag{5.6}
$$

Counting BPS operators of the SU(3) gauge theory with 6 flavours. In what follows, starting from (5.1), we count BPS operators in the SU(3) gauge theory with 6 flavours by computing the SU(2) gauge invariant spectrum. For now, let us first do this order by order for the operators of small scaling dimensions. In the later subsections, we present a method to count the operators to all orders.

- At level t^2, we expect the 35 to survive, as it is an SU(2) singlet. Denote the $2 - 1$ hypermultiplet in figure 15 by q and $\tilde{q}$. Set q to have fugacity tb^3 and $\tilde{q}$ to have fugacity t/b^3, where the normalization 3 is chosen for matching with the U(6) baryons. One can construct another SU(2) invariant which is a singlet under SU(6), by forming $q\tilde{q}$.

We therefore expect the SU(3) theory with 6 flavours to have $35_0 + 1_0$ at order t^2, where the subscript 0 refers to the $\mathrm{U}(1)_B$ baryonic charge. Indeed, in the SU(3) theory of figure 16 these are formed by the SU(3) mesons $\tilde{Q}Q$ that decompose as $35_0 + 1_0$.

- At level t^3, the $(2, 20)$ coupled to q or to $\tilde{q}$, leads to the SU(2) invariant operators which transform as $20_3 \oplus 20_{-3}$. This contributes the term $20(b^3 + 1/b^3)t^3$ to the U(6) Hilbert series.

- At level t^4 we have the singlets $1 + 189 + 405$, and the 35 from order t^2 multiplied by the SU(6)-singlet $q\tilde{q}$, for a total of 630 operators.

These are precisely the first few terms of the Hilbert series (4.8) of the Higgs Branch of SU(3) theory with 6 flavours:

$$g^{\mathrm{Higgs}}_{N_c=3,N_f=6}(t) = 1 + 36t^2 + 20(b^3 + b^{-3})t^3 + 630t^4 + \cdots \,. \tag{5.7}$$

5.1.2 Branching formula for Adj^k of E_6 to $\mathrm{SU}(2) \times \mathrm{SU}(6)$

In this subsection, we carry out the decomposition of the Adj^k-irreducible representations of E_6 into $\mathrm{SU}(2) \times \mathrm{SU}(6)$ to all order in k. This gives a useful check of the Argyres-Seiberg duality to all orders. The general form of the decomposition is as follows:

$$[0, k, 0, 0, 0, 0]_{E_6} = \sum_{m=0}^{2k} [m]_{\mathrm{SU}(2)} C^k_m \tag{5.8}$$

where C^k_m is a reducible representation of SU(6). The sets of irreps of SU(6) entering in C^k_m is constructed starting by the representation R^L_p, defined by:

$$\begin{aligned}
R^L_{p>0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=n} [i, j, k+p, j, i] \\
R^{2L}_{p=0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=2n} [i, j, k, j, i] \\
R^{2L+1}_{p=0} &= \sum_{n=0}^{L} \sum_{i+2j+3/2k=2n+1} [i, j, k, j, i]
\end{aligned} \tag{5.9}$$

Notice that only SU(6)-irreps whose Dynkin labels are symmetric enter the sum, and that R^k_n contains an irreducible representation at most one time. The C^k are given in terms of the R^L_p by

$$\begin{aligned}
C^k_{2m} &= \sum_{j=0}^{m} R^{k-m-j}_j \\
C^k_{2m+1} &= \sum_{j=0}^{m} R^{k-m-1-j}_j
\end{aligned} \tag{5.10}$$

In C^k_m the same irrep can appear multiple times. Summing these together we find the decomposition identity

$$\begin{aligned}
&(1-t^4)\sum_{k=0}^{\infty}[0,k,0,0,0,0]_{E_6}t^{2k} \qquad (5.11)\\
&=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}[n_1+2n_2]_{\mathrm{SU}(2)}[n_3,n_4,n_1+2n_5,n_4,n_3]_{\mathrm{SU}(6)}t^{2n_1+2n_2+2n_3+4n_4+6n_5}\\
&+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\\
&\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}[n_1+2n_2+1]_{\mathrm{SU}(2)}[n_3,n_4,n_1+2n_5+1,n_4,n_3]_{\mathrm{SU}(6)}t^{2n_1+2n_2+2n_3+4n_4+6n_5+4}.
\end{aligned}$$

Using these all order results, we can proceed to refine $g^{\mathrm{Irr}}_{E_6}(t)$ in (5.2) to a function of z and t (denoted as $g^{\mathrm{Irr}}_{E_6}(z,t)$), where z is the SU(2) fugacity.

5.1.3 The Hilbert series of the $6-\bullet-2-1$ quiver theory

As discussed earlier, the $6-\bullet-2-1$ quiver theory can be obtained by first decomposing the E_6 into $\mathrm{SU}(2)\times\mathrm{SU}(6)$, the SU(2) group is then gauged and is coupled as in the $2-1$ quiver. This process can also be described as a 'sewing' of two Riemann surfaces - one with 3 maximal punctures (corresponding to E_6) and the other with two simple puctures (corresponding to $\mathrm{U}(2)\times\mathrm{U}(1)$). The Hilbert series can be computed in analogy to the AGT relation [41, 42] as follows:

$$g_{6-\bullet-2-1}(t)=\int \mathrm{d}\mu_{\mathrm{SU}(2)}(z)\; g^{\mathrm{Irr}}_{E_6}(t,z)\; g_{\mathrm{glue}}(t,z)\; g_{2-1}(t,b,z)\,, \qquad (5.12)$$

where the Haar measure for SU(2) is given by

$$\int \mathrm{d}\mu_{\mathrm{SU}(2)}=\frac{1}{2\pi i}\oint dz\frac{1-z^2}{z}\,, \qquad (5.13)$$

the Hilbert series for the bi-fundmentals connecting the SU(2) and U(1) nodes is

$$\begin{aligned}
g_{2-1}(t,b,z)&=\mathrm{PE}\left[[1]_{\mathrm{SU}(2)}\left(b^3+b^{-3}\right)t\right]\\
&=\frac{1}{(1-tzb^3)(1-t\frac{z}{b^3})\left(1-\frac{tb^3}{z}\right)\left(1-\frac{t}{zb^3}\right)}\,,
\end{aligned} \qquad (5.14)$$

and the 'gluing factor' which keeps track of the 3 F-term relations that comes from differentiating the superpotential by the adjoint chiral field of SU(2) is

$$g_{\mathrm{glue}}(t,z)=\frac{1}{\mathrm{PE}\left[[2]_{\mathrm{SU}(2)}t^2\right]}=\left(1-t^2z^2\right)\left(1-t^2\right)\left(1-\frac{t^2}{z^2}\right)\,. \qquad (5.15)$$

The product of $g_{\text{fund}}(z,t)$ and $g_{\text{glue}}(z,t)$ can be written for $b=1$ as

$$g_{\text{glue}}(t,z)g_{2-1}(t,1,z) = \frac{\left(1-t^2z^2\right)\left(1-t^2\right)\left(1-\frac{t^2}{z^2}\right)}{(1-tz)^2\left(1-\frac{t}{z}\right)^2} \\ = \sum_{n=0}^{\infty}[n]t^n + \sum_{n=0}^{\infty}[n+1]t^{n+1} + t^2 - 2\sum_{n=0}^{\infty}[n]t^{n+4} \,. \tag{5.16}$$

If we restore the b dependence, this sum takes the form

$$g_{\text{glue}}(t,z)g_{2-1}(t,b,z) = \\ = \sum_{n=0}^{\infty}[n](tb^3)^n + \sum_{n=0}^{\infty}[n+1]\left(\frac{t}{b^3}\right)^{n+1} + t^2 - \sum_{n=0}^{\infty}[n]t^{n+4}(b^{3n+6}+b^{-3n-6}) \,. \tag{5.17}$$

From (5.12), one sees that the integral is computed by summing over two residues, one at $z=t$ and one at $z=t^2$. For $z=t$, the residue is a rational function with denominator $(1-t)^{21}(1+t+t^2)^{21}$. For $z=t^2$, the residue is a rational function with denominator $(1-t)^{21}(1+t)^{16}(1+t^2)^{37}(1+t+t^2)^{21}$. Summing these two residues gives precisely the unrefined Hilbert series $g^{\text{Higgs}}_{N_c=3,N_f=6}(t)$ of (4.6).

For the refined Hilbert series, it is better to exchange the integral in (5.12) with the sums and use the orthonormality relation

$$\oint_{|z|=1} \frac{dz(1-z^2)}{2\pi i z}[n][m] = \delta_{n,m} \tag{5.18}$$

to confirm that the fully refined Hilbert series coincides with (4.10).

5.2 E_7

The Hilbert series of one E_7-instanton on $\mathbb{R}^4$ is given by (2.1):

$$g^{\text{Irr}}_{E_7}(t;x_1,\ldots,x_6,x_7) = \sum_{k=0}^{\infty}[k,0,0,0,0,0,0]t^{2k}. \tag{5.19}$$

By setting the E_7 fugacities to 1, this equation can be resumed and written in the form of (2.5):

$$g^{\text{Irr}}_{E_7}(t;1,\ldots,1) = \frac{P_{E_7}(t)}{(1-t^2)^{34}}\,, \tag{5.20}$$

where the numerator is a palindromic polynomial of degree 17 in t^2,

$$P_{E_7}(t) = 1+99t^2+3410t^4+56617t^6+521917t^8+2889898t^{10}+10086066t^{12}+ \\ +22867856t^{14}+34289476t^{16}+\cdots \text{ (palindrome) } \ldots + t^{34}. \tag{5.21}$$

This is consistent with the fact that the Higgs branch is $2h_{E_7}-2=34$ complex dimensional, where $h_{E_7}=18$ is the dual Coxeter number of E_7.

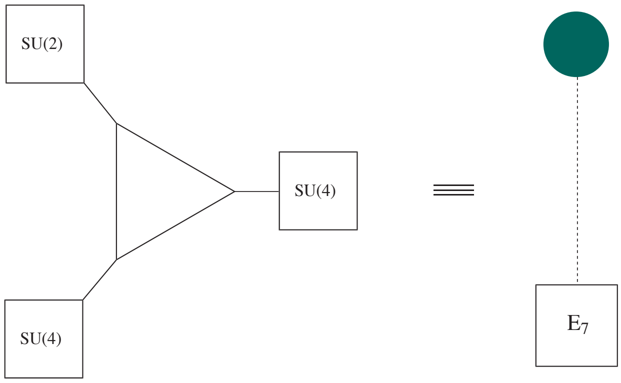

Figure 17. *Left:* The E_7 theory arising from 4 M5-branes wrapped over a sphere with 3 punctures of the type SU(4), SU(4), SU(2). *Right:* The quiver diagram representing the E_7 theory. The green blob denotes a theory with an unknown Lagrangian description. The E_7 global symmetry is indicated by the square node.

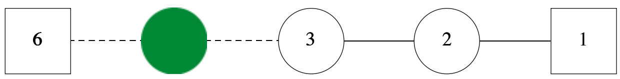

Figure 18. *The* $6 - \bullet - 3 - 2 - 1$ *quiver theory:* The global symmetry E_7 can be decomposed into the subgroup $\mathrm{SU}(3) \times \mathrm{SU}(6)$. The SU(3) symmetry is gauged and is coupled to the $3 - 2 - 1$ tail. The U(1) global symmetries are associated with the solid lines in the quiver diagram. The global symmetry is thus $\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

5.2.1 Duality between the $6 - \bullet - 3 - 2 - 1$ quiver theory and the $2 - 4 - 6$ quiver theory

In [23], it was realised that the E_7 theory can be realised as 4 M5-branes wrapped over a sphere with 3 punctures. The punctures are of the type SU(4), SU(4), SU(2). This theory is depicted in the left picture of figure 17. The Lagrangian description of this theory is unknown.

We denote the E_7 theory by a 'quiver diagram' analogue to those in previous sections. This is given in the right picture of figure 17. The green blob denotes the theory with unknown Lagrangian description. The E_7 global symmetry is indicated in the square node.

The E_7 theory can be used to construct a quiver gauge theory called the $6-\bullet-3-2-1$ theory, depicted in figure 18. The duality between this theory and the $2-4-6$ quiver theory (depicted in figure 19) is proposed by [23]. Our purpose of this section is to construct and match the Hilbert series of both sides of the duality.

Let us summarise a construction of the $6 - \bullet - 3 - 2 - 1$ quiver theory. The global symmetry E_7 can be decomposed into the subgroup $\mathrm{SU}(3) \times \mathrm{SU}(6)$. The SU(3) symmetry is gauged and is coupled to the $3 - 2 - 1$ tail, depicted in figure 18. The U(1) global symmetries are associated with the hypermultiplets and hence the solid lines in the quiver diagram. The global symmetry is thus $\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

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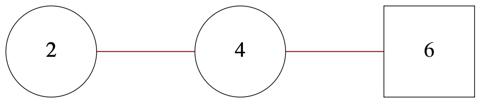

Figure 19. *The* $2-4-6$ *quiver theory.* This theory is dual to the $6-\bullet-3-2-1$ quiver theory.

A trick to obtain the $3-2-1$ tail is to consider the SU(2) theory with 4 flavours, whose flavour symmetry of is SO(8). The group SO(8) contains $SU(4)\times U(1) \supset SU(3)\times U(1)\times U(1)$ as subgroups. Gauging the SU(3) group in SO(8) and gluing it to the SU(3) group in E_7, we obtain the $6-\bullet-3-2-1$ quiver theory.

On the other side of the duality, we have the $2-4-6$ quiver theory, depicted in figure 19. The U(1) global symmetries are associated with the hypermultiplets and hence the solid lines in the quiver diagram. Therefore, the flavour symmetry is $U(6)\times U(1) \cong SU(6)\times U(1)\times U(1)$, in agreement with that of the $6-\bullet-3-2-1$ quiver theory. From the quiver diagram, it is clear that the $2-4-6$ quiver theory can also be obtained by gauging the SU(2) subgroup of the U(8) flavour group of the SU(4) gauge theory with 8 flavours.

5.2.2 The Hilbert series of the $2-4-6$ quiver theory

In this subsection, the refined and unrefined Hilbert series are computed. The former contains information about the global symmetries and how the gauge invariants transform under such symmetries, whereas the latter contains information about the dimension of the moduli space and the number of operators in the spectrum. In order to compute an exact form of the refined Hilbert series, general formulas involving branching rules need to be determined. However, such formulas can sometimes be very cumbersome and difficult to compute; in which case, what one can do is to compute the first few orders of the refined Hilbert series. Nevertheless, it may be possible that the unrefined Hilbert series can be computed exactly. We give an example below.

The $2-4-6$ quiver theory can be obtained by gauging the SU(2) subgroup of the U(8) flavour group of the SU(4) gauge theory with 8 flavours. The Hilbert series written in terms of SU(8) representations is given by (4.16). We first discuss a branching rule for SU(8) to $U(1)\times SU(2)\times SU(6)$.

A branching rule for SU(8) to U(1) × SU(2) × SU(6). A map from the SU(8) fugacities $x_1,\dots,x_7$ to the U(1) fugacity q, the SU(2) fugacity z and the SU(6) fugacities $y_1,\dots,y_5$ can be

$$\begin{array}{llll} x_1 = qy_1, & x_2 = q^2 y_2, & x_3 = q^3 y_3, & x_4 = q^4 y_4, \\ x_5 = q^5 y_5, & x_6 = q^6, & x_7 = q^3 z\ . & \end{array}$$

For example, we have

$$\begin{aligned} [1,0,0,0,0,0,0] &= [0;1,0,0,0,0]q + [1;0,0,0,0,0]q^{-3} \\ [1,0,0,0,0,0,1] &= [0;0,0,0,0,0] + [2;0,0,0,0,0] + [1;0,0,0,0,1]q^{-4} \\ &\quad +[1;1,0,0,0,0]q^4 + [0;1,0,0,0,1]\ . \end{aligned} \tag{5.22}$$

Using this decomposition, the Hilbert series of the SU(4) theory with 8 flavours can be written as

$$
\begin{aligned}
g^{\text{Higgs}}_{N_c=4,N_f=8} &= 1 + (2 + [2;0,0,0,0,0] + [1;0,0,0,0,1]\frac{1}{q^4} + [1;1,0,0,0,0]q^4 \\
&+[0;1,0,0,0,1])t^2 + \Big(4 + 2[2;0,0,0,0,0] + [4;0,0,0,0,0] + \frac{3[1;0,0,0,0,1]}{q^4} \\
&+\frac{[3;0,0,0,0,1]}{q^4} + \frac{[2;0,0,0,0,2]}{q^8} + \frac{[0;0,0,0,1,0]}{q^8} + \frac{q^4[0;0,0,0,1,0]}{b^2} \\
&+b^2q^4[0;0,0,0,1,0] + \frac{[1;0,0,1,0,0]}{b^2} + b^2[1;0,0,1,0,0] + \frac{[0;0,1,0,0,0]}{b^2q^4} \\
&+\frac{b^2[0;0,1,0,0,0]}{q^4} + q^8[0;0,1,0,0,0] + q^4[1;0,1,0,0,1] + [0;0,1,0,1,0] \\
&+3q^4[1;1,0,0,0,0] + q^4[3;1,0,0,0,0] + 3[0;1,0,0,0,1] + 2[2;1,0,0,0,1] \\
&+\frac{[1;1,0,0,0,2]}{q^4} + \frac{[1;1,0,0,1,0]}{q^4} + q^8[2;2,0,0,0,0] + q^4[1;2,0,0,0,1] \\
&+[0;2,0,0,0,2]\Big)t^4 + \cdots . \qquad (5.23)
\end{aligned}
$$

The refined Hilbert series of the $2-4-6$ theory. This can be computed by gauging the SU(2) symmetry. The gauging is done by integrating over the SU(2) Haar measure and Supersymmetry imposes additional adjoint valued F terms, which are written below as the glue factor,

$$
g_{2-4-6}(t;q;b;y_1,\dots,y_5) = \int \mathrm{d}\mu_{\mathrm{SU}(2)}\; g_{\text{glue}}\; g^{\text{Higgs}}_{N_c=4,N_f=8}\,, \qquad (5.24)
$$

where the gluing factor is given by

$$
g_{\text{glue}}(t;z) = \frac{1}{\mathrm{PE}\left[[2]_{\mathrm{SU}(2)}t^2\right]} = 1 - [2]t^2 + [2]t^4 - t^6 \,. \qquad (5.25)
$$

The integral in (5.24) projects out the SU(2) singlets. This gives

$$
\begin{aligned}
g_{2-4-6}(t;q,b;y_1,\dots,y_5) &= 1 + (2 + [1,0,0,0,1])t^2 + \Big(3 + \frac{1}{q^4}[0,0,0,1,0] \\
&+\frac{q^2}{b^2}[0,0,0,1,0] + b^2q^2[0,0,0,1,0] + \frac{1}{b^2q^2}[0,1,0,0,0] + \frac{b^2}{q^2}[0,1,0,0,0] \\
&+q^4[0,1,0,0,0] + [0,1,0,1,0] + 3[1,0,0,0,1] + [2,0,0,0,2]\Big)t^4 + \cdots . \qquad (5.26)
\end{aligned}
$$

The unrefined Hilbert series. The unrefined Hilbert series can be computed exactly. Setting $q = b = y_1 = \cdots = y_5 = 1$ in (5.24), it can be easily seen that the integrand is simply a rational function of t and z. Evaluating the integral, one obtains the closed form

$$
\begin{aligned}
g_{2-4-6}(t) &= \frac{P(t)}{(1-t^2)^{28}(1+t^2)^{14}} \\
&= 1 + 37t^2 + 792t^4 + 12180t^6 + 145838t^8 + 1422490t^{10} + \cdots . \qquad (5.27)
\end{aligned}
$$

where

$$
\begin{aligned}
P(t) = & \ 1 + 23t^2 + 351t^4 + 3773t^6 + 29904t^8 + 180648t^{10} + 855350t^{12} + \\
& + 3243202t^{14} + 10014534t^{16} + 25512281t^{18} + 54163863t^{20} + \\
& + 96566265t^{22} + 145392195t^{24} + 185575556t^{26} + 201252816t^{28} \\
& + \cdots \ (\text{palindrome}) \ \ldots + t^{56} \ .
\end{aligned} \tag{5.28}
$$

The plethystic logarithm of this Hilbert series is

$$
\mathrm{PL}[g_{2-4-6}(t)] = 37t^2 + 89t^4 - 252t^6 - 2800t^8 + 14720t^{10} + 124524t^{12} + \cdots \ . \tag{5.29}
$$

5.2.3 The Hilbert series of the $6 - \bullet - 3 - 2 - 1$ quiver theory

As described in section 5.2.1, the $6 - \bullet - 3 - 2 - 1$ quiver theory can be obtained by 'gluing' the SU(3) subgroup of the E_7 theory with the SU(3) subgroup of the SO(8) flavor symmetry for SU(2) with 4 flavors. The Hilbert series of the latter, written in terms of U(4) representations, is given in Equation (4.15). In order to gauge the SU(3) subgroup, one needs to find a branching rule for SU(4) to U(1) $\times$ SU(3).

A branching rule for SU(4) to U(1) $\times$ SU(3). A map from the SU(4) fugacities $x_1, \ldots, x_3$ to the U(1) fugacity q and the SU(3) fugacities z_1, z_2 can be

$$
x_1 = \frac{z_1}{q}, \quad x_2 = \frac{z_2}{q^2}, \quad x_3 = \frac{1}{q^3} \ . \tag{5.30}
$$

With this map, one can rewrite (4.15) in terms of SU(3) representations as

$$
\begin{aligned}
g^{\text{Higgs}}_{3-2-1} = & \ \frac{1}{1-t^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} [n_1, n_2+n_3, n_1]_{\mathrm{SU}(4)} t^{2n_1+2n_2+2n_3} b^{2n_2-2n_3} \\
= & \ \frac{1}{(1-t^2)^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} q^{2n_1-2n_2} \frac{b^{-2(n_1+n_2)}(1-b^{4(1+n_1+n_2)})}{(1-b^4)} \times \\
& \times \Bigg[[n_1+n_3, n_2+n_3] + \sum_{n_4=0}^{n_3-1} \big(q^{-4n_3+4n_4} [n_1+n_3, n_2+n_4] \\
& \quad + q^{4n_3-4n_4} [n_1+n_4, n_2+n_3] \big) \Bigg] t^{2(n_1+n_2+n_3)} \ .
\end{aligned} \tag{5.31}
$$

Since we need to gauge SU(3) $\subset E_7$, we also need to obtain the branching rule of E_7 representations to the subgroup SU(3) $\times$ SU(6).

Branching rule for E_7 to SU(3) $\times$ SU(6). The branching rules can be obtained by matching the characters on both sides. A map of the E_7 fugacities $u_1, \ldots, u_7$ to the SU(3) fugacities z_1, z_2 and the SU(6) fugacities $y_1, \ldots, y_5$ can be

$$
u_1 = z_1 y_2, \quad u_2 = y_1 y_2, \quad u_3 = z_2 y_2^2, \quad u_4 = y_2^3, \quad u_5 = \frac{y_2^2 y_3}{y_4}, \quad u_6 = \frac{y_2^2}{y_4}, \quad u_7 = \frac{y_2 y_5}{y_4} \ . \tag{5.32}
$$

For example, the decompositions of Adj^1 and Adj^2 of E_7 are given below. We use the notation $[a_1, a_2; b_1, \ldots, b_5]$ to denote the representations of $\mathrm{SU}(3) \times \mathrm{SU}(6)$.

$$\begin{aligned}
Adj^1 &= [1,1;0,0,0,0,0] + [1,0;0,1,0,0,0] + [0,1;0,0,0,1,0] + [0,0;1,0,0,0,1] \\
Adj^2 &= [2,2;0,0,0,0,0] + [2,0;0,2,0,0,0] + [0,2;0,0,0,2,0] + [0,0;2,0,0,0,2] \\
&\quad + [2,1;0,1,0,0,0] + [1,1;0,1,0,1,0] + [0,1;1,0,0,1,1] + [2,0;0,0,0,1,0] \\
&\quad + [1,2;0,0,0,1,0] + [1,0;1,1,0,0,1] + [1,1;0,0,0,0,0] + [0,2;0,1,0,0,0] \\
&\quad + [1,1;1,0,0,0,1] + [1,0;0,1,0,0,0] + [1,0;0,0,1,0,1] + [0,0;1,0,0,0,1] \\
&\quad + [0,0;0,1,0,1,0] + [0,1;0,0,0,1,0] + [0,1;1,0,1,0,0] + [0,0;0,0,0,0,0] \ .
\end{aligned} \tag{5.33}$$

The Hilbert series of the coherent component of the one E_7 instanton moduli space on $\mathbb{R}^4$ after using the fugacity map Equation (5.32) is

$$g^{\mathrm{Irr}}_{E_7}(t; z_1, z_2; y_1, \ldots, y_5) = \sum_{k=0}^{\infty} Adj^k(z_1, z_2; y_1, \ldots, y_5) t^{2k} \ . \tag{5.34}$$

Gluing process. We obtain the Hilbert series of the $6 - \bullet - 3 - 2 - 1$ quiver theory by using a similar 'gluing technique' to Equation (5.12):

$$g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) = \int \mathrm{d}\mu_{\mathrm{SU}(3)}\, g^{\mathrm{Irr}}_{E_7}\, g_{\mathrm{glue}}\, g^{\mathrm{Higgs}}_{3-2-1} \ , \tag{5.35}$$

where the gluing factor is given by the adjoint valued F terms,

$$g_{\mathrm{glue}}(t; z_1, z_2) = \frac{1}{\mathrm{PE}\left[[1,1]_{\mathrm{SU}(3)} t^2 \right]} \ . \tag{5.36}$$

Therefore, we obtain

$$\begin{aligned}
g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) &= 1 + (2 + [1,0,0,0,1])t^2 + \Big(3 + \frac{1}{q^8}[0,0,0,1,0] \\
&+ \frac{q^4}{b^2}[0,0,0,1,0] + b^2 q^4 [0,0,0,1,0] + \frac{1}{b^2 q^4}[0,1,0,0,0] + \frac{b^2}{q^4}[0,1,0,0,0] \\
&+ q^8 [0,1,0,0,0] + [0,1,0,1,0] + 3[1,0,0,0,1] + [2,0,0,0,2] \Big) t^4 + \cdots ,
\end{aligned} \tag{5.37}$$

in accordance with (5.26), up to a rescaling of q (which means simply that we use different units in counting charges):

$$g_{6-\bullet-3-2-1}(t; q, b; y_1, \ldots, y_5) = g_{2-4-6}(t; q^2, b; y_1, \ldots, y_5) \ . \tag{5.38}$$

Unrefining $b = q = y_1 = \cdots = y_5 = 1$, we obtain the unrefined Hilbert series up to the order t^8 as

$$g_{6-\bullet-3-2-1}(t) = 1 + 37t^2 + 792t^4 + 12180t^6 + 145838t^8 + \cdots \ . \tag{5.39}$$

This is in agreement with (5.27).

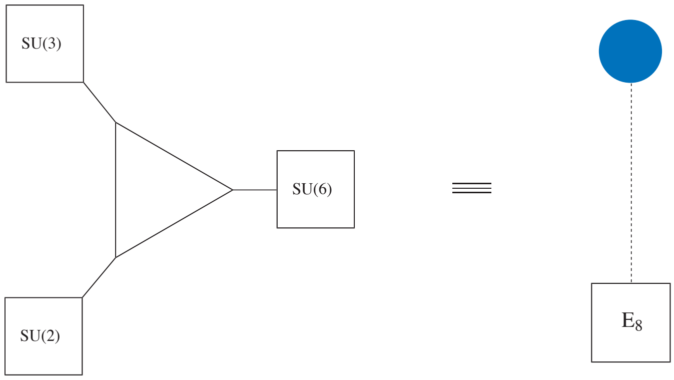

Figure 20. *Left:* The E_8 theory arises from 6 M5-branes wrapping a sphere with 3 punctures. The 3 punctures are of the type SU(6), SU(3), SU(2). *Right:* The quiver diagram representing the E_8 theory. The blue blob denotes a theory with an unknown Lagrangian description. The E_8 global symmetry is indicated in the square node.

5.3 E_8

The resummed Hilbert series for the coherent branch of one E_8 instanton is

$$g^{\mathrm{Irr}}_{E_8}(t;1,\ldots,1) = \frac{P_{E_8}(t)}{(1-t^2)^{58}}\,, \tag{5.40}$$

where the numerator is a palindromic polynomial of degree 58:

$$\begin{aligned} P_{E_8}(t) = {} & 1 + 190t^2 + 14269t^4 + 576213t^6 + 14284732t^8 + 234453749t^{10} + \\ & + 2675683550t^{12} + 21972715186t^{14} + 133126452657t^{16} + 606326972328t^{18} + \\ & + 2105555153625t^{20} + 5634990969615t^{22} + 11714759112330t^{24} + \\ & + 19025183027595t^{26} + 24223919026560t^{28} + \cdots \ (\text{palindrome}) \ \ldots + t^{58} \end{aligned} \tag{5.41}$$

This is consistent with the fact that the Higgs branch is $2h_{E_8} - 2 = 58$ complex dimensional, where $h_{E_8} = 30$ is the dual Coxeter number of E_8.

The E_8 theory arises from 6 M5-branes wrapping a sphere with 3 punctures. The 3 punctures are of the type SU(6), SU(3), SU(2). The quiver diagram is depicted in the left picture of figure 20. The Lagrangian description of this theory is unknown.

We denote the E_8 theory by a 'quiver diagram' analogue to those in previous sections. This is given in the right picture of figure 20. The blue blob denotes a theory with an unknown Lagrangian description. The E_8 global symmetry is indicated in the square node.

The E_8 theory can be used to construct a quiver gauge theory called the $5-\bullet-5-4-3-2-1$ theory, depicted in figure 21. The duality between this theory and the $3-6_{[5]}-4-2$ quiver theory (depicted in figure 22) is proposed by [23].

The $5-\bullet-5-4-3-2-1$ theory can be constructed as follows. The global symmetry E_8 can be decomposed into $\mathrm{SU}(5)\times\mathrm{SU}(5)$. One of the SU(5) is gauged and is coupled to the $5-4-3-2-1$ tail. The U(1) global symmetries are associated with the solid lines in the quiver diagram. Hence, the flavour symmetry is expected to be $\mathrm{SU}(5)\times\mathrm{U}(1)^4$.

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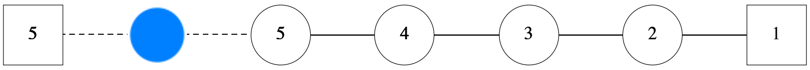

Figure 21. The $5-\bullet-5-4-3-2-1$ quiver theory. The U(1) global symmetries are associated with the solid lines in the quiver diagram. The flavour symmetry is expected to be $\mathrm{SU}(5)\times \mathrm{U}(1)^4$.

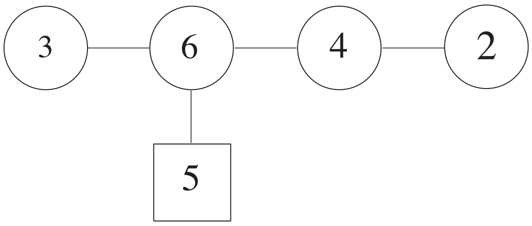

Figure 22. The $3-6_{[5]}-4-2$ quiver theory. This theory is dual to the $5-\bullet-5-4-3-2-1$ theory.

On the other side of the duality, we have the $3-6_{[5]}-4-2$ quiver theory depicted in figure 22. As in all previous quivers, the U(1) global symmetries are associated with the solid lines in the quiver diagram, and the flavour symmetry is expected to be $\mathrm{U}(5)\times\mathrm{U}(1)^3 \cong \mathrm{SU}(5)\times \mathrm{U}(1)^4$, in agreement with that of the $5-\bullet-5-4-3-2-1$ quiver theory.

The computations of Hilbert series of these theories are rather involved and technical. We leave such computations for future work.

5.4 One F_4 instanton on $\mathbb{C}^2$

There is no simple analog of the ADHM construction. Instead the conjecture of this paper is that the Hilbert series for the one instanton moduli space on $\mathbb{C}^2$ is a sum over symmetric adjoint representations. Explicitly, denote the adjoint representation of F_4 by $[1,0,0,0]$, and the symmetric adjoints by $[k,0,0,0]$, then the dimension of each representation is

$$\begin{aligned}&\dim\ [k,0,0,0] = \\ &= \frac{(k+1)(k+2)(k+3)^2(k+4)^3(k+5)^2(k+6)(k+7)(2k+5)(2k+7)(2k+9)(2k+11)}{4191264000},\end{aligned} \tag{5.42}$$

and the Hilbert series for the moduli space takes the form

$$g_{F_4}(t;x_1,x_2,x_3,x_4,x) = \frac{1}{(1-tx)(1-t/x)}\sum_{k=0}^{\infty}[k,0,0,0]t^{2k}\,, \tag{5.43}$$

Where as usual, the first term is the Hilbert series for $\mathbb{C}^2$, physically interpreted as the position of the instanton and the remaining function is the Hilbert series for the coherent component of the moduli space. By setting the F_4 fugacities to 1 one can get an explicit palindromic rational function for the coherent component of the moduli space,

$$g^{\mathrm{Irr}}_{F_4}(t) = \frac{1+36t^2+341t^4+1208t^6+1820t^8+1208t^{10}+341t^{12}+36t^{14}+t^{16}}{(1-t^2)^{16}} \tag{5.44}$$

giving a non-trivial check that the dimension of this moduli space is $2(h-1)=16$, where $h=9$ is the dual Coxeter number of F_4.

5.5 One G_2 instanton on $\mathbb{C}^2$

This case also has no known simple ADHM construction. Denote the character of the adjoint representation by $[0,1]$ and the character for the k-th symmetric adjoint by $[0,k]$, with dimension

$$\dim\ [0,k] = \frac{(k+1)(k+2)(2k+3)(3k+4)(3k+5)}{120}\ . \tag{5.45}$$

The Hilbert series takes the form

$$g_{G_2}(t;x_1,x_2,x) = \frac{1}{(1-tx)(1-t/x)}\sum_{k=0}^{\infty}[0,k]t^{2k}, \tag{5.46}$$

and setting the fugacities to 1 gives

$$g_{G_2}(t;1,1,1) = \frac{1}{(1-t)^2}\frac{1+8t^2+8t^4+t^6}{(1-t^2)^6}\ , \tag{5.47}$$

giving a non-trivial check that the dimension of this moduli space is $2(h_{G_2}-1)=6$, where $h_{G_2}=4$ is the dual Coxeter number of G_2. Since the rank of this gauge group is 2, it is possible to compute the sum explicitly and write the Hilbert series as a rational function with characters of G_2. Omitting the trivial $\mathbb{C}^2$ part we get

$$g^{\mathrm{Irr}}_{G_2}(t;x_1,x_2) = P_{G_2}(t;x_1,x_2)\mathrm{PE}\left[[0,1]t^2\right]\ , \tag{5.48}$$

where P_{G_2} is a palindromic polynomial of degree 11 in t^2 and has the form

$$\begin{aligned} P_{G_2}(t;x_1,x_2) \ =\ & 1-([2,0]+1)t^4+([1,1]+[2,0]+[0,1])t^6-([3,0]+[1,1]+[0,1]+[1,0])t^8 \\ & +([3,0]+[1,0])t^{10}+([3,0]+[1,0])t^{12}-([3,0]+[1,1]+[0,1]+[1,0])t^{14} \\ & +([1,1]+[2,0]+[0,1])t^8-([2,0]+1)t^{18}+t^{22}\ . \end{aligned} \tag{5.49}$$

Acknowledgments

We are indebted to Francesco Benini and Alberto Zaffaroni for useful discussions and to Sam Kitchen for his generosity in helping us in programming. A. H. would like to thank Ecole Polytechnique for their kind hospitality during the completion of this paper, and to Michael Douglas and Nikita Nekrasov for useful discussions. N. M. acknowledges Giuseppe Torri for a close collaboration and thanks John Davey, Ben Hoare and David Weir of their kind help. He is grateful to the following institutes and collaborators for their kind hospitality during the completion of this work: Max-Planck-Institut für Physik (Werner-Heisenberg-Institut); Rudolf Peierls Centre for Theoretical Physics, University of Oxford; DAMTP, University of Cambridge; Universiteit van Amsterdam; Frederik Beaujean, Francis Dolan, Yang-Hui He, Sven Krippendorf and Alexander Shannon. He also thanks his family for the warm encouragement and support. This research is supported by the DPST project, the Royal Thai Government. This work is partially supported under ERC Advanced Grant 226371.

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