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Taming open/closed string duality with a Losev trick

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Abstract: A target space string field theory formulation for open and closed B-model is provided by giving a Batalin-Vilkovisky quantization of the holomorphic Chern-Simons theory with off-shell gravity background. The target space expression for the coefficients of the holomorphic anomaly equation for open strings are obtained. Furthermore, open/closed string duality is proved from a judicious integration over the open string fields. In particular, by restriction to the case of independence on continuous open moduli, the shift formulas of [8] are reproduced and shown therefore to encode the data of a closed string dual.

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Contents

1 Introduction

A proper target space formulation of open plus closed topological strings is important for several reasons, the most compelling in our opinion being a better understanding of open/closed string duality which, once an off shell formulation of the theory is given, should become manifest. Actually, this is the main subject of this paper. Open/closed duality is commonly believed [27] to be the effect of integrating out open strings in the complete string field theory, leaving then a purely closed string theory on a suitably modified background. This program is very hard to be realized in the full string theory, but it becomes tractable in its truncation to its BPS protected sectors, namely in topological string theories [4, 36, 37]. This issue has been investigated by several authors in a first quantized or *on shell* framework. Actually, the first examples were discussed in terms of geometric transitions [11] which have been extended to the brane sector in [27]. Then, this picture has been refined in terms of a proper world-sheet analysis in [28, 29]. More advanced on-shell computations have been prompted by [7] and then further by [19–21] and [16, 17]. A distinctive feature of topological strings is that the non-holomorphic dependence of its amplitudes can be recursively computed by means of the holomorphic anomaly equations (HAE) [4]. It turns out that the target space formulation of the closed string in terms of the Kodaira-Spencer gravity is very effective in reproducing these recurrence relations from a Feynman diagram's expansion. This also provides a target space interpretation of the various coefficients appearing in the HAE. These latter have been more recently extended to open strings in [33] and [6]. These were further studied in [1, 2]. The topological open string target space formulation has been actually obtained long ago in [39] where it was shown to be given by the Chern-Simons theory for the A-model and its holomorphic version for the B-model. These are formulated for a fixed on shell background geometry, in particular for the B-model the holomorphic Chern-Simons is formulated with respect to an integrable complex structure on the Calabi-Yau target. Since the aim of this paper is to study a string field theory formulation of topological open plus closed strings on equal footing, we will extend this framework to non-integrable structures. The formulation of holomorphic anomaly equations and the target space interpretation of its structure functions are very important tools to obtain a well defined computational framework for open topological strings. D-branes sources for closed strings are actually represented in the HAE by the Walcher's term [33] whose target space interpretation has been given in terms of the Griffith's normal function (see also [25]). For the B-model this boils down to the on shell holomorphic Chern-Simons action. A remarkable observation [8] consists in the proof that the Walcher's term can be reabsorbed by a shift in the string coupling constant and the closed moduli. This indeed realizes an on shell proof of the open/closed duality, although at frozen open moduli.

In the following we will study this problem from a second quantized point of view, which turns out to be the most appropriate to study open/closed duality in particular for the B-model. We will work out the BV formulation of the holomorphic Chern-Simons theory by leaving the gravitational background (Kodaira-Spencer gravity field) off shell. This allows us to reformulate open-closed duality as a process of partial functional integration over the open string fields. From the BV viewpoint this procedure follows by partial integration of a proper subset of fields and anti-fields of a solution of the BV master equation by which one gets another solution depending on a reduced set of fields. This is known as Losev trick [22, 23]. In particular, at frozen open string moduli, we will show that this partial integration exactly reproduces the shift formulas proposed in [8, 26]. More in general, our BV formulation proves the existence of definite shift formulas also in presence of open moduli providing a computational set-up to determine them. Moreover, it yields a target space interpretation of the coefficients of the extended HAE for open string moduli as in [6, 34].

The paper is organized as follows. In section 2 we discuss the classical complete string field theory action for open plus closed B-model. In section 3 we proceed to its quantization using the BV formalism. In section 4 we discuss the target space interpretation of the coefficients in the open HAE from the string field theory. In section 5 we formulate and prove in general the open/closed duality or the B-model and apply it to the setting of [8, 33]. In section 6 we collect few concluding comments.

2 Open-closed effective field theory

It is well known from [4] that the effective space-time theory corresponding to the B-model for closed strings is given by the Kodaira-Spencer theory of gravity:

$$\lambda^2 S_{KS} = \int_X \frac{1}{2}A'\frac{1}{\partial}\overline{\partial}A' - \frac{1}{3}[(A+x)(A+x)]'(A+x)' \tag{2.1}$$

where λ is the string coupling, A and x are $(0,1)$ forms with values in the $(1,0)$ vector field that is, in coordinates, $A = A^j_{\bar{i}} dz^{\bar{i}} \frac{\partial}{\partial z^j}$ and similarly for x. In (2.1) $A' = i_A\Omega_0 =$

$3(\Omega_0)_{ijk}A^i_{\bar{i}}dz^j dz^k dz^{\bar{i}}$ and similarly for x' where Ω_0 is the holomorphic three form on the Calabi-Yau target space X.[1] $A+x$ is defined to be a deformation of the complex structure of X split into an infinitesimal part, x, and a finite one, A. The full deformation, $A+x$, is parametrized by the shift $\overline{\partial}_{\bar{i}} \to \overline{\partial}_{\bar{i}} - (x^j_{\bar{i}} + A^j_{\bar{i}})\partial_j$. By definition the coefficients of forms with barred indices transform in the same way: $w_{\bar{i}} \to w_{\bar{i}} - (x^j_{\bar{i}} + A^j_{\bar{i}})w_j$. In addition $dz^j \to dz^j + (x^j_{\bar{i}} + A^j_{\bar{i}})dz^{\bar{i}}$, while ∂ and $d\overline{z}$ are fixed (their shift refers to the antitopological theory). In this way real objects as the de Rham differential d or a real form $w_i dz^i + w_{\bar{i}} dz^{\bar{i}}$ remains unchanged. The condition of integrability of the modified complex structure is

$$0 = (\overline{\partial} - x - A)(\overline{\partial} - x - A) = -\overline{\partial}(A+x) + \frac{1}{2}[A+x, A+x] = 0$$

which can be rewritten, due to the fact that $\overline{\partial}x = 0$, x being the background parameter valued in $H^{0,1}_{\overline{\partial}}(TM)$, as

$$\overline{\partial}A' = \partial((A+x)\wedge(A+x))'. \tag{2.2}$$

(2.2) is the equation of motion of (2.1). Let us stress that it is *crucial* the fact that x does not appear in the kinetic term of (2.1). In addition A is required to satisfy the so called Tian's gauge, $\partial A' = 0$, in order to have a well defined kinetic term.

The symmetries of (2.1) are the Ω_0 preserving reparameterizations of the complex coordinates $z^i \to z^i + \chi^i(z, \overline{z})$ and $z^{\bar{i}} \to z^{\bar{i}}$ while the condition of being Ω_0 preserving reads $\partial\chi' = 0$. According to $\overline{\partial} \to \overline{\partial} - (A+x)$ and owing to the fact that x is a background, A transforms as

$$\delta A = -\overline{\partial}\chi - \mathcal{L}_\chi(A+x) = -\overline{\partial}\chi - [\chi, (A+x)] \tag{2.3}$$

Reinterpreting χ as a ghost field, this transformation can be promoted to a nilpotent BRST if

$$\delta\chi = -\frac{1}{2}\mathcal{L}_\chi \chi = -\chi^i \partial_i \chi. \tag{2.4}$$

The open effective theory has been analysed by Witten in [39] and for the B-model it is given by the holomorphic Chern-Simons action

$$\lambda S_{HCS} = \int_X \Omega_0 Tr\left(\frac{1}{2}B^{0,1}\overline{\partial}B^{0,1} + \frac{1}{3}B^{0,1}B^{0,1}B^{0,1}\right) \tag{2.5}$$

with $B^{0,1}$ a Lie algebra valued $(0,1)$-form.

The precise definition of the model has been presented in [32]. Indeed (2.5) is globally ill defined. From the Chern-Weil theorem we know that only the difference of two invariant polynomials with respect to two different connections $\hat{B}$ and B_0 (dropping for the moment the label $(0,1)$) is an exact form. So using the reference connection B_0 we can write

$$\begin{aligned} -\int_{K_4}\frac{\Theta}{2}Tr(\hat{F}^2 - F_0^2) &= -\int_{K_4}\Theta Tr\,\overline{\partial}\left(\frac{1}{2}\hat{B}\overline{\partial}\hat{B} + \frac{1}{3}\hat{B}^3 - \frac{1}{2}B_0\overline{\partial}B_0 - \frac{1}{3}B_0^3\right) = \\ &= \int_X \Omega_0 Tr\left(\frac{1}{2}\hat{B}\overline{\partial}\hat{B} + \frac{1}{3}\hat{B}^3 - \frac{1}{2}B_0\overline{\partial}B_0 - \frac{1}{3}B_0^3\right) \end{aligned} \tag{2.6}$$

[1] Factors may change depending on the conventions; we will use the ones of [31] and [18].

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where K_4 is a fourfold containing X as a divisor while Θ is a connection of the associated line bundle $\mathcal{L}_X$ so that $\overline{\partial}\Theta = \Omega_0\delta(X)$. We expand $\hat{B}$ with respect to the reference connection as

$$\hat{B} = B + B_0$$

so that (2.6) provides the globally well defined action

$$\lambda S_{HCS} = \int_X \Omega_0 Tr\left(\frac{1}{2}B^{0,1}\overline{\partial}_{B_0^{0,1}}B^{0,1} + \frac{1}{3}(B^{0,1})^3 + F_0^{0,2}B^{0,1}\right) \tag{2.7}$$

with $\overline{\partial}_{B_0^{0,1}}\varphi \equiv \overline{\partial}\varphi + [B_0^{0,1},\varphi]_\pm$ with $\pm$ depending on the grade of the form φ. B_0 is the open string background for the theory and as such it obeys the holomorphicity condition $F_0^{0,2} = 0$. The symmetries of (2.7) — at fixed background B_0 — are given by

$$\delta B^{0,1} = \overline{\partial}_{B_0^{0,1}}\epsilon + [B^{0,1},\epsilon]. \tag{2.8}$$

Now we want to explicitly couple the open theory to the closed field that is we want to deform the complex structure of X, over which the theory is defined, using the fields A and x. Of course the closed fields are in general not on shell so the new complex structure (better call it almost complex structure) is generically not integrable. In addition we want to write the new action with respect to the undeformed complex structure in order to keep the closed field explicit. Actually, under the deformation Ω_0 is mapped to [31]

$$\Omega = \Omega_0 + (A+x)' - [(A+x)(A+x)]' - [(A+x)(A+x)(A+x)]' \tag{2.9}$$

which is a $(\tilde{3},\tilde{0})$ form with respect to the new complex structure (from now on always indicated with a tilde) while with respect to the old one it decomposes in forms of total degree 3, namely (p,q) forms with $p+q=3$. We can now deform also the remaining $(0,3)$ part of the action, $L_{CS}^{0,3}$, with $L_{CS}^{0,3} \equiv Tr(\frac{1}{2}B^{0,1}\overline{\partial}_{B_0^{0,1}}B^{0,1} + \frac{1}{3}(B^{0,1})^3 + F_0^{0,2}B^{0,1})$, into a $(\tilde{0},\tilde{3})$ form. In order to keep into account the deformation of the complex structure of the full action the simplest way is to use a real form for the Chern-Simons term, rewriting

$$\int_X \Omega^{\tilde{3},\tilde{0}} L_{HCS}^{\tilde{0},\tilde{3}} = \int_X \Omega^{\tilde{3},\tilde{0}} L_{CS} = \int_X \Omega^{\tilde{3},\tilde{0}} Tr\left(\frac{1}{2}Bd_{B_0}B + \frac{1}{3}B^3 + F_0B\right) \tag{2.10}$$

where B is a real Lie algebra valued 1-form on X. Indeed, Ω being a $(\tilde{3},\tilde{0})$ form, the added piece is zero. However, from the path integral quantization viewpoint, we have to define a suitable measure for the new field component $B^{\tilde{1},\tilde{0}}$. We will discuss this issue in the next section by using the Batalin-Vilkovisky formalism. For the Kodaira-Spencer gravity in antifield formalism see [4]. Let us notice that the real form L_{CS} is completely independent from the closed field, while it is Ω which really takes care to project the action onto the new complex structure selecting the complementary form degree from L_{CS}.

Let us consider the symmetries of (2.10). As far as diffeomorphisms (2.3) are concerned, Ω in (2.9) transforms as $\mathcal{L}_\chi\Omega$ so that the whole action is invariant under the standard action on B, namely $\delta B = -\mathcal{L}_\chi B$.

The situation for the Chan-Paton gauge symmetry is more subtle. Indeed, the field A being off-shell, we do not have $d\Omega = 0$. In fact it can be shown [31] that $d\Omega = 0$ is equivalent to the equations of motion for the Kodaira-Spencer action, $\overline{\partial}A' = \partial((A+x)\wedge(A+x))'$. So we expect a variation of the action under the gauge transformations (2.8) proportional to it. We find

$$\delta S_{HCS} = \frac{1}{\lambda}\int_X \Omega Tr\, d\left(\frac{1}{2}\epsilon d_{B_0}B + F_0\epsilon\right) \tag{2.11}$$

We can save the day by adding to the action the term $-\frac{1}{2}\Omega db$, where b is a real 2-form field transforming as [3]:

$$\begin{aligned} \delta B &= d_{B_0}\epsilon + [B,\epsilon] \\ \delta b &= Tr(\epsilon d_{B_0}B + 2F_0\epsilon) \end{aligned} \tag{2.12}$$

The field b acts as a Lagrange multiplier enforcing the Kodaira-Spencer equations for the closed field A. However the role of implementation of the associated delta function requires also a determinant factor such that

$$\int \mathcal{D}A\mathcal{D}b e^{-\frac{1}{2}\int_X \Omega db} det_{FP} = 1 \tag{2.13}$$

This determinant measure has to be included in the very definition of the theory and will be explicitly derived in the next section.

This isn't really the end of the story as b has shift symmetries along its $(\tilde{2},\tilde{0})$ and $(\tilde{1},\tilde{1})$ components. In addition we should specify the full nilpotent symmetries and the gauge fixing. This will be the subject of the next section.

Summarizing, the classical action for open and closed B-model is

$$\begin{aligned} S_{\text{tot}} =& \frac{1}{\lambda^2}\int_X \left(\frac{1}{2}A'\frac{1}{\partial}\overline{\partial}A' - \frac{1}{3}[(A+x)(A+x)]'(A+x)'\right) + \\ &+ \frac{1}{\lambda}\int_X \Omega Tr\left(\frac{1}{2}Bd_{B_0}B + \frac{1}{3}B^3 + F_0B\right) - \frac{1}{2}\Omega db \end{aligned} \tag{2.14}$$

3 On the BV quantization of holomorphic Chern-Simons

In this section we provide the BV action for the holomorphic Chern-Simons theory and a non singular gauge fixing fermion. For simplicity in this section we will drop the tilde in the notation for forms in the new complex structure. Still the coupling with the closed field is always present.

The classical action is

$$\lambda S_o = \int_X \Omega^{(3,0)}\left[Tr\left(\frac{1}{2}Bd_{B_0}B + \frac{1}{3}B^3 + BF_0\right) - \frac{1}{2}db\right] \tag{3.1}$$

This is invariant under the infinitesimal gauge transformations

$$\begin{aligned} sB &= d_{B_0}\epsilon + [B,\epsilon] + \psi^{(1,0)} \\ sb &= Tr\left(Bd_{B_0}\epsilon + 2F_0\epsilon\right) + d\gamma + \eta^{(1,1)} + \eta^{(2,0)} \end{aligned} \tag{3.2}$$

where ϵ is the usual gauge symmetry ghost while $\psi^{(1,0)}$, $\eta^{(2,0)}$ and $\eta^{(1,1)}$ are the ghosts for the shift symmetries.

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By further defining

$$
\begin{aligned}
s\epsilon &= -\epsilon^2 \\
s\psi^{(1,0)} &= \left[\epsilon, \psi^{(1,0)}\right] \\
s\gamma^{(1,0)} &= n^{(1,0)} - Tr\left(\epsilon\partial^{(1,0)}_{B_0}\epsilon\right) \\
s\gamma^{(0,1)} &= \partial^{(0,1)}m - Tr\left(\epsilon\partial^{(0,1)}_{B_0}\epsilon\right) \\
s\eta^{(1,1)} &= -Tr\left(\psi^{(1,0)}\partial^{(0,1)}_{B_0}\epsilon\right) - \partial^{(1,0)}\partial^{(0,1)}m - \partial^{(0,1)}n^{(1,0)} \\
s\eta^{(2,0)} &= -Tr\left(\psi^{(1,0)}\partial^{(1,0)}_{B_0}\epsilon\right) - \partial^{(1,0)}n^{(1,0)} \\
sn^{(1,0)} &= 0 \\
sm &= 0
\end{aligned}
\tag{3.3}
$$

we get a *pseudo*-BRST operator. Actually the operator s defined by (3.2) and (3.3) is nilpotent only on shell. Explicitly, one gets

$$s^2 b^{(0,2)} = \left(\partial^{(0,1)}\right)^2 m \tag{3.4}$$

which is vanishing only on shell w.r.t. b. Actually, as discussed in [31], the differential of the shifted 3-form (2.9) is proportional to the Nijenhuis tensor. Thus (3.4) is proportional to the equation of motion of b. On all other fields one gets $s^2 = 0$.

The BV recipe is in this case still simple, since one can check that second order in the antifields already closes in this case. By labeling all the fields entering (3.2) and (3.3) as ϕ^i, we have therefore[2]

$$S_{BV} = S_o + \int_X \sum_i \phi^*_i s\phi^i + c\int_X \left((b^*)^{(2,2)}\partial^{(1,0)}m\right)^\vee (b^*)^{(3,1)} \tag{3.5}$$

where c is a non zero numerical constant which will not be relevant for our calculations (see later). One can explicitly show that S_{BV} satisfies $\Delta S_{BV} = 0$, where Δ is the BV-laplacian and $(S_{BV}, S_{BV}) = 0$ the corresponding bracket. In our conventions, all antifields have complementary form degree with respect to fields.

Let us notice that a parallel result has been obtained in [14] by C. Imbimbo for the A-model. Indeed, also in the case of the real Chern-Simons theory, the coupling with the gravitational background requires the use of the full BV formalism giving rise to quadratic terms in the anti-fields.

While gauge fixing, we need to add the anti-ghost multiplets for all gauge fixed parameters. Actually we are going to gauge fix our theory only partially, that is we will keep the (ϵ-)gauge freedom relative to the Chan-Paton bundle. By introducing the relevant anti-ghost multiplets, we define the gauge fixing fermion

$$\Psi = \int_X \left\{\bar\psi^{(1,3)}\left(d_{B_0}B + B^2 + F_0\right)^{(2,0)} + \bar\eta^{(2,2)}b^{(1,1)} + \bar\eta^{(1,3)}b^{(2,0)}\right. \tag{3.6}$$

$$\left. + \bar n^{(2,3)}\gamma^{(1,0)} + \bar m^{(3,3)}\left(\partial^{(0,1)}\right)^\dagger \gamma^{(0,1)} + \bar\gamma^{(3,2)}\left[\left(\partial^{(0,1)}\right)^\dagger b^{(0,2)} + \partial^{(0,1)}p\right]\right\} \tag{3.7}$$

[2]Here we use the $\vee$-operator as in [4] so that the $\vee$ of a $(3,p)$-form is a $(0,p)$-form.

by adding the anti-ghost (trivial) part of the BV action in the usual form. We extend therefore the s-operator action, that is the BV-bracket with the part of the BV action linear in the anti-fields, to the anti-ghosts in the trivial way, namely for any anti-ghost $\bar{\psi}$ we have $s\bar{\psi} = \Lambda_{\bar{\psi}}$ and $s\Lambda_{\bar{\psi}} = 0$. The anti-ghost gauge freedom is fixed by the addition of the relevant further sector.

Finally we can compute the (partially) gauge fixed action by specifying all anti-fields as derivatives with respect to their relative fields of gauge fermion Ψ. All in all, the (partially) gauge fixed action reads

$$S_{g.f.} = S_o + s\Psi + c \int_X \left(\bar{\eta}^{(2,2)} \partial^{(1,0)} m\right)^\vee \left(\partial^{(0,1)}\right)^\dagger \bar{\gamma}^{(3,2)} \tag{3.8}$$

Let us now perform the path-integral in the different sectors (by naming them by the relative anti-ghost as appearing in the gauge fermion).

- The $\bar{\psi}^{(1,3)}$ is seen to decouple since

  $$s\left\{d_{B_0} B + B^2 + F_0\right\}^{(2,0)} = \partial^{(1,0)}_{B_0} \psi^{(1,0)} + \left[B^{(1,0)}, \psi^{(1,0)}\right]_+$$

  Therefore we get the contribution

  $$\int \mathcal{D}[B^{(1,0)}] \delta\left(\partial^{(1,0)}_{B_0} B^{(1,0)} + B^{(1,0)} B^{(1,0)} + F_0^{(2,0)}\right) \det{}' \left\{\partial^{(1,0)}_{B_0} + \left[B^{(1,0)}, \cdot\right]_+\right\}$$

  which counts the volume of the space of holomorphic connections.

- The two $\bar{\eta}$-sectors are just algebraic and give a constant contribution to the path-integral. Notice that while integrating over $\bar{\eta}^{(2,2)}$ also the last term in (3.8) gets involved being reabsorbed in a shift of $\eta^{(1,1)}$. This gauge fixing of course restricts the field b to be a $(0,2)$-form only and set to zero $\eta^{(1,1)}$ and $\eta^{(2,0)}$.

- The $\bar{n}^{(2,3)}$ sector is algebraic too and simply sets to zero $\gamma^{(1,0)}$ and its partner.

- The last part is the standard term for higher form BV quantization (see for example [12]). The fermionic bilinear operator reduces to

  $$\mathcal{B} = \begin{pmatrix} -\partial^{(0,1)\dagger} \partial^{(0,1)} & -\partial^{(0,1)} \\ \partial^{(0,1)\dagger} & 0 \end{pmatrix}$$

  mapping $\Omega^{(0,1)}(X) \oplus \Omega^{(0,0)}(X)$ to itself. The bosonic bilinear operator is instead the anti-holomorphic laplacian $\Delta^{(0,0)} = \partial^{(0,1)\dagger} \partial^{(0,1)}$ on the scalars $\Omega^{(0,0)}(X)$. One therefore stays with the gauge fixed measure

  $$\int \mathcal{D}[Y] e^{-\frac{1}{2}\int_X Y\mathcal{C}Y + \int_X J^t Y} \tag{3.9}$$

  where $Y = \left(p, \Lambda_{\bar{\gamma}}, b^{(0,2)}\right)$,

  $$\mathcal{C} = \begin{pmatrix} 0 & -\partial^{(0,1)} & 0 \\ \partial^{(0,1)} & 0 & \partial^{(0,1)\dagger} \\ 0 & -\partial^{(0,1)\dagger} & 0 \end{pmatrix}$$

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and the source $J = (0, 0, d\Omega)$ takes into account the classical action. Eq. (3.9) can be integrated being a Gaussian.

Therefore, all in all, we find that the quantum measure for the holomorphic Chern-Simons theory is

$$\frac{det'[\mathcal{B}]}{det'[\Delta^{(0,0)}]\,(det'[\mathcal{C}])^{1/2}} e^{J^t(\mathcal{C})^{-1}J} \tag{3.10}$$

for a (generically non integrable) almost complex structure. The determinant of the operator $\mathcal{C}$ is easily obtained by noticing that

$$\{\mathcal{C}, \mathcal{C}^\dagger\} = \begin{pmatrix} \Delta^{(0,0)} & 0 & \left(\partial^{(0,1)}\right)^2 + \left(\partial^{(0,1)\dagger}\right)^2 \\ 0 & 2\Delta^{(3,2)} & 0 \\ \left(\partial^{(0,1)}\right)^2 + \left(\partial^{(0,1)\dagger}\right)^2 & 0 & \Delta^{(2,0)} \end{pmatrix}$$

We want to compare our open theory, defined as coupled to the closed field A, with the standard holomorphic Chern-Simons, defined for an integrable complex structure. In particular the two theories should match once we put on shell the closed field. So the integral of all the additional fields should contribute as one. Notice that, if the complex structure is integrable, then $d\Omega = 0$ and the source term is not contributing. On top of it, since $\left(\partial^{(0,1)}\right)^2 = 0$, the bosonic operator block-diagonalizes. Moreover, in this case, the determinant of the fermionic operator $\mathcal{B}$ can be easily computed[3] to be equal to $det'\Delta^{(0,2)} \left(det'\Delta^{(0,0)}\right)^{1/2}$.

All in all, we find an overall

$$\frac{det'[\Delta^{(0,2)}]\left(det'[\Delta^{(0,0)}]\right)^{1/2}}{det'[\Delta^{(0,0)}]\left\{\left(det'[\Delta^{(0,2)}]\right)^2 det'[\Delta^{(0,0)}]\right\}^{1/2}} = \frac{1}{det'[\Delta^{(0,0)}]} \tag{3.11}$$

This determines the value of the quantum measure introduced in (2.13). The factor (3.11) counts the extra degree of freedom introduced by the b field in the theory. Indeed the three components of $b^{(0,2)}$ are subject to the gauge freedom by the shift of an exact $\partial^{(0,1)}\gamma^{(0,1)}$ term up to the ghost-for-ghost shifting $\gamma^{(0,1)}$ by $\partial^{(0,1)}m$. Therefore the overall counting is $3 - 3 + 1 = 1$ complex modes.

4 String field theory as generating function of open and closed HAEs.

Our claim of having found the effective space-time theory for the open B-model should be checked explicitly. Because of tadpole cancellation, see [5] and [34], we know that the open theory is completely well defined only in its unoriented version (as in the case of usual string theories), so the most general case to consider is for open (and closed) unoriented strings. Closed moduli are known to be unobstructed and so expansions of the amplitudes

[3] This can be done by writing the eigenvector equation for $\mathcal{B}$ as $\mathcal{B}\binom{a}{b} = \lambda\binom{a}{b}$ and then expanding the 1-form $a = \partial^{(0,1)}x + \partial^{(0,1)\dagger}y$ in exact and co-exact parts. Then one finds that $b = \lambda x$ and that the eigenvalues of $\mathcal{B}$ coincide with those of $\Delta^{(0,2)}$ for $x = 0$ or with the square roots of those of $\Delta^{(0,0)}$ for $y = 0$.

in their value is always possible. For open moduli it is known that this is not always the case, and it is claimed that they are generically obstructed [33]. Actually this is expected only for Calabi-Yaus with holonomy strictly equal to SU(3) but not for the most general cases. Moreover, also when the open moduli are obstructed, the expansion in continuous parameters can be traded for a sum over discrete variables, which in principle could be recovered from a finite shift over closed moduli. An important result of [4] is that the partition function of Kodaira-Spencer theory encodes the recurrence relations of HAE via its Feynman diagram expansion. The generating function of the full HAE of [6] generalized to the unoriented case should be:

$$e^{W(x,u;t,\bar{t})} \sim \exp\left(\sum_{g,h,c,n,m} \frac{\lambda^{2g-2+h+c}}{2^{\frac{\chi}{2}+1}\, n!m!} \mathcal{F}^{(g,h,c)}_{i_1\dots i_n\alpha_1\dots\alpha_m} x^{i_1}\dots x^{i_n} u^{\alpha_1}\dots u^{\alpha_m}\right) \tag{4.1}$$

up to an overall λ dependent prefactor which encodes the contact terms in one loop calculations and will be discussed later. This prefactor $\lambda^{\cdots}$ is encoded, in the field theory side, in the measure of the path integral, namely as the multiplicative term weighting the regularized determinants with omitted zero modes. From now on, in any case, we will focus on the perturbative expansion in λ.

The notation is as follows: $\mathcal{F}^{(g,h,c)}_{i_1\dots i_n\alpha_1\dots\alpha_m}$ is the string amplitude with genus g, h boundaries, c crosscaps, n marginal operator insertions in the bulk and m on the boundary. The x^i's are the expansion coefficients of x in a base of Beltrami differentials, $x = x^i\mu_i$ and the u^α's are the expansion coefficients for B_0 in a basis $T_\alpha(x)$ of the open moduli $H^{(0,1)}(X, Ad_E)$, namely $B_0 = u^\alpha T_\alpha$. Thus the fields appearing as backgrounds in the field theory are the open and closed moduli themselves. The factor $\frac{1}{2^{\frac{\chi}{2}+1}}$ is explained in [34] and obviously $\chi = 2g-2+h+c$. If what we are doing is consistent it should be true that

$$\int \mathcal{D}A\mathcal{D}B\mathcal{D}b\dots e^{-S_{\rm tot}(x,B_0(x);t,\bar{t};A,B,b,\dots)} = e^{W(x,u(x);t,\bar{t})}. \tag{4.2}$$

We want to compare this at tree level, that is at $g=0, h=0,1, c=0$ and $g=0, h=0, c=1$, and obtain in this way some explicit expressions for all the basic objects entering the extended HAE of [6] computed at a generic background point. These amplitudes are already known and computed by worldsheet methods and the two results should of course match. To this end we will differentiate, at each order in λ, both members with respect to the moduli parameters x^i and u^α and identify the corresponding coefficients.

A comment is in order. We should remember that the expression (4.1) is the partition function for the unoriented theory. As explained in [34] this differs from the oriented one simply projecting the space of operators in the theory to the unoriented sector that is the ones with eigenvalue $+1$ under the parity operator $\mathcal{P}$. Being these operators nothing else than deformations of the moduli space of the theory, we have to consider only its invariant part under $\mathcal{P}$ and then parametrise with x^i and u^α its tangent space. This means that the x^i's and the u^α's appearing in (4.1) are really a subset of the ones in the oriented case. Specifically it implies a restriction on the space of complex structures for what matters x and a reduction to $\mathrm{Sp}(N)/\mathrm{SO}(N)$ groups for u. Still some amplitudes, as the sphere with three insertions, are perfectly meaningful also in the oriented case. This is why we will generically not specify to which space the x^i's and the u^α's belongs: it is possible to restrict their value depending on the case.

4.1 $g = 0$, $h = c = 0$

Here we start the comparison between the string theory partition function and the space-time path integral (4.2). We begin from the coefficients at lowest order in λ. From the point of view of (4.1) this is the amplitude at $g = h = c = 0$ with weight $\frac{1}{\lambda^2}$; on the field-theory side the contribution should come only from the Kodaira-Spencer action, also at weight $\frac{1}{\lambda^2}$. We know that the right-hand side of equation (4.2) at this order in λ has no dependence on open moduli (because without boundaries, $h = 0$, there is no space for open operator insertions) and the building block amplitude being $C_{ijk}(x)$:

$$C_{ijk}(x) = \mathcal{F}^{(0,0,0)}_{ijk}(x) = \sum_n \frac{1}{n!}\mathcal{F}^{(0,0,0)}_{ijki_1\ldots i_n} x^{i_1}\ldots x^{i_n} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^k} W \mid_{(order \lambda^{-2})}$$

Being at tree level and given (4.2), the same result can be obtained (see [4]) deriving the Kodaira-Spencer action on shell ($A = A(x)$) with respect to three x^i. The three derivative term gives[4]

$$-2\int_M \left[\left(\mu_i + \frac{\partial A(x)}{\partial x^i}\right)\wedge\left(\mu_j + \frac{\partial A(x)}{\partial x^j}\right)\right]'\left(\mu_k + \frac{\partial A(x)}{\partial x^k}\right)' = C_{ijk}(x)$$

The only point of possible confusion for the BCOV educated reader both here and in the subsequent computations, comes from the novel cross dependence of open and closed field on shell by each other by means of the field equations which are now modified with respect to the ones obtained with the open and closed actions separated. This might seem to carry on additional induced derivatives and contributions as, in this case, an induced open moduli dependence carried by the on shell closed field which would lead to the paradox of a non vanishing amplitude corresponding to a sphere with boundary insertions! Fortunately, integrating out the field b does the job of enforcing the closed field solutions that would be obtained from the Kodaira-Spencer action alone! It will be true instead that the on shell open fields will carry some closed field dependence as the B-field equation is: $F^{\tilde{0},\tilde{2}}_{B_0} \equiv (d_{B_0}B + B^2 + F_0) \mid^{\tilde{0},\tilde{2}} = 0$ which is both $B_0(u)$ and x dependent.

This is a good place to stop and discuss the connections between our result for the coupling between the open theory and the closed one, and the comments made by Witten in [39] about this point. In his paper Witten uses an argument from the fatgraph description of a string tree level amplitude to infer that, if one considers a diagram with n bulk and m boundary insertions, in general the bulk operators will reduce to exact (with respect to the topological charge) objects and so will decouple. This goes through even in the case $m = 0$ as long as some boundaries are present. The direct consequence is that the on-shell couplings between closed and open strings are zero. How can then one justify the non vanishing of the Δ_{ij} amplitudes of [26, 33] and [34]? Our answer is in a sense a weakened realization of Witten's idea, still allowing non zero amplitudes with bulk operators and

[4]The factor -2 depends on our conventions which are slightly different from [4].

boundaries. The key role is played by the field b, generated in the action to maintain the gauge symmetries in the Chern-Simons term. This field, once it is integrated over, fixes the closed field A to be on shell with respect to the original Kodaira-Spencer equations and so defining a shift of an integrable complex structure. This translates to the fact that the original genuine coupling between open and closed fields in the action reduces to a coupling between an open, integrated field and an on-shell closed field. That is it represent a new Chern-Simons expansion around a new shifted and fixed complex structure. So the path integration of the closed field A reduces to a single contribution coming from the unique deformation of the original complex structure with respect to which the Kodaira-Spencer action is written, this contribution being weighted by the corresponding Kodaira-Spencer on shell action. If closed strings are substantially decoupled by the open theory, what is then their role? This is the next point discussed by Witten in [39] where their crucial role in anomaly cancellation is pointed out. For example, in the A-model, whose effective theory is the real Chern-Simons, a well known topological anomaly is present. It comes from the η-invariant of [40], whose dependence by the metric is compensated by the addition of a gravitational Chern-Simons. Then an additional anomaly connected to the framing of the target space is well known. In the case of the B-model however, the η-invariant is simply zero because the spectrum of eigenvalues of the determinant whose phase is η, is symmetric around zero [32]. Instead we have one loop anomalies corresponding to a dependence by the wrong moduli [9] (Kähler moduli in this case) which is cured by tadpole cancellation, [34] and [5], involving unoriented contributions in the closed strings sector (Klein bottle).

4.2 $g = 0,\ h + c = 1$

In this subsection we want to compare the world-sheet and the target space perspective at order $1/\lambda$. From the string theory side the relevant amplitudes of weight $\frac{1}{\lambda}$ ($g = 0$, $h + c = 1$) entering the HAE were discussed in [6]. From the field theory perspective all of them should be reproduced by the holomorphic Chern-Simons action.

Let us start with purely closed moduli dependence. This can come either from both the explicit dependence by x in Ω and by the induced dependence in the $A(x)$ and $B(x, u)$ fields on shell, or implicitly through the background $B_0(x)$. We will find that the dependence w.r.t. closed moduli explicit and in the on shell fields, both closed and open, correspond to bulk insertion in the string amplitude, while the dependence w.r.t. closed moduli in the background open field corresponds to induced boundary insertions.[5]

The two operators will be indicated as ϕ_i and ψ_i (so for example $C_{ijk} = \langle\phi_i\phi_j\phi_k\rangle_{0,0,0}$ where the subscript denotes the triple g, h, c).

The first amplitude we want to derive is $\Delta_{ij} = \langle\phi_i\phi_j^{[1]}\rangle_{0,1,0\,+\,0,0,1}$ which was computed in [33] and [34] as additional building block for the extended HAE. This is the disk plus the crosscap with two bulk insertions. In particular ϕ_i is a local insertion while $\phi_j^{[1]}$ is an

[5]An additional closed moduli dependence in the worldsheet action would come also from the Warner term [35]. For the B-model this additional boundary term, needed to make the action invariant, vanishes under the usual boundary conditions [39] as discussed in [13].

integrated one being the second step of the descent equation. So, from (2.14) we get

$$\frac{1}{\sqrt{2}}\Delta_{ij}(x)=\int_X d_i d_j \Omega L_{CS}+\int_X d_i\Omega d_j L_{CS}+\int_X d_j\Omega d_i L_{CS}+\int_X \Omega d_i d_j L_{CS} \tag{4.3}$$

where all the fields are on shell; d_i is the derivative with respect to the closed modulus x^i, both explicitly and through the dependence induced by $A(x)$ and $B(u,x)$; the factor $\frac{1}{\sqrt{2}}$ comes from the normalization in (4.1). Using the field equations for B we obtain the identity

$$0=d_j\left(\int_X \frac{\delta S_{HCS}}{\delta B}\,|_{B=B(u,x)}\, d_i B(u,x)\right)=d_j\left(\int_X \Omega d_i L_{CS}\right)=\int_X d_j\Omega d_i L_{CS}+\int_X \Omega d_i d_j L_{CS}$$

that is, the last two terms in (4.3) cancel. This is nothing but Griffith's transversality condition for the normal function as stated in [33]. So we get

$$\frac{1}{\sqrt{2}}\Delta_{ij}(x)=\langle\phi_i\phi_j\rangle_{0,1,0\,+\,0,0,1}=\int_X d_i d_j\Omega L_{CS}+\int_X d_i\Omega d_j L_{CS} \tag{4.4}$$

This differs from the expression derived in [33, 34] by the first term. However notice that (4.4) is valid at a generic value x for closed string moduli, while the ones of [33, 34] are evaluated at $x=0$, where the double derivative of Ω is vanishing. This comes from expression (2.9) and from the fact that $A(x)=O(x^2)$ as follows by solving the Kodaira-Spencer equations iteratively.

Let us now consider the amplitudes with one bulk and one boundary insertion. The latter, as already stated, is obtained from the derivative with respect to the background open field B_0 which depends on x:

$$\frac{1}{\sqrt{2}}\Delta'_{ij}=\langle\phi_i\psi_j^{[1]}\rangle_{0,1,0}=\left(d_j B_0(x)\frac{\delta}{\delta B_0(x)}\right)d_i S_{HCS}$$

To compute this term from the space-time point of view it is easier to start from the action written in terms of $\hat{B}$ and B_0 (2.6). The result follows immediately as

$$\frac{1}{\sqrt{2}}\Delta'_{ij}=\langle\phi_i\psi_j^{[1]}\rangle_{0,1,0}=-\int_X d_i\Omega Tr(d_j B_0(x)F_0) \tag{4.5}$$

once the e.o.m. of the open field are imposed.

Now we pass to the purely open moduli derivatives. The only term is the one derived three times or, equivalently, the one with three boundary operator insertions: $\Delta_{\alpha\beta\gamma}$. Again using the form (2.6) we need only explicit derivatives with respect to u^α (remind that $B_0=u^\alpha T_\alpha$). The result is

$$\frac{1}{\sqrt{2}}C_{\alpha\beta\gamma}=\langle\Theta_\alpha\Theta_\beta\Theta_\gamma\rangle_{0,1,0}=-\int_X \Omega Tr(T_\alpha T_\beta T_\gamma) \tag{4.6}$$

which is the same that would be derived with worldsheet methods in analogy to C_{ijk}.

Finally we have mixed terms. These are similarly obtained giving

$$\frac{1}{\sqrt{2}}\Pi_{\alpha i}=\langle\Theta_\alpha\phi_i\rangle_{0,1,0}=-\int_X d_i\Omega Tr(T_\alpha F_0) \tag{4.7}$$

and

$$\frac{1}{\sqrt{2}}\Delta'_{\beta i\alpha}=\langle\Theta_\beta\psi_i^{[1]}\Theta_\alpha\rangle_{0,1,0}=-\int_X \Omega Tr(T_\beta d_i B_0 T_\alpha) \tag{4.8}$$

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5 Open-closed string duality as a Losev trick

Let us explain a basic argument about open-closed string duality in second quantization. This is referred to the topological string theory at hand (B-model), but in principle should hold in a more general setting.

The Losev trick, as explained in [22, 23], consists in a procedure to obtain solutions of the quantum Master Equation in Batalin-Vilkovisky quantization by partial gauge fixing. In its generality it reads as follows. Let $S(\Phi, \Phi^*)$ be a solution of the quantum Master equation

$$\Delta\left(e^{-S/\hbar}\right) = 0 \tag{5.1}$$

where $\Delta = \partial_\Phi \partial_{\Phi^*}$ is the nilpotent BV laplacian. Suppose that the fields/anti-fields space $\mathcal{F}$ is in the form of a fibration

$$\begin{array}{ccc} \mathcal{F}_2 & \hookrightarrow & \mathcal{F} \\ & & \downarrow \\ & & \mathcal{F}_1 \end{array}$$

so that one can choose a split coordinate system $(\Phi, \Phi^*) = (\Phi_1, \Phi_1^*, \Phi_2, \Phi_2^*)$ such that the BV laplacian splits consistently as $\Delta = \Delta_1 + \Delta_2$ with $\Delta_1^2 = 0$. Then, assuming the existence of a non singular gauge fermion Ψ, one can consider the partially gauge fixed BV effective action

$$e^{-\frac{1}{\hbar}S_{\text{eff}}(\Phi_1,\Phi_1^*)} = \int_{\mathcal{F}_2} \mathcal{D}\left[\Phi_2, \Phi_2^*\right] e^{-\frac{1}{\hbar}S(\Phi,\Phi^*)} \delta\left(\Phi_2^* - \partial_{\Phi_2}\Psi\right). \tag{5.2}$$

which can be readily seen to satisfy the reduced BV Master equation

$$\Delta_1 e^{-\frac{1}{\hbar}S_{\text{eff}}(\Phi_1,\Phi_1^*)} = 0 \tag{5.3}$$

Actually — the proof is two lines — one consider (5.1) partially gauge fixed on the fibers and integrated along the fiber $\mathcal{F}_2$

$$\begin{aligned} 0 = \int_{\mathcal{F}_2} \mathcal{D}\left[\Phi_2, \Phi_2^*\right] \left\{\Delta_1 + \Delta_2\right\} e^{-\frac{1}{\hbar}S(\Phi,\Phi^*)} \delta\left(\Phi_2^* - \partial_{\Phi_2}\Psi\right) = \Delta_1 e^{-\frac{1}{\hbar}S_{\text{eff}}(\Phi_1,\Phi_1^*)} + \\ + \int_{\mathcal{F}_2} \mathcal{D}\left[\Phi_2\right] \left\{ \frac{d}{d\Phi_2} \left(\left[\partial_{\Phi_2^*} e^{-\frac{1}{\hbar}S(\Phi_1,\Phi_2,\Phi_1^*,\Phi_2^*)}\right]_{\Phi_2^*=\partial_{\Phi_2}\Psi} \right) - \partial^2_{\Phi_2}\Psi \cdot \left(\partial^2_{\Phi_2^*} e^{-\frac{1}{\hbar}S(\Phi,\Phi^*)}\right) |_{\Phi_2^*=\partial_{\Phi_2}\Psi} \right\} \end{aligned}$$

Now, the last line vanishes because of translation invariance of the path-integral along the fiber and field/anti-field opposite statistics, so that we recover (5.3). Let us notice that the resulting BV effective action depends on the particular gauge fixing chosen to integrate the fiber degrees of freedom. This dependence is BV trivial in the effective action.

Let us now specify the above setup to open/closed string theory, namely we identify $\mathcal{F}$ with the open and closed string theory, $\mathcal{F}_2$ with the open strings and $\mathcal{F}_1$ with closed strings. The complete theory is given by the BV action

$$S_{c+o}(A, B; x, u, \lambda) = S_c(A; x, \lambda) + S_o(A, B; x, u, \lambda) \tag{5.4}$$

where $S_c(A; x, \lambda)$ is the closed string BV action, and $S_o(A, B; x, u, \lambda)$ completes the open and closed BV action. The BV laplacian takes the form $\Delta_{c+o} = \Delta_c + \Delta_o$. We assume that both closed and open plus closed strings have been BV formulated, so that the corresponding quantum Master equations hold. Moreover, the uniqueness of closed string field theory is taken to mean that all solutions of the quantum Master equation, with proper boundary conditions in the string coupling dependence — namely the background independence of the kinetic term, are given by $S_c(A;x,\lambda)$ for some background x and the choice of the string coupling constant λ. For the B-model, this is explicitly proved in [4].

Therefore, by specifying the Losev trick to our case, we obtain that the effective action obtained from (5.4) by partial gauge fixing and integration over the open string field, satisfies the quantum Master equation (5.3) that is the quantum master equation for the *closed* string field theory. Notice that, by definition,

$$e^{-S_{\text{eff}}(A,x,\lambda,u)} = e^{-S_c(A;x,\lambda)} \int_{\substack{gauge\\fixed}} \mathcal{D}[B] e^{-S_o(A,B;x,u,\lambda)} \tag{5.5}$$

approaches the required boundary condition in the string coupling constant dependence. The actions entering (5.5) are required to have a canonically normalized kinetic term. Therefore, we conclude that the effective action (5.5) has to be the closed string field action (in some gauge determined by the gauge fixing in the open string sector) for a shifted set of background moduli and a redefined string coupling constant, that is,

$$\mathcal{N}\ e^{-S_c(A;x^\star,\lambda^\star)} = e^{-S_c(A;x,\lambda)} \int_{\substack{gauge\\fixed}} \mathcal{D}[B] e^{-S_o(A,B;x,u,\lambda)} \tag{5.6}$$

up to a field independent normalization $\mathcal{N}$.

The particular case we have in mind is therefore the topological B-model, where S_c is the Kodaira-Spencer gravity action and S_o the holomorphic Chern-Simons action suitably coupled to the Kodaira-Spencer field as discussed in the previous sections. After passing to flat coordinates, (5.6) then specifies to

$$\mathcal{N}(u,x,\lambda^{-1}\Omega_0)\ e^{-\frac{1}{\lambda^{\star 2}}S_{KS}(A^*,x^\star)} = e^{-\frac{1}{\lambda^2}S_{KS}(A,x)} \int_{\substack{gauge\\fixed}} \mathcal{D}[B] e^{-\frac{1}{\lambda}S_{HCS}(A,B,x,u)} \tag{5.7}$$

where the closed string field gets renormalized as $A^\star/\lambda^\star = A/\lambda$. In (5.7) $\mathcal{N}$ is a normalization factor[6] and

$$\frac{1}{\lambda^\star} = \frac{1}{\lambda} + \delta(u,x,\lambda) \quad \text{and} \quad (x^\star)^i = x^i + \delta^i(u,x,\lambda) \tag{5.8}$$

are some shifted background and string coupling. All these are *to be determined* and can be perturbatively computed from (5.7) by Feynman diagrams expansion or with non perturbative techniques when available. The redefinition (5.8) is a generalization (with tunable open moduli) of the moduli shift in [8]. The aim of the next subsection is to show that, at frozen open moduli, the above formulas reproduce the shift of [8].

[6]The particular dependence on the ratio Ω_0/λ is due to the fact that we have chosen flat coordinates u,x for the moduli. See next section for a specific discussion on the relevance of the normalization factor in comparing with [26].

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5.1 Open-closed duality at frozen open moduli

In this subsection we want to apply the general arguments just explained in section 5 to the oriented string theory with frozen open moduli [8]. Indeed, since we will work just at tree level, we do not have to deal with unoriented amplitudes. The effect of freezing the open moduli is easily obtained by replacing the non abelian field B with N identical copies of an abelian one, reducing the trace simply to a Chan-Paton factor β, which takes into account the number of boundaries. Accordingly, we consider a slightly modified version of (4.1) which better fits our purposes:

$$e^{W(x,\lambda^{-1})} = \lambda^{\frac{\chi}{24}-1-\beta^2\frac{N}{2}} \exp\left(\sum_{g,h,n} \frac{\lambda^{2g-2+h+n}}{n!}\beta^h \mathcal{F}^{(g,h)}_{i_1\ldots i_n} x^{i_1}\ldots x^{i_n}\right) \tag{5.9}$$

(5.9) is obtained from (4.1) suppressing all the open moduli parameters u^α, rescaling $x^i \to \lambda x^i$ and considering the additional β-parameter dependence. The HAE for open strings of [33] are obtained as power expansion in x^i, λ and β of (5.9)

$$\left(-\overline{\partial}_{\bar{i}} + \frac{1}{2}C^{jk}_{\bar{i}}\frac{\partial^2}{\partial x^j \partial x^j} + G_{j\bar{i}}x^j \frac{\partial}{\partial \lambda^{-1}} - \beta \Delta^j_{\bar{i}} \frac{\partial}{\partial x^j}\right) e^{W(x,\lambda^{-1})} = 0. \tag{5.10}$$

In [8] it was shown that the above HAE (5.10) can be derived from the HAE of the closed theory by means of a suitable change of variables

$$\begin{aligned} x^i &\to x^i + \beta \Delta^i \\ \lambda^{-1} &\to \lambda^{-1} - \beta \Delta \end{aligned} \tag{5.11}$$

with $\overline{\partial}_{\bar{i}}\Delta = \Delta_{\bar{i}}$ and $\overline{\partial}_{\bar{i}}\Delta^i = \Delta^i_{\bar{i}}$ such that $G_{i\bar{i}}\Delta^i = \Delta_{\bar{i}}$ and explicitly

$$\Delta = g^{0\bar{0}} \int_X L_{CS} \wedge \overline{\Omega}_0 \quad g^{0\bar{0}} = \left(\int_X \Omega_0 \wedge \overline{\Omega_0}\right)^{-1}$$
$$\Delta^i = g^{i\bar{j}} \left(\int_X L_{CS} \wedge d_{\bar{j}}\overline{\Omega}\right)_{x=0} \qquad g^{i\bar{j}} = \left(\int_X d_i\Omega \wedge d_{\bar{j}}\overline{\Omega}\right)^{-1}_{x=0}$$

where all the fields are on shell and $x = 0$. Notice also that Δ and Δ^i have been computed starting from the antitopological theory. Finally the closed field does not appear because on shell it goes as $O(x^2)$. The shift (5.11) allows to rewrite (5.10) in the same form as the master equation for purely closed strings

$$\left(-\overline{\partial}_{\bar{i}} + \frac{1}{2}C^{jk}_{\bar{i}}\frac{\partial^2}{\partial x^j \partial x^k} + G_{j\bar{i}}x^j \frac{\partial}{\partial \lambda^{-1}}\right) e^{W(x+\beta\Delta^i,\lambda^{-1}-\beta\Delta)} = 0 \tag{5.12}$$

as follows from an easy application of the chain rule. Before going on let us mention that a refined shift was proposed in [26] in order to have a detailed matching of the open and closed string amplitudes. The crucial point is that the change of variables proposed in [26] takes covariantly into account the constraint over the amplitudes

$$D_{i_n}\mathcal{F}^{(g,h)}_{i_1\ldots i_{n-1}} = \mathcal{F}^{(g,h)}_{i_1\ldots i_n}.$$

However, since we are interested in checking the fact that the integration over the open string modes produces a wave function satisfying the shifted closed HAEs, we can restrict ourselves to (5.11). A more refined analysis of the boundary conditions would require the calculation of the normalization factor $\mathcal{N}$ in (5.7) corresponding to the rescaling in eq. (3.13) of [26].

It is now possible to postulate that an analog shift for x and λ^{-1} in the path integral with the Kodaira-Spencer action (corresponding to the closed partition function) would allow to obtain the full path integral with the complete action.

In order to reproduce the power expansion of (5.9) from the target space field theory we have to set $x \to \lambda x$, so that any bulk operator insertion carries a weight λx as in (5.9). To maintain our setting we translate (5.11) into a shift for the product λx

$$\begin{aligned} \lambda x^i &\to \lambda x^i + \lambda\beta\Delta^i - \lambda^2\beta\Delta x^i + o(\lambda^3, \beta^2) \\ \lambda^{-1} &\to \lambda^{-1} - \beta\Delta \end{aligned} \tag{5.13}$$

of which we will keep only the lowest order term for the first line, discarding the λ^2 piece induced by the transformation of λ. From now on λx will be denoted simply as x. We want to check that

$$\int \mathcal{D}A e^{-S_{KS}(x^i+\lambda\beta\Delta^i+\ldots,\lambda^{-1}-\beta\Delta;t,\bar{t};A)} \simeq \int \mathcal{D}A\mathcal{D}B\mathcal{D}b\ldots e^{-S_{\text{tot}}(x,B_0,\lambda^{-1};t,\bar{t};A,B,b,\ldots)} \tag{5.14}$$

Let us consider (5.14) at the tree level. Simply applying (5.13) to the Kodaira-Spencer action gives, at order β and λ^{-1}, and redefining S_{KS} in order to have the factor λ^{-2} explicit,

$$\begin{aligned} \frac{1}{\lambda^2} S_{KS}(x^i + \lambda\beta\Delta^i + \ldots, \lambda^{-1} - \beta\Delta; t, \bar{t}; A) &= \frac{1}{\lambda^2} S_{KS}(x^i, \lambda^{-1}; t, \bar{t}; A) - \\ &- \frac{\beta}{\lambda}\int_M [(A+x)(A+x)]'(\mu_i)'\Delta^i - \frac{2\beta\Delta}{\lambda} S_{KS}(x^i, \lambda^{-1}; t, \bar{t}; A) + O(\lambda^0, \beta^2) \end{aligned}$$

Going at tree level the $O(\lambda^0, \beta^2)$ are not taken into account; in addition the A field should be taken on shell with respect to the Kodaira-Spencer equation in the shifted background, that is

$$A \to A(x^i + \lambda\beta\Delta^i + \ldots) = A(x) + \lambda\beta\Delta^i\partial_i A(x) + O(\lambda^2, \beta^2) \tag{5.15}$$

Then, at order β, $\frac{1}{\lambda}$, the left side of (5.14) is the exponential of

$$\begin{aligned} \frac{1}{\lambda^2} S_{KS}(x^i, \lambda^{-1}; t, \bar{t}; A(x)) - \frac{\beta}{\lambda}\int_X [(A(x)+x)(A(x)+x)]'(\mu_i)'\Delta^i - \\ - \frac{2\beta\Delta}{\lambda} S_{KS}(x^i, \lambda^{-1}; t, \bar{t}; A(x)) + \\ + \frac{\beta}{\lambda}\int_X \Delta^i(\partial_i A(x))'\frac{1}{\partial}\bar{\partial}A(x) - [(A(x)+x)(A(x)+x)]'(\partial_i A(x))'\Delta^i \end{aligned} \tag{5.16}$$

where the last line is actually zero because of the equations obeyed by $A(x)$, and the second line reduces to

$$-\frac{\beta\Delta}{3\lambda}[(A(x)+x)(A(x)+x)]'(A(x)+x)' = -\frac{\beta\Delta}{\lambda}[(A(x)+x)(A(x)+x)(A(x)+x)]'\Omega_0$$

Remembering the expression (2.9) we can substitute the value of Δ^i in (5.16) and get, for the second term in the first line of (5.16),

$$\frac{\beta}{\lambda}\int_X \Omega^{(1,2)}_{A=A(x)} \wedge (d_i\Omega)^{(2,1)}_{x=0} \left(\int_X (d_i\Omega)^{(2,1)}_{x=0} \wedge (d_{\bar{j}}\overline{\Omega})^{(1,2)}_{\overline{x}=0}\right)^{-1} \cdot$$
$$\cdot \int_X L^{(2,1)}_{CS}\,|_{B=B(u,x)} \wedge (d_{\bar{j}}\overline{\Omega})^{(1,2)}_{\overline{x}=0} = \frac{\beta}{\lambda}\int_X \Omega^{(1,2)}_{A=A(x)} \wedge L^{(2,1)}_{CS}\,|_{B=B(u,x)} \quad (5.17)$$

The last equality has been obtained using the Riemann bilinear relations:

$$\int_X w \wedge \hat{w} = \sum_{a=0}^{h_{2,1}} \int_{\delta_a} w \int_{\delta_{a+h_{2,1}}} \hat{w} - \int_{\delta_{a+h_{2,1}}} w \int_{\delta_a} \hat{w}$$

where δ_a is a base of 3-cycles on X. First we express in this way the integrals containing $\Omega \wedge d_i\Omega$ and $L_{CS} \wedge d_{\bar{j}}\overline{\Omega}$. Then we can define X^i and $\overline{X}^j$ as three forms such that

$$\left(\int_X d_i\Omega \wedge d_{\bar{j}}\overline{\Omega}\right)^{-1} \equiv \int_X X^i \wedge \overline{X}^j = \sum_{a=0}^{h_{2,1}} \int_{\delta_a} X^i \int_{\delta_{a+h_{2,1}}} \overline{X}^j - \int_{\delta_{a+h_{2,1}}} X^i \int_{\delta_a} \overline{X}^j$$

respects the definition

$$\sum_{\bar{j}} \left(\int_X d_i\Omega \wedge d_{\bar{j}}\overline{\Omega}\right)^{-1} \int_X d_k\Omega \wedge d_{\bar{j}}\overline{\Omega} = \delta_{i,k}$$

that is

$$\sum_i \int_{\delta_a} d_i\Omega \int_{\delta_b} X^i \equiv \delta_{a,b} \qquad \sum_{a=0}^{2h_{2,1}+2} \int_{\delta_a} d_i\Omega \int_{\delta_a} X^j \equiv \delta_{i,j}$$

and similarly with the barred quantities. Substituting these expressions in (5.17) we obtain the result.

Equivalently for the term in Δ in the second line of (5.16) we get

$$\frac{\beta}{\lambda}\int_X \Omega^{(0,3)}_{A=A(x)} \wedge L^{(3,0)}_{CS}\,|_{B=B(u,x)} \quad (5.18)$$

In order to reconstruct the full integral $\int_X \Omega_{A=A(x)} \wedge L_{CS}\,|_{B=B(u,x)}$ from the above equation the $(0,3)$ and $(1,2)$ components of L_{CS} are still missing. Notice however that they can be recovered by requiring CPT invariance. In particular, we modify (5.17) as

$$\frac{\beta}{\lambda}\int_X \left(\Omega^{(1,2)}_{A=A(x)} + \Omega^{(2,1)}_{A=A(x)}\right) \wedge (d_i\Omega)^{(2,1)}_{x=0} \cdot g^{i\bar{j}} \cdot$$
$$\cdot \int_X \left(L^{(2,1)}_{CS}\,|_{B=B(u,x)} + L^{(1,2)}_{CS}\,|_{B=B(u,x)}\right) \wedge (d_{\bar{j}}\overline{\Omega})^{(1,2)}_{\overline{x}=0}$$

where the extra term actually vanishes due to form degree reasons. This lead to an additional term

$$\frac{\beta}{\lambda}\int_X \Omega^{(2,1)}_{A=A(x)} \wedge L^{(1,2)}_{CS}\,|_{B=B(u,x)}$$

An analogous modification has to be performed in order to obtain the $(0,3)$ component of L_{CS}.

The geometrical counterpart of the above is as follows. We know from the discussion of [33] that the coupling of the on-shell Chern-Simons action to Ω_0 can be translated in mathematical terms to the pairing with the related normal function, ν, dual to a suitable three-chain, Γ, such that

$$\int_X \Omega_0 \wedge L_{CS} \mid_{B=B(u,x)} = \int_\Gamma \Omega_0 = \langle \Omega_0, \nu \rangle$$

and similarly for a $(2,1)$ form. Then it exists a lift of ν such that the coupling with a $(0,3)$ and $(1,2)$ forms are defined to be obtained by CPT invariance, that is complex conjugation of the corresponding $(0,3)$ and $(2,1)$ couplings.

Summarizing we have shown that

$$\begin{aligned}\frac{1}{\lambda^2} S_{KS}(x^i + \beta\lambda\Delta^i, \lambda^{-1} - \beta\Delta; t, \bar{t}) \mid_{on\ shell} &= \\ &= \left(\frac{1}{\lambda^2} S_{KS}(x^i, \lambda^{-1}; t, \bar{t}) + \frac{\beta}{\lambda}\int_X \Omega \wedge L_{CS} - \Omega db\right) \mid_{on\ shell} \quad (5.19)\end{aligned}$$

in the gauge $F^{\tilde{2},\tilde{0}}_{B_0} = 0$. Notice that the completion of the solution via CPT invariance obtained by adding the classical solutions of the anti-topological theory is consistent with the fact that, in our gauge, the gauge fixing $F^{(2,0)} = 0$ and the equation of motion $F^{(0,2)} = 0$ of the topological theory are the same, up to a switch of role, as in the *on shell* anti-topological one which is then manifestly CPT conjugate.

6 Conclusions

In this paper we provided a target space string field theory formulation for open and closed B-model, by giving a BV quantization of the holomorphic Chern-Simons theory with off shell gravity background. This allowed us to design a target space interpretation of the coefficients in the HAE with open moduli in general. In this paper we applied our formalism to reproduce the results of [8] and interpret them as an open/closed string duality. It would be interesting to study other explicit examples to refine the details of the scheme that we have been elaborating so far: on the conifold the on shell results of [15] could be useful.

Moreover, the target space formulas we obtained for the structure coefficients of the HAE should complete the data needed to rephrase the latter as conditions of background independence of the open B-model wave-function extending [26, 38].

The picture we provided in this paper seems to allow an extension to generalized complex geometries. This should follow by the definition of an extended Chern-Simons functional where the 3-form Ω gets promoted to the relevant pure spinor as in [24]. Once this is done and the b field promoted to a multiform, this would extend to open strings the proposal in [30] to generalized complex geometry of an analog of the Kodaira-Spencer theory.

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