ON THE EULER-LAGRANGE EQUATION FOR A VARIATIONAL PROBLEM: THE GENERAL CASE II

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Abstract. We prove a regularity property for vector fields generated by the directions of maximal growth of the solutions to the variational problem

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{1}_D(\nabla u) + g(u))dx, \tag{0.1}$$

with D convex closed subset of $\mathbb{R}^n$ with non empty interior. This regularity property allows to disintegrate the Lebesgue measure along the maximal growth rays, and the equation satisfied by the divergence of this vector field allows to compute explicitly the disintegration. As an application, we show that the Euler-Lagrange equation can be reduced to an ODE along characteristics, and we deduce that there exists a positive solution to Euler-Lagrange different from 0 a.e. and satisfies a uniqueness property. These results prove a conjecture on the existence of variations on vector fields stated in [3].

1. Introduction

We consider the existence of a solution to the Euler-Lagrange equation for the minimization problem

$$\inf \left\{ \int_\Omega g(u), u \in \bar{u} + W_0^{1,\infty}(\Omega), \nabla u \in D \right\}, \tag{1.1}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ strictly monotone increasing and differentiable, Ω open set with compact closure in $\mathbb{R}^n$, and D convex closed subset of $\mathbb{R}^n$. Under the assumption that $\nabla \bar{u} \in D$ a.e. in Ω, there is a unique solution u to (1.1) and we can actually give an explicit representation of u is terms of a Hopf-Lax type formula. The solution is clearly Lipschitz continuous because $\nabla u \in \partial D$ a.e. in Ω.

The Euler-Lagrange equation for (1.1) can be written as

$$\mathrm{div}\,(\pi(x)) = g'(u(x)), \quad \pi(x) \cdot \nabla u(x) = \max \left\{ \pi(x) \cdot d, d \in D \right\}, \tag{1.2}$$

where π is a measurable function. The first equation is considered in the distribution sense, and the second relation follows by using the subdifferential to the convex function

$$\mathbb{1}_D(x) = \begin{cases} 0 & x \in D \\ +\infty & x \notin D \end{cases}$$

in the standard formulation of the Euler-Lagrange equations. It means that the vector $\pi(x)$ lies in the convex support cone of ∂D at the point $\nabla u(x)$.

In [5], the authors prove that under the assumption $D = B(0,1)$ (in which case u is the solution to the Eiconal equation), there is a solution to the Euler-Lagrange equation (1.2), which can be rewritten as

$$\mathrm{div}\,\big(p(x)\nabla u(x)\big) = g'(u(x)), \quad p \geq 0. \tag{1.3}$$

The main point in the proof is that in the region $\Omega \setminus J$, where J is the singularity set of u, the solution u is $C^{1,1}$, and thus the above equation can be reduced to an ODE for p along the characteristics. We recall that in this case u is locally semi convex, so that ∇u has many properties of monotone functions (see for example [1] for a survey on monotone functions).

Simple examples show that such differentiability properties do not hold for general sets D, see [4]. However, using some weaker continuity property of ∇u, in [4] the author proves that the Euler-Lagrange

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equation (1.3) holds (i.e. there exists a weak solution π in $L^\infty_{\mathrm{loc}}(\Omega)$) and in the case the dual D^* of D is strictly convex, an explicit representation formula of the solution can be given.

The results contained in [4] are not completely satisfactory, because they rest on a stability assumption on the flow of continuous vector fields, in particular Lemma 5.6 in Section 5. This stability condition yields the uniqueness of the solution to the Euler Lagrange equation (1.3), which is unclear in the general case.

In this paper we prove the results of [4] in the general case, i.e. under the only assumption that D, D^* are convex, and without any additional regularity assumption. The main results of this paper are the following two theorems:

Theorem 1.1. *There exists a measurable selection $d(x) \in \partial \mathbf{I}_D(\nabla u(x)) \cap \partial B(0,1)$, and a measurable function*

$$a : \partial\Omega \times \mathcal{S}^{n-1} \to \Omega$$

such that, defining the segments

$$(a(y,z), y) = \Big\{ x = y + (t - z \cdot y) z, \forall t \in (z \cdot a(y,z), z \cdot y), d(x) = z, y \in \partial\Omega \Big\},$$

the Lebesgue measure $\mathcal{H}^n|_\Omega$ can be disintegrated as

$$\int_\Omega \phi(x) d\mathcal{H}^n(x) = \int_{\{t \in (z \cdot a(y,z), z \cdot y)\}} \phi(y + (t - z \cdot y) z) c(t,y,z) d\mu(y,z) \times \mathcal{H}^1(t), \tag{1.4}$$

with $0 \leq c(t,y,z) \in L^\infty(\mathcal{H}^1 \times \mu)$, and, for μ a.e. (y,z), Lipschitz continuous in $t \in (z \cdot a(y,z), z \cdot y)$, uniformly positive in each compact subset of $(z \cdot a(y,z), z \cdot y)$ and absolutely continuous function of t in $[z \cdot a(y,z), z \cdot y]$.

Moreover, d has a locally bounded divergence in Ω and

$$\partial_t c(t,y,z) + \Big[(\operatorname{div} d)_{a.c.}(y + (t - z \cdot y) z) \Big] c(t,y,z) = 0, \qquad \int_{z \cdot a(y,z)}^{z \cdot y} c(t,y,z) dt = 1 \tag{1.5}$$

for μ a.e. (y,z).

Theorem 1.2. *There exists a solution to the transport equation*

$$\operatorname{div}(\rho(x) d(x)) = g(x), \tag{1.6}$$

such that in all measurable sets $\mathcal{Z}$ of the form

$$\mathcal{Z} = \Big\{ x = y + (t - e_1 \cdot y) z, (y,z) \in Z, t \in \big[e_1 \cdot a(y,z), e_1 \cdot y \big] \Big\},$$

$$Z(t) = \mathcal{Z} \cap \{ e_1 \cdot x = t \},$$

the divergence formula holds:

$$\int_{Z(t^-)} \rho(y) d(y) \cdot e_1 d\mathcal{H}^{n-1}(y) - \int_{Z(t^+)} \rho(y) d(y) \cdot e_1 d\mathcal{H}^{n-1}(y) = \int_{\cup_{(t^-,t^+)} Z(t)} g(x) d\mathcal{H}^n(x). \tag{1.7}$$

This solution is > 0 $\mathcal{H}^n$ a.e. in Ω if g is, and it is explicitly given by

$$\rho(y + (t - z \cdot y) z) = \frac{1}{c(t,y,z)} \int_{z \cdot a(y,z)}^t c(s,y,z) g(y + (t - z \cdot y) z) ds \tag{1.8}$$

for μ a.e. (y,z).

We recall that by using the same analysis of Section 7.1 in [4], it follows that the following conjecture of Bertone-Cellina [3] is true: let $u \in W^{1,\infty}(\Omega)$ be such that $\nabla u \in D$ $\mathcal{H}^n$ a.e. in Ω,

(1) either there exists a function $\eta \in W^{1,\infty}_0(\Omega)$ such that $\nabla u + \nabla \eta \in D$,
(2) or there exists a divergence free vector $\pi \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ such that $\pi \neq 0$ $\mathcal{H}^n$ a.e. in Ω and

$$\pi(x) \in \partial \mathbf{I}_D(\nabla u) \quad \mathcal{H}^n \text{ a.e. in } \Omega. \tag{1.9}$$

We just observe that the absence of variations means that u is the only solution to our variation problem (1.1).

The proof uses arguments which are not strictly related to the explicit form of the solution (1.1): in fact, we believe that the same approch can be used in many other situations, for which it is possible to approximate with suitable good vector fields the final vector field.

The proof is based on the following steps.

In Section 2, we recall the basic notation and the explicit formula of the solution u, Proposition 2.1:

$$u(x) = \max\Big\{u(\bar{x}) - |\bar{x} - x|_{D^*}, \bar{x} \in \partial\Omega, \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega}\ \forall \alpha \in [0,1]\Big\}. \tag{1.10}$$

In Section 3, we define the set valued function $\mathcal{B}(x) \subset \partial\Omega$ as the set of boundary data such that

$$u(y) - u(x) = |y - x|_{D^*}, \quad y \in \partial\Omega,$$

where $|\cdot|_{D*}$ is the pseudo norm generated by D^*, the dual of D. We also introduce the set valued function $\mathcal{D}(x)$ as the set of unit directions of the vectors $b - x$, $b \in \mathcal{B}(x)$. Following the analysis of [4], we prove the fundamental estimates on the continuity and measurability of $\mathcal{B}(x)$, $\mathcal{D}(x)$, Proposition 3.1 and Lemma 3.2. Finally we study the set where $\mathcal{D}(x)$ contains at least two directions d_1, d_2 which (after rescaling) belong to two different extremal faces of D^*. By repeating the analysis of [4], we show that this set is countably $n-1$ rectifiable, Proposition 3.6.

If D^* is strictly convex, it is known [4] that $\mathcal{B}(x)$, $\mathcal{D}(x)$ are single valued $\mathcal{H}^n$ a.e. in Ω, and we denote these functions by $b(x)$, $d(x)$: in Section 4 we analyze the case when D^* is strictly convex. First we introduce the single valued function $a(x)$, defined as the initial point of the segment $x + td(x)$ where $u(x + td(x)) = u(x) + t|d(x)|_{D^*}$. This function allows to represent the solution u also as

$$u(x) = \min\Big\{u(\bar{x}) + |x - \bar{x}|_{D^*}, \bar{x} \in \bigcup_{y\in\Omega} a(y), \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega}\ \forall \alpha \in [0,1]\Big\}. \tag{1.11}$$

The stability of the vector field d w.r.t. perturbations of the boundary data, Proposition 4.4, and the analysis of the particularly simple vector field d_I of Example 4.5 yield two important results: the estimate of the divergence of d, Proposition 5.6,

$$\operatorname{div} d + \frac{n-1}{\operatorname{dist}(\Omega', \partial\Omega)} d\mathcal{H}^n \geq 0, \quad \Omega' \subset\subset \Omega, \tag{1.12}$$

and the estimate on the push forward of the $\mathcal{H}^{n-1}$ measure on subsets of transversal planes by the vector field d, Lemma 4.7. In particular this lemma shows that the push forward remains equivalent to the $\mathcal{H}^{n-1}$ measure.

In Section 5 we show how to select an $L^\infty(\Omega, \mathbb{R}^n)$ function $b(x) \in \mathcal{B}(x)$ with good properties: the principal one is that it can be approximated $\mathcal{H}^n$ a.e. by functions $b_i(x)$ generated by the solution obtained using the strictly convex sets $D_i^* = (D + \frac{1}{i}K)^*$, with K strictly convex, Proposition 5.4. This allows to pass many properties of b_i to the limit, in particular that if we define the vector field

$$d(x) = \frac{b(x) - x}{|b(x) - x|_{D^*}}$$

then $\operatorname{div} d$ is a bounded measure, Proposition 5.6 and it is the limit of the vectors $d_i(x)$ constructed by considering the strictly convex sets D_i^*, Proposition 5.4. We also prove that this selection enjoys the same push forward estimates of the $\mathcal{H}^{n-1}$ measure on planes transversal to d proved in the strictly convex case, Lemma 5.7. A consequence of this estimate is that the set $\cup_{x\in\Omega} a(x)$ is $\mathcal{H}^n$ negligible, Proposition 5.8.

A deeper analysis of the vector field d is done in Section 6. In this section it is proved that on the sets $\mathcal{Z}$ of the form

$$Z(0) = B(0,r) \cap \{e_1 \cdot y = 0\} \cap \big\{e_1 \cdot a(y) \leq -h^-, e_1 \cdot b(y) \geq h + h^+\big\}, \tag{1.13}$$

such that b, d, a are continuous on $Z(0)$, $e_1 \cdot d > 1 - \epsilon$, and

$$\mathcal{Z} = \Big\{x = y + td(y), y \in Z(0), t \in [e_1 \cdot a(y), e_1 \cdot b(y)]\Big\}, \quad Z(t) = \mathcal{Z} \cap \{e_1 \cdot x = t\}, \tag{1.14}$$

the push forward of the $\mathcal{H}^{n-1}$ measure on $Z(t) = \mathcal{Z} \cap \{e_1 \cdot x = t\}$ defines a function $\alpha(t, s, y)$ with good properties: the inverse $1/\alpha$ remains uniformly bounded, different from 0 and absolutely continuous for $t \in (e_1 \cdot a(x), e_1 \cdot b(x))$, Lemma 6.2 and Corollary 6.3.

Since $1/\alpha$ is the factor appearing when writing the Lebesgue measure on $\mathcal{Z} \cap \Omega$ as an integral on $Z(0)$ along the lines $x + td(x)$, $t \in \mathbb{R}$, in Section 7 we use the uniform bound on $1/\alpha$ and its strict positivity to disintegrate the Lebesgue measure along the segments $(a(x), b(x))$, Theorem 7.5. Disintegrating the divergence formula

$$\int \phi \operatorname{div} d = -\int d \nabla \phi d\mathcal{H}^n, \qquad \phi \in C_c(\Omega', \mathbb{R}). \tag{1.15}$$

and using the estimates on the derivative $\partial_t(\alpha(x + td(x))^{-1})$, we prove that $1/\alpha$ satisfies the ODE

$$\partial_t(\alpha(x + td(x))^{-1}) + \Big[(\operatorname{div} d)_{\text{a.c.}}(y + (t - z \cdot y)z)\Big]\alpha(x + td(x))^{-1} = 0, \tag{1.16}$$

for almost all segments $(a(x), b(x))$, Proposition 7.8.

Finally, the last section, Section 8, shows how to construct a particular $L^\infty_{\text{loc}}(\Omega)$ solution to the PDE

$$\operatorname{div}(\rho(x)d(x)) = g(x), \quad g \in L^1_{\text{loc}}(\Omega), \tag{1.17}$$

with the property of satisfying the divergence formulation also in the sets $\mathcal{Z}$, Theorem 8.1. This solution is positive if g is. A uniqueness property of this solution is proved in Corollary 8.2.

2. Preliminaries

We consider the following variational problem

$$\inf_{\bar{u}+W_0^{1,\infty}} \int_\Omega (\mathbf{I}_D(\nabla u) + g(u))dx, \tag{2.1}$$

with $g : \mathbb{R} \mapsto \mathbb{R}$ strictly monotone increasing and differentiable, Ω open set with compact closure in $\mathbb{R}^n$. The function $\mathbf{I}_A$ is the indicative function of a set $A \subset \mathbb{R}^n$,

$$\mathbf{I}_A(x) = \begin{cases} 0 & x \in A \\ +\infty & x \notin A \end{cases} \tag{2.2}$$

Moreover, to have a finite infimum in (2.1), we assume that the function $\bar{u}$ satisfies

$$\nabla \bar{u} \in D. \tag{2.3}$$

As a consequence, the infimum is finite and it is attained.

To avoid degeneracies, in the following we assume that D is a bounded convex closed subset of $\mathbb{R}^n$, with non empty interior, and without loss of generality we suppose that

$$B(0, R_1) = \Big\{x \in \mathbb{R}^n, |x| \leq R_1\Big\} \subset D \subset B(0, R_2). \tag{2.4}$$

We then denote the dual convex set D^* by

$$D^* = \Big\{d \in \mathbb{R}^n : d \cdot \ell \leq 1 \ \forall \ell \in D\Big\}, \quad B(0, 1/R_2) \subset D^* \subset B(0, 1/R_1), \tag{2.5}$$

where the scalar product of two vectors $x, y \in \mathbb{R}^n$ is $x \cdot y$. The set D^* is closed, convex and $D^{**} = D$. We will write the support set at $\bar{\ell} \in \partial D$ as

$$\delta D(\bar{\ell}) = \Big\{d \in \partial D^* : d \cdot \bar{\ell} = \sup_{\ell \in D} d \cdot \ell = 1\Big\} = \partial \mathbf{I}_D(\bar{\ell}) \cap \partial D^*. \tag{2.6}$$

In the above formula, we have used the notation $\partial f(x)$ as the subdifferential of a convex function f evaluated at x.

Let $|\cdot|_D$ be the pseudo-norm given by the Minkowski functional

$$|x|_D = \inf\big\{k \in \mathbb{R} : x \in kD\big\} = \sup\big\{d \cdot x, d \in D^*\big\}, \tag{2.7}$$

and define the dual pseudo-norm by

$$|x|_{D^*} = \inf\big\{k \in \mathbb{R} : x \in kD^*\big\} = \sup\big\{\ell \cdot x, \ell \in D\big\}. \tag{2.8}$$

Note that due to convexity the triangle inequality holds,

$$|x + y|_{D^*} \leq |x|_{D^*} + |y|_{D^*}, \quad x, y \in R^n, \tag{2.9}$$

and that $|\cdot|_D$, $|\cdot|_{D^*}$ are the Legendre transforms of $\mathbf{I}_{D^*}$, $\mathbf{I}_D$ respectively.

In the following, we denote with $\mathcal{H}^{n-1}$ the $n-1$ dimensional Hausdorff measure [2], Definition 2.46 of page 72: for any $\Omega' \subset \Omega$,

$$|\Omega'|_{\mathcal{H}^{n-1}} = \mathcal{H}^{n-1}(\Omega') = \kappa \sup_{\delta>0}\biggl(\inf\biggl\{\sum_{i\in I}|\mathrm{diam}(B_i)|^{n-1}, \mathrm{diam}(B_i) \leq \delta, \Omega \subset \bigcup_{i\in I} B_i\biggr\}\biggr), \tag{2.10}$$

where κ is the constant such that $\mathcal{H}^{n-1}$ is equivalent to the Lebesgue measure on $n-1$ dimensional planes:

$$\kappa = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(1+\frac{n-1}{2})}, \quad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

We recall that $\mathcal{H}^n$ is the n dimensional Lebesgue measure $\mathcal{L}^n$, [2], Theorem 2.53 of page 76.

If $f : X \mapsto Y$ is a measurable map between the measure space $(X, \mathcal{S}, \mu)$ into the measurable space $(Y, \mathcal{T})$, we define the push forward measure $f_\sharp\mu$ as ([2], Definition 1.70 of page 32)

$$f_\sharp\mu(T) = \mu(f^{-1}(T)), \quad T \in \mathcal{T}. \tag{2.11}$$

The first proposition is the explicit representation of the solution by a Hopf-Lax type formula.

Proposition 2.1. *The solution of* (2.1) *is given explicitly by*

$$u(x) = \max\Bigl\{u(\bar{x}) - |\bar{x}-x|_{D^*}, \bar{x} \in \partial\Omega, \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega}\ \forall \alpha \in [0,1]\Bigr\}. \tag{2.12}$$

Moreover, u is Lipschitz continuous and $\nabla u \in \partial D$ a.e.. The existence of the maximum is part of the statement.

The proof of this proposition is standard, and can be found for example in [4], Proposition 2.1. The basic ideas are that the function defined by (2.12) has derivative still in D (because $\nabla|\cdot|_{D^*} \in \partial D\ \mathcal{H}^n$ a.e. on $\mathbb{R}^n$) and is the lowest possible function such that $\nabla u \in D$. The boundary data is certainly assumed because $\bar{u}$ satisfies $\nabla u \in D\ \mathcal{H}^n$ a.e. in Ω. Finally if $|\nabla u|_D \leq 1$ then u is not the lowest function, yielding a contradiction.

Remark 2.2. In the following we will only use the representation of u given by (2.12), i.e. we will not care if the boundary data $\bar{u}$ is assumed on the whole boundary $\partial\Omega$ or just in one point. Clearly we can always redefine the boundary data by means of (2.12) so that the boundary data is assumed.

3. Regularity estimates

Before studying the Euler-Lagrange equation for the variational problem (2.1), we introduce some important functions and prove some basic regularity estimates.

Define the set valued functions

$$\mathcal{B}(x) = \Bigl\{\bar{x} \in \partial\Omega : u(x) = u(\bar{x}) - |\bar{x}-x|_{D^*},\ [x,\bar{x}] \subset \bar{\Omega}\Bigr\} \subset \partial\Omega, \tag{3.1}$$

$$x \to \mathcal{D}(x) = \biggl\{\frac{\bar{x}-x}{|\bar{x}-x|}, \bar{x} \in \mathcal{B}(x)\biggr\} \subset \partial B(0,1). \tag{3.2}$$

Thus $\mathcal{D}(x)$ is the set of directions where u has the maximal growth in the norm $|\cdot|_{D^*}$. It is easy to prove that both sets $\mathcal{B}(x)$, $\mathcal{D}(x)$ are closed not empty subsets of $\partial\Omega$, $\partial B(0,1)$, respectively (see the proof of [4], Proposition 2.1). The normalization in (3.2) and $\nabla u \in D$ imply that

$$u(x+td) = u(x) + t|d|_{D^*} \tag{3.3}$$

for all $x \in \Omega$, $d \in \mathcal{D}(x)$. Roughly speaking, we can say that $\mathcal{B}(x)$ is the set where the half lines $x + td(x)$, with $d(x) \in \mathcal{D}(x)$ and $t \geq 0$, intersect $\partial\Omega$: this is perfectly correct in the case Ω is convex.

In the following, we will study the properties of $\mathcal{B}$, $\mathcal{D}$: since $\mathcal{D}$ is obtained by $\mathcal{B}$ by normalization, we will often prove that a property holds for one and only say that the same property holds for the other.

The first result is the upper continuity of the set valued maps $\mathcal{D}(x)$, $\mathcal{B}(x)$.

Proposition 3.1. *The function $\mathcal{D}(x)$ is closed graph and upper semicontinuous: more precisely for all $y \in \Omega$, for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\mathcal{D}(x) \subset \mathcal{D}(y) + B(0, \epsilon) \tag{3.4}$$

for $x \in B(y, \delta)$. Moreover, for all $d \in \mathcal{D}(y)$ there exists $x_n \to y$ ($x_n \neq y$), $d_n \in \mathcal{D}(x_n)$ such that $d_n \to d$.

The function $\mathcal{B}(x)$ has the same properties.

Proof. Fixed the point y, by rescaling we can restrict to the set of points whose distance from y is 1,

$$D^*(y, 1) = \big\{z : |z - y|_{D^*} = 1\big\},$$

and we can assume that $u(y) = 0$. By the explicit formula of solutions, the set $\mathcal{D}(y)$ is given by

$$\mathcal{D}(y) = \Big\{z - y : |z - y|_{D^*} = 1,\ u(z) = 1\Big\},$$

so that it follows from Lipschitz continuity that for all ϵ there is a δ such that

$$u(z) < 1 - \epsilon \qquad \forall z : |z - y|_{D^*} = 1, \mathrm{dist}(z, \mathcal{D}(y)) > \delta.$$

We thus have that for all x such that $|x - y|_{D^*} \leq \epsilon/2$, z as above,

$$u(x) \geq -\epsilon/2 > u(z) - 1 + \epsilon/2 \geq u(z) - |z - y|_{D^*} + |x - y|_{D^*} \geq u(z) - |z - x|_{D^*}.$$

Thus the set $\mathcal{D}(z)$ for such a z has a distance from $\mathcal{D}(y)$ less than $\mathcal{O}(\delta + \epsilon)$. The closed graph property follows from the fact that each $\mathcal{D}(x)$ is closed.

Since if $d \in \mathcal{D}(y)$, then $d \in \mathcal{D}(y + td(y))$ for all t such that $y + td(y) \in \Omega$, we have the last part of the statement.

The proof for $\mathcal{B}(x)$ is completely similar. $\square$

We next prove that the set valued map $\mathcal{B}$ is measurable: we repeat the computations of [4], Proposition 3.3. We recall that if F is a set valued function, then

$$F^{-1}(A) = \Big\{x : F(x) \cap A \neq \emptyset\Big\}. \tag{3.5}$$

Lemma 3.2. *The function $\mathcal{B}(x)$ is Borel measurable, i.e. the inverse image of open sets are Borel measurable. More precisely, the inverse image of a compact set is compact in Ω.*

Proof. We have to prove that for all open set O in $\partial\Omega$, the inverse image

$$\big\{x : \mathcal{B}(x) \in O\big\}$$

is Borel. Take a sequence of closed set $\bar{O}_i \subset O$, $i \in \mathbb{N}$, on the boundary $\partial\Omega$ such that $\cup_i \bar{O}_i = O$. The measurability of $\mathcal{B}^{-1}(\bar{O}_i)$ is trivial for the function

$$u_0 = \max\Big\{u(\bar{x}) - |\bar{x} - x|_{D^*}, \bar{x} \in \bar{O}_i, \alpha x + (1 - \alpha)\bar{x} \in \bar{\Omega}\ \forall \alpha \in [0, 1]\Big\},$$

since $\mathcal{B}^{-1}(\bar{O}_i) = \Omega$. Then, one only observes that

$$\mathcal{B}^{-1}(\bar{O}_i) = \Big\{x \in \Omega : u_0(x) = u(x)\Big\},$$

where $u(x)$ is the solution to the variational problem. Since u_0, u are Lipschitz function, it follows that $\mathcal{B}^{-1}(\bar{O}_i)$ is a closed set in Ω, hence $\mathcal{B}^{-1}(O) = \cup_i \mathcal{B}^{-1}(\bar{O}_i)$ is Borel. $\square$

We now prove the following relation among the derivative of the Lipschitz function u and the function $\mathcal{D}$.

Lemma 3.3. *If x is a point of differentiability of u, then*

$$\nabla u \in \delta D^*(d/|d|_{D^*}) = \Big\{\ell \in D : \ell \cdot d/|d|_{D^*} = 1\Big\}, \tag{3.6}$$

where $d \in \mathcal{D}(x)$. Similarly,

$$\Big\{d/|d|_{D^*}, d \in \mathcal{D}(x)\Big\} \subset \delta D(\nabla u(x)). \tag{3.7}$$

Proof. This follows from the equation $u(x+td) = u(x) + t|d|_{D^*}$ for all $d \in \mathcal{D}(x)$, which gives

$$\nabla u \cdot d/|d|_{D^*} = 1 \qquad \Longrightarrow \qquad \nabla u \in \delta\mathcal{D}^*(d/|d|_{D^*})$$

by definition of $\delta D^*(d)$ and the fact that $\nabla u \in \partial D$ (Proposition 2.1). The same equation shows also the second part of the lemma, since $d/|d|_{D^*} \in \partial D^*$. □

We finally show the countably $n-1$ rectifiability of the singular set

$$J = \Big\{ x \in \Omega : \exists d_1, d_2 \in \mathcal{D}(x), \delta D^*(d_1/|d_1|_{D^*}) \cap \delta D^*(d_2/|d_2|_{D^*}) = \emptyset \Big\}. \tag{3.8}$$

We can write

$$J = \bigcup_{m\in\mathbb{N}} J^m, \tag{3.9}$$

where, for $m \in \mathbb{N}$ J_m, is the compact set

$$J^m = \Big\{ x \in \Omega : \exists d_1, d_2 \in \partial B(0,1), \mathrm{dist}_H\big(\delta D^*(d_1/|d_1|_{D^*}), \delta D^*(d_2/|d_2|_{D^*})\big) \geq \frac{1}{m} \Big\}, \tag{3.10}$$

where d_H is the Hausdorff distance among compact sets.

We begin with a simple geometrical lemma, taken from [4]. Given a compact convex set K with $0 \notin K$, define

$$\begin{aligned} C^+(K) &= \Big\{ x \in \mathbb{R}^n : x \cdot \ell > 0 \ \forall \ell \in K \Big\}, \\ C(K) = C^+(K) \cup C^+(-K) &= \Big\{ x \in \mathbb{R}^n : x \cdot \ell \neq 0 \ \forall \ell \in K \Big\}. \end{aligned} \tag{3.11}$$

Clearly $C(K)$ is an open non empty cone, because K is convex and compact and does not contains the origin: $C^+(K)$, $C(K)$ can be represented by means of duality.

Lemma 3.4. *Let $d_1, d_2 \in \mathcal{D}(x_0)$, and assume that $\mathrm{dist}_H\big(\delta D^*(d_1/|d_1|_{D^*}), \delta D^*(d_2/|d_2|_{D^*})\big) > 0$. Let x_i be a sequence of points converging to x_0 such that there exists $d_i \in \mathcal{D}(x_i)$ with $d_i \to d_1$. Then, if Y is the derived set of $\frac{x_i - x}{|x_i - x|}$,*

$$Y \cap C^+\big(\delta D^*(d_2/|d_2|_{D^*}) - \delta D^*(d_1/|d_1|_{D^*})\big) = \emptyset, \tag{3.12}$$

where $\delta D^*(x)$ is the support cone of D^* at x.

We recall that the derived set of a sequence a_i is the closed set of limits of all subsequences.

Proof. Without any loss of generality, assume $x_0 = 0$, $u(x_0) = 0$ and consider the set $D^* = \{|x|_{D^*} = 1\}$ of D^*-radius 1, so that $u(d_1/|d_1|_{D^*}) = u(d_2/|d_2|_{D^*}) = 1$. Moreover $u < 1$ on $\partial D^* \setminus \cup_{\alpha>0} \alpha\mathcal{D}(x)$.

Since d_i is close to d_1, then for some $\epsilon > 0$ it holds

$$1 - |y - x_i|_{D^*} \geq u(y_i) - |y_i - x_i|_{D^*} \geq u(d_2/|d_2|_{D^*}) - |d_2/|d_2|_{D^*} - x_i|_{D^*} = 1 - |d_2/|d_2|_{D^*} - x_i|_{D^*},$$

where y is the closest point to x in the ϵ neighborhood of $d_1/|d_1|_{D^*}$ and y_i is the point such that $u(x_i) = u(y_i) - |y_i - x_i|_{D^*}$. We can approximate the two distances as

$$|d_2/|d_2|_{D^*} - x_i|_{D^*} = 1 - \ell_i(d_2/|d_2|_{D^*})x_i + o(|x_i|), \quad |y - x_i|_{D^*} = 1 - \ell_i(y)x_i + o(|x_i|), \tag{3.13}$$

with $\ell_i(d_2/|d_2|_{D^*}) \in \delta D^*(d_2/|d_2|_{D^*})$, $\ell_i(y) \in \delta D^*(y)$. From upper continuity continuity of $\delta D^*(d)$ (as the derivative of a convex function), it follows that $\delta D^*(y) \subset \delta D^*(d_1/|d_1|_{D^*}) + B(0,\delta)$, with $\delta \to 0$ as $y \to d_1/|d_i|_{D^*}$, so that $\delta D^*(y) \cap \delta D^*(d_2/|d_2|_{D^*}) = \emptyset$ for ϵ sufficiently small, because of the assumption that $d_H(\delta D^*(d_1/|d_1|_{D^*}), \delta D^*(d_2/|d_2|_{D^*})) > 0$. It thus follows that there exists $\ell_i(y) \neq \ell_i(d_2/|d_2|_{D^*})$ such that $x_i \cdot (\ell_i(d_2/|d_2|_{D^*}) - \ell_i(y)) \leq o(|x_i|)$, so that by taking a converging subsequence of $x_i/|x_i|$ and $\ell_i(y)$ we obtain

$$\lim_{i\to\infty} \frac{x_i}{|x_i|} \notin C^+(\delta D^*(d_2/|d_2|_{D^*}) - \delta D^*(d_i/|d_1|_{D^*})).$$

Since this holds for all converging subsequences, we obtain (3.12). □

Up to subsequences x_{i_j}, we can assume that the vectors $\ell_i(d_2/|d_2|_{D^*})$, $\ell_i(y)$ converge to some limits $\ell(d_2/|d_2|_{D^*})$, $\ell(d_1/|d_1|_{D^*})$, $\ell(d_2/|d_2|_{D^*}) \neq \ell(d_1/|d_1|_{D^*})$, so that it follows that the sequence of x_{i_j} asymptotically belongs to the half space $\{x \cdot (\ell(d_2/|d_2|_{D^*}) - \ell(d_1/|d_1|_{D^*})) \leq 0\}$.

If we have a sequence $x_i \to x$ in J^m, it follows that we can extract a subsequence such that $(x_i - x)/|x_i - x|$ and the vectors $\ell_i(d_k/|d_k|_{D^*})$ defined by

$$\big|d_k/|d_k|_{D^*} - (x_i - x)\big|_{D^*} = 1 - \ell_{ik}(d_k/|d_k|_{D^*})(x_i - x) + o(|x_i - x|_{D^*}),$$

converge to some vectors e, $\ell(d_k/|d_k|_{D^*})$, where d_k, $k = 1, 2$, are independent directions in $\mathcal{D}(x)$ in the sense of the definition (3.10). From Lemma 3.4 we have that any limit

$$\begin{aligned} \lim_{i\to\infty} \frac{x_i - x}{|x_i - x|} &= e \notin C^+(\delta D^*(d_2/|d_2|_{D^*}) - \delta D^*(d_1/|d_1|_{D^*})) \cup C^+(\delta D^*(d_1/|d_1|_{D^*}) - \delta D^*(d_2/|d_2|_{D^*})) \\ &= C(\delta D^*(d_2/|d_2|_{D^*}) - \delta D^*(d_1/|d_1|_{D^*})). \end{aligned} \tag{3.14}$$

The rectifiability of J^m follows thus from the following rectifiability criterion [2], Theorem 2.61 page 82:

Proposition 3.5. *Let $E \subset \mathbb{R}^n$ be such that for all $x \in E$ there exists $B_x = B(x, r_x)$, and a cone*

$$C_x = \Big\{ y : |d_x \cdot (y - x)| \geq m_x |(\mathbb{I} - d_x \otimes d_x) \cdot (y - x)|, d_x \in \mathbb{R}^n, |d_x| = 1 \Big\}$$

such that $E \cap B_x \cap C_x = \emptyset$. Then E is $n - 1$ rectifiable, i.e. it can be covered with a countable number of images of Lipschitz functions.

We thus have

Proposition 3.6. *The set $J = \cup J^m$ is countably $n - 1$ rectifiable.*

It can be proved that the set where there exists k directions whose support cones are pairwise disjoint is countably $n - k + 1$ rectifiable, [4] Proposition 6.4.

4. The strictly convex case

In this generality we cannot say much on the functions $\mathcal{B}$, $\mathcal{D}$. Even if we can apply selection principle to the measurable set valued functions $\mathcal{B}$, $\mathcal{D}$, in general these selections do not show any particular good property apart from being measurable. Following [4], in this section we make the following assumption:

the conjugate set D^* is strictly convex.

By duality, this implies that ∂D is differentiable. Since for smooth ∂D the support $\partial D(\ell)$ reduces to a single point for all $\ell \in \partial D$, using (3.7), the continuity of $\mathcal{D}$ and Proposition 3.6 it follows that

Corollary 4.1. *The functions $\mathcal{D}$ is single valued in each differentiability point of u.*

Proof. Let x be a point where $\mathcal{D}(x)$ is not single valued, i.e. there exists d_1, d_2 such that $d_1, d_2 \in \mathcal{D}(x)$. Then from the strict convexity of D^* it follows that the support cones of d_1, d_2 are disjoint, so that we conclude from Proposition 3.6 that the set of points where $\mathcal{D}$ is not single valued is countably $n - 1$ rectifiable.

□

We consider the set of segments

$$\begin{aligned} \Sigma(x) = \bigcup_{d\in\mathcal{D}(x)} \Big\{ x + td : d \in \mathcal{D}(x), t \in [t^-(x,d), t^+(x,d)], \\ t^-(x,d), t^+(x,d) \text{ are the minimal, maximal values such that} \\ u(x + td) = u(x) + t|d|_{D^*} \ \forall t \in (t^-(x,d), t^+(x,d)) \Big\}. \end{aligned} \tag{4.1}$$

The set $\mathcal{B}(x)$ is the set where $x + td(x) \in \partial\Omega$, while by considering the end points for $t \leq 0$, we define the function

$$a(x) = \Big\{ x + td : t = t^-(x,d), d \in \mathcal{D}(x) \Big\} \tag{4.2}$$

From the strict convexity of D^*, it follows in fact that

Lemma 4.2. *$a(x)$ is single valued.*

Proof. Assume that $\mathcal{D}(x)$ contains two different directions d_1, d_2, and let $a_1(x) \neq x$ be defined by (4.2). We denote as usual by b_1, b_2 the two boundary data corresponding to d_1, d_2. Then

$$u(b_2) = u(x) + |b_2 - x|_{D^*} = u(a_1) + |x - a_1|_{D^*} + |b_2 - x|_{D^*} > u(a_1) + |b_2 - a_1|_{D^*},$$

which is a contradiction with $\nabla u \in D^*$. We have used the fact

$$|x + y|_{D*} < |x|_{D^*} + |y|_{D^*}$$

if x and y are not parallel. □

This lemma is clearly not true when D^* is not strictly convex. As a corollary of the explicit form of the solution and the above definitions, we have

Corollary 4.3. *The solution u can be written as*

$$u(x) = \min\biggl\{u(\bar{x}) + |x - \bar{x}|_{D^*}, \bar{x} \in \bigcup_{y\in\Omega} a(y), \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega}\ \forall\alpha \in [0,1]\biggr\},$$

$$u(x) = \max\biggl\{u(\bar{x}) - |\bar{x} - x|_{D^*}, \bar{x} \in \bigcup_{y\in\Omega} \mathcal{B}(y), \alpha x + (1-\alpha)\bar{x} \in \bar{\Omega}\ \forall\alpha \in [0,1]\biggr\}. \tag{4.3}$$

The formula based on $\cup_y a(y)$ follows from the Lipschitz continuity and the fact that the minimum is assumed because $a(x)$ exists for all $x \in \Omega$: the strict convexity of D^* implies that this minimum is unique.

In the set $S = \Omega \setminus J$ where $\mathcal{B}(x)$, $\mathcal{D}(x)$ are single valued, we will use the notation

$$\mathcal{B}(x)\Big|_S = b(x), \quad \mathcal{D}(x)\Big|_S = d(x). \tag{4.4}$$

An important consequence of the continuity property of $\mathcal{B}$ (Proposition 3.1) and Corollary 4.1 is that we have some stability of the vector d w.r.t. perturbation of the boundary data, of the set D and approximation by smooth vector fields.

Proposition 4.4. *The function $x \to d(x)$ is continuous w.r.t. the inherited topology on the differentiability set of u. Moreover, the following holds:*

1. *If $u_i(\partial\Omega) \to u(\partial\Omega)$ in $L^\infty(\partial\Omega)$, then $d_i(x) \to d(x)$ $\mathcal{H}^n$ a.e. in Ω, where $d_i/|d_i|_{D^*} = \delta D(\nabla u_i)$.*
2. *If ρ_ϵ is a convolution kernel, then $\rho_\epsilon * d$ converges to d $\mathcal{H}^n$ a.e. in Ω.*
3. *If D_i is a sequence of convex sets converging to D w.r.t. the Hausdorff distance, with D_i^* strictly convex and $D \subset D_i$, then the vector field $d_i(x)$ corresponding to the solution u_i to*

$$\inf_{\bar{u}+W_0^{1,\infty}(\Omega)} \int_\Omega (\mathbb{I}_{D_i}(\nabla u) + g(u))dx, \tag{4.5}$$

converges to the vector field d corresponding to u $\mathcal{H}^n$ a.e. in Ω.

Proof. From the continuity of the set valued map $\mathcal{D}$ and the fact that for D^* strictly convex $\mathcal{D}(x)$ is single valued $\mathcal{H}^n$ a.e. in Ω, the continuity on the differentiability set follows, as well as the second point.

For point (1) and (3), one has only to repeat the proof of Proposition 3.1 to obtain that, fixed $x \in \Omega$, $\epsilon > 0$, then

$$\mathcal{D}_i(x) \subset \mathcal{D}(x) + B(0,\epsilon),$$

for $i \geq i(x,\epsilon) \gg 1$, where $\mathcal{D}_i$ is the set corresponding to u_i is both cases. □

4.1. **Approximation of the vector field d.** In the following we will use as a fundamental tool the following construction, i.e. the possibility to approximate d with vector field d_i which can be studied more easily.

Example 4.5. Consider the functions

$$u_I(x) = \max\biggl\{u(\bar{x}_i) - |\bar{x}_i - x|_{D^*} : i = 1,\dots,I, \alpha x + (1-\alpha)\bar{x}_i \in \bar{\Omega}\ \forall\alpha \in [0,1]\biggr\}. \tag{4.6}$$

for a dense sequence of points $\{\bar{x}_i\}_{i=1}^\infty$ in $\partial\Omega$ (it suffices that $\{\bar{x}_i\}_{i=1}^\infty$ is dense in $\cup_x\mathcal{B}(x)$, or in a selection $\cup_x b(x)$, with $b(x) \in \mathcal{B}(x)$).

We can split the set Ω into at most I open regions Ω_i, $i = 1, \dots, I$, which are defined by

$$\Omega_i = \text{interior of } \Big\{x : u_I(x) = u(\bar{x}_i) - |\bar{x}_i - x|_{D^*}\Big\}, \tag{4.7}$$

together with the negligible set

$$\begin{aligned} J_I &= \bigcup_{i \neq j} \Big(\bar{\Omega}_i \cap \bar{\Omega}_j\Big) \\ &= \Big\{x : \exists i, j,\ i \neq j, u_I(x) = u(\bar{x}_i) - |\bar{x}_i - x|_{D^*} = u(\bar{x}_j) - |\bar{x}_j - x|_{D^*}\Big\}. \end{aligned}$$

In fact, J_I is a countably $n-1$ rectifiable set by Corollary 4.1: a simple argument shows that it is actually the image of a finite number (at most $I(I-1)/2$) of Lipschitz functions from $\mathbb{R}^{n-1}$ into Ω, since the number of points $\bar{x}_i$ is finite. In each open region Ω_i, the function $d_I(x)$ is given by

$$d_I(x) = \frac{x_i - x}{|x_i - x|}.$$

Its divergence can be written as

$$\operatorname{div} d_i = (\operatorname{div} d_i)_{\mathrm{s}} + (\operatorname{div} d_i)_{\mathrm{a.c.}}, \tag{4.8}$$

with $(\operatorname{div} d_i)_{\mathrm{s}}$ supported on J_I and positive, while using the explicit form of d_i and the fact that $\mathcal{H}^n(J_I) = 0$ it follows

$$(\operatorname{div} d_i)_{\mathrm{a.c.}}(x) \geq -\frac{n-1}{\operatorname{dist}(x, \partial\Omega)}.$$

4.2. **Basic estimates.** The vector field $d(x)$ enjoys good properties. The first property is that, while it is not BV, its divergence is still a measure. The strong convergence of d_i to d yields the following result.

Proposition 4.6. *The divergence of the vector field $d(x)$ is a locally finite Radon measure, satisfying*

$$\operatorname{div} d + \frac{n-1}{\operatorname{dist}(\Omega', \partial\Omega)} \mathcal{H}^n \geq 0 \tag{4.9}$$

for all $x \in \Omega' \subset\subset \Omega$. Moreover, we have the estimate

$$|\operatorname{div} d|(B(x,r)) \leq |\partial B(0,r)| + \frac{2(n-1)|B(x,r)|}{\operatorname{dist}(B(x,r), \partial\Omega)}, \quad B(x,r) \subset\subset \Omega. \tag{4.10}$$

Finally, the singular part is positive in Ω.

Proof. The first inequality follows by the convergence of d_I to d pointwise (Proposition 4.4), and the fact that positive definite distributions are positive locally finite Radon measures ([6], page 29, Theorem 5).

The inequality (4.9) implies that the singular part is positive. In fact, for a test function $\phi \in C_c^\infty(\Omega)$, consider for each $\epsilon > 0$ a cut-off function $\psi_\epsilon \in C_c^\infty(\Omega)$, supported in an ϵ neighbourhood of $J \cap \operatorname{supp}(\phi)$ and which is identically 1 on $J \cap \operatorname{supp}(\phi)$. Then for $\phi_\epsilon = \phi\psi_\epsilon$, one has $\langle (\operatorname{div} d)_s, \phi_\epsilon \rangle = \langle (\operatorname{div} d)_s, \phi \rangle$, while the absolutely continuous terms in (4.9) tend to zero as $\epsilon \to \infty$.

It is clear that since $d \in L^\infty(\Omega)$, for a.e. $r \in (0, \infty)$

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{B(x,r+\delta) \setminus B(x,r)} d(y) \cdot \frac{y}{|y|} dy = \int_{\partial B(x,r)} d(y) \cdot \frac{y}{|y|} dy \in L^\infty(\mathbb{R}),$$

so that we can write by taking the limit of the test function $\rho_\delta * B(x,r)$

$$\begin{aligned} \operatorname{div} d(B(x,r)) &= \operatorname{div} d^+(B(x,r)) - \operatorname{div} d^-(B(x,r)) \\ &= \int_{\partial B(x,r)} d(y) \cdot \frac{y}{|y|} dy \leq |\partial B(x,r)| \end{aligned}$$

for a.e. $r > 0$. From the first estimate (4.9), we have

$$\operatorname{div} d^-(B(x,r)) \leq \frac{(n-1)|B(x,r)|}{\operatorname{dist}(B(x,r), \partial\Omega)},$$

so that (5.13) follows. □

We next study the push forward of the $\mathcal{H}^{n-1}$ dimensional Lebesgue measure on hyperplanes by the vector field d. Let $x \in \Omega$ such that, and by translation and rotation we assume

$$\bar{x} = 0, \qquad d(0) = e_1 = (1,0), \quad e_1 \cdot a(0) \leq -h^-, \ e_1 \cdot b(0) \geq h + h^+.$$

where we denote with e_i the unit vector along the i-th coordinate axis.

Let K_ϵ be a compact subset of $B(0,r) \subset \Omega$ such that

$$d|_{K_\epsilon}, a|_{K_\epsilon}, b|_{K_\epsilon} \text{ are continuous}, \quad e_1 \cdot d|_{K_\epsilon} \geq 1-\epsilon, \ e_1 \cdot a|_{K_\epsilon} \leq -h^- + \epsilon, \ e_1 \cdot b|_{K_\epsilon} \geq h + h^+ - \epsilon. \tag{4.11}$$

We assume that $\mathcal{H}^n(K_\epsilon) > 0$, otherwise the next inequalities are trivially satisfied.

Define the compact set

$$\mathcal{Z}_\epsilon = \Big\{ (1-s)a(x) + sb(x), x \in K_\epsilon, s \in [0,1] \Big\}.$$

This set is compact because a, b are continuous on K_ϵ. Define the slices

$$Z_\epsilon(s) = \mathcal{Z}_\epsilon \cap \{e_1 \cdot x = s\}$$

and the $n-1$ dimensional vector field $d^\perp(t,y)$ by

$$\frac{d(t,y)}{e_1 \cdot d(t,y)} = \big(1, d^\perp(t,y)\big).$$

Lemma 4.7. *For all $0 \leq s \leq t \leq h$ the following estimate holds*

$$\left(\frac{h + h^+ - \epsilon - t}{h + h^+ - \epsilon - s} \right)^{n-1} \mathcal{H}^{n-1}(Z_\epsilon(s)) \leq \mathcal{H}^{n-1}(Z_\epsilon(t)) \leq \left(\frac{t + h^- - \epsilon}{s + h^- - \epsilon} \right)^{n-1} \mathcal{H}^{n-1}(Z_\epsilon(s)). \tag{4.12}$$

Proof. Let d_I be the vector field constructed as in Example 4.5, by taking only a dense sequence in $B = \cup_{y \in Z_\epsilon} b(y)$. This vector field d_I is single valued outside the $\mathcal{H}^n$ negligible set J_I: since the points b_i, $i = 1, \dots, I$, are on the same side of $Z_\epsilon(s)$ for all $s \in [0,h]$, then

$$\mathcal{H}^{n-1}(Z_\epsilon(s) \cap J_I) = 0$$

for all $I \in \mathbb{N}$, $s \in [0,h]$.

By Proposition 4.4, $\mathcal{D}_i$ converges to d pointwise on $\mathcal{Z}_\epsilon$, because d is single valued on $\mathcal{Z}_\epsilon$: in particular, since the set $Z_\epsilon(s) \setminus J_I$ is of full $\mathcal{H}^{n-1}$ measure for all $I \in \mathbb{N}$, d_i converges to d $\mathcal{H}^{n-1}$ a.e. on $Z_\epsilon(s)$, for all $s \in [0,h]$.

Fixed $\bar{s}$, by Egoroff and Lusin theorems, we can assume that d_I, b_I are continuous and converge uniformly to d on a compact set $A_\eta(\bar{s}) \subset Z_\epsilon(\bar{s})$ with $\mathcal{H}^{n-1}(A_\eta(\bar{s})) \geq \mathcal{H}^{n-1}(Z_\epsilon(\bar{s})) - \eta$. This implies in particular that d_I is single valued on $A_\eta(\bar{s})$, i.e. $J_I \cap A_\eta(\bar{s}) = \emptyset$ for all $I \in \mathbb{N}$.

If we define the vector field

$$\frac{d_I(x)}{e_1 \cdot d(x)} = (1, d_I^\perp(x)),$$

and the compact sets

$$\mathcal{A}_{\eta,I} = \Big\{ (t, y + (t-\bar{s})d_I^\perp(y)), t \geq \bar{s}, (\bar{s}, y) \in A_\eta \Big\}, \ A_{\eta,I}(t) = \mathcal{A}_{\eta,I} \cap \{e_1 \cdot x = t\},$$

then from $d_I^\perp \to d^\perp$ uniformly on A_η, the sets $\mathcal{A}_{\eta,I}$, $A_{\eta,I}(t)$ converge in Hausdorff metric to

$$\mathcal{A}_\eta = \Big\{ (t, y + (t-\bar{s})d^\perp(y)), t \geq \bar{s}, (\bar{s}, y) \in A_\eta(\bar{s}) \Big\}, \ A_\eta(t) = \Big\{ (t, y + (t-\bar{s})d^\perp(y)), (\bar{s}, y) \in A_\eta(\bar{s}) \Big\}.$$

Moreover, from the explicit form of the vector field d_I we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}^{n-1}(A_{\eta,I}(t)) &= \sum_{i=1}^I \int_{A_{\eta,I}(t)\cap\Omega_i} \frac{1-n}{e_1 \cdot b_i - t} d\mathcal{H}^{n-1}(y) = \sum_{i=1}^I \frac{1-n}{e_1 \cdot b_i - t} \mathcal{H}^{n-1}(A_{\eta,I}(t) \cap \Omega_i) \\
&\geq \frac{1-n}{h + h^+ - \epsilon - t} \sum_{i=1}^I \mathcal{H}^{n-1}(A_{\eta,I}(t) \cap \Omega_i) = \frac{1-n}{h + h^+ - \epsilon - t} \mathcal{H}^{n-1}(A_{\eta,I}(t)),
\end{aligned}$$

so that

$$\mathcal{H}^{n-1}(A_{\eta,I}(t)) \geq \left(\frac{h + h^+ - \epsilon - t}{h + h^+ - \epsilon - \bar{s}} \right)^{n-1} \mathcal{H}^{n-1}(A_{\eta,I}(\bar{s})) = \left(\frac{h + h^+ - \epsilon - t}{h + h^+ - \epsilon - \bar{s}} \right)^{n-1} \mathcal{H}^{n-1}(A_\eta(\bar{s})).$$

By the Hausdorff convergence of the compact set $A_{\eta,I}(t)$ to the compact set $A_\eta(t)$, we obtain that

$$\limsup_{I\to\infty} \mathcal{H}^{n-1}(A_{\eta,I}(t)) \leq \mathcal{H}^{n-1}(A_\eta(t)) \leq \mathcal{H}^{n-1}(Z_\epsilon(t)),$$

so that by letting $\eta \to 0$, the above inequality holds also for the limit $Z_\epsilon(t)$, for all $t \geq \bar{s}$.

Repeating the computation by using a dense sequence in $\cup_{x\in K_\epsilon} a(x)$ and using (4.3), one obtains the symmetric inequality for $t \leq \bar{s}$:

$$\mathcal{H}^{n-1}(Z_\epsilon(s)) \geq \left(\frac{s+h^- - \epsilon}{t+h^- - \epsilon}\right)^{n-1} \mathcal{H}^{n-1}(Z_\epsilon(t)).$$

□

5. The general case

This section is devoted to finding a measurable selection of the set valued $\mathcal{D}(x)$ such that we can prove the same regularity estimates of the previous section.

We consider a sequence of strictly convex sets D_i^* converging to D^*: natural candidates are the sets D_i obtained by the inf-convolution,

$$D_i = D \square \frac{1}{i} K = D + \frac{1}{i} K = \Big\{ x : \exists x_1 \in D, x_2 \in K, x = x_1 + \frac{1}{i} x_2 \Big\}, \tag{5.1}$$

where $K \subset \mathbb{R}^n$ is a convex bounded subset containing the origin and such that K^* is strictly convex.

Remark 5.1. To prove the results of this section, it is sufficient to have $(D+K)^*$ strictly convex, i.e. we just need to assume that $D+K$ is smooth.

By construction, D_i is smooth and its Legendre transform is

$$|x|_{D_i^*} = |x|_{D^*} + \frac{1}{i}|x|_{K^*}. \tag{5.2}$$

We thus have that D_i^* is strictly convex, and $D_i^* \subset D^*$. By computations similar to the proof of Proposition 3.1, it follows that for all $x \in \Omega$, $\epsilon > 0$,

$$\mathcal{D}_i(x) \subset \mathcal{D}(x) + B(0,\epsilon), \qquad \mathcal{B}_i(x) \subset \mathcal{B}(x) + B(0,\epsilon), \tag{5.3}$$

if $i \geq i(\epsilon, x)$ sufficiently large. We now show that the set valued function defined as

$$\mathcal{K}(x) = \Big\{ b \in \mathcal{B}(x) : |b-x|_{K^*} \text{ is minimal} \Big\} \tag{5.4}$$

is single valued $\mathcal{H}^n$ a.e. in Ω.

Lemma 5.2. *The set valued function $\mathcal{K}(x)$ is measurable.*

Proof. Let C be a closed subset of $\partial\Omega$, and consider

$$\mathcal{K}^{-1}(C) = \Big\{ x \in \Omega : \exists y \in C \cap \mathcal{B}(x) \text{ such that } |y-x|_{K^*} = \min_{b\in\mathcal{B}(x)} |b-x|_{K^*} \Big\},$$

We obtain $\mathcal{K}^{-1}(C)$ by countable operations of measurable sets. For $i, k \in \mathbb{N}$, define the sets

$$\begin{aligned}
A_{ik}(C) &= \Big\{x \in \Omega : \exists y \in C \cap \mathcal{B}(x) \text{ such that } |y-x|_{K^*} \in [2^{-i}k, 2^{-i}(k+1)]\Big\}, \\
B_{ik} &= \Big\{x \in \Omega : \exists z \in \mathcal{B}(x) \text{ such that } |z-x|_{K^*} \le 2^{-i}k\Big\} \\
C_{ik}(C) &= A_{ik}(C) \setminus B_{ik} \\
&= \Big\{x \in \Omega : \exists y \in C \cap \mathcal{B}(x) \text{ such that } |y-x|_{K^*} \in (2^{-i}k, 2^{-i}(k+1)] \\
&\qquad \text{and } \forall z \in \mathcal{B}(x)\ |x-z|_{K^*} > 2^{-i}k\Big\}, \\
D_i(C) &= \bigcup_k C_{ik}(C) \\
&= \Big\{x \in \Omega : \exists k \in \mathbb{N}, \exists y \in C \cap \mathcal{B}(x) \text{ such that } |x-y|_{K^*} \in (2^{-i}k, 2^{-i}(k+1)], \\
&\qquad \text{and } \forall z \in \mathcal{B}(x)\ |x-z|_{K^*} > 2^{-i}k\Big\} \\
&\subset \Big\{x \in \Omega : \exists y \in C \cap \mathcal{B}(x) \text{ such that } \forall z \in \mathcal{B}(x)\ |y-x|_{K^*} < |z-x|_{K^*} + 2^{-i}\Big\} \\
&=: \tilde{D}_i(C).
\end{aligned}$$

On the one hand,

$$\mathcal{K}^{-1}(C) \subset D_i(C) \quad \text{for all } i.$$

On the other hand,

$$\begin{aligned}
\bigcap_i \tilde{D}_i(C) &= \Big\{x \in \Omega : \forall i\ \exists y_i \in C \cap \mathcal{B}(x)\ \forall z \in \mathcal{B}(x)\ |y_i - x|_{K^*} < |z-x|_{K^*} + 2^{-i}\Big\} \\
&= \Big\{x \in \Omega : \exists y \in C \cap \mathcal{B}(x)\ \forall z \in \mathcal{B}(x)\ |y-x|_{K^*} \le |z-x|_{K^*}\Big\},
\end{aligned}$$

because $C \cap \mathcal{B}(x)$ is compact. So we have

$$\bigcap_i D_i(C) = \bigcap_i \tilde{D}_i(C) = \mathcal{K}^{-1}(C).$$

The sets B_{ik} are closed, due to the upper semicontinuity of $\mathcal{B}$.

It remains to prove that $A_{ik}(C)$ is measurable. For an interval $[a,b] \subset \mathbb{R}$, consider a set of the form

$$A = \Big\{x \in \Omega : \exists y \in C \cap \mathcal{B}(x)\ |y-x|_{K^*} \in [a,b]\Big\}.$$

The set

$$\begin{aligned}
\Gamma &= \Big\{(x,y,z) \in \Omega \times \partial\Omega \times \mathbb{R} : y \in \mathcal{B}(x), z = |y-x|_{K^*}\Big\} \\
&= \Big\{(x,y,z) \in \Omega \times \partial\Omega \times \mathbb{R} : (x,y) \in \operatorname{graph}(\mathcal{B}), z = |y-x|_{K^*}\Big\}
\end{aligned}$$

is the graph of the continuous function $|\cdot|$ on the closed set graph$(\mathcal{B})$, thus it is closed. Since Ω is bounded, Γ is compact. Thus

$$A = \Big\{x \in \Omega : \exists (y,z) \in \partial\Omega \times \mathbb{R}\ (x,y,z) \in \Gamma \cap (\Omega \times C \times [a,b])\Big\}$$

is the projection of a compact set, and is itself compact. □

Lemma 5.3. *Let $x \in \Omega$, $y \in \mathcal{K}(x)$. Then $\mathcal{K}$ is single valued on the segment*

$$]x,y[= \big\{tx + (1-t)y : 0 < t < 1\big\}.$$

Proof. Suppose by contradiction that there exists $x' \in]x,y[$ such that $\mathcal{K}(x')$ is not single valued. Clearly, $y \in \mathcal{B}(x')$. Let $y' \in \mathcal{K}(x')$, $y' \neq y$.

1) Assume that the directions of $y - x'$ and $y' - x'$ lie on the same extremal face of D^*. If $[x, y'[\subset \Omega$, then

$$\begin{aligned} u(x) &= \bar{u}(y) - |y - x|_{D^*} = \bar{u}(y) - |y - x'|_{D^*} - |x' - x|_{D^*} \\ &= u(x') - |x' - x|_{D^*} = \bar{u}(y') - |y' - x'|_{D^*} - |x' - x|_{D^*} = \bar{u}(y') - |y' - x|_{D^*}, \end{aligned}$$

i.e. $y' \in \mathcal{B}(x)$. But by the strict convexity of K^*, we have

$$|y' - x|_{K^*} < |y' - x'|_{K^*} + |x' - x|_{K^*} \leq |y - x'|_{K^*} + |x' - x|_{K^*} = |y - x|_{K^*},$$

in contradiction to $y \in \mathcal{K}(x)$.

2) If $[x, y'[\not\subset \Omega$, one considers the shortest convex curve γ connecting x to y' in the intersection of $\overline{\Omega}$ with the triangle with vertices $\{x, x', y'\}$. All tangent vectors $\dot{\gamma}/|\dot{\gamma}|_{D^*}$ of this curve belong to the same convex face of D^*. It thus follows that, if z is the closest point of $\gamma \cap \partial\Omega$ to x ($z \neq x$, otherwise we are in case 1), then z is in $\mathcal{B}(x)$ and it is closer to x than y.

3) Assume now that the directions of $y - x'$ and $y' - x'$ lie on different extremal faces of D^*. If $[x, y'[\subset \Omega$, then by a similar computation to 1),

$$u(x) = \bar{u}(y') - |y' - x'|_{D^*} - |x' - x|_{D^*} < \bar{u}(y') - |y' - x|_{D^*},$$

in contradiction to Proposition 2.1.

4) If $[x, y'[\not\subset \Omega$, one considers a point $z \in]x', y'[$ such that $[x, z] \subset \Omega$. By analogous computations to the previous steps, one gets

$$u(x) = u(z) - |z - x'|_{D^*} - |x' - x|_{D^*} < u(z) - |z - x|_{D^*},$$

which again contradicts Proposition 2.1. $\square$

Proposition 5.4. *The set valued function $\mathcal{K}$ is single valued $\mathcal{H}^n$ a.e. in Ω.*

Proof. 1) Let $\tilde{J} \subset \Omega$ be the set where $\mathcal{K}$ is not single valued, and assume by contradiction $\mathcal{H}^n(\tilde{J}) > 0$. Cover $\partial\Omega$ with a finite family of balls $\{B(y_j, \rho)\}$, $y_j \in \partial\Omega$, $\rho > 0$, and let

$$\tilde{J}_j = \tilde{J} \cap \mathcal{K}^{-1}(\partial\Omega \cap B(y_j, \rho)).$$

By assumption, at least one of the $\tilde{J}_j$ is not negligible. Let x be a density point of $\tilde{J}_j$. Assume for simplicity $x = 0$ and $(x - y_j)/|x - y_j| = e_1$. From the definition of density point it follows that

$$\frac{1}{\kappa r^{n-1}} \mathcal{H}^{n-1}\Big\{\tilde{J}_j \cap B(te_1, r) \cap \{e_1 \cdot x = t\}\Big\} \geq 1 - \epsilon \tag{5.5}$$

for $t \in A \subset [-r, r]$ with $\mathcal{H}^1(A) \geq 2r(1 - \epsilon)$, for r sufficiently small. Consider now a dense countable sequence $\{b_i\}_{i \in \mathbb{N}}$ in $\partial\Omega \cap B(y_j, \rho)$, and let u_I be the solution to

$$u_I(x) = \sup\Big\{\bar{u}(y) - |y - x|_{D_I^*}, y \in \{b_i\}_{i=1}^I\Big\}. \tag{5.6}$$

Then by the explicit formula for the solution, $u_I \to u$ pointwise on $\tilde{J}_j$.

2) We now prove that for any $\epsilon > 0$

$$\mathcal{B}_I(x) \subset \mathcal{K}(x) + B(0, \epsilon) \quad \text{for all } I \geq J(\epsilon, x).$$

Let $y \in \partial\Omega \setminus (\mathcal{K}(x) + B(0, \epsilon))$. We need to prove

$$u_I(x) > \bar{u}(y) - |y - x|_{D_I^*}$$

for $I \geq J(\epsilon, x)$.

By definition of $\mathcal{K}(x)$, there exists η such that

$$|y - x|_{K^*} > 2\eta + |\tilde{y} - x|_{K^*}$$

for all $y \in \mathcal{B}(x) \setminus (\mathcal{K}(x) + B(0, \epsilon))$, $\tilde{y} \in \mathcal{K}(x)$. Since $y \mapsto |y - x|_{K^*}$ is Lipschitz continuous, then there exists ϵ' such that

$$|y - x|_{K^*} > \eta + |\tilde{y} - x|_{K^*}$$

for all $y \in (\mathcal{B}(x) + B(0, 2\epsilon')) \setminus (\mathcal{K}(x) + B(0, \epsilon))$, $\tilde{y} \in \mathcal{K}(x)$.

By definition of $\mathcal{B}(x)$, there exists η' such that

$$|y - x|_{D^*} - \bar{u}(y) > \eta' + |\tilde{y} - x|_{D^*} - \bar{u}(\tilde{y})$$

for all $y \in \partial\Omega \setminus (\mathcal{B}(x) + B(0,\epsilon'))$, $\tilde y \in \mathcal{K}(x)$.

We thus have for $y \in \partial\Omega \setminus (\mathcal{B}(x) + B(0,\epsilon'))$, $\tilde y \in \mathcal{K}(x)$,

$$\begin{aligned}\bar u(y) - |y-x|_{D^*} - \frac{1}{I}|y-x|_{K^*} &< \bar u(\tilde y) - |\tilde y - x|_{D^*} - \eta' - \frac{1}{I}|y-x|_{K^*} \\ &\leq u_I(x) - \eta + \frac{1}{I}\mathrm{diam}_{K^*}(\Omega) \leq u_I(x) - \frac{\eta}{2}\end{aligned}$$

for $I \gg 1$ (so that there is a point b_i sufficiently close to $\mathcal{K}(x)$). For $y \in (\mathcal{B}(x)+B(0,\epsilon')) \setminus (\mathcal{K}(x)+B(0,\epsilon))$

$$\bar u(y) - |y-x|_{D^*} - \frac{1}{I}|y-x|_{K^*} \leq \bar u(\tilde y) - |\tilde y - x|_{D^*} - \frac{2\eta}{I} - \frac{1}{I}|\tilde y - z|_{K^*} < u_I(x) - \frac{\eta}{I}$$

for $I \gg 1$.

3) Let $t \in [-r,r]$ be such that (5.5) holds, and consider a compact set K_ϵ such that

$$K_\epsilon \subset \tilde J_j \cap B(te_1, r) \cap \{e_1 \cdot x = t\}, \quad \mathcal{H}^{n-1}(K_\epsilon) \geq (1-2\epsilon) r^{n-1}\kappa,$$

and the vector fields $d_I(x)$ are continuous (hence single valued) on K_ϵ for all $I \in \mathbb{N}$. Let $\mathcal{Z}_I$ be the compact set

$$\mathcal{Z}_I = \bigcup_{x \in K_\epsilon} [x, b_I(x)]$$

The same analysis of Lemma 4.7 yields the estimates

$$\mathcal{Z}_I \cap \{e_1 \cdot x \in [0, r\delta]\} \subset B(0, r(1+\epsilon)), \quad \mathcal{H}^n\Big\{\mathcal{Z}_I \cap \{e_1 \cdot x \in [0,r]\}\Big\} \geq (1-3\epsilon) r^n, \tag{5.7}$$

for $r \ll 1$, δ fixed sufficiently small. Since $b_i(x) \to \mathcal{K}(x)$ pointwise, by Egorov we can choose the compact K_ϵ such that the convergence on it is uniform, so by passing to the limit,

$$\mathcal{H}^{n-1}\Big\{\cup_{x\in K_\epsilon}[x, \mathcal{K}(x)] \cap \{e_1 \cdot x = t\}\Big\} \geq (1-3\epsilon) r^{n-1}\kappa,$$

for $t \in [0, r\delta]$, $r \ll 1$, δ fixed sufficiently small.

However this is in contradition with the estimate (5.5) and Lemma 5.3, because for $r \ll 1$ it implies that $[x, \mathcal{K}(x)] \cap \tilde J_j \cap \{e_1 \cdot x = t\} \neq \emptyset$ for $t \in A \cap [0,r]$. ☐

Let us define

$$\mathfrak{D}(x) = \bigg\{\frac{y-x}{|y-x|}, y \in \mathcal{K}(x)\bigg\}. \tag{5.8}$$

We have the corollary:

Corollary 5.5. *The set valued function $\mathfrak{D}(x)$ is single valued in $\Omega \setminus \tilde J$: let $\mathfrak{D}(x) = \{d(x)\}$, for $x \notin \tilde J$. Then the function $d_I \in L^\infty(\Omega, \partial B(0,1))$ converges pointwise to $d \in L^\infty(\Omega, \partial B(0,1))$ $\mathcal{H}^n$ a.e. in Ω.*

Also in this case, we can consider the set of lines

$$\Sigma(x) = \bigcup_{d \in \mathfrak{D}(x)} \Big\{x + td : t \in \mathbb{R}, u(x+td) = u(x) + t|d|_{D^*}\Big\}. \tag{5.9}$$

This set reduces to a segment $\mathcal{H}^n$ a.e.: when $\Sigma(x)$ is not a segment, then $x \in \tilde J$ is one of the end points, because by the proof of Lemma 5.3 we have shown that on $x + td(x)$, $d \in \mathfrak{D}(x)$, $t > 0$, the set valued function $\mathfrak{D}$ is single valued ($x + td(x)$, $t > 0$ cannot intersect J for the same reasons of the strictly convex case). We can introduce the function

$$x \to \mathfrak{A}(x) = \Big\{x + td(x), t = \inf\{s : \mathfrak{D}(x) \subset \mathfrak{D}(x + sd(x))\}, d \in \mathfrak{D}(x)\Big\}. \tag{5.10}$$

Since $\mathfrak{D}$ is Borel measurable, then also $\mathfrak{A}(x)$ is: if $x \in \tilde J$, then $\mathfrak{A}(x) = \{x\}$ so that $\mathfrak{A}(x)$ is single valued on Ω. In the following we will denote with $b(x)$, $d(x)$, $a(x)$ the single valued representatives of $\mathcal{K}(x)$, $\mathfrak{D}(x)$ (defined outside $\tilde J$) and of $\mathfrak{A}(x)$ (defined everywhere in Ω).

The definition of $a(x)$ implies that $a(a(x)) = a(x)$. As before, we will denote by $\hat J$ the set of the initial points of Σ:

$$\hat J = \bigcup_{x \in \Omega} a(x). \tag{5.11}$$

5.1. **Analysis of the vector field d in the general case.** Using the $\mathcal{H}^n$ a.e. pointwise convergence of d_i to d and following the same proof of Proposition 4.6, we can prove that

Proposition 5.6. *The divergence of the vector field $d(x)$ is a locally finite Radon measure, satisfying*

$$\text{div } d + \frac{n-1}{\text{dist}(\Omega', \partial\Omega)} \geq 0 \tag{5.12}$$

for all $x \in \Omega' \subset\subset \Omega$. Moreover, we have the estimate

$$|\text{div } d|(B(x,r)) \leq |\partial B(0,r)| + \frac{2(n-1)|B(x,r)|}{\text{dist}(B(x,r), \partial\Omega)}, \quad B(x,r) \subset\subset \Omega. \tag{5.13}$$

Finally, the singular part is strictly positive in Ω.

Proof. The idea of the proof is that the above estimates holds for the approximating sequence d_i, so that they pass to the limit $\text{div}\, d_i \rightharpoonup \text{div}\, d$. □

We prove that also in this case a uniform estimate on the push forward of the $n-1$ dimension Lebesgue measure by the map $x \to x + td(x)$ holds. Let $\bar{x}$ be a Lebesgue point of d, b, a: without any loss of generality we assume

$$\bar{x} = 0, \qquad d(0) = e_1 = (1, 0),$$

where as before we denote with e_i the unit vector along the i-th coordinate axis.

Consider the Borel measurable sets

$$Z(0) = B(0,r) \cap \{e_1 \cdot y = 0\} \cap \big\{e_1 \cdot a(y) \leq -h^-, e_1 \cdot b(y) \geq h + h^+\big\}, \tag{5.14}$$

such that b, d, a are continuous on $Z(0)$. Define

$$\mathcal{Z} = \Big\{x = y + td(y), y \in Z(0), t \in [e_1 \cdot a(y), e_1 \cdot b(y)]\Big\}, \quad Z(t) = \mathcal{Z} \cap \{e_1 \cdot x = t\}. \tag{5.15}$$

Since d, a, b are continuous, $\mathcal{Z}$ is compact: it may as well happen that $\mathcal{H}^{n-1}(Z(0)) = 0$ if $h^- > 0$, which means that $a(x) = x$ $\mathcal{H}^n$ a.e. in Ω. However for $h^- = 0$ we can assume that $\mathcal{H}^{n-1}(Z(0)) > 0$.

Define the vector field

$$d(x) = e_1 \cdot d(x)(1, d^\perp(x)), \quad d^\perp(x) \in R^{n-1}. \tag{5.16}$$

We now show that $(\mathbb{I} + td^\perp(0,\cdot))_\sharp \mathcal{H}^{n-1}|_{Z(0)}$, $t \geq 0$, remains equivalent to $\mathcal{H}^{n-1}|_{Z(t)}$.

Lemma 5.7. *For all $0 \leq s \leq t \leq h$ the following estimate holds*

$$\bigg(\frac{h + h^+ - t}{h + h^+ - s}\bigg)^{n-1} \mathcal{H}^{n-1}(Z(s)) \leq \mathcal{H}^{n-1}(Z(t)) \leq \bigg(\frac{t + h^-}{s + h^-}\bigg)^{n-1} \mathcal{H}^{n-1}(Z(s)). \tag{5.17}$$

Proof. The proof follows the same steps of the proof of Lemma 4.7 for the first inequality, because of the pointwise convergence of the vector field d_I, corresponding to the solution

$$u_I(x) = \sup\bigg\{u(b_i) - |b_i - x|_{D_I^*}, i = 1, \dots, I, (1-s)x + sb_i \in \bar{\Omega}, s \in [0,1]\bigg\}, \tag{5.18}$$

to the vector field d on $\mathcal{Z} \setminus \mathfrak{A}$: this is the consequence of point 1) of the proof of Proposition 5.4. As before b_i is a dense sequence in $\partial\Omega$.

In the second inequality we can assume $s = 0$ and (restricting $Z(0)$ in case) than

$$e_1 \cdot a(y) \leq -h^- < 0, \quad h + h^+ \leq e_1 \cdot b(y) \leq h + h^+ + \epsilon.$$

Moreover, again by restricting $Z(0)$, we can assume that for $\epsilon > 0$

$$e_1 \cdot \Big(b(y) - (h + h^+ + h^-)d(y)\Big) \leq -h^- + 2\epsilon.$$

For simplicity, we assume also that Ω is convex. The general case can be treated as in the proof of the third point of Proposition 5.4.

Define the new set $\mathtt{A}$ of initial data $\mathtt{a}(y)$ as

$$\mathtt{A} = \Big\{\mathtt{a}(y) = b(y) - (h + h^+ + h^-)d(y), y \in Z(0)\Big\}.$$

The set

$$\mathtt{A}(x) = \Big\{\mathtt{a}(y) : u(x) = u(\mathtt{a}(y)) + |x - \mathtt{a}(y)|_{D^*}, |\mathtt{a}(y) - x| \text{ is minimal}\Big\} \tag{5.19}$$

is single valued on the set of lines

$$\Sigma = \bigcup_{y \in Z(0)} \Sigma(y) = \bigcup_{y \in Z(0)} \big[\mathtt{a}(y), b(y)\big]. \tag{5.20}$$

In fact, if $\mathtt{a}(y_1), \mathtt{a}(y_2) \in \mathtt{A}(x)$, with $x \in [\mathtt{a}(y_1), b(y_1)]$, then by the explicit formula of the solution

$$\begin{aligned} u(b(y_1)) = u(x) + |b(y_1) - x|_{D^*} &= u(\mathtt{a}(y_2)) + |x - \mathtt{a}(y_2)|_{D^*} + |b(y_1) - x|_{D^*} \\ &\geq u(\mathtt{a}(y_2)) + |b(y_1) - \mathtt{a}(y_2)|_{D^*}. \end{aligned}$$

Thus from the convexity of Ω, it follows that $b(y_1), b(y_2) \in \mathcal{B}(\mathtt{a}(y_2))$ and since $b(y_2) \in \mathfrak{B}(\mathtt{a}(y_2))$,

$$\begin{aligned} h + h^- + h^+ &= |b(y_2) - \mathtt{a}(y_2)| \leq |b(y_1) - \mathtt{a}(y_2)| \leq |b(y_1) - x| + |x - \mathtt{a}(y_2)| \\ &= |b(y_1) - x| + |x - \mathtt{a}(y_1)| = |b(y_1) - \mathtt{a}(y_1)| = h + h^- + h^+. \end{aligned} \tag{5.21}$$

It thus follows that $x - \mathtt{a}(y_2) = x - \mathtt{a}(y_1)$, i.e. $\mathtt{a}(y_2) = \mathtt{a}(y_1)$.

If $\{\mathtt{a}_i\}_{i \in \mathbb{N}}$ is a dense sequence in $\mathtt{A}$, define the functions

$$\begin{aligned} \mathtt{u}_I(x) &= \min \Big\{u(\mathtt{a}_i) + |x - \mathtt{a}_i|_{D^*} + \frac{1}{I}|x|, i = 1, \dots, I\Big\}, \\ \mathtt{u}(x) &= \min \Big\{u(\mathtt{a}(y)) + |x - \mathtt{a}(y)|_{D^*}, y \in Z(0)\Big\}, \end{aligned} \tag{5.22}$$

and the vector field

$$\mathtt{d}_I(x) = \frac{x - \mathtt{a}_i(x)}{|x - \mathtt{a}_i(x)|}, \tag{5.23}$$

where $\mathtt{a}_i(x)$ is the point where the minimum is reached in (5.22).

By construction $\mathtt{u} = u$ on $\mathcal{Z} \cap \{e_1 \cdot x \geq 0\}$, and $\mathtt{u}_I \to \mathtt{u}$ for all $x \in \Omega$. Moreover $d_i \to d$ $\mathcal{H}^{n-1}$ a.e. for all $Z(t)$, $t \geq 0$, by repeating the part 1) of the proof of Proposition 5.4 and showing that

$$\mathcal{D}_I(x) \subset \left\{\frac{x - \mathtt{a}}{|x - \mathtt{a}|}, \mathtt{a} \in \mathtt{A}(x)\right\} + B(0, \epsilon),$$

for $I \gg 1$.

One can thus repeat the analysis of Lemma 4.7 to obtain

$$\operatorname{div} \mathtt{d}_i, \operatorname{div} \mathtt{d} \leq \frac{n-1}{\operatorname{dist}(x, \mathtt{A})},$$

and

$$\mathcal{H}^{n-1}(Z(t)) \leq \left(\frac{t + h^- - \epsilon}{s + h^- - \epsilon}\right)^{n-1} \mathcal{H}^{n-1}(Z(s)).$$

Since ϵ can be taken as small as we want by restricting $Z(0)$, one obtains (5.17) by covering $Z(0)$ with a countable number of disjoint compact sets up to a set of 0 measure. □

A particular case is the estimate even when $h^- = 0$, i.e.

$$\left(\frac{h + h^+ - t}{h + h^+ - s}\right)^{n-1} \mathcal{H}^{n-1}(Z(s)) \leq \mathcal{H}^{n-1}(Z(t)). \tag{5.24}$$

This shows that the $\mathcal{H}^{n-1}$ measure will not shrink to 0 if the distance of $b(x)$ from the set $Z(s)$ is not 0. As a consequence, by using the same argument used in [4, 5] we have the following proposition, which shows that for $\mathcal{H}^n$ a.e. $x \in \Omega$ the function $a(x)$ does not coincide with x: in fact, the set $\hat{J} = \cup_{x \in \Omega} a(x)$ is negligible. Note that $\hat{J}$ is larger than the set J defined in (3.8).

Proposition 5.8. *The set*

$$\hat{J} = \bigcup_{x \in \Omega} a(x) \tag{5.25}$$

has Lebesgue measure 0.

Proof. The argument of the proof is the same used in part 5) of the proof of Proposition 5.4. If $\bar{x}$ is a Lebesgue point for $A = \cup_{x\in\Omega} a(x)$ and a Lebesgue point for d, then one considers the images of the intersection of A with the planes orthogonal to d. By using (5.24) and the fact that d is close to $d(0)$ on a set of large measure, one obtains that the number of planes orthogonal to $d(0)$ which have intersection with A of $\mathcal{H}^{n-1}$ positive measure are countable, so that by Fubini one reaches a contradiction. □

This proposition shows that for $\mathcal{H}^n$ a.e. $x \in \Omega$ we can take the set $Z(0)$ with positive $\mathcal{H}^{n-1}$ measure and with $h^- > 0$.

6. Properties of the sets $\mathcal{Z}$

The results of Lemma 5.7 yield that the push forward of $\mathcal{H}^{n-1}|_{Z(s)}$ by the map $y + (t-s)d^\perp(s,y)$ can be written as

$$(y + (t-s)d^\perp(s,y))_\sharp \mathcal{H}^{n-1}|_{Z(s)} = \alpha(t,s,y)\mathcal{H}^{n-1}|_{Z(s)}, \quad t \geq s,$$

with α bounded above and below by a constant depending only on

$$h^- = \inf\{h^-(y), y \in Z(0)\}, \quad h^+ = \inf\{h^+(y), y \in Z(0)\},$$

and $n-1$.

Lemma 6.1. *A set of $\mathcal{H}^n$ measure* 0 *in* $Z(0)\times[0,h]$ *is negligible in* $\mathcal{Z}\cap\{e_1\cdot x\in[0,h]\}$*, and viceversa.*

Proof. The map

$$[0,h]\times Z(0) \ni (t,y) \to \left(\begin{array}{c} t \\ y + td^\perp(0,y) \end{array} \right) \in \mathcal{Z}\cap\{e_1\cdot x\in[0,h]\}$$

is continuous and invertible. If N is a $\mathcal{H}^n$ negligible set in $Z(0)\times[0,h]$, then denoting with N_t its t slice

$$\begin{aligned}\mathcal{H}^n\Big\{x = (t, y + td^\perp(0,y)), (t,y)\in N\Big\} &= \int_0^h \mathcal{H}^{n-1}\Big\{x = (t, y+td^\perp(0,y)), y \in N_t\Big\}dt \\ &\leq C(h^-)\int_0^h \mathcal{H}^{n-1}(N_t)dt = 0.\end{aligned}$$

Similarly, if $\mathcal{N}$ is negligible in $\mathcal{Z}\cap\{e_1\cdot x\in[0,h]\}$, then

$$\mathcal{H}^n\Big\{(y,t) : (t, y+td(0,y)) \in \mathcal{N}\Big\} \leq C(h^+)\mathcal{H}^n(\mathcal{N}) = 0.$$

□

This lemma allows to pass from $\mathcal{H}^n$ measurable functions on $Z(0)\times[0,h]$ to $\mathcal{H}^n$ measurable functions on $\mathcal{Z}\cap\{e_1\cdot x\in[0,h]\}$. In fact, for all $\mathcal{H}^n$ measurable functions $f(x)$, we can define the function $\tilde{f}_s$ on the set $Z(s)\times[0,h]$, $s\in[0,h]$, as

$$\tilde{f}_s(t,y) = f\big(t, y + (t-s)d^\perp(s,y)\big). \tag{6.1}$$

In particular, we will consider the function

$$\tilde{\alpha}_s(t,y) = \alpha\big(t,s,y+(t-s)d^\perp(s,y)\big).$$

Note that $\tilde{d}_s(t,y) = d(s,y)$, since $d(x+td(x)) = d(x)$, and similarly $\tilde{a}_s(t,y) = a(s,y)$, $\tilde{b}_s(t,y) = b(s,y)$.

We observe that $\alpha(t,s,y)$ is a measurable function: in fact, from the formula

$$\int_{Z(t)} \phi(t,y)\alpha(t,s,y)d\mathcal{H}^{n-1}(y) = \int_{Z(s)} \phi(t, y+(t-s)d(s,y))d\mathcal{H}^{n-1}(y), \quad \phi\in C_c(\Omega),$$

one can verify that it is the weak limit of the functions $\alpha_I(t,s,y)$ in Ω, and then also $\tilde{\alpha}_s(t,y)$ is measurable in the measure $\mathcal{H}^n|_{Z(0)\times[0,h]}$. Moreover $\mathcal{H}^{n-1}$ almost all Lebesgue points of $\mathcal{H}^{n-1}|_{Z(0)}$ remain Lebesgue points of $\mathcal{H}^{n-1}|_{Z(t)}$, $t\in[0,h]$, under the map $y\to y+td^\perp(0,y)$: this in fact it is true for a dense sequence of times t_i, and the estimates of Lemma 5.7 yield the results for all t.

The first result is an estimate of the function $\tilde{\alpha}_s$ as a function of t.

Lemma 6.2. *The function $\tilde\alpha_s(t,y)$ is a Lipschitz function of t satisfying the estimate*

$$\frac{d}{dt}\left(\frac{1}{\tilde\alpha_s(t,y)}\right) \in \frac{1}{\tilde\alpha_s(t,y)}\left[-\frac{n-1}{e_1\cdot b(s,y)-t}, \frac{n-1}{t-e_1\cdot a(s,y)}\right] \qquad \mathcal{H}^{n-1} \text{ a.e. } y \in Z(s), \tag{6.2}$$

where $a(s,y)$, $b(s,y)$ are the initial and final point of the segment $(s,y)+td(s,y)$.

In particular, using the parameterization $t \to (s,y)+(t-s)d(y)$, we obtain the formula

$$\frac{d}{dt}\frac{1}{\tilde\alpha_s(t,y)} \in \frac{1}{\tilde\alpha_s(t,y)}\left[-\frac{n-1}{|b(s,y)-(s,y)-(t-s)d(s,y)|}, \frac{n-1}{|(s,y)+(t-s)d(s,y)-a(s,y)|}\right] \tag{6.3}$$

for $\mathcal{H}^{n-1}$ a.e. $y \in Z(s)$.

Proof. The definition of push forward $\alpha(t,s,y)$ gives for an arbitrary $\bar s$ and compact set $F \subset Z(\bar s)$

$$\begin{aligned}\left(\frac{h+h^+-t}{h+h^+-s}\right)^{n-1}\int_{Z(\bar s)\cap F}\frac{1}{\tilde\alpha_{\bar s}(s,y)}\mathcal{H}^{n-1}(y) &\leq \int_{Z(\bar s)\cap F}\frac{1}{\tilde\alpha_{\bar s}(t,y)}\mathcal{H}^{n-1}(y)\\ &\leq \left(\frac{t+h^-}{s+h^-}\right)^{n-1}\int_{Z(\bar s)\cap F}\frac{1}{\tilde\alpha_{\bar s}(s,y)}\mathcal{H}^{n-1}(y).\end{aligned} \tag{6.4}$$

This implies that for $\mathcal{H}^{n-1}$ a.e. $y \in Z(s)$, and for a countable dense sequence $\{t_i\}_{i\in N}$ in $(e_1\cdot a(y), e_1\cdot b(y))$,

$$\left[\left(\frac{h+h^+-t_j}{h+h^+-t_i}\right)^{n-1}-1\right]\frac{1}{\tilde\alpha_{\bar s}(t_i,y)} \leq \frac{1}{\tilde\alpha_{\bar s}(t_j,y)}-\frac{1}{\tilde\alpha_{\bar s}(t_i,y)} \leq \left[\left(\frac{t_i+h^-}{t_j+h^-}\right)^{n-1}-1\right]\frac{1}{\tilde\alpha_{\bar s}(t_j,y)}. \tag{6.5}$$

It follows that $1/\bar\alpha_{\bar s}(t_j,y)$ is Lipschitz on $\{t_i\}_{i\in N}$, for $\mathcal{H}^n$ a.e. $y \in Z(s)$, so that it can be extended to a Lipschitz function on the whole $(e_1\cdot a(y), e_1\cdot b(y))$. Using again (6.4), one sees that this extension satisfies

$$\mathcal{H}^{n-1}\Big\{y+(t-\bar s)d(y), y \in Z(\bar s)\cap F\Big\} = \int_{Z(\bar s)\cap F}\frac{1}{\tilde\alpha_{\bar s}(t,y)}\mathcal{H}^{n-1}(y)$$

for all t, F compact, which means that this extension is a Lipschitz representative in t of $\tilde\alpha_{\bar s}(t,y)$, defined for $\mathcal{H}^{n-1}$ a.e. $y \in Z(s)$.

By taking the derivative in (6.5)

$$-\left(\frac{n-1}{h+h^+-t}\right)\frac{1}{\tilde\alpha_{\bar s}(t,y)} \leq \frac{d}{dt}\left(\frac{1}{\tilde\alpha_{\bar s}(t,y)}\right) \leq \left(\frac{n-1}{t+h^-}\right)\frac{1}{\tilde\alpha_{\bar s}(t,y)}.$$

Since b, a are continuous on $Z(\bar s)$, one can improve the above estimate to (6.2) by repeating the estimates of Lemma 5.7 in a small compact set around y, thus obtaining (6.2). $\square$

The following corollary is very important because it tells us that the function $\tilde\alpha_s(t,y)^{-1}$ is uniformly bounded and different from 0 on the segment $(a(x),b(x))$, and that it is an absolutely continuous function of t in $(a(x),b(x))$.

Corollary 6.3. *The function $\tilde\alpha_s(t,y)^{-1}$ satisfies the following estimates:*

$$\frac{1}{\tilde\alpha_s(t,y)} \in \left[\left(\frac{|b(s,y)-(t,y+(t-s)d^\perp(s,y))|}{|b(s,y)-(s,y)|}\right)^{n-1}, \left(\frac{|(t,y+(t-s)d^\perp(s,y))-a(s,y)|}{|(s,y)-a(s,y)|}\right)^{n-1}\right], \quad s \leq t, \tag{6.6}$$

$$\frac{1}{\tilde\alpha_s(t,y)} \in \left[\left(\frac{|(t,y+(t-s)d^\perp(s,y))-a(s,y)|}{|(s,y)-a(s,y)|}\right)^{n-1}, \left(\frac{|b(s,y)-(t,y+(t-s)d^\perp(s,y))|}{|b(s,y)-(s,y)|}\right)^{n-1}\right], \quad t \leq s, \tag{6.7}$$

for $\mathcal{H}^{n-1}$ a.e. $y \in Z(s)$. Moreover

$$\int_{e_1\cdot a(s,y)}^{e_1\cdot b(s,y)}\frac{d}{dt}\frac{1}{\tilde\alpha_s(t,y)} \leq C(s,a(s,y),b(s,y)), \tag{6.8}$$

for some constant depending only s, $a(s,y)$, $b(s,y)$.

Proof. By the estimates on the derivative of $1/\tilde{\alpha}_s(t,y)$ w.r.t. t it follows that

$$\frac{d}{dt}\bigg(\frac{1}{|e_1\cdot b(s,y)-t|^{n-1}}\frac{1}{\tilde{\alpha}_s(t,y)}\bigg) = \frac{n-1}{|e_1\cdot b(s,y)-t|^{n}}\frac{1}{\tilde{\alpha}_s(t,y)} + \frac{1}{|e_1\cdot b(s,y)-t|^{n-1}}\frac{d}{dt}\frac{1}{\tilde{\alpha}_s(t,y)} \geq 0,$$

$$\frac{d}{dt}\bigg(\frac{1}{|t-e_1\cdot a(s,y)|^{n-1}}\frac{1}{\tilde{\alpha}_s(t,y)}\bigg) = -\frac{n-1}{|t-e_1\cdot a(s,y)|^{n}}\frac{1}{\tilde{\alpha}_s(t,y)} + \frac{1}{|t-e_1\cdot a(s,y)|^{n-1}}\frac{d}{dt}\frac{1}{\tilde{\alpha}_s(t,y)} \leq 0,$$

from which (6.6), (6.7) follow by the parameterization $t\to(s,y)+(t-s)d(y)$. Thus $1/\tilde{\alpha}_s(t,y)$ is uniformly bounded, and from the estimates

$$\frac{d}{dt}\frac{1}{\tilde{\alpha}_s(t,y)} \geq -\frac{n-1}{|e_1\cdot b(s,y)-t|}\frac{1}{\tilde{\alpha}_s(t,y)},$$

using the bound

$$\frac{1}{\tilde{\alpha}_s(t,y)} \leq \bigg(\frac{|b(s,y)-(t,y+(t-s)d^{\perp}(s,y))|}{|b(s,y)-(s,y)|}\bigg)^{n-1}, \quad t\leq s,$$

it follows

$$\begin{aligned}
\mathrm{Tot.Var.}\bigg(\frac{1}{\tilde{\alpha}_s(t,y)},(e_1\cdot a(s,y),s]\bigg) &\leq \int_{e_1\cdot a(s,y)}^{s}\frac{d}{dt}\frac{1}{\tilde{\alpha}_s(t,y)} + \frac{n-1}{|e_1\cdot b(s,y)-t|}\frac{1}{\tilde{\alpha}_s(t,y)}dt \\
&\quad + \int_{e_1\cdot a(s,y)}^{s}\frac{n-1}{|e_1\cdot b(s,y)-t|}\frac{1}{\tilde{\alpha}_s(t,y)}dt \\
&= 1-\frac{1}{\tilde{\alpha}_s(e_1\cdot a(s,y),y)} + 2\int_{e_1\cdot a(s,y)}^{s}\frac{n-1}{e_1\cdot b(s,y)-t}\frac{1}{\tilde{\alpha}_s(t,y)}dt \\
&\leq 1+2\int_{e_1\cdot a(s,y)}^{s}\frac{(n-1)(e_1\cdot b(s,y)-t)^{n-2}}{(e_1\cdot b(s,y)-s)^{n-1}}dt \\
&= 1+2\bigg(\frac{(e_1\cdot(b(s,y)-a(s,y)))^{n-1}}{(e_1\cdot b(s,y)-s)^{n-1}}-1\bigg).
\end{aligned}$$

The symmetric estimate is

$$\mathrm{Tot.Var.}\bigg(\frac{1}{\tilde{\alpha}_s(t,y)},[s,e_1\cdot b(s,y))\bigg) \leq 1+2\bigg(\frac{(e_1\cdot(b(s,y)-a(s,y)))^{n-1}}{(s-e_1\cdot a(s,y))^{n-1}}-1\bigg).$$

from which we obtain the total variation estimate of $1/\tilde{\alpha}_s$. □

In the next section we will use this result to show an explicit formula for the disintegration of the Lebesgue measure on Ω.

7. Disintegration of the Lebesgue measure

Consider now the Borel maps

$$\Omega\ni x\to b(x)\in\partial\Omega, \qquad \Omega\ni x\to d(x)\in\partial B(0,1),$$

and the Borel measure m on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ defined by

$$\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n}\phi(x,y,z)dm = \int_{\Omega}\phi(x,b(x),d(x))dx, \tag{7.1}$$

for all continuous function $\phi:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\mapsto\mathbb{R}$. We can write equivalently

$$m = (\mathbb{I},b,d)_{\sharp}\mathcal{H}^n|_{\Omega}. \tag{7.2}$$

Denote with π_2 the projection on the second set of coordinates, i.e.

$$\begin{array}{rccc}\pi_2 & : & \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n & \to & \mathbb{R}^n\times\mathbb{R}^n \\ & & (x,y,z) & \mapsto & \pi_2(x,y,z)=(y,z)\end{array}$$

We recall the following disintegration theorem [2], Theorem 2.28 of page 57:

Theorem 7.1. *Let m be a positive Radon measure on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that, if π is the projection on $\mathbb{R}^{n_2}$, the measure $\mu = \pi_\sharp m$ is Radon. Then there exist finite positive Radon measures ν_{x_2} on $\mathbb{R}^{n_1}$ such that $x_2 \mapsto \int \phi(x_1) d\nu_{x_2}(x_1)$ is measurable for all $\phi \in C_c(\mathbb{R}^{n_1})$, $\nu_{x_2}(\mathbb{R}^{n_1}) = 1$ for μ a.e. x_2, and for all measurable functions $f \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, m)$, the function $x_1 \mapsto f(x_1, x_2)$ is measurable for μ a.e. x_2*

$$x_2 \mapsto \int f(x_1, x_2) d\nu_{x_2}(x_1) \in L^1(\mathbb{R}^{n_2}, \mu) \tag{7.3}$$

and the following disintegration formula holds:

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} f(x_1, x_2) dm(x_1, x_2) = \int_{\mathbb{R}^{n_2}} \left(\int_{\mathbb{R}^{n_1}} f(x_1, x_2) d\nu_{x_2}(x_1) \right) d\mu(x_2). \tag{7.4}$$

This decomposition is unique, in the sense that if there is another μ measurable map ν'_{x_2} such that (7.3) and (7.4) hold for every Borel function f with compact support, and such that $\nu'_{x_2}(\mathbb{R}^{n_1}) \in L^1_{\mathrm{loc}}(\mathbb{R}^{n_2}, \mu)$, then $\nu'_{x_2} = \nu_{x_2}$ for μ a.e. x_2.

In our case, since $|m|(\mathbb{R}^{3n}) = |\Omega| < +\infty$, the measure m is clearly Radon, and also the projection $\mu = (\pi_2)_\sharp m$

$$\int_{\mathbb{R}^{2n}} \phi(y, z) d\mu = \int_\Omega \phi(b(x), d(x)) dx$$

is clearly a Radon measure. Moreover, μ is concentrated in the set

$$\Upsilon = (b, d)(\Omega) = \left\{ (y, z) : b^{-1}(y) \cap d^{-1}(z) \neq \emptyset \right\}. \tag{7.5}$$

Remark 7.2. We use the disintegration w.r.t. $(b(x), d(x))$ because we need not only to control the direction $d(x)$, but also the end point $b(x)$: in fact the coordinates $(b(x), d(x))$ parameterize the set $\cup_{x \in \Omega} \Sigma(x)$, and to all (b, d) there corresponds at most one segment $x + td(x)$.

Applying Theorem 7.1 to our case, we obtain the following proposition:

Proposition 7.3. *There exist probability measures $\nu_{(y,z)}$ such that for all functions $f \in L^1(\mathbb{R}^{3n}, m)$ it holds*

$$\int_{\mathbb{R}^{3n}} f(x, y, z) dm(x, y, z) = \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} f(x, y, z) d\nu_{(y,z)}(x) \right) d\mu(y, z), \tag{7.6}$$

where $\mu = (\pi_2)_\sharp m$. Moreover μ is concentrated on Υ, and $\nu_{(y,z)}$ is concentrated on the line $(b, d)^{-1}(y, z)$ for μ a.e. (b, d).

Using the results of the previous section, we obtain the following explicit formula for the disintegration of the Lebesgue measure on each compact $\mathcal{Z}$. We recall that $a(z, y)$ is the end point of the segment $y + tz$, $t \leq 0$, while in these notations $b(y, z) = y$.

Lemma 7.4. *The disintegration of the Lebesgue measure on $\mathcal{Z}$ is*

$$\int_{\mathcal{Z}} \phi(x, y, z) dm(x, y, z) = \int_{(b,d)(\mathcal{Z})} d\mu(y, z) \int_{z \cdot a(y,z)}^{z \cdot y} \phi\big(y + (t - z \cdot y)z, y, z\big) c(t, y, z) dt, \tag{7.7}$$

where

$$0 \leq c(t, y, z) \chi\{t \in (z \cdot a(y, z), z \cdot y)\} \in L^\infty(\mu \times \mathcal{H}^1) \tag{7.8}$$

and strictly positive for all compact sets in $(z \cdot a(y, z), z \cdot y)$ for μ a.e. (y, z).

Proof. We first decompose the measure m on the compact set $\mathcal{Z} \cap \{e_1 \cdot x \in [0, h]\}$, with the notations of Section 5.1. We have

$$\begin{aligned}
\int_{\mathcal{Z} \cap \{e_1 \cdot x \in [0,h]\}} \phi(x, b(x), d(x)) d\mathcal{H}^n(x) &= \int_0^h dt \int_{Z(t)} \phi((t, w), b(t, w), d(t, w)) d\mathcal{H}^{n-1}(w) \\
&= \int_0^h dt \int_{Z(0)} \phi((t, w + td^\perp(0, w)), b(0, w), d(0, w)) \frac{d\mathcal{H}^{n-1}(w)}{\tilde{\alpha}_0(t, w)} \\
&= \int_0^h \int_{Z(0)} \frac{\phi((t, w + td^\perp(0, w)), b(0, w), d(0, w))}{\tilde{\alpha}_0(t, w)} dt \times d\mathcal{H}^{n-1}(w).
\end{aligned}$$

Since $\cup_{x\in Z(0)} a(x) \subset \hat{J}$ is $\mathcal{H}^n$ negligible, for all $\epsilon > 0$ we can cover a subset $\mathcal{Z}_\epsilon$ of $\cup_{x\in Z(0)}(a(x), b(x))$, $\mathcal{H}^n(\mathcal{Z} \setminus \mathcal{Z}_\epsilon) < \epsilon$, with a countable number of disjoint compact sets $\mathcal{K}_i$ of the form

$$\Big\{ y + td(0,y)/(e_1 \cdot d(0,y)), t \in [h_i^-, h_i^+], y \text{ in a compact subset } K_i \ of \ Z(0) \Big\}.$$

It thus follows that the above formula holds for the whole $\mathcal{Z}_\epsilon \cap \Omega$:

$$\int_{\mathcal{Z}_\epsilon \cap \Omega} \phi(x, b(x), d(x)) d\mathcal{H}^n(x) = \int_{\cup_i K_i} \int_{h_i^-}^{h_i^+} \frac{\phi((t, w + td^\perp(0,w)), b(0,w), d(0,w))}{\tilde{\alpha}_0(t,w)} dt \times d\mathcal{H}^{n-1}(w).$$

Since $\tilde{\alpha}_0$ is uniformly bounded, we can pass to the limit $\epsilon \to 0$ in the above formula to obtain

$$\int_{\mathcal{Z}\cap\Omega} \phi(x, b(x), d(x)) d\mathcal{H}^n(x) = \int_{Z_0} \int_{e_1\cdot a(0,w)}^{e_1 \cdot b(0,w)} \frac{\phi((t, w + td^\perp(0,w)), b(0,w), d(0,w))}{\tilde{\alpha}_0(t,w)} dt \times d\mathcal{H}^{n-1}(w).$$

By using the continuous rescaling $t \to t/(e_1 \cdot d(0,y))$, we can rewrite as

$$\int_{\mathcal{Z}\cap\Omega} \phi(x, b(x), d(x)) d\mathcal{H}^n(x) = \int_{Z(0)} \int_{d(0,w)\cdot a(0,w)}^{d(0,w)\cdot b(0,w)} \frac{\phi((0,w) + td(0,w), b(0,w), d(0,w))}{(\tilde{\alpha}_0(t,w)/(e_1 \cdot d(0,w))} dt \times d\mathcal{H}^{n-1}(w).$$

By means of the push forward

$$(t, y) \to (t, b(0,y), d(0,y))$$

the integral takes the form (7.7), where

$$d\mu(y,z) = \bigg(\int_{z\cdot a(y,z)}^{z\cdot y} \frac{1}{\tilde{\alpha}_0(a(y,z) + tz)/(e_1 \cdot z)} dt \bigg) d\big[(b,d)_\sharp \mathcal{H}^{n-1}\llcorner_{Z(0)}\big],$$

$$c(t,y,z) = \bigg(\int_{z\cdot a(y,z)}^{z\cdot y} \frac{1}{\tilde{\alpha}_0(a(y,z)+tz)} dt \bigg)^{-1} \frac{1}{\tilde{\alpha}_0(a(y,z)+tz)}.$$

From Corollary 6.3, we have that for μ a.e. (y,z) the integrand is not 0, being μ the image measure of $\mathcal{H}^{n-1}\llcorner_{Z(0)}$, so that $c(t,y,z)$ is $\mu \times \mathcal{H}^1$ a.e. bounded and strictly positive in each compact of $(a(y,z), y)$. The measurability of $c(t,y,z)$, extended to 0 outside $t \in (z \cdot a(y,z), z \cdot y)$, follows because the set

$$\Big\{ (t,y,z) : y, z \in (b,d)(Z), t \in (z \cdot a(y,z), z \cdot y) \Big\}$$

is Borel, since $(b,d)(Z)$ is compact and $a(y,z)$ is continuous ((b,d) is continuous and invertible on $Z(0)$). □

We can now disintegrate the Lebesgue measure.

Theorem 7.5. *The measure $m = (\mathbb{I}, b, d)_\sharp \mathcal{H}^n\llcorner_\Omega$ can be disintegrated as*

$$\int_{\mathbb{R}^{3n}} \phi(x,y,z) dm(x,y,z) = \int_{\{t\in(z\cdot a(y,z), z\cdot y)\}} \phi(y + (t - z\cdot y)z, y, z) c(t,y,z) d\mu(y,z) \times \mathcal{H}^1(t), \tag{7.9}$$

with $0 \leq c(t,y,z) \in L^\infty(\mu \times \mathcal{H}^1)$, and, for μ a.e. (y,z), Lipschitz continuous in $t \in (z \cdot a(y,z), z\cdot y)$, uniformly positive in each compact subset of $(z \cdot a(y,z), z \cdot y)$ and absolutely continuous function of t in $[z \cdot a(y,z), z \cdot y]$.

Proof. We first show that Ω can be covered $\mathcal{H}^n$ a.a. by a countable number of sets $\mathcal{Z}_i$, such that

$$\mathcal{H}^n\big(\mathcal{Z}_i \cap \mathcal{Z}_j\big) = 0.$$

Clearly the number of sets $\mathcal{Z}$ with positive $\mathcal{H}^n$ measure is countable.

Define the finite covering S_j^{n-1} of S^{n-1} by

$$S_j^{n-1} = \Big\{ d \in S^{n-1} : d \cdot \mathfrak{e}_j \geq 1 - \epsilon \Big\} \setminus \bigcup_{i=1}^{j-1} S_i^{n-1},$$

where $\mathfrak{e}_j$ is a finite sequence of points in S^{n-1} such that

$$S^{n-1} \subset \bigcup_{j=0}^{J} B(\mathfrak{e}_j, \epsilon).$$

Set thus for $j \in \{0, \dots, J\}$, $k, \ell \in \mathbb{Z}$

$$\Omega_{jk\ell} = \Big\{ x \in \Omega : d(x) \in S^{n-1}_j, \mathfrak{e}_j \cdot (b(x) - a(x)) \in (2^{-k}, 2^{-k+1}], \mathfrak{e}_j \cdot a(x) \in 2^{-k-2}(\ell, \ell+1] \Big\}.$$

This family is countable and each point of $\Omega \setminus J$ belongs to only one $\Omega_{jk\ell}$: thus it is a countable covering of $\Omega \setminus J$.

For each $x \in \Omega_{jk\ell}$ the line $(a(x), b(x))$ intersects the plane $\{x \cdot \mathfrak{e}_j = (\ell+2)2^{-k-2}\}$ in one point, and let $K_{jk\ell m}$ be a countable sequence of compact subsets of $\Omega_{ik\ell} \cap \{x \cdot \mathfrak{e}_j = (\ell+2)2^{-k-2}\}$ such that a, d, b are continuous on $K_{jk\ell m}$ and

$$\mathcal{H}^{n-1} \bigg(\Big(\Omega_{ik\ell} \cap \{x \cdot \mathfrak{e}_j = (\ell+2)2^{-k-2}\} \Big) \setminus \bigcup_m K_{jk\ell m} \bigg) = 0.$$

Let $\mathcal{Z}_{jk\ell m}$ be the compact set

$$\mathcal{Z}_{jk\ell m} = \Big\{ [a(x), b(x)], x \in K_{jk\ell m} \Big\}.$$

If

$$\mathcal{H}^n \bigg(\Omega_{jk\ell} \setminus \bigcup_m \mathcal{Z}_{jk\ell m} \bigg) > 0,$$

then there exists a plane $\{\mathfrak{e}_j \cdot x = t\}$ such that

$$\mathcal{H}^{n-1} \bigg(\{\mathfrak{e}_j \cdot x = t\} \cap \bigg(\Omega_{jk\ell} \setminus \bigcup_m \mathcal{Z}_{jk\ell m} \bigg) \bigg) > 0.$$

We can thus take a compact subset $K_{jk\ell m,t}$ of

$$\{\mathfrak{e}_j \cdot x = t\} \cap \bigg(\Omega_{jk\ell} \setminus \bigcup_m \mathcal{Z}_{jk\ell m} \bigg)$$

with positive Lebesgue measure and such that a, b, d are continuous. It thus follows from Lemma 4.7 that the image of $\mathcal{H}^{n-1}|_{K_{jk\ell m,t}}$ on the plane $\{x \cdot \mathfrak{e}_j = (\ell+2)2^{-k-2}\}$ is of positive $\mathcal{H}^{n-1}$ measure, yielding a contradition with the choice of the $K_{jk\ell m}$.

The result thus follows by applying Lemma 7.4 to each $\mathcal{Z}_{jk\ell m}$, and using Lemma 6.2 for the Lipschitz continuity and Corollary 6.3 for the absolutely continuous estimate in t, since the length of $|a(x) - b(x)| > 0$ $\mathcal{H}^n$ a.e. (and hence μ a.e.). $\square$

Since $c(t, y, z)$ is different from 0 for all $t \in (z \cdot a(y, z), z \cdot y)$, for μ a.e. (y, z), we obtain the following corollary:

Corollary 7.6. *A function $\phi(x)$ is $\mathcal{H}^n|_\Omega$ measurable if and only if the corresponding $\phi(t, y, z)$ is measurable in $\mathcal{H}^1 \times \mu$.*

In the next result we prove that $\partial_t c(t, y, z)/c(t, y, z)$ belongs to $L^1_{\mathrm{loc}}(\Omega)$.

Proposition 7.7. *For all $\Omega' \subset\subset \Omega$ there exists a constant $C(\Omega')$ such that*

$$\int d\mu(y, z) \int_{z \cdot a(y,z)}^{z \cdot y} \chi_{\Omega'}(y + (t - z \cdot y)z) |\partial_t c(t, y, z)| d\mathcal{H}^1(t) \leq C(\Omega'). \tag{7.10}$$

Proof. From the proof of Corollary 6.3 we have that for $s = z \cdot (y + a(y, z)/2$

$$\int_{z \cdot a(z,y)}^{z \cdot y} |\partial_t c(t, y, z)| dt \leq (2 + 2^{n-1}) \bigg(\int_{z \cdot a(y,z)}^{y \cdot z} \frac{1}{\alpha_{z \cdot (a(y,z)+y)/2}(t, y, z)} dt \bigg)^{-1} \leq \mathcal{O}(1) |y - a(y, z)|^{-n}.$$

Since for all $\Omega' \subset\subset \Omega$

$$|y - a(y, z)| \geq \mathrm{dist}(\Omega', \partial\Omega) \geq C > 0,$$

then we have that $\partial_t c \in L^1(\mu \times \mathcal{H}^1|_{\Omega'})$. $\square$

We observe that since from Lemma 6.2 it follows that in Ω'

$$\partial_t c + \frac{n-1}{\mathrm{dist}(\Omega', \partial\Omega)} c \geq 0,$$

then

$$\int_{\Omega'} \frac{|\partial_t c|}{c} d\mathcal{H}^n \in \bigg[- \frac{n-1}{\mathrm{dist}(\Omega', \partial\Omega)} \mathcal{H}^n(\Omega'), +\infty \bigg]$$

is meaningful and from the disintegration formula it follows that $\frac{\partial_t c}{c} \in L^1_{\mathrm{loc}}(\Omega)$.

We conclude this section by relating the function $c(t,y,z)$ with the measure $\mathrm{div}\, d$. We can consider the disintegration of the divergence formula in each $\Omega' \subset\subset \Omega$

$$\int \phi \mathrm{div}\, d = - \int d\nabla\phi d\mathcal{H}^n, \qquad \phi \in C_c(\Omega', \mathbb{R}). \tag{7.11}$$

Applying again the disintegration Theorem 7.5 to the measure $\mathrm{div}\, d$, we obtain

$$\begin{aligned}
\int \phi(x) \mathrm{div}\, d &= \int \phi(x) (\mathrm{div}\, d)_{\mathrm{a.c.}} d\mathcal{H}^n(x) + \int \phi (\mathrm{div}\, d)_{\mathrm{s}} \\
&= \int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} \phi(y + (t - z\cdot y)z)(\mathrm{div}\, d)_{\mathrm{a.c.}} c(t,y,z) dt + \int \phi (\mathrm{div}\, d)_{\mathrm{s}} \\
&= - \int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} z \cdot \nabla\phi(y + (t - z\cdot y)z) c(t,y,z) dt \\
&= \int d\mu(y,z) \bigg[c(z\cdot a(y,z), y, z)\phi(a(y,z)) + \int_{z\cdot a(y,z)}^{z\cdot y} \partial_t c(t,y,z) \phi(y + (t - z\cdot y)z) dt \bigg]. \qquad (7.12)
\end{aligned}$$

where we used the fact that c is absolutely continuous w.r.t. t in $[z\cdot a(y,z), z\cdot y]$.

By taking $\psi_\epsilon \in C^1$ to be 0 on a compact set $Z(0)$ such that $(\mathrm{div}\, d)_s(\Omega' \setminus Z(0)) \leq \epsilon$, and 1 outside an open set O_ϵ of measure $\mathcal{H}^n(O_\epsilon) \leq \epsilon$, with $\hat{J} \subset O_\epsilon$ and $(\mathrm{div}\, d)_{\mathrm{s}}(\Omega \setminus O_\epsilon) = 0$, we have thus

$$\begin{aligned}
&\int d\mu(y,z) \bigg[c(z\cdot a(y,z), y, z) \psi_\epsilon(a(y,z)) \phi(a(y,z)) \\
&\qquad + \int_{z\cdot a(y,z)}^{z\cdot y} \partial_t c(t,y,z) \psi_\epsilon(y + (t - z\cdot y)z) \phi(y + (t - z\cdot y)z) dt \bigg] \\
&+ \int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} c(t,y,z)(\mathrm{div}\, d)_{\mathrm{a.c.}} \psi_\epsilon(y + (t - z\cdot y)z)\phi(y + (t - z\cdot y)z) dt + \int \psi_\epsilon \phi (\mathrm{div}\, d)_{\mathrm{s}} = 0.
\end{aligned} \tag{7.13}$$

We now compute the limit as ψ_ϵ converges to the characteristic function $\chi_{\Omega\setminus O_\epsilon}$: since $\mathrm{div}\, d$ is a locally bounded positive measure, it follows that

$$\int (\mathrm{div}\, d)_{\mathrm{s}} \phi\psi_\epsilon \to \int_{\Omega\setminus O_\epsilon} (\mathrm{div}\, d)_{\mathrm{s}} \phi = 0.$$

Similarly, using the fact that $c \in L^1(\Omega)$, $c_t \in L^1_{\mathrm{loc}}(\Omega)$, we have by the dominated convergence theorem

$$\begin{aligned}
&\int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} c(t,y,z)(\mathrm{div}\, d)_{\mathrm{a.c.}} \psi_\epsilon(y + (t - z\cdot y)z)\phi(y + (t - z\cdot y)z) dt \\
&\qquad \to \int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} c(t,y,z)(\mathrm{div}\, d)_{\mathrm{a.c.}} \chi_{\Omega\setminus O_\epsilon}(y + (t - z\cdot y)z)\phi(y + (t - z\cdot y)z) dt,
\end{aligned}$$

$$\begin{aligned}
&\int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} \partial_t c(t,y,z) \psi_\epsilon(y + (t - z\cdot y)z)\phi(y + (t - z\cdot y)z) dt \\
&\qquad \to \int d\mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} \partial_t c(t,y,z) \chi_{\Omega\setminus O_\epsilon}(y + (t - z\cdot y)z)\phi(y + (t - z\cdot y)z) dt.
\end{aligned}$$

Since $c(t,x,y)$ is bounded, then

$$c(z\cdot a(y,z), y, z)\psi_\epsilon(a(y,z))\phi(a(y,z)) \to 0$$

for μ a.e. y, z, so that we conclude that by the dominated convergence theorem

$$\int \mu(y,z) \int_{z\cdot a(y,z)}^{z\cdot y} \big(\partial_t c(t,y,z) + c(t,y,z)(\operatorname{div} d)_{\text{a.c.}}\big)\phi(y + (t - z\cdot y)z)dt = 0$$

for $\phi \in C^1_c(\Omega)$. Again, by means of $\partial_t c \in L^1_{\text{loc}}(\Omega)$ and the Lebesgue dominated convergence theorem, it follows that the above equation holds for ϕ continuous with compact support in Ω. We thus obtain

Proposition 7.8. *The density $c(t, y, z)$ satisfies the equation*

$$\partial_t c(t,y,z) + \Big[(\operatorname{div} d)_{a.c.}(y + (t - z\cdot y)z)\Big]c(t,y,z) = 0, \quad \int_{z\cdot a(y,z)}^{z\cdot y} c(t,y,z)dt = 1 \tag{7.14}$$

for μ a.e. (y, z). As a consequence, the absolutely continuous divergence satisfies

$$(\operatorname{div} d)_{a.c.}(y + (t - z\cdot y)z) \in \Big[-\frac{n-1}{z\cdot y - t}, \frac{n-1}{t - z\cdot a(y,z)}\Big], \quad \mathcal{H}^1 \text{ a.e. } t, \ \mu \text{ a.e. } (y,z) \tag{7.15}$$

Moreover in each set $\mathcal{Z}$ of the form (5.15) *the following divergence formula holds:*

$$\int_{Z(0)} d(y)\cdot e_1 d\mathcal{H}^{n-1}(y) - \int_{Z(h)} d(y)\cdot e_1 d\mathcal{H}^{n-1}(y) = \int_{\mathcal{Z}} (\operatorname{div} d)_{a.c.}(x) d\mathcal{H}^n(x). \tag{7.16}$$

Proof. The estimate on $(\operatorname{div} d)_{\text{a.c.}}$ follows from Lemma 6.2 and (7.14), while the divergence formula follows from the definition of $c(t, x, y)$ as the inverse of the push forward of the Lebesgue measure and integrating (7.14) along the line $y + (t - z \cdot y)z$. □

8. Solution to the transport equation

We prove the following theorem.

Theorem 8.1. *There exists a solution to the transport equation*

$$\operatorname{div}(\rho(x)d(x)) = g(x), \tag{8.1}$$

such that in all sets $\mathcal{Z}$ of the form (5.15) *the divergence formula holds:*

$$\int_{Z(0)} \rho(y)d(y)\cdot e_1 d\mathcal{H}^{n-1}(y) - \int_{Z(h)} \rho(y)d(y)\cdot e_1 d\mathcal{H}^{n-1}(y) = \int_{\mathcal{Z}} g(x)d\mathcal{H}^n(x). \tag{8.2}$$

This solution is > 0 $\mathcal{H}^n$ a.e. in Ω if g is.

Proof. Assume first that a solution to (8.1) exists and satisfies the divergence formula (8.2) in each $\mathcal{Z}$. By using the disintegration of the Lebesgue measure and the weak formulation of (8.1) we obtain

$$\int d\mu\chi_{\mathcal{Z}}(y,z)\int d\mathcal{H}^1(t)c(t,y,z)\Big(\rho(y+(t-z\cdot y)z)\partial_t\phi(y+(t-z\cdot y)z) - g(y+(t-z\cdot y)z)\phi(y+(t-z\cdot y)z)\Big) = 0.$$

Since the set $\mathcal{Z}$ is arbitrary, it follows that

$$\int d\mathcal{H}^1(t)\Big(\rho(y+(t-z\cdot y)z)c(t,y,z)\partial_t\phi(y+(t-z\cdot y)z) - g(y+(t-z\cdot y)z)c(t,y,z)\phi(y+(t-z\cdot y)z)\Big) = 0$$

for μ a.e. (y, z). By integrating by parts, a solution to (8.1) satisfies

$$\partial_t(\rho(y + (t - z\cdot y)z)c(t,y,z)) = g(y + (t - z\cdot y)z)c(t,y,z), \quad \rho(a(y,z)) = \rho_0(y,z)$$

for μ a.e. (y, z): a solution to the above equation is given by

$$\rho(y + (t - z\cdot y)z) = \rho_0(y,z)\frac{c(z\cdot a(y,z), y, z)}{c(t,y,z)} + \frac{1}{c(t,y,z)}\int_{z\cdot a(y,z)}^{t} c(s,y,z)g(y + (t - z\cdot y)z)ds. \tag{8.3}$$

If $\rho_0(y, z) \in L^1(\mu)$, then $\rho(y + (t - z \cdot y)z) \in L^1_{\text{loc}}(\Omega)$, because of Corollary 7.6 and

$$\int d\mu(y,z)\int_{z\cdot a(y,z)}^{z\cdot y} d\mathcal{H}^1(t)c(t,y,z)\chi_{\Omega'}|\rho_0(y,z)|\frac{c(z\cdot a(y,z),y,z)}{c(t,y,z)} \leq \frac{n2^{n-1}}{\operatorname{dist}(\Omega',\partial\Omega)}\int d\mu(y,z)|\rho_0(y,z)|.$$

We have used the estimates of Corollary 6.3 with $s = (b + a)/2$ to evaluate

$$c(z\cdot a(y,z), y, z) \leq \frac{n2^{n-1}}{\operatorname{dist}(\Omega',\partial\Omega)}.$$

Since it is clear that a function of the form (8.3) is a solution to (8.1), we have proved the first part of the theorem, as well as an explicit formula for the solutions to (8.1) for which the divergence formula holds in each $\mathcal{Z}$.

A particular solution is obtained for $\rho_0 = 0$, and the strict positivity of ρ follows from the explicit formula (8.3) and Theorem 7.5. □

By integrating by parts, we obtain

$$-\int \big(\rho(x)d(x)\cdot\nabla\phi(x) + g(x)\phi(x)\big)dx = \int d\mu(y,z)\rho_0(y,z)c(a(y,z),y,z)\phi(a(y,z)) = 0, \tag{8.4}$$

from which it follows that $\rho_0(y,z) = 0$ μ a.e. (y,z) when $c(a(y,z),y,z) \neq 0$, if the solution ρ constructed in the above theorem is positive.

When $c(a(y,z),y,z) = 0$, by integrating

$$\partial_t(\rho(y+(t-z\cdot y)z)c(t,y,z)) = g(y+(t-z\cdot y)z)c(t,y,z)$$

as

$$\rho(y+(t-z\cdot y)z) = \rho(y+(\bar{t}-z\cdot y)z)\frac{c(\bar{t},y,z)}{c(t,y,z)} + \frac{1}{c(t,y,z)}\int_{\bar{t}}^{t} c(s,y,z)g(y+(t-z\cdot y)z)ds, \tag{8.5}$$

and assuming that $\rho \in L^\infty_{\mathrm{loc}}(\Omega)$, then for μ a.e. (y,z) it follows that

$$\rho(y+(t-z\cdot y)z) = \frac{1}{c(t,y,z)}\int_{z\cdot a(y,z)}^{t} c(s,y,z)g(y+(t-z\cdot y)z)ds, \tag{8.6}$$

also when $c(a(y,z),y,z) = 0$, i.e. $\rho_0(y,z) = 0$ for μ a.e. (y,z) such that $a(y,z) \in \Omega$. We thus obtain the following uniqueness result:

Corollary 8.2. *The solution $\rho \in L^\infty_{\mathrm{loc}}(\Omega)$ to* (8.1) *constructed in Theorem 8.1 satisfies $\rho(a(y,z)) = 0$ μ a.e. on the the set $\{y,z : a(y,z) \in \Omega\}$.*

We recall that the existence of a weak $L^\infty_{\mathrm{loc}}(\Omega)$ solution can be proved directly by means of the convergence of the vector fields d_I to d, see Section 4 of [4].

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