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Why 'scaffolding' is the wrong metaphor[1]

There is an important difference between mathematical representations and writing in natural language. It is an essential feature of mathematical representations that they invite actions of a disciplined sort that encode rules of rigour. In Euclidean plane geometry[2], there is a short list of actions that one is explicitly permitted to do to a drawn figure (the first three postulates). To this explicit list may be added other moves that are implicit in the practice, such as superposition. Sticking to this list of implicitly and explicitly permitted actions is part of what it means to work rigorously in Euclid's system. (The rest of the specification of rigour in Euclid books I-IV concerns the interaction between the diagram and the accompanying text.[3]) Similarly, the difference between number-words[4] in natural language (whether written or spoken) and numerals in a system of arithmetic is that the arithmetic numerical system has rules for the manipulation of its symbols that (mostly) ensure that the reckoning is accurate. With Hindu-Arabic numerals, the algorithms for adding, subtracting, multiplying and dividing large numbers work even in the hands of someone who does not know how or why they work. The task of tracking all the single-digit calculations and recombining them correctly has been off-loaded to the numeral system and its associated practices (such as long multiplication and long division). High-school algebra has the same feature. In addition to rules for forming expressions, algebra at this elementary level has rules for transforming them (such as the rules for multiplying out brackets or gathering terms). These transformation rules embody a good deal of the rigour of the practice.[5] This is not true of natural language (whether spoken or written). Grasping that "Mary and John are in the kitchen" means the same as "Mary is in the kitchen and John is in the kitchen" is part of knowing the language, but understanding such equivalences does very little to aid inference and calculation. In a slogan: we do things *with* words, but we do (disciplined, rule-governed) things *to* mathematical representations. We extend their baselines, we join their points with lines, we gather their terms and factorise their expressions. In using these representations, we offload much of the cognitive load of calculation and inference.

Explaining this in more detail will supply examples and embellishment to Richard Menary's (2015) argument for his claim that enculturation does not merely refine and ripen our natural mathematical abilities, but transforms them and embeds them in a world of representations, practices and artefacts without which they could not function. More precisely, we will add detail to the arguments

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[1] I should like to record my gratitude to the philosophy department at the Bristol University for hosting the meeting where these ideas were first presented, to Catarina Dutilh Novaes encouraging me to write them up, and to two anonymous and diligent referees who saw this paper through three rounds of significant improvements.

[2] That is, the system expounded in Euclid books I-IV.

[3] See Manders (2008) for a comprehensive analysis.

[4] Note that a language with precise number-words that are not limited in size is already a significant cognitive tool. According to Ferreirós, "With counting, the imprecise innate grasp of cardinality is joined by a notion of order (ordinality), which refines the notion of cardinality into a precise concept, with the aid of specific words, so that they combine to obtain the number concept." (2016, p. 186). By 'mathematical representations' in this paper is meant inscriptions, mental images, computer models, etc. in addition to the words for counting numbers that one has simply as a speaker of a language that expresses the counting-number concept.

[5] Though not all of it. Following the expression-transformation rules of the language of high school algebra is not quite a perfect guarantee of rigour, because they can't always prevent you from accidentally dividing by zero or from dividing an inequality by a negative number.

that Menary offers in sections five and six of his paper (Menary 2015). As a philosopher of mathematical practices, I have nothing to say directly about brains.[6] The contribution of this paper to Menary's argument is that mathematics must be enculturated because mathematical reason is encoded in and offloaded to external representations to a much greater degree than other discourses.[7]

For this reason, 'scaffolding' is a poor metaphor for the cognitive assistance that we get from mathematical representations. First, we need to see where this scaffolding metaphor came from, and in which respects it is and is not accurate. As we will see, it has never been a very good metaphor, even for the purpose it had when it was first introduced in educational psychology. Nor is it especially apt for expressing scaffolding theory in cognitive science. Despite its shortcomings, it is now so entrenched that it is best regarded as a dead metaphor. There is no point in campaigning for its removal. The value of assessing it as a metaphor is to throw into relief some of the particular advantages of mathematical representations.

Scaffolding—the origins of a metaphor

The scaffolding metaphor originates in an educational psychology paper from 1976, 'The role of tutoring in problem solving' by Wood, Bruner & Ross.[8] Just as cognitive science, at a corresponding stage of its development, modelled the human mind as an isolated system that receives inputs from and sends outputs to its environment but remains separate from it, so educational psychology treated children as isolated units. This paper by Wood et al. was part of a critical reaction to that approach:

> Discussions of problem solving or skill acquisition are usually premised on the assumption that the learner is alone and unassisted. If the social context is taken into account, it is usually treated as an instance of modelling and imitation. But the intervention of a tutor may involve much more than this. More often than not, it involves a kind of "scaffolding" process that enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts. This scaffolding consists essentially of the adult "controlling" those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence. (Wood, Bruner & Ross, 1976 p. 2)

Note that the presence of scaffolding is signalled by success in some task that would at first be beyond the bare, isolated individual. The most accurate element in the scaffolding metaphor here is that the tutor's assistance is temporary. As the child achieves mastery, the scaffolding is removed.

Note also that the tutor is active in helping the child to learn. Wood *et al.* offer six functions that the 'scaffolding' tutor performs while the child attempts the set task, including direction maintenance, marking critical features and frustration control. At every stage, the tutor seeks to minimise her interventions, so she constantly monitors and reacts to the child's propensity to wander off task,

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[6] Doing philosophy of mathematical practices means treating mathematics as a human activity rather than a body of established knowledge. Whether or not this is a replacement for, a challenge to or a friend of longer established approaches to the philosophy of mathematics is matter of dispute. See Mancosu (2008) p. 18.

[7] This does not entail that other discourses are not enculturated.

[8] See (Stone 1998) for the history of the scaffolding metaphor from its origin to 1998. See also following footnote.

overlook critical features and get frustrated. In this sense, 'scaffolding' was always a poor metaphor for what Wood *et al.* had in mind. Literal scaffolding is inert and indifferent. It does not vary the support it offers precisely according to need. Given that it is not an especially apt metaphor, one might wonder why it was chosen. It may be that the scaffolding metaphor suggested itself as a development of constructivist approaches to educational psychology. If the child constructs the edifice of its own knowledge, then 'scaffolding' is a natural way of extending the architectural-construction metaphor to include the role of the tutor in facilitating (but not contributing to) that cognitive building-work. Whatever its shortcomings, the scaffolding metaphor in this educational context denotes something precise, namely the teacher's six functions. It captures the temporary nature of the support and the constructivist idea that the principal agent is the child, not the teacher.

The word 'scaffolding' seems to have entered the discourse of cognitive science just over twenty years later, thanks to Andy Clark:

> We may call an action 'scaffolded' to the extent that it relies on some kind of external support. Such support could come from the use of tools, or the knowledge and skills of others; that is to say, scaffolding (as I shall use the term) denotes a broad class of physical, cognitive and social augmentations—augmentations which allow us to achieve some goal which would otherwise be beyond us. Simple examples include the use of a compass and pencil to draw a perfect circle, the role of other crew members in enabling a ship's pilot to steer a course and the infant's ability to take its first steps only while suspended in the enabling grip of its parents. (Andy Clark, 1998)[9]

On this definition, almost everything we do is scaffolded action, except the most elementary bodily functions including singing, whistling and silent cogitation. Notice that the scaffolding metaphor is sustained in the word 'support'. Even when we confine ourselves to epistemic actions[10], the scope of the term 'scaffolded action' is broad. Turning to an early reviewer of Clark's book:

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[9] Menary and others routinely present Clark as the originator of this usage in cognitive science. This is corroborated by Google Scholar searches: prior to 1976 (the year of Wood et al.), scaffolding is not mentioned in educational theory or developmental psychology. From 1976 to 1998, there are many references to scaffolding in educational theory but none in cognitive science. After 1998, scaffolding appears in papers in cognitive science, mostly with direct citations of Clark's use. See (Stone, 1998) for a history of the term in educational theory from Wood et al. onwards. Writing in 1998, the year of Clark's paper, Stone makes no mention of cognitive science. Clark does not cite Wood et al., but there is a common ancestor in Vygotsky. Wood, Bruner and Ross do not mention Vygotsky in their article but (as Stone explains) Bruner wrote on Vygotsky and in educational psychology the scaffolding metaphor was taken up as a way to articulate Vygotsky's model of teaching as management of the pupil's zone of proximal development. The quotation from Clark in the present paper is taken from a section in which he discusses Vygotsky with approval as a source of his own view.

[10] Epistemic actions are "*external* actions that an agent performs to change his or her own computational state." (Kirsh & Maglio, 1994, p. 514). They elaborate thus, "a physical action whose primary function is to improve cognition by: 1. reducing the memory involved in mental computation, that is, space complexity; [or] 2. reducing the number of steps involved in mental computation, that is, time complexity; [or] 3. reducing the probability of error of mental computation, that is, unreliability. Typical epistemic actions found in everyday activities… include familiar memory-saving actions such as… placing a key in a shoe, or tying a string around a finger; time-saving actions such as preparing the workplace, for example, partially sorting nuts and bolts before beginning an assembly task in order to reduce later search…; and information gathering activities such

> Both these categories, epistemic action and external scaffolding, Clark points out, are extremely large: maps, models, tools, language and culture can all act as external scaffolding; using any of these pieces of scaffolding, for example, writing one large number above another to multiply them with pen on paper, is epistemic action. In all these cases, we act so as to simplify cognitive tasks by "leaning on" the structures in our environment.[11]

Here, the metaphor is more explicitly maintained; 'scaffolding' means external structures on which we lean. It invokes a temporary structure of spars and planks erected around a building. It is worth pausing to consider this metaphor in the context of cognitive science. Literal scaffolding is immobile—it is vital to its function that it should not move much beneath the feet of the workmen standing on it. It is not open to manipulation. So long as it does not wobble when you stand on it, it has done its job. The tools and practices that make mathematics possible are nothing like this, as we shall see. Even before we consider mathematical examples, it is clear that much of what Clark would regard as cognitive scaffolding is unlike literal scaffolding. Language and culture, and the assistance of crew-mates, can be dynamic and refined. Scaffolding of the literal sort, in addition to being temporary, is strictly speaking separate from the building it abuts. However close it is, it remains a separate structure with a separate function. In contrast, the principal claim of the extended mind hypothesis is that there is a continuity of function between the embodied brain and the environmental[12] affordances with which it is coupled. Literal scaffolding does not usually share its function with the building it abuts. As we shall see, there is an intimate back-and-forth between the body of the mathematician and the inscriptions of mathematical practices.

Scaffolding in Clark's sense is a much broader category than in Wood et al. In contrast to the temporary assistance provided by teachers, much of the scaffolding in Clark's sense is permanent (though not the parental support of the toddler learning to walk). I will always need a compass to draw a circle; even champions at mental arithmetic cannot compute large or complex calculations; the pilot will always need a crew; and so on. On the other hand, the tutor in the experiment reported by Wood et al. is not a passive artefact like a compass.

In Clark's sense, 'scaffolding' is any object (natural or fabricated) or practice that permits us to achieve something that we could not manage alone and naked.[13] That is not a problem for Clark. It helps his project in cognitive science if almost all human action is 'scaffolded' in his sense, because he needs to show that pattern-matching neural nets can do everything we can do. From his point of view, the more help they have, the better. On the other hand, it does mean that the language of scaffolding is inapt to explore the variety of kinds of environmental cognitive assistance, and the specific advantages of mathematical representations.

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as exploring, for example, scouting unfamiliar terrain to help decide where to camp for the night." (*Op. Cit.* pp. 514-5).

[11] Chemero, 1998. Of course, Clark is not responsible for what others have made of his metaphor. Nevertheless, Chemero is pertinent because the reception and use of the metaphor by the cognitive science and philosophy of mind communities is our target, rather than Clark's personal take on it.

[12] For the most part, the environmental affordances for mathematics are artefacts, but nothing hangs on this. One might use pebbles as counters or draw diagrams in wet sand with a stick.

[13] Naked, because even the most rudimentary clothing is rich in affordances. With a loincloth, one might carry water, signal for help, trap a small animal, bind a wound, etc..

Menary's argument and the philosophy of mathematical practices

The conclusion of Menary's argument in his (2015) is about brain development. He argues against the view that "we are born with our primary cognitive faculties intact and they simply need to mature, or be finetuned by learning mechanisms" (Menary 2015, abstract). Rather, he claims that, "a process of enculturation transforms our basic biological faculties." (*ibid.*)—right down to the level of brain-structure. I have nothing to say about brains, but I am interested in Menary's argument, which is indirect and takes a route through the philosophy of mathematical practices. He offers as evidence for his enculturation view of mathematical cognition, "A long period of development, learning-driven plasticity, and a cultural environment suffused with practices, symbols, and complex social interactions" (*ibid.*). If the learning mechanisms for mathematical cognition do no more than fine-tune our inborn cognitive faculties, then (he asks) how come it takes so long and involves so much that lies outside the brain? On Menary's view, the education of a competent human is more a matter of establishing dynamic couplings between that person and the inherited cognitive practices in the environment.[14]

From the point of view of the philosophy of mathematical practices, Menary's description of enculturation sounds entirely familiar:

> Practices govern how we deploy tools, writing systems, number systems, and other kinds of representational systems to complete cognitive tasks. These are not simply static vehicles that have contents; they are active components embedded in dynamical patterns of cultural practice. (p. 4)

Philosophers of mathematical practices have been making versions of this point for some time. Consider ordinary algebraic notation, as learned in secondary school. This was invented and developed to the point where it became an accepted tool of mathematical argument within the lifetime of one of its developers, René Descartes (1596-1650). When he was born, many mathematicians were still writing algebraic procedures out in something close to ordinary prose. This mathematical prose had ready abbreviations for 'the thing sought', 'square' and so on, and these were eventually replaced by symbols. The crucial breakthrough, however, was the introduction of brackets.[15] Then, the same quantity could be written in two ways (that is to say, brackets can be multiplied out, or conversely terms can be gathered). At first, the manipulation of these new symbols was regarded as a useful calculating aid, but such manipulations quickly (that is, in a couple of decades) gained the status of proof-procedures. Mathematicians had invented syntactic argumentation, in the sense that the rules for transforming the symbols coincide with most

[14] In his 1998 paper, Clark rejects the suggestion that the effect of mathematical training is to reprogram the brain (which he believes to be a pattern-completing, massively parallel neural net) so that it starts to behave like a serial computer. 'Dynamic couplings' is my gloss on Menary's paper.

[15] The earliest algebraic notations, such as that of Thomas Harriot, did not use brackets quite as we have them now. The methods that we now recognise as algebraic emerged in Europe from arithmetic with Hindu-Arabic numerals, and the work of brackets was done by devices such as arranging expressions in vertical columns as in long multiplication. There is some obscurity over who developed symbolic algebra first, because Renaissance mathematicians did not always publish their methods when these contravened the official standards of rigour or if they saw a commercial advantage in keeping them secret. John Wallis was convinced that Descartes must have seen and plagiarised Thomas Harriot's book, because he heard that a French nobleman was struck by the similarity between Harriot's system and Descartes's. Wallis's argument has not persuaded many historians. What matters here, philosophically, is not the origins or details of this story but how it ends, that is, with symbolic algebra governed by syntactic rules for transforming equations.

of the rules of arithmetic.[16] When Newton, late in the 17th century, retained the practice of offering geometrical proofs for results that he had found algebraically, this made him something of a conservative. For almost everyone else, manipulating symbols according to syntactic rules without worrying about what the symbols meant was a perfectly rigorous way of doing mathematics. With the new notation, more of the rigour was offloaded into the symbolism than ever before.

Turning from the history of mathematics to cognitive science, the first point to note is that the actions on algebraic symbols that constitute the process of proof are not actions that one could perform on the same content represented by means available to mathematically untrained but otherwise cognitively normal adult humans. To see this, consider the following initial step in a proof by George Pólya:

> Define the real numbers $c_1, c_2, c_3, \ldots c_n, \ldots$ by:
>
> $$c_1 c_2 c_3 \ldots c_n = (n+1)^n$$
>
> Then trivially, for any real numbers $a_1, a_2, a_3, \ldots a_n, \ldots$ it follows that:
>
> $$\sum_1^\infty (a_1 a_2 \ldots a_n)^{1/n} = \sum_1^\infty \frac{(a_1 c_1 a_2 c_2 \ldots a_n c_n)^{1/n}}{n+1}$$

This is the first step in a proof that consists entirely of re-arrangements and replacements of terms (Pólya, 1954, II, 147). Given the definition of the c_i, this first step is relatively easy for a trained mathematical eye to see—provided we have the algebraic notation available to us, for this is a mathematical inference made evident and compelling by the notation. We could write this out in mathematical prose, but then the correctness of the equation would be lost in a mass of words. One way to see the importance of the notation is to consider why Pólya used the recursive definition of the c_i. Formally, he might just as well have defined c_n as ${(n+1)^n}/{n^{n-1}}$, but then Pólya's 'trivial' step would not have been as obvious. Even setting up the simpler left-hand side would baffle many people who can easily understand the operation as written here ("Think of an infinite set of real numbers and put them in some order. It doesn't matter what order. Now consider the sequence of initial segments of that order. Then form the sum of the geometric means of those initial segments. Now hold that thought while we define another infinite set of real numbers…"). Suppose that a reader has understood such a prose equivalent to the expression on the left-hand side of Pólya's equation and further understood the definition of the c_i without making any use of algebraic notation. The whole equation written in prose would be baffling, and it would be very difficult to see why it is true. Perhaps a Renaissance mathematician who was used to reading mathematical arguments in prose might manage it—but it's worth noting that even this relatively simple expression is more complex than anything in Renaissance mathematics.

In his paper, Menary makes it sound as though the necessity for mathematical notations arises simply because proofs are too long and detailed for human working memory:

> Our cognitive capacities cannot cope with long sequences of complex symbols and operations on them. This is why we must learn strategies and methods for writing out proofs. (2015, p. 16)

[16] For a brief account of this development, see Larvor (2005). For exhaustive detail, see Serfati (2005).

If the length and complexity of the operations were the whole story, then writing them down in ordinary prose would solve the problem. After all, legal arguments can be long and complex. As already noted, an important difference with mathematical notations and diagrams is that they have rigour built in to their use. Sticking to the rules for manipulating the symbols or the diagram goes a long way towards guaranteeing the correctness of the result (and in systems that are known to be sound, it goes all the way to guaranteeing correctness). This allows us to offload much of the cognitive effort and reduces the work that the human performs to much smaller units. Instead of grasping an entire argument, it is enough to check that each step in a proof or a calculation follows the rules of the practice. Mathematical symbol-systems and diagrammatic practices do not merely summarise and represent information efficiently.[17] They embody mathematical thinking. Pólya offers the example cited above as a case in which it is easy to check each step of the proof (they are all trivial) without understanding why the result is true or how it might have been discovered. Of course, we want understanding as well as correctness. That is the ultimate point of Pólya's example. Nevertheless, checking the correctness of this proof largely consists in checking that the successive versions of the equation are achieved by following the rules for transforming one algebraic expression into another. Broadly speaking, syntactically[18] encoded reasoning goes much farther and is much more central in mathematics than in other disciplines.

The essential role of symbol-manipulation in mathematical thinking is the reason why blackboards (and now, whiteboards) are central to mathematical material culture.[19] One way to see how mathematicians think is to watch them teach the next generation. In teaching, the rules of the practices are made explicit and the characteristic techniques and tools exhibited. Every discipline has its signature pedagogy, its teaching occasion when novices learn to think, perform and act as members of their profession. Shulman (who originated the expression 'signature pedagogy') offers as examples the quasi-Socratic exchanges in law school and the teaching rounds when medical students accompany a senior physician visiting patients in a hospital (Shulman 2005, p. 52). In mathematics, the signature pedagogy is the proof performed and discussed on the blackboard. The instructor does not simply print out the proof as a literature professor might write out a poem ready for analysis—for if that were the case, the mathematics instructor has no reason not to follow the literature professor in switching from the blackboard to pre-prepared slides.[20] The mathematics instructor has to teach the strategic use of inscription-space, the separation of the blackboard into a working area and an area for recording the developing proof. The students must learn to pay attention selectively to the symbols before them, to see possible manipulations and to find a series of permitted inscriptions that leads to a solution. To return to our equation from Pólya, the first

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[17] In his 1998 paper, Clark lists six kinds of help that 'scaffolding' can offer to help humans think, but ensuring the correctness of results is not one of them.
[18] Reading 'syntax' broadly so that, for example, the rules for adding elements to diagrams in Euclidean geometry are the syntax of that system.
[19] The use of a large, wall-hung blackboard seems to have originated at the start of the 19th century. Before that, mathematics was taught and learned on hand-held slates that had the same features (ease of inscription and erasure) that endear blackboards to mathematicians (Wylie, 2012). The significant point here is not how or when mathematicians started using blackboards, but rather their reluctance to give them up now that other technologies are available. See Barany & MacKenzie (2014) for a fuller exploration of the place of chalk, blackboards and spontaneous inscription in mathematical practice.
[20] See Artemeva & Fox (2011) on the persistence of 'chalk talk' as a pedagogical genre in university level mathematics teaching.

thing an experienced reader of mathematics[21] does with it is to check that there is no difference in the summations. They both go from one to infinity, so the eye can blank them out and focus only on the expressions being summed. The next thing is to check the indices of the a_i and the c_i. They just run from 1 to *n*, so we need pay no more attention to those fiddly little subscripted numbers. What about the exponents? These are $^1/_n$ on both sides, so again one can withdraw a little of one's attention. At this point, one is ready to spot the structural similarities between these expressions and the recursive definition of the c_i, and then the equation becomes trivial. Controlled selective attention to inscriptions is essential to mathematics. It works together with rule-governed symbol manipulation to constitute the practice of algebraic argument. This explains the centrality of blackboards and whiteboards in mathematical life. You cannot manipulate expressions written on pre-prepared slides. You cannot perform the characteristic epistemic actions of mathematics on a fixed image.

Menary gestures towards a third feature of mathematical representations with his observation that, "A further issue is how novelty comes about from the ability to abstractly combine symbols and functions that apply to the symbols." (*Op. cit.* p. 15). This has many dimensions. Sometimes, the symbolic expressions are suggestive (as in, for example, the structural similarity between a binomial expansion and Leibniz's general rule for repeated differentiation of a product).[22] Sometimes they become objects of mathematical enquiry in their own right.

To see something of the deeper creative role of symbols in mathematics, note that equations written in algebraic notation became an object of study almost immediately. Descartes noticed that there is a relationship between the number of positive roots of a polynomial equation with one variable and the number of times the sign of the coefficients changes (assuming the equation is written in the usual way, with all the terms on one side, and the powers of the variable in descending order). The rule states that the number of positive roots of the equation is either equal to the number of sign changes between consecutive non-zero coefficients or is less than it by an even number. For example,

$$0 = x^3 + 3x^2 - x - 7$$

has just one change of sign (between the x^2 and the x), so it has precisely one positive root. There is a corresponding rule to work out how many negative roots the equation has. It is unlikely that this would have been discovered before the invention of the new notation.[23] It is typical of mathematics that tools, notations and processes become objects of mathematical study.

Descartes's rule calculates the number of real roots, but some equations have complex roots. If we consider all the solutions to a polynomial equation in one variable, real and complex, it is possible to write any such polynomial equation like this: $f(x) = (x-\alpha)(x-\beta)(x-\gamma)\dots(x-\sigma)$ where the Greek letters are the roots. The symmetry of this expression is obvious and because multiplication of complex numbers is commutative, we could re-order the factors without altering the equation.

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[21] On the difference between expert and novice readers of mathematics, see Inglis & Alcock (2012).

[22] See (Larvor 2010) pp. 198-201 for Leibniz's heuristic use of structural similarities between algebraic expressions.

[23] Until the end of the 16th century, mathematicians tended to re-arrange equations of a given degree so that all the coefficients were positive. Perhaps a Renaissance mathematician might have seen a general relation between the number of roots and the distribution of terms either side of the equation. After all, al-Khwārizmī knew that quadratics in which squares plus numbers equal roots give two results (see Heeffer 2013). It's not categorically impossible to understand the rule of signs without the symbolism—but no-one did.

The tool for studying symmetries and permutations is group theory, because the set of permutations of some collection of objects is an algebraic group. For example, if we have three objects, there are six ways of shuffling them (including the null-shuffle that leaves them all exactly where they are). These six possible shuffles of three objects form an algebraic group, because two shuffles can be combined (that is, carried out in sequence) to create a third shuffle. In the 19th century, Évariste Galois (1811-1832) developed an approach to the study of equations based on group-theoretic study[24] of the permutations of the roots of polynomials that led to (among other things) the solution to a long-standing question. Mathematicians had found general solutions in radicals for polynomials of degree two, three, and four—but then the progress stopped. Galois's work on the group structure of permutations of roots of polynomials resulted in a proof that there can be no solutions in radicals for polynomials of degree higher than four. Here again, a tool has become an object of study, and the notation plays an essential role in that process.

For another example, also from 19th century algebra, consider Cayley graphs of finitely generated groups. This is a way of picturing groups in diagrams that are themselves mathematical objects. Each element of the group appears as a node in the graph, and the edges of the graph are the generating elements that get you from one node to the next (distinguished by colours and arrows). For example, the rotations of an equilateral triangle form a group called C_3. This group has a Cayley graph (which is itself a triangle, but this coincidence does not hold generally). The group can be generated by a single twist of the triangle through 120 degrees, and we can represent that generating element by the arrowed line.

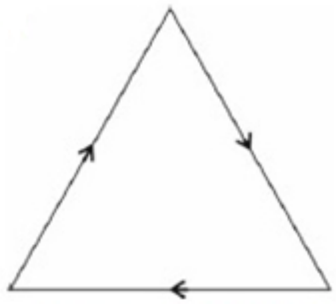

Figure 1: Cayley Graph of the group C_3

Now think of another operation: flipping the triangle along a vertical axis. This gives us an even simpler group, C_2. If we combine these two groups, we have six ways of rearranging our triangle, and these are displayed in this Cayley graph, where the dotted line represents the vertical flip:

[24] As is common in mathematics, Galois did not invent Galois theory as it is now taught to undergraduates. The group-concept was not isolated and captured in axioms until after his death. The details of this story do not affect the philosophical point, which is that once expressed in modern notation, equations became objects of systematic mathematical enquiry. There is even a glimmer of this in Harriot, who had a classification of equations into canonical and non-canonical forms (Harriot 2007).

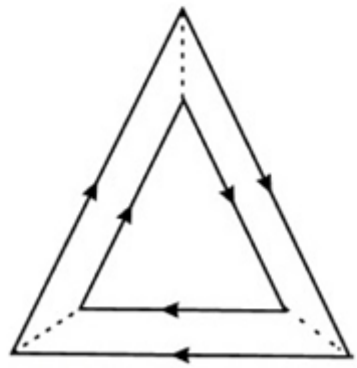

Figure 2: A Cayley graph for the abelian group $C_3 \times C_2$

So far, we have simply found a new way of depicting finitely generated groups. But these Cayley diagrams are directed graphs and as such are the objects of study of another body of mathematics, namely, graph theory. By using theorems from graph theory, mathematicians can prove results about Cayley graphs and from these derive information about the groups that the Cayley graphs represent.[25] The Cayley graphs themselves become objects of epistemic actions—one of the central mathematical papers in this area is entitled "Cutting up Graphs" (see Starikova 2010).

Thus, the novelty arising in this way from mathematical notations is a special advantage of mathematical representations. It is not simply a special case of the fact that in any language there are indefinitely many sentences that have never been uttered or written. Nor is it simply the fact that writing something down in an efficient form permits a synoptic overview that sparks new insights (this is after all true of a wiring diagram or an architect's blue-print). Mathematical symbol systems and diagrams can become objects of mathematical enquiry in their own rights, which can generate whole new areas of mathematics and the development of powerful techniques for resolving the original questions and problems that prompted the enquiry.

Conclusion

Consideration of some examples from elementary algebra permitted us to amplify and elaborate Menary's remarks about the uniqueness of the cognitive help that we get from mathematical notations and other mathematical representations, and about their role in the generation of new mathematics. Early in this paper, we quoted Chemero's explication of scaffolding in Clark's sense, "In all these cases, we act so as to simplify cognitive tasks by "leaning on" the structures in our environment." Consideration of mathematical cases show that this is false. Mathematical representations do not merely simplify cognitive tasks that we might otherwise have faced unsuccessfully. Mathematics is full of cognitive tasks that would not exist without the notations and representations that create the environment in which those tasks present themselves. Calculating the Galois group of a polynomial is one such example. Investigating finitely generated groups through their Cayley graphs is another. Examples could be multiplied endlessly. We do not simply 'lean on' mathematical notations; we manipulate them according to truth-preserving syntactic rules, we offload thinking to them and we think about them mathematically using other diagrams and

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[25] See (Starikova 2010) for examples of the way that graph theory can supply methods to group theory via Cayley graphs. This relies on the fact that although there are often several graphs for any given finitely generated group, all the graphs of a given group share certain properties and these properties can therefore be regarded as properties of the generating group.

notations that allow us to offload some of that thinking. Some of the most effective applications of mathematics are to mathematics itself.

It is of course open to Menary's opponents to insist that, however useful the external resources of mathematics might be, the thinking is still all in the head. This raises a point that returns us to the short-comings of the scaffolding-metaphor. The notations and diagrams of mathematics are mostly external to the human organism (though it is possible to do some limited mathematics with symbols and diagrams imagined inwardly). But these marks and pictures only become mathematical representations when they are part of a practice. The practice is partly constituted by the rules for using the symbols and diagrams, and it is the fact that these rules embody mathematical rigour that gives mathematical representations their special utility. So where are the practices? Are they inside or outside the human organism? In one sense, they are outside. They are socially agreed ways of doing things with symbols and diagrams. Someone learning mathematics meets these practices as part of the environment and learns them by doing them in public—on a chalkboard or in an exercise book. On the other hand, they have to be internalised, and because mathematics is essentially inscribed, internalising its practices includes developing new muscular habits and new ways of reading.

Recent studies emphasise the intimate relationship between the mathematically trained hand, eye and brain. In a series of papers, De Toffoli and Giardino have elaborated a concept of 'manipulative imagination' to articulate the skill of using diagrams in knot theory and low-dimensional topology.[26] To learn knot theory, it is not enough to read knot diagrams. One must learn to imagine loops in the knot being slid around or flipped across other loops. Roi Wagner (2017) borrows the term 'haptic vision' from Deleuze, and characterises it thus:

> Initially, the hand only drew, and the eye only observed the result. But the emergent Figure involves a *man*ipulation or *hand*ling without resorting back to the hand, using only the associated enriched mode of vision. Indeed we see in the diagram a sequence of past drawings, intended drawings and possible integration of noise into a new drawing—all without any actual manual re-drawing. (Wagner 2017, p. 170)

'Haptic vision' is the result of coupling between hand, eye and brain so intimate that the eye is able to do the handling on its own, and ultimately, in some cases, need not have recourse to a physical diagram or written expression. Menary writes:

> Mastery of [mathematical] symbol systems results in changes to cortical circuitry, altering function and sensitivity to a new, public, representational system. However, it also results in new sensori-motor capacities for manipulating symbols in public space.

The philosophy of mathematical practice can have nothing to say about exactly what these changes to cortical circuitry are. Nevertheless, mastering mathematical practices does require the development of 'haptic vision', and that must be somehow registered in the brain. It's worth noting that Clark mentions just one mathematical example in his 1998 paper. As always, his aim is to argue that a massively parallel pattern-matching machine can do everything that we can do, because that (in his opinion) is what we are:

[26] De Toffoli, S. & Giardino, V. (2013, 2014, 2016); Giardino (2017).

> …experience with drawing and using Venn diagrams allows us to train a neural network which subsequently allows us to manipulate imagined Venn diagrams in our heads. Such imaginative manipulations require a specially trained neural resource to be sure. But there is no reason to suppose that such training results in the installation of a different kind of computational device. (1998 p. 166-7)

The structure and logical order of the brain is a matter for other disciplines. Nevertheless, it should be evident in view of the examples and arguments presented here that the scaffolding metaphor radically misdescribes the help that we get from mathematical inscriptions and other elements of mathematical material culture (such as cardboard models, computer-generated images and shapes drawn in the air). It does not readily express the to-and-fro between inward cogitation and the manipulation of symbols and diagrams, nor the process of internalising shared, materially mediated mathematical practices. Therefore, it begs the question in favour of those who would confine thinking to the inside of the skull. Given Clark's and Menary's philosophical aims, this is an unfortunate effect of this metaphor's unbudgeable presence in the literature.

References

Artemeva, N., & Fox, J. (2011). The Writing's on the Board: The Global and the Local in Teaching Undergraduate Mathematics Through Chalk Talk. *Written Communication*, *28*(4), 345–379. https://doi.org/10.1177/0741088311419630

Barany, M., & MacKenzie, D. (2014-01-24). Chalk: Materials and Concepts in Mathematics Research. In *Representation in Scientific Practice Revisited*. The MIT Press.

Bruner, J. S. (1966). *Toward a theory of instruction*, Cambridge, Mass.: Belkapp Press.

Chemero, Anthony (1998) 'A Stroll Through the Worlds of Animats and Humans: Review of *Being There: Putting Brain, Body and World Together Again'* by Andy Clark. *PSYCHE*, 4(14), October, 1998

Clark, Andy (1998) "Magic Words: How Language Augments Human Computation" in P. Carruthers and J. Boucher (Eds) *Language and Thought: Interdisciplinary Themes* (Cambridge University Press: Cambridge, pp.162-183.

De Toffoli, S. & Giardino, V. (2013), 'Forms and Roles of Knot Diagrams'. *Erkenntnis*, 79(4), 829 – 842.

De Toffoli, S. & Giardino, V. (2014), 'An Inquiry into the Practice of Proving in Low-Dimensional Topology'. *Boston Studies in the History and the Philosophy of Science*, 308, 315 – 336.

De Toffoli, S. & Giardino, V. (2016), 'Envisioning Transformations. The Practice of Topology'. In: *Mathematical Cultures*, edited by B. Larvor, Birkhäuser Science, Springer.

Ferreirós, José (2016) *Mathematical Knowledge and the Interplay of Practices* Princeton University Press.

Giardino, Valeria (2017) «L'imagination manipulatoire en mathématique», *Bulletin d'Analyse Phénoménologique* [En ligne], Volume 13, Numéro 2: L'acte d'imagination: Approches phénoménologiques (Actes n°10)

Harriot, Thomas (2007) *Artis analyticae praxis*: an English translation with commentary / Muriel Seltman, Robert Goulding, editors and translators, New York: Springer.

Heeffer, A. (2013). 'Why the quadratic equation has only one root (according to abbaco masters)'. *Mathematics Today*, (2), 140–141.

Ifrah, Georges (1998) *The Universal History of Numbers from prehistory to the invention of the computer*. London: Harvill Press. Tr. Bellos, Harding, Wood & Monk.

Inglis, M. & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education, 43*, 358-390.

Kirsh, David & Maglio, Paul (1994) 'On Distinguishing Epistemic from Pragmatic Action' *Cognitive Science* 18, 513-549

Larvor, Brendan (2010) "Syntactic Analogies and Impossible Extensions" in Löwe, B., & Müller, T. [2010] *PhiMSAMP. Philosophy of Mathematics: Sociological Aspects and Mathematical Practice* (Texts in Philosophy volume 11). College Publications.

Mancosu, P. (2008) editorial introduction, *The Philosophy of Mathematical Practice*, Oxford University Press, pages 1–21.

Manders, Ken (2008) The Euclidean diagram. In Mancosu, P., *The Philosophy of Mathematical Practice*, Oxford University Press 2008, pages 80–133.

McLeod, S. A. (2008). Bruner. Retrieved from www.simplypsychology.org/bruner.html

Menary, R. (2015). Mathematical Cognition - A Case of Enculturation. In T. Metzinger & J. M. Windt (Eds). *Open MIND:* 25(T). Frankfurt am Main: MIND Group.

Pólya, G. (1954). *Mathematics and Plausible Reasoning*. Two Volumes. Princeton University Press, Princeton, N.J.

Shulman, Lee S. (2005) Signature pedagogies in the professions. *Daedalus* 134:3, 52-59

Starikova, Irina 2010 "Why do Mathematicians Need Different Ways of Presenting Mathematical Objects? The Case of Cayley Graphs" *Topoi* Volume 29, Issue 1 , pp 41-51

Stone, C. A. (1998). The metaphor of scaffolding: Its utility for the field of learning disabilities. *Journal of Learning Disabilities*, 31(4), 344-364.

Vygotsky, L.S. (1962). *Thought and Language*. Cambridge, MA: MIT Press.

Wagner, Roi (2017) *Making and Breaking Mathematical Sense: Histories and Philosophies of Mathematical Practice.* Princeton University Press.

Wood, David, Bruner, Jerome S. & Ross, Gail (1976) 'The role of tutoring in problem solving' J . *Child Psychol. Psychiat.,* Vol. 17, pp. 89 to 100. Pergamon Press. Printed in Great Britain.

Wylie, C.D. (2012) Teaching manuals and the blackboard: accessing historical classroom practices. *History of Education* 41(2), 257-272.