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BPS states of D=4 N=1 supersymmetry

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ABSTRACT

We find the combinations of momentum and domain-wall charges corresponding to BPS states preserving 1/4, 1/2 or 3/4 of D=4 N=1 supersymmetry, and we show how the supersymmetry algebra implies their stability. These states form the boundary of the convex cone associated with the Jordan algebra of 4×4 real symmetric matrices, and we explore some implications of the associated geometry. For the Wess-Zumino model we derive the conditions for preservation of 1/4 supersymmetry when one of two parallel domain-walls is rotated and in addition show that this model does not admit any classical configurations with 3/4 supersymmetry. Our analysis also provides information about BPS states of N=1 D=4 anti-de Sitter supersymmetry.

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1 Introduction

Although N=1 supersymmetric field theories in $3+1$ dimensions have been extensively investigated for more than twenty five years, most of these investigations have been based on the standard supersymmetry algebra. It has been known for some time, however, that p-brane solitons in supersymmetric theories carry p-form charges that appear as central charges in the spacetime supertranslation algebra [1]. Allowing for all such charges, the D=4 N=1 supertranslation algebra is spanned by a four component Majorana spinor charge Q, the 4-vector P_μ and a Lorentz 2-form charge $Z_{\mu\nu}$. The only non-trivial relation is the anticommutator

$$\{Q,Q\} = C\gamma^\mu P_\mu + \frac{1}{2}C\gamma^{\mu\nu}Z_{\mu\nu}, \tag{1}$$

where C is the charge conjugation matrix and $\gamma_\mu = (\gamma_0, \gamma_i)$ are the four Dirac matrices. Our metric convention is 'mostly plus' so that we may choose a real representation of the Dirac matrices. In this representation the Majorana spinor charges Q are real, so $\{Q,Q\}$ is a symmetric 4×4 matrix with a total of ten real entries. The number of components of P_μ and $Z_{\mu\nu}$ is also ten, so that we have indeed included all possible bosonic central charges. Note that the automorphism group of this algebra is $GL(4;\mathbb{R})$.

The components of $Z_{\mu\nu}$ can be interpreted as charges carried by domain walls [1], while P_μ is (in general) a linear combination of the momentum and a string charge. In the case of a domain wall, the tension is bounded by the charge, and saturation of this bound implies preservation of 1/2 of the N=1 D=4 supersymmetry. This is one example in the class of '1/2 supersymmetric' configurations allowed by the supersymmetry algebra[1] Such 1/2 supersymmetric domain walls were shown to occur in [2] in the Wess-Zumino (WZ) model, for an appropriate superpotential, and also arise in the $SU(n)$ SQCD [3] because the low-energy effective Lagrangian is that of a WZ model with a superpotential admitting n discrete vacua [4]. More recently, it was shown that the WZ model also admits (again for an appropriate superpotential) 1/4 supersymmetric configurations that can be interpreted as intersecting domain walls [6, 7]. More precisely, it was established that such configurations must solve a certain 'Bogomol'nyi' equation for which earlier mathematical studies had made the existence of appropriate solutions plausible (especially in view of the results of [8] which were recently brought to our attention). Domain wall junctions of the WZ model have since been studied further in [9, 10, 11, 12] and an explicit 1/4 supersymmetric domain wall junction of a related model has recently been found [13].

It was pointed out in [6] that the possibility of 1/4 supersymmetric intersecting domain walls is inherent in the supersymmetry algebra. If we choose $C=\gamma^0$ and $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, then (1) becomes

$$\{Q,Q\} = H + \gamma^{0i}P_i + \frac{1}{2}\gamma^{0ij}U_{ij} + \frac{1}{2}\gamma^{0ij}\gamma_5 V_{ij} \tag{2}$$

where $H = P^0$, $U_{ij} = Z_{ij}$ and $V_{ij} = -\varepsilon_{ijk}Z_{0k}$. One is thus led to expect 'electric' type domain walls with non-zero 2-form U_{ij} but vanishing V_{ij} and 'magnetic' type domain walls with non-zero 2-form V_{ij} but vanishing U_{ij}. In general, a domain wall will be specified not only by its tension and orientation but also by an angle in the electric-magnetic charge space; the domain wall is 'dyonic' when this angle is not a multiple of π. It is not difficult to show that the algebra (2) allows for dyonic charge configurations preserving 1/4 supersymmetry. In this paper we determine the model-independent restrictions on such configurations that are implied by the supersymmetry algebra.

As pointed out in [6], the charge associated with the stringlike junctions of domain walls in the WZ model appears in the supersymmetry algebra in the same way as the

[1] An analysis of 1/2 supersymmetric combinations of charges in $N>1$ D=4 theories, $N=2$ in particular, can be found in [5].

3-momentum, so for a static 1/4 supersymmetric configuration of the WZ model the 3-vector $\mathbf{P}$ must be interpreted as a string charge carried by the domain wall junction. It was further shown in [6] that this junction charge contributes positively to the energy of the 1/4 supersymmetric configuration as a whole. In contrast, the charge associated to domain wall junctions of the model considered in [13] was shown there to contribute *negatively* to the total energy. As we shall see, this apparent disagreement is due to a different central charge structure for the two models. There is therefore more than one field theory realization of static intersecting domain walls preserving 1/4 supersymmetry, but as yet no example that exploits the most obvious possibility in which $\mathbf{P}$ vanishes but U_{ij} and V_{ij} do not.

These observations underscore the importance of the model-independent analysis of 1/4 supersymmetric configurations based only on the N=1 D=4 supersymmetry algebra, but our aim is to understand the implications of the supersymmetry algebra for *all* supersymmetric configurations, not just those preserving 1/4 supersymmetry. Since the matrix $\{Q, Q\}$ is a positive definite real symmetric one, it can be brought to diagonal form with real non-negative eigenvalues. The number of zero eigenvalues is the number of supersymmetries preserved by the configuration. The 'supersymmetric' configurations are those for which this number is $1, 2, 3$ or 4. There is a unique 'vacuum' charge configuration preserving all four supersymmetries. Configurations preserving two supersymmetries are 1/2 supersymmetric while those preserving one supersymmetry are 1/4 supersymmetric. Configurations preserving three supersymmetries are 3/4 supersymmetric, but there is no known field theoretic realization of this possibility. Indeed, we will show here that there is no classical field configuration of the WZ model that preserves 3/4 supersymmetry. However, possible string-theoretic realizations of exotic supersymmetry fractions such as 3/4 supersymmetry were recently explored in [14], and this possibility has been considered previously in a variety of other contexts [15, 16, 17, 18]. In particular, the $OSp(1|8;\mathbb{R})$-invariant superparticle model of [16] provides a simple realization in the context of particle mechanics. The fundamental representation of $OSp(1|8;\mathbb{R})$ is spanned by $(\rho^\alpha, \lambda_\alpha, \zeta)$, where ρ and λ are two 4-component real commuting spinors of $Spin(1,3)$, and ζ is a real anticommuting scalar. The action

$$S = \int dt \left[\rho^\alpha \dot{\lambda}_\alpha + \zeta\dot{\zeta}\right] \tag{3}$$

is manifestly $OSp(1|8)$ invariant; in particular, it is supersymmetric with supersymmetry charge $Q = \lambda\zeta$. The canonical (anti)commutation relations imply that $\{Q_\alpha, Q_\beta\} = \lambda_\alpha\lambda_\beta$, which is a matrix of rank one, corresponding to 3/4 supersymmetry.

Thus, there exist models of one kind or another in which all possible fractions of D=4 N=1 supersymmetry are preserved. This fact provides further motivation for the general model-independent analysis of the possibilities allowed by the supersymmetry algebra that we present here. As we shall explain, the space of supersymmetric charge configurations, or 'BPS states', is the boundary of the convex cone of 4×4 real symmetric matrices and this has an interpretation in terms of Jordan algebras. In analogy with the way that the conformal group acts on massless states on the light-cone $P^2 = 0$, there is a group $Sp(8,\mathbb{R})$ that acts on the 'BPS cone' of supersymmetric configurations and which has an interpretation in this context as the Möbius group of the Jordan algebra [19]. Another purpose of this paper is to explore some of the geometrical ideas underlying this interpretation of supersymmetric charge configurations.

It is generally appreciated that BPS states are stable states, this being the main reason for their importance, but some "standard" arguments for stability rely on physical intuition derived from special cases. For example, a massive charged particle that minimises the energy for given charge cannot radiate its energy away in the form of uncharged photons because this would leave behind a particle with the same charge but lower energy, contradicting the statement that the original particle minimised the energy in its charge sector. However, this heuristic argument is not conclusive. For instance, the stability against radiative relaxation to a lower energy state of the same 'charge vector' assumes that the radiated energy carries away no momentum because momentum is one of the charges, and this assumption would be violated by a decay in which just one photon is emitted. It is also implicit in the heuristic argument that prior to decay one can go to the rest frame, but the supersymmetry algebra allows BPS states for which this is not possible, a massless particle being an obvious, but by no means the only, example. These considerations show that it is not quite as obvious as generally supposed that BPS states are stable. Here we provide a complete analysis, for the general D=4 N=1 supersymmetry algebra, based on a combination of the Minkowski reverse-triangle inequality for positive-definite matrices and the ordinary triangle inequality for BPS energies.

The supertranslation algebra for which (1) is the only non-trivial (anti)commutator is a contraction of the superalgebra $osp(1|4;\mathbb{R})$, which is the D=4 N=1 anti-de Sitter (adS) superalgebra. The anticommutator of the 4 real supercharges of the latter is

$$\{Q,Q\} = C\gamma^\mu P_\mu + \frac{1}{2}C\gamma^{\mu\nu}M_{\mu\nu}, \tag{4}$$

where $M_{\mu\nu}$ are the Lorentz generators. This is formally equivalent to (1), although the charges on the right hand side are no longer central because they generate the adS group $SO(3,2)$. However the positivity conditions on these charges are the same, as are the conditions for preservation of supersymmetry. This fact means that much of our analysis of the centrally-extended supertranslation algebra can be immediately applied to the adS case. A related analysis has been considered previously for D=5 in [20], where the D=4 case was briefly mentioned, and BPS states in D=4 adS have also been analysed by other methods in [21].

We begin with an analysis of the N=1 D=4 supersymmetry algebra, determining the charge configurations that preserve the various possible fractions of supersymmetry, and we show how the positivity of $\{Q,Q\}$ implies the stability of BPS states carrying these charges. We also show how the supersymmetry algebra determines, in a model-independent way, some properties of the 1/4 supersymmetric intersecting domain walls that are realized by the WZ model, but show also that 3/4 supersymmetry is not realized by classical WZ field configurations. We then turn to an exposition of the geometry associated with the supersymmetric configurations, which is that of self-dual homogeneous convex cones, and review their relation to Jordan algebras. We then discuss how our results apply to D=4 N=1 adS supersymmetry, and conclude with comments on implications and generalizations of our work, in particular to M-theory.

2 BPS states

The anticommutator (2) can be rewritten as

$$\{Q,Q\} = H + \gamma^{0i}P_i + \gamma_5\gamma^i U_i + \gamma^i V_i \tag{5}$$

where

$$U_i = \frac{1}{2}\varepsilon_{ijk}U_{jk} \qquad V_i = \frac{1}{2}\varepsilon_{ijk}V_{jk}\,. \tag{6}$$

As mentioned above, a charge configuration is supersymmetric if the matrix $\{Q,Q\}$ has at least one zero eigenvalue. Thus, supersymmetric charge configurations are those for which $\{Q,Q\}$ has vanishing determinant. We see from (5) that this determinant must be expressible in terms of H and the three 3-vectors $\mathbf{P}$, $\mathbf{U}$ and $\mathbf{V}$.

Now $\det\{Q,Q\}$ is manifestly $SL(4;\mathbb{R})$ invariant, but the subgroup that keeps H fixed is its maximal compact $SO(4) \cong [SU(2) \times SU(2)_R]/\mathbb{Z}_2$ subgroup. Ignoring the quotient by $\mathbb{Z}_2$, the first $SU(2)$ factor can be identified with the 3-space rotation group while the $SU(2)_R$ group rotates the three 3-vectors $\mathbf{P}$, $\mathbf{U}$ and $\mathbf{V}$ into each other, i.e. these three 3-vectors form a triplet of $SU(2)_R$. The notation chosen here reflects the fact that $SU(2)_R \supset U(1)_R$, where $U(1)_R$ is the R-symmetry group[2] rotating $\mathbf{U}$ into $\mathbf{V}$ keeping $\mathbf{P}$ fixed (this is the automorphism group of the standard supersymmetry algebra, including Lorentz generators). We conclude from this that $\det\{Q,Q\}$ is a fourth-order polynomial in H with coefficients that are homogeneous polynomials in the three algebraically-independent $SU(2) \times SU(2)_R$ invariants that can be constructed from $\mathbf{P}$, $\mathbf{U}$ and $\mathbf{V}$. These are

$$\begin{aligned} a &= U^2 + V^2 + P^2 \\ b &= \mathbf{P} \cdot \mathbf{U} \times \mathbf{V} \\ c &= |\mathbf{U} \times \mathbf{V}|^2 + |\mathbf{P} \times \mathbf{U}|^2 + |\mathbf{P} \times \mathbf{V}|^2 \,. \end{aligned} \tag{7}$$

An explicit computation shows that

$$\det\{Q,Q\} = P(H) \tag{8}$$

where $P(H)$ is the quartic polynomial

$$P(H) = H^4 - 2aH^2 - 8bH + a^2 - 4c \tag{9}$$

The fact that $\{Q,Q\}$ is a positive real symmetric matrix imposes a bound on H in terms of the invariants a, b, c. Specifically,

$$H \geq E(a,b,c) \tag{10}$$

where $E(a,b,c)$ is the largest root of $P(H) = (H-\lambda_1)(H-\lambda_2)(H-\lambda_3)(H-\lambda_4)$. Since the sum of the roots vanishes, the largest root E is necessarily non-negative. The number of supersymmetries preserved is then the number of roots equal to E. The vacuum configuration has all roots equal with $E = 0$. In all other cases E is strictly positive and the number of roots equal to it is 1,2 or 3, corresponding to 1/4,1/2 or 3/4 supersymmetry.

Our first task, to be undertaken below, is to analyse the conditions required for the realization of each of these possibilities. We will then show how the stability of states preserving supersymmetry, alias 'BPS states', is guaranteed by the supersymmetry algebra. Although all model-independent consequences of supersymmetry are encoded in the supersymmetry algebra, the extraction of these consequences for BPS states is facilitated by methods that involve only the constraints on the Killing spinors associated with these states, and we show in the subsequent subsection how these methods can be used to learn about restrictions imposed by the preservation of supersymmetry on intersecting domain walls. We conclude with a discussion of 3/4 supersymmetry, and a proof that this fraction is not realized in the WZ model.

2.1 Supersymmetry fractions

The analysis of the conditions on the invariants a, b, c required for the preservation of the various possible fractions of supersymmetry is fairly straightforward when the polynomial $P(H)$ has at least two equal roots, and is especially simple when there are three equal roots. We shall therefore begin with the case of three equal roots, followed by the case of two equal roots, arriving finally at the generic case.

[2] This symmetry is usually broken in D=4 N=1 QFTs, either by the superpotential or by anomalies. We shall comment on this fact in the conclusions, but it is not relevant to the purely algebraic analysis presented here.

The quartic polynomial $P(H)$ has three equal roots if

$$c = \frac{a^2}{3}, \qquad b = \mp(\frac{a}{3})^{3/2}, \tag{11}$$

and the roots are

$$\lambda_1 = \lambda_2 = \lambda_3 \equiv \lambda = \pm(\frac{a}{3})^{1/2}, \qquad \lambda_4 = -3\lambda\,. \tag{12}$$

If λ is positive then we have the BPS bound $H \geq \lambda$, and charge configurations saturating this bound preserve 3/4 supersymmetry. If λ is negative then we instead find the BPS bound $H \geq -3\lambda$, with only 1/4 supersymmetry being preserved by charge configurations that saturate it.

Charge configurations preserving 1/2 supersymmetry can occur only when $P(H)$ has two equal roots. The conditions for the special case in which $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$ are

$$\begin{aligned} b = c &= 0 \\ \lambda_1 = -\lambda_3 &= \pm\sqrt{a} \end{aligned} \tag{13}$$

In the more general case when $\lambda_1 = \lambda_2 \equiv \lambda$ and $\lambda_3 \equiv \rho$ we have $\lambda_4 = -(2\lambda + \rho)$. If $\lambda = 0$ we have $a^2 = 4c$, $b = 0$ and $\rho^2 = 2a$, with 1/4 supersymmetry when $H = |\rho|$. Otherwise we find the condition

$$4a^3b^2 + 27b^4 - 18ab^2c - a^2c^2 + 4c^3 = 0 \tag{14}$$

with

$$3\lambda^2 = a \pm 2(a^2 - 3c)^{1/2}\,, \qquad \rho^2 + 2\lambda\rho + 3\lambda^2 = 2a\,. \tag{15}$$

with 1/2 supersymmetry possible when λ is the largest root.

The general case of four unequal roots is quite complicated, unless $b = 0$, in which case

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\sqrt{a + 2\sqrt{c}}, \sqrt{a - 2\sqrt{c}}, -\sqrt{a - 2\sqrt{c}}, -\sqrt{a + 2\sqrt{c}}\right). \tag{16}$$

One way to achieve $b = 0$ is to set $\mathbf{P} = \mathbf{0}$. In this case the bound on H becomes

$$H \geq \sqrt{U^2 + V^2 + 2|\mathbf{U} \times \mathbf{V}|}\,. \tag{17}$$

Note that this becomes $H \geq |\mathbf{U}| + |\mathbf{V}|$ when $\mathbf{U} \cdot \mathbf{V} = \mathbf{0}$, which is typical of 1/4 supersymmetric orthogonal intersections of branes. The four eigenvalues of $\{Q, Q\}$ are, in order of increasing magnitude,

$$H - \sqrt{a + 2\sqrt{c}}, \qquad H - \sqrt{a - 2\sqrt{c}}, \qquad H + \sqrt{a - 2\sqrt{c}}, \qquad H + \sqrt{a + 2\sqrt{c}}\,. \tag{18}$$

The first of these vanishes when the bound is saturated. The last two are never zero unless all four vanish, which is the vacuum charge sector. The second eigenvalue equals the first only when $c = 0$, so in this case there are two zero eigenvalues when the bound is saturated and we have 1/2 supersymmetry. Otherwise we have 1/4 supersymmetry.

As emphasized earlier, static configurations need not have $\mathbf{P} = \mathbf{0}$ because $\mathbf{P}$ may have an interpretation as a domain-wall junction charge, rather than 3-momentum (in general it must be interpreted as a sum of the 3-momentum and a string junction charge). Nevertheless, one may still have $b = 0$ if $\mathbf{U} \times \mathbf{V}$ vanishes, which it will do if, say, $\mathbf{V} = 0$. In this case, the results are exactly as in the $\mathbf{P} = \mathbf{0}$ case just analysed but with $\mathbf{V}$ replaced by $\mathbf{P}$. In particular, if $\mathbf{P} \cdot \mathbf{U} = 0$ we then have $H \geq |\mathbf{P}| + |\mathbf{U}|$, and static 1/4 supersymmetric configurations have $H \geq |\mathbf{P}| + |\mathbf{U}|$. For this case, we can bring the charges to the form

$$\mathbf{P} = (0, 0, Q)\,, \qquad \mathbf{U} = (u_1, u_2, 0)\,, \qquad \mathbf{V} = (0, 0, 0)\,. \tag{19}$$

where Q is a junction charge. This case is the one analysed in [6], with $T = u_1 + iu_2$ being the complex scalar charge in the D=3 supersymmetry algebra obtained by dimensional reduction on the 3-direction. In agreement with [6] we find that $H = |T| + |Q|$, so the junction charge contributes positively to the energy of the whole configuration.

More generally, we might have

$$\mathbf{P} = (0,0,Q)\,, \qquad \mathbf{U} = (u_1,u_2,0)\,, \qquad \mathbf{V} = (v_1,v_2,0)\,. \tag{20}$$

This case was analysed in [13], and an explicit realization of it was found in a model with several chiral superfields; in this model the charge Q is again associated with a domain wall junction. In agreement with [13] we find the four roots to be

$$\begin{aligned}
\lambda_1 &= -Q + \sqrt{(u_2+v_1)^2 + (u_1-v_2)^2} \\
\lambda_2 &= -Q - \sqrt{(u_2+v_1)^2 + (u_1-v_2)^2} \\
\lambda_3 &= Q - \sqrt{(u_2-v_1)^2 + (u_1+v_2)^2} \\
\lambda_4 &= Q + \sqrt{(u_2-v_1)^2 + (u_1+v_2)^2}\,.
\end{aligned} \tag{21}$$

Note that the four roots are distinct, in general, and (in contrast to the previous case) $b \neq 0$. If Q is positive and λ_1 is the largest root, the junction charge Q contributes negatively to the total energy as in [13].

The case just considered is a special case of the larger class of configurations with $b \neq 0$ for which $P(H)$ has four distinct roots. At this point the analysis becomes quite complicated, and we shall not pursue it further.

2.2 Stability of BPS states

Our aim in this subsection is to prove the stability of BPS states. We begin by considering the possible decay of a general state, not necessarily BPS, with energy H_3 into two other states, not necessarily BPS, with energies H_1 and H_2. This can be represented schematically as

$$(state)_3 \to (state)_1 + (state)_2\,. \tag{22}$$

Let us write

$$\{Q,Q\} = H + K(a,b,c)\,, \tag{23}$$

where K is a traceless symmetric matrix, and (a,b,c) are the three $SU(2)\times SU(2)_R$ invariants introduced previously. Conservation of charges and energy requires that

$$H_3 = H_1 + H_2 \tag{24}$$

$$K_3 = K_1 + K_2 \tag{25}$$

where $K_i = K(a_i,b_i,c_i)$, with (a_i,b_i,c_i) being the values of the invariants (a,b,c) for the ith state. Since the matrices $H_i + K_i$ are positive definite they are subject to the Minkowski reverse triangle inequality (see e.g. [22])

$$[\det(H_3+K_3)]^{\frac{1}{4}} \geq [\det(H_1+K_1)]^{\frac{1}{4}} + [\det(H_2+K_2)]^{\frac{1}{4}}\,. \tag{26}$$

We now want to see the consequences of supposing state 3 to be BPS. We observe that the left hand side of (26) vanishes if state 3 is BPS, but the right hand side can vanish only if both states 1 and 2 are also BPS. The extension to more than two decay products is immediate so we conclude that any unstable BPS state would have to decay into other BPS states.

To complete the proof of stability we now show that a BPS state cannot decay into other BPS states. A BPS state has an energy $H = E(K) \equiv E(a,b,c)$ where $E(K)$ is the largest value of H for which $\det(H+K)=0$. An equivalent characterization of $E(K)$ is as the smallest eigenvalue of K. It follows that $E(K) = min\left(\zeta^T K\zeta\right)$, where ζ is a commuting spinor normalized such that $\zeta^T\zeta = 1$ but otherwise arbitrary. From this and the fact that $min(a+b) \leq min(a) + min(b)$, we deduce the triangle inequality

$$E(K_1+K_2) \leq E(K_1) + E(K_2)\,. \tag{27}$$

Generic models will have a spectrum of BPS states for which this inequality is never saturated. In such cases BPS states are absolutely stable. In those cases for which there are BPS energies saturating the inequality (27) there may be states of marginal stability[3]. The inequality (27) is saturated when K_1 and K_2 are proportional, with positive constant of proportionality, but this is only a sufficient condition for equality. Another sufficient condition, which we believe to be necessary, is the coincidence, up to normalization, of the eigenvectors of K_1 and K_2 with lowest eigenvalue.

It is instructive to see how the above comments apply to the special case in which $H+K = C\gamma^\mu P_\mu$. The Minkowski inequality becomes

$$\sqrt{-(P_1+P_2)^2} \geq \sqrt{-P_1^2} + \sqrt{-P_2^2}\,. \tag{28}$$

Since $\sqrt{-P^2}$ is the rest mass m of a particle with 4-momentum P, we learn that

$$m_3 \geq m_1 + m_2\,. \tag{29}$$

This is the familiar rule that the sum of the masses of the decay products cannot exceed the mass of the particle undergoing decay. Given that $m_3 = 0$ we deduce that $m_1 = m_2 = 0$, so if a massless particle decays into two other particles those two particles must also be massless. For this special case the triangle inequality (27) reduces to

$$|\mathbf{P}_1 + \mathbf{P}_2| \leq |\mathbf{P}_1| + |\mathbf{P}_2|\,, \tag{30}$$

which is saturated if and only if $\mathbf{P}_1$ and $\mathbf{P}_2$ are parallel, and in this case there is no phase space for the decay.

2.3 Domain Walls at Angles

Each supersymmetric configuration is associated with a set of Killing spinors ϵ which span the kernel of $\{Q,Q\}$. With the exception of the vacuum configuration, these spinors are subject to constraints that reduce the dimension of the space that they span. Some properties of supersymmetric configurations follow directly from the nature of these constraints. In particular, intersecting brane configurations can be considered as configurations obtained from parallel branes by rotation of one or more of them. The constraints can be similarly obtained, and then analysed to determine the dimension of the space of Killing spinors they allow [23]. We shall apply this analysis here to intersecting domain walls of N=1 D=4 theories.

We begin with two coincident domain walls, corresponding to the constraint

$$\gamma_{013}\epsilon = \epsilon\,. \tag{31}$$

We then rotate one of them around the 3-axis until it makes an angle β in the 12-plane, and simultaneously rotate by some angle α in the electric-magnetic charge space. This operation is represented by the matrix

$$R = e^{\frac{1}{2}\alpha\gamma_5} e^{\frac{1}{2}\beta\gamma_{12}}\,, \tag{32}$$

[3]It is well known that marginal stability is the mechanism by which BPS states 'decay' as one moves in the space of parameters defining certain theories, but this is a discontinuity of the BPS spectrum as a function of parameters and not a process within a given theory.

which satisfies

$$\gamma_{013}R^{-1} = R\gamma_{013}\,. \tag{33}$$

The constraint on the Killing spinor ϵ imposed by the rotated brane is

$$R\gamma_{013}R^{-1}\epsilon = \epsilon\,. \tag{34}$$

Using (33) and (31), one easily verifies that this second constraint is equivalent to

$$\left(R^2 - 1\right)\epsilon = 0\,. \tag{35}$$

It is not difficult to show that this equation has no non-zero solutions for ϵ unless $\alpha \pm \beta = 0$. We thus have

$$R = e^{\alpha\Sigma}\,, \qquad\qquad \Sigma = \frac{1}{2}\left(\gamma_5 \pm \gamma_{12}\right)\,. \tag{36}$$

Using the identity $\Sigma^3 = -\Sigma$ one can establish that

$$R^2 - 1 = (2R)(\sin\alpha\,\Sigma)\,. \tag{37}$$

Since $2R$ is invertible, it follows that (35) is equivalent to

$$\sin\alpha\,\Sigma\,\epsilon = 0\,. \tag{38}$$

This is trivially satisfied if $\sin\alpha = 0$. Otherwise it reduces to $\Sigma\epsilon = 0$, which is equivalent to

$$\gamma_{03}\epsilon = \pm\epsilon\,. \tag{39}$$

If this is combined with (31) we deduce that

$$\gamma_5\gamma_{023}\epsilon = \mp\epsilon\,, \tag{40}$$

which is the constraint associated with a purely magnetic domain wall in the 23-plane. We may take any two of these three constraints as the independent ones; the choice (31) and (40) have an obvious interpretation as the constraints associated with the orthogonal intersection of an electric wall with a magnetic one. This constitutes the special $\alpha = \pi/2$ case of the more general configuration of rotated intersecting branes that we have been studying. But we have now derived these constraints for *any* angle $\alpha \neq 0, \pi$. The fraction of supersymmetry preserved by the general rotated brane configuration is therefore the same as the fraction preserved in the special case of orthogonal intersection. Standard arguments can now be used to show that this fraction is 1/4.

We have thus shown that starting from a 1/2 supersymmetric configuration of two parallel coincident domain walls with normal $\mathbf{n}$, one of them may be rotated relative to the other by an arbitrary angle in a plane containing $\mathbf{n}$, preserving 1/4 supersymmetry, provided that the charge of the rotated wall is simultaneously rotated by the same angle in the 'electric-magnetic' charge space. In practice it may not be possible for the domain walls to intersect at arbitrary angles (preserving supersymmetry). For example, in the $\mathbb{Z}_3$-invariant model discussed in [6], supersymmetric intersections are necessarily at $2\pi/3$ angles. But such additional restrictions are model-dependent. What we learn from the supersymmetry algebra is the model-independent result that *the angle separating 1/4 supersymmetric intersecting domain walls must equal the angle between them in the 'electric/magnetic' charge space*.

Since the constraint (39) is associated with non-zero P_3 we also learn from the above analysis that we can include this charge, provided it has the appropriate sign, which is determined by the sign in (36), without affecting the constraints imposed by 1/4 supersymmetry, although we then leave the class of configurations for which $b = 0$. Setting

$P_3 \neq 0$ might be considered as performing a boost along the 3-direction except for the previously noted fact that P_3 is not necessarily to be interpreted as momentum. Nevertheless, as a terminological convenience we shall *call* $\mathbf{P}$ the '3-momentum' in what follows. Consider the charge configuration obtained by adding the charges of an electric brane in the 13-plane with a brane rotated in the 12-plane, preserving 1/4 supersymmetry, and then adding momentum in the 3 direction:

$$\begin{aligned}
\mathbf{U} &= v\cos\alpha(\sin\alpha, -\cos\alpha, 0) + (0, -u, 0) \\
\mathbf{V} &= v\sin\alpha\,(\sin\alpha, -\cos\alpha, 0) \\
\mathbf{P} &= (0, 0, p)
\end{aligned} \tag{41}$$

We now have

$$\begin{aligned}
a &= u^2 + v^2 + 2uv\cos^2\alpha + p^2 \\
b &= puv\sin^2\alpha \\
c &= u^2v^2\sin^4\alpha + p^2(u^2 + v^2 + 2uv\cos^2\alpha)
\end{aligned} \tag{42}$$

One can show that the eigenvalues of $\{Q, Q\}$ are

$$H + p \pm \sqrt{u^2 + v^2 + 2uv\cos 2\alpha}\,, \qquad H - p \pm (u + v) \tag{43}$$

For $u, v, p \geq 0$, we conclude that $H \geq p + u + v$ and that 1/4 supersymmetry is preserved when the bound is saturated. Note that in this case

$$\{Q, Q\} = u\,(1 - \gamma_{013}) + v\left(1 - \gamma_{013}R^2\right) + p\,(1 - \gamma_{03})\ , \tag{44}$$

for the upper sign in (36), confirming that the projections remain unchanged by the inclusion of momentum.

2.4 3/4 Supersymmetry

Continuing the above analysis, we now turn to the case in which u, v, p are not necessarily all positive because this case includes the possibility of domain wall configurations preserving 3/4 supersymmetry [14]. Consider the case $\alpha = \pi/2$ for an electric wall and a magnetic wall intersecting at right angles, so that the eigenvalues (43) are

$$H + p \pm (u - v)\,, \qquad H - p \pm (u + v)\,. \tag{45}$$

It follows that H is bounded below by each of the eigenvalues

$$\begin{aligned}
\lambda_1 &= p - u - v \\
\lambda_2 &= v - u - p \\
\lambda_3 &= u - v - p \\
\lambda_4 &= u + v + p\,.
\end{aligned} \tag{46}$$

If only one of the charges is non-zero, u say, then we obtain the standard BPS bound, $H \geq |u|$, which is saturated by the electrically charged BPS domain wall. With two charges, u and v say, we obtain $H \geq |u + v|$ and $H \geq |u - v|$, and when the stronger of these is saturated we have the intersecting domain wall configuration preserving 1/4 supersymmetry. With all three charges, there are four bounds corresponding to the four eigenvalues and 1/4 supersymmetry is preserved, generically, when the strongest bound is saturated. There are then two subcases to consider according to whether or not λ_4 is the largest eigenvalue. If λ_4 *is* the largest eigenvalue, as happens, for example, when u, v, p

are all positive, then we recover the standard 1/4 supersymmetric case considered above, unless two of the three charges u, v, p vanish in which case 1/2 supersymmetry is preserved. If λ_4 is *not* the largest eigenvalue then one of the others is, and we may choose it to be λ_1 because the other possibilities are related to this one by $SU(2)_R$ transformations. Given this, H is bounded below by $p - u - v$ and if there is a state saturating this bound with $H = p - u - v$ then the eigenvalues of $\{Q, Q\}$ are

$$0\,, \qquad 2(p-v)\,, \qquad 2(p-u)\,, \qquad -2(u+v)\,. \tag{47}$$

It follows that 1/4 supersymmetry is preserved *generically* but more supersymmetry is preserved for special values of the charges. The possibility of this kind of enhancement of supersymmetry, including the possibility of 3/4 supersymmetry, was recently discussed in [14] and the case under consideration here is very similar. If $p = v$ or $p = u$ or $u = -v$, then a charge configuration saturating the BPS bound will preserve 1/2 supersymmetry and if $p = u = v$ or $u = -v = \pm p$ then 3/4 supersymmetry will be preserved. Thus, a charge configuration saturating the bound $H \geq \lambda_1$ will preserve 1/4 supersymmetry for *generic* values of the charges, but 1/2 or 3/4 supersymmetry for certain special values.

We should stress that the above analysis is purely algebraic and it is an open question whether there exists a physical model with domain wall configurations preserving 3/4 supersymmetry. As we now show, this possibility is not realized by the WZ model.

2.5 BPS Solutions of the Wess-Zumino Model

The WZ model is known to admit both 1/4 and 1/2 supersymmetric classical solutions, which (at least potentially) correspond to states in the quantum theory. We shall show here that there are no classical solutions preserving 3/4 supersymmetry. We shall begin by considering purely bosonic field configurations and then extend the result to arbitrary classical configurations.

The fields of the WZ model belong to a single chiral superfield, the components of which are a complex physical scalar $\phi = A + iB$, a complex two-component spinor, which is equivalent to a 4-component Majorana spinor λ, and a complex auxiliary field $F = f + ig$. We will continue to use a real representation of the four Dirac matrices γ^μ. For purely bosonic field configurations we need only consider fermion supersymmetry transformations. Our starting point will therefore be the (off-shell) supersymmetry transformation of λ, which takes the form $\delta\lambda = M\epsilon$, where ϵ is a real constant spinor parameter and M is the real 4×4 matrix

$$M = \gamma^\mu(\partial_\mu A + \gamma_5 \partial_\mu B) + f + \gamma_5 g\,. \tag{48}$$

This transformation is valid for the spinor component of any chiral superfield. The Wess-Zumino model is characterised by the fact that the auxiliary field equation is

$$F \equiv f + ig = W'(\phi)\,, \tag{49}$$

where $W'(\phi)$ is the derivative with respect to ϕ of the holomorphic superpotential $W(\phi)$.

A bosonic field configuration of the WZ model will be supersymmetric if there is a spinor field ϵ that is both annihilated by $M(x)$, for all x, and covariantly constant with respect to a metric connection on $\mathbb{E}^{(1,3)}$. Thus, for there to be n preserved supersymmetries it is a necessary condition that $M(x)$ has an n dimensional kernel for each x. Our strategy for showing that there are no 3/4 supersymmetric field configurations will be to analyse necessary conditions for the matrix $M_0 \equiv M(x_0)$ at a fixed point x_0 to have an n-dimensional kernel.

We begin by noting that a WZ field configuration can preserve 1/4 supersymmetry only if $\det M_0$ vanishes, which is equivalent to

$$\left[(\partial A)^2 + (\partial B)^2 - f^2 - g^2\right]^2 = 4\left[(\partial A)^2(\partial B)^2 - (\partial A\cdot\partial B)^2\right] . \tag{50}$$

This condition is necessary for the preservation of at least 1/4 supersymmetry in any model with a single chiral superfield, and in particular in the WZ model. Configurations preserving more than 1/4 supersymmetry are characterized by additional constraints on the fields. Necessary constraints can be found very easily by making use of the fact that M_0 can be brought to (real) upper-triangular form by a similarity transformation. We may therefore assume that M_0 is upper triangular. If, in addition, it has a 2-dimensional kernel then it may be brought to the form

$$\begin{pmatrix} 0 & 0 & * & * \\ & 0 & * & * \\ & & * & * \\ & & & * \end{pmatrix} \tag{51}$$

where $*$ indicates an entry that is not zero (or not necessarily zero in the case of the off-diagonal entries). This matrix has the property that

$$2\mathrm{tr}\, M_0^3 - 3\mathrm{tr}\, M_0\, \mathrm{tr}\, M_0^2 + (\mathrm{tr}\, M_0)^3 = 0\,, \tag{52}$$

and substituting (48) we learn that

$$f\left[f^2 + g^2 - (\partial A)^2 - (\partial B)^2\right] = 0\,. \tag{53}$$

This condition is therefore necessary for a field configuration to preserve 1/2 supersymmetry.

Similarly, any upper-triangular matrix with a 3-dimensional kernel can be brought to the form

$$\begin{pmatrix} 0 & 0 & 0 & * \\ & 0 & 0 & * \\ & & 0 & * \\ & & & * \end{pmatrix} \tag{54}$$

This matrix satisfies both (52) and $\mathrm{tr}\, M_0^2 = (\mathrm{tr}\, M_0)^2$, in addition to (50). These conditions, which are therefore necessary for 3/4 supersymmetry, are equivalent to the joint conditions

$$\begin{aligned} f &= 0 \\ g^2 &= (\partial A)^2 + (\partial B)^2 \\ (\partial A)^2(\partial B)^2 &= (\partial A\cdot\partial B)^2\,. \end{aligned} \tag{55}$$

We are now in a position to show that there are no 3/4 supersymmetric WZ field configurations (other than the vacuum which has 4/4 supersymmetry). The conditions (55) must be satisfied by such a field configuration. We shall analyse these conditions at a fixed point $x = x_0$ and consider separately the cases in which $g = 0$ and $g \neq 0$ at that point. If $g = 0$ then the second condition in (55) implies that at x_0 *either* the 4-vectors ∂A and ∂B are both null *or* one is spacelike and the other is timelike. The latter option contradicts the third of eqs (55) so both are null. It then follows from (55) that ∂A and ∂B are parallel, so that

$$f = g = 0\,, \qquad \partial A = \alpha\, v\,, \qquad \partial B = \beta\, v \tag{56}$$

where α and β are constants and v is a null 4-vector. This field configuration is therefore a candidate for 3/4 supersymmetry, but because the conditions leading to it were not sufficient for 3/4 supersymmetry this must be checked. In fact, it is readily shown that the matrix M corresponding to the configuration (56) has only a two-dimensional kernel so that at most 1/2 supersymmetry can be preserved.

The remaining candidates for 3/4 supersymmetry in the WZ model arise from field configurations in which f vanishes but g is non-zero. Then (55) implies that at x_0 either ∂A and ∂B are both spacelike, or one is spacelike and the other is null. Suppose first that either ∂A or ∂B is null. In the case in which ∂B is null we have

$$f = 0 \qquad \partial A = gs\,, \qquad \partial B = v\,, \tag{57}$$

where v is a null vector orthogonal to a spacelike vector s normalized such that $s^2 = 1$. For this configuration one can check that the matrix M generically has a one dimensional kernel, and has a two dimensional kernel when either $g = 0$ or $\beta = 0$. The case in which ∂A is null is similar, with the same result that at most 1/2 of the supersymmetry is preserved.

If neither ∂A nor ∂B is null then they are both spacelike and we can arrange for them to take the form

$$\begin{aligned} \partial B &= \beta(0,1,0,0) \\ \partial A &= \alpha(\sin\theta, \cos\theta, 0, \sin\theta) \end{aligned} \tag{58}$$

with $g^2 = \alpha^2 \cos^2\theta + \beta^2$. One then finds that the kernel of $M(x_0)$ is 2-dimensional if $\alpha\beta\sin\theta = 0$ and otherwise 1-dimensional. Configurations of the form (58) can therefore preserve at most 1/2 supersymmetry.

We have now shown that there are no non-vacuum *bosonic* WZ field configurations that preserve 3/4 supersymmetry. We now wish to consider whether this remains true when we consider general configurations that are not necessarily bosonic. This question is perhaps best posed in the context of the quantum theory, which we will not consider here, but it can also be posed classically by taking all fields to be supernumbers with a 'body' and a nilpotent 'soul'. Any general field configuration of this kind preserving 3/4 supersymmetry must have a body preserving at least 3/4 supersymmetry and, as we have just seen, the vacuum configuration is the only candidate. It follows that the only remaining way in which a classical field configuration could be 3/4 supersymmetric is if the 4/4 supersymmetry of the bosonic vacuum configuration is broken to 3/4 by fermions. Preservation of any fraction of supersymmetry in a fermionic background requires the vanishing of the supersymmetry transformations of the bosons. For the WZ model this implies ($\bar\lambda \equiv \lambda^T C$)

$$\bar\lambda\epsilon = 0\,, \qquad \bar\lambda\gamma_5\epsilon = 0\,, \tag{59}$$

and for 3/4 supersymmetry there must be a three-dimensional space of parameters ϵ for which this condition holds. At a given point in space we may choose, without loss of generality, a basis in spinor space such that $C\epsilon = (0,*,*,*)^T$, where an asterisk indicates an entry that may be non-zero. The first equation then implies that $\lambda^T = (*,0,0,0)$ and the second that $\lambda^T\gamma_5 = (*,0,0,0)$. But since γ_5 is both real and satisfies $\gamma_5^2 = -1$ these conditions are not mutually compatible. This concludes our proof that the WZ model has no non-vacuum classical configurations, bosonic or otherwise, that preserve 3/4 supersymmetry

3 The geometry of supersymmetry

We now turn to a discussion of the geometry associated with BPS representations of the algebra (2), which we may re-write in terms of a positive semi-definite symmetric bispinor

Z as $\{Q, Q\} = Z$. The positivity of $\{Q, Q\}$ implies that Z is a vector in a convex cone, with the boundary of the cone corresponding to the BPS condition $\det Z = 0$. We shall first explain some of the geometry associated with convex cones, and how it relates to BPS states. We will then explain how this ties in with the theory of Jordan algebras.

3.1 Convex cones

Let us begin with the standard D=4 N=1 supersymmetry algebra, in which case $Z = \gamma \cdot P$ and the positivity of $\{Q, Q\}$ implies that P lies *either* in the forward lightcone of D=4 Minkowski momentum-spacetime *or* on its boundary, the lightfront. In the latter case, $P^2 = 0$ and any states with this 4-momentum are BPS, preserving 1/2 supersymmetry. The forward lightcone in momentum space and the forward lightcone in position space are both examples of convex cones. An n-dimensional cone $\mathcal{C}$ is a subspace of an n-dimensional vector space V with the property that $\lambda x \in \mathcal{C}$ for all $x \in \mathcal{C}$ and all real positive λ. The cone is convex if the sum of any two vectors in the cone is also in it. The dual cone is then defined as follows. Let y be a vector in the dual vector space V^* and let $y \cdot x$ be a bilinear map from $V \times V^*$ to $\mathbb{R}$. The dual cone $\mathcal{C}^*$ is the subspace of V^* for which $y \cdot x > 0$ for all $x \in \mathcal{C}$.

Given a translation-invariant measure on V we can associate with each convex cone in V a characteristic function ω defined by

$$\omega^{-1}(x) = \int_{\mathcal{C}^*} e^{-y \cdot x} d^n y \,. \tag{60}$$

As all translation-invariant measures are multiples of any given translation-invariant measure, this formula defines ω up to a scale factor, but this ambiguity will not affect the statements to follow. The cone is foliated by hypersurfaces of constant ω, with the limiting hypersurface $\omega = 0$ being the boundary of the cone. In the case of the forward light cone in D=4 Minkowski spacetime the vector space V is $\mathbb{R}^4$ and $\omega = \mathcal{N}^2$, where $\mathcal{N}(x) = -\eta_{\mu\nu} x^\mu x^\nu$ is the quadratic form defined by the Minkowski metric η (we adopt a 'mostly plus' metric convention). The hypersurfaces of constant ω are therefore hyperboloids homothetic to $SO(1,3)/SO(3)$. Note that this is a symmetric space; this is a general feature of self-dual homogeneous convex cones, of which the forward lightcone in Minkowski space is an example. Homogeneous convex cones that are not self-dual are foliated by homogeneous spaces that are not symmetric spaces.

Because, in this example, ω is determined by a *quadratic* function $\mathcal{N}$, the vector space $V = \mathbb{R}^4$ can be viewed as a metric space, with Minkowski metric η. More generally, ω is not quadratic and hence does not furnish V with a metric. Nevertheless, ω does provide a positive definite metric for $\mathcal{C}$ (obviously, this differs from the Minkowski metric of the 'quadratic' case discussed above). Let us first note that, by the definition of a cone, the map $D : x \mapsto \lambda x$ is an automorphism, in that $Dx \in \mathcal{C}$ if $x \in \mathcal{C}$. It follows immediately that $\omega(x)$ is a homogeneous function of degree n. A corollary of this is that $\pi(x) \cdot x = 1$ where

$$\pi(x) = \frac{1}{n} \frac{\partial \log \omega}{\partial x} \,. \tag{61}$$

Thus, $\pi \in \mathcal{C}^*$, and as x ranges over all vectors in $\mathcal{C}$ so π ranges over all vectors in $\mathcal{C}^*$. One can now introduce a metric g on $\mathcal{C}$ with components[4]

$$g_{ij} = -\frac{1}{n} \partial_i \partial_j \log \omega(x) \,. \tag{62}$$

[4]For the forward light-cone in Minkowski spacetime with Minkowski metric η, we have $g_{ij} = (x^2)^{-2}(2x_i x_j - x^2 \eta_{ij})$ where $x^2 = \eta_{ij} x^i x^j$ and $x_i = \eta_{ij} x^j$, so that $\pi_i = (x^2)^{-1} x_i$.

One may verify that
$$\pi_j = x^i g_{ij} \,. \tag{63}$$
The map from $\mathcal{C}$ to $\mathcal{C}^*$ provided by the metric (62) has a natural interpretation in terms of Hamilton-Jacobi theory: if $\log\omega$ is interpreted as a characteristic function in the sense of Hamilton, then π as defined by (61) is the conjugate momentum.

A feature of the metric g is that it is invariant under automorphisms of $\mathcal{C}$. For example it follows from the homogeneity of ω that the linear map D is an isometry of g. The group of automorphisms will generally be a semi-direct product of D with some group G that acts on the leaves of the foliation. The cone is homogeneous if G acts transitively. A homogeneous cone is foliated by homogeneous hypersurfaces of the form G/H for some isotropy group H. For a self-dual cone this homogeneous space is also a symmetric space. As already mentioned, the forward light cone in $\mathbb{E}^{(1,3)}$ is foliated by hyperboloids homothetic to $SO(1,3)/SO(3)$, so G is the (proper orthochronous) Lorentz group. The metric induced on each leaf of the foliation by the metric g_{ij} of the cone is the positive-definite $SO(1,3)$-invariant metric on $SO(1,3)/SO(3)$.

Let us now turn to the general D=4 N=1 supersymmetry algebra $\{Q,Q\} = Z$. The bispinor charge Z can be interpreted as a vector in the convex cone of positive-definite real 4×4 symmetric matrices. This is a cone in $\mathbb{R}^{10}$ which, since Z includes the 4-momentum, we may consider as a 'momentum-space' cone $\mathcal{C}^*$. We set aside to the following subsection consideration of the corresponding 'position space' cone $\mathcal{C}$. The characteristic function of $\mathcal{C}^*$ is[5]
$$\omega(Z) = (\det Z)^{\frac{5}{2}} \,. \tag{64}$$
The cone is again a self-dual homogeneous one, and is foliated by symmetric spaces that are homothetic to $SL(4;\mathbb{R})/[SO(4)]$. Of principal interest here is the boundary of $\mathcal{C}^*$, defined by $\det Z = 0$, because this is the condition for preservation of supersymmetry. The geometry of this boundary is now rather more complicated than it was before.

The basic observation required to understand this geometry is that the cone is a stratified space with strata $\mathcal{S}_n$, $n = 0,1,2,3,4$, where $\mathcal{S}_n$ is the subspace in which at least n of the four eigenvalues vanish, corresponding to at least n supersymmetries being preserved, and $\mathcal{S}_{n+1}$ is the boundary of $\mathcal{S}_n$. The boundary of the cone is the space $\mathcal{S}_1$, which is the 9-dimensional space of matrices of rank 3 or less. The boundary of this is the space $\mathcal{S}_2$ of matrices of rank 2 or less which make up a 7 dimensional space. To see why it is 7 dimensional recall that to specify a matrix of rank 2 it suffices to give the normalised eigenvectors with non-vanishing eigenvalues together with their eigenvalues. The two eigenvectors define a 2-plane in $\mathbb{R}^4$, corresponding to an element of the 4-dimensional Grassmannian $SO(4)/(SO(2)\times SO(2))$. Giving the orientation of the eigenvectors within the 2-plane means specifying one of the $SO(2)$ factors. In other words the basis of 2 eigenvectors corresponds to the 5 dimensional Stiefel manifold $SO(4)/SO(2)$. Taking into account the two eigenvalues we have a 7-dimensional space, as claimed. The boundary of this stratum is the set $\mathcal{S}_3$ of matrices of rank 1 or less. These span a 4-dimensional space, since a rank 1 matrix is specified by the direction, up to a sign, of its eigenvector with non-zero eigenvalue together with the eigenvalue. This is a point in $\mathbb{R}P^3 \times \mathbb{R}^+$. Finally, the boundary of $\mathcal{S}_3$ is the stratum $\mathcal{S}_4$ consisting of the zero matrix, which is the vertex of the cone.

3.2 Reverse triangle inequalities

The Minkowski inequality that we used previously to establish the stability of BPS states is a special case of a reverse-triangle inequality valid for all convex cones. Let us define

[5] Note that ω^2 is a polynomial. A theorem of Koecher states that ω^2 is a polynomial for all self-dual homogeneous convex cones.

the 'length' of a vector in an n-dimensional convex cone with characteristic function ω as

$$L(x) = \omega^{1/n}(x)\,. \tag{65}$$

This is a homogeneous function of degree 1. Because the hypersurfaces of constant ω are *concave*, this 'length' satisfies the *reverse* triangle inequality

$$L(x + x') \geq L(x) + L(x')\,. \tag{66}$$

with equality if and only if x and x' are proportional. In the case of the cone of $m \times m$ positive definite hermitian matrices we have $L(x) = (\det x)^{1/m}$ and the reverse triangle inequality is the Minkowski inequality

$$[\det(x+y)]^{\frac{1}{m}} \geq [\det x]^{\frac{1}{m}} + [\det y]^{\frac{1}{m}}\,, \tag{67}$$

with equality if the two matrices are proportional. In the special case of diagonal matrices, the cone becomes the positive orthant $\mathbb{R}^m_+$ in $\mathbb{E}^m$. The length of a vector $x = diag(x_1, \ldots, x_m)$ in $\mathbb{R}^m_+$ is $L(x) = (x_1 \ldots x_n)^{1/n}$, and Minkowski's inequality for positive definite matrices reduces to a form of Holder's inequality (see e.g. [22]). The metric g on $\mathbb{R}^m_+$ is the flat metric $dl^2 = (1/n)\sum(d\log x^i)^2$. The automorphism group is the permutation group S_m, which is clearly an invariance of the length.

3.3 Conformal Invariance

For the standard D=4 N=1 supersymmetry algebra without central charges all BPS states have $P^2 = 0$. This is the momentum space version of the massless wave-equation, which is invariant under the action of the conformal group $SU(2,2)$ on compactified Minkowski spacetime. Our aim here is to show how this generalizes when the domain wall charges are included. This will turn out to be a straightforward extension of the standard case, appropriately formulated, so we consider that first.

It is convenient to identify a point in Minkowski spacetime with a matrix $X = X^\mu \sigma_\mu$, where $\sigma_\mu = (1, \sigma_1, \sigma_2, \sigma_3)$ are the 2×2 Hermitian sigma-matrices. The conjugate momentum P is then similarly a 2×2 Hermitian matrix and $-P^2$ becomes $\det P$. (The momentum P should not be confused with the dual variable π introduced in the previous subsection.) Let us now consider the massless particle action

$$I = \int [\mathrm{tr} P dX - e \det P] \tag{68}$$

where e (the einbein) is a Lagrange multiplier for the mass-shell constraint $\det P = 0$. The conformal group $SU(2,2)$ acts on the compactification of Minkowski space via the fractional linear transformation

$$X \to X' = (AX + B)(CX + D)^{-1} \tag{69}$$

where the hermiticity of X' requires that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2,2)\,. \tag{70}$$

This implies that

$$dX'(CX + D) = (A - X'C)dX\,. \tag{71}$$

We deduce from this that the PdX part of the action I is invariant (up to a surface term) if

$$P \to P' = (CX + D)P(A - X'C)^{-1}\,. \tag{72}$$

This transformation implies

$$\det P \to \det P' = \Omega^{-1} \det P \tag{73}$$

where

$$\Omega = \frac{\det(A - X'C)}{\det(CX + D)} \,. \tag{74}$$

The action I is therefore invariant if we assign to the einbein the transformation $e \to e' = \Omega e$.

We now wish to determine the analogous symmetry group of the more general BPS condition $\det Z = 0$. The matrix Z can be viewed as a vector in a 10-dimensional vector space. Let X be coordinates of the dual space and consider the particle action

$$I = \int [\mathrm{tr} Z dX - e \det Z] \,. \tag{75}$$

Special cases of actions of this type were considered previously by Cederwall [24], with a motivation derived from Jordan algebra considerations that we shall explain in the following subsection (see also [25, 16]). Now consider the fractional linear transformation

$$X \to X' = (AX + B)(CX + D)^{-1} \,, \tag{76}$$

which acts on the compactification of the space of symmetric matrices [26]. The matrix X' will also be real and symmetric provided that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(8; \mathbb{R}) \,. \tag{77}$$

That is,

$$A^T D - C^T B = 1 \,, \qquad A^T C = C^T A \,, \qquad B^T D = D^T B \,. \tag{78}$$

As before, we deduce (71) and from this that the ZdX term is invariant up to a surface term if

$$Z \to Z' = (CX + D) Z (A - X'C)^{-1} \,. \tag{79}$$

This implies

$$\det Z \to \det Z' = \Omega^{-1} \det Z \tag{80}$$

where Ω has form of (74). We may again take $e \to e' = \Omega e$ to achieve an invariance of the action I. In this case, the invariance group is $Sp(8; \mathbb{R})$.

Note that this conclusion rests on an interpretation of the 4-dimensional compactified Minkowski spacetime as a subspace of a ten-dimensional vector space of the 4×4 real symmetric matrices X. A field theory realization of Sp(8;R) would require fields defined on this larger space. For example, the analogue of the massless wave equation on Minkowski space is the fourth-order equation

$$\det(-i\partial/\partial X)\, \Psi = 0 \tag{81}$$

The symmetry group of this equation is $Sp(8; \mathbb{R})$. By analogy with the Minkowski case, we expect this to be the maximal symmetry group of this equation.

3.4 Jordan algebras

The results of the previous subsections have an interpretation in terms of Jordan algebras. A Jordan algebra J of dimension n and degree ν is an n-dimensional real vector space with a commutative, power associative, bilinear product, and a norm $\mathcal{N}$ that is a homogeneous polynomial of degree ν (see e.g. [27]). There are four infinite series of simple Jordan algebras, realizable as matrices with the Jordan product being the anticommutator: the degree 2 algebras $\Sigma(n)$ to be discussed below, and the series $J_k^{\mathbb{R}}$, $J_k^{\mathbb{C}}$, $J_k^{\mathbb{H}}$, which are realized by $k \times k$ hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, with norm given by the determinant, $\mathcal{N}(x) = det(x)$. In addition, there is one 'exceptional' Jordan algebra $J_3^{\mathbb{O}}$ realizable by 3×3 hermitian matrices over the octonions.

Associated with any Jordan algebra J with product $x \circ y$ is a self-dual homogeneous convex cone $\mathcal{C}(J)$. This is the subspace of J consisting of elements e^x with $x \in J$ (where e^x is defined by the usual power series with $x^{n+1} = x^n \circ x$). The characteristic function is

$$\omega = \mathcal{N}^{n/\nu}\,, \tag{82}$$

so the boundary of the cone corresponds to $\mathcal{N} = 0$. The cone is foliated by copies of the homogeneous space $Str(J)/Aut(J)$, where $Str(J)$ is the invariance group of $\mathcal{N}$ (the 'structure group' of the algebra) and $Aut(J)$ is the automorphism group of the algebra (the subgroup of $Str(J)$ that fixes the identity element in J).

The relation of self-dual homogeneous convex cones to Jordan algebras has similarities to the relation between Lie groups and Lie algebras. Recall that a Lie group is parallelizable but has a non-zero torsion given by the structure constants of its Lie algebra. A self-dual homogeneous convex cone $\mathcal{C}$, on the other hand, is not parallelizable (in general) but its torsion-free affine connection is determined by the structure constants of a Jordan algebra. Because of homogeneity it suffices to know the connection at the 'base' point $c \in \mathcal{C}$ defined by[6]

$$g_{ij}|_c = \delta_{ij}\,. \tag{83}$$

Let $f_{ij}{}^k$ be the structure constants of J in a basis $e^i = (c, e_a)$. Then

$$\Gamma_{ij}{}^k|_c = f_{ij}{}^k\,. \tag{84}$$

Although Jordan algebras are commutative they are nonassociative. Define the *associator*

$$\{a, b, c\} \equiv (a \circ b) \circ c - a \circ (b \circ c)\,. \tag{85}$$

The curvature tensor of the cone at the base point is then given by the relation

$$\{e_i, e_j, e_k\} = R_{ijk}{}^l|_c e_l\,. \tag{86}$$

In addition to the automorphism and structure groups, there is a larger 'Möbius group' associated with any Jordan algebra J, acting on elements of J by fractional linear transformations. We therefore have the sequence of groups

$$Aut(J) \subset Str(J) \subset Mo(J)\,, \tag{87}$$

associated with any Jordan algebra J. These can be interpreted as generalized, rotation, Lorentz and conformal groups, respectively [19]. To motivate this interpretation, we return to the representation of a Minkowski 4-vector as the 2×2 Hermitian matrix X. This is an element in the degree 2 Jordan algebra $J_2^{\mathbb{C}}$. The dimension is 4 and the norm

[6] There is only one such point, even in those cases for which $\mathcal{C}$ is flat. It corresponds to the identity element in the algebra. We use the notation c to indicate both the identity element of J and the base point of the cone $\mathcal{C}(J)$.

is the determinant, which is the $SL(2;\mathbb{C})$ invariant Minkowski norm $\mathcal{N}$ on $\mathbb{R}^4$. The group $SL(2;\mathbb{C})$ acts on 2×2 matrices by conjugation so the subgroup leaving invariant the identity matrix is its maximal compact $SU(2)$ subgroup. The convex cone associated with this Jordan algebra is the forward light-cone of D=4 Minkowski spacetime. As we saw in the previous subsection, the group of fractional linear transformations of X is $SU(2,2)$, so the sequence (87) is, in this case,

$$SU(2) \subset SL(2;\mathbb{C}) \subset SU(2,2)\,. \tag{88}$$

These are the standard rotation, Lorentz and conformal groups.

The inclusion of domain wall charges means that we should replace $J_2^{\mathbb{C}}$ by $J_4^{\mathbb{R}}$, the algebra of 4×4 symmetric real matrices. One can see that $J_2^{\mathbb{C}}$ is a subalgebra of $J_4^{\mathbb{R}}$ from the fact that $J_2^{\mathbb{C}} \cong \Sigma(4)$, where $\Sigma(n)$ is the n-dimensional Jordan algebra with basis $(1, \sigma_1, \ldots \sigma_{n-1})$ and Jordan product $\sigma_a \circ \sigma_b = 2\delta_{ab}$; this has a realization in which σ_a are sigma-matrices of an n-dimensional Minkowski spacetime, with the Jordan product being the anticommutator; it follows that the standard supersymmetry algebra in D dimensions is naturally associated with $\Sigma(D)$. For $D = 4$ one can choose the σ_a to be the three 2×2 hermitian Pauli matrices, hence the isomorphism $J_2^{\mathbb{C}} \cong \Sigma(4)$. All simple Jordan algebras of degree 2 are isomorphic to $\Sigma(n)$ for some n. Having replaced $J_2^{\mathbb{C}}$ by $J = J_4^{\mathbb{R}}$ we find that the sequence (88) is generalized to [19]

$$SU(2) \times SU(2) \subset SL(4;\mathbb{R}) \subset Sp(8;\mathbb{R}) \tag{89}$$

We now turn to the Jordan algebraic interpretation of the boundary of the convex cone $\mathcal{C}(J)$. This consists of elements $\lambda P \in J$ where λ is a positive real number and P is an idempotent of J with less than maximal rank, i.e. its trace, defined by $\operatorname{tr} X = \log \mathcal{N}(e^X)$, is less than ν. An idempotent is a non-zero element $P \in J$ satisfying $P \circ P = P$, and two idempotents P and P' are said to be orthogonal if $P \circ P' = 0$. The idempotents with unit trace are called the primitive idempotents, and the number of mutually orthogonal primitive idempotents equals the degree ν of the algebra. For a Jordan algebra of degree 2 all idempotents of less than maximal rank have unit trace and are therefore primitive. This is true of $J_2^{\mathbb{C}}$, in particular, corresponding to the fact that the only supersymmetric states other than the vacuum permitted by the standard D=4 N=1 supersymmetry algebra are 1/2 supersymmetric states associated with massless particles (for which the 4-momentum lies on the positive light-front). Note that although at most two primitive idempotents of a degree 2 Jordan algebra can be orthogonal in the above sense, the space of primitive idempotents of $\Sigma(D)$ is $(D-1)$-dimensional. The boundary of the associated convex cone is therefore $(D-1)$-dimensional. For $\Sigma(4) \cong J_2^{\mathbb{C}}$, in particular, this boundary is the three-dimensional forward light-front of the origin of 4-dimensional Minkowski momentum space.

For a Jordan algebra J of degree $\nu > 2$, there are idempotents of less than maximal rank that are not primitive. For an algebra of degree 3, these non-primitive idempotents generate faces of the boundary of e^J which themselves have a boundary generated by the primitive idempotents. An example is the (non-simple) Jordan algebra $J = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ for which e^J is the positive octant in $\mathbb{E}^3$; its boundary consists of three faces that meet on the three axes generated by the three primitive idempotents (in this case there are only three primitive idempotents, which are therefore orthogonal; details can be found in [28]). More generally, for Jordan algebras of higher degree, the boundary of the associated convex cone is a stratified set of faces. In particular, $J_4^{\mathbb{R}}$ has degree 4 so the faces of the boundary of its associated convex cone are generated by idempotents of trace 1,2 and 3, corresponding to 3/4,1/2 and 1/4 supersymmetry respectively. The primitive idempotents, of unit trace, correspond to 3/4 supersymmetry.

3.5 Entropy of BPS fusion

In a quantum field theory realization of the D=4 N=1 supersymmetry algebra the central charges Z are labels of quantum states. We have now seen that the set of these charges naturally carries the structure of a Jordan algebra. This algebra may itself be regarded as a finite-dimensional state space (not to be confused with infinite-dimensional space of states of the field theory that carry these charges). This interpretation is of course how Jordan algebras originally arose (see [29] for a review). The exceptional Jordan algebra provides a state space more general than conventional quantum mechanics but for all other Jordan algebras the formalism is equivalent to one in which a state is represented by a density matrix. The general state is therefore a mixed state. The pure states correspond to the primitive idempotents; these lie on the boundary of the convex cone $\mathcal{C}(J)$ but do not in general exhaust it. Rather, the boundary is stratified by sets of states of successively less purity, corresponding in our application to states with successively less supersymmetry. Thus, the pure states in this sense are the charge configurations that preserve 3/4 supersymmetry, the remaining supersymmetric configurations corresponding to states on the boundary of the cone that are not pure.

We previously showed that a BPS state is stable against decay into any other pair of states; in particular it cannot decay into two BPS states. Consider now the reverse process, i.e. fusion of two BPS states to form a third via the inverse of the reaction (22), i.e.

$$(BPS)_1 + (BPS)_2 \to (BPS)_3 \, , \tag{90}$$

If the first two states preserve 3/4 supersymmetry then the third one will generally preserve less supersymmetry. This is like passing from a pure to a mixed state. There is also a formal resemblance here to classical thermodynamics. The Jordan algebra J, now viewed as vector space V containing the convex cone $\mathcal{C}(J)$, is spanned by the extensive quantities while the dual vector space V^* is spanned by the intensive variables. The function

$$S(x) = \log \omega(x) \tag{91}$$

of the extensive variables may be interpreted as entropy. Because it is convex

$$S(\mu x + (1-\mu)x') \geq \mu S(x) + (1-\mu) S(x') \, , \tag{92}$$

with equality when x is proportional to x', the entropy can not decrease as a result of a fusion process such as (90). Conversely, the (marginal) stability of a single BPS state against decay into two other BPS states can now be understood as being forbidden by a version of the second law of thermodynamics.

4 BPS states for adS

The N=1 D=4 adS anticommutator (4) may be written as

$$\{Q_\alpha, Q_\beta\} = \frac{1}{2} M_{AB} \left(\mathcal{C}\Gamma^{AB}\right)_{\alpha\beta} \, , \tag{93}$$

where

$$\Gamma^A = (\gamma^\mu, \gamma_5) \tag{94}$$

and $M_{AB} = -M_{BA}$ are the generators of the adS group $SO(3,2)$ (and so are no longer central). The matrix $\mathcal{C}$ is the $SO(3,2)$ charge conjugation matrix; we can choose a representation in which

$$\mathcal{C} = \gamma_0 \gamma_5 \tag{95}$$

and this choice will be implicit in what follows. Note that

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}\,, \tag{96}$$

where η is a flat metric on $\mathbb{E}^{(2,4)}$, such that $\eta = \mathrm{diag}(-1,1,1,1,-1)$ in cartesian coordinates. Although (4) is preserved by $GL(4;\mathbb{R})$, the automorphism group of the adS supergroup $OSp(1|4;\mathbb{R})$ is $Sp(4;\mathbb{R}) \subset SL(4;\mathbb{R}) \subset GL(4;\mathbb{R})$.

The anticommutator (4) can also be written in the form (2), with

$$M_{04} = H, \qquad M_{i4} = -P_i, \qquad M_{0i} = -U_i, \qquad J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk} = -V_i \tag{97}$$

where H is the hamiltonian, $\mathbf{P}$ the 3-momentum, $\mathbf{J}$ the angular momentum while the 3-vector $\mathbf{U}$ generates boosts. The analysis of supersymmetric charge configurations is then exactly the same as in the super-Poincaré case considered earlier, and in particular requiring $\frac{1}{4}, \frac{1}{2}$ or $\frac{3}{4}$ supersymmetry gives exactly the same conditions on the charges $H, \mathbf{U}, \mathbf{V}, \mathbf{P}$ as were found earlier.

The condition for preservation of supersymmetry can be expressed in terms of the $SO(3,2)$ Casimirs. We will first show how the values of these Casimirs are constrained by the physical state condition, and then turn to the supersymmetric states.

4.1 Physical States in adS

Physical states lie either in the convex cone for which $Z = \frac{1}{2}M_{AB}C\Gamma^{AB}_{\alpha\beta}$ is positive, or on its boundary, for which $\det Z = 0$. This cone is a subspace of the 10-dimensional vector space spanned by 5×5 skew-symmetric matrices M with entries M_{AB}. The matrix commutator turns this space into the Lie algebra $so(3,2)$. This algebra has rank 2, with quadratic Casimir[7]

$$c_2 = \frac{1}{2}M_{AB}M^{AB}\,, \tag{98}$$

and quartic Casimir

$$c_4 = M^A{}_B M^B{}_C M^C{}_D M^D{}_A. \tag{99}$$

Since $\det Z$ is both a quartic polynomial of the charges and $SO(3,2)$ invariant it must be a linear combination of c_4 and c_2^2. In fact

$$\det Z = c_4 - c_2^2\,, \tag{100}$$

and hence

$$c_4 \geq c_2^2 \tag{101}$$

for physical states.

There is a further constraint on the Casimirs required by physical states. To see this, we begin by noting that the vacuum is the only physical state for which the energy M_{04} vanishes. This follows from the fact that $\{Q,Q\}$ is positive semi-definite, with a trace equal to $4M_{04}$. We next prove that M_{04} must vanish if the kernel of M contains a timelike 5-vector. Suppose that such a 5-vector exists. By an $SO(3,2)$ transformation, we can arrange for it to have only one non-vanishing component, in the 4-direction. It then follows that the only non-vanishing components of M are $M_{\mu\nu}$. In particular, the energy M_{04} vanishes. Thus, *for any non-vacuum physical state the kernel of M contains no timelike vectors*. Note that the kernel of M has dimension 1, 3 or 5, according to

[7]The quadratic Casimir provides a metric of signature $(4,6)$ on the 10-dimensional vector space, but this metric (which is inherited from the metric η on $\mathbb{E}^{(3,2)}$) does not play a crucial role in the following analysis.

whether M has rank 4, 2 or 0, respectively. The vacuum is the only physical state for which M has rank 0.

Now consider the Pauli-Lubanski 5-vector

$$s^A = \frac{1}{8}\epsilon^{ABCDE} M_{BC} M_{DE} \,. \tag{102}$$

This satisfies the identity

$$M_{AB} s^B \equiv 0 \,, \tag{103}$$

which shows that, unless it vanishes, s is in the kernel of M. A timelike s would therefore be in the kernel of M but, as we have just seen, the kernel of M cannot contain timelike vectors unless M vanishes, but in that case s also vanishes. Thus, s cannot be timelike. Now,

$$s^2 \equiv \eta^{AB} s_A s_B = \frac{1}{4}(2c_2^2 - c_4) \,, \tag{104}$$

so s will be non-timelike if and only if

$$c_4 \leq 2c_2^2 \,. \tag{105}$$

This bound implies (for physical states) that $c_4 = 0$ when $c_2 = 0$.

4.2 Supersymmetric States

Our main interest is in BPS states, i.e. the subset of physical states that are supersymmetric. These must saturate the bound (101), so BPS states are those for which

$$c_4 = c_2^2 \,. \tag{106}$$

Using this in (104) we see that

$$s^2 = \frac{1}{4} c_2^2 \tag{107}$$

for *supersymmetric* states. We will organise our discussion of the supersymmetric states according to whether s is zero, spacelike or non-vanishing null.

If s vanishes then M has either a 3-dimensional or a 5-dimensional kernel. M will have a 5-dimensional kernel only if it vanishes. If the kernel is 3-dimensional then, as we have seen, it cannot contain timelike vectors. It may contain null vectors but any such null vector must be orthogonal to all other vectors in the kernel, spacelike or null, because we could otherwise find a timelike linear combination. Since the maximum number of mutually orthogonal null 5-vectors is 2, a 3-dimensional kernel must contain at least one spacelike vector. There are three possible choices for the other two linearly independent 5-vectors: (i) both spacelike, (ii) one spacelike and one null, or (iii) both null. In all cases M can be brought to a form in which $M_{04} = E \geq 0$ is its only *independent* entry. In case (i) M_{04} and M_{40} are the only entries, and the only *supersymmetric* state with this property is the vacuum, with $E = 0$. In case (ii) M can be brought to a form for which the only non-zero upper-triangular entries are $M_{04} = M_{02} = E$. It then follows from the discussion of section 2.4, on which we will elaborate below, that *all* such states are 1/2 supersymmetric. In case (iii) M can brought to a form for which the only non-zero upper-triangular entries are $M_{04} = -M_{02} = M_{23} = M_{34}$; *all* such states are 3/4 supersymmetric.

Consider now spacelike s. In this case we may choose the only non-vanishing component of s to be its 1-component. Since s now spans the kernel of M, this 5×5 matrix M then reduces to a 4×4 matrix F acting on the 4-dimensional (0234) subspace orthogonal to s, on which η restricts to a metric $\tilde{\eta}$ of signature $(2, 2)$. The matrix F is equivalent to a second-rank antisymmetric tensor in $\mathbb{E}^{(2,2)}$ that can be written uniquely as $F = F^+ + F^-$ where F^+ is real and self-dual while F^- is real and anti-self-dual matrix. Now

$$c_4 - c_2^2 = \left[\operatorname{tr}(\tilde{\eta}F^+)^2\right]\left[\operatorname{tr}(\tilde{\eta}F^-)^2\right] . \tag{108}$$

We can write F as

$$F = \begin{pmatrix} 0 & u & b & E \\ -u & 0 & -v & c \\ -b & v & 0 & -p \\ -E & -c & p & 0 \end{pmatrix} \tag{109}$$

provided that

$$vE + bc + up \neq 0\,, \tag{110}$$

since s would otherwise vanish. Now

$$-\operatorname{tr}(\tilde{\eta}F^\pm)^2 = (E \mp v)^2 - (u \pm p)^2 - (b \pm c)^2 . \tag{111}$$

Configurations with self-dual or anti-self-dual F, for which $E = \mp v$, $u = \pm p$ and $b = \pm c$, are 1/2 supersymmetric. However, any configuration for which

$$(E \mp v)^2 = (u \pm p)^2 + (b \pm c)^2 \tag{112}$$

is also supersymmetric. In fact

$$\{Q,Q\} = \left[(E \mp v) - (b \pm c)\gamma^{012} + (u \pm p)\gamma^{013}\right] + \left(v - c\gamma^{02} + p\gamma^{03}\right)\left(1 \pm \gamma^1\right) . \tag{113}$$

Given (112), the term in square brackets is proportional to a 1/2 supersymmetry projector that commutes with the 1/2 supersymmetry projector $(1/2)\left(1 \pm \gamma^1\right)$ which leads generically to 1/4 supersymmetry.

The final case to consider is s null but non-zero. By means of an $SO(3,2)$ transformation we may choose

$$s \propto (1,0,1,0,0) \tag{114}$$

This choice is preserved by an $SO(1,2)$ 'stability' subgroup, and by a transformation in the $SO(2)$ subgroup of this group we can bring M to the standard form

$$M = E\begin{pmatrix} 0 & 0 & -a & 0 & 1 \\ 0 & 0 & a & 0 & -1 \\ a & -a & 0 & t & -q \\ 0 & 0 & -t & 0 & -r \\ -1 & 1 & q & r & 0 \end{pmatrix} \tag{115}$$

One then finds that

$$c_2 = E^2(t^2 - q^2 - r^2)\,, \tag{116}$$

so that supersymmetric states are those with

$$t = \pm\sqrt{q^2 + r^2}\,. \tag{117}$$

Actually, in arriving at the above form of M we have used only that the null 5-vector $(1,1,0,0,0)$ is in the kernel of M. To ensure that this 5-vector is proportional to s (with non-zero constant of proportionality) we require that

$$t + ra \neq 0\,. \tag{118}$$

This condition also ensures (as it must) that M has rank 4. When combined with (117) it implies that

$$t \neq 0\,. \tag{119}$$

For M of the form (115) we have

$$\{Q,Q\} = E\left\{(1-a\gamma_3)(1-\gamma_{01}) - t\gamma_1\left[1-(q/t)\gamma_{012}-(r/t)\gamma_{013}\right]\right\} \tag{120}$$

A spinor ϵ is in the kernel of $\{Q,Q\}$ if

$$\left[(q/t)\gamma_{012}+(r/t)\gamma_{013}\right]\epsilon = \epsilon\,, \tag{121}$$

and

$$\gamma_{01}\epsilon = \epsilon\,, \tag{122}$$

and these two constraints imply 1/4 supersymmetry. Note that when $a = \pm 1$ and $q = 0$ and hence $t = \pm r$, the latter constraint can be replaced by $\gamma_3\epsilon = \pm\epsilon$, which again yields 1/4 supersymmetry.

4.3 Examples

Many of the possibilities for BPS configurations just noted are illustrated by the class of examples considered in section 2.4. This means, in the language of this section, that the non-zero upper-triangular components of M_{AB} are taken to be $M_{04} = E$, $M_{34} = -p$, $M_{02} = u$ and $M_{23} = -v$. The Pauli-Lubanski 5-vector is then

$$s = (0, Ev + up, 0, 0, 0)\,, \tag{123}$$

so s is spacelike unless it vanishes. The Casimirs for this class are given by

$$\begin{aligned} c_2 &= E^2 + v^2 - p^2 - u^2 && (124)\\ c_4 &= 2[E^4 + u^4 + v^4 + p^4 - 2(v^2+E^2)(u^2+p^2) - 4Euvp] && (125)\end{aligned}$$

The BPS condition $c_4 = c_2^2$ becomes

$$(E-u-v-p)(E-u-v+p)(E-u+v-p)(E+u-v-p) = 0\,, \tag{126}$$

in agreement with (45).

Let us first consider vanishing s. We have seen above that M can be brought to a standard form in which all charges are determined in terms of $M_{04} = E$. The non-vacuum BPS states occured for cases (ii) and (iii) discussed above. An example of case (ii) within the class of configurations now under discussion is found by setting $v = p = 0$ and $E = |u|$[8]. Finally, an example of case (iii), with 3/4 supersymmetry, is obtained by setting $u = v = p = -E < 0$, although there is no known field theoretic realization of this case.

We next to turn to examples with s spacelike. Let us first consider $u = p = 0$ and set $v = -J$, where J is the spin about the 1-axis. We then have

$$c_2 = E^2 + J^2\,, \qquad c_4 = 2E^4 + 2J^4\,, \tag{127}$$

which is equivalent to

$$\begin{aligned} E &= \sqrt{\frac{c_2}{2} + \frac{1}{2}\sqrt{c_4 - c_2^2}}\,,\\ J &= \pm\sqrt{\frac{c_2}{2} - \frac{1}{2}\sqrt{c_4 - c_2^2}}\,. && (128)\end{aligned}$$

[8]The charge u can be interpreted as a membrane charge. To see this note that there is a static planar solution of the equations of motion of a test membrane in adS_4 at a fixed radial distance, in horospherical coordinates, from the Killing horizon [30]. This solution must preserve 1/2 supersymmetry of the adS_4 supersymmetry because adS_4 can itself be interpreted as a membrane, at the horizon, to which the test membrane is parallel. Because this test membrane remains at a fixed distance from the horizon, the worldline of a point on it is uniformly accelerated, and therefore naturally associated with a non-zero boost u.

The physical states satisfy $E \geq |J|$ and states that saturate this bound preserve 1/2 supersymmetry. For these configurations the matrix F of (109) is either self-dual or anti-self-dual. An example of states with s spacelike and F neither self-dual or anti-self-dual can be obtained by taking u, v, p to be positive and solving (112) via $E = u + v + p$. We then have

$$\{Q, Q\} = u(1 + \gamma^{013}) + p(1 + \gamma^{03}) + v(1 + \gamma^{1}) \tag{129}$$

and 1/4 of the supersymmetry is preserved.

5 Comments

We have seen that a full analysis of the D=4 N=1 supersymmetry algebra not only confirms the existence of 1/2 and 1/4 supersymmetric states, realizable within the WZ model, and determines some of their properties, but it also permits states with 3/4 supersymmetry [14, 16, 18] which, as we have shown, cannot be realized by solutions of the WZ model. However, it has been argued that such 'exotic' fractions might play a role in other contexts, and with this in mind we have provided a detailed analysis of the BPS states of D=4 N=1 supersymmetry. We have also seen that these states can be understood in terms of the geometry associated with the convex cone of the Jordan algebra $J_4^{\mathbb{R}}$, and that this leads to a natural generalization of the rotation, Lorentz and conformal groups.

In general, the $U(1)_R$ symmetry will be broken to at most a discrete subgroup. For theories with domain walls (e.g. the WZ model), the R-symmetry will be explicitly broken by the scalar potential. In theories with only massless particles, and no domain walls, the $U(1)_R$ symmetry will be generically broken to a discrete subgroup by chiral anomalies. For theories in which the domain wall charges are quantized, the $U(1)_R$ symmetry will be broken to the discrete subgroup preserving the quantization condition. An example of this is given by M-theory compactified on a 7-manifold of G_2 holonomy, yielding a D=4 N=1 theory in which the domain walls are M2-branes and wrapped M5-branes, with the M2-brane and M5-brane charges quantized. Given that only a discrete subgroup of $U(1)_R$ survives the same is true of the larger group $SU(2)_R$.

We noted that, in the classical theory, the automorphism group of the full supertranslation algebra is $GL(4, \mathbb{R})$, but it seems that any realization of this on fields, and any realisation of the generalized conformal group $Sp(8, \mathbb{R})$, requires an enlargement of 3-space to include coordinates conjugate to the 'domain-wall' charges $\mathbf{U}$ and $\mathbf{V}$. Of course, the domain wall interpretation is probably no longer appropriate in this case. Other interpretations are certainly possible in the context of particle mechanics [17]. In such one-dimensional field theories it is possible to realize the $SU(2)_R$ symmetry between the three 3-vector 'charges' $\mathbf{P}, \mathbf{U}, \mathbf{V}$ as an internal symmetry. For such models that arise from the toroidal compactification of some D=4 theory with quantized $\mathbf{U}$ and $\mathbf{V}$, the 3-momentum will also be quantized and the classical $GL(4; \mathbb{R})$ symmetry will be broken to the discrete $GL(4; \mathbb{Z})$ subgroup preserving the 9-dimensional charge lattice.

Many of the observations made here for $N = 1$ $D = 4$ can of course be generalized to $N > 1$ or to $D > 4$. For example the general N extended $D = 4$ supersymmetry algebra has automorphism group $GL(4N; \mathbb{R})$ and $\det\{Q, Q\}$ is preserved by the subgroup $SL(4N, \mathbb{R})$. This leads to the sequence

$$SO(4N) \subset SL(4N; \mathbb{R}) \subset Sp(8N; \mathbb{R})\,. \tag{130}$$

for the Jordan algebra $J_{4N}^{\mathbb{R}}$ of $4N \times 4N$ symmetric matrices over the reals. The generalised conformal symmetry of the BPS condition is then $Sp(8N; \mathbb{R})$, as deduced from a different analysis in [20].

A $D > 4$ case of particular interest is the D=11 'M-theory algebra' $\{Q, Q\} = Z$ where Q is now a 32 component real spinor of the D=11 Lorentz group and Z is a 32×32

real symmetric matrix containing the Hamiltonian and 527 central charges carried by M-branes [31]. This supersymmetry algebra has automorphism group $GL(32;\mathbb{R})$, as noted independently in [32], and Z takes values in the convex cone associated with the Jordan algebra $J_{32}^{\mathbb{R}}$. The sequence (87) of groups associated with this algebra is

$$SO(32) \subset SL(32;\mathbb{R}) \subset Sp(64;\mathbb{R})\,, \tag{131}$$

so that $Sp(64;\mathbb{R})$ is the M-theoretic generalisation of the D=11 conformal group. As in the D=4 case, the realization of any of these larger 'spacetime' symmetry groups, or discrete subgroups such as $GL(32;\mathbb{Z})$, would seem to require consideration of an enlarged space of 527 coordinates, as considered for other reasons in [33].

Finally, we have found many possibilities for new BPS states in anti de Sitter space. It seems likely that some of these, in particular those with 1/4 supersymmetry, will have a realization in the context of N=1 D=4 supersymmetric field theories in an adS spacetime.

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