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COMPLEX POWERS OF ABSTRACT PSEUDODIFFERENTIAL OPERATORS

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Abstract. Under suitable assumptions, we show that the abstract pseudodifferential operators introduced by Connes and Moscovici possess complex powers that belong to this class of operators. We analyse several spectral functions obtained via the (super)trace including the zeta function and the heat trace. We present examples showing that the analysis is explicit and tractable.

1. Introduction and key results

In their seminal paper on the local index theorem [20], Connes and Moscovici introduced an algebra of abstract pseudodifferential operators. These are operators that have a certain asymptotic expansion in powers of $|D|$ where D is a Dirac operator. In a sense, these operators may be seen as an analogue of a class of classical pseudodifferential operators between vector bundles on manifolds.

More precisely, let (A, H, D) be a spectral triple and denote by $\mathcal{B}$ the algebra generated by $\{\delta^k(a), \delta^k([D, a])\}$ for $a \in A$ and $k = 0, 1, 2, \ldots$ where δ is the derivation $\delta(\cdot) = [|D|, \cdot]$. Abstract pseudodifferential operators are then formally given as an asymptotic series

$$P \simeq p_\mu |D|^\mu + p_{\mu-1}|D|^{\mu-1} + p_{\mu-2}|D|^{\mu-2} + \cdots , \tag{1}$$

where p_μ is a positive real number and $p_{\mu-k} \in \mathcal{B}$ for $k \geq 1$. We call μ the *order* of the operator. The space of such operators of order μ is denoted by Ψ^μ_{cl}.

The main application of this class of operators is to prove the local index theorem as shown in [20, 34]. However, these operators also provide a very transparent framework in which to study classical constructions such as complex powers, spectral functions and traces. Indeed, zeta-regularised traces with regulariser $|D|$ and a canonical trace on Ψ^μ_{cl} were studied in [51].

In this paper we construct complex powers of suitable abstract pseudodifferential operators and use these to investigate spectral functions. The study of complex powers of positive elliptic pseudodifferential operators goes back to the investigations of Seeley [59, 60, 61] and was then further extended for example in [42]. These approaches define the complex powers as a Dunford integral and replace the resolvent by the parametrix. Related constructions for various classes of operators can be found in [7, 30, 44, 57] to name but a few. An axiomatic approach was introduced in [31] and generalised in [2]. Applications of complex powers of operators are rather broad including spectral theory [62], index theory [4] and evolution equations [21]. For the complete picture we refer the reader to the references in the cited papers and also to the references in [45, 50, 58].

It is intriguing that much of the richness of the "commutative" results carries over to the abstract setting, see below for a summary of our key results. Note, however, that the definition (1) already shows the limitations of abstract pseudodifferential operators.

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They are expressed in terms of the basic building block given by the operator $|D|$, which makes the framework rather rigid. This rigidity is also observed from the point of view of $|D|$-regularised traces in [51].

Our key results can roughly be described as follows. Assume that the spectral triple (A, H, D) is d-summable, i.e. that $|D|^{-1}$ belongs to the Schatten class $\mathcal{L}^d$. If the spectral triple is even, let γ be the grading operator. Also denote by $⨍ b = \operatorname{res}_{z=0} \operatorname{Trace}(b|D|^{-z})$ the noncommutative integral. We assume for simplicity that the dimension spectrum Sd is simple. We then construct two operators:

(i) *Complex powers.* For suitable $P \in \Psi^{\mu}_{cl}$ we define the complex powers P^z by a Dunford integral. The P^z form a holomorphic family of abstract pseudodifferential operators in $\Psi^{\mu z}_{cl}$ whose asymptotic expansion can be computed explicitly.

(ii) *Heat operator.* For suitable $P \in \Psi^{\mu}_{cl}$ the heat operator e^{-tP} belongs to the intersection $\cap_{\mu} \Psi^{\mu}_{cl}$, the analogue of smoothing operators.

Using the operators, we study several spectral functions for $Q \in \Psi^{\nu}_{cl}$ and suitable $P \in \Psi^{\mu}_{cl}$:

(i) *Zeta function.* The zeta function $\zeta(z, Q, P) = \operatorname{Trace}(\gamma Q P^{-z})$ can be meromorphically continued to the whole complex plane with at most simple poles in a discrete set $\mathcal{P}$ that can be explicitly given in terms of Sd. As in the classical case, we find the singularity structure to be

$$\Gamma(z)\zeta(z, Q, P) \sim \sum_{\beta \in \mathcal{P} \cup -\mathbb{N}_0} \sum_{l=0}^{1} \frac{a_{\beta,l}}{(z-\beta)^{l+1}}.$$

The residues $a_{\beta,l}$ are expressed in terms of $⨍$.

Since an important purpose of these zeta functions is to define a P-regularised trace of Q, our results form an extension of part of the analysis in [51].

(ii) *Heat trace.* The heat trace $\operatorname{Trace}(\gamma Q e^{-tP})$ has an asymptotic expansion

$$\operatorname{Trace}\left(\gamma Q e^{-tP}\right) \sim \sum_{\beta \in \mathcal{P} \cup -\mathbb{N}_0} \sum_{l=0}^{1} a_{\beta,l} t^{-\beta} \log^l(t)$$

for $t \to 0^+$ with $\mathcal{P}$ and coefficients as above.

(iii) *Weyl asymptotics.* Denote by $N(\lambda)$ the number of eigenvalues of P less than or equal to λ. Then

$$N(\lambda) \sim \frac{a_{d/\mu,0}}{\Gamma(1 + d/\mu)} \lambda^{-d/\mu},$$

where $a_{d/\mu,0}$ is as above.

(iv) *Determinant.* If the zeta function $\operatorname{Trace}(|D|^{-z})$ is regular at $z = 0$, then one can define the determinant

$$\det P = \exp\left(-\frac{d}{dz}\operatorname{Trace}(P^{-z})|_{z=0}\right)$$

for any $P \in \Psi^{\mu}_{cl}$ for which the complex powers exist in Ψ^{μ}_{cl}.

Clearly, these functions provide different ways of aggregating the spectrum so that in particular the pole structure of zeta function and the short-time asymptotics of the heat trace contain equivalent information. It is, however, instructive to see how properties of the spectral triple, namely the singularity structure of the zeta functions $\operatorname{Trace}(b|D|^{-z})$, can be translated into more complex situations.

The paper is organised as follows. For the reader's benefit we give a very brief introduction to the basic ideas of noncommutative geometry in the following section. Section 3 formally introduces the abstract pseudodifferential operators and develops the corresponding calculus. This allows the construction of the complex powers and the heat operator within that class. In Section 4 we apply the (super)trace to the operators to construct certain spectral functions and investigate their properties.

2. A brief survey of noncommutative geometry

In this section we give a brief overview of the basic ideas of noncommutative geometry as initiated by Alain Connes. This overview is limited by the requirements of the main part of this paper so that we only touch upon some aspects of the theory. There are several excellent detailed introductions to noncommutative geometry so that the reader is referred to [5, 13, 15, 29, 43, 67] for a comprehensive presentation. The presentation in this chapter also benefited from unpublished lecture notes by Antony Wassermann.

The main structure and contents of our exposition are strongly inspired by [15, 17]. We will be imprecise on some of the topics and give only key ideas without proofs. The technical depth of the description depends on whether an idea is needed later in the paper or not. We only give selected references and do not aspire to produce a historically accurate account.

The basic idea of noncommutative geometry is to exploit the correspondence between "spaces" and "algebras". In the classic case, the Gelfand-Naimark Theorem tells us that a unital commutative C^*-algebra corresponds to the algebra of continuous functions on a compact Hausdorff space. One then views a noncommutative algebra as encoding a "noncommutative space". Standard examples of noncommutative spaces are:

(i) the space of Penrose tilings,
(ii) the space of leaves of a foliation,
(iii) the phase space in quantum mechanics, and
(iv) the noncommutative torus (the irrational rotation algebra).

One of the aims of noncommutative geometry is to develop tools that allow to analyse such spaces with a noncommutative algebra of coordinates. Analogues of classical "global" tools include de Rham cohomology which corresponds to cyclic cohomology and the K-theory of vector bundles which corresponds to the K-theory of C^*-algebras. Another strand of ideas in noncommutative geometry is motivated by index theory where geometric information is encoded using elliptic operators, in K-homology etc.

The following subsections consider different generalisations of the commutative world. The notation in sections 2.1, 2.2 and 2.3 is borrowed from the respective disciplines and may not fully agree with the notation used in the rest of the paper.

2.1. **Noncommutative measure theory: von Neumann algebras.** Let H be a separable infinite-dimensional Hilbert space and denote by $\mathcal{B}(H)$ the algebra of bounded linear operators on H. A *von Neumann algebra* A is a unital $*$-subalgebra of $\mathcal{B}(H)$ that is closed in the weak (or strong) operator topology. Denote the commutant of A by $A' = \{y \in \mathcal{B}(H) | yx = xy \text{ for all } x \in A\}$. The famous Double Commutant Theorem of von Neumann [65] asserts that A is a von Neumann algebra precisely if $A = A''$, i.e. if A is equal to the commutant of A'.

Any commutative von Neumann algebra is isomorphic to the algebra of bounded measurable functions on a measure space. The surprising feature of a noncommutative von Neumann algebra is that it automatically carries a one-parameter group of automorphisms which one can view as representing time, see below.

Murray and von Neumann [47, 48, 49, 66] showed that any von Neumann algebra can be written as a direct integral of factors. A *factor* is a von Neumann algebra with trivial centre. As any von Neumann algebra is generated by its projections, Murray and von Neumann classified factors into three types by comparing and ordering projections using a Schroeder-Bernstein-type theorem. The types are also distinguished by the range of a countably additive dimension function D defined on projections:

(i) *Type I:* Each Type I factor is either of type I_n for $n \in \mathbb{N}$ (isomorphic to $\mathcal{B}(H_n)$ for some $n \in \mathbb{N}$ where H_n is a Hilbert space of dimension n) with D taking values in $\{1, \ldots, n\}$ or of type I_∞ (isomorphic to $\mathcal{B}(H)$) with D taking values in $\{1, \ldots, \infty\}$.
(ii) *Type II:* This leads to a continuous infinity of factors, any two of them nonisomorphic. We have types II_1 with D taking values in $[0, 1]$ and II_∞ where D has range $[0, \infty]$.
(iii) *Type III:* Here, the dimension function takes only values 0 and ∞, and all nonzero projections have infinite dimension. Powers [53] showed that there is an infinity of such factors R_λ for $\lambda \in (0, 1)$ with any two nonisomoprhic.

The dimension function can be linearised to give a trace on the factor. Factors of Types I_n (n finite) and II_1 allow for a tracial state on the algebra, in type I_∞ and II_∞ some elements have infinite trace and in type III all nonzero elements have infinite trace.

A complete classification of the Type III factors appeared elusive until the 1970s when Connes [11, 12] applied the modular theory of Tomita and Takesaki [63] to this problem. The basic idea is as follows: let Ω be a vector that is cyclic for both A and A', i.e. $A\Omega$ and $A'\Omega$ are dense in H. Define a closable unbounded operator $S : H \to H$ by $x\Omega \mapsto x^*\Omega$ and let $\Delta_\Omega = S^*S$ be the modular operator. This gives rise to a one-parameter group of automorphisms $\sigma_t(x) = \Delta_\Omega^{it} x \Delta_\Omega^{-it}$ for $t \in \mathbb{R}$. If A is commutative, then this is just the identity but for A noncommutative we find the "time evolution" mentioned above. Connes then defined the essential spectrum as $S(A) = \cap \mathrm{Sp}(\Delta_\Omega)$ with intersection over all vectors that are cyclic for A and A'. This is of course hard to compute in concrete situations as one must find all cyclic vectors. However, the so-called 2×2-matrix trick shows that the class of σ_t in the group of outer automorphisms does not depend on Ω. One obtains a group structure on $\Gamma = S(A) \cap \mathbb{R}_+^*$ (the star means that $\mathbb{R}_+$ has the multiplicative group operation) that allows to classify type III factors: III_0 with $\Gamma = \{1\}$, III_λ for $\lambda \in (0, 1)$ with $\Gamma = \{\lambda^k | k \in \mathbb{Z}\}$ and III_1 with $\Gamma = \mathbb{R}_+^*$. Further properties of III_1 were established by Haagerup [32], finalising the classification.

Subsequent major developments in von Neumann algebras include the index theory of subfactors by V. Jones [37] (see [38, 39, 71] for example for its impact on other mathematical subdisciplines) and the programme of free probability theory developed by D. Voiculescu, cf. [70] for an overview of early results.

2.2. **Noncommutative topology: K-theory.** Recall that one of the origins of noncommutative geometry is the Gelfand-Naimark Theorem which one can view as a duality of the category of (locally) compact spaces with (proper) continuous maps and the category of unital commutative C^*-algebras and $*$-homomorphisms.

A second observation allowing to transfer tools from topology to the noncommutative world is the fact that K-theory and its dual theory K-homology allow the use of Hilbert space techniques with the key idea being the idea of a "Fredholm representation" of a C^*-algebra in KK-theory. For example, topological K-theory of a space X can be described in terms of the algebra $C(X)$. Moreover, Atiyah and Bott gave an algebraic proof of the Bott periodicity theorem. Another example is the formulation of K-homology as an *Ext*-functor by Brown, Douglas and Fillmore [6].

These are special cases of the bivariant KK-Theory of Kasparov. This is a powerful tool that allowed a proof of the Novikov conjecture for discrete subgroups of Lie groups. Other key advances in this area include the Pimsner-Voiculescu exact sequence allowing the computation of K-groups of crossed products of C^*-algebras or the excision theorem.

Let us look at these ideas in slightly more detail, cf. [5, 22, 56] for more comprehensive treatments with further references. For any C^*-algebra we define $K_0(A)$ to be the Grothendieck group obtained from equivalence classes of projections in matrix algebras over A. This gives a covariant functor from C^*-algebras to Abelian groups satisfying three main axioms:

(i) *Homotopy invariance.* If A, B are C^*-algebras and φ_1, φ_2 are homotopic homomorphisms $A \to B$, then the induced maps $\varphi_{1*}, \varphi_{2*} : K_0(A) \to K_0(B)$ are the same.

(ii) *Half-exactness.* Given an exact sequence of C^*-algebras $0 \to J \to A \to B \to 0$, the sequence of groups $K_0(J) \to K_0(A) \to K_0(B)$ is exact.

(iii) *Stability.* Let $\mathcal{K}$ be the algebra of compact operators. Then $K_0(A)$ is isomorphic to $K_0(\mathcal{K} \otimes A)$, where one thinks of $\mathcal{K} \otimes A$ as infinitely large matrices with entries from A.

One can view this as an extension of the dimension function introduced by Murray and von Neumann, cf. Section 2.1. The group $K_1(A)$ is also a covariant functor from C^*-algebras to Abelian groups and morally gives an index group for maps from the invertible elements in matrix algebras over A. The prototype is the Fredholm index map from the Calkin algebra $\mathcal{B}/\mathcal{K}$ to the integers. The K-groups have two principal properties:

(i) *Bott periodicity.* Recall that the suspension of a C^*-algebra A is defined as $SA = C_0(\mathbb{R}, A) \simeq A \otimes C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ denotes the algebra of functions on $\mathbb{R}$ vanishing at infinity. Then $K_1(A) \simeq K_0(SA)$ and $K_0(A) \simeq K_1(SA)$.

(ii) *6-term exact sequence.* Let J be a closed two-sided ideal in A. Then the following sequence is exact:

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J) \end{array}$$

There are similar exact sequences that allow the computation of K-groups for crossed products, in the case of crossed products with $\mathbb{Z}$ this is the Pimsner-Voiculescu exact sequence [52]; for an overview of the general situation we refer to Chapter 5 of [24]. By their very constructions, K-groups are also the natural receptacles for index maps.

The basis of K-homology, i.e. the dual theory to K-theory, lies in the classification of short exact sequences $0 \longrightarrow J \longrightarrow E \longrightarrow A \longrightarrow 0$ up to equivalence. Brown, Douglas and Fillmore [6] considered the special case $J = \mathcal{K}$ and $A = C(X)$ for a suitable topological space X. They defined an object $Ext(A)$ which can under certain circumstances be given a group structure. This leads to a contravariant functor from C^*-algebras to Abelian groups that has similar properties to the K-groups. In the case $A = C(X)$, they showed that $Ext(C(X))$ was isomorphic to the first K-homology group of the space X. This was further refined by Kasparov [40] who constructed a bivariant functor $Ext(A, B)$ by considering $J = B \otimes \mathcal{K}$ for certain C^*-algebras B.

Kasparov [41] then constructed another homotopy-invariant bivariant theory $KK(A, B)$ on pairs of C^*-algebras (contravariant in the first variable, covariant in the second). In fact, he defined two groups $KK_0(A, B) = KK(A, B)$ and $KK_1(A, B) = KK(SA, B) \simeq KK(A, SB)$: the group $KK_0(A, B)$ can be seen as "K-homomorphisms" from A to B, whereas $KK_1(A, B)$ is interpreted as equivalence classes of extension of A by B.

Selected key properties of this functor are:

(i) $KK_*(\mathbb{C}, B) \simeq K_*(B)$, and $KK(A, \mathbb{C}) = K^*(A)$, the K-homology.

(ii) There is a product (sometimes called "intersection product") $KK(A, B) \times KK(B, C) \to KK(A, C)$ which corresponds to the composition of "K-homomorphisms".

(iii) $KK(A, \mathbb{C})$ is related to $Ext(A)$ and $KK(SA, B) \simeq Ext(A, B)$.

(iv) In many cases, KK satisfies a 6-term cyclic exact sequence in each variable.

The two basic approaches (or "pictures") of KK-Theory are: the Kasparov picture that has cycles called "Kasparov modules" which themselves are generalisations of the Fredholm modules suggested in [3], where they appear as abstract elliptic operators that are candidates for cycles in K-Homology. And the Cuntz picture as homotopy classes of $*$-homomorphisms from a universal algebra qA, an ideal in the free product $A * A$, to $\mathcal{K} \otimes B$, cf. [22, 23]. An axiomatic characterisation of KK-theory is provided by Higson in [33].

2.3. **Noncommutative differential topology: cyclic cohomology.** We address this in somewhat greater detail as it is more relevant to the rest of the paper. The idea is to generalise classical concepts such as de Rham homology, curvature and connections to a noncommutative setting and the appropriate theory is cyclic cohomology. It also serves as a receptacle for the Chern character $K(A) \to HC(A)$ and allows a pairing between the K-theory of an algebra A and its cyclic cohomology: this generalises the pairing of the Chern character and de Rham currents in the classical theory. For a comprehensive presentation of cyclic (co)homology we refer the reader to [43]. Cyclic homology was independently discovered by Tsygan [64] in the context of Lie algebras.

The cyclic cohomology $HC^*(A)$ of an algebra A is defined (cf. part II of [13]) as the cohomology of the complex of cyclic cochains i.e., functionals satisfying

$$\varphi(a_1, \ldots, a_n, a_0) = (-1)^n \varphi(a_0, a_1, \ldots, a_n)$$

under the coboundary operator

$$\begin{aligned}(b\varphi)(a_0, \ldots, a_{n+1}) &= \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\ &\quad +(-1)^{n+1} \varphi(a_{n+1}a_0, \ldots, a_n).\end{aligned}$$

Equivalently it can be given in terms of the second filtration $Z^*(F^*C)$ of the (b, B)-complex of arbitrary cochains with coboundary operators b and $B = AB_0$ where

$$\begin{aligned}(A\varphi)(a_0, \ldots, a_n) &= \sum_{j=0}^{n} (-1)^j \varphi(a_j, \ldots, a_{j-1}), \\ (B_0\varphi)(a_0, \ldots, a_n) &= \varphi(1, a_0, \ldots, a_n) - (-1)^{n+1} \varphi(a_0, \ldots, a_n, 1).\end{aligned}$$

To switch between these two perspectives on cyclic cohomology note that to an n-dimensional cyclic cocycle φ there corresponds a (b, B)-cocycle $\psi \in Z^p(F^qC)$ with $n = p - 2q$ defined by

$$\psi_{p,q} = \frac{(-1)^{[n/2]}}{n!} \varphi,$$

where $\psi_{p,q}$ is the only nonzero component of ψ (cf. [18] and Remark 30 in Chapter III.1.γ of [15] for details).

The tensor product of algebras descends to a linear periodicity operator $S : HC^n(A) \to HC^{n+2}(A)$ whence the periodic cyclic cohomology $HP^*(A)$ can be defined as the inductive limit $HP^*(A) = \varinjlim_S HC^{2n+*}(A)$.

The pairing between $HP^0(A)$ and $K_0(A)$ is given as follows. Choose $\varphi = (\varphi_{2k}) \in HP^0(A)$ and a selfadjoint idempotent $e \in M_q(A)$. The pairing is given as

$$\langle [\varphi], [e] \rangle = \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} \varphi_{2k} \# \mathrm{Trace}(e, \ldots, e), \tag{2}$$

where $\varphi_{2k}\#\mathrm{Trace}$ is defined on $M_q(A) = M_q(\mathbb{C}) \otimes A$ by

$$\varphi_{2k}\#\mathrm{Trace}(\mu_0 \otimes a_0, \ldots, \mu_{2k} \otimes a_{2k}) = \mathrm{Trace}(\mu_0\mu_1 \cdots \mu_{2k}) \varphi_{2k}(a_0, a_1, \ldots, a_{2k}).$$

For an A-bimodule M recall that the Hochschild cohomology $H^*(A, M)$ is the cohomology of n-linear maps $A \to M$ under the coboundary operator b. Cyclic cohomology is linked to Hochschild cohomology via a long exact sequence

$$\cdots \to HC^n(A) \xrightarrow{I} H^n(A, A^*) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \to \cdots, \tag{3}$$

where A^* denotes the dual of A as a linear space and I is the inclusion map.

To see that periodic cyclic cohomology is the "right" generalisation of de Rham homology, we quote the

Theorem 2.1 ([13], Theorem 46)**.** *Let M be a smooth compact manifold, and let $A = C^\infty(M)$. Then $HP^*(A)$ is canonically isomorphic to the de Rham homology $H_*^{dR}(M, \mathbb{C})$.*

Note that to a homomorphism $\varphi : A \to B$ there correspond morphisms in the complexes we have studied inducing maps on homology denoted by $HC_*(\varphi), HC^*(\varphi)$, etc.

2.4. **Noncommutative manifolds: Spectral triples and the Dirac operator.** A Riemannian geometry is characterised by a tuple (M, g) consisting of a manifold M and a metric g. Points on the manifold are labelled by coordinates which form a commutative algebra. The noncommutative notion of a manifold is given by a spectral triple (A, H, D) consisting of an algebra A represented as bounded operators on a Hilbert space H and an unbounded operator D, the *Dirac operator*. The algebra A corresponds to the manifold M and we think of A as noncommutative coordinates. The metric corresponds to the Dirac operator in the sense that the line element ds becomes the inverse D^{-1}. The metric aspect means that distance between two points on a manifold is generalised using a dual formula based on states. The dimension of a manifold is replaced by the dimension spectrum reflecting the growth of the eigenvalues of $|D|$. The link between the spectral triple and (M, g) is that spectral triples coming from Riemannian manifolds are characterised by commutation relations between D and A and Poincaré duality, cf. Theorem 4.1 of [16] for example.

The link between local and global properties in ordinary manifolds is given by differential and integral calculus. In noncommutative geometry, this is replaced by a local index formula. To describe this, we need a brief excursion into a quantized calculus. The typical "dictionary" of noncommutative geometry translates classical notions into the noncommutative world based on a spectral triple (verbatim from [17]):

Classical	Noncommutative
Complex variable	Linear operator on H
Real variable	Selfadjoint operator on H
Infinitesimal	Compact operator
Infinitesimal of order α	Compact operator with characteristic values μ_n satisfying $\mu_n = O(n^{-\alpha})$ as $n \to \infty$
Integral of an infinitesimal of order 1	$⨍ T$= Coefficient of logarithmic divergence in the trace of T

A metric noncommutative geometry is based on the notion of a spectral triple. We specialise straight to the "regular" case as this is the important situation for the rest of the paper.

Definition 2.2 ([19], Definition 1.120)**.** A *spectral triple* (A, H, D) is given by an involutive algebra A represented on a Hilbert space H and a selfadjoint operator D with compact resolvent such that all commutators $[D, a]$ are bounded for $a \in A$.

Let δ be the derivation $\delta(a) = [|D|, a]$. A spectral triple (A, H, D) is called *regular* if for each $k \in \mathbb{N}_0$ the operators $\delta^k(a)$ and $\delta^k([D, a])$ are bounded. We denote by $\mathcal{B}$ the algebra generated by $\delta^k(a), \delta^k([D, a])$ for $a \in A$ and $k \in \mathbb{N}_0$.

We tacitly assume that D is invertible. The general case can be treated with minor modifications, cf. Section IV.2.γ of [15] or Section 6.1 of [34].

We sometimes assume that there is a grading operator γ on H that commutes with any $a \in A$ and anticommutes with D. The Hilbert space H then splits as $H = H^+ \oplus H^-$. With respect to this splitting we have the representation $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, an element $a \in A$ acts diagonally as $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ and the Dirac operator can be decomposed as $\begin{pmatrix} 0 & D_{-+} \\ D_{+-} & 0 \end{pmatrix}$. The selfadjointness of D forces $D_{-+}^* = D_{+-}$.

The metric aspect of the Dirac operator can be seen as follows. Given a C^*-algebra A, one can define a metric on its space of states by

$$d(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : ||[D, a]|| \leq 1\},$$

which agrees with the geodesic distance when evaluated on a Riemannian manifold and its commutative coordinate algebra. The reader is referred to [15], Chapter VI.1 for further details.

There is also the notion of dimension of a noncommutative space.

Definition 2.3 (cf. [19], Definition 1.122)**.** A spectral triple (A, H, D) is *finitely summable* or *d-summable* when $|D|^{-1}$ belongs to the Schatten ideal $\mathcal{L}^d$ for some $d \geq 1$. We call d the *degree of summability* of the spectral triple.

One can interpret d as the *metric dimension* of the noncommutative space defined by the spectral triple. A more refined notion of dimension is played by a discrete set of singularities of a zeta function.

Definition 2.4 ([20], Definition II.1)**.** A regular spectral triple (A, H, D) has discrete *dimension spectrum* Sd if $Sd \subset \mathbb{C}$ is discrete and for any element of the algebra $\mathcal{B}$ the function

$$\zeta_b(z) = \mathrm{Trace}(b|D|^{-z})$$

extends holomorphically to $\mathbb{C} \setminus Sd$. We say that the dimension spectrum is *simple* if ζ_b hast at most simple poles for any $b \in \mathcal{B}$.

Note that for ease of presentation we assume throughout this paper that any dimension spectrum is simple. The results can easily be adjusted to the case when the poles are of higher order.

Example 2.5. We give some examples of regular spectral triples.

(i) In the case of classical pseudodifferential operators on a closed manifold M one typically considers zeta functions of the form $z \mapsto \mathrm{Trace}(QP^{-z})$. Here P is a positive differential operator of order μ and Q is a differential operator of order ν. Standard results (e.g., Theorem 1.12.2 of [28]) show that the function $\Gamma(z)\mathrm{Trace}(QP^{-z})$ can be extended to a meromorphic function with at most isolated simple poles at the points $Sd = \{(d + \nu - j)/\mu | j \in \mathbb{N}_0\}$ where d is the dimension of M. This example suggests the interpretation of Sd as the correct notion of dimension in noncommutative geometry.

(ii) The noncommutative n-torus with dimension spectrum $\{n, n-1, \ldots\}$, cf. [26, 27] for an analysis of the associated spectral triple.

(iii) The quantum sphere $SU_q(2)$ with dimension spectrum $\{1, 2, 3\}$, cf. [35].

(iv) The spectral triple from the triangular structure in [20].
(v) A general construction for obtaining regular spectral triples with finite and simple dimension spectrum was proposed in [8].

An example of a non-regular spectral triple but with a meaningful calculus of abstract pseudodifferential operators is the standard Podleś sphere [25].

A key role will be played by the residues of ζ_b, see above.

Definition 2.6 (cf. [19], Theorem 1.134)**.** We denote the residues at the poles of the function $\zeta_b(z)$ by

$$⨍ b = \operatorname*{res}_{z=0} \operatorname{Trace}(b|D|^{-z})$$

and call this the *noncommutative integral*.

This deserves further explanation and context as it is used throughout this paper. We briefly recall the definition of the Dixmier trace and list its main properties. To any bounded sequence $(\alpha_n)_{n\in\mathbb{N}}$ we assign the bounded function f_α given by $f_\alpha(\lambda) = \alpha_n$ for $\lambda \in (n-1, n]$. The Cesàro mean of a function $f : [1,\infty) \to \mathbb{C}$ with respect to the multiplicative group $\mathbb{R}_+^*$ with Haar measure $d\lambda/\lambda$ is given by

$$M(f)(\lambda) = \frac{1}{\log \lambda} \int_1^\lambda f(u)\, \frac{du}{u} \,.$$

Now take a positive linear form L on the bounded continuous functions $C_b(\mathbb{R}_+^*)$ such that $L(1) = 1$ and which is zero on the subspace $C_0(\mathbb{R}_+^*)$ of functions vanishing at ∞.

In the following definition of the Dixmier trace, denote the linear map on $l^\infty(\mathbb{N})$ given by

$$\alpha \longmapsto L(M(f_\alpha))$$

by $\operatorname{Lim}_\omega(\alpha)$ for $\alpha \in l^\infty(\mathbb{N})$.

Definition 2.7 (cf. [15], Chapter IV.2.β, Definition 2)**.** For any $T \geq 0, T \in \mathcal{L}^{(1,\infty)}(H)$, the *Dixmier trace* is defined by

$$\operatorname{Trace}_\omega(T) = \operatorname{Lim}_\omega \left(\frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T) \right)$$

with $\mu_n(T)$ being the eigenvalues of $|T|$ in decreasing order with multiplicity.

One can prove that $\operatorname{Trace}_\omega$ is additive (cf. [15], IV.2.β),

$$\operatorname{Trace}_\omega(T_1 + T_2) = \operatorname{Trace}_\omega(T_1) + \operatorname{Trace}_\omega(T_2)$$

for $T_1, T_2 \geq 0$ and $T_1, T_2 \in \mathcal{L}^{(1,\infty)}(H)$. This shows that one can extend $\operatorname{Trace}_\omega$ linearly to the whole of $\mathcal{L}^{(1,\infty)}(H)$.

The next proposition collects some properties of $\operatorname{Trace}_\omega$:

Proposition 2.8 ([15], Chapter IV.2.β, Proposition 3)**.** *Denote by* $\operatorname{Trace}_\omega$ *the Dixmier trace on the ideal* $\mathcal{L}^{(1,\infty)}(H)$.

(i) If $T \geq 0$ *then* $\operatorname{Trace}_\omega(T) \geq 0$.
(ii) If S *is any bounded operator and* $T \in \mathcal{L}^{(1,\infty)}(H)$, *then* $\operatorname{Trace}_\omega(ST) = \operatorname{Trace}_\omega(TS)$.
(iii) $\operatorname{Trace}_\omega(T)$ *is independent of the choice of the inner product on* H, *i.e., it depends only on the Hilbert space* H *as a topological vector space.*
(iv) $\operatorname{Trace}_\omega$ *vanishes on the ideal* $\mathcal{L}_0^{(1,\infty)}(H)$ *which is the closure, for the* $\|\cdot\|_{1,\infty}$*-norm, of the ideal of finite rank operators.*

Here, $\|x\|_{p,\infty} = \sup_{N\geq 1} \frac{1}{N^{1-\alpha}}\sigma_N(x)$ with $\alpha = 1/p, \sigma_N(x) = \sum_0^{N-1}\mu_n(x)$. The last claim implies that $\mathrm{Trace}_\omega(T) = 0$ if $T \in \mathcal{L}^1(H)$. We shall employ that fact later.

So far, we have not dealt with the dependence of Trace_ω on the choice of the linear functional L. For a positive operator $T \in \mathcal{L}^{(1,\infty)}(H)$, the complex powers T^z are well-defined on the set $\{z \in \mathbb{C}|\mathrm{Re}\ (z) > 1\}$ and are of trace class, so that

$$\zeta(z) = \mathrm{Trace}(T^z) = \sum_0^\infty \mu_n(T)^z$$

defines a holomorphic function. Here, $\mu_n^z = e^{iz\log\mu_n}$ and log is analytic on the cut plane $\mathbb{C}\setminus\{x \in \mathbb{R} \mid x \leq 0\}$.

The next proposition allows us to compute the Dixmier trace under certain circumstances.

Proposition 2.9 ([15], Chapter IV.2.β, Proposition 4)**.** *For any $T \geq 0, T \in \mathcal{L}^{(1,\infty)}(H)$, the following conditions are equivalent*

(i) $(z-1)\zeta(z) \longrightarrow k$ *as* $z \to 1^+$,
(ii) $\frac{1}{\log N}\sum_0^{N-1}\mu_n \longrightarrow k$ *as* $N \to \infty$.

Under these conditions, the value of $\mathrm{Trace}_\omega(T)$ is of course independent of the functional L, and if $\zeta(z)$ has a simple pole at $z = 1$, this value is just the residue of ζ at $s = 1$, i.e.

$$\mathrm{Trace}_\omega(T) = \operatorname*{res}_{z=1}\zeta(z)\ .$$

In 1988, Connes [14] established the link between the Dixmier trace and the Wodzicki residue [72]. He proved the following theorem where the Wodzicki residue is denoted by RES.

Theorem 2.10 ([14], Theorem 1)**.** *Let M be a compact n-dimensional manifold, E a complex vector bundle on M and P a pseudodifferential operator of order $-n$ acting on sections of E. Then the corresponding operator P on $H = L^2(M)$ belongs to the Macaev ideal $\mathcal{L}^{(1,\infty)}(H)$ and one has*

$$\mathrm{Trace}_\omega(P) = \frac{1}{n}\mathrm{RES}(P)$$

for any invariant mean ω.

The notion of "locality" in noncommutative geometry means that we use the "local trace" $⨍$ which vanishes in infinitesimals of order greater than 1. There is a related sense of locality in that the quantity is expressed as the residue of an operator zeta function. In the commutative case, the Wodzicki residue can indeed be computed locally in an integral by means of a density.

To finish this introduction, we briefly summarise two index theorems. The second is the Connes-Moscovici index theorem whose original proof uses the abstract pseudodifferential operators that we study later. The Hochschild class of the Chern character of a d-summable spectral triple (A, H, D) can then be expressed locally with the Dixmier trace Trace_ω.

Theorem 2.11 ([15], Chapter IV.2.γ, Theorem 8)**.** *Denote by (A, H, D) a d-summable spectral triple such that for any $a \in A$ the operators a and $[D, a]$ are in the domain of all powers of the derivation $\delta(x) = [|D|, x]$. Then the following expression defines a Hochschild d-cocycle on A by*

$$\varphi_\omega(a_0, \ldots, a_d) = \lambda_d \mathrm{Trace}_\omega(\gamma a_0[D, a_1]\ldots[D, a_p]|D|^{-d})$$

for a constant λ_p. This yields the Hochschild class of the Chern character i.e., its image under the map I in the exact sequence (3) in a pointwise sense: for every d-dimensional Hochschild cycle c the pairing between Hochschild homology and cohomology satisfies

$$\langle \varphi_\omega, c\rangle = \langle \varphi_d, c\rangle,$$

where $\varphi_d \in HC^d(A)$ is the Chern character of (H, F).

This theorem is rather powerful as it is local in the sense of noncommutative geometry and does not need the high regularity assumptions on the spectral triple that the next theorem requires.

We now state the Connes-Moscovici local index theorem. Let (A, H, D) be a finitely summable spectral triple with discrete simple dimension spectrum. Set $da = [D, a]$ for a an operator on H, let

$$\begin{aligned} \nabla a &= [D^2, a], \\ a^{(k)} &= \nabla^k(a), \end{aligned}$$

and define trace-like functionals through residues as

$$\tau_k(P) = \operatorname*{res}_{z=0} z^k \mathrm{Trace}(\gamma P|D|^{-2z})$$

for operators $P \in \mathcal{B}$.

In the even case, the local index theorem of Connes and Moscovici reads

Theorem 2.12 ([20], Theorem II.3)**.** *Under the above assumptions, the following assertions hold.*

(i) One can construct an even cocycle in the (b, B) bicomplex of A by

$$\varphi_n(a_0, \ldots, a_n) = \sum_{k_j \geq 0, q \geq 0} c_{k,\alpha} \tau_q \left(\gamma a_0 (da_1)^{(k_1)} \ldots (da_n)^{(k_n)} |D|^{-(2|k|+n)}\right) \tag{4}$$

for $n \neq 0$ even, where

$$\varphi_0(a_0) = \tau_{-1}(\gamma a_0)$$

with $\tau_{-1}(b) = \mathrm{res}_{s=0}\, s^{-1}\mathrm{Trace}(b|D|^{-2s})$. Here, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $|k| = k_1 + \ldots + k_n$ and the $c_{k,\alpha}$ are suitable coefficients.

(ii) The cohomology class of the cocycle (φ_n) in $HP^0(A)$ coincides with the periodic cyclic cohomology class of the Chern character of (A, H, D).

(iii) We have $\langle[\varphi], [e]\rangle = Ind(e(D \otimes I_N)e)$ for a projection $e \in M_N(A)$. where I_N is the $N \times N$ identity matrix.

3. The calculus of abstract pseudodifferential operators

In this section we recall the basic properties of abstract pseudodifferential operators from [19, 20, 34] and develop a calculus that allows the construction of complex powers as in the case of classical pseudodifferential operators between vector bundles on a closed manifold (the "commutative case").

3.1. **Abstract pseudodifferential operators.** Let (A, H, D) be a regular d-summable spectral triple. We define the Sobolev spaces H^s for $s \in \mathbb{R}$ as $H^s = \mathrm{Dom}\, |D|^s$. These can be completed to Hilbert spaces under the natural norm $||\xi||_s = ||\,|D|^s\xi||$. For $s > t$ we have a continuous inclusion $H^s \hookrightarrow H^t$. Clearly, $H^0 = H$ and we define $H^\infty = \cap_{s\in\mathbb{R}} H^s$ and $H^{-\infty} = \cup_{s\in\mathbb{R}} H^s$.

An immediate consequence of $|D|^{-1}$ being compact is the compact inclusion of Sobolev spaces.

Lemma 3.1. *If $s > t$, then the inclusion $H^s \hookrightarrow H^t$ is compact.*

Proof. We split the inclusion $H^s \hookrightarrow H^t$ as $H^s \xrightarrow{|D|^s} H^0 \xrightarrow{|D|^{t-s}} H^0 \xrightarrow{|D|^{-t}} H^t$ and note that the first and third operators are bounded and the middle one is compact. □

We will consider continuous operators between these spaces. Thus, for $z \in \mathbb{C}$ we introduce the class op^z of linear operators $H^\infty \to H^\infty$ that extend continuously to maps $H^s \to H^{s-\operatorname{Re} z}$ for any $s \in \mathbb{R}$. We first introduce a notion of asymptotic expansion of operators.

Definition 3.2. Let $P \in op^{\mu_0}$ with $\mu_0 \in \mathbb{C}$. Then we say

$$P \simeq p_{\mu_0}|D|^{\mu_0} + p_{\mu_1}|D|^{\mu_1} + p_{\mu_2}|D|^{\mu_2} + \cdots ,$$

where $p_{\mu_k} \in \mathcal{B}$ and $\mu_k \in \mathbb{C}$ with $\operatorname{Re} \mu_k \downarrow -\infty$ if for any $N = 1, 2, \ldots$ there is an $l \in \mathbb{N}$ such that $P - \sum_{k=0}^{l} p_{\mu_k}|D|^{\mu_k} \in op^{-N}$.

This allows to define spaces of abstract (classical) pseudodifferential operators.

Definition 3.3. We say that such a $P \in op^{\mu}$ is an *abstract pseudodifferential operator* if P has the asymptotic expansion

$$P \simeq p_\mu|D|^\mu + p_{\mu-1}|D|^{\mu-1} + p_{\mu-2}|D|^{\mu-2} + \cdots ,$$

where $p_{\mu-k} \in \mathcal{B}$. We call μ the *order* of the operator. The space of abstract pseudodifferential operators of order μ is denoted by Ψ^μ_{cl}. We define the Ψ^∞_{cl} to be the union $\cup_{\mu\in\mathbb{C}}\Psi^\mu_{cl}$ of all abstract pseudodifferential operators and $\Psi^{-\infty}_{cl}$ their intersection $\cap_{\mu\in\mathbb{C}}\Psi^\mu_{cl}$.

Remark 3.4. Two brief remarks on this definition.

(i) We can compare the above notion of abstract pseudodifferential operators with the usual notion of classical pseudodifferential operators. This also makes the rigidity of the abstract framework transparent. For simplicity consider the algebra of *SG*-operators that allow a global calculus on $\mathbb{R}^n$, cf. [44]. Given $m \in \mathbb{C}$, let $S^m(\mathbb{R}^n_x)$ be the space of smooth functions $a(x)$ of a variable x that obey the growth condition

$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha_x a(x)|\, \langle x\rangle^{-m-|\alpha|} < \infty$$

for any multi-index $\alpha \in \mathbb{N}^n_0$ where $\langle x\rangle = \sqrt{1+|x|^2}$. Denote by $S^m_{cl}(\mathbb{R}^n_x)$ the space of *classical* symbols. They are characterised by having an asymptotic expansion into homogeneous terms of certain orders of homogeneity, i.e. there are functions a_{m-j} with

$$a(x) - \sum_{j=0}^{N-1} \chi(x)a_{m-j}(x) \in S^{m-N}(\mathbb{R}^n_x)$$

and $a_{m-j}(tx) = t^{m-j}a_{m-j}(x)$ for $t > 0$ where χ is an arbitrary excision function. Loosely speaking, the space of symbols of *SG*-operators is then given as the tensor product $S^m_{cl}(\mathbb{R}^n_x) \otimes S^\mu_{cl}(\mathbb{R}^n_\xi)$ for $m, \mu \in \mathbb{C}$, cf. [44] for details. The abstract pseudodifferential operators of Ψ^μ_{cl} can then be seen as the analogues of the symbol subclass

$$S^0_{cl}(\mathbb{R}^n_x) \otimes \left\{ a(\xi) \in S^\mu_{cl}(\mathbb{R}^n_\xi) \,\middle|\, a(\xi) \sim \sum_{j=0}^{\infty} \lambda_{\mu-j}|\xi|^{\mu-j}, \lambda_{\mu-j} \in \mathbb{C} \right\}.$$

(ii) A typical example of an abstract pseudodifferential operator is the curvature operator $P = \Delta + [D, a]$ where $\Delta = |D|^2$. This operator appears in the context of cyclic cohomology (Section 8 of [54], cf. also [34]).

We collect several simple properties of these operators that will be useful later.

Lemma 3.5. *Let (A, H, D) be a regular d-summable spectral triple.*

(i) *For* $\mu \in \mathbb{C}$, *any* $P \in \Psi^{\mu}_{cl}$ *belongs to* op^{μ}, *i.e. it extends continuously to a bounded operator* $H^s \to H^{s-\operatorname{Re}\mu}$ *for any* $s \in \mathbb{R}$.

(ii) *If* $P \in op^{\mu}$ *with* $\operatorname{Re}\mu < -d$, *then* P *is trace class. In particular, any* $P \in \Psi^{\mu}_{cl}$ *is trace class for* $\operatorname{Re}\mu < -d$.

(iii) *If the resolvent* $(\lambda - |D|^{\mu})^{-1}$ *exists, then* $(\lambda - |D|^{\mu})^{-k} \in op^{-\mu k}$ *for all* $k \in \mathbb{N}$.

(iv) *If* $b \in \mathcal{B}$ *denote by* $b^{(n)}$ *the* n*-fold commutator with* $|D|^{\mu}$:

$$b^{(0)} = b, \quad b^{(1)} = [|D|^{\mu}, b], \quad b^{(n)} = \left[|D|^{\mu}, b^{(n-1)}\right]$$

for $n = 2, 3, \ldots$ *We then have* $b^{(n)} \in \Psi^{n(\mu-1)}_{cl}$.

(v) *Let* $\mu \in \mathbb{C}$ *and* $b \in \mathcal{B}$. *Then we have the asymptotic expansion*

$$(|D|^{\mu})^z\, b \simeq \sum_{k=0}^{\infty} \binom{z}{k} b^{(k)} \,(|D|^{\mu})^{z-k} \tag{5}$$

for any $z \in \mathbb{C}$, *with* $b^{(k)}$ *as above. In particular,*

$$|D|^z b \simeq \sum_{k=0}^{\infty} \binom{z}{k} \delta^k(b) |D|^{z-k}, \tag{6}$$

where $\delta(b) = [|D|, b]$ *and* δ^k *is the* k*-fold application of* δ.

(vi) *The space* Ψ^{∞}_{cl} *is a graded algebra: if* $P \in \Psi^{\mu}_{cl}$ *and* $Q \in \Psi^{\nu}_{cl}$, *then* $PQ \in \Psi^{\mu+\nu}_{cl}$.

(vii) *Let* $\mu \in \mathbb{C}$, $b \in \mathcal{B}$ *and* $\lambda \notin \operatorname{Sp}(|D|^{\mu})$. *Then*

$$\begin{aligned}(\lambda - |D|^{\mu})^{-k} b \simeq\, & b(\lambda - |D|^{\mu})^{-k} + k b^{(1)} (\lambda - |D|^{\mu})^{-(k+1)} \\ & + \tfrac{k(k+1)}{2!} b^{(2)} (\lambda - |D|^{\mu})^{-(k+2)} \\ & + \tfrac{k(k+1)(k+2)}{3!} b^{(3)} (\lambda - |D|^{\mu})^{-(k+3)} + \cdots\end{aligned} \tag{7}$$

for any $k = 1, 2, 3, \ldots$

Proof of Lemma 3.5. (i) This follows from the definition of the spaces op^{μ} and Definition 3.3.

(ii) The d-summability means that $|D|^{-d}$ is of trace class. Now let $P \in op^{\mu}$ with $\operatorname{Re}\mu < -d$. Then $P = P|D|^d|D|^{-d}$. Here, $P|D|^d$ is bounded and $|D|^{-d}$ is trace class so that the claim follows. The same assertion holds for $P \in \Psi^{\mu}_{cl}$ since such a P belongs to op^{μ} by (i).

(iii) The assertion is clear if we can show that $(\lambda - |D|^{\mu})^{-1} \in op^{-\mu}$. To see this we must show that the resolvent extends to a continuous map $H^s \to H^{s+\operatorname{Re}\mu}$ for any $s \in \mathbb{R}$. But this is clear from the definition of the norm in H^s.

(iv) First recall equation (11) from [20] (which is also equation (6) above)

$$|D|^z b \simeq \sum_{k=0}^{\infty} \binom{z}{k} \delta^k(b) |D|^{z-k}, \tag{8}$$

where $\delta(b) = [|D|, b]$ and δ^k is the k-fold application of δ.

We then construct a suitable asymptotic expansion of $b^{(n)}$ inductively. The claim for $n = 1$ follows from (8) since

$$b^{(1)} = [|D|^{\mu}, b] \simeq \sum_{k=1}^{\infty} \binom{\mu}{k} \delta^k(b) |D|^{\mu-k}.$$

Now assume that the claim holds for some $n > 1$, i.e. there is an asymptotic expansion

$$b^{(n)} \simeq \sum_{k=0}^{\infty} b_k |D|^{n(\mu-1)-k}$$

for some operators $b_k \in \mathcal{B}$. We then obtain the formal expansion

$$\begin{aligned} b^{(n+1)} &= [|D|^\mu, b^{(n)}] \\ &\simeq \sum_{k=0}^{\infty} \left[|D|^\mu, b_k |D|^{n(\mu-1)-k} \right] \\ &= \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \binom{\mu}{l} \delta^l(b_k) |D|^{n(\mu-1)+\mu-(l+k)}, \end{aligned}$$

where we used (8) again. The highest order in $|D|$ is $(n+1)(\mu-1)$ since $k+l \geq 1$. This is the desired asymptotic expansion of $b^{(n+1)}$ in the sense of Definition 3.2.

(v) Equation (5) follows from the proof of Lemma 4.30 in [34] with $\lambda - \Delta$ replaced by $|D|^\mu$. For the reader's convenience we sketch the argument which is based on Cauchy's theorem and commutator identities. We start with the identity

$$[(w - |D|^\mu)^{-1}, b] = (w - |D|^\mu)^{-1} b^{(1)} (w - |D|^\mu)^{-1}$$

for $w \in \mathbb{C}$ and $b \in \mathcal{B}$. Applying this again on the right hand side to push $b^{(1)}$ to the left leads to

$$[(w - |D|^\mu)^{-1}, b] = b^{(1)} (w - |D|^\mu)^{-2} + (w - |D|^\mu)^{-1} b^{(2)} (w - |D|^\mu)^{-2}.$$

We apply this procedure N times to obtain an expression of the form

$$[(w - |D|^\mu)^{-1}, b] = \sum_{k=1}^{N} b^{(k)} (w - |D|^\mu)^{-(k+1)} + R_N(w), \tag{9}$$

where $R_N(w) \in op^{-(N+1)}$ and operator norm $||R_N(w)||_{s,s+N+1} \sim |w|^{-(N+1)}$. The formal series for $[(|D|^\mu)^z, b]$ is then obtained by a Cauchy-type formula

$$\binom{z}{k} (|D|^\mu)^{z-k} = \frac{1}{2\pi i} \int_\Gamma w^z (w - |D|^\mu)^{-(k+1)} dw,$$

which can be proved by integrating by parts. The contour Γ is taken as in [34] as a straight line in the complex plane separating the origin and the spectrum of $|D|$.

The claim (v) follows by applying the Cauchy formula to (9). Note that the Dunford integral converges for Re z suitably negative but can be extended to the whole of the complex plane by increasing N thanks to the estimates on the operator norm of R_N and the order of $b^{(k)}$ from (iv).

(vi) This is clear from (6) as it allows us to write any product $b|D|^{\mu-k} c |D|^{\nu-l}$ with $b, c \in \mathcal{B}$ as a formal asymptotic sum

$$b|D|^{\mu-k} c |D|^{\nu-l} \simeq \sum_{n=0}^{\infty} \binom{\mu-k}{n} b \delta^n(c) |D|^{(\mu+\nu)-(k+l+n)},$$

which belongs to $\Psi_{cl}^{\mu+\nu}$.

(vii) This is Lemma 4.20 in [34] with Δ replaced by $|D|^\mu$, cf. the proof of (v) above. □

3.2. **The parameter-dependent resolvent.** Our first task is to find an asymptotic expansion of the resolvent of an operator $P \in \Psi_{cl}^\mu$ for $\mu > 0$. To this end we consider the operator $\lambda - P$ for λ in a certain sector in the complex plane. The construction is analogous to the construction of parameter-dependent parametrices for classical pseudodifferential operators in a global calculus, cf. [44, 50].

In the rest of this section we make the following standing assumption on abstract pseudodifferential operators.

Hypothesis 3.6. Let $P \in \Psi^{\mu}_{cl}$ for some $\mu > 0$ with asymptotic expansion

$$P \simeq p_\mu |D|^\mu + p_{\mu-1}|D|^{\mu-1} + p_{\mu-2}|D|^{\mu-2} + \cdots ,$$

where p_μ is a positive real number and $p_{\mu-k} \in \mathcal{B}$. Let Λ be a sector in the right half of the complex plane with apex at the origin: $\Lambda = \left\{re^{i\varphi} \middle| \pi - \delta < \varphi < \pi + \delta\right\}$ for a $\delta \in (0, \pi/4)$. Given $\epsilon > 0$ define $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} | |z| \le \epsilon\}$. We then assume:

(i) The operator $\lambda - P$ is invertible for all $\lambda \in \Lambda$ with $\lambda \neq 0$ and $\lambda = 0$ is at most an isolated spectral point.
(ii) For suitably small $\epsilon > 0$, the resolvent of the analogue of the principal symbol $(\lambda - p_\mu|D|^\mu)^{-1}$ exists for any $\lambda \in \Lambda_\epsilon$. For convenience we choose ϵ such that 2ϵ is smaller than any eigenvalue of $p_\mu|D|^\mu$.
(iii) The operator $|D|^\mu(\lambda - p_\mu|D|^\mu)^{-1}$ is bounded in operator norm on H independently of $\lambda \in \Lambda_\epsilon$.

Remark 3.7. These conditions mimic the classical conditions for the construction of complex powers. The first part of assumption (i) is almost verbatim condition (A) of [44]. Also, (ii) and (iii) replace the Λ-ellipticity assumption from [44]; condition (iii) is mentioned for completeness only since considering the eigenvalues of $|D|$ we see that it holds automatically. The condition on the sector Λ as expressed by δ is of technical nature and used in Case (ii) in the proof of Lemma 3.9.

We first note that the resolvent $(\lambda - P)^{-1}$ is of order $-\mu$ on the scale of Sobolev spaces and bounded in operator norm in terms of $|\lambda|$ uniformly for $\lambda \in \Lambda_\epsilon$.

Theorem 3.8. *Assume Hypothesis 3.6. Then for any $l \in \mathbb{R}$ with $0 \le l \le \mu$ there is a constant $c_{\mu,l}$ such that*

$$\left|\left|(\lambda - P)^{-1}\right|\right|_{s,s+l} \le \frac{c_{\mu,l}}{(1 + |\lambda|)^{1-l/\mu}} \tag{10}$$

for any $\lambda \in \Lambda_\epsilon$. Here $|| \cdot ||_{s,s+l}$ denotes the operator norm for maps $H^s \to H^{s+l}$ for any $s \in \mathbb{R}$.

The assertion and proof are analogous to Theorem 9.1 in [62]. We start with an auxiliary result.

Lemma 3.9. *Under the assumptions of Theorem 3.8 define the operator $T = 1 + |D| + |\lambda|^{1/\mu}$. Then $T^\mu(\lambda - |D|^\mu)^{-1}$ belongs to op^0 and has operator norms $H^s \to H^s$ bounded independently of $\lambda \in \Lambda_\epsilon$.*

Proof. The claim follows by considering the action of this operator on the eigenvectors of $|D|$ once we can show that there is a $C > 0$ such that we have

$$(1 + x + |\lambda|^{1/\mu})^\mu(\lambda - x^\mu)^{-1} \le C$$

for all $x \ge 2\epsilon$ and $\lambda \in \Lambda_\epsilon$.

Fix $x > 0$, let $\lambda \in \Lambda_\epsilon$ and decompose $\lambda = \lambda_1 + i\lambda_2$ into real and imaginary parts. We distinguish two cases by the sign of λ_1.

Case (i) $\lambda_1 \ge 0$. The part of Λ_ϵ with $\lambda_1 \ge 0$ is a compact set and hence of no concern for the inequalities. Note that we have $\lambda - x^\mu$ bounded below as $x \ge 2\epsilon$.

Case (ii) $\lambda_1 < 0$. Note that by elementary calculus there is a $C > 0$ such that

$$(1 + t)^\mu \le C(1 + t^\mu) \tag{11}$$

for $t > 0$. By the design of Λ_ϵ we know that if $\lambda_1 < 0$, then $|\lambda_2| < -\lambda_1$. So

$$\begin{aligned}\left|(1 + x + |\lambda|^{1/\mu})^\mu\, (\lambda - x^\mu)^{-1}\right| &= \left(1 + x + \sqrt{\lambda_1^2 + \lambda_2^2}^{\,1/\mu}\right)^\mu \sqrt{(\lambda_1 - x^\mu)^2 + \lambda_2^2}^{\,-1} \\ &\le \left(1 + x + \sqrt{2}^{\,1/\mu}|\lambda_1|^{1/\mu}\right)^\mu (|\lambda_1| + x^\mu)^{-1}.\end{aligned}$$

By repeated application of (11) we find for the first factor that

$$\begin{aligned}\left(1+x+\sqrt{2}^{1/\mu}|\lambda_1|^{1/\mu}\right)^{\mu} &\le C\left(1+\left(x+\sqrt{2}^{1/\mu}|\lambda_1|^{1/\mu}\right)^{\mu}\right)\\ &= C\left(1+x^{\mu}\left(1+x^{-1}\sqrt{2}^{1/\mu}|\lambda_1|^{1/\mu}\right)^{\mu}\right)\\ &\le C\left(1+x^{\mu}C\left(1+x^{-\mu}\sqrt{2}|\lambda_1|\right)\right)\\ &= C+C^2x^{\mu}+C^2\sqrt{2}|\lambda_1|.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\left(1+x+|\lambda_1|^{1/\mu}\right)^{\mu}(\lambda-x^{\mu})^{-1}\right| &\le (C+C^2x^{\mu}+C^2\sqrt{2}|\lambda_1|)/(|\lambda_1|+x^{\mu})\\ &= \frac{C}{|\lambda_1|+x^{\mu}}+C^2\frac{x^{\mu}}{|\lambda_1|+x^{\mu}}+\sqrt{2}C^2\frac{|\lambda_1|}{|\lambda_1|+x^{\mu}},\end{aligned}$$

which can be bounded uniformly in $x \ge 2\epsilon$ and $\lambda \in \Lambda_\epsilon$. □

We are now ready to prove Theorem 3.8.

Proof of Theorem 3.8. Let P be as in Hypothesis 3.6. Without loss of generality $p_\mu = 1$. To prove (10) we write

$$\begin{aligned}\left|\left|(\lambda-P)^{-1}\right|\right|_{s,s+l} &= \left|\left|T^{-\mu}T^{\mu}(\lambda-P)^{-1}\right|\right|_{s,s+l}\\ &\le \left|\left|T^{-\mu}\right|\right|_{s,s+l}\left|\left|T^{\mu}(\lambda-P)^{-1}\right|\right|_{s,s}\end{aligned}$$

and bound the two norms independently of λ in separate steps.

1. To estimate the first norm let $\xi \in H^s$:

$$\begin{aligned}||T^{-\mu}\xi||_{s+l} &= \left|\left||D|^{s+l}(1+|D|+|\lambda|^{1/\mu})^{-\mu}\xi\right|\right|\\ &= \left|\left||D|^{l}(1+|D|+|\lambda|^{1/\mu})^{-\mu}|D|^s\xi\right|\right|\\ &\le \left|\left||D|^{l}(1+|D|+|\lambda|^{1/\mu})^{-\mu}\right|\right|_{0,0}||\xi||_s.\end{aligned}$$

Note that there is a constant $c_{\mu,l}$ such that

$$\sup_{x\ge 0} x^l(1+x+t)^{-\mu} = c_{\mu,l}t^{l-\mu},$$

cf. Step 3 in the proof of Theorem 9.1 of [62]. Considering the eigenvalues of the diagonal operator $|D|^{s+l}(1+|D|+|\lambda|^{1/\mu})^{-\mu}$ and using this bound we find

$$\left|\left||D|^{l}(1+|D|+|\lambda|^{1/\mu})^{-\mu}\right|\right| \le c_{\mu,l}|\lambda|^{(l-\mu)/\mu},$$

which is bounded independently of λ as $0 \le l \le \mu$.

2. By the asymptotic expansion of P we can write $P = |D|^{\mu} + R$ for some $R \in op^{\mu-1}$. Then in the notation of Lemma 3.9 we have

$$T^{\mu}(\lambda-P)^{-1} = T^{\mu}(\lambda-|D|^{\mu})^{-1}\left(I+(\lambda-|D|^{\mu})^{-1}R\right)^{-1}.$$

By Lemma 3.9 the first operator is bounded on norm independently of $\lambda \in \Lambda_\epsilon$. Writing $(\lambda-|D|^{\mu})^{-1}R = (\lambda-|D|^{\mu})^{-1}|D|^{\mu}\ |D|^{-\mu}R$ shows that this is the product of two bounded operators on H with bound controllable in terms of $|\lambda|^{-1}$. Hence, $T^{\mu}(\lambda-P)^{-1}$ is bounded independently of λ.

We know that $T^{\mu}(\lambda-P)^{-1}$ maps $H^s \to H^s$ continuously for any $s \in \mathbb{R}$. It is bounded for $s = 0$, so it must be bounded for any s: consider $|D|^{-s}T^{\mu}(\lambda-P)^{-1}|D|^s$, the composition of bounded operators $H^s \to H \to H \to H^s$. □

We are now ready to investigate parameter-dependent resolvents.

Theorem 3.10. *Under Hypothesis 3.6 there are operators $b_{-\mu-k}(\lambda) \in op^{-\mu-k}$ for $k = 0, 1, 2, \dots$ such that the following holds with $B_N(\lambda) = \sum_{k=0}^{N-1} b_{-\mu-k}(\lambda)$.*

(i) Uniformly in $\lambda \in \Lambda_\epsilon$ we have

$$|\lambda|\,[(\lambda - P)B_N(\lambda) - I] \in op^{-N}, \quad |\lambda|\,[B_N(\lambda)(\lambda - P) - I] \in op^{-N}$$

for any $N = 0, 1, 2, \dots$. By "uniformly" we mean that the operator norm $H^s \to H^{s+N}$ is bounded uniformly in λ.

(ii) Uniformly in $\lambda \in \Lambda_\epsilon$ we have

$$|\lambda|^2 \left[(\lambda - P)^{-1} - B_N(\lambda)\right] \in op^{-N} \tag{12}$$

for any $N = 0, 1, 2, \dots$ so that in particular $(\lambda - P)^{-1} - B_N(\lambda) \in op^{-N}$.

Moreover, the $b_{-\mu-k}$ can be computed explicitly as

$$\left.\begin{aligned} b_{-\mu}(\lambda) &= (\lambda - p_\mu|D|^\mu)^{-1}, \\ b_{-\mu-k}(\lambda) &= \sum_{n=1}^{k} \sum_{\substack{|j|=k \\ j.\geq 1}} \begin{pmatrix} (\lambda - p_\mu|D|^\mu)^{-1} p_{\mu-j_1}|D|^{\mu-j_1}(\lambda - p_\mu|D|^\mu)^{-1} \times \cdots \\ \cdots \times (\lambda - p_\mu|D|^\mu)^{-1} p_{\mu-j_n}|D|^{\mu-j_n}(\lambda - p_\mu|D|^\mu)^{-1} \end{pmatrix} \end{aligned}\right\} \tag{13}$$

for $k \geq 1$. Here, $j = (j_1, \dots, j_n) \in \mathbb{N}^n$ is a multi-index and $j. \geq 1$ means that all components of j are greater than or equal to 1.

If $p_\mu = 1$, the terms read in lowest orders

$$\begin{aligned} b_{-\mu}(\lambda) =&(\lambda - |D|^\mu)^{-1}, \\ b_{-\mu-1}(\lambda) =&(\lambda - |D|^\mu)^{-1} p_{\mu-1}|D|^{\mu-1}(\lambda - |D|^\mu)^{-1} + \cdots, \\ b_{-\mu-2}(\lambda) =&(\lambda - |D|^\mu)^{-1} p_{\mu-1}|D|^{\mu-1}(\lambda - |D|^\mu)^{-1} p_{\mu-1}|D|^{\mu-1}(\lambda - |D|^\mu)^{-1} \\ &+ (\lambda - |D|^\mu)^{-1} p_{\mu-2}|D|^{\mu-2}(\lambda - |D|^\mu)^{-1} + \cdots \end{aligned}$$

with $\lambda \in \Lambda_\epsilon$.

Proof. Without loss of generality $p_\mu = 1$. Part of the argument is similar to Section 3.2 of [44].

1. The construction of the $B_N(\lambda)$ is done as in the commutative case by formally solving the equation

$$\left[(\lambda - |D|^\mu) - p_{\mu-1}|D|^{\mu-1} - \cdots\right]\left[b_{-\mu}(\lambda) + b_{-\mu-1}(\lambda) + \cdots\right] = 1,$$

where $b_{-\mu-k}(\lambda)$ is an operator of order $-\mu-k$ to be determined. By order we mean that the term is homogeneous of order $-\mu-k$ upon replacing $|D|$ by $t|D|$ and λ by $t^\mu\lambda$ for $t > 0$. Solving the formal equation iteratively in each order yields

$$\begin{aligned} b_{-\mu}(\lambda) &= (\lambda - |D|^\mu)^{-1}, \\ b_{-\mu-k}(\lambda) &= (\lambda - |D|^\mu)^{-1}\left[p_{\mu-1}|D|^{\mu-1} b_{-\mu-(k-1)}(\lambda) + \cdots + p_{\mu-k}|D|^{\mu-k} b_{-\mu}(\lambda)\right] \end{aligned}$$

for $k \geq 1$. In closed form this yields (13).

By construction of the $b_{-\mu-k}$ and by Theorem 3.8 we have

$$||b_{-\mu}(\lambda)|| \leq C/(1 + |\lambda|), \quad ||b_{-\mu-k}(\lambda)|| \leq C/(1 + |\lambda|)^2 \tag{14}$$

for some constant C and all $\lambda \in \Lambda_\epsilon$. We finally set

$$B_N(\lambda) = b_{-\mu}(\lambda) + \cdots + b_{-\mu-(N-1)}(\lambda) \tag{15}$$

for $N = 1, 2, \dots$

2. We now prove assertion (i) in two steps.

a) Note that if we can show the claim for $(\lambda - P)B_N(\lambda) - I$, then the reverse order follows automatically. For suppose that $R_{-N}(\lambda) = (\lambda - P)B_N(\lambda) - I$ with $|\lambda|R_{-N}(\lambda) \in op^{-N}$ uniformly in λ. Then $B_N(\lambda)(\lambda - P) - I = S_{-N}(\lambda)$ with $S_{-N}(\lambda) = (\lambda - P)^{-1}R_{-N}(\lambda)(\lambda - P)$. To show that $|\lambda|S_{-N}(\lambda) \in op^{-N}$ uniformly in λ we write

$$\begin{aligned}|\lambda|S_{-N}(\lambda) &= (\lambda - P)^{-1}|\lambda|R_{-N}(\lambda)(\lambda - P)\\ &= \lambda(\lambda - P)^{-1}|\lambda|R_{-N}(\lambda) - (\lambda - P)^{-1}|\lambda|R_{-N}(\lambda)P.\end{aligned}$$

Consider this expression term by term. We know that $|\lambda|R_{-N}(\lambda) \in op^{-N}$ uniformly in λ. Also by Theorem 3.8, the operator $\lambda(\lambda - P)^{-1}$ is in op^0 uniformly in λ so that the summand $\lambda(\lambda - P)^{-1}|\lambda|R_{-N}(\lambda)$ belongs to op^{-N} uniformly in λ. The same holds for the second summand $(\lambda - P)^{-1}|\lambda|R_{-N}(\lambda)P$ by the estimates

$$\begin{aligned}&\left|\left|(\lambda - P)^{-1}|\lambda|R_{-N}(\lambda)P\right|\right|_{s,s+N}\\ &\le ||(\lambda - P)^{-1}||_{s-\mu+N,s+N} \cdot |||\lambda|R_{-N}(\lambda)||_{s-\mu,s-\mu+N} \cdot ||P||_{s,s-\mu}\end{aligned}$$

and again invoking Theorem 3.8 to see that the right hand side is bounded independently of λ.

b) We prove assertion (i) for $R_{-N}(\lambda) = (\lambda - P)B_N(\lambda) - I$. Set $P_N = \sum_{k=0}^{N-1} p_{\mu-k}|D|^{\mu-k}$ which belongs to op^{μ} so that $P - P_N \in op^{\mu-N}$. Then

$$\begin{aligned}(\lambda - P)B_N(\lambda) &= (\lambda - P_N - (P - P_N))\, B_N(\lambda)\\ &= (\lambda - P_N)B_N(\lambda) - (P - P_N)B_N(\lambda).\end{aligned}$$

The first summand can be written as

$$(\lambda - P_N)B_N(\lambda) = \sum_{k=0}^{N-1}\sum_{j=0}^{k} p_{\mu-j}|D|^{\mu-j}b_{-\mu-j}(\lambda) + \sum_{k=N}^{2N-2}\sum_{j=0}^{k} p_{\mu-j}|D|^{\mu-j}b_{-\mu-j}(\lambda).$$

By construction of the $b_{-\mu-k}$ we find

$$\sum_{k=0}^{N-1}\sum_{j=0}^{k} p_{\mu-j}|D|^{\mu-j}b_{-\mu-j}(\lambda) = 1$$

and

$$|\lambda|\left[\sum_{k=N}^{2N-2}\sum_{j=0}^{k} p_{\mu-j}|D|^{\mu-j}b_{-\mu-j}(\lambda)\right] \in op^{-N}$$

uniformly in $\lambda \in \Lambda_\epsilon$. By (14) we have $|\lambda|\left[(P - P_N)B_N(\lambda) - I\right] \in op^{-N}$ also uniformly in $\lambda \in \Lambda_\epsilon$ which proves the claim for $(\lambda - P)B_N(\lambda) - I$.

3. To show (ii) we use the identity $(\lambda - P)^{-1} - B_N(\lambda) = -B_N(\lambda)R_{-N}(\lambda)$ and note that $|\lambda|B_N(\lambda)$ and $|\lambda|R_{-N}(\lambda)$ are uniformly bounded in λ. □

The above form of the resolvent expansion (13) is highly symmetric. For later use we need a different yet more unpleasant representation of the $b_{-\mu-k}$.

Corollary 3.11. *Under the assumptions of Theorem 3.10 and with $p_\mu = 1$, the parameter-dependent operator $(\lambda - P)^{-1}$ has the following explicit asymptotic representation. For $k \ge 1$ we have*

$$b_{-\mu-k}(\lambda) \simeq \sum_{n=1}^{k}\sum_{\substack{|j|=k,\\ j\ge 1\\ l\ge 0\\ m\ge 0}} c(l)c'(j,m)p(j,m,l)|D|^{\mu(n-|m|)-|j|}(\lambda - |D|^{\mu})^{-(|l|+n+1)} \tag{16}$$

with

$$p(j,m,l) = p_{\mu-j_1}^{(l_1)}p_{\mu-j_2}^{(l_2+m_2)}\cdots p_{\mu-j_n}^{(l_n+m_n)}.$$

Here, $l=(l_1,\dots,l_n)\in\mathbb{N}_0^n$ *and* $m=(m_2,\dots,m_n)\in\mathbb{N}_0^{n-1}$ *for constants* $c(l)$ *and* $c'(j,m)$

$$c(l)=\frac{(l_1+\cdots+l_n+n)!}{l_1!\cdots l_n!(l_1+1)\cdots(l_1+\cdots+l_n+n)} \tag{17}$$

and

$$\begin{aligned} c'(j,m)=\binom{1-\frac{j_1}{\mu}}{m_2}\binom{2-\frac{j_1+j_2}{\mu}-m_2}{m_3}\times\cdots \\ \times\binom{(n-1)-\frac{j_1+\cdots+j_{n-1}}{\mu}-(m_2+\cdots+m_{n-1})}{m_n}. \end{aligned} \tag{18}$$

Here, $p^{(l)}$ *denotes the* l*-fold commutator with* $|D|^\mu$*.*

In lowest orders this can be made explicit

$$\begin{aligned} b_{-\mu}(\lambda)=&(\lambda-|D|^\mu)^{-1},\\ b_{-\mu-1}(\lambda)=&p_{\mu-1}|D|^{\mu-1}(\lambda-|D|^\mu)^{-2}+p^{(1)}_{\mu-1}|D|^{\mu-1}(\lambda-|D|^\mu)^{-3}+\cdots,\\ b_{-\mu-2}(\lambda)=&p_{\mu-2}|D|^{\mu-2}(\lambda-|D|^\mu)^{-2}+p^{2}_{\mu-1}|D|^{2\mu-2}(\lambda-|D|^\mu)^{-3}+\cdots \end{aligned}$$

in the above notation.

Proof. Set $X_i=p_{\mu-j_i}|D|^{\mu-j_i}$ for $i=1,\dots,n$ and recall that

$$b_{-\mu-k}(\lambda)=\sum_{n=1}^{k}\sum_{\substack{|j|=k,\\ j_\cdot\geq 1}}(\lambda-|D|^\mu)^{-1}X_1(\lambda-|D|^\mu)^{-1}\cdots(\lambda-|D|^\mu)^{-1}X_n(\lambda-|D|^\mu)^{-1}.$$

Using rules (5) and (7) we move all powers of $|D|$ and $(|D|^\mu-\lambda)^{-1}$ to the right.

1. First pushing all powers of $(|D|^\mu-\lambda)^{-1}$ to the right using (7) we find

$$\begin{aligned} &(\lambda-|D|^\mu)^{-1}X_1(\lambda-|D|^\mu)^{-1}\cdots(\lambda-|D|^\mu)^{-1}X_n(\lambda-|D|^\mu)^{-1}\\ \simeq&\sum_{l\geq 0}c(l)p^{(l_1)}_{\mu-j_1}|D|^{\mu-j_1}p^{(l_2)}_{\mu-j_2}|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}(\lambda-|D|^\mu)^{-(|l|+n+1)}, \end{aligned}$$

where $c(l)$ as given in (17). We have used the algorithm and constants defined in the proof of Proposition 4.14 of [34] noting that $|D|$ and $(\lambda-|D|^\mu)^{-1}$ commute.

2. In each term

$$p^{(l_1)}_{\mu-j_1}|D|^{\mu-j_1}p^{(l_2)}_{\mu-j_2}|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}$$

we now move the powers of $|D|$ to the right using rule (5) incurring further commutators with $|D|^\mu$. The first step is

$$\begin{aligned} &p^{(l_1)}_{\mu-j_1}|D|^{\mu-j_1}p^{(l_2)}_{\mu-j_2}|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}\\ =&p^{(l_1)}_{\mu-j_1}(|D|^\mu)^{1-j_1/\mu}p^{(l_2)}_{\mu-j_2}|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}\\ \simeq&p^{(l_1)}_{\mu-j_1}\left[\sum_{m_2\geq 0}\binom{1-j_1/\mu}{m_2}p^{(l_2+m_2)}_{\mu-j_2}(|D|^\mu)^{1-j_1/\mu-m_2}\right]|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}\\ =&\sum_{m_2\geq 0}\binom{1-j_1/\mu}{m_2}p^{(l_1)}_{\mu-j_1}p^{(l_2+m_2)}_{\mu-j_2}|D|^{2\mu-(j_1+j_2)-\mu m_2}p^{(l_3)}_{\mu-j_3}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n}. \end{aligned}$$

Now note that

$$\begin{aligned}
&|D|^{2\mu-(j_1+j_2)-\mu m_2} p^{(l_3)}_{\mu-j_3} \\
&= (|D|^{\mu})^{2-(j_1+j_2)/\mu-m_2}\, p^{(l_3)}_{\mu-j_3} \\
&\simeq \sum_{m_3=0}^{\infty} \binom{2-\frac{j_1+j_2}{\mu}-m_2}{m_3} p^{(l_3+m_3)}_{\mu-j_3} |D|^{2\mu-(j_1+j_2)-\mu(m_2+m_3)}.
\end{aligned}$$

We then repeat this to finally obtain

$$\begin{aligned}
&p^{(l_1)}_{\mu-j_1}|D|^{\mu-j_1} p^{(l_2)}_{\mu-j_2}|D|^{\mu-j_2}\cdots p^{(l_n)}_{\mu-j_n}|D|^{\mu-j_n} \\
&\simeq \sum_{m\geq 0} c'(j,m) p^{(l_1)}_{\mu-j_1} p^{(l_2+m_2)}_{\mu-j_2}\cdots p^{(l_n+m_n)}_{\mu-j_n} |D|^{n\mu-|j|-\mu|m|}
\end{aligned}$$

for constants $c'(j,m)$ as in (18) where $m=(m_2,\ldots,m_n)\in\mathbb{N}_0^{n-1}$. □

Example 3.12. An important special case of this result concerns the operator $P=\Delta+p$ where $p\in\mathcal{B}$ and $\Delta=|D|^2$. This operator arises in Quillen's interpretation of the JLO cocycle [36] in the framework of superconnections [54]. We find $b_{-2-k}=0$ if k is odd and for k even we have

$$b_{-2-k}(\lambda)=(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}\cdot\ldots\cdot(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}$$

with p appearing $k/2$ times.

Example 3.13. A nontrivial example is given by the operator $P=|D|^{\mu}+p|D|^{\mu-1}$. A similar calculation as above yields

$$\begin{aligned}
b_{-\mu}(\lambda) &= (\lambda-|D|^{\mu})^{-1}, \\
b_{-\mu-1}(\lambda) &= (\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}, \\
b_{-\mu-2}(\lambda) &= (\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}.
\end{aligned}$$

One can expand these terms further. In the lowest non-trivial order we find

$$(\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}\simeq\sum_{l=0}^{\infty}p^{(l)}|D|^{\mu-1}(\lambda-|D|^{\mu})^{-(l+2)}$$

with $p^{(l)}$ the l-fold commutator with $|D|^{\mu}$.

For the construction of the complex powers we will need a finer analysis of the coefficients $p^{(l_1)}_{\mu-j_1}p^{(l_2+m_2)}_{\mu-j_2}\cdots p^{(l_n+m_n)}_{\mu-j_n}$. These are expressed as commutators with $|D|^{\mu}$ whereas our class of pseudodifferential operators requires coefficients given in terms of commutators with $|D|$. Thanks to Lemma 3.5 we can switch between these representations.

Proposition 3.14. *The operators* $p^{(l_1)}_{\mu-j_1}p^{(l_2+m_2)}_{\mu-j_2}\cdots p^{(l_n+m_n)}_{\mu-j_n}$ *from* (16) *belong to* $\Psi^{(|l|+|m|)(\mu-1)}_{cl}$. *Indeed they can be expressed as asymptotic sums of the form*

$$p^{(l_1)}_{\mu-j_1}p^{(l_2+m_2)}_{\mu-j_2}\cdots p^{(l_n+m_n)}_{\mu-j_n}\simeq\sum_{k=0}^{\infty}b_k(j,l,m)|D|^{(|l|+|m|)(\mu-1)-k}$$

for some coefficients $b_k(j,l,m)\in\mathcal{B}$.

In terms of commutators with $|D|$ we can write the resolvent components in lowest orders as

$$\begin{aligned}
b_{-\mu}(\lambda) =& (\lambda-|D|^{\mu})^{-1}, \\
b_{-\mu-1}(\lambda) =& p_{\mu-1}|D|^{\mu-1}(\lambda-|D|^{\mu})^{-2}+\mu\delta(p_{\mu-1})|D|^{2\mu-2}(\lambda-|D|^{\mu})^{-3}+\cdots, \\
b_{-\mu-2}(\lambda) =& p_{\mu-1}^2|D|^{2\mu-2}(\lambda-|D|^{\mu})^{-3}+p_{\mu-2}|D|^{\mu-2}(\lambda-|D|^{\mu})^{-2}+\cdots
\end{aligned}$$

Proof. It suffices to prove this for $n = 1$ since Ψ^{∞}_{cl} is a graded algebra by Lemma 3.5 (vi). The claim for $n = 1$ follows from Lemma 3.5 (v). □

3.3. **The complex powers.** We construct the complex powers of suitable operators in Ψ^{μ}_{cl}, show that these also belong to this class and construct explicit asymptotic expansions. Our construction is parallel to the situation in a global calculus of classical pseudodifferential operators on $\mathbb{R}^n$ as in [44].

Let Λ_ϵ be as in Hypothesis 3.6. For Re $z < 0$ we define the complex powers of P by a Dunford integral

$$P_z = \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z(\lambda - P)^{-1}d\lambda, \tag{19}$$

where $\partial\Lambda_\epsilon$ is a parametrisation of the boundary of Λ_ϵ. The power $\lambda^z = e^{z\log\lambda}$ is given by the main branch of the logarithm. The integral converges for Re $(z) < 0$ to a bounded operator by the decay properties of the resolvent in Theorem 3.8.

In general we define $P^z = P^k P_{z-k}$ for arbitrary $z \in \mathbb{C}$ by choosing $k \in \mathbb{N}$ sufficiently large so that Re $z < k$. A simple argument (as in Theorem 10.1 of [62]) shows that this definition is independent of k and that the complex powers so defined have the group property.

We now show that P^z also belongs to $\Psi^{\mu z}_{cl}$, i.e. has a suitable asymptotic expansion that can be computed explicitly in principle.

Theorem 3.15. *Assume Hypothesis 3.6. Then the operator P^z belongs to $\Psi^{\mu z}_{cl}$. More precisely, for* Re $z < 0$ *we have the following asymptotic expansions.*

(i) On a symbolic level of the parameter-dependent resolvent we have in the notation of Hypothesis 3.6 that

$$\begin{aligned}P^z &\simeq |D|^{\mu z}\\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{\substack{|j|=k\\ j.\geq 1}}\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon}\lambda^z(\lambda-p_\mu|D|^\mu)^{-1}p_{\mu-j_1}|D|^{\mu-j_1}(\lambda-p_\mu|D|^\mu)^{-1}\times\cdots\\ &\qquad\times(\lambda-p_\mu|D|^\mu)^{-1}p_{\mu-j_n}|D|^{\mu-j_n}(\lambda-p_\mu|D|^\mu)^{-1}d\lambda.\end{aligned}$$

Here, $j. \geq 1$ means that all components of j are greater than or equal to 1.

(ii) Expressing this with coefficients in terms of commutators with $|D|^\mu$ one has

$$P^z \simeq |D|^{\mu z} + \sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{\substack{|j|=k\\ j.\geq 1\\ l\geq 0\\ m\geq 0}}\binom{z}{|l|+n}c(l)c'(j,m)p(j,l,m)|D|^{\mu(z-|l|-|m|)-|j|},$$

where

$$p(j,l,m) = p^{(l_1)}_{\mu-j_1}p^{(l_2+m_2)}_{\mu-j_2}\cdots p^{(l_n+m_n)}_{\mu-j_n}$$

with constants c, c' as in Corollary 3.11. Here, $l = (l_1, \ldots, l_n)$ and $m = (m_2, \ldots, m_n)$ are multi-indices.

(iii) And with coefficients in $\mathcal{B}$ we find,

$$\begin{aligned}P^z &\simeq |D|^{\mu z}\\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{\substack{|j|=k\\ j.\geq 1\\ l\geq 0\\ m\geq 0}}\sum_{r=0}^{\infty}\binom{z}{|l|+n}c(l)c'(j,m)b_r(j,m,l)|D|^{\mu z-(|j|+r+|l|+|m|)}\end{aligned}\tag{20}$$

for operators $b_r(j,m,l) \in \mathcal{B}$ as in Proposition 3.14.

The lowest-order terms in the asymptotic expansion of P^z are given as

$$\left.\begin{aligned} b_{\mu z} &= |D|^{\mu z}, \\ b_{\mu z-1} &= z p_{\mu-1} |D|^{\mu z-1}, \\ b_{\mu z-2} &= \left[\tfrac{z(z-1)}{2} p_{\mu-1}^2 + z p_{\mu-2} + \mu\delta(p_{\mu-1})\right] |D|^{\mu z-2}, \end{aligned}\right\} \tag{21}$$

where we assumed $p_\mu = 1$ and Re $z < 0$.

Proof. Without loss of generality $p_\mu = 1$. We only consider the case Re $z < 0$, the other case follows by recalling that $P^z = P^k P_{z-k}$ for any k satisfying Re $z < k$, where P_{z-k} is defined by a Dunford integral. As usual, the idea is to replace the resolvent in (19) by the operators B_N from Proposition 3.10.

1. In order to construct the asymptotic expansion term by term, we write

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z (\lambda - P)^{-1} d\lambda = \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z \left[(\lambda-P)^{-1} - B_N(\lambda)\right] d\lambda + B_N^{(z)}, \tag{22}$$

where

$$B_N^{(z)} = \sum_{k=0}^{N-1} \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z b_{-\mu-k}(\lambda) d\lambda.$$

We first show that the operators $\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z b_{-\mu-k}(\lambda)d\lambda$ belong to $op^{\mu z-k}$. For $k=0$ we have $b_{-\mu}(\lambda) = (\lambda - |D|^\mu)^{-1}$ so that by Cauchy's integral formula $b_{\mu z} = |D|^{\mu z}$. Now, from (16) we must consider

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z p_{\mu-j_1}^{(l_1)} p_{\mu-j_2}^{(l_2+m_2)} \cdots p_{\mu-j_n}^{(l_n+m_n)} |D|^{n\mu-|m|\mu-|j|} (\lambda - |D|^\mu)^{-(|l|+n+1)} d\lambda \\ =& p_{\mu-j_1}^{(l_1)} p_{\mu-j_2}^{(l_2+m_2)} \cdots p_{\mu-j_n}^{(l_n+m_n)} |D|^{n\mu-|m|\mu-|j|} \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^{-z} (\lambda - |D|^\mu)^{-(|l|+n+1)} d\lambda. \end{aligned}$$

Since Theorem 3.8 ensures that the functions

$$\begin{aligned} \Lambda \to op^0 &: \lambda \mapsto |\lambda|^k (\lambda - |D|^\mu)^{-k}, \\ \Lambda \to op^{-\mu k} &: \lambda \mapsto (\lambda - |D|^\mu)^{-k} \end{aligned}$$

are bounded uniformly in λ we can integrate by parts to find

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z (\lambda - |D|^\mu)^{-(|l|+n+1)} d\lambda = \binom{z}{|l|+n} |D|^{\mu(z-(|l|+n))}$$

belonging to $op^{\mu(z-(|l|+n))}$. Moreover,

$$p_{\mu-j_1}^{(l_1)} p_{\mu-j_2}^{(l_2+m_2)} \cdots p_{\mu-j_n}^{(l_n+m_n)} \in op^{(|l|+|m|)(\mu-1)}$$

by Proposition 3.14.

This shows that overall

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z p_{\mu-j_1}^{(l_1)} p_{\mu-j_2}^{(l_2+m_2)} \cdots p_{\mu-j_n}^{(l_n+m_n)} |D|^{n\mu-|m|\mu-|j|} (\lambda - |D|^\mu)^{-(|l|+n+1)} d\lambda \tag{23}$$

belongs to $op^{\mu z-|l|-|m|-|j|}$ so that the claim follows since $|j| = k$ and $|l|, |m| \geq 0$.

2. We then claim that $P^z \in \Psi_{cl}^{\mu z}$, i.e. has the asserted asymptotic expansion. From (12) we deduce that the integral in (22) converges and yields an operator in op^{-N}. Thus, the difference $P^z - \sum_{k=0}^N b_{\mu z-k}$ belongs to op^{-N} and the asymptotic expansion of P^z holds.

3. The expansions in (ii) and (iii) follow immediately from Corollary 3.11 and Proposition 3.14. □

We illustrate this theorem in two examples.

Example 3.16. Consider the operator $P = \Delta + p$. We computed the asymptotic expansion of the resolvent in Example 3.12. The expansion of the complex powers is an immediate consequence.

$$\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}d\lambda = \Delta^{-z},$$

$$\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}d\lambda \simeq \sum_{l=0}^{\infty}\binom{-z}{l+1}p^{(l)}\Delta^{-z-(l+1)}$$

and the subsequent term reads

$$\begin{aligned}&\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}d\lambda\\ \simeq&\sum_{l_1,l_2=0}^{\infty} c(l_1,l_2)p^{(l_1)}p^{(l_2)}\binom{-z}{l_1+l_2+2}\Delta^{-z-(l_1+l_2)-2}.\end{aligned}$$

The expansion of the next term is

$$\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}d\lambda \simeq \binom{-z}{3}p^3\Delta^{-z-3}+\cdots$$

Collecting terms according to powers in Δ, we find the lowest orders in the asymptotic expansion of $(\Delta+p)^{-z}$ to be

$$\begin{aligned}(\Delta+p)^{-z} \simeq&\Delta^{-z}+\binom{-z}{1}p\Delta^{-z-1}+\binom{-z}{2}\left[p^{(1)}+c(0,0)p^2\right]\Delta^{-z-2}\\ &+\binom{-z}{3}\left[p^{(2)}+\left(c(0,1)pp^{(1)}+c(1,0)p^{(1)}p\right)+p^3\right]\Delta^{-z-3}+\cdots\end{aligned}$$

This can be further simplified using $c(0,0)=1$, $c(0,1)=2$ and $c(1,0)=3/2$.

Example 3.17. We consider the operator $P = |D|^{\mu}+p|D|^{\mu-1}$. We find in the lowest order that

$$\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-|D|^{\mu})^{-1}d\lambda = |D|^{-\mu z},$$

using Example 3.13. The next order is computed as follows

$$\begin{aligned}&\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}d\lambda\\ \simeq&\sum_{l=0}^{\infty}p^{(l)}|D|^{\mu-1}\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-|D|^{\mu})^{-(l+2)}d\lambda\\ =&\sum_{l=0}^{\infty}p^{(l)}\binom{-z}{l+1}|D|^{-\mu z-\mu l-1}.\end{aligned}$$

The highest order in the subsequent term is

$$\begin{aligned}&\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}p|D|^{\mu-1}(\lambda-|D|^{\mu})^{-1}d\lambda\\ \simeq&p^2|D|^{2\mu-2}\frac{1}{2\pi i}\int \lambda^{-z}(\lambda-|D|^{\mu})^{-3}d\lambda+\cdots\\ =&\binom{-z}{2}p^2|D|^{-\mu z-2}+\cdots\end{aligned}$$

Collecting terms according to powers of $|D|$ we find

$$(|D|^\mu + p|D|^{\mu-1})^{-z} \simeq |D|^{-\mu z} + \binom{-z}{1} p|D|^{-\mu z-1} + \binom{-z}{2} p^{(1)}|D|^{-\mu z-\mu-1} + \cdots + \binom{-z}{2} p^2|D|^{-\mu z-2} + \cdots$$

Using (6) we express $p^{(1)}$ as

$$p^{(1)} = [|D|^\mu, p] \simeq \mu\delta(p)|D|^{\mu-1} + \tfrac{\mu(\mu+1)}{2}\delta^2(p)|D|^{\mu-2} + \cdots$$

Thus,

$$(|D|^\mu + p|D|^{\mu-1})^{-z} \simeq |D|^{-\mu z} - zp|D|^{-\mu z-1} + \tfrac{z(z+1)}{2}(\mu\delta(p) + p^2)|D|^{-\mu z-2} + \cdots$$

in lowest orders.

The operator family P^z is holomorphic in z. For a domain $U \subset \mathbb{C}$ and a Banach space E denote by $\mathcal{O}(U, E)$ the space of holomorphic functions from U to E.

Proposition 3.18. *For any $N \in \mathbb{N}$ let B_N be as in Theorem 3.10 and set*

$$B_N^{(z)} = \begin{cases} \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z B_N(\lambda)d\lambda & \textit{for } \mathrm{Re}\ z < 0 \\ P^k \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^{z-k} B_N(\lambda)d\lambda & \textit{for } \mathrm{Re}\ z > 0 \end{cases},$$

where k is any natural number such that $k > \mathrm{Re}\ z$. Then $P^z - B_N^{(z)} \in \mathcal{O}(\mathrm{Re}\ z < k, op^{\mu k - N})$.

Proof. 1. We first consider the case $k = 0$. Here we set $R_N^{(z)} = P^z - B_N^{(z)}$ and want to show analyticity of the map $z \mapsto R_N^{(z)}$ on the domain $\{z \in \mathbb{C} | \mathrm{Re}\ z < 0\}$. Now choose $z \in \mathbb{C}$ with $\mathrm{Re}\ z \leq -\delta$ for $\delta > 0$ small. We know that by construction $R_N^{(z)}$ is given from (22) as

$$R_N^{(z)} = \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z \left[(\lambda - P)^{-1} - B_N(\lambda)\right] d\lambda$$

and the operator norm of the integral can be bounded by means of Theorem 3.10 so that $R_N^{(z)}$ yields a bounded operator in op^{-N}. Formal differentiation of the integral k times with respect to z yields

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} \lambda^z (\log\lambda)^k \left[(\lambda - P)^{-1} - B_N(\lambda)\right] d\lambda$$

and this integral converges for $\mathrm{Re}\ z \leq -\delta$ by Theorem 3.10 to an operator in op^{-N}. The claim now follows as δ was arbitrary.

2. For $k > 0$ recall that P^z was defined as $P^z = P^k P_{z-k}$. Thus, we look at $z \mapsto R_N^{(z)} = P^k P_{z-k} - P^k B_N^{(z-k)}$ which we can rewrite as $R_N^{(z)} = P^k(P_{z-k} - B_N^{(z-k)})$. The last expression shows that $R_N^{(z)}$ belongs to $op^{\mu k - N}$ by step 1. The arguments of step 1. also show that this function is analytic in z since composition with P^k does not destroy analyticity. □

3.4. **The heat operator.** Under Hypothesis 3.6 we define the heat operator by a Dunford integral

$$e^{-tP} = \frac{1}{2\pi i}\int_{\Lambda_\epsilon} e^{-t\lambda}(\lambda - P)^{-1} d\lambda, \tag{24}$$

which converges to a bounded operator for $t > 0$. The resulting operator is of order $-\infty$ on the scale of Sobolev spaces. This is analogous to the construction in Theorem 4.1 of [45].

Theorem 3.19. *Under Hypothesis 3.6 the heat operator as just defined belongs to* $\Psi_{cl}^{-\infty}$ *and has the following formal asymptotic expansions.*

(i) On the level of the symbolic resolvent expansion we find

$$\begin{aligned} e^{-tP} &\simeq e^{-t|D|^\mu} \\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{\substack{|j|=k \\ j.\geq 1}} \frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}(\lambda-p_\mu|D|^\mu)^{-1}p_{\mu-j_1}|D|^{\mu-j_1}(\lambda-p_\mu|D|^\mu)^{-1}\times\cdots \\ &\qquad\times(\lambda-p_\mu|D|^\mu)^{-1}p_{\mu-j_n}|D|^{\mu-j_n}(\lambda-p_\mu|D|^\mu)^{-1}d\lambda. \end{aligned}$$

Here, $j.\geq 1$ *means that all components of* j *are greater than or equal to* 1.

(ii) On the level of commutators with $|D|^\mu$ *one has*

$$\begin{aligned} e^{-tP} \simeq & e^{-t|D|^\mu} \\ &+\sum_{k=1}^{\infty}\sum_{n=1}^{k}\sum_{\substack{|j|=k \\ j.\geq 1 \\ l\geq 0 \\ m\geq 0}} \frac{(-t)^{|l|+n}}{(|l|+n)!}c(l)c'(j,m)p(j,l,m)|D|^{(n-|m|)\mu-|j|}e^{-t|D|^\mu}, \end{aligned}$$

where

$$p(j,l,m)=p_{\mu-j_1}^{(l_1)}p_{\mu-j_2}^{(l_2+m_2)}\cdots p_{\mu-j_n}^{(l_n+m_n)}$$

with constants c, c' *as in Corollary 3.11.*

The expansions are in a formal sense as the right hand sides are of order $-\infty$ *on the scale of Sobolev spaces.*

Proof. We want to show that e^{-tP} belongs to Ψ_{cl}^{-N} for any $N\in\mathbb{N}$. In the Dunford integral (24) we replace the resolvent by the operators $B_N(\lambda)$ yielding

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}(\lambda-P)^{-1}d\lambda \\ =&\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}\left[(\lambda-P)^{-1}-B_N(\lambda)\right]d\lambda+\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}B_N(\lambda)d\lambda \end{aligned} \tag{25}$$

with $B_N(\lambda)$ be as in Theorem 3.10. This allows us to construct the formal asymptotic expansion. Without of loss of generality we set $p_\mu=1$.

1. By the definition of the B_N, the second integral in (25) becomes the sum

$$\sum_{k=0}^{N-1}\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}b_{-\mu-k}(\lambda)d\lambda.$$

So the first claim is that the summands belong to $op^{-\infty}$. For $k=0$ we have $b_{-\mu}(\lambda)=(\lambda-|D|^\mu)^{-1}$ so that by Cauchy's integral formula

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}(\lambda-|D|^\mu)^{-1}=e^{-t|D|},$$

which clearly belongs to $op^{-\infty}$.

For $k\geq 1$ we employ the representation of $b_{-\mu-k}$ from (16) and consider

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}p_{\mu-j_1}^{(l_1)}p_{\mu-j_2}^{(l_2+m_2)}\cdots p_{\mu-j_n}^{(l_n+m_n)}|D|^{(n-|m|)\mu-|j|}(\lambda-|D|^\mu)^{-(|l|+n+1)}d\lambda \\ =&p_{\mu-j_1}^{(l_1)}p_{\mu-j_2}^{(l_2+m_2)}\cdots p_{\mu-j_n}^{(l_n+m_n)}|D|^{(n-|m|)\mu-|j|}\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}(\lambda-|D|^\mu)^{-(|l|+n+1)}d\lambda. \end{aligned}$$

Integration by parts yields

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}\left(\lambda-|D|^\mu\right)^{-(|l|+n+1)}d\lambda=\frac{(-t)^{|l|+n}}{(|l|+n)!}e^{-t|D|^\mu}$$

and this belongs to $op^{-\infty}$. This shows that overall

$$\frac{1}{2\pi i}\int_{\partial\Lambda_\epsilon} e^{-t\lambda}p_{\mu-j_1}^{(l_1)}p_{\mu-j_2}^{(l_2+m_2)}\cdots p_{\mu-j_n}^{(l_n+m_n)}|D|^{(n-|m|)\mu-|j|}(\lambda-|D|^\mu)^{-(|l|+n+1)}d\lambda$$

belongs to $op^{-\infty}$ so that the claim follows.

2. We then claim that e^{-tP} has the asserted formal asymptotic expansion. From (12) we deduce that the first integral in (25) converges and yields an operator in op^{-N} for any N. The asymptotic expansions follow immediately from Corollary 3.11 and the proof of Theorem 3.15. □

Example 3.20. As a first example we consider the operator $P=\Delta+p$ with $p\in\mathcal{B}$. The expansion of the heat operator in powers of t can be computed as follows. In the lowest order we find

$$\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-\Delta)^{-1}d\lambda=e^{-t\Delta}.$$

The next term is

$$\begin{aligned}\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-\Delta)^{-1}p(\lambda-\Delta)^{-1}d\lambda &\simeq\sum_{l=0}^{\infty}p^{(l)}\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-\Delta)^{-j-2}d\lambda\\ &=\sum_{l=0}^{\infty}p^{(l)}\frac{(-t)^{l+1}}{(l+1)!}e^{-t\Delta}.\end{aligned}$$

So we have the formal expansion

$$e^{-t(\Delta+p)}\simeq e^{-t\Delta}+\left[-tp+\frac{t^2}{2!}p^{(1)}-\frac{t^3}{3!}p^{(2)}+\cdots\right]e^{-t\Delta}$$

in lowest orders.

Example 3.21. Now we consider the operator $P=|D|^\mu+p|D|^{\mu-1}$. Using the resolvent expansions from Example 3.13 we obtain

$$\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-|D|^\mu)^{-1}d\lambda=e^{-t|D|^\mu}.$$

The next term reads

$$\begin{aligned}&\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-|D|^\mu)^{-1}p|D|^{\mu-1}(\lambda-|D|^\mu)^{-1}d\lambda\\ \simeq&\sum_{l=0}^{\infty}p^{(l)}|D|^{\mu-1}\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-|D|^\mu)^{-(j+2)}d\lambda\\ =&\left(\sum_{l=0}^{\infty}\frac{(-t)^{l+1}}{(l+1)!}p^{(l)}\right)|D|^{\mu-1}e^{-t|D|^\mu}.\end{aligned}$$

The top order in the next term is

$$\begin{aligned}&\frac{1}{2\pi i}\int e^{-t\lambda}(\lambda-|D|^\mu)^{-1}p|D|^{\mu-1}(\lambda-|D|^\mu)^{-1}p|D|^{\mu-1}(\lambda-|D|^\mu)^{-1}d\lambda\\ \simeq& p^2|D|^{2\mu-2}\frac{(-t)^2}{2!}e^{-t|D|^\mu}+\cdots\end{aligned}$$

Overall we find the formal expansion

$$e^{-t(|D|^\mu+p|D|^{\mu-1})} \simeq e^{-t|D|^\mu} + \left(-tp + \frac{(-t)^2}{2!}p^{(1)} + \cdots\right)|D|^{\mu-1}e^{-t|D|^\mu} + p^2|D|^{2\mu-2}\frac{(-t)^2}{2!}e^{-t|D|^\mu} + \cdots$$

in lowest orders.

4. Spectral functions

This section considers spectral functions of the abstract pseudodifferential operators. We consider in particular the zeta function and the heat trace but also the regularised determinant and Weyl-type eigenvalue asymptotics. In line with [34] we consider graded and ungraded Hilbert spaces simultaneously. If we are interested in the "ordinary" spectral functions with the trace we set $\gamma = 1$.

4.1. **The zeta function.** The construction of the complex powers allows us to find the singularity structure of the zeta function $\mathrm{Trace}(\gamma Q P^z)$, which can also be viewed as the P-regularised trace of Q. The case $P = |D|$ was extensively analysed in [51]. We employ the notation of [28] for zeta functions depending on two operators.

Theorem 4.1. *Let (A, H, D) be a regular spectral triple with degree of summability d and grading operator γ. Let $P \in \Psi^\mu_{cl}$ satisfy Hypothesis 3.6 and let $Q \in \Psi^\nu_{cl}$. Then the function*

$$\zeta(z, Q, P) = \mathrm{Trace}(\gamma Q P^{-z})$$

is analytic for $\mathrm{Re}\, z > d/\mu$. It can be extended to a meromorphic function on the whole complex plane with at most simple poles which are located in the set

$$\mathcal{P} = \{z \in \mathbb{C} | \mu z - \nu + j \in Sd, j = 0, 1, 2, \ldots\} = \bigcup_{j=0}^{\infty} \tfrac{1}{\mu}\left(Sd + \nu - j\right),$$

where Sd is the dimension spectrum of (A, H, D). The residue at the poles can be expressed in terms of the noncommutative integral $⨍$.

Remark 4.2. Of course this agrees with the commutative case of classical pseudodifferential operators on a closed manifold. Here, the poles of the function $\zeta(z, Q, P)$ are located at the points $\{(d + \nu - j)/\mu | j \in \mathbb{N}_0\}$ where d is the dimension of the manifold, ν is the order of Q and μ is the order of P.

Proof. We consider the ungraded case $\gamma = 1$, the graded case follows similarly. Without loss of generality we have $p_\mu = 1$. Let $Q = 1$ for the time being.

1. Recall that by Theorem 3.15 (iii), P^{-z} is a classical abstract pseudodifferential operator which for $\mathrm{Re}\, z > 0$ has an asymptotic expansion of the form

$$P^{-z} \simeq |D|^{-\mu z} + p_{-\mu z-1}|D|^{-\mu z-1} + p_{-\mu z-2}|D|^{-\mu z-2} + \cdots$$

for operators $p_{-\mu z-k} \in \mathcal{B}$. Note that $\mathrm{Trace}(p_{-\mu z-k}|D|^{-\mu z-k})$ is holomorphic except for poles at points for which $\mu z + k \in Sd$. So the set of poles is a subset of $\frac{1}{\mu}(Sd - k)$. The residue at these poles can obviously be expressed in terms of the noncommutative integral.

2. The meromorphic extension of $\mathrm{Trace}(P^{-z})$ is then obtained as follows. Fix $N \in \mathbb{N}$ and let

$$P_N(z) = \sum_{k=0}^{N-1} p_{-\mu z-k}|D|^{-\mu z-k}$$

be the terms of order up to $-\mu z - (N-1)$ in the asymptotic expansion of P^{-z}. Then $\mathrm{Trace}(P^{-z}) = \mathrm{Trace}(P^{-z} - P_N(z)) + \mathrm{Trace}(P_N(z))$. The operator $P^{-z} - P_N(z)$ belongs to

$op^{-\mu z - N}$. Since the spectral triple is d-summable, this operator is trace-class if $\mu z + N > d$, cf. Lemma 3.5 (ii). Hence, the function $\text{Trace}(P^{-z} - P_N(z))$ is analytic on the half-plane $\{z \in \mathbb{C} | \text{Re } z > d - N\}$. By step 1, the poles of the second map $\text{Trace}(P_N(z))$ are located in the set $\cup_{j=0}^{\infty} \frac{1}{\mu}(Sd - j)$.

3. For general $Q \in \Psi_{cl}^{\nu}$ we have an asymptotic expansion

$$Q \simeq q_\nu |D|^\nu + q_{\nu-1}|D|^{\nu-1} + q_{\nu-2}|D|^{\nu-2} + \cdots$$

so that the product QP^{-z} has an expansion given by

$$\left(q_\nu |D|^\nu + q_{\nu-1}|D|^{\nu-1} + \cdots\right)\left(|D|^{-\mu z} + p_{-\mu z-1}|D|^{-\mu z-1} + \cdots\right)$$

leading to and expansion of the form

$$QP^{-z} \simeq \sum_{k=0}^{\infty} r_{-\mu z+\nu-k}|D|^{-\mu z+\nu-k}$$

for some $r_{-\mu z+\nu-k} \in \mathcal{B}$. The arguments presented in step 2 prove the assertion. □

Corollary 4.3. *Under the assumptions of Theorem 4.1, the map* $\Gamma(z)\zeta(z, Q, P)$ *has the singularity structure*

$$\Gamma(z)\zeta(z, Q, P) \sim \sum_{\beta \in \mathcal{P} \cup -\mathbb{N}_0} \sum_{l=0}^{1} \frac{a_{\beta,l}}{(z-\beta)^{l+1}},$$

where the relation $\sim$ *has the following meaning: the left hand side minus a finite sum on the right hand side over* $\{\beta \in \mathcal{P} \cup -\mathbb{N}_0 | \text{Re } \beta > r\}$ *is holomorphic on the half-plane* $\{z \in \mathbb{C} | \text{Re } z > r\}$ *for any* $r \in \mathbb{R}$.

The poles located in the set $-\mathbb{N}_0$ are due to the Gamma-function. We have $a_{\beta,1} \neq 0$ only if both Γ and ζ have a pole at β. These double poles can occur precisely at points $\beta \in \mathcal{P} \cap -\mathbb{N}_0$.

Remark 4.4. It is natural to consider the eta function

$$\eta(z) = \zeta(z, D, |D|) = \text{Trace}(D|D|^{-z-1}).$$

However, the operator D itself is not in Ψ_{cl}^{∞}, so that the eta function does not fit into our framework.

Example 4.5. Higson's approach [34] to the Connes-Moscovici local index theorem involves another kind of zeta function, namely multilinear functionals on Ψ_{cl}^{μ} parametrised by $z \in \mathbb{C}$

$$\begin{aligned}&\langle X_0, X_1, \ldots, X_n \rangle_z \\ =&(-1)^n \frac{\Gamma(z)}{2\pi i} \text{Trace}\left(\gamma \int_{\partial \Lambda_\epsilon} \lambda^{-z} X_0 (\lambda - |D|^\mu)^{-1} X_1 \cdots X_n (\lambda - |D|^\mu)^{-1} d\lambda\right).\end{aligned}$$

One can then expand the function $\Gamma(z)\text{Trace}(\gamma X_0 P^{-z})$ in terms of the simpler functions $\langle \cdot \rangle_z$. Indeed, if P satisfies Hypothesis 3.6. Then

$$\Gamma(z)\text{Trace}\left(\gamma X_0 P^{-z}\right) \sim \sum_{k=0}^{\infty} \sum_{n=1}^{k} \sum_{\substack{|j|=k \\ j. \geq 1}} (-1)^n \langle X_0, X_1, \ldots, X_n \rangle_z,$$

where $X_i = p_{\mu - j_i}|D|^{\mu - j_i}$.

4.2. **The heat trace.** An alternative formulation of the spectral information encoded by the zeta function is the short-time asymptotic expansion of the heat trace. For a comprehensive review of the applications of heat kernel asymptotics in mathematics and physics, we refer to [68]. The case of generalised Laplacians on certain noncommutative spaces is treated in [69].

Theorem 4.6. *For $P \in \Psi^{\mu}_{cl}$ satisfying Hypothesis 3.6 and $Q \in \Psi^{\nu}_{cl}$, the asymptotics of the heat trace* $\mathrm{Trace}\left(\gamma Q e^{-tP}\right)$ *are*

$$\mathrm{Trace}\left(\gamma Q e^{-tP}\right) \sim \sum_{\beta \in \mathcal{P} \cup -\mathbb{N}_0} \sum_{l=0}^{1} a_{\beta,l} t^{-\beta} \log^l(t)$$

as $t \to 0^+$. Here, the coefficients $a_{\beta,l}$ are taken from Corollary 4.3. The summation is over the discrete set $\mathcal{P} \cup -\mathbb{N}_0$ where $\mathcal{P}$ is as in Corollary 4.3.

Proof. The existence of e^{-tP} as a trace-class operator is clear. The asymptotics are fairly standard and we refer to the detailed account in Section 4.1 of [45] or to Chapter 3.3.3 of [58]. The arguments relating the resolvent, zeta and heat traces go through as Theorem 3.8 ensures that the functions

$$\Lambda \to op^0 : \lambda \mapsto |\lambda|^k (p_\mu |D|^\mu - \lambda)^{-k},$$
$$\Lambda \to op^{-\mu k} : \lambda \mapsto (p_\mu |D|^\mu - \lambda)^{-k}$$

are bounded uniformly in λ. □

Example 4.7 (Lowest-order terms)**.** Let (A, H, D) be an ungraded spectral triple whose dimension spectrum is of the form $Sd = \{d, d-1, d-2, \ldots\}$ with $d > 2$. Choose a $P \in op^\mu$ with $P \simeq |D|^\mu + p_{\mu-1}|D|^{\mu-1} + \cdots$ satisfying Hypothesis 3.6 so that the poles of $\zeta(z, 1, P)$ are located in the set of points $\left\{\frac{d}{\mu}, \frac{d-1}{\mu}, \ldots\right\}$. For simplicity we assume that μ is irrational, meaning that $\Gamma(z)\mathrm{Trace}(P^{-z})$ has no double poles and the heat trace expansion contains no logarithmic terms.

From (21), the zeta function $\zeta(z, 1, P) = \mathrm{Trace}(P^{-z})$ can be written as

$$\mathrm{Trace}\left(|D|^{-\mu z}\right) + \mathrm{Trace}\left(\phi_1(z)|D|^{-\mu z-1}\right) + \mathrm{Trace}\left(\phi_2(z)|D|^{-\mu z-2}\right) + \text{mero}$$

with coefficient functions

$$\phi_1(z) = z p_{\mu-1},$$
$$\phi_2(z) = \tfrac{z(z+1)}{2} p_{\mu-1}^2 - z p_{\mu-2} + \mu\delta(p_{\mu-1})$$

from (21) where mero stands for a meromorphic function with potential poles in the half plane $\{z \in \mathbb{C} | \mathrm{Re} \ \ < (d-2)/\mu\}$.

We consider the three right-most poles separately:

$z_0 = \frac{d}{\mu}$: here only the function $\mathrm{Trace}\left(|D|^{-\mu z}\right)$ can have a pole with residue $⨍|D|^{-d}$ as $-\mu z - 1 = -(d+1)$ and $-\mu z - 2 = -(d+2)$ do not belong to Sd. By assumption, none of these points is a nonpositive integer so that $\Gamma(z)$ is regular at each point. Thus,

$$a_{d/\mu,0} = \operatorname*{res}_{z=z_0} \Gamma(z)\zeta(z, 1, P) = \Gamma\left(\tfrac{d}{\mu}\right) ⨍ |D|^{-d}. \tag{26}$$

$z_1 = \frac{d-1}{\mu}$: here both $\mathrm{Trace}\left(|D|^{-\mu z}\right)$ and $\mathrm{Trace}\left(p_{\mu-1}|D|^{-\mu z-1}\right)$ can have poles with residues $⨍|D|^{-(d-1)}$ and $⨍ p_{\mu-1}|D|^{-d}$, respectively. Hence

$$a_{(d-1)/\mu,0} = \operatorname*{res}_{z=z_1} \Gamma(z)\zeta(z, 1, P)$$
$$= \Gamma\left(\tfrac{d-1}{\mu}\right)\left(⨍ |D|^{-(d-1)} + \tfrac{d-1}{\mu} ⨍ p_{\mu-1}|D|^{-d}\right). \tag{27}$$

$z_2 = \frac{d-2}{\mu}$: in this case all traces may have poles and we find

$$\begin{aligned} a_{(d-2)/\mu,0} &= \operatorname*{res}_{z=z_2} \Gamma(z)\zeta(z,1,P) \\ &= \Gamma(z_2)\left(⨍ |D|^{-(d-2)} + ⨍ \phi_1(z_2)|D|^{-(d-1)} + ⨍ \phi_2(z_2)|D|^{-d}\right). \end{aligned} \tag{28}$$

The singularity structure of $\Gamma(z)\zeta(z,1,P)$ can thus be expressed as

$$\Gamma(z)\zeta(z,1,P) = \sum_{j=0}^{2} \frac{a_{z_j,0}}{z - z_j} + \text{mero},$$

where mero stands for a meromorphic function analytic at z_0, z_1 and z_2. We obtain the heat kernel expansion

$$\operatorname{Trace}\left(e^{-tP}\right) \sim a_{d/\mu,0}t^{-d/\mu} + a_{(d-1)/\mu,0}t^{-(d-1)/\mu} + a_{(d-2)/\mu,0}t^{-(d-2)/\mu} + \cdots$$

In analogy with the commutative situation (cf. [46] and also [19], Chapter 10.1) we can view $⨍ b|D|^{-d}$ as the volume form and $⨍ b|D|^{-(d-2)}$ as the scalar curvature form (or Einstein-Hilbert action).

Example 4.8 (The JLO cocycle)**.** We touch upon the relationship with the JLO cocycle by considering the operator $P = \Delta + [D,a]$. Let $a_0, a_1 \in A$. Following Appendix 1 of [34] we introduce functionals parametrised by $t \geq 0$ given by

$$\langle a_0, [D,a_1], \ldots, [D,a_1]\rangle_t^{\text{JLO}} = t^{\frac{n}{2}} \operatorname{Trace}\left(\gamma a_0 \int_{\Sigma_n} [D,a_1]e^{-u_0 t\Delta} \cdots [D,a_1]e^{-u_n t\Delta} du\right)$$

with n arguments $[D,a]$. The integration is over the standard n-simplex

$$\Sigma_n = \left\{(u_0, \ldots, u_n) \in \mathbb{R}^{n+1} | u_i \geq 0, u_0 + \cdots + u_n = 1\right\}.$$

The usual JLO cocycle as defined in [36] is obtained for $t = 1$. Lemma A.2 in [34] shows that we can express the functionals as

$$\begin{aligned} &\langle a_0, [D,a_1], \ldots, [D,a_1]\rangle_t^{\text{JLO}} \\ &= t^{-\frac{n}{2}} \frac{(-1)^n}{2\pi i} \operatorname{Trace}\left(\gamma a_0 \int_{\Lambda_\epsilon} e^{-t\lambda}(\lambda - \Delta)^{-1}[D,a_1] \cdots [D,a_1](\lambda - \Delta)^{-1} d\lambda\right). \end{aligned}$$

We recognise the integral as coming from the asymptotic expansion of the heat trace $\operatorname{Trace}(\gamma a_0 e^{-t[D,a_1]})$ which was also observed in Section 8 of [54].

4.3. **The regularised determinant.** Following [55] one can investigate the regularised determinant of an operator P as defined by

$$\det P = \exp\left(-\frac{d}{dz}\zeta(z,1,P)|_{z=0}\right), \tag{29}$$

where $\zeta(z,1,P) = \operatorname{Trace}(P^{-z})$ provided that this zeta function is regular at the origin. This notion of regularised determinant has been investigated in various calculi of (pseudodifferential) operators and we refer to [45, 58] for a discussion of the literature, alternative definitions of operator determinants and applications.

In our setting we can make sense of the above definition if the dimension spectrum of (A,H,D) does not contain 0.

Theorem 4.9. *Let (A,H,D) be a spectral triple with simple dimension spectrum such that $0 \notin Sd$. For $P \in \Psi^\mu_{cl}$ satisfying Hypothesis 3.6, the zeta function* $\operatorname{Trace}(P^{-z})$ *is regular at the origin and hence the definition* (29) *makes sense.*

Remark 4.10. Unfortunately, there is no closed-form formula for the determinant even in the commutative case. The condition $0 \notin Sd$ must be checked in concrete cases and it is for example satisfied in the case of quantum spheres $SU_q(2)$ where $Sd = \{1, 2, 3\}$. Even if the condition is not satisfied (recall that this entails the regularity of $\mathrm{Trace}(b|D|^{-z})$ at the origin for all $b \in \mathcal{B}$), we may still have regularity of $\zeta(z, 1, |D|)$ at the origin so that $\det |D|$ make sense.

Proof. This follows from a closer inspection of the proof of Theorem 4.1. Let P have the asymptotic expansion $P \simeq \sum_{k=0}^{\infty} p_{\mu-k}|D|^{\mu-k}$ where without loss of generality $p_\mu = 1$. By (20) we can write

$$P^{-z} \simeq |D|^{-z} + p_{-\mu z-1}|D|^{-\mu z-1} + p_{-\mu z-2}|D|^{-\mu z-2} + \cdots ,$$

where the point is that the $p_{-\mu z-k}$ are linear combinations of operators in $\mathcal{B}$ with coefficients of the form $\binom{z}{r}$ for $r \in \mathbb{N}$. Now each $\binom{z}{r}$ is a polynomial of degree r in z with leading term z. The consequence is that any simple pole that $\mathrm{Trace}(b|D|^{-\mu z-k})$ may have at $z = 0$ is cancelled out by this coefficient.

So if $\mathrm{Trace}(P^{-z})$ has a pole at 0, then this must be caused by the term $|D|^{-\mu z}$ in the asymptotic expansion of P^{-z}. But this pole only exists if $0 \in Sd$. □

4.4. **Weyl asymptotics.** The pole structure of the zeta function or equivalently the short-time expansion of the heat trace yield the Weyl asymptotics of the eigenvalue growth of an abstract pseudodifferential operator.

Theorem 4.11. *Let (A, H, D) be a d-summable spectral triple and let $P \in \Psi^\mu_{cl}$ satisfy Hypothesis 3.6. Denote by $N(\lambda)$ the number of eigenvalues of P less than or equal to λ. Then*

$$N(\lambda) \sim \frac{a_{d/\mu,0}}{\Gamma(1 + d/\mu)} \lambda^{-d/\mu},$$

where $a_{d/\mu,0}$ is a coefficient from Corollary 4.3 or Theorem 4.6.

Proof. The usual proof of this goes via a Tauberian theorem. We can relate the eigenvalue growth to the poles of the zeta function $\zeta(z, 1, P)$, cf. the argument following Theorem 6.1.1 of [1]. We can write

$$\Gamma(z)\zeta(z, 1, P) = \frac{a_{d/\mu,0}}{z - d/\mu} + \mathrm{mero},$$

where mero denotes a meromorphic function analytic at d/μ. The claim then follows from Ikehara's Tauberian Theorem.

Alternatively, one could employ Karamata's Tauberian Theorem on the heat trace asymptotics

$$\mathrm{Trace}(e^{-tP}) \sim a_{d/\mu,0}t^{-d/\mu} + \text{higher orders}.$$

Formally, the result also follows by applying the spectral action principle [9, 10] with the indicator function on the interval $[0, 1]$. □

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References

1. M. S. Agranovich, *Elliptic operators on closed manifolds.* In Yu. V. Egorov and M. A. Shubin, editors. Partial Differential Equations VI, vol. 63 of Encyclopaedia Math. Sci., pp. 1–130. Springer-Verlag, Berlin, 1994.
2. B. Ammann, R. Lauter, V. Nistor, and A. Vasy, *Complex powers and non-compact manifolds*, Comm. Partial Differential Equations **29** (2004), no. 5–6, 671–705.
3. M. F. Atiyah, *Global theory of elliptic operators.* In Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), pp. 21–30. Univ. of Tokyo Press, Tokyo, 1970.

4. M. F. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.
5. B. Blackadar, *K-Theory for Operator Algebras*, vol. 5 of Math. Sci. Res. Inst. Publ. Cambridge University Press, Cambridge, 1986.
6. L. G. Brown, R. G. Douglas, and P. A. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. **105** (1977), 265–324.
7. E. Buzano and F. Nicola, *Complex powers of hypoelliptic pseudodifferential operators*, J. Funct. Anal. **245** (2007), no. 2, 353–378.
8. P. S. Chakraborty and S. Sundar, *Quantum double suspension and spectral triples*, J. Funct. Anal. **260** (2011), no. 9, 2716–2741.
9. A. H. Chamseddine and A. Connes, *Universal formula for noncommutative geometry actions: Unification of gravity and the standard model*, Phys. Rev. Lett. **77** (1996), no. 24, 4868–4871.
10. A. H. Chamseddine and A. Connes, *The spectral action principle*, Commun. Math. Phys. **186** (1997), no. 3, 731–750.
11. A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup. (4) **6** (1973), 133–252.
12. A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. of Math. **104** (1976), no. 1, 73–115.
13. A. Connes, *Non-commutative differential geometry*, Publ. Math. Inst. Hautes Études Sci. **62** (1985), no. 2, 41–144.
14. A. Connes, *The action functional in non-commutative geometry*, Commun. Math. Phys. **117** (1988), 673–683.
15. A. Connes, *Noncommutative Geometry.* Academic Press, San Diego, 1994.
16. A. Connes, *Noncommutative differential geometry and the structure of space time*, In Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., pp. 17–42. Amer. Math. Soc., Providence, RI, 1997.
17. A. Connes, *A short survey of noncommutative geometry*, J. Math. Phys. **41** (2000), no. 6, 3832–3866.
18. A. Connes, *Cyclic cohomology, quantum group symmetries and the local index formula for* $\mathrm{SU}_q(2)$, J. Inst. Math. Jussieu **3** (2004), no. 1, 17–68.
19. A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, vol. 55 of Colloquium Publications. Amer. Math. Soc., Providence, RI, 2008.
20. A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, GAFA **5** (1995), no. 2, 174–243.
21. S. Coriasco, E. Schrohe, and J. Seiler, *Bounded imaginary powers of differential operators on manifolds with conical singularities*, Math. Z. **244** (2003), no. 2, 235–269.
22. J. Cuntz, *K-theory and C^*-algebras*, In Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), vol. 1046 of Lecture Notes in Math., pp. 55–79. Springer, Berlin, 1984.
23. J. Cuntz, *A new look at KK-theory*, K-Theory **1** (1987), no. 1, 31–51.
24. J. Cuntz, R. Meyer, and J. M. Rosenberg, *Topological and bivariant K-theory*, vol. 36 of Oberwolfach Seminars. Birkhäuser, Basel, 2007.
25. M. Eckstein, B. Iochum, and A. Sitarz, *Heat trace and spectral action on the standard Podleś sphere*, Comm. Math. Phys. **332** (2014), no. 2, 627–668.
26. D. Essouabri, B. Iochum, C. Levy, and A. Sitarz, *Spectral action on noncommutative torus*, J. Noncommut. Geom. **2** (2008), 53–123.
27. M. Fahrenwaldt, *The Connes-Moscovici local index theorem for the non-commutative 2-torus and the meromorphic extendibility of certain Dirichlet series.* Issue 304, preprint series SFB 478, Universität Münster, 2002.
28. P B. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem.* 2nd edition, CRC Press, Boca Raton, 1995.
29. J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry.* Birkhäuser Advanced Texts. Birkhäuser, Boston, MA, 2001.
30. G. Grubb, *Functional Calculus of Pseudodifferential Boundary Problems*, vol. 65 of Progr. Math. 2nd edition, Birkhäuser, Boston, 1996.
31. V. Guillemin, *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. in Math. **55** (1985), no. 2, 131–160.
32. U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1, Acta Math. **158** (1987), no. 1–2, 95–148.
33. N. Higson, *A characterization of KK-theory*, Pacific J. Math. **126** (1987), no. 2, 253–276.

,

34. N. Higson, *The residue index theorem of Connes and Moscovici*, In Surveys in Noncommutative Geometry, vol. 6 of Clay Math. Proc., pp. 71–126. Amer. Math. Soc., Providence, RI, 2006.
35. B. Iochum, C. Levy, and A. Sitarz, *Spectral action on $SU_q(2)$*, Commun. Math. Phys. **289** (2009), no. 1, 107–155.
36. A. Jaffe, A. Lesniewski, and K. Osterwalder, *Quantum K-Theory* I. *The Chern character*, Commun. Math. Phys. **118** (1988), no. 1, 1–14.
37. V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25.
38. V. F. R. Jones, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111.
39. V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. **126** (1987), no. 2, 335–388.
40. G. G. Kasparov, *Topological invariants of elliptic operators.* I. *K-homology*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 4, 796–838.
41. G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636.
42. H. Kumano-go and C. Tsutsumi, *Complex powers of hypoelliptic pseudo-differential operators with applications*, Osaka J. Math. **10** (1973), 147–174.
43. J.-L. Loday, *Cyclic Homology*, vol. 301 of Grundlehren Math. Wiss. 2nd edition, Springer-Verlag, Berlin, 1998.
44. L. Maniccia, E. Schrohe, and J. Seiler, *Complex powers of classical SG-pseudodifferential operators*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **52** (2006), no. 2, 353–369.
45. L. Maniccia, E. Schrohe, and J. Seiler, *Determinants of classical SG-pseudodifferential operators*, Math. Nachr. **287** (2014), no. 7, 782–802.
46. H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geom. **1** (1967), 43–69.
47. F. J. Murray and J. Von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), no. 1, 116–229.
48. F. J. Murray and J. von Neumann, *On rings of operators.* II, Trans. Amer. Math. Soc. **41** (1937), no. 2, 208–248.
49. F. J. Murray and J. von Neumann, *On rings of operators.* IV, Ann. of Math. **44** (1943), 716–808.
50. F. Nicola and L. Rodino, *Global Pseudo-Differential Calculus on Euclidean Spaces*, vol. 4 of Pseudo Diff. Oper. Birkhäuser, Basel, 2010.
51. S. Paycha, *A canonical trace associated with certain spectral triples*, SIGMA Symmetry Integrability Geom. Methods Appl. **6** (2010), Paper 077, 17.
52. M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras*, J. Operator Theory **4** (1980), no. 1, 93–118.
53. R. T. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Ann. of Math. **86** (1967), 138–171.
54. D. Quillen, *Algebra cochains and cyclic cohomology*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), no. 1, 139–174.
55. D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. **7** (1971), no. 2, 145–210.
56. M. Rørdam, F. Larsen, and N. Laustsen, *An Introduction to K-theory for C^*-algebras*, vol. 49 of London Math. Soc. Stud. Texts. Cambridge University Press, Cambridge, 2000.
57. E. Schrohe, *Complex powers of elliptic pseudodifferential operators*, Integr. Equ. Oper. Theory **9** (1986), no. 3, 337–354.
58. S. Scott, *Traces and Determinants of Pseudodifferential Operators*, Oxford Math. Monogr. Oxford University Press, Oxford, 2010.
59. R. T. Seeley, *Complex powers of an elliptic operator.* In Singular Integrals, (Proc. Sympos. Pure Math., Chicago, Ill., 1966), pp. 288–307. Amer. Math. Soc., Providence, RI, 1967.
60. R. T. Seeley, *Analytic extension of the trace associated with elliptic boundary problems*, Amer. J. Math. **91** (1969), 963–983.
61. R. T. Seeley, *The resolvent of an elliptic boundary problem*, Amer. J. Math. **91** (1969), 889–920.
62. M. A. Shubin, *Pseudodifferential Operators and Spectral Theory.* 2nd edition, Springer-Verlag, Berlin, 2010.
63. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, vol. 128 of Lecture Notes in Math. Springer-Verlag, Berlin, 1970.
64. B. L. Tsygan, *The homology of matrix Lie algebras over rings and the Hochschild homology*, Russian Math. Surveys **38** (1983), no. 2, 198–199.
65. J. v. Neumann, *Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren*, Math. Ann. **102** (1930), no. 1, 370–427.

66. J. v. Neumann, *On rings of operators.* III, Ann. of Math. **41** (1940), 94–161.
67. W. D. Van Suijlekom, *Noncommutative Geometry and Particle Physics*, Mathematical Physics Studies. Springer, Dordrecht, 2015.
68. D. V. Vassilevich, *Heat kernel expansion: user's manual*, Phys. Rep. **388** (2003), no. 5–6, 279–360.
69. D. V. Vassilevich, *Heat trace asymptotics on noncommutative spaces*, SIGMA Symmetry Integrability Geom. Methods Appl. **3** (2007), Paper 093, 11.
70. D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free Random Variables.* vol. 1 of CRM Monograph Series. Amer. Math. Soc., Providence, RI, 1992.
71. E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), no. 3, 351–399.
72. M. Wodzicki, *Spectral asymmetry and zeta functions*, Invent. Math. **66** (1982), no. 1, 115–135.

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