Reliability of systems with randomly varying parameters by the path integration method

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ARTICLE INFO

ABSTRACT

The paper considers a first passage time reliability problem for systems subjected to multiplicative and additive white noises. For numerical calculations of the reliability function and the first passage time the path integration method is properly adapted and used. Some results of numerical calculations are compared to approximate analytical results, obtained by the stochastic averaging method.

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1. Introduction

The reliability may often be considered as a major concern when a dynamic system is being designed. Among different reliability criteria the *first passage problem* [1,2] is widely used and studied by a number of authors. It may be defined as the probability that a system's response stays within a prescribed domain, an outcrossing of which leads to immediate failure. It has been shown that the first passage problem is directly related to a solution of the corresponding Pontryagin equation, written with respect to the first excursion time T. An exact analytical solution to this problem, even for a linear system, is yet to be found. During the last decades a few strategies have been proposed to deal with this type of problems. The averaging procedure with further problem reformulation for the system's response amplitude or energy has been used for a linear system [3], systems with nonlinear stiffness [4] or nonlinear damping [5].

Solving the corresponding Potryagin equation numerically has been proposed in [6–8], whereas a numerical solution to the backward Kolmogorov–Feller equation, for a system subjected to a Poisson driven train of impulses, can been found in [9]. Different novel analytical as well as numerical strategies have been proposed in recent years by a number of authors [10–12].

It has been shown recently in [13] that it is possible to adapt the path integration (PI) method [14,15] for problems of reliability, including the first passage problem. This paper focuses on a numerical investigation of the reliability of systems, subjected to multiplicative and additive white noise. The PI method is used to construct the reliability function and calculate the first passage time for two systems. The first one is a *half-degree* or the first order system subjected to multiplicative and additive uncorrelated white noises. This system possesses a very special quality — its higher order moments are always unstable no matter how you select the corresponding parameter for stability of lower order moments. The second system considered is a single-degree-of-freedom (SDOF) system, considered earlier in [16,1]. Using the averaging procedure it was possible to estimate the first passage time of the system's amplitude. In this paper the first passage time of the systems' displacement and velocity is investigated.

2. Path integration approach to reliability

The motion of a stochastic dynamic system may be expressed as an Itô stochastic differential equation (SDE):

$$dZ(t) = h(Z(t))dt + b dB(t), \qquad (1)$$

where the state space vector process $Z(t) = (X(t), Y(t))^T = (X(t), \dot{X}(t))^T$ has been introduced; $h = (h_1, h_2)^T$ with $h_1(Z) = Y$ and $h_2(Z) = -g(X, Y)$; $b = (0, \sqrt{D})^T$, and $B(t)$ denotes a standard Brownian motion process. From Eq. (1) it follows immediately that $Z(t)$ is a Markov process, and it is precisely the Markov property that will be used in the formulation of the PI procedure.

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The reliability is defined in terms of the displacement response process $X(t)$ in the following manner, assuming that all events are well defined,

$$R(T \mid x_0, 0, t_0)$$
$$= \text{Prob}\{x_l < X(t) < x_c; t_0 < t \leq T | X(t_0) = x_0, Y(t_0) = 0\}, \quad (2)$$

where x_l, x_c are the lower and upper threshold levels defining the safe domain of operation. Hence the reliability $R(T|x_0, 0, t_0)$, as we have defined it here, is the probability that the system's response $X(t)$ stays above the threshold x_l and below the threshold x_c throughout the time interval (t_0, T) given that it starts at time t_0 from x_0 with zero velocity ($x_l < x_0 < x_c$). In general, it is impossible to calculate the reliability exactly as it has been specified here since it is defined by its state in continuous time, and for most systems the reliability has to be calculated numerically, which inevitably will introduce a discretization of the time. Assuming that the realizations of the response process $X(t)$ are piecewise differentiable with bounded slope with probability one, the following approximation is introduced

$$R(T|x_0, 0, t_0)$$
$$\approx \text{Prob}\{x_l < X(t_j) < x_c, j = 1, \ldots, n | X(t_0) = x_0, Y(t_0) = 0\}, \quad (3)$$

where $t_j = t_0 + j\Delta t, j = 1, \ldots, n$, and $\Delta t = (T - t_0)/n$. With the assumptions made, the right-hand side of this equation can be made to approximate the reliability as closely as desired by appropriately choosing Δt, or, equivalently, n. Within the adopted approximation, it is realized that the reliability can now be expressed in terms of the joint conditional PDF $f_{X(t_1)\ldots X(t_n)|X(t_0),Y(t_0)}(\cdot, \ldots, \cdot|x_0, 0)$ as follows, which is just a rephrasing of Eq. (3),

$$R(T|x_0, 0, t_0) \approx \int_{x_l}^{x_c} \cdots \int_{x_l}^{x_c} f_{(\ldots)}(x_1, \ldots, x_n|x_0, 0)dx_1 \cdots dx_n. \quad (4)$$

Due to the Markov property of the state space vector process $Z(t) = (X(t), Y(t))^T$, we may express the joint PDF of $Z(t_1), \ldots, Z(t_n)$ in terms of the transition probability density function

$$p(z, t|z', t') = f_{Z(t)|Z(t')}(z|z')$$
$$= f_{Z(t)Z(t')}(z, z')/f_{Z(t')}(z'), \quad (f_{Z(t')}(z') \neq 0) \quad (5)$$

in the following way

$$f_{Z(t_1)\ldots Z(t_n)|Z(t_0)}(z_1, \ldots, z_n|z_0) = \prod_{j=1}^{n} p(z_j, t_j|z_{j-1}, t_{j-1}). \quad (6)$$

This leads to the expression ($z_0 = (x_0, 0)^T$, $dz_j = dx_j dy_j$, $j = 1, \ldots, n$)

$$R(T|x_0, 0, t_0)$$
$$\approx \int_{-\infty}^{\infty} \int_{x_l}^{x_c} \cdots \int_{-\infty}^{\infty} \int_{x_l}^{x_c} \prod_{j=1}^{n} p(z_j, t_j|z_{j-1}, t_{j-1})dz_1 \cdots dz_n, \quad (7)$$

which is the path integration formulation of the reliability problem. The numerical calculation of the reliability is done iteratively in an entirely analogous way as in standard path integration. To show that, let us introduce a reliability density function (RDF) $q(z, t|z_0, t_0)$ as follows,

$$q(z_2, t_2|z_0, t_0) = \int_{-\infty}^{\infty} \int_{x_l}^{x_c} p(z_2, t_2|z_1, t_1) p(z_1, t_1|z_0, t_0)dz_1, \quad (8)$$

and ($n > 2$)

$$q(z_k, t_k|z_0, t_0) = \int_{-\infty}^{\infty} \int_{x_l}^{x_c} p(z_k, t_k|z_{k-1}, t_{k-1})$$
$$\times q(z_{k-1}, t_{k-1}|z_0, t_0)dz_{k-1}, \quad k = 3, \ldots, n. \quad (9)$$

The reliability is then finally calculated approximately as ($T = t_n$)

$$R(T|x_0, 0, t_0) \approx \int_{-\infty}^{\infty} \int_{x_l}^{x_c} q(z_n, t_n|z_0, t_0)dz_n. \quad (10)$$

The complementary probability distribution of the time to failure T_e, i.e. the first passage time, is given by the reliability function. The mean time to failure $\langle T_e \rangle$ can thus be calculated by the equation

$$\langle T_e \rangle = \int_0^{\infty} R(\tau|x_0, 0, t_0)d\tau. \quad (11)$$

To evaluate the reliability function it is required to know the transition probability density function $p(z, t|z', t')$, which is unknown for the considered nonlinear systems. However, from Eq. (1) it is seen that for small $t - t'$ it can be determined approximately, which is what is needed for the numerical calculation of the reliability. A detailed discussion of this, and the iterative integrations of Eqs. (8) and (9), is given in [13,17]. Concerning the integrations, there is, however, one small difference between the present formulation and that described in these references. In Eqs. (8) and (9), the integration in the x-variable only extends over the interval (x_l, x_c). The infinite upper and lower limits on the y-variable are replaced by suitable constants determined by e.g. an initial Monte Carlo simulation.

If the system response $Z(t)$ has a stationary response PDF $f_Z(z)$ as $t \to \infty$, it follows that the conditional response PDF $f_{\{Z(t_n)|Z(0), x_l < X(t_j) < x_c; 0 \leq j \leq n-1\}}(z)$ also reaches a stationary density, say $q^*(z)$, when $t_n \to \infty$. This means that the reliability process eventually becomes memoryless, and hence the RDF converges $q(z, t_n|z_0, t_0) \to q^*(z)Ke^{-\nu t_n}$ for some constants K and ν as $t_n \to \infty$. Also the numerical method should reach stationarity in the conditional density. This also implies that the numerically estimated reliability function must be exponential, since the same relative amount of probability mass leaves the system at every iteration. So in the end, the only thing one should need for a good reliability estimate is the behavior in the transient phase, and the exponential decay thereafter.

3. Numerical examples

3.1. First order system under multiplicative and additive noises

Consider the first order stochastic system, subjected to multiplicative and additive uncorrelated, zero mean Gaussian white noises:

$$\dot{X} = -aX + X\chi(t) + \xi(t), \quad 0 \leq t \leq t_f, \quad (12)$$

where $E[\xi(t)\xi(t+\tau)] = D_\xi\delta(\tau)$ and $E[\chi(t)\chi(t+\tau)] = D_\chi\delta(\tau)$. Interpreting Eq. (12) as a limiting case of broadband noise processes, then it must be interpreted as an SDE in the Stratonovich sense. Since PI assumes an interpretation in the Ito sense, the equation has to be rewritten as an SDE in the following form ($0 \leq t \leq t_f$),

$$dX = -\left(a - \frac{D_\chi}{2}\right)Xdt + X\sqrt{D_\chi}dB_1(t) + \sqrt{D_\xi}dB_2(t), \quad (13)$$

where $B_1(t)$ and $B_2(t)$ are two independent standard Brownian motion processes.

The stability condition for the n-th order moment can be written as follows, assuming $\langle X^n \rangle = D_n$:

$$\dot{D}_n = -anD_n + \frac{n^2 D_n}{2}D_\chi + \frac{n(n-1)D_{n-2}}{2}D_\xi,$$
$$\Longrightarrow D_n = \frac{(n-1)D_{n-2}D_\xi}{2a - nD_\chi}, \quad D_0 = 1. \quad (14)$$
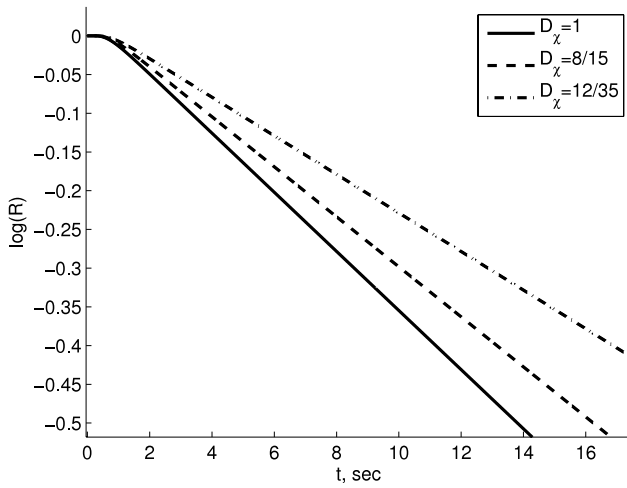
Fig. 1. One-sided reliability function of system (12) for different values of D_χ and $a = 1$.

Note, that the system (12) has the following stability property, namely, it is possible to select such a value of multiplicative noise intensity, so that the system will be stable with respect to the second order moment ($\sigma^2 = D_2$), and unstable for higher order moments ($D_n, n > 2$). Following this logic, one can derive the following stability condition for the n-th order moment:

$$D_\chi < \frac{2a}{n}. \tag{15}$$

Fig. 1 presents the numerically calculated reliability function for critical level $x_c = 3.5\sigma$ for three different cases: (1) $D_\chi = 1.0$ and $a = 1.5$ (stable in D_2, unstable in D_4); (2) $D_\chi = 8/15$ and $a = 1 + 4/15$ (stable in D_4, unstable in D_6); (3) $D_\chi = 12/35$ and $a = 1 + 6/35$ (stable in D_6, unstable in D_8). The expected time to failure has been calculated using Eq. (11):

$\langle T_e \rangle = 26.9$ s   for $D_\chi = 1$, $D_\xi = 1$

$\langle T_e \rangle = 31.7$ s   for $D_\chi = 8/15$, $D_\xi = 22/15$

$\langle T_e \rangle = 46.0$ s   for $D_\chi = 12/35$, $D_\xi = 58/35$.

The values of D_χ and D_ξ have been chosen so that $D_2 = 1.0$, that is, the standard deviation of the response is the same for all three examples. It can be seen from these numbers that higher order moment instability may significantly influence the system's reliability. Namely, the first passage time for the system with unstable 4th order moment is significantly smaller than that for the systems with unstable 6th order moments.

3.2. Second order system under multiplicative and additive noise

The focus is on the following linear oscillator under both additive and multiplicative random excitations:

$$\ddot{X} + \omega_0[2\zeta + W_2(t)]\dot{X} + \omega_0^2[1 + W_1(t)]X = W_3(t), \tag{16}$$

where $W_j(t)$, $j = 1, 2, 3$, are wide band stationary processes with zero mean values. This model was studied by Ariaratnam and Tam [16] under the assumption that ζ is of order ϵ and the $W_j(t)$ are of order $\sqrt{\epsilon}$, where ϵ is a small parameter. By applying the stochastic averaging procedure, it was argued that the amplitude process $A(t) = (X^2 + \dot{X}^2/\omega_0^2)^{1/2}$ is approximately a Markov diffusion process governed by the (Itô) stochastic differential equation (SDE)

$$dA = m(A)dt + \sigma(A)dB(t). \tag{17}$$

The drift coefficient $m(A)$ and the diffusion coefficient $\sigma(A)$ are given by the equations,

$$m(A) = -\alpha A + \frac{\delta}{2A}, \tag{18}$$

$$\sigma(A) = (\gamma A^2 + \delta)^{1/2}, \tag{19}$$

in which

$$\begin{aligned}\alpha &= \zeta\omega_0 - \frac{\pi\omega_0^2}{8}\big[2\Phi_{22}(0) + 3\Phi_{22}(2\omega_0) \\ &\quad + 3\Phi_{11}(2\omega_0) - 6\Psi_{12}(2\omega_0)\big],\end{aligned} \tag{20}$$

$$\delta = \frac{\pi}{\omega_0^2}\Phi_{33}(\omega_0), \tag{21}$$

$$\gamma = \frac{\pi\omega_0^2}{4}\big[2\Phi_{22}(0) + \Phi_{22}(2\omega_0) + \Phi_{11}(2\omega_0) + 2\Psi_{12}(2\omega_0)\big], \tag{22}$$

and

$$\begin{aligned}\Phi_{ij}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E[W_i(t)W_j(t+\tau)]\cos(\omega\tau)d\tau, \\ &\quad i, j = 1, 2, 3, \end{aligned} \tag{23}$$

$$\begin{aligned}\Psi_{ij}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} E[W_i(t)W_j(t+\tau)]\sin(\omega\tau)d\tau, \\ &\quad i, j = 1, 2, 3. \end{aligned} \tag{24}$$

Ariaratnam and Tam [16] showed that the expected time $\langle T_f \rangle$ to first failure of the amplitude process $A(t)$ is given by the formulas

$$\langle T_f \rangle = \frac{1}{\eta\gamma}\int_{a_0}^{a_c}\frac{1}{u}\left[\left(1 + \frac{\gamma}{\delta}u^2\right)^\eta - 1\right]du, \qquad \eta = \frac{\alpha}{\gamma} + \frac{1}{2} \neq 0 \tag{25}$$

$$\langle T_f \rangle = \frac{1}{\gamma}\int_{a_0}^{a_c}\frac{1}{u}\ln\left(1 + \frac{\gamma}{\delta}u^2\right)du, \qquad \eta = 0. \tag{26}$$

Here a_0 denotes the initial condition and a_c the critical level ($a_0 < a_c$). This approach would usually represent an approximation in the sense that failure for the original problem would typically be when $X(t)$ exceeds a critical region bounded by the thresholds $\pm x_c$. An approximate solution for this is obtained by studying the exceedance of $a_c = x_c$ by the amplitude process $A(t)$.

For the numerical calculations in this paper the $W_j(t)$ are assumed to be independent Gaussian white noise processes, with $E[W_j(t)W_j(t+\tau)] = \sigma_j^2\delta(\tau)$. Using numerical PI we have calculated the reliability function associated with the linear oscillator model in Eq. (16) for three case studies with different values of the ω_0 parameter. Since PI calculations can be done for any choice of parameter values, it provides a means of studying the limitations of the amplitude diffusion model adopted in [16], and thereby also the limitations of stochastic averaging in this context.

To provide a means for verification of the PI results, we have calculated the stationary part of the reliability function by the ACER method [18]. This method makes it possible to estimate the exact extreme value distribution, and hence the reliability function, of the response process provided the transient response can be neglected. From Eq. (10) and the following discussion, it is obtained that the tail behavior of the reliability function is given as,

$$R(t) = R(t_0)\exp\{-\nu(x_c)(t - t_0)\}, \quad t \geq t_0, \tag{27}$$

for a suitable t_0. The ACER method provides an estimate and a 95% confidence interval of the parameter $\nu(x_c)$ for each critical level x_c. An approximate mean time to failure is then given by $1/\nu(x_c)$.

Table 1
Mean time to failure for $\omega_0 = 0.1$, $\zeta = 0.01$.

x_c/σ_X	$\langle T_{SA} \rangle$ (s)	$\langle T_{PI} \rangle$ (s)	ν_{PI}	ν_{ACER}	95% CI
2.5	$1.67 \cdot 10^6$	$5.90 \cdot 10^3$	$1.88 \cdot 10^{-4}$	$2.08 \cdot 10^{-4}$	$(1.81 \cdot 10^{-4}, 2.36 \cdot 10^{-4})$
3.0	$4.22 \cdot 10^6$	$1.58 \cdot 10^4$	$6.64 \cdot 10^{-5}$	$7.56 \cdot 10^{-5}$	$(5.64 \cdot 10^{-5}, 9.48 \cdot 10^{-5})$
3.5	$1.20 \cdot 10^7$	$4.87 \cdot 10^4$	$2.10 \cdot 10^{-5}$	$2.21 \cdot 10^{-5}$	$(1.32 \cdot 10^{-5}, 3.09 \cdot 10^{-5})$
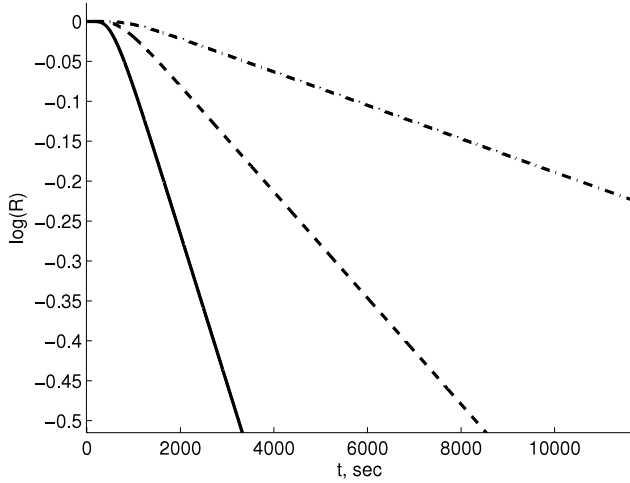
Fig. 2. Two-sided reliability function of system (16). −−: $x_c/\sigma_X = 2.5$, − − −: $x_c/\sigma_X = 3.0$, − · −: $x_c/\sigma_X = 3.5$.

Fig. 3. Two-sided reliability function of system (16). −−: $x_c/\sigma_X = 2.5$, − − −: $x_c/\sigma_X = 3.0$, − · −: $x_c/\sigma_X = 3.5$.
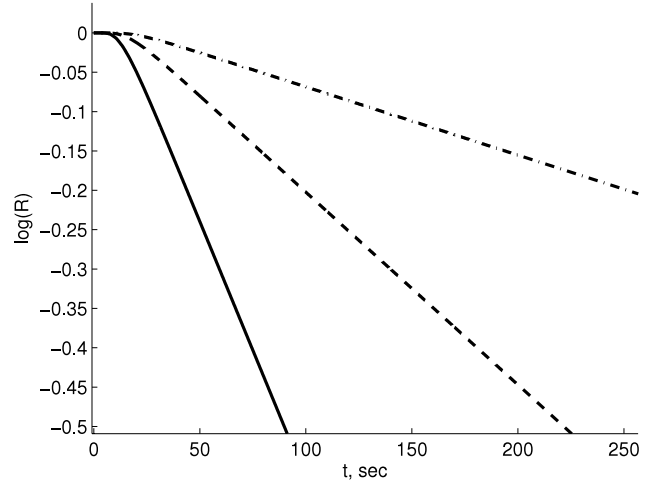
Fig. 4. Two-sided reliability function of system (16). −−: $x_c/\sigma_X = 2.5$, − − −: $x_c/\sigma_X = 3.0$, − · −: $x_c/\sigma_X = 3.5$.
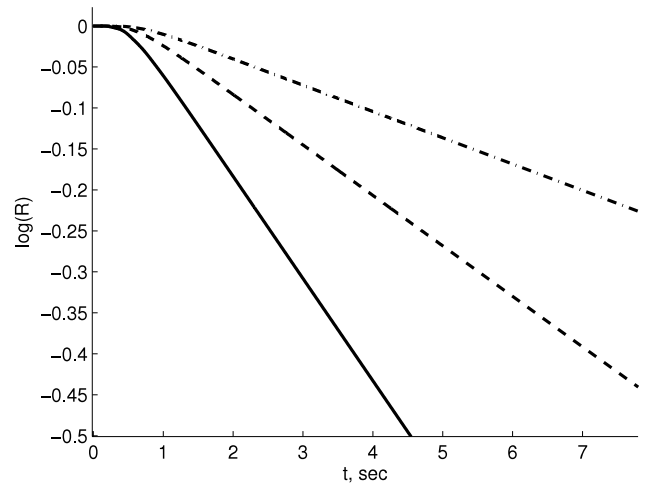
3.2.1. Case 1

In this case the system's parameters are $\omega_0 = 0.1$, $\zeta = 0.01$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. The standard deviation of the response was found to be $\sigma_X = 16.271$, while $\sigma_{\dot{X}} = 1.628$. The reliability functions calculated by the numerical PI method for three different critical levels are shown in Fig. 2. Some numerical results are summarized in Table 1. It is seen that the mean time to failure $\langle T_{PI} \rangle$ calculated by PI agrees with the approximate time to failure provided by $1/\nu(x_c)$ for all three critical levels, while the corresponding failure times $\langle T_{SA} \rangle$ obtained by stochastic averaging are orders of magnitude wrong. This seems to be caused by a large value of the parameter η, cf. Eq. (25), which assumes the value 19.5 for this case.

3.2.2. Case 2

In this case the system's parameters are $\omega_0 = 1.0$, $\zeta = 0.05$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. The standard deviation of the response was found to be $\sigma_X = 1.832$, while $\sigma_{\dot{X}} = 1.836$. The reliability functions calculated by the numerical PI method for three different critical levels are shown in Fig. 3. Numerical results are summarized in Table 2. Again it is seen that the mean time to failure calculated by PI agrees with the approximate time to failure provided by $1/\nu(x_c)$ for all three critical levels, while the corresponding failure times obtained by stochastic averaging are approximately two orders of magnitude wrong. In this case $\eta = 9.5$, which is still quite large.

3.2.3. Case 3

In this case the system's parameters are $\omega_0 = 10.0$, $\zeta = 0.15$, and $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. The standard deviation of the response was found to be $\sigma_X = 0.0410$, while $\sigma_{\dot{X}} = 0.4261$. The reliability functions calculated by the numerical PI method for three different critical levels are shown in Fig. 4. Numerical results are summarized in Table 3. Also in this case it is seen that the mean time to failure calculated by PI agrees with the approximate time to failure provided by $1/\nu(x_c)$ for all three critical levels, while the corresponding failure times obtained by stochastic averaging are now more on the same level as the correct values. In this case $\eta = 2.5$.

4. Conclusions

In the paper the authors have considered a first passage type reliability problem for two types of systems: first order and second order systems with parametric and additive white noises. The numerical results presented in the paper are obtained by the path integration method, which was reformulated from its standard form to handle reliability problems. The results were verified by Monte-Carlo simulations through the use of the ACER method.

For the first order system the numerical results have demonstrated that the higher order moments instability may significantly influence the system's reliability. It has been demonstrated that the first passage time for the system with unstable 4th order moment is half as large as that with an unstable 6th order moment.

For the second order system it has been shown that the use of stochastic averaging has its limitations especially for calculating the reliability. The results calculated by numerical PI were verified by using Monte-Carlo simulations in combination with the ACER

Table 2
Mean time to failure for $\omega_0 = 1.0$, $\zeta = 0.05$.

x_c/σ_X	$\langle T_{SA} \rangle$ (s)	$\langle T_{PI} \rangle$ (s)	ν_{PI}	ν_{ACER}	95% CI
2.5	$7.45 \cdot 10^3$	$1.66 \cdot 10^2$	$6.54 \cdot 10^{-3}$	$6.15 \cdot 10^{-3}$	$(5.83 \cdot 10^{-3}, 6.48 \cdot 10^{-3})$
3.0	$1.69 \cdot 10^4$	$4.26 \cdot 10^2$	$2.45 \cdot 10^{-3}$	$2.37 \cdot 10^{-3}$	$(2.19 \cdot 10^{-3}, 2.54 \cdot 10^{-3})$
3.5	$4.03 \cdot 10^4$	$1.17 \cdot 10^3$	$8.68 \cdot 10^{-4}$	$7.98 \cdot 10^{-4}$	$(7.31 \cdot 10^{-4}, 8.65 \cdot 10^{-4})$

Table 3
Mean time to failure for $\omega_0 = 10.0$, $\zeta = 0.15$.

x_c/σ_X	$\langle T_{SA} \rangle$ (s)	$\langle T_{PI} \rangle$ (s)	ν_{PI}	ν_{ACER}	95% CI
2.5	16.8	8.54	$1.25 \cdot 10^{-1}$	$1.16 \cdot 10^{-1}$	$(1.09 \cdot 10^{-1}, 1.24 \cdot 10^{-1})$
3.0	29.4	16.9	$6.16 \cdot 10^{-2}$	$5.85 \cdot 10^{-2}$	$(5.33 \cdot 10^{-2}, 6.36 \cdot 10^{-2})$
3.5	48.9	31.9	$3.21 \cdot 10^{-2}$	$2.89 \cdot 10^{-2}$	$(2.50 \cdot 10^{-2}, 3.28 \cdot 10^{-2})$

method, which allows the estimation of the exact extreme value distribution for the stationary part of the response process. This provides a means of determining an approximate value of the mean time to failure. In all the case studies investigated there was agreement between the results calculated by PI and estimated by the ACER method.

Acknowledgement

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