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Lattice SUSY for the DiSSEP at $\lambda^2 = 1$ (and $\lambda^2 = -3$)

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Abstract

We investigate whether the dynamical lattice supersymmetry discussed for various Hamiltonians, including one-dimensional quantum spin chains, by (Fendley *et al* 2003 *J. Phys. A* **36**, 12399424; Yang and Fendley 2004 *J. Phys. A* **37**, 893748; Hagendorf and Fendley 2012 *J. Stat. Phys.* **146**, 112255) and (Hagendorf *et al* 2013 *J. Stat. Phys.* **150**, 60957; Hagendorf and Liénardy 2017 *J. Phys. A* **50**, 185202; Hagendorf and Liénardy 2018 *J. Stat. Mech.* 033106) might also exist for the Markov matrices of any one-dimensional exclusion processes, which have the additional constraint of zero column sums by comparison with the spin chains. We find that the DiSSEP (Dissipative Symmetric Simple Exclusion Process), introduced by Crampe *et al* in (Crampe *et al* 2016 *Phys. Rev. E* **94**, 032102; Vanicat [arXiv:1708.02440]), provides one such example for suitably chosen parameters. The DiSSEP Markov matrix admits the supersymmetry in these cases because it is conjugate to Ising ($\lambda^2 = 1$) and $\Delta = -1/2$ XXZ ($\lambda^2 = -3$) spin chain Hamiltonians which also possess the supersymmetry. The consequences for the spectrum of the DiSSEP Markov matrix are discussed. We also note that the length-changing supersymmetry relation for the DiSSEP Markov matrix M^L and the supercharge $Q^{L\dagger}$ for L sites, $M^L Q^{L\dagger} = Q^{L\dagger} M^{L-1}$, is reminiscent of a 'transfer matrix' symmetry that has been observed in other exclusion processes and discuss the similarity.

1. Lattice SUSY

A dynamical, exact lattice supersymmetry in one dimensional lattice fermion systems and spin chains was first observed by Fendley *et al* [1–3]. A lattice Hamiltonian for L sites with such a supersymmetry can be written as

$$H^L = Q^{L\dagger} Q^L + Q^{L+1} Q^{L+1\dagger} \tag{1}$$

where H^L acts on the vector space $V^{\otimes L}$, with $V \simeq \mathbb{C}^2$. The lattice supercharges Q^L, $Q^{L\dagger}$ act on chains of length L and $L-1$ respectively as Q^L: $V^{\otimes L} \to V^{\otimes (L-1)}$ and $Q^{L\dagger}$: $V^{\otimes (L-1)} \to V^{\otimes L}$ [1]. For an open chain, these may be expressed in terms of *local* supercharges as

$$Q^L = \sum_{k=1}^{L-1} (-1)^{k+1} \mathfrak{q}_{k,k+1}, \qquad Q^{L\dagger} = \sum_{k=1}^{L-1} (-1)^{k+1} \mathfrak{q}_k^\dagger \tag{2}$$

where $\mathfrak{q}$: $V \otimes V \to V$ and $\mathfrak{q}^\dagger$: $V \to V \otimes V$ and the subscripts denote the lattice sites on which the operators act [4]. Q^L is defined for $L \geqslant 2$. In a matrix representation $\mathfrak{q}$ and $\mathfrak{q}^\dagger$ are thus 2×4 and 4×2 matrices respectively. Satisfying the standard nilpotency conditions for the global supercharges

$$Q^{L-1} Q^L = 0, \qquad Q^{L+1\dagger} Q^{L\dagger} = 0, \tag{3}$$

gives the following associativity condition on the local supercharge $\mathfrak{q}$ for open chains [5, 6]

$$\mathfrak{q}(\mathfrak{q} \otimes \mathbb{I})|\psi\rangle = \mathfrak{q}(\mathbb{I} \otimes \mathfrak{q})|\psi\rangle, \quad \forall |\psi\rangle \in V \tag{4}$$

[1] The choice of $Q^{L\dagger}$ and $\mathfrak{q}^\dagger$ to be creation operators, which seems appropriate in this context, is the opposite of that used in [4, 5, 7] but agrees with that in [6].

or the equivalent coassociativity condition on $\mathrm{q}^\dagger$

$$(\mathrm{q}^\dagger \otimes \mathbb{I})\mathrm{q}^\dagger|\psi\rangle = (\mathbb{I} \otimes \mathrm{q}^\dagger)\mathrm{q}^\dagger|\psi\rangle, \quad \forall|\psi\rangle \in V. \tag{5}$$

The condition on $\mathrm{q}^\dagger$ (and similarly for $\mathfrak{q}$) for closed chains of length L is modified to

$$[(\mathrm{q}^\dagger \otimes \mathbb{I})\mathrm{q}^\dagger - (\mathbb{I} \otimes \mathrm{q}^\dagger)\mathrm{q}^\dagger]|\psi\rangle = |\chi\rangle \otimes |\psi\rangle - |\psi\rangle \otimes |\chi\rangle, \quad \forall|\psi\rangle \in V \tag{6}$$

where $|\chi\rangle \in V \otimes V$ is some fixed vector and $L \geqslant 3$ as a consequence of the definition of Q.

If a supercharge of the form equation (2) satisfying equations (4), (5) or equation (6) is inserted into equation (1) all the non-nearest-neighbour terms in the anticommutator cancel due to the alternating sign factors and the resulting nearest-neighbour bulk Hamiltonian is of the form

$$h = -(\mathbb{I} \otimes \mathrm{q})(\mathrm{q}^\dagger \otimes \mathbb{I}) - (\mathfrak{q} \otimes \mathbb{I})(\mathbb{I} \otimes \mathrm{q}^\dagger) + \mathrm{q}^\dagger\mathfrak{q} + \frac{1}{2}(\mathfrak{q}\mathrm{q}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{q}\mathrm{q}^\dagger), \tag{7}$$

supplemented by boundary terms $(1/2)\mathfrak{q}\mathrm{q}^\dagger$ for open chains. For a closed spin chain of length L an additional local charge $\mathrm{q}_0^\dagger$ is defined which either shifts the action of $\mathrm{q}_1^\dagger$ to the left or $\mathrm{q}_L^\dagger$ to the right. Demanding the compatibility of these two possibilities restricts the supersymmetric construction of the Hamiltonian to particular eigenspaces of the translation operator in this case. The supersymmetry for the open chains is thus somewhat simpler than for the closed chains since no restriction to special subsectors in necessary to implement it.

Using the nilpotency conditions in equation (3) shows that the supercharges relate the Hamiltonians of chains of different length, i.e.

$$H^{L-1}Q^L = Q^L H^L, \quad H^L Q^{L\dagger} = Q^{L\dagger} H^{L-1}. \tag{8}$$

Various choices of q, $\mathrm{q}^\dagger$ leading to well-known Hamiltonians have been explored. Fendley and Yang [2] noted that

$$\mathrm{q}^\dagger|0\rangle = \varnothing \quad \mathrm{q}^\dagger|1\rangle = |00\rangle \tag{9}$$

gave (up to a constant term) the XXZ Hamiltonian at its combinatorial point with diagonal boundary conditions

$$H_{comb} = -\frac{1}{2}\sum_{k=1}^{L-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y - \frac{1}{2}(\sigma_k^z\sigma_{k+1}^z - \mathbb{I})\right) - \frac{1}{4}(\sigma_1^z + \sigma_L^z) \tag{10}$$

with $\sigma^{x,\,y,\,z}$ being the standard Pauli matrices. We have dropped the superscript L on the Hamiltonian above, and henceforward, for notational conciseness. In the above, the basis vectors are $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, $|00\rangle = |0\rangle \otimes |0\rangle$ and the local supercharge $\mathfrak{q}$ is

$$\mathfrak{q} = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}. \tag{11}$$

Hagendorf *et al* [5] observed that this supercharge can be combined with its image under spin reversal ($|0\rangle \to |1\rangle$)

$$\bar{\mathfrak{q}} = \begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{12}$$

and a gauge supercharge which acts on any vector $|\psi\rangle \in V$ as

$$\mathrm{q}_\phi^\dagger|\psi\rangle = |\phi\rangle \otimes |\psi\rangle + |\psi\rangle \otimes |\phi\rangle, \tag{13}$$

where $|\phi\rangle$ is some vector in V, to give a one parameter family of supercharges

$$\mathfrak{q}(y) = x\begin{bmatrix}-2y & -y^2 & -y^2 & y^3\\ 1 & -y & -y & -2y^2\end{bmatrix} \tag{14}$$

with $x = (1 + |y|^6)^{-1/2}$. These still produced the same XXZ bulk Hamiltonian when inserted into equation (7) but gave identical left and right, now non-diagonal, boundary terms that depended explicitly on y. The supercharge Q^L resulting from equation (14) can be further elaborated to give a limited class of non-identical boundary terms [5]. The approach readily generalises to higher spin models [4, 8] and $\mathfrak{gl}(N|M)$ Hamiltonians [9].

There is a close relation between one-dimensional quantum spin chains and various one-dimensional exclusion processes, so a natural question to pose is whether the dynamical lattice supersymmetry might also exist in such models. The Markov matrices of the exclusion processes possess the additional constraint by comparison with the spin chain Hamiltonians of having zero column sums (for continuous processes), both for

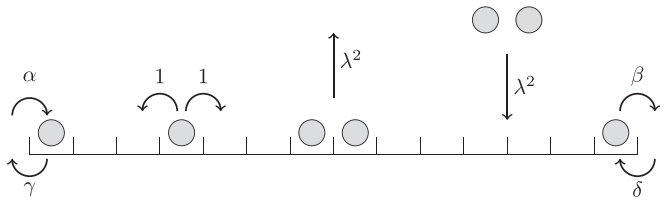

Figure 1. Allowed DiSSEP moves and their rates.

the bulk and boundary matrices, so this must also be satisfied. Even with the additional constraint at least one model, the Dissipative Symmetric Simple Exclusion Process (DiSSEP) can be shown to possess the supersymmetry for a particular choice of the model parameters. The DiSSEP is described in the next section and its local supercharges are explicitly constructed in the sequel. The consequences of the supersymmetry for the DiSSEP spectrum are then outlined and parallels drawn with a 'transfer matrix' symmetry that has been observed in other exclusion processes.

2. The DiSSEP

The DiSSEP was presented in [10] as an integrable deformation of the Symmetric Simple Exclusion Process (SSEP) which still allowed a solution via the matrix product ansatz [11]. A concise way to describe the dynamics in such systems is to use Dirac braket notation to describe the state. For an open system with L sites, introduce an indicator variable $n_i \in \{0, 1\}$ at each site i to denote the presence or absence of a particle and denote the probability of finding a configuration n_1..., n_L at time t by $P_t(n_1,\ldots,n_L)$. The evolution of the ket vector $|P_t\rangle$

$$|P_t\rangle = \sum_{n_1,\ldots,n_L \in \{0,1\}} P_t(n_1,\ldots,n_L)\ \ |n_1 \ldots n_L\rangle, \tag{15}$$

where $|n_1 \ldots n_L\rangle = |n_1\rangle \otimes \ldots \otimes |n_L\rangle$ is given by master equation

$$\frac{d|P_t\rangle}{dt} = M\ |P_t\rangle. \tag{16}$$

The basis vectors are again, as for the spin chains, $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$. The Markov matrix M appearing in the master equation is given for the DiSSEP by

$$M(\lambda^2) = B_1 + \sum_{k=1}^{L-1} m_{k,k+1} + \overline{B}_L \tag{17}$$

with boundary transition matrices B, $\overline{B}$ and bulk transition matrix m given by

$$B = \begin{pmatrix}-\alpha & \gamma\\ \alpha & -\gamma\end{pmatrix}, \quad m = \begin{pmatrix}-\lambda^2 & 0 & 0 & \lambda^2\\ 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0\\ \lambda^2 & 0 & 0 & -\lambda^2\end{pmatrix}, \quad \overline{B} = \begin{pmatrix}-\delta & \beta\\ \delta & -\beta\end{pmatrix}. \tag{18}$$

The bulk Markov matrix $m_{k,k+1}$ acts between nearest neighbour sites k, $k+1$, giving forward and backward hops and pair addition and annihilation in the bulk, while the boundary matrices B, $\overline{B}$ allow the addition and removal of particles at both ends of the system. The stochastic nature of the model is evident from the column sums of the various matrices being zero, since they describe rates. This is the distinguishing feature of this class of models. The various allowed processes for particle moves and their associated rates are shown in figure 1.

The Markov matrices of such one-dimensional exclusion processes and one-dimensional quantum spin chains can be related by conjugation. For the case of the DiSSEP, the Markov matrix $M(\lambda^2)$ is conjugate to the Hamiltonian $H_{XXZ}(\lambda^2)$ of an open XXZ spin chain with upper diagonal boundary conditions, both with L sites, via [12]

$$H_{XXZ}(\lambda^2) = -U \otimes U \ldots \otimes U\ M(\lambda^2)\ U^{-1} \otimes U^{-1} \ldots \otimes U^{-1} \tag{19}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \tag{20}$$

and

$$H_{XXZ}(\lambda^2) = -(\alpha-\gamma)\sigma_1^+ + \frac{\alpha+\gamma}{2}(\sigma_1^z + \mathbb{I}) - (\delta-\beta)\sigma_L^+ + \frac{\delta+\beta}{2}(\sigma_L^z + \mathbb{I}) + \frac{\lambda^2-1}{2}\sum_{k=1}^{L-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y - \frac{\lambda^2+1}{\lambda^2-1}(\sigma_k^z\sigma_{k+1}^z - \mathbb{I})\right) \tag{21}$$

where $\sigma^{+,-} = \sigma^x \pm i\sigma^y$ are raising and lowering matrices[2]. It is clear from equation (21) that $\lambda^2 = 1$ is a particularly simple, diagonal Ising limit for the bulk Hamiltonian in the model. Similarly, if $\alpha = \gamma$ and $\beta = \delta$ the XXZ Hamiltonian boundary conditions also become diagonal. The simplicity is reflected in the solution of the conjugate DiSSEP when $\lambda^2 = 1$ [10].

3. The DiSSEP at $\lambda^2 = 1$ and lattice SUSY

It is straightforward to see that

$$\mathrm{q} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

and its image under spin reversal

$$\bar{\mathrm{q}} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \tag{23}$$

satisfy equations (4), (5) and that both generate the negative of the bulk DiSSEP Markov matrix

$$-m = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = -(\sigma^x \otimes \sigma^x - \mathbb{I}) \tag{24}$$

for $\lambda^2 = 1$ when employed in equation (7). Inserting an overall minus into the relation between the supercharges and the Hamiltonian, now Markov matrix, in equation (1) does not change any of the ensuing discussion, so the change in sign is immaterial for the existence of the lattice supersymmetry. The boundary matrix $(1/2)\mathrm{q}\mathrm{q}^\dagger$, however, obtained from both of these supercharges is diagonal

$$B = \bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

and therefore non-stochastic.

For the XXZ Hamiltonian of equation (10) $\mathrm{q}^\dagger$ and $\bar{\mathrm{q}}^\dagger$ anti-commute up to boundary terms

$$[(-\mathrm{q}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \mathrm{q}^\dagger)\bar{\mathrm{q}}^\dagger + (-\bar{\mathrm{q}}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \bar{\mathrm{q}}^\dagger)\mathrm{q}^\dagger]|\psi\rangle = |\chi\rangle \otimes |\psi\rangle - |\psi\rangle \otimes |\chi\rangle \tag{26}$$

$\forall\, |\psi\rangle \in V$ and where $|\chi\rangle$ is explicitly calculable, so additional gauge terms are needed to combine them into equation (14) to give a $\mathrm{q}(y)$ that will satisfy equations (4), (5). In the case of the DiSSEP $\mathrm{q}^\dagger$ and $\bar{\mathrm{q}}^\dagger$ from equations (22), (23) anti-commute without boundary terms

$$[(-\mathrm{q}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \mathrm{q}^\dagger)\bar{\mathrm{q}}^\dagger + (-\bar{\mathrm{q}}^\dagger \otimes \mathbb{I} + \mathbb{I} \otimes \bar{\mathrm{q}}^\dagger)\mathrm{q}^\dagger]|\psi\rangle = 0. \tag{27}$$

This allows them to be directly combined without introducing any gauge terms to give a one-parameter family of supercharges $\mathrm{q}(y)$ which continue to satisfy the (co)associativity conditions of equations (4), (5)

$$\mathrm{q}(y) = x\begin{bmatrix} 1 & y & y & 1 \\ y & 1 & 1 & y \end{bmatrix}, \tag{28}$$

where $x = (1 + |y|^2)^{-1/2}$ in this case. When inserted into equation (7) $\mathrm{q}(y)$ still gives the (negative) DiSSEP Markov matrix in the bulk of equation (24) but the boundary terms are modified to

$$B(y) = \bar{B}(y) = \begin{bmatrix} 1 & \dfrac{2\Re(y)}{1+|y|^2} \\ \dfrac{2\Re(y)}{1+|y|^2} & 1 \end{bmatrix}. \tag{29}$$

[2] We have included minus signs in both the conjugation in equation (19) and the Hamiltonian in equation (21) by comparison with [12] (in a similar manner to [13]) to facilitate comparison with various H_{XXZ} Hamiltonians and Markov matrices later, where the natural choice is to take the minus sign in front of the Hamiltonians. We have also chosen U to be unitary to simplify some of the ensuing numerical factors when discussing conjugating supercharges.

We are thus able to obtain stochastic boundary matrices by taking $y = -1$, corresponding to the zero bias case of $\alpha = \beta = \gamma = \delta = 1$ when the overall minus sign is taken into account.

The DiSSEP supercharge $\mathrm{q}(y)$ can be translated to its conjugate, $\mathrm{q}_c(y)$, using the U matrix from equation (20)

$$\begin{aligned}\mathrm{q}_c(y) &= U\ \mathrm{q}(y)\ U^{-1} \otimes U^{-1} \\ \mathrm{q}_c^{\dagger}(y) &= U \otimes U\ \mathrm{q}^{\dagger}(y)\ U^{-1},\end{aligned} \tag{30}$$

which gives the supercharge for the $\lambda^2 = 1$ XXZ Hamiltonian (i.e. Ising Hamiltonian H_Z) that is conjugate to $\lambda^2 = 1$ DiSSEP Markov matrix. We find

$$\mathrm{q}_c(y) = \sqrt{2}x\begin{bmatrix} y-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y+1 \end{bmatrix} \tag{31}$$

and

$$\mathrm{q}_c^{\dagger}(y) = \sqrt{2}x\begin{bmatrix} \bar{y}-1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \bar{y}+1 \end{bmatrix}. \tag{32}$$

When $\mathrm{q}_c(y)$, $\mathrm{q}_c^{\dagger}(y)$ are inserted into equation (7) they give the simple diagonal bulk and boundary Hamiltonians

$$B_c(y) = \bar{B}_c(y) = \begin{pmatrix} 1 - \dfrac{2Re(y)}{1+|y|^2} & 0 \\ 0 & 1 + \dfrac{2Re(y)}{1+|y|^2} \end{pmatrix}, \quad m_c = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

so $m_c = -(\sigma^z \otimes \sigma^z - \mathbb{I})$. Consistently, this is the bulk term for $H_{XXZ}(\lambda^2 = 1)$ in equation (21), which is just the Ising Hamiltonian, or H_Z.

The results of this section could thus equivalently be construed as stating that $\mathrm{q}_c(y)$, $\mathrm{q}_c^{\dagger}(y)$ of equation (31), (32) provide a one parameter family of supercharges for the diagonal Ising Hamiltonian H_Z

$$H_Z(y) = -\sum_{k=1}^{L-1}(\sigma_k^z\sigma_{k+1}^z - \mathbb{I}) + B_{c,1}(y) + \bar{B}_{c,L}(y). \tag{34}$$

This Hamiltonian is conjugate to the (negative of the) $\lambda^2 = 1$ DiSSEP Markov matrix, $H_X(y)$, generated by supercharges $\mathrm{q}(y)$, $\mathrm{q}^{\dagger}(y)$

$$H_X(y) = -\sum_{k=1}^{L-1}(\sigma_k^x\sigma_{k+1}^x - \mathbb{I}) + B_1(y) + \bar{B}_L(y), \tag{35}$$

via

$$H_Z(y) = U \otimes U ... \otimes U\ H_X(y)\ U^{-1} \otimes U^{-1} ... \otimes U^{-1} \tag{36}$$

(since $H_X(y) = -\mathrm{M}$) and both therefore display the supersymmetry.

Regarding equation (34) purely as an Ising chain there is no reason to fix a particular value of y, so the supersymmetry exists for the entire one-parameter family of Hamiltonians. However, demanding that the boundary terms $B_1(-1)$ and $\bar{B}_L(-1)$ in the conjugate DiSSEP Hamiltonian of equation (35), H_X, are stochastic forces $y = -1$ since then

$$B_1(-1) = \bar{B}_L(-1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{37}$$

These stochastic boundary terms are conjugate to

$$B_{c,1}(-1) = \bar{B}_{c,L}(-1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \sigma_z + \mathbb{I}, \tag{38}$$

which can be seen to be the boundary terms in equation (21) when $\alpha = \beta = \delta = \gamma = 1$.

4. The DiSSEP at $\lambda^2 = 1$, lattice SUSY and the spectrum

The principal physical consequences of the dynamical lattice supersymmetry are a singlet zero energy state and a spectral degeneracy between chains of length L and $L + 1$ for either a Hamiltonian H or a Markov matrix M.

These features are a direct consequence of the definition of the Hamiltonian in equation (1)

$$H^L = Q^{L\dagger}Q^L + Q^{L+1}Q^{L+1\dagger} \ (=-M^L)$$

and the observation [5] that $H|\psi\rangle = E|\psi\rangle$ then implies that (dropping L superscripts for brevity)

$$||Q|\psi\rangle||^2 + ||Q^\dagger|\psi\rangle||^2 = E||\psi||^2 \tag{39}$$

giving zero and positive (negative) eigenstates for the Hamiltonian (Markov matrix). If $E \neq 0$ these states are doublets (superpartners) of the form

$$|\psi\rangle,\ Q^\dagger|\psi\rangle \tag{40}$$

with $Q|\psi\rangle = 0$, as is standard in supersymmmetric theories. The unusual feature for the dynamical lattice supersymmetry is that these states are for chains of different length. There are thus spectral degeneracies between chains of length L and $L+1$. A zero energy state, on the other hand, requires

$$Q|\psi\rangle = 0, \quad Q^\dagger|\psi\rangle = 0 \tag{41}$$

which implies that it must be a singlet [5].

It is possible to see both of these features straightforwardly in short DiSSEP chains at $\lambda^2 = 1$ and $\alpha = \beta = \delta = \gamma = 1$ by using equations (1), (2) to construct the Markov matrix M explicitly to obtain its eigenvalues. Taking $L = 2, 3$ as illustrative examples, the respective Markov matrices are given by

$$M^2 = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \tag{42}$$

with eigenvalues 0(1), $-$ 4(3), where the degeneracy is indicated in parentheses, and

$$M^3 = \begin{pmatrix} -4 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -4 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -4 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & -4 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & -4 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & -4 \end{pmatrix} \tag{43}$$

with eigenvalues 0(1), $-8(1)$, $-4(6)$. Both M^2 and M^3 possess a zero eigenvalue singlet and the eigenvalues -4 for $L = 2$ have partners in the $L = 3$ spectrum. M^L is diagonalized by the conjugation relation in equation (19) when $\lambda^2 = \alpha = \beta = \gamma = \delta = 1$. For instance, when $L = 2$ the conjugation gives

$$U \otimes U\ M^2\ U^{-1} \otimes U^{-1} = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

It is also possible to build the Markov matrices by hand for short chains by simply putting the rates for the allowed transitions into the correct positions in the matrix. The process can be automated for larger L[3]. In this construction the boundary rates are not restricted to $\alpha = \beta = \delta = \gamma = 1$, but can take on generic values. In principle λ^2 can also take on generic values when building the Markov matrices in such a manner, but we limit the discussion here to the simpler case of $\lambda^2 = 1$ (it is also unclear if the supersymmetry is present for $\lambda^2 \neq 1$). The Markov matrix for $L = 2$, M^2, is given by

$$\begin{pmatrix} -1-\alpha-\delta & \beta & \gamma & 1 \\ \delta & -1-\alpha-\beta & 1 & \gamma \\ \alpha & 1 & -1-\gamma-\delta & \beta \\ 1 & \alpha & \delta & -1-\beta-\gamma \end{pmatrix} \tag{45}$$

with eigenvalues 0, $-2-\beta-\delta$, $-2-\alpha-\gamma$ and $-\alpha-\beta-\gamma-\delta$. The conjugation transformation in equation (19) now *upper* diagonalizes M^2

[3] We would like to thank Arvind Ayyer for pointing this out, and doing it.

$$U \otimes U\, M^2\, U^{-1} \otimes U^{-1} = \begin{pmatrix} -\alpha-\beta-\delta-\gamma & \delta-\beta & \alpha-\gamma & 0 \\ 0 & -2-\alpha-\gamma & 0 & \alpha-\gamma \\ 0 & 0 & -2-\delta-\beta & \delta-\beta \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{46}$$

but still allows the eigenvalues to be read off directly. Similarly, for L^3 the Markov matrix M^3 is given by

$$\begin{pmatrix} -2-\alpha-\delta & \beta & 0 & 1 & \gamma & 0 & 1 & 0 \\ \delta & -2-\alpha-\beta & 1 & 0 & 0 & \gamma & 0 & 1 \\ 0 & 1 & -2-\alpha-\delta & \beta & 1 & 0 & \gamma & 0 \\ 1 & 0 & \delta & -2-\alpha-\beta & 0 & 1 & 0 & \gamma \\ \alpha & 0 & 1 & 0 & -2-\gamma-\delta & \beta & 0 & 1 \\ 0 & \alpha & 0 & 1 & \delta & -2-\beta-\gamma & 1 & 0 \\ 1 & 0 & \alpha & 0 & 0 & 1 & -2-\gamma-\delta & \beta \\ 0 & 1 & 0 & \alpha & 1 & 0 & \delta & -2-\beta-\gamma \end{pmatrix}$$

with eigenvalues $-4, 0, -2-\beta-\delta$ (2), $-2-\alpha-\gamma$ (2), $-4-\alpha-\beta-\gamma-\delta$ and $-\alpha-\beta-\gamma-\delta$ and is again upper diagonalized by the conjugation of equation (19).

Since the DiSSEP is a free fermion model when $\lambda^2 = 1$ even for generic $\alpha, \beta, \gamma, \delta$, the eigenvalues and the eigenvectors can be computed for any L [10, 14]. For $L \geqslant 3$, the 2^L eigenvectors and eigenvalues are characterized by $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_L)$ with $\epsilon_i = \pm 1$, giving the eigenvalues

$$\Lambda(\boldsymbol{\epsilon}) = f(\epsilon_1, \epsilon_2, \alpha+\gamma) + \sum_{j=2}^{L-2} (\epsilon_j \epsilon_{j+1} - 1) + f(\epsilon_L, \epsilon_{L-1}, \delta+\beta), \tag{47}$$

where the boundary terms are

$$f(\epsilon, \epsilon', \tau) = \epsilon\epsilon' - 1 - \frac{\tau}{2}(1-\epsilon). \tag{48}$$

This can be seen to be in agreement when $L = 3$ with the explicit calculation for M^3 above. $\Lambda(\boldsymbol{\epsilon})$ can also be written as

$$\Lambda(\boldsymbol{\epsilon}) = \frac{\alpha+\gamma}{2}(\epsilon_0\epsilon_1 - 1) + \sum_{j=1}^{L-1} (\epsilon_j \epsilon_{j+1} - 1) + \frac{\beta+\delta}{2}(\epsilon_L \epsilon_{L+1} - 1) \tag{49}$$

where additional fictitious boundary spins $\epsilon_0 = \epsilon_{L+1} = 1$ have been introduced. The DiSSEP spectrum for length L is thus equivalent to that of an Ising model of length $L+2$ with fixed boundary conditions and an inhomogenous bond at each end.

equations (47), (49) make it clear that the eigenvalues for a chain of length L will have partners in a chain of length $L+1$ even for generic $\alpha, \beta, \gamma, \delta$. The boundary terms containing $\alpha, \beta, \gamma, \delta$ are, as one might expect, unaffected by the length of the chain (for $L \geqslant 3$) and the bulk bond terms add minus two or zero to these. The discussion in this section shows that this can be interpreted as the consequence of a dynamical lattice supersymmetry when $\alpha = \beta = \gamma = \delta = 1$ by putting the DiSSEP in correspondence via conjugation with an open Ising chain which manifests the supersymmetry. For generic $\alpha, \beta, \gamma, \delta$, on the other hand, it can be put it into correspondence with a longer Ising chain with fixed boundary conditions. In this case a construction of the Hamiltonian from local supercharges, if it exists, is not known. Nonetheless, the spectral degeneracies between chains of different lengths persist. Since the origin of these when $\alpha = \beta = \gamma = \delta = 1$ is the local supersymmetry, their presistence for generic $\alpha, \beta, \gamma, \delta$ is suggestive of supersymmetry in that case also.

For a non-equilibrium model the fact that there is a zero energy singlet with (unbroken) supersymmetry guarantees the existence of a unique non-equilibrium steady state. The spectral degeneracy visible for both $\alpha = \beta = \gamma = \delta = 1$ and generic $\alpha, \beta, \gamma, \delta$ means that the possible values of first excited state eigenvalue, which gives the relaxation rate of the system, are already visible in very short chains. For the DiSSEP the greatest non-vanishing eigenvalue is $-4, -2-\beta-\delta, -2-\alpha-\gamma$ or $-\alpha-\beta-\gamma-\delta$, depending on the parameter values and all of these are already present in M^3.

5. The DiSSEP at $\lambda^2 = -3$ and lattice SUSY

The exact dynamical lattice supersymmetry also exists in the DiSSEP at the unphysical value of $\lambda^2 = -3$, since the conjugate Hamiltonian in this case is a multiple of the XXZ Hamiltonian at its combinatorial point, which possesses the supersymmetry.

If we define

$$\hat{\mathrm{q}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{50}$$

the (co)associativity conditions equations (4), (5) are satisfied and the corresponding bulk Markov matrix obtained from equation (7) is

$$m = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 5 & -1 & 0 \\ 0 & -1 & 5 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} 3 & 0 & 0 & -3 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -3 & 0 & 0 & 3 \end{pmatrix} + 4\mathbb{I} \tag{51}$$

which is minus the DiSSEP Markov matrix at $\lambda^2 = -3$ along with a constant term, together with stochastic boundary matrices $(1/2)\hat{\mathrm{q}}\hat{\mathrm{q}}^\dagger$

$$B = \bar{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{52}$$

When $\lambda^2 = -3$ the bulk XXZ Hamiltonian conjugate to the DiSSEP

$$H_{XXZ}(\lambda^2) = \frac{\lambda^2 - 1}{2} \sum_{k=1}^{L-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \frac{\lambda^2 + 1}{\lambda^2 - 1} (\sigma_k^z \sigma_{k+1}^z - \mathbb{I}) \right) \tag{53}$$

is four times the XXZ Hamiltonian at its combinatorial point, H_{comb}, in equation (10), i.e.

$$H_{XXZ}(-3) = 4H_{comb} = -2 \sum_{k=1}^{L-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \frac{1}{2} (\sigma_k^z \sigma_{k+1}^z - \mathbb{I}) \right). \tag{54}$$

On the other hand, the conjugates of the supercharge $\hat{\mathrm{q}}$ and $\hat{\mathrm{q}}^\dagger$ from equation (50) which give the $\lambda^2 = -3$ DiSSEP are

$$\hat{\mathrm{q}}_c = U\ \hat{\mathrm{q}}\ U^{-1} \otimes U^{-1} = 2\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{55}$$

and

$$\hat{\mathrm{q}}_c^\dagger = U \otimes U\ \hat{\mathrm{q}}^\dagger\ U^{-1} = 2\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{56}$$

$\hat{\mathrm{q}}_c$, $\hat{\mathrm{q}}_c^\dagger$ are multiples of the spin reversed supercharge $\bar{\mathrm{q}}$, $\bar{\mathrm{q}}^\dagger$ for H_{comb} in equation (12), so substituting them into equation (7) gives $4H_{comb}$, consistently with equation (54).

Thus, just as for the $\lambda^2 = 1$ DiSSEP, the supersymmetry observed in the $\lambda^2 = -3$ DiSSEP is a consequence of the Markov matrix being conjugate to a spin chain Hamiltonian which displays the supersymmetry. As a DiSSEP at an unphysical value of λ^2 this is simply a curiousity, but it might be interesting to change perspective and inquire if results from the DiSSEP side had any implications for the conjugate XXZ spin chain at its combinatorial point.

6. Conclusions

A brute force scan by computer of possible integer entries $\{\ldots \pm 2, \pm 1, 0, \pm 1, \pm 2 \ \ldots\}$ in q reveals that while it is relatively easy to generate solutions of equations (4), (5), demanding that these should represent bulk stochastic matrices (column sum zero, up to a possible constant term) and that the boundary matrices also be stochastic leaves only the two DiSSEP cases discussed here, $\lambda^2 = 1$ and the unphysical value of $\lambda^2 = -3$. It is possible that exploring conjugations and equivalences systematically along the lines of [15] might show that other stochastic Markov matrices are accessible from known supersymmetric Hamiltonians.

The investigations here were motivated by the observation that a 'transfer matrix' symmetry which takes the form

$$M^L T^{L\dagger} = T^{L\dagger} M^{L-1} \tag{57}$$

exists in several stochastic models, which is analogous to the length changing SUSY relation of equation (8). $T^{L\dagger}$ was explicitly presented via a recursion relation for the asymmetric annihilation process (ASAP), whose bulk and boundary Markov matrices are given by

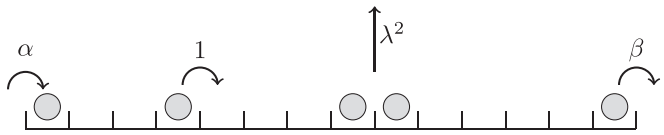

Figure 2. Allowed ASAP moves and rates.

$$B = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & 0 & 0 & \lambda^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\lambda^2 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}. \tag{58}$$

in [16]. The allowed moves for the ASAP are shown in figure 2. It is tempting to regard the transfer matrix symmetry as evidence for a similar dynamical lattice supersymmetry to the one discussed here for the DiSSEP. However, the bulk Markov matrix in equation (58) is not amongst those generated by scanning through various potential $\mathfrak{q}$'s here. The algorithm for determining $T^{L\dagger}$ in [16] is based on the recursive properties of the Markov matrix and is a global construction rather than a local formulation.

A similar situation exists for the Totally Asymmetric Exclusion Process (TASEP) [17]. For this a relation between the Markov matrices for systems of different lengths is of the form

$$M^L \tilde{T}^{L\dagger} = T^{L\dagger} M^{L-1}, \tag{59}$$

where $\tilde{T}^{L\dagger}$ and $T^{L\dagger}$ are now two different matrices. Again, the Markov matrix for the TASEP

$$m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60}$$

is not produced by the paricular class of $\mathfrak{q}$'s we have examined here.

In summary, we have shown that the DiSSEP possesses a dynamical lattice supersymmetry in the sense of [1–5, 7] for $\lambda^2 = 1, -3$ and $\alpha = \beta = \gamma = \delta = 1$. Both λ^2 values represent simplifying values for the model parameters since the bulk Markov matrices for $\lambda^2 = 1, -3$ are conjugate to a diagonal Ising Hamiltonian and an *XXZ* Hamiltonian at its combinatorial point respectively, which are themselves supersymmetric. As we saw in the section 4, the physical consequences of the dynamical lattice supersymmetry are a zero energy singlet and degeneracies in the spectra between chains of length L and $L+1$. These were observed explicitly in the spectrum of the DiSSEP Markov matrix at the supersymmetric point $\lambda^2 = 1$ and $\alpha = \beta = \gamma = \delta = 1$. These spectral degeneracies still appear for generic boundary rate values, although in this case a supersymmetric formulation of the corresponding Ising spin chain is not known. A consequence of the degeneracy is that the possible values of the relaxation rate are already visible in short chains.

The formal similarity between the length changing supersymmetry for various spin chains in equation (8) and the global transfer matrix symmetry in equations (57), (59) in the ASAP [16] and TASEP [17] is intriguing and it would be an interesting challenge to see if this was evidence of a hidden supersymmetry in these models too.

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