

# Asymmetric solitary vortex evolution in two-dimensional Bose-Einstein condensate in harmonic trap

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## ABSTRACT

We studied the evolution of an asymmetric solitary vortex in a two-dimensional Bose-Einstein condensate starting from a quasi-one dimensional state with external harmonic trapping potential modulation. We derived the analytical solution of the asymmetric solitary vortex based on the two-dimensional Gross-Pitaevskii equation model and variational method. Furthermore, we identified the oscillation mode between the extreme asymmetric state and circular symmetric state for the derived solitary vortex. The obtained theoretical results can be used as guides for relevant experimental studies of asymmetric solitary vortices under similar physical conditions.

## Introduction

Nonlinear physical phenomena in ultracold atomic systems have been popular research topics even since the first experimental realization of Bose-Einstein condensation (BEC). The typical nonlinear features of solitons and vortices have been extensively investigated [1] in ultracold systems because of the flexible controllability of the system. Ultracold atomic systems can be easily modulated by controlling the external trapping potential. In addition, the nonlinear interactions can be modulated via the Feshbach resonance experimental technique [2,3]. It has been demonstrated theoretically as well as experimentally that stable solitons can exist only in one-dimensional settings. In contrast, solitary vortices typically exist in two-dimensional settings. Prior theoretical [4] and experimental investigations [5] have revealed that, in systems with dominant leading-order nonlinear interactions, modulational instabilities [6] are suppressed. Furthermore, when the nonlinear interaction strength is less than a certain threshold value, stable solitary vortices [5] are obtained that exist in quasi-stable, breathing, or rotating states.

The Gross-Pitaevskii equation (GPE) [7–14] model has been proven to be reliable for a quantitative description of the two-dimensional dynamics of BECs. For the one-dimensional system, analytical soliton solutions have been strictly derived, while in the two-dimensional setting, circular symmetric solitary vortices have been analytically studied in prior investigations. However, circular asymmetric solitary vortices have rarely been investigated. Asymmetric solitary vortices can be generated from axial disturbances of initially circular symmetric settings, or by

cross-dimensional modulations that can change the external trapping potentials of quasi-one dimensional matter waves. In this study, we have investigated the evolution of asymmetric solitary vortices in isotropic harmonic trapping potentials from initial quasi-one dimensional states. Using the two-dimensional GPE model and variational method [15,16], we have derived analytical solutions for the solitary vortices that exhibit circular asymmetric features, with the resulting solitary vortex oscillating between the extreme asymmetric and circular symmetric states under typical leading-order nonlinear interactions. The derived theoretical results can be used as guides for relevant experimental studies of asymmetric solitary vortices in two-dimensional ultracold atomic systems.

The remainder for this paper is organized as follows. In section II, we first formulate the physical settings that initiates the evolution of an asymmetric solitary vortex. Next, we describe the procedural details of the derivation of the asymmetric solitary vortex solution based on the two-dimensional GPE model. The theoretical results and possible extensions for the analysis are also discussed. Finally, we present the concluding remarks in the final section.

## Asymmetric solitary vortex and its evolutionary patterns based on the two-dimensional GPE model

### Generation of asymmetric solitary vortex state

We first consider the two-dimensional Bose-Einstein condensate system under the quasi-one dimensional setting in anisotropic harmonic

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trap. The harmonic trap take the form:  $V(x, y) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2)$  ( $\omega_y \gg \omega_x$ ). The quasi-one dimensional system is assumed to stay at first excited state along the  $x$  direction described by the following wave function:

$$\psi_0(x, y, t = t_{10}) = C_1 \exp\left[-\left(\frac{x^2}{2\sigma_{x0}^2} + \frac{y^2}{2\sigma_{y0}^2}\right)\right] \quad (1)$$

To generate the asymmetric vortex, we rotate the system within very short time so that the system achieve the angular speed  $\Omega = \Omega_0 = \frac{\hbar}{m}$ . The system's initial wave function changes to:

$$\begin{aligned} \psi'_0(x, y, t = t_{20}) &= C_2(x + iy)\exp\left[-\left(\frac{x^2}{2\sigma_{x0}^2} + \frac{y^2}{2\sigma_{y0}^2}\right)\right] \\ &= C_2 r \exp\left[-\left(\frac{x^2}{2\sigma_{x0}^2} + \frac{y^2}{2\sigma_{y0}^2}\right) + i\phi\right] \end{aligned} \quad (2)$$

The wave function (2) is in the format of asymmetric solitary vortex with asymmetric distribution in the  $x$  and  $y$  direction. Also, we switch the external trapping strength  $\omega_y$  in the  $y$  direction to  $\omega_x$  so that external trapping is isotropic starting from time  $t = t_{20}$ . We will investigate the evolution of the asymmetric solitary vortex starting from state (2). This is the topic for the next subsection.

*The solitary vortex solution of the two-dimensional GPE for the ensuing evolution of the asymmetric solitary vortex*

Starting from time  $t_{20} = 0$ , we study the evolution of the state described by (2) based on the following two-dimensional GPE,

$$i\hbar \frac{\partial \psi(x, y, t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \nabla_{\perp}^2 + \frac{1}{2}m\omega_x^2(x^2 + y^2) + g|\psi|^2\right] \psi(x, y, t) \quad (3)$$

The derivation of the analytical solitary vortex solution of Eq. (3) is based on the variational methodology by working on the action  $S = \int \mathcal{L} dr dt$  with the Lagrangian density taking the following format,

$$\mathcal{L} = \psi^* \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_{\perp}^2 - \frac{1}{2}m\omega_x^2(x^2 + y^2) - g|\psi|^2 \right] \psi \quad (4)$$

In order to identify the analytical solitary vortex solution for  $\psi(x, y, t)$ , we assume the following variational ansatz for the wave function,

$$\begin{aligned} \psi(x, y, t) &= A(t) \exp\left[-\left(\frac{x^2}{2\sigma_x^2(t)} + \frac{\delta(t)xy}{\sigma_x(t)\sigma_y(t)} + \frac{y^2}{2\sigma_y^2(t)}\right) + i\phi + i(\gamma_x(t)x^2 + \gamma'(t)xy + \gamma_y(t)y^2)\right] \\ &= A(t)(x + iy)\exp\left[-\left(\frac{x^2}{2\sigma_x^2(t)} + \frac{\delta(t)xy}{\sigma_x(t)\sigma_y(t)} + \frac{y^2}{2\sigma_y^2(t)}\right) + i(\gamma_x(t)x^2 + \gamma'(t)xy + \gamma_y(t)y^2)\right] \end{aligned} \quad (5)$$

which takes the form of the solitary vortex with vorticity number  $S = 1$  [In ]. The inherent asymmetry of the solitary vortex is incorporated in the formulation by redefining  $\sigma_x(t)$ ,  $\sigma_y(t)$  and  $\delta(t)$  as follows;

$$\sigma_x(t) = \sigma(t) \sqrt{1 + \cos\nu(t)} \quad (6a)$$

$$\sigma_y(t) = \sigma(t) \sqrt{1 - \cos\nu(t)} \quad (6b)$$

$$\delta(t) = \frac{1}{2} \cos\vartheta(t) \quad (6c)$$

where the system evolves from extreme asymmetric state starting from time  $t = 0$ ,  $\sigma_x(t = t_{20}) = \sigma_{x0}$ ,  $\sigma_y(t = t_{20}) = \sigma_{y0}$ ,  $\delta(t = t_{20}) = 0$ . It is not hard to see that the system's ensuing dynamic evolution are determined by the time dependent parametric functions  $\sigma(t)$ ,  $\nu(t)$  and  $\vartheta(t)$ . We will show immediately that the three parametric functions are not actually independent and two constraint formulae are reached showing that  $\nu(t)$  and  $\vartheta(t)$  are functions of  $\sigma(t)$ . The two constraint formulae were derived by plugging Eq. (5) into Eq. (3) and consider the imaginary portion [15] (proportional to  $i\psi$  or  $i(x + iy)E$ ,  $E$  is defined in the

ensuing steps) of Eq. (3) first, which require that all the coefficients of these imaginary terms are zero as

$$ix^2(x + iy)E \text{ term:} \quad \hbar \frac{\dot{\sigma}_x(t)}{\sigma_x^3(t)} - \frac{\hbar^2}{m} \left(2\frac{\gamma'_x(t)}{\sigma_x^2(t)} + \frac{\delta(t)\gamma'(t)}{\sigma_x(t)\sigma_y(t)}\right) = 0 \quad (7a)$$

$$iy^2(x + iy)E \text{ term:} \quad \hbar \frac{\dot{\sigma}_y(t)}{\sigma_y^3(t)} - \frac{\hbar^2}{m} \left(2\frac{\gamma'_y(t)}{\sigma_y^2(t)} + \frac{\delta(t)\gamma'(t)}{\sigma_x(t)\sigma_y(t)}\right) = 0 \quad (7b)$$

$$\begin{aligned} ixy(x + iy)E \text{ term:} \\ -\hbar \left(\frac{1}{\sigma_x(t)\sigma_y(t)} \frac{d\delta(t)}{dt} + \delta(t) \frac{d}{dt} \left(\frac{1}{\sigma_x\sigma_y}\right)\right) - \frac{\hbar^2}{m} \left(\frac{2\delta(t)}{\sigma_x\sigma_y} (\gamma_x + \gamma_y) + \gamma'(t) \left(\frac{1}{\sigma_x^2} + \frac{1}{\sigma_y^2}\right)\right) \\ = 0 \end{aligned} \quad (7c)$$

$$i(x + iy)E \text{ and } i(x - iy)E \text{ term:} \quad \left[\frac{1}{C(t)} \frac{dC(t)}{dt} + \frac{2\hbar}{m} (\gamma_x + \gamma_y)\right] i(x + iy) + (\gamma_x - \gamma_y - \frac{\delta(t)}{\sigma_x\sigma_y}) i(x - iy) = 0 \quad (7d)$$

$$(x - iy)E \text{ term:} \quad \frac{\hbar^2}{2m} \left[(2\gamma'(t) + \frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2})\right] (x - iy) = 0 \quad (7e)$$

where  $E = \exp\left[-\left(\frac{x^2}{2\sigma_x^2(t)} + \frac{\delta(t)xy}{\sigma_x(t)\sigma_y(t)} + \frac{y^2}{2\sigma_y^2(t)}\right) + i\phi + i(\gamma_x(t)x^2 + \gamma'(t)xy + \gamma_y(t)y^2)\right]$ , and  $A(t) = (1 - \cos^2\vartheta(t))^{3/4} (A_0 + B_0 \cos\nu(t) \sin\vartheta(t))^{-1/2} \sigma^{-2}(t) \sin^{-1/2}\nu(t)$  is the normalization factor with  $A_0 = 4\Gamma(1/2)\Gamma(2)$ , and  $B_0 = 4\Gamma^2(3/2)$ .

Equations  $\sigma_x^2 \times (7a) + \sigma_y^2 \times (7b)$  reaches

$$\frac{\hbar}{m} \frac{\gamma'(t)}{\sigma_x\sigma_y} (\sigma_x^2 + \sigma_y^2) = \frac{1}{\delta(t)} \left(\frac{\dot{\sigma}_x}{\sigma_x} + \frac{\dot{\sigma}_y}{\sigma_y}\right) - \frac{2\hbar}{\delta(t)m} (\gamma_x(t) + \gamma_y(t)) \quad (8)$$

and plugging Eq. (7c) into  $\sigma_x\sigma_y \times$  Eq. (8) gives,

$$\frac{d\delta(t)}{dt} - \left(\delta(t) - \frac{1}{\delta(t)}\right) \left(\frac{\dot{\sigma}_x}{\sigma_x} + \frac{\dot{\sigma}_y}{\sigma_y}\right) + \frac{2\hbar}{m} \left(\delta(t) - \frac{1}{\delta(t)}\right) (\gamma_x(t) + \gamma_y(t)) = 0 \quad (9)$$

Making the equation's imaginary part equal to zero also establishes that the coefficient formula of the 2nd asymmetric term of Eqs. (7d) and (7e) take zero value, this gives

$$\gamma'(t) = -\frac{1}{2} \left(\frac{1}{\sigma_x^2} - \frac{1}{\sigma_y^2}\right) \quad (10)$$

$$\delta(t) = \frac{1}{2} \cos\vartheta(t) = \sigma_x(t)\sigma_y(t)(\gamma_x(t) - \gamma_y(t)) \quad (11)$$

Combining Eqs. (7d) and (9) generates the 1st constraint formula

$$[B_0 \cos\nu(t) + G_0 \sin\nu(t)] \sin\vartheta(t) = A_0 \quad (12)$$

where  $G_0 \simeq A_0$  is the integral constant.

Combining Eq. (10) and (11), and utilizing Eq. (7a) and (7b) produces the second constraint formula

$$\begin{aligned} \frac{\cos\nu(t) \sin^2\nu(t)}{1 + \sin^2\nu(t)} \frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{\sin 2\nu(t) (\cos\nu(t) - 3)}{4(1 + \cos\nu(t))(1 + \sin^2\nu(t))} \dot{\nu}(t) \\ = \frac{\hbar \cos\vartheta(t)}{2m\sigma^2(t) \sin\nu(t)} \end{aligned} \quad (13)$$

The two derived constraint Eqs. (12) and (13) formulate  $\vartheta(t)$  and  $\nu(t)$  as functions of  $\sigma(t)$

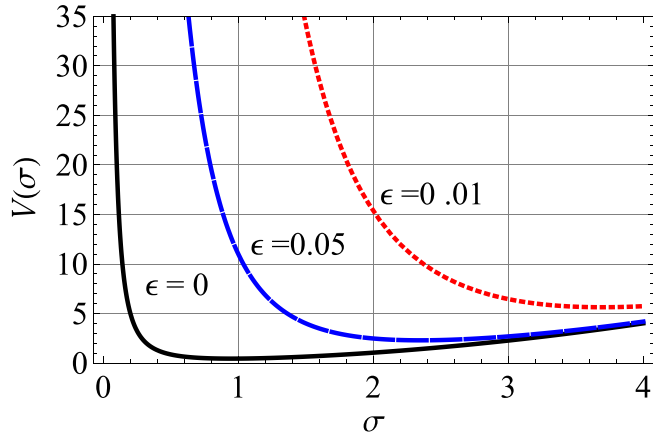
$$\nu(t) = \nu(\sigma(t)), \quad \vartheta(t) = \vartheta(\sigma(t)) \quad (14)$$

In combination with Eq. (7), we reach the following formulae:

$$\sigma_x(t) = \sigma_x(\sigma(t)) \quad (15a)$$

$$\sigma_y(t) = \sigma_y(\sigma(t)) \quad (15b)$$

$$\gamma_x(t) = \gamma_x(\dot{\sigma}(t), \sigma(t)) \quad (15c)$$



**Fig. 1.** Variation of  $V(\sigma)$  with  $\sigma$  (in units of  $\frac{20}{2}$ ) for nonlinear interaction constants with three different strength values:  $\epsilon = 0, 0.01, 0.05$  in units of  $\frac{4\pi\hbar^2 a_s}{m}$ , and  $a_s$  is the interaction s-wave scattering length.

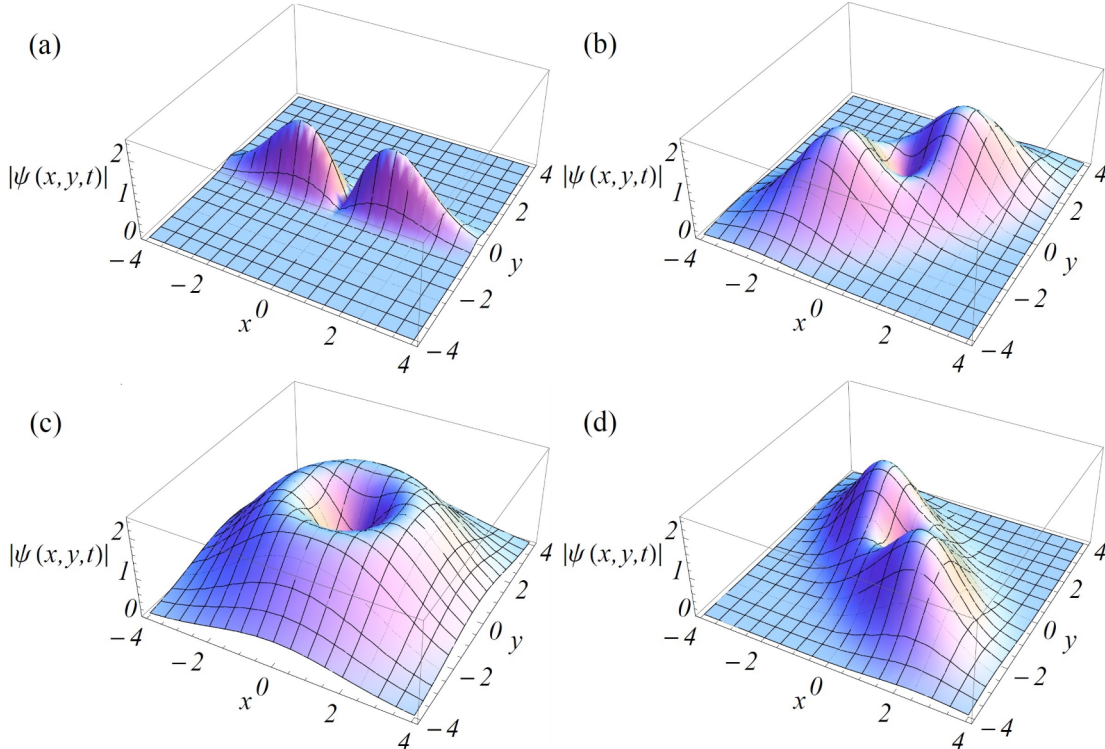
$$\gamma_y(t) = \gamma_y(\dot{\sigma}(t), \sigma(t)) \quad (15d)$$

$$\gamma'(t) = \gamma'(\sigma(t)) \quad (15e)$$

After plugging Eq. (14 and (15) into the Eq. (5), then integrating the Lagrangian density Eq. (4), over the spatial variables, we reach

$$\begin{aligned} \mathcal{L}(\dot{\sigma}, \sigma) &= \iint \mathcal{L} dx dy \\ &= \iint \left[ -\hbar(\dot{\gamma}_x(\dot{\sigma}, \sigma)x^2 + \dot{\gamma}_y(\dot{\sigma}, \sigma)y^2 + \dot{\gamma}'(\sigma)xy) - \frac{\hbar^2}{2m}(\gamma_x^2(\dot{\sigma}, \sigma)x^2 + \gamma_y^2(\dot{\sigma}, \sigma)y^2) \right. \\ &\quad \left. + \frac{\hbar^2}{2m} \left( \frac{x^2}{\sigma_x^2(\sigma)} + \frac{\cos\theta(\sigma)(x^2+y^2)}{2\sigma_x(\sigma)\sigma_y(\sigma)} + \frac{y^2}{\sigma^4(\sigma)} - \frac{2}{\sigma_x^2(\sigma)} - \frac{2}{\sigma_y^2(\sigma)} \right) \right. \\ &\quad \left. - \frac{m\omega_x^2}{2}(x^2 + y^2) - g|\psi|^2 \right] |\psi(x, y, t)|^2 dx dy \end{aligned} \quad (16a)$$

Working on the action  $S = \int \mathcal{L}(\dot{\sigma}, \sigma) dt$ , the Euler-Lagrangian equation



**Fig. 2.** Three dimensional plots of the solitary vortex evolution with snapshot images recorded at four timings (when  $t = 0, \cos(\varpi) \simeq \cos(\tau) \simeq 0$  holds): (a)  $t = 0$ , (b)  $t = T/4$ , (c)  $t = T/2$ , and (d)  $t = 3T/4$  ( $T$  is the evolution period of the vortex, both the horizontal and vertical axes are in units of  $\sigma_0/2$ ).

for  $\sigma(t) \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} - \frac{\partial \mathcal{L}}{\partial \sigma} = 0$  reads

$$m\ddot{\sigma} = -\frac{\partial V(\sigma)}{\partial \sigma} \quad (17)$$

We identify that the ensuing evolutionary dynamics of the system depends on the explicit formulation of  $V(\sigma)$ . The evolution of  $\sigma(t)$  (and also for  $\nu(t)$  and  $\vartheta(t)$ ) will be determined by Eq. (17) with the boundary condition  $\sigma_x(t = t_{20}) = \sigma_{x0}$ ,  $\sigma_y(t = t_{20}) = \sigma_{y0}$ ,  $\delta(t = t_{20}) = 0$ .

We proceed by making a prior analysis of the analytical form of  $V(\sigma)$ . Assume that  $\sigma_0$  is the local minimum of  $V(\sigma)$  and oscillation amplitude according to Eq. (17) is small, the resultant Lagrangian density can be expressed as,

$$\begin{aligned} \mathcal{L}(\dot{\sigma}(t), \sigma(t)) \\ = \pi C_0^2 m (\dot{\sigma}(t))^2 - \frac{\hbar^2 C_0^2 \pi}{m} \frac{1}{\sigma^2(t)} + C_0^2 \left[ \frac{g\pi}{8\sigma^2(t)} - (\pi m \omega_x^2 + \epsilon) \sigma^2(t) \right] \end{aligned} \quad (18)$$

where  $C_0$  is the normalization constant of the wave function,  $\epsilon$  is coefficient of the first-order expansion of  $\sigma(t)$  arising from the parametric functions  $\nu(t)$  and  $\vartheta(t)$ . The  $V(\sigma)$  in the corresponding Euler-Lagrangian of Eq. (17) takes the following form

$$V(\sigma) = \frac{\hbar^2}{m} \frac{1}{\sigma^2(t)} + (m\omega_x^2 + \epsilon)\sigma^2(t) - \frac{g}{8\sigma^2(t)} \quad (19)$$

The existence of  $\sigma_0$  with the local minimum of  $V(\sigma)$  for small  $g$  is shown in Fig. 1. For typical BEC system where the inter-particle nonlinear interaction and the system deviation from isotropic parametric setting is very small ( $g \simeq 0$ ,  $\epsilon \simeq 0$ ), by plugging Eq. (19) into Eq. (17), we can get the approximate analytical solution of  $\sigma(t)$  as

$$\sigma^2(t) = P_1 \sin \omega t + P_2 \quad (20)$$

where

$$P_1 = \sqrt{\left(\frac{\sigma_0^2}{2} + \frac{\hbar^2}{8m^2\omega_x^2\sigma_0^2}\right)^2 - \frac{2\hbar^2}{8m^2\omega_x^2}},$$

$$P_2 = \frac{\sigma_0^2}{2} + \frac{\hbar^2}{4m^2\omega_x^2\sigma_0^2},$$

$$\omega = 2\omega_x,$$

which show the oscillatory behavior of  $\sigma(t)$  with period  $T = \frac{2\pi}{\omega}$  around  $\sigma_0$ . The evolutionary behaviors for  $\nu(t)$  and  $\vartheta(t)$ , which are determined by the Eqs. (12) and (13), are not generally analytically solvable. By numerically solving Eqs. (12) and (13) for typical small  $g = 0.02$  (in units of  $\frac{4\pi\hbar^2 a_0}{m}$ , and  $a_0$  is the s-wave scattering length). We identify the periodic variation of  $\nu(t)$  and  $\vartheta(t)$  in this situation and the evolutionary patterns are shown in Fig. 2 at four timing locations ( $t_i = \frac{i\pi}{2\omega}$ ,  $i = 0, 1, 2, 3$ ). We can see the solitary vortex takes the circular symmetric shape at time  $t = t_2$ . We can see that solitary vortex oscillates between the extreme asymmetric shape (quasi-one dimensional soliton pair) and the circular symmetric case periodically.

The periodic feature of the asymmetric vortex arises from the oscillation of  $\sigma(t)$  around the local minimum  $\sigma_0$  of  $V(\sigma)$ . For asymmetric vortex with higher vorticity  $N \geq 2$ , the wave function is proportional to  $r^N \exp(iN\phi) = (x + iy)^N$  with other components in ansatz (5) unchanged. In this case, we will also reach the dynamical evolution equation for  $\sigma$  that is similar to Eq. (17) with potential function  $V(\sigma)$ . If local minimum  $\sigma_{00}$  of  $V(\sigma)$  exists and  $\sigma$  oscillates around  $\sigma_{00}$ , we anticipate the occurrence of asymmetric vortex with periodic evolution in this scenario. We leave these as extended work for future investigation.

The theoretical results derived in this work can be used to guide the experimental observation of the dynamical vortex modes for two-dimensional Bose-Einstein condensate with variable harmonic trapping strength, specifically the evolution of the dynamical vortex mode from quasi-one dimensional setting to the isotropic two-dimensional setting via the adjustment of harmonic strength  $\omega_y$  from  $\omega_y \gg \omega_x$  to the setting  $\omega_y = \omega_x$ .

## Conclusion

We studied the evolutionary pattern of an asymmetric vortex in a two-dimensional BEC; the vortex was initiated from a quasi-one dimensional state via external harmonic trapping potential modulation. Based on the two-dimensional GPE model and variational method, we analytically derived the solution of the asymmetric solitary vortex. Furthermore, we established that the asymmetric vortex exhibits

periodic features and oscillates between and extreme asymmetric and circular symmetric states. The theoretical results derived here can be used as guides for corresponding experimental investigations of the generation and evolution of asymmetric solitary vortices under similar physical settings.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at <https://doi.org/10.1016/j.rinp.2019.102716>.

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