HOMOGENIZATION OF DEGENERATE CROSS-DIFFUSION SYSTEMS

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Abstract. Two-scale homogenization limits of parabolic cross-diffusion systems in a heterogeneous medium with no-flux boundary conditions are proved. The heterogeneity of the medium is reflected in the diffusion coefficients or by the perforated domain. The diffusion matrix is of degenerate type and may be neither symmetric nor positive semi-definite, but the diffusion system is assumed to satisfy an entropy structure. Uniform estimates are derived from the entropy production inequality. New estimates on the equicontinuity with respect to the time variable ensure the strong convergence of a sequence of solutions to the microscopic problems defined in perforated domains.

1. Introduction

Multicomponent systems are ubiquitous in nature; examples are as various as gas mixtures, bacterial colonies, lithium-ion battery cells, and animal crowds. On the diffusive level, these systems can be described by cross-diffusion equations taking into account multicomponent diffusion and reaction [14]. When the mass transport occurs in a domain with periodic microstructure or in a porous medium, macroscopic models can be derived from the microscopic description of the processes by homogenization techniques. In this paper, we consider cross-diffusion systems defined in a heterogeneous medium, where the heterogeneity is reflected in spatially periodic diffusion coefficients or by the perforated domain. The corresponding macroscopic equations are derived by combining, for the first time, two-scale convergence techniques and entropy methods.

The problem of reducing a heterogenous material to a homogenous one has been investigated in the literature since many decades. The research started in the 19th century by Maxwell and Rayleigh and was developed later by engineers leading to asymptotic expansion techniques. Homogenization became a topic in mathematics in the 1960s and 1970s. For instance, the Γ-convergence was introduced by De Giorgi [9] with the aim to describe the asymptotic behavior of functionals and their minimizers. The G-convergence of Spagnola [25] and its generalization to nonsymmetric problems, the H-convergence of Tartar and Murat [19], are related to the convergence of the Green kernel of the corresponding elliptic operator. The two-scale convergence [2, 20] combines formal asymptotic expansion

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and test function methods. Nguetseng introduced an extension of two-scale convergence to almost periodic homogenization, called Σ-convergence [21, 22]. Another extension concerns the two-scale convergence in spaces of differentiable functions [26], which is important in nonlinear problems [17]. A classical reference for the homogenization theory of periodic structures is [4].

In spite of the huge amount of literature on homogenization problems, there are not many studies on the homogenization of nonlinear parabolic systems. Most of the results concern weakly coupled equations like periodic homogenization of reaction-diffusion systems or of thermal-diffusion equations in periodically perforated domains [3, 5, 23]. Particular cross-diffusion systems – of triangular type – were investigated in [16]. However, up to our knowledge, there are no results on more general cross-diffusion systems.

In this paper, we investigate strongly coupled parabolic cross-diffusion systems with a formal gradient-flow or entropy structure by combining two-scale convergence and the boundedness-by-entropy method [13]. The difficulty is the handling of the degenerate structure of the equations. We investigate two classes of degeneracies: a local one of porous-medium type and a nonlocal one; see Section 2 for details.

The paper is organized as follows. In Section 2, the microscopic models are formulated and the main results are stated. The main theorems are proved in Sections 3 and 4. For the convenience of the reader, the definition and some properties of two-scale convergence are recalled in Appendix A. The technical Lemma 9 is proved in Appendix B. Finally, two cross-diffusion systems from applications which satisfy our assumptions are presented in Appendix C.

2. Formulation of the microscopic models and main results

We investigate two types of homogenization problems. The first homogenization limit is performed in cross-diffusion systems with spatially periodic coefficients,

$$\partial_t u_i^\varepsilon - \operatorname{div}\left(\sum_{j=1}^n P\left(\frac{x}{\varepsilon}\right) a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\right) = f_i(u^\varepsilon) \quad \text{in } \Omega,\ t>0,\ i=1,\ldots,n, \tag{1}$$

in a bounded domain $x \in \Omega \subset \mathbb{R}^d$ ($d \geq 1$), together with no-flux boundary and initial conditions

$$\sum_{j=1}^n P\left(\frac{x}{\varepsilon}\right) a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega,\ t>0, \quad u_i^\varepsilon(0) = u_i^0 \ \text{ in } \Omega. \tag{2}$$

Here, $u^\varepsilon = (u_1^\varepsilon, \ldots, u_n^\varepsilon)$ is the vector of concentrations or mass fractions of the species depending on the spatial variable $x \in \Omega$ and on time $t > 0$, and $\varepsilon > 0$ is a characteristic length scale. Furthermore, $P(y) = \operatorname{diag}(P_1(y), \ldots, P_d(y))$ is a diagonal matrix, where the periodic functions $P_j : Y \to \mathbb{R}$ describe the heterogeneity of the medium and $Y = (0, b_1) \times \cdots \times (0, b_d)$ with $b_i > 0$ is the "periodicity cell", $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ are the density-dependent diffusion coefficients, $f_i : \mathbb{R}^n \to \mathbb{R}$ models the reactions, and $\nu(x)$ is the exterior unit normal vector to $\partial\Omega$. The theory works also for reaction terms depending on x/ε, but we do not consider this dependence to simplify the presentation. The divergence operator is understood in the following sense:

$$\operatorname{div}\left(\sum_{j=1}^n P\left(\frac{x}{\varepsilon}\right)a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\right) = \sum_{k=1}^d \frac{\partial}{\partial x_k}\left(\sum_{j=1}^n P_k\left(\frac{x}{\varepsilon}\right)a_{ij}(u^\varepsilon)\frac{\partial u_j^\varepsilon}{\partial x_k}\right).$$

The second homogenization limit is shown in cross-diffusion systems solved in a perforated domain. A perforated domain Ω^ε is obtained by removing a subset Ω_0^ε from Ω, which gives $\Omega^\varepsilon = \Omega\backslash\Omega_0^\varepsilon$. The set Ω_0^ε may consist of periodically distributed holes in the original domain. More precisely, we introduce the reference set $Y \subset \mathbb{R}^d$ and the set $Y_0 \subset Y$ (the reference hole) with Lipschitz boundary $\Gamma = \partial Y_0$, satisfying $\overline{Y}_0 \subset Y$, and $Y_1 = Y \setminus \overline{Y}_0$. Then Ω_0^ε and the corresponding boundary are defined by

$$\Omega_0^\varepsilon = \bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(Y_0+\xi), \quad \Gamma^\varepsilon = \bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\Gamma+\xi),$$

where $\Xi^\varepsilon = \{\xi \in \mathbb{R}^d : \varepsilon(\overline{Y}+\xi) \subset \Omega\}$, and the microscopic model in the perforated domain Ω^ε reads as

$$\partial_t u_i^\varepsilon - \operatorname{div}\left(\sum_{j=1}^n a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\right) = f_i(u^\varepsilon) \quad \text{in } \Omega^\varepsilon,\ t>0,\ i=1,\ldots,n, \tag{3}$$

together with the boundary and initial conditions

$$\sum_{j=1}^n a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon \cdot \nu = 0 \text{ on } \partial\Omega \cup \Gamma^\varepsilon,\ t>0, \quad u_i^\varepsilon(0) = u_i^0 \text{ in } \Omega^\varepsilon. \tag{4}$$

A key feature of (1) and (3) is that the diffusion matrix $A(u) = (a_{ij}(u))$ is generally neither symmetric nor positive semi-definite; see [13, 14] for examples from applications in physics and biology. Two examples are presented in Appendix C. To ensure the global existence of weak solutions of problem (1)-(2) or (3)-(4), we assume that the diffusion system has an *entropy structure*, i.e., there exists a convex function $h \in C^2(\mathcal{G};\mathbb{R})$ with $\mathcal{G} \subset \mathbb{R}^n$ such that the matrix product $h''(u)A(u)$, where $h''(u)$ denotes the Hessian of h, is positive semi-definite. Then the so-called entropy $H(u) = \int_\Omega h(u)dx$ is a Lyapunov functional if $f_i \equiv 0$:

$$\frac{dH}{dt} = -\int_\Omega \nabla u : h''(u)A(u)\nabla u dx \le 0, \tag{5}$$

where ":" denotes the Frobenius matrix product. Gradient estimates, needed for the analysis, are obtained by making a stronger condition on $h''(u)A(u)$ than just positive semi-definiteness. Since strict positive definiteness cannot be expected from the applications, we assume that $h''(u)A(u)$ is "degenerate" positive definite. We investigate two types of degeneracies, a local and a nonlocal one.

Locally degeneracy structure. We assume that $h''(u)A(u) \geq \alpha \operatorname{diag}((u_i)^{2s_i})_{i=1}^n$ in the sense of symmetric matrices and with $\alpha > 0$, $s_i > -1$. Then (5) becomes (still with $f_i \equiv 0$)

$$\frac{dH}{dt} + \alpha \sum_{i=1}^n \int_\Omega u_i^{2s_i} |\nabla u_i|^2 dx \leq 0,$$

leading to L^2-estimates for $\nabla u_i^{s_i+1}$. Gradient estimates of such a type are well known in the analysis of the porous-medium equation. The analysis requires a further assumption: The domain $\mathcal{G}$ is bounded and the derivative $h' : \mathcal{G} \to \mathbb{R}^n$ is invertible. Examples are Boltzmann-type entropies containing expressions like $u_i \log u_i$. As shown in [13], this leads to $u_i(x,t) \in \overline{\mathcal{G}}$ for $x \in \Omega$, $t > 0$, and hence to L^∞-estimates for u_i (without the use of a maximum principle). Using a nonlinear Aubin-Lions lemma, the global existence of bounded weak solutions was proved in [13] under the condition that the domain $\mathcal{G}$ is bounded. Even when $\mathcal{G}$ is not bounded, the entropy method can be applied, giving global weak solutions (but possibly not bounded) [14, Section 4.5].

Nonlocally degeneracy structure. As an example of a nonlocally degenerate structure, we consider cross-diffusion systems with coefficients

$$a_{ij}(u) = D_i(\delta_{ij} u_{n+1} + u_i), \quad i,j = 1,\ldots,n, \tag{6}$$

where δ_{ij} is the Kronecker delta symbol, $u_{n+1} = 1 - \sum_{i=1}^n u_i$, and $D_i > 0$ for $i = 1,\ldots,n$ are diffusion coefficients. Such models are used for the transport of ions through biological channels, where u_i are the ion volume fractions and u_{n+1} is the solvent concentration. The entropy density is given by

$$h(u) = \sum_{i=1}^{n+1} u_i(\log u_i - 1) \quad \text{for } u = (u_1,\ldots,u_n) \in \mathcal{G}, \tag{7}$$

where $\mathcal{G} = \{(u_1,\ldots,u_n) \in \mathbb{R}^n : u_1,\ldots,u_n, u_{n+1} > 0\}$. Then $h''(u)A(u)$ is positive semi-definite and if $f_i \equiv 0$, it holds that (see [14, Section 4.6] and [28, Theorem 1])

$$\frac{dH}{dt} + \sum_{i=1}^n D_i \int_\Omega \big(u_{n+1}|\nabla u_i^{1/2}|^2 + |\nabla u_{n+1}|^2\big) dx \leq 0.$$

This gives an L^2-estimate for ∇u_{n+1}, but generally not for ∇u_i because of the factor u_{n+1} which may vanish. We call this a nonlocal degeneracy since the degeneracy u_{n+1} depends on u_i in a nonlocal way through the other components u_j for $j \neq i$.

We note that our results can be extended to more general coefficients of the form

$$a_{ij}(u) = sD_i u_i^{s-1} q(u_{n+1})\delta_{ij} + D_i u_i^s q'(u_{n+1}), \quad i,j = 1,\ldots,n,$$

where $s = 1$ or $s = 2$ and $q \in C^2([0,1])$ is a positive and nondecreasing function satisfying $q(0) = 0$ and $q'(\xi) \geq \gamma q(\xi)$ for some $\gamma > 0$ and all $\xi \in (0,1)$.

To prove the convergence of solutions of the microscopic problems to a solution of the corresponding macroscopic equations, we derive some a priori estimates for (u_i^ε) independent of ε. Compared to [13], the main novelty is the derivation of equicontinuous estimates for (u_i^ε) with respect to the time variable. This will allow us to obtain compactness properties for a sequence of solutions of the microscopic problem defined in a perforated domain. Notice that estimates for a discrete time derivative of (u^ε) in $L^2(0,T;H^1(\Omega^\varepsilon)')$ do not ensure a priori estimates uniform in ε for the discrete time derivative for an extension of u^ε from Ω^ε into Ω. Another important step of the analysis presented here is the proof of an existence result for the degenerate unit-cell problem, which determines the macroscopic diffusion matrix. Here, we apply a regularization technique and use the structure and assumptions on the matrix $A(u)$.

For the first main result on locally degenerate systems, we impose the following assumptions:

A1. Entropy: There exists a convex function $h \in C^2(\mathcal{G};\mathbb{R})$ such that $h' : \mathcal{G} \to \mathbb{R}^n$ is invertible, where $\mathcal{G} \subset (0,1)^n$ is open and $n \geq 1$.

A2. "Degenerate" positive definiteness: There exist numbers $s_i > -1$ $(i = 1,\ldots,n)$ and $\alpha > 0$ such that for $z = (z_1,\ldots,z_n) \in \mathbb{R}^n$, $u = (u_1,\ldots,u_n) \in \mathcal{G}$,

$$z^\top h''(u)A(u)z \geq \alpha \sum_{i=1}^n |u_i|^{2s_i} z_i^2.$$

A3. Diffusion coefficients: Let $A(u) = (a_{ij}(u)) \in C^0(\mathcal{G};\mathbb{R}^{n\times n})$. There exists a constant $C_A > 0$ such that for all $u \in \mathcal{G}$ and for those $j = 1,\ldots,n$ such that $s_j > 0$, it holds that

$$|a_{ij}(u)| \leq C_A u_j^{s_j} \quad \text{for } i = 1,\ldots,n.$$

Furthermore, $P \in L^\infty(Y;\mathbb{R}^{d\times d})$ with $P(y) = \mathrm{diag}(P_1(y),\ldots,P_d(y))$ satisfies $P_i(y) \geq d_0 > 0$ in Y for some $d_0 > 0$ and for all $i = 1,\ldots,d$.

A4. Reaction terms: $f \in C^0(\overline{\mathcal{G}};\mathbb{R}^n)$ and there exists $C_f > 0$ such that $f(u)\cdot h'(u) \leq C_f(1+h(u))$ for $u \in \mathcal{G}$.

A5. Initial datum: $u^0 : \Omega \to \mathbb{R}^n$ is measurable and $u^0(x) \in \mathcal{G}$ for $x \in \Omega$.

A6. Bound for the matrix $h''(u)A(u)$: There exists a constant $C > 0$ such that for all $u \in \mathcal{G}$ and $i,j = 1,\ldots,n$,

$$(h''(u)A(u))_{ij} \leq C u_i^{s_i} u_j^{s_j}.$$

Let us discuss these assumptions. As mentioned above, Assumption A1 guarantees the L^∞ boundedness of the solutions. Assumption A2 is needed for the compactness argument. For the existence analysis, it can be weakened to continuous functions instead of power-law functions [18], but the convergence $\varepsilon \to 0$ is more delicate. The growth estimate for $a_{ij}(u)$ in Assumption A3 is crucial for the proof of the equicontinuity property with respect to the time variable. The growth condition on f_i in Assumption A4 allows us to handle the reaction terms. The latter condition generally rules out quadratic growth of the concentrations; we refer to [11] for reaction-diffusion systems with diagonal diffusion matrices but quadratic reaction terms. Assumption A5 guarantees that the initial datum is bounded; it can be relaxed to $u^0(x) \in \overline{\mathcal{G}}$. Finally, Assumption A6 is a technical condition to ensure the solvability of the unit-cell problems. In Appendix C, we give two examples from applications, for which the assumptions are satisfied.

To simplify the presentation, we introduce some notation:

$$P_k^\varepsilon(x) = P_k(x/\varepsilon) \text{ for } x \in \Omega,\ k = 1,\ldots,d, \quad \Omega_T = \Omega\times(0,T), \quad \Omega_T^\varepsilon = \Omega^\varepsilon\times(0,T).$$

Definition 1. *A weak solution of problem* (1)-(2) *is a function* $u^\varepsilon \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^n))$ *with* $(u_i^\varepsilon)^{s_i+1} \in L^2(0,T;H^1(\Omega))$ *and* $\partial_t u_i^\varepsilon \in L^2(0,T;H^1(\Omega)')$ *for* $i = 1,\ldots,n$, *satisfying*

$$\int_0^T \sum_{i=1}^n \langle \partial_t u_i^\varepsilon, \varphi_i\rangle dt + \int_0^T\int_\Omega \bigg(\sum_{i,j=1}^n P^\varepsilon(x) a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\cdot\nabla\varphi_i - \sum_{i=1}^n f_i(u^\varepsilon)\varphi_i\bigg)dxdt = 0,$$

for all $\varphi \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$, *and the initial conditions are satisfied in the* L^2 *sense.*

A weak solution of problem (3)-(4) *is defined in a similar way by replacing* Ω *by* Ω^ε.

Here, $\langle\psi,\varphi\rangle$ denotes the dual product between $\psi \in H^1(\Omega)'$ and $\varphi \in H^1(\Omega)$ and the expression $P^\varepsilon\nabla u_j^\varepsilon\cdot\nabla\varphi_i$ is the sum $\sum_{k=1}^d P_k^\varepsilon \partial_{x_k} u_j^\varepsilon \partial_{x_k}\varphi_i$.

Theorem 1 (Homogenization limit for problems with local degeneracy)**.** *Let Assumptions A1-A6 hold.*

(i) *Let* u^ε *be a weak solution of the microscopic system* (1)-(2). *Then there exists a subsequence of* (u^ε), *which is not relabeled, such that* $u^\varepsilon \to u$ *strongly in* $L^p(\Omega_T;\mathbb{R}^n)$ *for all* $p < \infty$ *as* $\varepsilon \to 0$, *and the limit function* $u \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^n))$, *with* $u_i^{s_i+1} \in L^2(0,T;H^1(\Omega))$ *and* $\partial_t u_i \in L^2(0,T;H^1(\Omega)')$ *for* $i = 1,\ldots,n$, *solves the macroscopic system*

$$\begin{aligned} &\partial_t u_i - \sum_{k,m=1}^d\sum_{\ell=1}^n \frac{\partial}{\partial x_m}\bigg(B_{mk}^{i\ell}(u)\frac{\partial u_\ell}{\partial x_k}\bigg) = f_i(u) \quad \text{in } \Omega,\ t>0,\ i=1,\ldots,n, \\ &\sum_{k,m=1}^d\sum_{\ell=1}^n \nu_m B_{mk}^{i\ell}(u)\frac{\partial u_\ell}{\partial x_k} = 0 \text{ on } \partial\Omega,\ t>0, \quad u_i(0) = u_i^0 \text{ in } \Omega, \end{aligned} \tag{8}$$

where $(B_{mk}^{i\ell}(u))$ *is the homogenized diffusion matrix defined in* (28).

(ii) *Let* u^ε *be a weak solution of the microscopic system* (3)-(4). *Then, up to a subsequence and by identifying* u^ε *with its extension from* Ω^ε *into* Ω, $u^\varepsilon \to u$ *strongly in* $L^p(\Omega_T;\mathbb{R}^n)$ *for* $p<\infty$, *where* u, *with* $u_i^{s_i+1} \in L^2(0,T;H^1(\Omega))$ *and* $\partial_t u_i \in L^2(0,T;H^1(\Omega)')$ *for* $i=1,\ldots,n$, *is a solution of* (8) *with the macroscopic diffusion matrix* $(B_{mk}^{i\ell}(u))$ *defined in* (30).

For nonlocally degenerate systems (1) or (3) with diffusion coefficients (6), the weak solution is defined in a slightly different way than usually, since the regularity $u_i^\varepsilon \in L^2(0,T;H^1(\Omega))$ may not hold. We recall the definition from [13].

Definition 2. *A weak solution of* (1)-(2) *with diffusion coefficients* (6) *are functions* $u_1^\varepsilon,\ldots,u_n^\varepsilon$ *and* $u_{n+1}^\varepsilon = 1-\sum_{i=1}^n u_i^\varepsilon$ *satisfying* $u_i^\varepsilon \ge 0$, $u_{n+1}^\varepsilon \ge 0$ *in* Ω_T, $u_i^\varepsilon \in L^\infty(0,T;L^\infty(\Omega))$, $(u_{n+1}^\varepsilon)^{1/2}$, $(u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon \in L^2(0,T;H^1(\Omega))$, $\partial_t u_i^\varepsilon \in L^2(0,T;H^1(\Omega)')$ *for* $i=1,\ldots,n$, *and*

$$\begin{aligned} \int_0^T\sum_{i=1}^n\langle\partial_t u_i,\varphi_i\rangle dt + &\int_0^T\int_\Omega\sum_{i=1}^n P^\varepsilon(x)D_i(u_{n+1}^\varepsilon)^{1/2} \\ &\times\Big(\nabla\big((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon\big) - 3u_i^\varepsilon\nabla(u_{n+1}^\varepsilon)^{1/2}\Big)\cdot\nabla\varphi_i dxdt = 0 \end{aligned} \tag{9}$$

for all $\varphi \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$, *and the initial conditions are satisfied in the* $H^1(\Omega)'$ *sense.*

A weak solution of problem (3)-(4) *with diffusion coefficients* (6) *is defined analogously by replacing* Ω *by* Ω^ε.

Theorem 2 (Homogenization limit for problems with nonlocal degeneracy)**.** *Let Assumptions A1 and A5 hold.*

(i) *A subsequence* (u^ε) *of solutions of the microscopic problem* (1)-(2), *with the matrix* A *defined in* (6), *converges to a solution* $u \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^n))$, *with* $u_{n+1}^{1/2}$, $u_{n+1}^{1/2}u_i \in L^2(0,T;H^1(\Omega))$, $\partial_t u_i \in L^2(0,T;H^1(\Omega)')$ *for* $i=1,\ldots,n$, *of the macroscopic equations*

$$\begin{aligned} &\partial_t u - \operatorname{div}\big(D_{\rm hom}A(u)\nabla u\big) = 0 \quad && \text{in } \Omega,\ t>0, \\ &D_{\rm hom}A(u)\nabla u\cdot\nu = 0 && \text{on } \partial\Omega,\ t>0, \quad u(0)=u_0 \text{ in } \Omega, \end{aligned} \tag{10}$$

where the macroscopic matrix $D_{\rm hom}$ *is defined in* (46).

(ii) *In the case of the microscopic problem* (3)-(4), *we obtain the same macroscopic equations as in* (10) *with a different macroscopic diffusion matrix given by* (47).

3. Proof of Theorem 1

For the proof the theorem, we show some a priori estimates uniform in ε for solutions of the microscopic problems (1)-(2) and (3)-(4). We suppose throughout the section that Assumptions A1-A6 hold. First, we recall the following elementary inequalities.

Lemma 3 (Hölder-type inequalities)**.** *Let* a, $b \geq 0$ *and* $p \geq 1$. *Then*

$$|a-b|^p \leq |a^p - b^p| \leq p(a^{p-1}+b^{p-1})|a-b|.$$

The a priori estimates for problem (1)-(2) are as follows.

Lemma 4 (A priori estimates)**.** *For any* $\varepsilon > 0$, *there exists a bounded weak solution* u^ε *of problem* (1)-(2) *such that* $u^\varepsilon(x,t) \in \overline{\mathcal{G}}$ *for* $x \in \Omega$, $t>0$ *and*

$$\|(u_i^\varepsilon)^{s_i+1}\|_{L^2(0,T;H^1(\Omega))} \leq C \qquad \text{for } i=1,\ldots,n, \tag{11}$$

$$\|u_i^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C \qquad \text{for } -1<s_i\leq 0, \tag{12}$$

$$\|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon\|_{L^2((0,T-\tau)\times\Omega)} \leq C\tau^{1/4} \qquad \text{for } -1<s_i\leq 0, \tag{13}$$

$$\|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon\|_{L^{2+s_i}((0,T-\tau)\times\Omega)} \leq C\tau^{1/(4+2s_i)} \qquad \text{for } s_i>0, \tag{14}$$

where $\vartheta_\tau u_i^\varepsilon(x,t) = u_i^\varepsilon(x,t+\tau)$ *for* $x\in\Omega$ *and* $t\in(0,T-\tau)$, *for* $\tau\in(0,T)$, *and the constant* $C>0$ *is independent of* ε.

Proof. Theorem 2 in [13] shows that there exists a bounded weak solution u^ε to (1)-(2) satisfying $u^\varepsilon(x,t)\in\overline{\mathcal{G}}$ for $x\in\Omega$, $t>0$. Estimates (11)-(12) are a consequence of the entropy production inequality, which is obtained by taking an approximation of $(\partial h/\partial u_i)(u^\varepsilon)$ as a test function in (1). Notice that the dependence on $x\in\Omega$ is via multiplication by a diagonal matrix $P^\varepsilon(x)$, so the entropy $h(u)$ does not depend explicitly on x. Since the entropy h is generally undefined on $\partial\mathcal{G}$, the equations in [13] have been approximated, and the existence of a family of approximate solutions satisfying (11) has been proved.

Then the convergence of the approximate solutions in appropriate spaces for vanishing approximation parameters directly leads to (11). Thanks to the positive lower bound for P (uniform in ε), we see that estimate (11) is independent of ε.

Estimate (12) for $-1 < s_i \leq 0$ follows from (11) and the boundedness of u^ε:

$$\|\nabla u_i^\varepsilon\|_{L^2(\Omega_T)} = \frac{1}{s_i+1}\|u_i^\varepsilon\|_{L^\infty(\Omega_T)}^{-s_i}\|\nabla(u_i^\varepsilon)^{s_i+1}\|_{L^2(\Omega_T)} \leq C,$$

for $i = 1,\ldots,n$, where $C > 0$ is here and in the following a generic constant independent of ε. The boundedness of (u^ε) (uniform in ε) is ensured by the assumptions on h, see Assumption A1.

It remains to show (13) and (14). For this, we use the (admissible) test function $\phi = (\phi_1,\ldots,\phi_n)$ with

$$\phi_i(x,t) = \int_{t-\tau}^t (\vartheta_\tau u_i^\varepsilon(x,\sigma) - u_i^\varepsilon(x,\sigma))\kappa(\sigma)d\sigma \qquad \text{if } s_i \leq 0,$$

$$\phi_i(x,t) = \int_{t-\tau}^t \big((\vartheta_\tau u_i^\varepsilon(x,\sigma))^{s_i+1} - (u_i^\varepsilon(x,\sigma))^{s_i+1}\big)\kappa(\sigma)d\sigma \qquad \text{if } s_i > 0,$$

where $\tau \in (0,T)$, $i = 1,\ldots,n$, $\kappa(\sigma) = 1$ for $\sigma \in (0, T-\tau)$ and $\kappa(\sigma) = 0$ for $\sigma \in [-\tau, 0] \cup [T-\tau, T]$. This gives

$$0 = \int_0^T \sum_{i=1}^n \langle \partial_t u_i^\varepsilon, \phi_i\rangle dt + \int_0^T\int_\Omega \sum_{i,j=1}^n P^\varepsilon(x) a_{ij}(u^\varepsilon)\nabla u_j^\varepsilon\cdot\nabla\phi_i dxdt$$
$$- \int_0^T\int_\Omega\sum_{i=1}^n f_i(u^\varepsilon)\phi_i dxdt =: I_1 + I_2 + I_3.$$

We integrate by parts in the first integral, taking into account that $\phi_i(0) = \phi_i(T) = 0$. Then, for all $i = 1,\ldots,n$ such that $s_i \leq 0$,

$$\begin{aligned}\int_0^T \langle\partial_t u_i^\varepsilon, \phi_i\rangle dt &= -\int_0^T\int_\Omega u_i^\varepsilon \partial_t\phi_i dxdt \\ &= -\int_0^{T-\tau}\int_\Omega u_i^\varepsilon(\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)dxdt + \int_\tau^T\int_\Omega u_i^\varepsilon(u_i^\varepsilon - \vartheta_{-\tau}u_i^\varepsilon)dxdt \\ &= -\int_0^{T-\tau}\int_\Omega u_i^\varepsilon(\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)dxdt + \int_0^{T-\tau}\int_\Omega \vartheta_\tau u_i^\varepsilon(\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)dxdt \\ &= \int_0^{T-\tau}\int_\Omega(\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)^2 dxdt.\end{aligned}$$

In a similar way, for those $i = 1,\ldots,n$ such that $s_i > 0$,

$$\int_0^T\langle\partial_t u_i^\varepsilon,\phi_i\rangle dt = \int_0^{T-\tau}\int_\Omega(\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)\big((\vartheta_\tau u_i^\varepsilon)^{s_i+1} - (u_i^\varepsilon)^{s_i+1}\big)dxdt.$$

Lemma 3 with $p = s_i + 1$ gives

$$|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon|^{s_i+1} \leq \big|(\vartheta_\tau u_i^\varepsilon)^{s_i+1} - (u_i^\varepsilon)^{s_i+1}\big|.$$

Thus, still in the case $s_i > 0$,

$$\int_0^T \langle \partial_t u_i^\varepsilon, \phi_i \rangle dt \geq \int_0^{T-\tau} \int_\Omega (\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)^{s_i+2} dx dt.$$

We conclude that

$$I_1 \geq \int_0^{T-\tau} \int_\Omega \bigg(\sum_{i=1,\, s_i>0}^n (\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)^{s_i+2} + \sum_{i=1,\, s_i \leq 0}^n (\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon)^2 \bigg) dx dt.$$

For the second integral I_2, we use the relation

$$\int_0^T w(t) \int_{t-\tau}^t v(\sigma)\, d\sigma dt = \int_0^{T-\tau} \int_t^{t+\tau} w(\sigma) d\sigma\, v(t) dt,$$

where $v(t) = 0$ for $t \in [-\tau, 0] \cup [T-\tau, T]$, to infer that

$$\begin{aligned} I_2 = &\int_{\Omega_{T-\tau}} \sum_{i,j=1,\, s_i>0}^n P^\varepsilon(x) \nabla \big((\vartheta_\tau u_i^\varepsilon)^{s_i+1} - (u_i^\varepsilon)^{s_i+1} \big) \cdot \int_t^{t+\tau} a_{ij}(u^\varepsilon) \nabla u_j^\varepsilon d\sigma dx dt \\ &+ \int_{\Omega_{T-\tau}} \sum_{i,j=1,\, s_i \leq 0}^n P^\varepsilon(x) \nabla (\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon) \cdot \int_t^{t+\tau} a_{ij}(u^\varepsilon) \nabla u_j^\varepsilon d\sigma dx dt. \end{aligned}$$

Again, we distinguish between the cases $s_i \leq 0$ and $s_i > 0$. Employing the Cauchy-Schwarz inequality we have

$$\begin{aligned} |I_2| \leq \tau^{1/2} C &\sum_{i,j=1,\, s_i,s_j>0}^n \left\| \frac{a_{ij}(u^\varepsilon)}{(u_j^\varepsilon)^{s_j}} \right\|_{L^\infty(\Omega_T)} \|\nabla (u_i^\varepsilon)^{s_i+1}\|_{L^2(\Omega_T)} \|\nabla (u_j^\varepsilon)^{s_j+1}\|_{L^2(\Omega_T)} \\ + \tau^{1/2} C &\sum_{i,j=1,\, s_i>0,\, s_j\leq 0}^n \|a_{ij}(u^\varepsilon)\|_{L^\infty(\Omega_T)} \|\nabla (u_i^\varepsilon)^{s_i+1}\|_{L^2(\Omega_T)} \|\nabla u_j^\varepsilon\|_{L^2(\Omega_T)} \\ + \tau^{1/2} C &\sum_{i,j=1,\, s_i\leq 0,\, s_j>0}^n \left\| \frac{a_{ij}(u^\varepsilon)}{(u_j^\varepsilon)^{s_j}} \right\|_{L^\infty(\Omega_T)} \|\nabla u_i^\varepsilon\|_{L^2(\Omega_T)} \|\nabla (u_j^\varepsilon)^{s_j+1}\|_{L^2(\Omega_T)} \\ + \tau^{1/2} C &\sum_{i,j=1,\, s_i,s_j\leq 0}^n \|a_{ij}(u^\varepsilon)\|_{L^\infty(\Omega_T)} \|\nabla u_i^\varepsilon\|_{L^2(\Omega_T)} \|\nabla u_j^\varepsilon\|_{L^2(\Omega_T)} \leq C\tau^{1/2}, \end{aligned}$$

in view of Assumption A3 and estimates (11)-(12).

It remains to estimate I_3. The boundedness of u^ε yields

$$\begin{aligned} |I_3| \leq &\int_0^{T-\tau} \int_\Omega \sum_{i=1,\, s_i>0}^n \int_t^{t+\tau} |f_i(u^\varepsilon)| ds \left| (\vartheta_\tau u_i^\varepsilon)^{s_i+1} - (u_i^\varepsilon)^{s_i+1} \right| dx dt \\ &+ \int_0^{T-\tau} \int_\Omega \sum_{i=1,\, s_i\leq 0}^n \int_t^{t+\tau} |f_i(u^\varepsilon)| ds\, |\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon|\, dx dt \leq C\tau. \end{aligned}$$

Putting these estimates together, we infer that (13) for $s_i \leq 0$ and (14) for $s_i > 0$ holds, concluding the proof. □

Lemma 5 (A priori estimates)**.** *For any $\varepsilon > 0$, there exists a bounded weak solution u^ε to* (3)-(4) *such that $u^\varepsilon(x,t) \in \overline{\mathcal{G}}$ for $x \in \Omega$, $t > 0$ and*

$$\|(u_i^\varepsilon)^{s_i+1}\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \leq C \qquad \textit{for } i = 1,\ldots,n, \tag{15}$$

$$\|u_i^\varepsilon\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \leq C \qquad \textit{for } -1 < s_i \leq 0, \tag{16}$$

$$\|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon\|_{L^2((0,T-\tau)\times\Omega^\varepsilon)} \leq C\tau^{1/4} \qquad \textit{for } -1 < s_i \leq 0, \tag{17}$$

$$\|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon\|_{L^{2+s_i}((0,T-\tau)\times\Omega^\varepsilon)} \leq C\tau^{1/(4+2s_i)} \qquad \textit{for } s_i > 0, \tag{18}$$

where $\vartheta_\tau u_i^\varepsilon(x,t) = u_i^\varepsilon(x,t+\tau)$ for $x \in \Omega^\varepsilon$, $t \in (0,T-\tau)$, and the constant $C > 0$ is independent of ε.

Proof. The proof of a priori estimates (15)-(18) follows the same steps as in the proof of Lemma 4. Thanks to the structure of the proof, all estimates in Lemma 4 can be obtained for Ω^ε instead of Ω, independently of ε. □

Remark 6 (Extension)**.** Our assumptions on the microscopic structure of Ω^ε ensure that there exists an extension $\overline{u_i^\varepsilon}$ of u_i^ε and $\overline{(u_i^\varepsilon)^{s_i+1}}$ of $(u_i^\varepsilon)^{s_i+1}$ from Ω^ε to Ω with the properties

$$\|\overline{u_i^\varepsilon}\|_{L^2(\Omega)} \leq \mu\|u_i^\varepsilon\|_{L^2(\Omega^\varepsilon)}, \quad \|\nabla\overline{u_i^\varepsilon}\|_{L^2(\Omega)} \leq \mu\|\nabla u_i^\varepsilon\|_{L^2(\Omega^\varepsilon)} \text{ for } -1 < s_i \leq 0,$$
$$\|\overline{(u_i^\varepsilon)^{s_i+1}}\|_{L^2(\Omega)} \leq \mu\|(u_i^\varepsilon)^{s_i+1}\|_{L^2(\Omega^\varepsilon)}, \quad \|\nabla\overline{(u_i^\varepsilon)^{s_i+1}}\|_{L^2(\Omega)} \leq \mu\|\nabla(u_i^\varepsilon)^{s_i+1}\|_{L^2(\Omega^\varepsilon)}$$

for $t > 0$, where $\mu > 0$ is some constant independent of ε; see, e.g., [6] or Appendix A for details. □

Lemma 7 (Convergence)**.** *Let u^ε be a weak solution of* (1)-(2) *or* (3)-(4)*. Then there exists a subsequence of (u^ε), which is not relabeled, and functions $u \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^n))$, with $u_i^{s_i+1} \in L^2(0,T;H^1(\Omega))$ for $i = 1,\ldots,n$, $V_1,\ldots,V_n \in L^2(\Omega_T;H^1_{\rm per}(Y)/\mathbb{R})$ such that, as $\varepsilon \to 0$,*

$$u_i^\varepsilon \to u_i \qquad \textit{strongly in } L^p(\Omega_T),\ p < \infty, \tag{19}$$

$$(u_i^\varepsilon)^{s_i+1} \to u_i^{s_i+1} \qquad \textit{strongly in } L^2(\Omega_T) \textit{ for } s_i > 0, \tag{20}$$

$$\nabla(u_i^\varepsilon)^{s_i+1} \rightharpoonup \nabla u_i^{s_i+1} \qquad \textit{weakly in } L^2(\Omega_T), \tag{21}$$

$$\nabla(u_i^\varepsilon)^{s_i+1} \rightharpoonup \nabla u_i^{s_i+1} + \nabla_y V_i \qquad \textit{two-scale},\ i = 1,\ldots,n. \tag{22}$$

In the case of solutions (u^ε) of (3)-(4)*, convergence results* (19)-(22) *hold for a subsequence of the extension of $(u_i^\varepsilon)^{s_i+1}$ and of u_i^ε from Ω_T^ε into Ω_T, for $i = 1,\ldots,n$, considered in Remark 6.*

Proof. For $s_i \leq 0$, estimates (12) and (13) allow us to apply the Aubin-Lions lemma in the version of [24], giving the existence of a subsequence, not relabeled, such that $u_i^\varepsilon \to u_i$ strongly in $L^2(\Omega_T)$. Since (u_i^ε) is bounded in $L^\infty(\Omega_T)$, by construction, this convergence even holds in $L^p(\Omega_T)$ for any $p < \infty$.

For $s_i > 0$, we apply Lemma 3 with $p = s_i + 1$ and the bounds $|\vartheta_\tau u_i^\varepsilon| \le 1$, $|u_i^\varepsilon| \le 1$:

$$\|\vartheta_\tau (u_i^\varepsilon)^{s_i+1} - (u_i^\varepsilon)^{s_i+1}\|_{L^{2+s_i}(\Omega_{T-\tau})} \le C\|\vartheta_\tau u_i^\varepsilon - u_i^\varepsilon\|_{L^{2+s_i}(\Omega_{T-\tau})} \le C\tau^{1/(4+2s_i)}.$$

Hence, by applying the Aubin-Lions lemma of [24] to $(u_i^\varepsilon)^{s_i+1}$, we deduce the strong convergence $(u_i^\varepsilon)^{s_i+1} \to w_i$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$ for some $w_i \in L^2(\Omega_T)$ with $w_i \ge 0$. In particular, up to a subsequence, we have $(u_i^\varepsilon)^{s_i+1} \to w_i$ a.e. in Ω_T and consequently, $u_i^\varepsilon \to u_i := w_i^{1/(s_i+1)}$ a.e. in Ω_T. Since (u_i^ε) is bounded in $L^\infty(\Omega_T)$, it follows that $u_i^\varepsilon \to u_i$ strongly in $L^p(\Omega_T)$ for any $p < \infty$ and also $(u_i^\varepsilon)^{s_i+1} \to (u_i)^{s_i+1}$ strongly in $L^2(\Omega_T)$, which proves (20).

Convergence (21) follows from the bound (11), possibly after extracting another subsequence. Finally, the two-scale convergence (22) is a consequence of the boundedness of $\nabla(u_i^\varepsilon)^{s_i+1}$ in $L^2(\Omega_T)$; see, e.g., [2, 20] or Lemma 15 in Appendix A.

In the case of solutions (u^ε) of problem (3)-(4), we consider extensions of u_i^ε and $(u_i^\varepsilon)^{s_i+1}$ from Ω^ε into Ω as in Remark 6, for $i = 1, \ldots, n$. The properties and the linearity of the extension and the a priori estimates from Lemma 5 imply the corresponding estimates for $\overline{u_i^\varepsilon}$, for those i such that $-1 < s_i \le 0$, and $\overline{(u_i^\varepsilon)^{s_i+1}}$ in $L^2(0,T;H^1(\Omega))$, for $\vartheta_\tau\overline{u_i^\varepsilon} - \overline{u_i^\varepsilon}$ in $L^2(\Omega_T)$ if $-1 < s_i \le 0$, and for $\vartheta_\tau\overline{(u_i^\varepsilon)^{s_i+1}} - \overline{(u_i^\varepsilon)^{s_i+1}}$ in $L^{2+s_i}(\Omega_T)$ for $s_i > 0$.

We conclude from the estimates for $\overline{u_i^\varepsilon}$ that there exists $u_i \in L^2(0,T;H^1(\Omega))$ such that, up to a subsequence, $\overline{u_i^\varepsilon} \to u_i$ strongly in $L^2(\Omega_T)$ for $-1 < s_i \le 0$. Furthermore, the estimates for $\overline{(u_i^\varepsilon)^{s_i+1}}$ ensure that there exists $w \in L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ such that, up to a subsequence, $w_i^\varepsilon := \overline{(u_i^\varepsilon)^{s_i+1}}$ converges strongly to w_i in $L^2(\Omega_T)$. This ensures also the a.e. pointwise convergence of $(w_i^\varepsilon)^{1/(s_i+1)} = (\overline{(u_i^\varepsilon)^{s_i+1}})^{1/(s_i+1)}$ to $w_i^{1/(s_i+1)}$ in Ω_T as $\varepsilon \to 0$. It remains to prove that $w_i^{1/(s_i+1)} = u_i$.

The properties of the extension imply that $(w_i^\varepsilon)^{1/(s_i+1)}$ is uniformly bounded as u_i^ε is uniformly bounded. We deduce that, up to a subsequence, $(w_i^\varepsilon)^{1/(s_i+1)} \to w_i^{1/(s_i+1)}$ strongly in $L^2(\Omega_T)$. Notice that, due to the construction of the extension, we have $(\overline{(u_i^\varepsilon)^{s_i+1}})^{1/(s_i+1)} = u_i^\varepsilon$ in Ω_T^ε. For $-1 < s_i \le 0$, we apply Lemma 3 to $a = (u_i^{\varepsilon_m})^{s_i+1}$, $b = (u_i^{\varepsilon_\ell})^{s_i+1}$ and $p = 1/(s_i+1) \ge 1$ and use the properties of the extension:

$$\begin{aligned}\big\|\overline{(u_i^{\varepsilon_m})^{s_i+1}} - \overline{(u_i^{\varepsilon_\ell})^{s_i+1}}\big\|^2_{L^2(\Omega_T)} &\le \mu\big\|(u_i^{\varepsilon_m})^{s_i+1} - (u_i^{\varepsilon_\ell})^{s_i+1}\big\|^2_{L^2(\Omega_T^\varepsilon)} \\ &\le \mu_1\|u_i^{\varepsilon_m} - u_i^{\varepsilon_\ell}\|^{2(s_i+1)}_{L^2(\Omega_T^\varepsilon)} \le \mu_2\big\|\overline{u_i^{\varepsilon_m}} - \overline{u_i^{\varepsilon_\ell}}\big\|^{2(s_i+1)}_{L^2(\Omega_T)},\end{aligned}$$

whereas for $s_i > 0$ we obtain

$$\begin{aligned}\big\|\overline{u_i^{\varepsilon_m}} - \overline{u_i^{\varepsilon_\ell}}\big\|^2_{L^2(\Omega_T)} &\le \mu\|u_i^{\varepsilon_m} - u_i^{\varepsilon_\ell}\|^2_{L^2(\Omega_T^\varepsilon)} \le \mu_1\big\|(u_i^{\varepsilon_m})^{s_i+1} - (u_i^{\varepsilon_\ell})^{s_i+1}\big\|^{2/(s_i+1)}_{L^2(\Omega_T^\varepsilon)} \\ &\le \mu_2\big\|\overline{(u_i^{\varepsilon_m})^{s_i+1}} - \overline{(u_i^{\varepsilon_\ell})^{s_i+1}}\big\|^{2/(s_i+1)}_{L^2(\Omega_T)}.\end{aligned}$$

Hence, if $-1 < s_i \le 0$, the strong convergence of $\overline{u_i^\varepsilon}$ implies the strong convergence of $\overline{(u_i^\varepsilon)^{s_i+1}}$, while for $s_i > 0$, strong convergence of $\overline{(u_i^\varepsilon)^{s_i+1}}$ ensures the strong convergence of

$\overline{u_i^\varepsilon}$. Therefore, denoting by $\chi_{\Omega^\varepsilon}$ the characteristic function of Ω^ε,

$$\frac{|Y_1|}{|Y|}\int_{\Omega_T} u_i\phi dxdt = \lim_{\varepsilon\to 0}\int_{\Omega_T}\overline{u_i^\varepsilon}\chi_{\Omega^\varepsilon}\phi dxdt = \lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^\varepsilon} u_i^\varepsilon \phi dxdt$$
$$= \lim_{\varepsilon\to 0}\int_{\Omega_T}(w_i^\varepsilon)^{1/(s_i+1)}\chi_{\Omega^\varepsilon}\phi dxdt = \frac{|Y_1|}{|Y|}\int_{\Omega_T} w_i^{1/(s_i+1)}\phi dxdt,$$

for any $\phi\in C_0^\infty(\Omega_T)$. We deduce that $w_i^{1/(s_i+1)} = u_i$ and consequently $w_i = u_i^{s_i+1}$ a.e. in Ω_T. The boundedness of $\nabla(u_i^\varepsilon)^{s_i+1}$ and the properties of the extension ensure the convergence results in (21) and (22). Hence, we obtain convergence results (19)-(22) for a subsequence of $(\overline{u_i^\varepsilon})$ and $(\overline{(u_i^\varepsilon)^{s_i+1}})$, respectively, finishing the proof. $\square$

Proof of Theorem 1. Using the convergence results in Lemma 7, we are now able to derive the macroscopic equations for microscopic problems (1)-(2) and (3)-(4).

The strong convergence of (u^ε) in $L^p(\Omega_T)$ for any $p<\infty$ and Assumption A3 imply that, for those j satisfying $s_j>0$,

$$\frac{a_{ij}(u^\varepsilon)}{(u_j^\varepsilon)^{s_j}} \to \frac{a_{ij}(u)}{(u_j)^{s_j}} \quad \text{strongly in } L^p(\Omega_T)$$

and also weakly* in $L^\infty(\Omega_T)$, where we set $a_{ij}(u)/u_j^{s_j} := 0$ if $u_j=0$. For those j with $s_j\le 0$, it follows that

$$a_{ij}(u^\varepsilon)\to a_{ij}(u), \quad \frac{a_{ij}(u^\varepsilon)}{(u_j^\varepsilon)^{s_j}} \to \frac{a_{ij}(u)}{(u_j)^{s_j}} \quad \text{strongly in } L^p(\Omega_T).$$

Furthermore, $f_i(u^\varepsilon)\to f_i(u)$ strongly in $L^p(\Omega_T)$, for $p<\infty$. Notice that we use the same notation for u_i^ε or $(u_i^\varepsilon)^{s_i+1}$ and the corresponding extensions from Ω^ε into Ω when considering problem (3)-(4) defined in the perforated domain Ω^ε.

Step 1: problem (1)-(2). We use the admissible test function $\phi^\varepsilon = (\phi_1^\varepsilon,\ldots,\phi_n^\varepsilon)$ in the weak formulation of (1)-(2), where

$$\phi_i^\varepsilon(x,t) = \phi_i^0(x,t) + \varepsilon\phi_i^1(x,t,x/\varepsilon), \quad i=1,\ldots,n,$$

with $\phi_i^0\in C^1([0,T];H^1(\Omega))$ such that $\phi_i^0(x,T)=0$ and $\phi_i^1\in C_0^1(\Omega_T;C^1_{\rm per}(Y))$. This gives

$$0 = \int_0^T\langle\partial_t u^\varepsilon,\phi^\varepsilon\rangle dt + \int_{\Omega_T}\sum_{i,j=1}^n P^\varepsilon(x)\frac{a_{ij}(u^\varepsilon)}{(s_j+1)(u_j^\varepsilon)^{s_j}}\nabla(u_j^\varepsilon)^{s_j+1}\cdot\nabla\phi_i^\varepsilon dxdt$$
$$- \int_{\Omega_T} f(u^\varepsilon)\cdot\phi^\varepsilon dxdt =: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon. \tag{23}$$

We perform the limit $\varepsilon\to 0$ in the integrals I_k^ε term by term, for $k=1,2,3$. Using the strong convergence of (u^ε), we obtain

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = -\lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega u^\varepsilon\cdot(\partial_t\phi^0 + \varepsilon\partial_t\phi^1)dxdt + \int_\Omega u^\varepsilon(0)\cdot\big(\phi^0(0)+\varepsilon\phi^1(0)\big)dx\right)$$

$$= -\int_0^T \int_\Omega u \cdot \partial_t \phi^0 dxdt - \int_\Omega u^0 \cdot \phi^0(0)dx,$$

$$\lim_{\varepsilon\to 0} I_3^\varepsilon = -\int_0^T \int_\Omega f(u) \cdot \phi^0 dxdt.$$

The limit $\varepsilon \to 0$ in I_2^ε is more involved. By (22), we have $\nabla(u_j^\varepsilon)^{s_j+1} \rightharpoonup \nabla u_j^{s_j+1} + \nabla_y V_j$ two-scale. Furthermore, we deduce from the definition of P^ε, the strong convergence of (u^ε), and the strong two-scale convergences of $(P(x/\varepsilon))$ and $(\nabla\phi_i^\varepsilon)$ that

$$\lim_{\varepsilon\to 0} \left\| P^\varepsilon(x) \frac{a_{ij}(u^\varepsilon)}{(u_j^\varepsilon)^{s_j}} \nabla \phi_i^\varepsilon \right\|_{L^2(\Omega_T)} = |Y|^{-1/2} \left\| P(y) \frac{a_{ij}(u)}{u_j^{s_j}} (\nabla\phi_i^0 + \nabla_y \phi_i^1) \right\|_{L^2(\Omega_T\times Y)},$$

for $i, j = 1, \ldots, n$. Therefore, by Lemma 17 in Appendix A,

$$I_2^\varepsilon \to \int_0^T \int_\Omega ⨍_Y \sum_{i,j=1}^n P(y) \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \big(\nabla u_j^{s_j+1} + \nabla_y V_j\big) \cdot \big(\nabla\phi_i^0 + \nabla_y\phi_i^1\big) dydxdt,$$

as $\varepsilon \to 0$, where $⨍_Y(\cdots)dy = |Y|^{-1}\int_Y(\cdots)dy$. Hence, the limit $\varepsilon \to 0$ in (23) leads to

$$\begin{aligned} &-\int_0^T \int_\Omega u \cdot \partial_t\phi^0 dxdt - \int_\Omega u^0 \cdot \phi^0(0)dx \\ &\quad + \int_0^T \int_\Omega ⨍_Y \sum_{i,j=1}^n P(y) \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \big(\nabla u_j^{s_j+1} + \nabla_y V_j\big) \cdot \big(\nabla\phi_i^0 + \nabla_y\phi_i^1\big) dydxdt \\ &= \int_0^T \int_\Omega f(u) \cdot \phi^0 dxdt. \end{aligned} \tag{24}$$

Next, we need to identify V_j. For this, let first $\phi_i^0 = 0$ for $i = 1, \ldots, n$ in (24). Then

$$0 = \int_0^T \int_\Omega \int_Y \sum_{i,j=1}^n P(y) \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \big(\nabla u_j^{s_j+1} + \nabla_y V_j\big) \cdot \nabla_y \phi_i^1 dydxdt. \tag{25}$$

We insert the ansatz

$$V_j(t,x,y) = \sum_{k=1}^d \sum_{\ell=1}^n \frac{\partial}{\partial x_k} u_\ell^{s_\ell+1}(t,x) W_j^{k\ell}(t,x,y), \quad j = 1, \ldots, n,$$

with functions $W_j^{k\ell}$, which need to be determined, in (25):

$$\begin{aligned} 0 &= \int_{\Omega_T} \int_Y \sum_{i,j=1}^n \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \sum_{m=1}^d P_m(y) \bigg(\frac{\partial u_j^{s_j+1}}{\partial x_m} + \sum_{k=1}^d \sum_{\ell=1}^n \frac{\partial u_\ell^{s_\ell+1}}{\partial x_k} \frac{\partial W_j^{k\ell}}{\partial y_m} \bigg) \frac{\partial \phi_i^1}{\partial y_m} dydxdt \\ &= \int_{\Omega_T} \int_Y \sum_{k=1}^d \sum_{\ell=1}^n \frac{\partial u_\ell^{s_\ell+1}}{\partial x_k} \sum_{i,j=1}^n \sum_{m=1}^d P_m(y) \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \bigg(\frac{\partial W_j^{k\ell}}{\partial y_m} + \delta_{km}\delta_{j\ell} \bigg) \frac{\partial \phi_i^1}{\partial y_m} dydxdt, \end{aligned}$$

where δ_{km} is the Kronecker symbol. By the linear independence of $(\partial u_\ell^{s_\ell+1}/\partial x_k)_{k\ell}$, we infer that

$$0 = \int_Y \sum_{i,j=1}^n \sum_{m=1}^d P_m(y) \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \bigg(\frac{\partial W_j^{k\ell}}{\partial y_m} + \delta_{km}\delta_{j\ell} \bigg) \frac{\partial \phi_i^1}{\partial y_m} dy,$$

for $k = 1, \dots, d$ and $\ell = 1, \dots, n$. This means that the functions $W_j^{k\ell}$ are solutions, if they exist, of the linear elliptic cross-diffusion equations

$$\sum_{m=1}^d \sum_{i,j=1}^n \frac{\partial}{\partial y_m} \bigg(P_m(y) \widehat{A}_{ij}(u(x,t)) \bigg(\frac{\partial W_j^{k\ell}}{\partial y_m} + \delta_{j\ell}\delta_{km} \bigg) \bigg) = 0,$$

where

$$\widehat{A}_{ij}(u) = \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}}, \quad i,j = 1, \dots, n.$$

More precisely, $W_j^{k\ell}$ are the solutions, if they exist, of the elliptic problem

$$\begin{aligned} &\operatorname{div}_y \big(P(y) \widehat{A}(u(x,t)) (\nabla_y W^{k\ell} + e_k e_\ell) \big) = 0 \quad \text{in } Y, \\ &\int_Y W^{k\ell}(x,y,t) dy = 0, \quad W_j^{k\ell} \text{ is } Y\text{-periodic}, \ j = 1, \dots, n, \end{aligned} \tag{26}$$

for $k = 1, \dots, d$ and $\ell = 1, \dots, n$, parametrized by $(x,t) \in \Omega_T$, where e_k and e_ℓ are the standard basis vectors of $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively, and $e_k e_\ell$ is the matrix in $\mathbb{R}^{d\times n}$ with the elements $\delta_{km}\delta_{j\ell}$. The solvability of (26) is proved in Lemma 8 below.

Setting $\phi^1 = 0$ and arguing similarly as above, we can write the macroscopic equations (24) as

$$\begin{aligned} &-\int_0^T \int_\Omega \sum_{i=1}^n u_i \partial_t \phi_i^0 dx dt + \int_0^T \int_\Omega \sum_{k,m=1}^d \sum_{i,\ell=1}^n B_{mk}^{i\ell}(u) \frac{\partial u_\ell}{\partial x_k} \frac{\partial \phi_i^0}{\partial x_m} dx dt \\ &\qquad = \int_0^T \int_\Omega \sum_{i=1}^n f_i(u) \phi_i^0 dx dt + \int_\Omega \sum_{i=1}^n u_i^0 \phi_i^0(0) dx, \end{aligned} \tag{27}$$

where

$$B_{mk}^{i\ell}(u) = \sum_{j=1}^n \bigg(a_{ij}(u) \delta_{km}\delta_{j\ell} ⨍_Y P_m(y) dy + \frac{a_{ij}(u)(s_\ell+1)u_\ell^{s_\ell}}{(s_j+1)u_j^{s_j}} ⨍_Y P_m(y) \frac{\partial W_j^{k\ell}}{\partial y_m} dy \bigg). \tag{28}$$

From equation (27) and $u_i^{s_i+1} \in L^2(0,T;H^1(\Omega))$, we obtain $\partial_t u \in L^2(0,T;H^1(\Omega;\mathbb{R}^n)')$. This, together with the boundedness of u_i, implies that $\partial_t u_i^{s_i+1} \in L^2(0,T;H^1(\Omega)')$ for $s_i > 0$. Hence, $u_i \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)')$ for those i satisfying $-1 < s_i \le 0$ and $u_i^{s_i+1} \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)')$ if $s_i > 0$. Therefore, u_i, $u_j^{s_j+1} \in C^0([0,T];L^2(\Omega))$ for those $i,j = 1, \dots, n$ satisfying $-1 < s_i \le 0$ and $s_j > 0$. Consequently, the initial datum is satisfied in the sense of $L^2(\Omega)$.

Step 2: problem (3)-(4). We use the two-scale convergence of $\nabla(u_i^\varepsilon)^{s_i+1}$ and take the limit $\varepsilon \to 0$ in the weak formulation of (3), i.e.

$$\int_0^T \langle \partial_t u^\varepsilon, \phi^\varepsilon \rangle dt + \int_0^T \int_{\Omega^\varepsilon} \sum_{i,j=1}^n \frac{a_{ij}(u^\varepsilon)}{(s_j+1)(u_j^\varepsilon)^{s_j}} \nabla (u_j^\varepsilon)^{s_j+1} \cdot \nabla \phi_i^\varepsilon dx dt = \int_{\Omega_T^\varepsilon} f(u^\varepsilon) \cdot \phi^\varepsilon dx dt$$

to obtain the macroscopic equation

$$\begin{aligned} &\int_0^T \int_\Omega \fint_{Y_1} \sum_{i,j=1}^n \frac{a_{ij}(u)}{(s_j+1)u_j^{s_j}} \big(\nabla u_j^{s_j+1} + \nabla_y V_j\big) \cdot \big(\nabla \phi_i^0 + \nabla_y \phi_i^1\big) dy dx dt \\ &\quad - \int_0^T \int_\Omega u \cdot \partial_t \phi^0 dx dt - \int_\Omega u^0 \cdot \phi^0(0) dx = \int_0^T \int_\Omega f(u) \cdot \phi^0 dx dt. \end{aligned} \tag{29}$$

Repeating the calculations from Step 1, we arrive at the macroscopic problem (8) with the macroscopic diffusion matrix

$$B_{mk}^{i\ell}(u) = \sum_{j=1}^n \bigg(a_{ij}(u)\delta_{km}\delta_{j\ell} + \frac{a_{ij}(u)(s_\ell+1)u_\ell^{s_\ell}}{(s_j+1)u_j^{s_j}} \fint_{Y_1} \frac{\partial \widehat{W}_j^{k\ell}}{\partial y_m} dy \bigg). \tag{30}$$

where $\widehat{W}^{k\ell}$ for $k = 1, \ldots, d$ and $\ell = 1, \ldots, n$ are the solutions of the unit-cell problem

$$\begin{aligned} &\operatorname{div}_y \big(\widehat{A}(u(x,t))(\nabla_y \widehat{W}^{k\ell} + e_\ell e_k)\big) = 0 \quad \text{in } Y_1, \quad \int_{Y_1} \widehat{W}^{k\ell}(x,y,t) dy = 0, \\ &\widehat{A}(u(x,t))(\nabla_y \widehat{W}^{k\ell} + e_\ell e_k) \cdot \nu = 0 \quad \text{on } \Gamma, \quad \widehat{W}_j^{k\ell} \text{ is } Y\text{-periodic}, \end{aligned} \tag{31}$$

where $j = 1, \ldots, n$. This finishes the proof. □

It remains to prove the solvability of the unit-cell problems.

Lemma 8 (Solvability of the unit-cell problem)**.** *There exist weak solutions of the unit-cell problems* (26) *and* (31)*, respectively. The solutions are unique on* $\{u_i > 0 : i = 1, \ldots, n\}$.

Proof. Let us first consider problem (26). Since $\widehat{A}(u(x,t))$ may vanish, the unit-cell problem is of degenerate type. Therefore, we introduce the regularization

$$\begin{aligned} &\operatorname{div}_y \big(P(y)\widehat{A}(u_\delta(x,t))(\nabla_y W_\delta^{k\ell} + e_k e_\ell)\big) = 0 \quad \text{in } Y, \\ &\int_Y W_{\delta,j}^{k\ell}(x,y,t) dy = 0, \quad W_{\delta,j}^{k\ell} \text{ is } Y\text{-periodic}, \ j = 1, \ldots, n, \end{aligned} \tag{32}$$

where $u_{\delta,j}(x,t) = (u_j(x,t) + \delta/2)/(1+\delta)$ for $j = 1, \ldots, n$. Since $0 \le u_j(x,t) \le 1$, it follows that $0 < \delta/(2+2\delta) \le u_{\delta,j} \le (2+\delta)/(2+2\delta) < 1$, which avoids the degeneracy in Assumption A2. Furthermore, we define

$$\widetilde{W}_{\delta,j}^{k\ell}(x,y,t) := \frac{W_{\delta,j}^{k\ell}(x,y,t)}{u_{\delta,j}^{s_j}(x,t)}, \quad j = 1, \ldots, n.$$

Then $\widetilde{W}_\delta^{k\ell}$ satisfies the problem

$$\operatorname{div}_y \big(P(y)(A(u_\delta(x,t))\nabla_y \widetilde{W}_\delta^{k\ell} + \widehat{A}(u_\delta(x,t))e_k e_\ell)\big) = 0 \quad \text{in } Y,$$

$$\int_Y \widetilde{W}^{k\ell}_{\delta,j}(x,y,t)dy = 0, \quad \widetilde{W}^{k\ell}_{\delta,j} \text{ is } Y\text{-periodic}, \ j = 1,\ldots,n. \tag{33}$$

Notice that $u_\delta(x,t)$ is independent of $y \in Y$. The weak formulation of the elliptic problem reads as

$$\begin{aligned} 0 &= \int_Y \sum_{i,j=1}^n \sum_{m=1}^d P_m(y)\Big(a_{ij}(u_\delta(x,t))\partial_{y_m}\widetilde{W}^{k\ell}_{\delta,j} + \widehat{a}_{ij}(u_\delta(x,t))\delta_{j\ell}\delta_{km}\Big)\partial_{y_m}\psi_i dy \\ &= \int_Y \bigg(\sum_{i,j=1}^n P(y)a_{ij}(u_\delta(x,t))\nabla_y\widetilde{W}^{k\ell}_{\delta,j}\cdot\nabla_y\psi_i + \sum_{i=1}^n P_k(y)\widehat{a}_{i\ell}(u_\delta(x,t))\frac{\partial\psi_i}{\partial y_k}\bigg)dy. \end{aligned}$$

We take the test function $\psi_i(x,y,t) = \sum_{m=1}^n \partial_{im}h(u_\delta(x,t))\phi_m(y)$, where $\partial_{im}h = \partial^2 h/(\partial\xi_i \partial\xi_m)$ and ϕ_m is another test function:

$$0 = \int_Y \bigg(\sum_{i,j,m=1}^n P(y)\partial_{im}h(u_\delta)a_{ij}(u_\delta)\nabla_y\widetilde{W}^{k\ell}_{\delta,j}\cdot\nabla_y\phi_m + \sum_{i,m=1}^n P_k(y)\partial_{im}h(u_\delta)\widehat{a}_{i\ell}(u_\delta)\frac{\partial\phi_m}{\partial y_k}\bigg)dy.$$

We rename $m \mapsto i$ and $i \mapsto m$ and use the symmetry of the Hessian $(\partial_{im}h)$:

$$\begin{aligned} 0 &= \int_Y \bigg(\sum_{i,j,m=1}^n P(y)\partial_{im}h(u_\delta)a_{mj}(u_\delta)\nabla_y\widetilde{W}^{k\ell}_{\delta,j}\cdot\nabla_y\phi_i + \sum_{i,m=1}^n P_k(y)\partial_{im}h(u_\delta)\widehat{a}_{m\ell}(u_\delta)\frac{\partial\phi_i}{\partial y_k}\bigg)dy \\ &= \int_Y \bigg(\sum_{i,j=1}^n P(y)\big(h''(u_\delta)A(u_\delta)\big)_{ij}\nabla_y\widetilde{W}^{k\ell}_{\delta,j}\cdot\nabla_y\phi_i + \sum_{i=1}^n P_k(y)\big(h''(u_\delta)\widehat{A}(u_\delta)\big)_{i\ell}\frac{\partial\phi_i}{\partial y_k}\bigg)dy. \end{aligned} \tag{34}$$

The assumptions on $A(u_\delta)$ and $h(u_\delta)$ imply that $h''(u_\delta)A(u_\delta)$ is positive definite in Ω_T, giving coercivity of the elliptic problem. Furthermore, for any fixed $\delta > 0$, the coefficients of $h''(u_\delta)A(u_\delta)$ are uniformly bounded. Therefore, we can apply the Lax-Milgram lemma to conclude the existence of a unique solution $\widetilde{W}^{k\ell}_\delta(x,\cdot,t) \in H^1_{\rm per}(Y;\mathbb{R}^n)$ of problem (34). As $h''(u_\delta)$ is invertible, we may consider $\phi = h''(u_\delta)^{-1}\psi$ as a test function in (34), which means that the function $W^{k\ell}_{\delta,j}(x,\cdot,t) = u^{s_j}_{\delta,j}(x,t)\widetilde{W}^{k\ell}_{\delta,j}(x,\cdot,t)$ for $j = 1,\ldots,n$ also solves (32).

The next step is the derivation of bounds uniform in δ. To this end, we take the test function $\widetilde{W}^{k\ell}_\delta(x,\cdot,t)$ in (34), take into account the lower bound $P_k(y) \ge d_0 > 0$ for $k = 1,\ldots,d$, and the definition of $\widehat{A}(u_\delta(x,t))$, and apply the Cauchy-Schwarz inequality. This leads for any $\sigma > 0$ to

$$d_0\int_Y \sum_{j=1}^n u^{2s_j}_{\delta,j}\big|\nabla_y\widetilde{W}^{k\ell}_{\delta,j}\big|^2 dy \le C_\sigma\int_Y \sum_{i,\ell=1}^n \frac{(h''(u_\delta)A(u_\delta))^2_{i\ell}}{u^{2s_i}_{\delta,i}u^{2s_\ell}_{\delta,\ell}}dy + \sigma\int_Y \sum_{j=1}^n u^{2s_j}_{\delta,j}|\nabla_y\widetilde{W}^{k\ell}_{\delta,j}|^2 dy.$$

Choosing $\sigma = d_0/2$ and using Assumption A6, we find that

$$\frac{d_0}{2}\int_Y \sum_{j=1}^n \big|\nabla_y W^{k\ell}_{\delta,j}\big|^2 dy \le C_{d_0/2}\int_Y \sum_{i,\ell=1}^n \frac{(h''(u_\delta)A(u_\delta))^2_{i\ell}}{u^{2s_i}_{\delta,i}u^{2s_\ell}_{\delta,\ell}}dy \le C,$$

where $C > 0$ does not depend on δ. As the mean of $W^{k\ell}_{\delta,j}$ vanishes, the Poincaré-Wirtinger inequality gives a uniform estimate in $H^1(Y)$.

The uniform estimate for $W^{k\ell}_\delta$ implies the existence of a subsequence, which is not relabeled, such that $W^{k\ell}_\delta \rightharpoonup W^{k\ell}$ weakly in $H^1(Y;\mathbb{R}^n)$ as $\delta \to 0$, for $x \in \Omega$ and $t > 0$. Hence, we can pass to the limit $\delta \to 0$ in (32) to conclude that $W^{k\ell}$ is a solution of (26).

We claim that the solution is unique on the set $\{(x,t) : u_i(x,t) > 0 \text{ for } i = 1,\ldots,n\}$. Indeed, taking two solutions $W^{k\ell}_{(1)}$ and $W^{k\ell}_{(2)}$ of (26), choosing (x,t) such that $u_i(x,t) > 0$ and arguing as before, we obtain

$$\|\nabla_y (W^{k\ell}_{(1),i} - W^{k\ell}_{(2),i})\|_{L^2(Y;\mathbb{R}^d)} \le 0$$

for $k = 1,\ldots,d$ and $i,\ell = 1,\ldots,n$. This implies that $W^{k\ell}_{(1)} = W^{k\ell}_{(2)}$ and proves the claim.

The same arguments ensure also the existence of a solution of the unit-cell problem (31) and its uniqueness for those $x \in \Omega$ and $t > 0$ satisfying $u_i(t,x) > 0$ for all $i = 1,\ldots,n$. ☐

4. Proof of Theorem 2

First, we state an existence result which follows from [28].

Lemma 9 (Entropy inequality)**.** *There exists a weak solution $u^\varepsilon = (u^\varepsilon_1,\ldots,u^\varepsilon_n)$ of problem* (1)-(2) *with the diffusion matrix* (6) *in the sense of Definition 2. This solution satisfies the entropy inequality*

$$\int_\Omega h(u^\varepsilon)dx + C\int_0^T\int_\Omega \sum_{i=1}^n \big(u^\varepsilon_{n+1}|\nabla (u^\varepsilon_i)^{1/2}|^2 + |\nabla (u^\varepsilon_{n+1})^{1/2}|^2\big)dxdt \le \int_\Omega h(u^0)dx, \tag{35}$$

where $C = d_0 \min_{i=1,\ldots,n} D_i$. A similar estimate with Ω replaced by Ω^ε holds for solutions of problem (3)-(4) *with the diffusion matrix* (6)*.*

Proof ideas. The existence of a weak solution u^ε follows from Theorem 1 in [28] for $p_i(u) = D_i$ $(i = 1,\ldots,n)$ and $q(s) = s$. The entropy inequality (35) follows from inequality (33) in [28] in the regularization limit. A direct proof of estimate (35) using the definition of a weak solution of (1)-(2) or (3)-(4) with the diffusion matrix (6) can be found in Appendix B. ☐

Lemma 10 (A priori estimates)**.** *Weak solutions of* (1)-(2) *with diffusion matrix* (6) *satisfy*

$$\begin{aligned}
&\|(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\|_{L^2(0,T;H^1(\Omega))} \le C,\\
&\|(u^\varepsilon_{n+1})^{1/2}\|_{L^2(0,T;H^1(\Omega))} + \|(u^\varepsilon_{n+1})^{3/2}\|_{L^2(0,T;H^1(\Omega))} \le C,\\
&\|\vartheta_\tau u^\varepsilon_{n+1} - u^\varepsilon_{n+1}\|_{L^{5/2}(\Omega_T)} \le C\tau^{1/5},\\
&\|\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\|_{L^2(\Omega_{T-\tau})} \le C\tau^{1/10},
\end{aligned} \tag{36}$$

for all $\varepsilon > 0$ and $i = 1,\ldots,n$, where $\vartheta_\tau v(x,t) = v(x,t+\tau)$ for $t \in (0,T-\tau)$ and $\tau \in (0,T)$ and the constant $C > 0$ is independent of ε.

Proof. The entropy production inequality (35) shows that there exists $C>0$ independent of ε such that for all $i=1,\ldots,n$,

$$\|(u_{n+1}^\varepsilon)^{1/2}\|_{L^2(0,T;H^1(\Omega))}+\|(u_{n+1}^\varepsilon)^{1/2}\nabla(u_i^\varepsilon)^{1/2}\|_{L^2(\Omega_T)}\le C. \tag{37}$$

Because of

$$\nabla\big((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon\big)=2(u_i^\varepsilon u_{n+1}^\varepsilon)^{1/2}\nabla(u_i^\varepsilon)^{1/2}+u_i^\varepsilon\nabla(u_{n+1}^\varepsilon)^{1/2},$$

estimate (37), and the boundedness of u_i^ε for $i=1,\ldots,n$, we conclude that

$$\|\nabla((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon)\|_{L^2(\Omega_T)}\le C,\quad i=1,\ldots,n.$$

Adding this inequality for $i=1,\ldots,n$ and recalling that $\sum_{i=1}^n\nabla u_i^\varepsilon=-\nabla u_{n+1}^\varepsilon$, it follows that

$$\|(u_{n+1}^\varepsilon)^{3/2}\|_{L^2(0,T;H^1(\Omega))}\le C.$$

It remains to verify the uniform estimates on the equicontinuity of u^ε with respect to the time variable. For this, we define similarly as in the proof of Lemma 4

$$\phi(x,t)=\int_{t-\tau}^t\big((\vartheta_\tau u_{n+1}^\varepsilon)^{3/2}-(u_{n+1}^\varepsilon)^{3/2}\big)\kappa(\sigma)d\sigma$$

for some $\tau\in(0,T)$, where $\kappa(\sigma)=1$ for $\sigma\in(0,T-t)$ and $\kappa(\sigma)=0$ for $\sigma\in[-\tau,0]\cup[T-\tau,T]$. We take ϕ as a test function in the sum of equations (9) for $i=1,\ldots,n$ and use Lemma 3 with $p=3/2$ and the Cauchy-Schwarz inequality to infer that

$$\begin{aligned}
&\|\vartheta_\tau u_{n+1}^\varepsilon-u_{n+1}^\varepsilon\|_{L^{5/2}(\Omega_{T-\tau})}^{5/2}\le\int_0^{T-\tau}\int_\Omega|\vartheta_\tau u_{n+1}^\varepsilon-u_{n+1}^\varepsilon|^{5/2}dxdt\\
&\le\int_0^{T-\tau}\int_\Omega(\vartheta_\tau u_{n+1}^\varepsilon-u_{n+1}^\varepsilon)\big(\vartheta_\tau(u_{n+1}^\varepsilon)^{3/2}-(u_{n+1}^\varepsilon)^{3/2}\big)dxdt\\
&\le C\bigg|\int_0^{T-\tau}\int_\Omega\bigg(\int_t^{t+\tau}P^\varepsilon(x)\sum_{i=1}^n(A(u^\varepsilon)\nabla u^\varepsilon)_i\,d\sigma\bigg)\cdot\nabla\big(\vartheta_\tau(u_{n+1}^\varepsilon)^{3/2}-(u_{n+1}^\varepsilon)^{3/2}\big)dxdt\bigg|\\
&\le C\bigg\{\int_{\Omega_{T-\tau}}\bigg[\int_t^{t+\tau}\sum_{i=1}^n(A(u^\varepsilon)\nabla u^\varepsilon)_i d\sigma\bigg]^2dxdt\bigg\}^{\frac12}\bigg\{\int_{\Omega_{T-\tau}}\big|\nabla\big(\vartheta_\tau(u_{n+1}^\varepsilon)^{\frac32}-(u_{n+1}^\varepsilon)^{\frac32}\big)\big|^2dxdt\bigg\}^{\frac12}.
\end{aligned}$$

The second factor on the right-hand side is uniformly bounded since $\nabla(u_{n+1}^\varepsilon)^{3/2}$ is bounded in $L^2(\Omega_T)$. The first factor can be estimated from above by using definition (6) of $A(u^\varepsilon)$ and the uniform estimates for $(u_{n+1}^\varepsilon)^{1/2}\nabla u_i^\varepsilon$ as well as $\nabla(u_{n+1}^\varepsilon)^{1/2}$:

$$\begin{aligned}
&\int_0^{T-\tau}\int_\Omega\bigg(\int_t^{t+\tau}\sum_{i=1}^n(A(u^\varepsilon)\nabla u^\varepsilon)_i d\sigma\bigg)^2dxdt\\
&\le C\tau\int_0^{T-\tau}\int_\Omega u_{n+1}^\varepsilon\sum_{i=1}^n\Big(\big|\nabla\big((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon\big)\big|^2+|\nabla(u_{n+1}^\varepsilon)^{1/2}|^2(u_i^\varepsilon)^2\Big)dxdt\le C\tau.
\end{aligned}$$

We conclude that

$$\|\vartheta_\tau u_{n+1}^\varepsilon-u_{n+1}^\varepsilon\|_{L^{5/2}(\Omega_{T-\tau})}\le C\tau^{1/5}.$$

To prove the remaining estimate in (36), we take the test function

$$\phi_i(x,t) = \int_{t-\tau}^{t} \vartheta_\tau((u^\varepsilon_{n+1})^{1/2})\big(\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u_i^\varepsilon) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big)\kappa(\sigma)d\sigma$$

in (9) for $i=1,\ldots,n$. A computation shows that

$$\begin{aligned}
&\int_0^{T-\tau}\int_\Omega \sum_{i=1}^n \big(\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big)^2 dxdt \\
&\quad + \int_0^{T-\tau}\int_\Omega\int_t^{t+\tau}\sum_{i=1}^n P^\varepsilon(x)D_i\Big[(u^\varepsilon_{n+1})^{1/2}\nabla\big((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big) - 3u^\varepsilon_i(u^\varepsilon_{n+1})^{1/2}\nabla(u^\varepsilon_{n+1})^{1/2}\Big] \\
&\quad \times \nabla\Big[\vartheta_\tau((u^\varepsilon_{n+1})^{1/2})\big(\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big)\Big]d\sigma dxdt \\
&= \int_0^{T-\tau}\int_\Omega\sum_{i=1}^n u^\varepsilon_i\big(\vartheta_\tau(u^\varepsilon_{n+1})^{1/2} - (u^\varepsilon_{n+1})^{1/2}\big)\big(\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big)dxdt \\
&\le \sum_{i=1}^n \big\|\vartheta_\tau(u^\varepsilon_{n+1})^{1/2} - (u^\varepsilon_{n+1})^{1/2}\big\|_{L^2(\Omega_{T-\tau})}\big\|\vartheta_\tau((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big\|_{L^2(\Omega_{T-\tau})}.
\end{aligned}$$

The second integral on the left-hand side is bounded by $C\tau^{1/2}$ in view of the gradient estimates in (36). We infer from Lemma 3 with $p=2$, $a=\vartheta_\tau(u^\varepsilon_{n+1})^{1/2}$, $b=(u^\varepsilon_{n+1})^{1/2}$ and the third estimate in (36) that

$$\|\vartheta_\tau(u^\varepsilon_{n+1})^{1/2} - (u^\varepsilon_{n+1})^{1/2}\|_{L^2(\Omega_{T-\tau})} \le C\|\vartheta_\tau u^\varepsilon_{n+1} - u^\varepsilon_{n+1}\|^{1/2}_{L^2(\Omega_{T-\tau})} \le C\tau^{1/10},$$

finishing the proof. □

Remark 11. Similar uniform estimates as in Lemma 10 hold for the solutions of problem (3)-(4) with the diffusion matrix (6) defined in a perforated domain with the only difference that the domain Ω has to be replaced by Ω^ε:

$$\begin{aligned}
&\big\|(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \le C, \\
&\|(u^\varepsilon_{n+1})^{1/2}\|_{L^2(0,T;H^1(\Omega^\varepsilon))} + \|(u^\varepsilon_{n+1})^{3/2}\|_{L^2(0,T;H^1(\Omega^\varepsilon))} \le C, \\
&\big\|\vartheta_\tau u^\varepsilon_{n+1} - u^\varepsilon_{n+1}\big\|_{L^{5/2}(\Omega^\varepsilon_T)} \le C\tau^{1/5}, \\
&\big\|\vartheta_\tau\big((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big) - (u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big\|_{L^2(\Omega^\varepsilon_{T-\tau})} \le C\tau^{1/10},
\end{aligned} \tag{38}$$

for $i=1,\ldots,n$ and $\Omega^\varepsilon_T = \Omega^\varepsilon\times(0,T)$. □

The uniform estimates in Lemma 10 yield the following convergence results.

Lemma 12 (Convergence)**.** *Let u^ε be a solution of* (1)-(2) *with diffusion matrix* (6) *satisfying estimates* (36)*. Then there exist functions $u_1,\ldots,u_n \in L^\infty(0,T;L^\infty(\Omega))$, with $u^{1/2}_{n+1}u_i$, $u^{1/2}_{n+1} \in L^2(0,T;\ H^1(\Omega))$, and functions $V_1,\ldots,V_{n+1} \in L^2(\Omega_T;H^1_{\rm per}(Y)/\mathbb{R})$ such that, up to subsequences,*

$$u^\varepsilon_{n+1} \to u_{n+1} \qquad \textit{strongly in } L^p(\Omega_T),\ p\in(1,\infty),$$

$$
\begin{aligned}
& u_i^\varepsilon \rightharpoonup u_i && \textit{weakly in } L^p(\Omega_T),\ p\in(1,\infty),\\
& (u_{n+1}^\varepsilon)^{1/2} \rightharpoonup (u_{n+1})^{1/2} && \textit{weakly in } L^2(0,T;H^1(\Omega)),\\
& (u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon \to (u_{n+1})^{1/2}u_i && \textit{strongly in } L^2(\Omega_T),\\
& (u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon \rightharpoonup (u_{n+1})^{1/2}u_i && \textit{weakly in } L^2(0,T;H^1(\Omega)),\\
& \nabla((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon) \rightharpoonup \nabla((u_{n+1})^{1/2}u_i) + \nabla_y V_i && \textit{two-scale},\\
& \nabla(u_{n+1}^\varepsilon)^{1/2} \rightharpoonup \nabla(u_{n+1})^{1/2} + \nabla_y V_{n+1} && \textit{two-scale},
\end{aligned}
\tag{39}
$$

as $\varepsilon \to 0$, *where* $i = 1,\dots,n$ *and* $u_{n+1} = 1 - \sum_{i=1}^n u_i$.

Proof. The estimates for u_{n+1}^ε in (36) and Lemma 3 with $p = 2$ show that

$$\|\vartheta_\tau(u_{n+1}^\varepsilon)^{1/2} - (u_{n+1}^\varepsilon)^{1/2}\|_{L^2(\Omega_{T-\tau})} \le \|\vartheta_\tau u_{n+1}^\varepsilon - u_{n+1}^\varepsilon\|_{L^2(\Omega_{T-\tau})}^{1/2} \le C\tau^{1/10}.$$

Thus, together with the uniform bound for u_{n+1}^ε in $L^2(0,T;H^1(\Omega))$, the Aubin-Lions lemma [24] implies the existence of a function $w \in L^2(\Omega_T)$ and a subsequence (not relabeled) such that $(u_{n+1}^\varepsilon)^{1/2} \to w$ strongly in $L^2(\Omega_T)$ as $\varepsilon \to 0$. In particular, possibly for another subsequence, $(u_{n+1}^\varepsilon)^{1/2} \to w$ a.e. in Ω_T. Then, defining $u_{n+1} := w^2 \ge 0$, it follows that $u_{n+1}^\varepsilon \to u_{n+1}$ a.e. in Ω_T and, because of the boundedness of u_{n+1}^ε, also $u_{n+1}^\varepsilon \to u_{n+1}$ in $L^p(\Omega_T)$ for any $p < \infty$.

The weak convergence of (u_i^ε) to u_i in $L^p(\Omega_T)$ for $p < \infty$ is a consequence of the uniform L^∞-bound of u_i^ε. As a consequence, $(u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon \rightharpoonup u_{n+1}^{1/2}u_i$ weakly in $L^2(\Omega_T)$. By the first estimate in (36), a subsequence of $((u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon)$ is weakly converging in $L^2(0,T;H^1(\Omega))$, and we can identify the limit by $u_{n+1}^{1/2}u_i$. In fact, this limit is strong because the first and last estimate in (36) allow us to apply the Aubin-Lions lemma again to conclude that, for a subsequence, $(u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon \to u_{n+1}^{1/2}u_i$ strongly in $L^2(\Omega_T)$.

Using the first three estimates in (36) and the compactness theorem for two-scale convergence (see Lemma 15 in Appendix A), we obtain the two-scale convergences in (39). □

The uniform estimates (38) lead to the following convergences for the extensions $\overline{u_i^\varepsilon}$, $\overline{(u_{n+1}^\varepsilon)^{1/2}}$, and $\overline{(u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon}$ from Ω^ε to Ω of u_i^ε, $(u_{n+1}^\varepsilon)^{1/2}$, and $(u_{n+1}^\varepsilon)^{1/2}u_i^\varepsilon$, respectively, where $i = 1,\dots,n$ and u^ε is a weak solution of problem (3) and (4) with the diffusion matrix (6).

For any $\psi \in L^p(\Omega_T^\varepsilon)$, we denote by $[\psi]^\sim$ the extension of ψ by zero from Ω_T^ε to Ω_T.

Lemma 13 (Convergence)**.** *Let* u^ε *be a solution of* (3) *and* (4) *with the diffusion matrix* (6)*, satisfying estimates* (38)*. Then there exist* $u_1,\dots,u_n \in L^\infty(0,T;L^\infty(\Omega))$ *with* $u_{n+1}^{1/2}u_i$, $u_{n+1}^{1/2} \in L^2(0,T;H^1(\Omega))$ *and functions* $V_1,\dots,V_{n+1} \in L^2(\Omega_T;H^1_{\rm per}(Y_1)/\mathbb{R})$ *such that, up to subsequences,*

$$
\begin{aligned}
& \overline{u_{n+1}^\varepsilon} \to u_{n+1} && \textit{strongly in } L^2(\Omega_T),\\
& \overline{(u_{n+1}^\varepsilon)^{1/2}} \rightharpoonup u_{n+1}^{1/2} && \textit{weakly in } L^2(0,T;H^1(\Omega)),\\
& [u_i^\varepsilon]^\sim \rightharpoonup \theta u_i && \textit{weakly in } L^p(\Omega_T),\ p\in(1,\infty),
\end{aligned}
$$

$$
\begin{aligned}
&\overline{(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i} \to u^{1/2}_{n+1}u_i && \textit{strongly in } L^2(\Omega_T),\\
&\overline{(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i} \rightharpoonup u^{1/2}_{n+1}u_i && \textit{weakly in } L^2(0,T;H^1(\Omega)),\\
&[(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i]^\sim \rightharpoonup \chi_{Y_1}u^{1/2}_{n+1}u_i && \textit{two-scale},\\
&[\nabla((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i)]^\sim \rightharpoonup \chi_{Y_1}(\nabla(u^{1/2}_{n+1}u_i)+\nabla_y V_i) && \textit{two-scale},\\
&[\nabla(u^\varepsilon_{n+1})^{1/2}]^\sim \rightharpoonup \chi_{Y_1}(\nabla u^{1/2}_{n+1}+\nabla_y V_{n+1}) && \textit{two-scale},
\end{aligned} \tag{40}
$$

for $i=1,\dots,n$, $u_{n+1}=1-\sum_{i=1}^n u_i$, $\theta=|Y_1|/|Y|$, *and* χ_{Y_1} *is the characteristic function of* Y_1*.*

Proof. As in the proof of Lemma 10, we obtain the uniform estimate

$$\|\vartheta_\tau(u^\varepsilon_{n+1})^{1/2}-(u^\varepsilon_{n+1})^{1/2}\|_{L^2(\Omega^\varepsilon_{T-\tau})}\le C\tau^{1/10}.$$

Then, together with the uniform bound on $(u^\varepsilon_{n+1})^{1/2}$ in $L^2(0,T;H^1(\Omega^\varepsilon))$, the properties of the extension of $(u^\varepsilon_{n+1})^{1/2}$ from Ω^ε to Ω, and the Aubin-Lions lemma [24], we conclude the strong convergence (up to a subsequence)

$$\overline{(u^\varepsilon_{n+1})^{1/2}}\to\overline{w}\quad\text{strongly in } L^2(\Omega_T)$$

as $\varepsilon\to0$. To identify the limit, we use the properties of the extension, the boundedness of u^ε_{n+1}, and the elementary inequality $|a-b|\le 2|\sqrt{a}-\sqrt{b}|$ for $0\le a,b\le1$ (also see Lemma 3) to find that

$$
\begin{aligned}
\big\|\overline{u^{\varepsilon_m}_{n+1}}-\overline{u^{\varepsilon_k}_{n+1}}\big\|_{L^2(\Omega_T)} &\le C\|u^{\varepsilon_m}_{n+1}-u^{\varepsilon_k}_{n+1}\|_{L^2(\Omega^\varepsilon_T)}\le C\big\|(u^{\varepsilon_m}_{n+1})^{1/2}-(u^{\varepsilon_k}_{n+1})^{1/2}\big\|_{L^2(\Omega^\varepsilon_T)}\\
&\le C\big\|\overline{(u^{\varepsilon_m}_{n+1})^{1/2}}-\overline{(u^{\varepsilon_k}_{n+1})^{1/2}}\big\|_{L^2(\Omega_T)},
\end{aligned}
$$

for a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Thus, the strong convergence of $\overline{(u^\varepsilon_{n+1})^{1/2}}$ in $L^2(\Omega_T)$ implies the strong convergence $\overline{u^\varepsilon_{n+1}}\to u_{n+1}$ in $L^2(\Omega_T)$. Then the weak convergence

$$
\begin{aligned}
\theta\int_{\Omega_T}u^{1/2}_{n+1}\phi dxdt &= \lim_{\varepsilon\to0}\int_{\Omega_T}(\overline{u^\varepsilon_{n+1}})^{1/2}\chi_{\Omega^\varepsilon}\phi dxdt=\lim_{\varepsilon\to0}\int_{\Omega_T}(u^\varepsilon_{n+1})^{1/2}\chi_{\Omega^\varepsilon}\phi dxdt\\
&=\lim_{\varepsilon\to0}\int_{\Omega_T}\overline{(u^\varepsilon_{n+1})^{1/2}}\chi_{\Omega^\varepsilon}\phi dxdt=\theta\int_{\Omega_T}\overline{w}\phi dxdt
\end{aligned}
$$

for any $\phi\in C_0(\Omega_T)$ shows that $\overline{w}=u^{1/2}_{n+1}$ a.e. in Ω_T. We have proved the first two convergences in (40).

The uniform estimate for $(\nabla(u^\varepsilon_{n+1})^{1/2})$ and the compactness results for the two-scale convergence, see, e.g., [2] or Lemma 16 in Appendix A, imply the last convergence in (40). Moreover, by the first and last estimate in (38) for $(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i$, the properties of its extension from Ω^ε to Ω, and the Aubin-Lions lemma, it follows that, up to a subsequence, $\overline{(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i}\to v_i$ strongly in $L^2(\Omega_T)$ and weakly in $L^2(0,T;H^1(\Omega))$. We need to identify this limit. To this end, we first observe that, thanks to the boundedness of u^ε_i in Ω^ε_T, it follows that

$$[u^\varepsilon_i]^\sim\rightharpoonup\chi_{Y_1}u_i\quad\text{two-scale}$$

for some function $u_i \in L^p(\Omega_T \times Y)$, where $p \in (1,\infty)$ and $i = 1,\dots,n$. The a priori estimates and the compactness properties for sequences defined in perforated domains, see [2] or Lemma 16 in Appendix A, yield the existence of functions $V_1,\dots,V_n \in L^2(\Omega_T; H^1_{\rm per}(Y_1)/\mathbb{R})$ such that, up to subsequences,

$$[(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i]^\sim \rightharpoonup \chi_{Y_1} v_i \qquad \text{two-scale,}$$
$$[\nabla((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i)]^\sim \rightharpoonup \chi_{Y_1}(\nabla v_i + \nabla_y V_i) \qquad \text{two-scale.}$$

The strong convergence of $\overline{(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i}$ and the identity

$$\int_{\Omega_T}[(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i]^\sim \phi dxdt = \int_{\Omega_T}(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i \chi_{\Omega^\varepsilon}\phi dxdt = \int_{\Omega_T}\overline{(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i}\chi_{\Omega^\varepsilon}\phi dxdt$$

for any $\phi \in C_0(\Omega_T)$ imply that

$$[(u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i]^\sim \rightharpoonup \theta v_i \quad \text{weakly in } L^2(\Omega_T).$$

By Proposition 18 and Theorem 19 in Appendix A, this gives

$$\mathcal{T}^\varepsilon_{Y_1}\big((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big) \rightharpoonup v_i \qquad \text{weakly in } L^2(\Omega_T \times Y_1), \tag{41}$$
$$\mathcal{T}^\varepsilon_{Y_1}\big(\nabla((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i)\big) \rightharpoonup \nabla v_i + \nabla_y V_i \qquad \text{weakly in } L^2(\Omega_T \times Y_1).$$

The strong convergence of $\overline{(u^\varepsilon_{n+1})^{1/2}}$, the two-scale convergence of $[u^\varepsilon_i]^\sim$, and the fact that $\overline{(u^\varepsilon_{n+1})^{1/2}}\chi_{\Omega^\varepsilon} = (u^\varepsilon_{n+1})^{1/2}\chi_{\Omega^\varepsilon}$, imply

$$\mathcal{T}^\varepsilon_{Y_1}\big((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i\big) = \mathcal{T}^\varepsilon_{Y_1}\big((u^\varepsilon_{n+1})^{1/2}\big)\mathcal{T}^\varepsilon_{Y_1}(u^\varepsilon_i) \rightharpoonup u^{1/2}_{n+1}u_i \quad \text{weakly in } L^2(\Omega_T \times Y_1).$$

By the convergence (41) and the fact that u_{n+1} and v_i are independent of y, we infer that $u_i(x,y,t) = u_i(x,t)$ and $v_i = u^{1/2}_{n+1}u_i$, proving the claim. □

Proof of Theorem 2. Let $\phi^0 \in C^1_0([0,T]; C^1(\overline{\Omega};\mathbb{R}^n))$ and $\phi^1 \in C^1_0(\Omega_T; C^1_{\rm per}(Y;\mathbb{R}^n))$ and set $\phi(x,t) = \phi^0(x,t) + \varepsilon\phi^1(x,x/\varepsilon,t)$. We take this function as a test function in (9) and pass to the limit $\varepsilon \to 0$, using the two-scale convergence of $\nabla((u^\varepsilon_{n+1})^{1/2}u^\varepsilon_i)$ and $\nabla(u^\varepsilon_{n+1})^{1/2}$ (the last two convergences in (39)):

$$\begin{aligned}0 = &-\int_0^T\int_\Omega u\cdot\partial_t\phi^0 dxdt + \int_0^T\int_\Omega \fint_Y \sum_{i=1}^n P(y)D_i\, u^{1/2}_{n+1} \\ &\times \Big(\nabla(u^{1/2}_{n+1}u_i) + \nabla_y V_i - 3u_i\big(\nabla u^{1/2}_{n+1} + \nabla_y V_{n+1}\big)\Big)\cdot(\nabla\phi^0_i + \nabla_y\phi^1_i)\, dydxdt.\end{aligned} \tag{42}$$

Choosing $\phi^0 = 0$ and setting $W_i = V_i - 3u_iV_{n+1}$, this gives

$$0 = \sum_{i=1}^n\int_0^T\int_\Omega\fint_Y P(y)D_iu^{1/2}_{n+1}\big(\nabla(u^{1/2}_{n+1}u_i) - 3u_i\nabla u^{1/2}_{n+1} + \nabla_y W_i\big)\cdot\nabla_y\phi^1_i dydxdt.$$

This is a linear equation for $W_1,\dots,W_n$ and a weak formulation of a system of uncoupled elliptic equations for $W = (W_1,\dots,W_n)$. Since for $x \in \Omega$ and $t > 0$ such that $u_{n+1}(t,x) >$

0, we have a unique (up to a constant) solution of the system for W, each W_i is defined by

$$0 = \int_0^T \int_\Omega \fint_Y P(y) D_i u_{n+1}^{1/2} \big(\nabla (u_{n+1}^{1/2} u_i) - 3u_i \nabla u_{n+1}^{1/2} + \nabla_y W_i \big) \cdot \nabla_y \phi_i^1 dy dx dt. \tag{43}$$

This motivates the following ansatz:

$$W_i(x,y,t) = \sum_{\ell=1}^d \left(\frac{\partial}{\partial x_\ell} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_\ell} u_{n+1}^{1/2} \right) w_i^\ell (x,y,t) \tag{44}$$

for some functions w_i^ℓ for $\ell = 1, \ldots, d$ and $i = 1, \ldots, n$. Substituting the ansatz (44) into (43), we find that w_i^ℓ solves

$$\begin{aligned} 0 &= D_i \int_0^T \int_\Omega \fint_Y u_{n+1}^{1/2} \sum_{k=1}^d P_k(y) \bigg\{ \frac{\partial}{\partial x_k} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_k} u_{n+1}^{1/2} \\ &\quad + \sum_{\ell=1}^d \left(\frac{\partial}{\partial x_\ell} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_\ell} u_{n+1}^{1/2} \right) \frac{\partial w_i^\ell}{\partial y_k} \bigg\} \frac{\partial \phi_i^1}{\partial y_k} dy dx dt \\ &= D_i \sum_{\ell=1}^d \int_{\Omega_T} \fint_Y u_{n+1}^{1/2} \left(\frac{\partial}{\partial x_\ell} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_\ell} u_{n+1}^{1/2} \right) \sum_{k=1}^d P_k(y) \left(\frac{\partial w_i^\ell}{\partial y_k} + \delta_{k\ell} \right) \frac{\partial \phi_i^1}{\partial y_k} dy dx dt. \end{aligned}$$

Since the functions u_i are independent of y, we see that w_i^ℓ is in fact a solution of the unit-cell problem

$$\operatorname{div}_y \big(P(y)(\nabla_y w_i^\ell + e_\ell) \big) = 0 \quad \text{in } Y, \qquad \int_Y w_i^\ell (y,t) dy = 0, \quad w_i^\ell \text{ is } Y\text{-periodic},$$

where $i = 1, \ldots, n$, $\ell = 1, \ldots, d$, and recalling that $(e_1, \ldots, e_d)$ is the canonical basis of $\mathbb{R}^d$. These problems do not depend on i, so we may set $w^\ell := w_i^\ell$ for $i = 1, \ldots, n$.

Next, we choose $\phi_i^1 = 0$ for $i = 1, \ldots, n$ in (42):

$$\begin{aligned} 0 &= -\int_0^T \int_\Omega u \cdot \partial_t \phi^0 dx dt \\ &\quad + \int_0^T \int_\Omega \fint_Y \sum_{i=1}^n P(y) D_i u_{n+1}^{1/2} \big(\nabla (u_{n+1}^{1/2} u_i) - 3u_i \nabla u_{n+1}^{1/2} + \nabla_y W_i \big) \cdot \nabla \phi_i^0 dy dx dt. \end{aligned}$$

Inserting the ansatz (44) and rearranging the terms leads to

$$\begin{aligned} 0 &= -\int_0^T \int_\Omega u \cdot \partial_t \phi^0 dx dt + \sum_{i=1}^n D_i \int_0^T \int_\Omega \fint_Y P(y) u_{n+1}^{1/2} \bigg\{ \nabla (u_{n+1}^{1/2} u_i) - 3u_i \nabla u_{n+1}^{1/2} \\ &\quad + \sum_{\ell=1}^d \left(\frac{\partial}{\partial x_\ell} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_\ell} u_{n+1}^{1/2} \right) \nabla_y w^\ell \bigg\} \cdot \nabla_y \phi_i^0 dy dx dt \qquad (45) \\ &= -\int_0^T \int_\Omega u \cdot \partial_t \phi^0 dx dt + \sum_{i=1}^n D_i \int_0^T \int_\Omega u_{n+1}^{1/2} \sum_{\ell=1}^d \bigg\{ \left(\frac{\partial}{\partial x_\ell} (u_{n+1}^{1/2} u_i) - 3u_i \frac{\partial}{\partial x_\ell} u_{n+1}^{1/2} \right) \end{aligned}$$

$$\times \fint_Y \left(P_l(y) \frac{\partial \phi_i^0}{\partial x_\ell} + \sum_{k=1}^d P_k(y) \frac{\partial w^\ell}{\partial y_k} \frac{\partial \phi_i^0}{\partial x_k} \right) dy \Big\} dxdt.$$

Then, defining the macroscopic matrix $D_{\rm hom} = (D_{{\rm hom},k\ell})_{k,\ell=1}^d$ by

$$D_{{\rm hom},k\ell} = \fint_Y P_k(y) \left(\delta_{k\ell} + \frac{\partial w^\ell}{\partial y_k} \right) dy, \quad \text{for} \quad k, \ell = 1, \ldots, d, \tag{46}$$

we obtain the macroscopic problem (10). We deduce from equation (45) and the regularity of u that $\partial_t u \in L^2(0,T; H^1(\Omega;\mathbb{R}^n)')$ and consequently, the initial conditions are satisfied in the sense of $H^1(\Omega;\mathbb{R}^n)'$.

In the case of the macroscopic problem (3) with the diffusion matrix (6) defined in the perforated domain Ω^ε, the convergence results of Lemma 13 lead to the following two-scale problem:

$$\begin{aligned} 0 = &- \int_0^T \int_\Omega u \cdot \partial_t \phi \, dxdt + \int_0^T \int_\Omega \fint_{Y_1} \sum_{i=1}^n u_{n+1}^{1/2} D_i \Big(\nabla (u_{n+1}^{1/2} u_i) + \nabla_y V_i \\ &- 3u_j \big(\nabla u_{n+1}^{1/2} + \nabla_y V_{n+1} \big) \Big) \cdot (\nabla \phi_i^0 + \nabla_y \phi_i^1) \, dydxdt. \end{aligned}$$

We can calculate as above to find similar macroscopic equations for the microscopic problem (3) with the only difference that the unit-cell problem for $\widehat{w}^\ell$ is given by

$$\begin{aligned} &{\rm div}_y (\nabla_y \widehat{w}^\ell + e_\ell) = 0 \quad \text{in } Y_1, \quad \int_{Y_1} \widehat{w}^\ell (y,t) = 0, \\ &(\nabla_y \widehat{w}^\ell + e_\ell) \cdot \nu = 0 \quad \text{on } \Gamma, \quad \widehat{w}^\ell \text{ is } Y\text{-periodic}, \end{aligned}$$

and the macroscopic diffusion coefficients are

$$D_{{\rm hom},k\ell} = \fint_{Y_1} \left(\delta_{k\ell} + \frac{\partial \widehat{w}^\ell}{\partial y_k} \right) dy, \quad \text{for} \quad k, \ell = 1, \ldots, d, \tag{47}$$

Observe that the specific structure of the microscopic problem implies a separation of variables in the two-scale problems and that consequently, scalar unit-cell problems determine the macroscopic diffusion matrix. □

Appendix A. Two-scale convergence

We recall the definition and some properties of two-scale convergence. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $Y \subset \mathbb{R}^d$ be the "periodicity cell" identified with the d-dimensional torus with measure $|Y|$. Consider also the perforated domain Ω^ε and the corresponding subsets $\overline{Y}_0 \subset Y$ and $Y_1 = Y \setminus \overline{Y}_0$.

Definition 3 (Two-scale convergence)**.** (i) *A sequence* (u^ε) *in* $L^2(\Omega)$ *is* two-scale convergent *to* $u \in L^2(\Omega \times Y)$ *if for any smooth* Y*-periodic function* $\phi : \Omega \times Y \to \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \phi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \fint_Y u(x,y) \phi(x,y) dxdy.$$

(ii) *The sequence* (u^ε) *is* strongly two-scale convergent *to* $u \in L^2(\Omega \times Y)$ *if*

$$\lim_{\varepsilon\to 0}\int_\Omega \left| u^\varepsilon(x) - u\left(x, \frac{x}{\varepsilon}\right)\right|^2 dx = 0.$$

Remark 14. Let $[\cdot]^\sim$ denote the extension by zero in the domain $\Omega \setminus \Omega^\varepsilon$ and $\chi_{\Omega^\varepsilon}$ be the characteristic function of Ω^ε.

(i) If $\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C$, then $\|[u^\varepsilon]^\sim\|_{L^2(\Omega)} \le C$ and there exists $u \in L^2(\Omega \times Y)$ such that, up to a subsequence, $[u^\varepsilon]^\sim \rightharpoonup \chi_{Y_1} u$ two-scale:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon} u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)dx &= \lim_{\varepsilon\to 0}\int_\Omega [u^\varepsilon(x)]^\sim \phi\left(x, \frac{x}{\varepsilon}\right)dx \\ &= \lim_{\varepsilon\to 0}\int_\Omega [u^\varepsilon(x)]^\sim \chi_{\Omega^\varepsilon}(x)\phi\left(x, \frac{x}{\varepsilon}\right)dx = \int_\Omega \fint_Y \chi_{Y_1}(y)u(x,y)\phi(x,y)dxdy.\end{aligned}$$

(ii) If $u^\varepsilon \rightharpoonup u$ two-scale with $u \in L^p(\Omega \times Y)$ then

$$u^\varepsilon \rightharpoonup \fint_Y u(x,y)dy \quad \text{weakly in } L^p(\Omega) \text{ for } p \in [1,\infty).$$

The following results hold.

Lemma 15 ([2, 20])**.** (i) *If* (u^ε) *is bounded in* $L^2(\Omega)$*, there exists a subsequence (not relabeled) such that* $u^\varepsilon \rightharpoonup u$ *two-scale as* $\varepsilon \to 0$ *for some function* $u \in L^2(\Omega \times Y)$*.*

(ii) *If* $u^\varepsilon \rightharpoonup u$ *weakly in* $H^1(\Omega)$ *then* $\nabla u^\varepsilon \rightharpoonup \nabla u(x) + \nabla_y u_1(x,y)$ *two-scale, where* $u_1 \in L^2(\Omega; H^1_{\rm per}(Y)/\mathbb{R})$*.*

Lemma 16 ([2])**.** *Let* $\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|\nabla u^\varepsilon\|_{L^2(\Omega^\varepsilon)} \le C$*. Then, up to a subsequence,* $[u^\varepsilon]^\sim$ *and* $[\nabla u^\varepsilon]^\sim$ *two-scale converge to* $\chi_{Y_1}(y)u(x)$ *and* $\chi_{Y_1}(y)[\nabla u(x) + \nabla_y u_1(x,y)]$ *as* $\varepsilon \to 0$*, respectively, where* $u \in H^1(\Omega)$ *and* $u_1 \in L^2(\Omega; H^1_{\rm per}(Y_1)/\mathbb{R})$*.*

Lemma 17 ([2, 20])**.** *Let* $(u^\varepsilon) \subset L^2(\Omega)$ *converges two-scale to* $u \in L^2(\Omega\times Y)$*,* $\|u^\varepsilon\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega\times Y)}$ *as* $\varepsilon \to 0$*, and let* $(v^\varepsilon) \subset L^2(\Omega)$ *converges two-scale to* $v \in L^2(\Omega \times Y)$*. Then, as* $\varepsilon \to 0$*,*

$$\int_\Omega u^\varepsilon v^\varepsilon dx \to \int_\Omega \fint_Y u(x,y)v(x,y)dxdy.$$

To define the unfolding operator, let $[z]$ for any $z \in \mathbb{R}^d$ denotes the unique combination $\sum_{i=1}^d k_i e_i$ with $k \in \mathbb{Z}^d$, such that $z - [z] \in Y$, where e_i is the ith canonical basis vector of $\mathbb{R}^d$.

Definition 4 ([8])**.** *Let* $p \in [1,\infty]$ *and* $\phi \in L^p(\Omega)$*. Then the unfolding operator* $\mathcal{T}^\varepsilon$ *is defined by* $\mathcal{T}^\varepsilon(\phi) \in L^p(\mathbb{R}^d \times Y)$*, where*

$$\mathcal{T}^\varepsilon(\phi)(x,y) = \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{for a.e. } (x,y) \in \Omega \times Y.$$

Furthermore, for $\psi \in L^p(\Omega^\varepsilon)$*, the unfolding operator* $\mathcal{T}^\varepsilon_{Y_1}$ *is defined by*

$$\mathcal{T}^\varepsilon_{Y_1}(\psi)(x,y) = \psi\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right) \quad \text{for a.e. } (x,y) \in \Omega \times Y_1.$$

For any function ψ defined on Ω^ε, we have $\mathcal{T}^\varepsilon_{Y_1}(\psi) = \mathcal{T}^\varepsilon([\psi]^\sim)|_{\Omega\times Y_1}$, whereas for ϕ defined on Ω, it holds that $\mathcal{T}^\varepsilon_{Y_1}(\phi|_{\Omega^\varepsilon}) = \mathcal{T}^\varepsilon(\phi)|_{\Omega\times Y_1}$. The following result relates the two-scale convergence and the weak convergence involving the unfolding operator.

Proposition 18 ([7]). *Let (ψ^ε) be a bounded sequence in $L^p(\Omega)$ for some $1 < p < \infty$. Then the following assertions are equivalent:*

(i) *$(\mathcal{T}^\varepsilon(\psi^\varepsilon))$ converges weakly to ψ in $L^p(\Omega \times Y)$.*
(ii) *(ψ^ε) converges two-scale to ψ.*

Theorem 19 ([8]). *Let (ψ^ε) be a bounded sequence in $W^{1,p}(\Omega^\varepsilon)$ for some $1 \le p < \infty$. Then there exist functions $\psi \in W^{1,p}(\Omega)$ and $\psi_1 \in L^p(\Omega; W^{1,p}_{\rm per}(Y_1)/\mathbb{R})$ such that as $\varepsilon \to 0$, up to a subsequence,*

$$\begin{aligned} &\mathcal{T}^\varepsilon_{Y_1}(\psi^\varepsilon) \rightharpoonup \psi && \text{weakly in } L^p(\Omega; W^{1,p}(Y_1)),\\ &\mathcal{T}^\varepsilon_{Y_1}(\psi^\varepsilon) \to \psi && \text{strongly in } L^p_{\rm loc}(\Omega; W^{1,p}(Y_1)),\\ &\mathcal{T}^\varepsilon_{Y_1}(\nabla\psi^\varepsilon) \rightharpoonup \nabla\psi + \nabla_y\psi_1 && \text{weakly in } L^p(\Omega \times Y_1). \end{aligned}$$

Lemma 20 ([6, 10]). (i) *For $u \in H^1(Y_1)$, there exists an extension $\overline{u}$ into Y_0 and thus onto Y such that*

$$\|\overline{u}\|_{L^2(Y)} \le C\|u\|_{L^2(Y_1)}, \quad \|\nabla\overline{u}\|_{L^2(Y)} \le C\|\nabla u\|_{L^2(Y_1)}.$$

(ii) *For $u \in H^1(\Omega^\varepsilon)$ there exists an extension $\overline{u}$ into Ω such that*

$$\|\overline{u}\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega^\varepsilon)}, \quad \|\nabla\overline{u}\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega^\varepsilon)},$$

where the constant C is independent of ε.

Sketch of the proof. We can write $u = ⨍_{Y_1} u dy + \psi$, where $⨍_{Y_1} \psi dy = 0$. By standard extension results, we obtain an extension $\overline{\psi} \in H^1(Y)$ of ψ. The definition $\overline{u} = ⨍_{Y_1} u dy + \overline{\psi}$ and the Poincaré inequality imply the results stated in (i). The results in (i) and a scaling argument ensure the existence of an extension from Ω^ε into Ω and estimates in (ii) uniform in ε. □

The same results hold also for $u \in W^{1,p}(\Omega^\varepsilon)$, with $1 \le p < \infty$, see, e.g., [1].

Notice that the corresponding extension operator is linear and continuous from $H^1(\Omega^\varepsilon)$ to $H^1(\Omega)$ and by the construction of the extension, we have $\overline{u} = u$ in Ω^ε.

Appendix B. Proof of Lemma 9.

Consider the entropy density

$$h(u) = \sum_{i=1}^{n+1}(u_i \log u_i - u_i + 1) \quad \text{for } u = (u_1, \ldots, u_n) \in \mathcal{G}, \tag{48}$$

where $u_{n+1} = 1 - \sum_{i=1}^n u_i$. Since $h'(u) = (\log(u_1/u_{n+1}), \ldots, \log(u_n/u_{n+1}))$ is invertible on $\mathcal{G}$, the solutions of the microscopic problem are bounded, $u \in \overline{\mathcal{G}}$. By Lemma 7 in [28], it holds for all $z \in \mathbb{R}^n$ and $u \in \mathcal{G}$ that

$$z^\top h''(u)A(u)z \geq p_0 u_{n+1} \sum_{i=1}^n \frac{z_i^2}{u_i} + \frac{p_0}{2}\frac{1}{u_{n+1}}\bigg(\sum_{i=1}^n z_i\bigg)^2,$$

where $p_0 = \min_{i=1,\dots,n} D_i > 0$. This shows that for suitable functions $u = (u_1, \dots, u_n)$,

$$\nabla u : h''(u)A(u)\nabla u \geq 4p_0 u_{n+1} \sum_{i=1}^n |\nabla u_i^{1/2}|^2 + 2p_0 |\nabla u_{n+1}^{1/2}|^2.$$

The entropy inequality is derived formally from the weak formulation of (3) by choosing the test function $w^\varepsilon = h'(u^\varepsilon)$. Since this function is not in $L^2(0,T;H^1(\Omega))$, we need to consider a regularization. We define

$$w_\delta^\varepsilon(u^\varepsilon) = h'(u_\delta^\varepsilon) \quad \text{and} \quad \phi_{\sigma,\delta}^\varepsilon = \frac{(u_{n+1}^\varepsilon)^{1/2}}{(u_{n+1}^\varepsilon)^{1/2} + \sigma} w_\delta^\varepsilon(u^\varepsilon), \quad \text{where}$$

$$u_{\delta,j}^\varepsilon = \frac{u_j^\varepsilon + \delta_1}{1+\delta}, \quad u_{\delta,n+1}^\varepsilon = \frac{u_{n+1}^\varepsilon + \frac{\delta}{2}}{1+\delta} \quad \text{for } \delta > 0,\ \delta_1 = \frac{\delta}{2n},\ j = 1, \dots, n.$$

Thanks to the regularity properties of u_i^ε, the function

$$\begin{aligned}\nabla \phi_{\sigma,\delta,i}^\varepsilon &= \frac{(u_{n+1}^\varepsilon)^{1/2}}{(u_{n+1}^\varepsilon)^{1/2} + \sigma}\bigg(\frac{\nabla u_i^\varepsilon}{u_i^\varepsilon + \delta_1} + \frac{\nabla u_{n+1}^\varepsilon}{u_{n+1}^\varepsilon + \delta/2}\bigg) \\ &\quad + w_{\delta,i}^\varepsilon(u^\varepsilon)\bigg(\frac{\nabla (u_{n+1}^\varepsilon)^{1/2}}{(u_{n+1}^\varepsilon)^{1/2} + \sigma} - \frac{(u_{n+1}^\varepsilon)^{1/2}\nabla (u_{n+1}^\varepsilon)^{1/2}}{((u_{n+1}^\varepsilon)^{1/2} + \sigma)^2}\bigg)\end{aligned}$$

is in $L^2(\Omega_T)$ for each fixed σ, $\delta > 0$. Thus, we can use $\phi_{\sigma,\delta}^\varepsilon$ as a test function in (9):

$$\begin{aligned}&\sum_{i=1}^n \int_{\Omega_T} P^\varepsilon(x) D_i (u_{n+1}^\varepsilon)^{1/2}\big(\nabla (u_i^\varepsilon (u_{n+1}^\varepsilon)^{1/2}) - 3u_i^\varepsilon \nabla (u_{n+1}^\varepsilon)^{1/2}\big) \cdot \nabla \phi_{\sigma,\delta,i}^\varepsilon dxdt \\ &\quad + \sum_{i=1}^n \int_0^T \langle \partial_t u_i^\varepsilon, \phi_{\sigma,\delta,i}^\varepsilon \rangle dt = 0. \qquad (49)\end{aligned}$$

The nonnegativity of u_{n+1}^ε and u_j^ε yields the pointwise monotone convergences

$$\begin{aligned}&\frac{(u_{n+1}^\varepsilon)^{1/2}}{(u_{n+1}^\varepsilon)^{1/2} + \sigma} \to 1, \quad \frac{u_{n+1}^\varepsilon}{[(u_{n+1}^\varepsilon)^{1/2} + \sigma]^2} \to 1 \quad \text{as } \sigma \to 0, \\ &\frac{u_i^\varepsilon}{u_i^\varepsilon + \delta/2n} \to 1, \qquad \frac{u_{n+1}^\varepsilon}{u_{n+1}^\varepsilon + \delta/2} \to 1 \qquad \text{as } \delta \to 0.\end{aligned}$$

As these four sequences are uniformly bounded by 1, they converge strongly in $L^p(\Omega_T)$ for any $1 < p < \infty$. Thus, the definition of $w_\delta^\varepsilon(u^\varepsilon)$ and the L^2-regularity of $(u_{n+1}^\varepsilon)^{1/2}\nabla u_i^\varepsilon$, $(u_{n+1}^\varepsilon)^{1/2}\nabla (u_i^\varepsilon)^{1/2}$, $\nabla (u_{n+1}^\varepsilon)^{3/2}$, and $\nabla (u_{n+1}^\varepsilon)^{\frac{1}{2}}$ ensure that

$$(u_{n+1}^\varepsilon)^{1/2}\nabla \phi_{\sigma,\delta,i}^\varepsilon \to \frac{(u_{n+1}^\varepsilon)^{1/2}\nabla u_i^\varepsilon}{u_i^\varepsilon + \delta/2n} + \frac{(u_{n+1}^\varepsilon)^{1/2}\nabla u_{n+1}^\varepsilon}{u_{n+1}^\varepsilon + \delta/2} \quad \text{strongly in } L^2(\Omega_T),$$

as $\sigma \to 0$, and the sequences

$$\frac{u_i^\varepsilon}{u_i^\varepsilon + \delta/2n}(u_{n+1}^\varepsilon)^{1/2}\nabla u_i^\varepsilon \cdot \nabla (u_{n+1}^\varepsilon)^{1/2}, \quad \frac{u_i^\varepsilon}{u_i^\varepsilon + \delta/2n} u_{n+1}^\varepsilon |\nabla (u_i^\varepsilon)^{1/2}|^2,$$

$$\frac{u_{n+1}^\varepsilon}{u_{n+1}^\varepsilon + \delta/2} u_i^\varepsilon |\nabla (u_{n+1}^\varepsilon)^{1/2}|^2, \qquad \frac{u_{n+1}^\varepsilon}{u_{n+1}^\varepsilon + \delta/2}(u_{n+1}^\varepsilon)^{1/2}\nabla u_i^\varepsilon \cdot \nabla (u_{n+1}^\varepsilon)^{1/2}$$

convergence, up to a subsequence, strongly in $L^1(\Omega_T)$ as $\delta \to 0$, for $i = 1, \dots, n$. The pointwise convergence of $u_{\delta,j}^\varepsilon$ as $\delta \to 0$ and the boundedness of the function $s \mapsto s \log s$ for $s \in [0,1]$ ensure the convergence of $h(u_\delta^\varepsilon)$ in $L^1(\Omega_T)$. Rearranging the terms in (49) and letting first $\sigma \to 0$ and then $\delta \to 0$ yields the entropy inequality (35).

The same calculations yield entropy estimate for solutions of problem (3)-(4) with diffusion matrix (6).

Appendix C. Examples satisfying Assumption A6

We present two cross-diffusion systems whose diffusion matrix and associated entropy density satisfy Assumption A6. The first example appears in biofilm modeling. A biofilm is an aggregate of microorganisms consisting of several subpopulations of bacteria, algae, protozoa, etc. We assume that the biofilm consist of three subpopulations and that it is saturated, i.e., the volume fractions of the subpopulations u_i sum up to one. Therefore, the volume fraction of one subpopulation can be expressed by the remaining ones, $u_3 = 1-u_1-u_2$. A heuristic approach to define the diffusion fluxes [27] leads to the cross-diffusion system (1) with diffusion matrix

$$A(u) = \begin{pmatrix} D_1(1-u_1) & -D_2 u_1 \\ -D_1 u_2 & D_2(1-u_2) \end{pmatrix},$$

where $D_1 > 0$ and $D_2 > 0$ are some diffusion coefficients. Taking the entropy density

$$h(u) = \sum_{i=1}^{2} u_i(\log u_i - 1) + (1-u_1-u_2)(\log(1-u_1-u_2) - 1), \tag{50}$$

we compute

$$h''(u)A(u) = \begin{pmatrix} D_1/u_1 & 0 \\ 0 & D_2/u_2 \end{pmatrix}.$$

This shows that Assumptions A2 and A6 are satisfied with $s_1 = s_2 = -1/2$.

The second example is a model that describes the evolution of an avascular tumor. During the avascular stage, the tumor remains in a diffusion-limited, dormant stage with a diameter of a few millimeters. We suppose that the tumor growth can be described by the volume fraction u_1 of tumor cells, the volume fraction of the extracellular matrix u_2 (a mesh of fibrous proteins and polysaccharides), and the volume fraction of water/nutrients $u_3 = 1 - u_1 - u_2$. Jackson and Byrne [12] have derived by a fluiddynamical approach the cross-diffusion model (1) with diffusion matrix

$$A(u) = \begin{pmatrix} 2u_1(1-u_1) - \beta\theta u_1 u_2^2 & -2\beta u_1 u_2(1+\theta u_1) \\ -2u_1 u_2 + \beta\theta(1-u_2)u_2^2 & 2\beta u_2(1-u_2)(1+\theta u_1) \end{pmatrix},$$

where the parameters $\beta > 0$ and $\theta > 0$ model the strength of the partial pressures. With the entropy (50), we find that [15, (32)]

$$h''(u)A(u) = \begin{pmatrix} 2 & 0 \\ \beta\theta u_2 & 2\beta(1+\theta u_1) \end{pmatrix}.$$

Assuming that $\theta < 4\sqrt{\beta}$, it follows for $0 \le u_1, u_2 \le 1$ and $z \in \mathbb{R}^2$ that

$$z^\top h''(u)A(u)z \ge (2-\varepsilon)z_1^2 + 2\beta\left(1 - \frac{\beta\theta^2}{8\varepsilon}\right)z_2^2 \ge \kappa|z|^2,$$

where $\kappa = \min\{2-\varepsilon, 2\beta(1-\beta\theta^2/(8\varepsilon)\} > 0$ if we choose $0 < \varepsilon < 2$. Then Assumption A2 is fulfilled with $s_1 = s_2 = 0$, and Assumption A6 holds as well since $(h''(u)A(u))_{21}$ is bounded from above by $\beta\theta$.

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