Accepted Manuscript

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PII: S0167-2789(18)30559-1
DOI: https://doi.org/10.1016/j.physd.2019.06.003
Reference: PHYSD 32137

To appear in: *Physica D*

Received date : 23 November 2018
Revised date : 25 March 2019
Accepted date : 2 June 2019

Please cite this article as: I. Findlay, A dual construction of the isotropic Landau-Lifshitz model, *Physica D* (2019), https://doi.org/10.1016/j.physd.2019.06.003

A Dual Construction of the Isotropic Landau-Lifshitz Model

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Abstract

By interchanging the roles of the space and time coordinates, we describe a dual construction of the isotropic Landau-Lifshitz model, providing equal-space Poisson brackets and dual Hamiltonians conserved with respect to space-evolution. This construction is built in the Lax/zero-curvature formalism, where the duality between the space and time dependencies is evident.

Keywords: isotropic Landau-Lifshitz model, Lax pair, r-matrix, zero-curvature condition, dual integrable model, integrable boundary conditions

1. Introduction

The idea of considering 1+1 dimensional integrable models in terms of their "space-evolution", as governed by some equal-space Poisson brackets found by interchanging the roles of the space and time coordinates was systematically introduced in [1], following the suggestion in [2] for the purposes of identifying integrable defects (that lie in the spatial axis) with Darboux-Bäcklund transformations. This concept was applied rigorously to the Lax/zero-curvature construction [3, 4] of the non-linear Schrödinger (NLS) model in [1], and then later proven for the general NLS hierarchy in [5].

In this paper, we apply this equal-space construction to the isotropic Landau-Lifshitz model [6, 7], which is also known as the continuous classical Heisenberg magnet (HM) model:

$$\partial_t \vec{S} = \frac{\mathrm{i}}{c^2} \vec{S} \times (\partial_x^2 \vec{S}), \tag{1.1}$$

which depends on the vector $\vec{S} = (S_x, S_y, S_z)^T$. These fields will also be written in the combinations $S_\pm = S_x \pm \mathrm{i} S_y$, which satisfy the sl_2 exchange relations:

$$\{S_\pm(x), S_z(y)\} = \pm S_\pm \, \delta(x - y), \qquad \{S_+(x), S_-(y)\} = -2 S_z \, \delta(x - y). \tag{1.2}$$

These Poisson brackets are found through the r-matrix construction [8]. The HM model has the same underlying r-matrix as the NLS model, namely the Yangian r-matrix (2.2), so hence it arises as a natural next step in the development of this dual approach.

Because this equal-space picture follows in parallel to the usual method for building conserved quantities and higher systems (see [9]), we also introduce reflective time-like boundary conditions [10] to the HM model by following an equivalent procedure to the development of reflective space-like boundary conditions, [12, 13], which have been applied to the isotropic Landau-Lifshitz equation in [14].

The HM model is also of recent practical interest as a simple model of 1 dimensional ferromagnetism (due to being the continuum limit of the classical analogue of the quantum XXX spin chain, see [9, 15, 16] for details), [17, 18, 19]. This paper therefore sheds new light on this model by approaching it from a time-like perspective, analogous to the standard description in terms of time-evolution.

The paper is laid out as follows: The remainder of Section 1 defines the basic terms that we will be using throughout. Then, in Section 2 we describe the standard (equal-time) construction of the hierarchy of conserved quantities and their associated Lax pairs and integrable systems, applying these to the HM model for later comparison. This section starts by constructing the Poisson brackets between the fields in Subsection 2.1, before building the hierarchy of conserved quantities that guarantee the integrability of the HM model. This is done for both closed (periodic) boundary conditions in Subsection 2.2 and open (reflective) boundary conditions in Subsection 2.3. Subsection 2.3 recalls the results of [14], except using notation that will be consistent with the sections that follow. Finally, we repeat these same steps for the dual (equal-space) construction of the HM model in Section 3, with the dual Poisson structure constructed in Subsection 3.1, and the hierarchies of dual Hamiltonians (and the corresponding Lax pairs) for both closed and open boundary conditions are constructed in Subsections 3.2 and 3.4, respectively.

1.1. *Preliminaries*

In terms of the fields $S_\pm$ and S_z, the equations of motion (1.1) become:

$$\partial_t S_\pm = \pm\frac{1}{c^2}\big(S_\pm(\partial_x^2 S_z) - (\partial_x^2 S_\pm)S_z\big), \qquad \partial_t S_z = \frac{1}{2c^2}\big((\partial_x^2 S_+)S_- - S_+(\partial_x^2 S_-)\big). \tag{1.3}$$

When referencing the three fields $S_\pm$ and S_z, we will use the subscript $\sigma \in \{+,-,z\}$ to collectively refer to them as S_σ. We will also use $\dot{S}_\sigma = \partial_{t_k} S_\sigma$ to denote the derivative of S_σ with respect to the appropriate time flow[1] t_k, and $S'_\sigma = \partial_{x_k} S_\sigma$ for the derivative with respect to the contextually appropriate space flow. Where there is likely ambiguity however, we will explicitly use either ∂_{t_k} or ∂_{x_k}.

It was shown in [7] that the system of equations (1.3) appear as the compatibility condition of the auxiliary linear problem:

$$\Psi' \equiv \partial_x \Psi = U\Psi, \qquad \dot{\Psi} \equiv \partial_t \Psi = V\Psi, \tag{1.4}$$

where Ψ is an arbitrary vector field, and the 2×2 matrices U and V, depending on the fields S_σ as well as some free complex parameter λ, comprise the Lax pair [3, 4] of the system, given by:

$$U = \frac{1}{2\lambda}S, \qquad V = \frac{1}{2\lambda^2}S - \frac{1}{2c^2\lambda}S'S, \tag{1.5}$$

where:

$$S = \begin{pmatrix} S_z & S_- \\ S_+ & -S_z \end{pmatrix}.$$

Cross-differentiating the auxiliary linear problem gives rise to the following compatibility condition (called the zero-curvature condition) between the matrices of the Lax pair:

$$0 = \dot{U} - V' + [U, V], \tag{1.6}$$

such that when the matrices U and V are inserted into this, and the resulting equations are split about powers of λ, the equations of motion, (1.3), are returned.

2. The Standard Picture

2.1. *Poisson Brackets*

Before we introduce the dual picture for (1.3) we first recap the method for constructing the hierarchy of integrable equations and their Hamiltonians. The core objects in this construction are the spatial component of the Lax pair, U, and an associated r-matrix that satisfies the classical Yang-Baxter equation [20]:

$$0 = [r_{ab}(\lambda-\mu), r_{ac}(\lambda)] + [r_{ab}(\lambda-\mu), r_{bc}(\mu)] + [r_{ac}(\lambda), r_{bc}(\mu)], \tag{2.1}$$

[1] These distinct time flows will arise from considering the tower of conserved quantities that define the system as integrable, and treating each of the quantities as the Hamiltonian for a distinct integrable system, describing the evolution of the fields along the associated time flow t_k. When we consider the dual picture, we will likewise have a hierarchy of dual Hamiltonians that govern the space-evolution of the fields along a tower of space flows x_k.

where $\lambda, \mu \in \mathbb{C}$ are some free parameters and the subscripts denote which vector spaces the matrices act on (e.g. $r_{ab} = r \otimes \mathbb{I}$ and $r_{bc} = \mathbb{I} \otimes r$, with $r : V \otimes V \to V \otimes V$, so that the whole equation acts on $V_a \otimes V_b \otimes V_c$, where the subscripts attached to the vector spaces are merely used to denote which index corresponds to them, e.g. r_{ab} would act only on the first two). For the HM model, the relevant solution is:

$$r(\lambda) = \frac{1}{2\lambda}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

This r-matrix is connected to the U-matrix and the equations of motion for the system (1.3) through the linear algebraic relation[2] [8]:

$$\{U_a(x,\lambda), U_b(y,\mu)\}_S = [r_{ab}(\lambda-\mu), U_a(x,\lambda) + U_b(y,\mu)]\,\delta(x-y), \tag{2.3}$$

which provides an ultra-local Poisson bracket between the fields. Inserting the U-matrix (1.5) and r-matrix (2.2) into this relation returns the $\mathfrak{sl}_2$ exchange relations, (1.2). From these Poisson brackets we can read off a Casimir element that restricts the vector $\vec{S}$ to the surface of the sphere of radius c, where we have labelled the Casimir c^2:

$$c^2 = S_z^2 + S_+S_- = S_x^2 + S_y^2 + S_z^2. \tag{2.4}$$

2.2. Periodic Boundary Conditions

In order to find conserved quantities that commute with respect to this Poisson bracket, we start by considering the (spatial) transport matrix, which is a path-ordered exponential solution to the spatial component of the auxiliary linear problem (1.4) in place of Ψ:

$$T_S(x,y;\lambda) = \mathcal{P}\exp\int_y^x U(\xi)\mathrm{d}\xi. \tag{2.5}$$

For a periodic system on the interval $[-L, L]$, i.e. where $S_\sigma(L) = S_\sigma(-L)$, the full monodromy matrix is $T_S(\lambda) = T_S(L,-L;\lambda)$. Due to the U-matrix satisfying the linear algebraic relation, (2.3), the monodromy matrix can be seen to satisfy a quadratic algebraic relation [21, 22]:

$$\{T_{S,a}(\lambda), T_{S,b}(\mu)\}_S = [r_{ab}(\lambda-\mu), T_{S,a}(\lambda)T_{S,b}(\mu)]. \tag{2.6}$$

Consequently, if we define a new object, called the transfer matrix $\mathfrak{t}_S(\lambda)$, as the trace of the monodromy matrix:

$$\mathfrak{t}_S(\lambda) = \mathrm{tr}\,\{T_S(\lambda)\}, \tag{2.7}$$

then this can be shown to Poisson commute with itself for different values of the spectral parameter λ. Because of this, if we expand $\mathfrak{t}_S$ as a formal power series in λ, $\mathfrak{t}_S = \sum_k \lambda^k \mathfrak{t}_S^{(k)}$, then these coefficients commute:

$$\{\mathfrak{t}_S^{(k)}, \mathfrak{t}_S^{(j)}\}_S = 0. \tag{2.8}$$

As such, the terms in this expansion $\mathfrak{t}_S^{(k)}$ can be seen as "Hamiltonians" governing the evolution of the system along distinct time flows t_k. Further to this, the evolution along each time flow t_k will be integrable *à la* Liouville, as the $\mathfrak{t}_S^{(j)}$ with $j \neq k$ will provide the infinite tower of conserved quantities.

[2]The subscript S is used here and in what follows to denote that we are building this system out of the Spatial component of the Lax pair (U). This will be important later when we construct the dual model out of the Temporal component of the Lax pair (V), where we will use a T subscript.

Unfortunately, the "Hamiltonians" generated in this manner will be non-local. To circumvent this, we will consider the coefficients in the expansion of the logarithm of this, $\mathcal{G}_S(\lambda) = \ln(\mathfrak{t}_S(\lambda))$. The logarithm is chosen as it acts to remove the non-locality introduced by the exponential in (2.5) and in the diagonalisation below, (2.9).

The task is therefore to find the expansion of $\mathfrak{t}_S(\lambda)$ in some limit of λ. For the Lax pair (1.5) the appropriate limit is $\lambda \to 0^+$. In order to avoid evaluating the path-ordered exponential, we consider a diagonalisation of the transport matrix [9]:

$$T_S(x, y; \lambda) = \big(\mathbb{I} + W_S(x;\lambda)\big)\mathrm{e}^{Z_S(x,y;\lambda)}\big(\mathbb{I} + W_S(y;\lambda)\big)^{-1}, \tag{2.9}$$

where W_S and Z_S are wholly anti-diagonal and diagonal matrices, respectively. If we insert this diagonalisation into the spatial half of the auxiliary linear problem, the diagonal and anti-diagonal components can be separated into two relations:

$$\begin{aligned} 0 &= W_S' + [W_S, U_D] + W_S U_A W_S - U_A, \\ Z_S' &= U_D + U_A W_S, \end{aligned} \tag{2.10}$$

where U_D and U_A are the diagonal and anti-diagonal components of the U-matrix, respectively. If we expand W_S and Z_S in powers of λ, with coefficients $W_S^{(k)}$ and $Z_S^{(k)}$ [9]:

$$W_S(\lambda) = \sum_{k=0}^{\infty} \lambda^k W_S^{(k)}, \qquad Z_S(\lambda) = \sum_{k=-1}^{\infty} \lambda^k Z_S^{(k)},$$

we can split (2.10) into a series of recurrence relations (making use of how U only depends on λ^{-1}):

$$0 = [W_S^{(0)}, U_D] + W_S^{(0)} U_A W_S^{(0)} - U_A, \qquad 0 = (W_S^{(k)})' + [W_S^{(k+1)}, U_D] + \sum_{j=0}^{k+1} W_S^{(k+1-j)} U_A W_S^{(j)},$$
$$(Z_S^{(-1)})' = U_D + U_A W_S^{(0)}, \qquad (Z_S^{(k)})' = U_A W_S^{(k+1)},$$

which we can recursively solve to find ever higher coefficients in the series expansions of W_S and Z_S. The first few terms in the Z_S-series are:

$$\begin{aligned} Z_S^{(-1)} &= cL\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ Z_S^{(0)} &= \frac{1}{4c}\int_{-L}^{L} \frac{S_+S_-' - S_+'S_-}{c + S_z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathrm{d}x, \\ Z_S^{(1)} &= \frac{-1}{4c^3}\int_{-L}^{L} \left(S_+'S_-' + (S_z')^2\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathrm{d}x. \end{aligned} \tag{2.11}$$

The reason for doing this is that if we insert the decomposition into the definition of the transfer matrix, (2.7), the explicit W dependence cancels out, leaving:

$$\mathfrak{t}_S(\lambda) = \mathrm{tr}\left\{\mathrm{e}^{Z_S(\lambda)}\right\} = \mathrm{e}^{Z_{11,S}(\lambda)} + \mathrm{e}^{Z_{22,S}(\lambda)}.$$

We are actually instead interested in the expansion of $\mathcal{G}_S = \ln(\mathfrak{t}_S)$, which is then:

$$\mathcal{G}_S(\lambda) = \ln\left(\mathrm{e}^{\lambda^{-1}Z_{11,S}^{(-1)} + Z_{11,S}^{(0)} + \lambda Z_{11,S}^{(1)} + \cdots} + \mathrm{e}^{\lambda^{-1}Z_{22,S}^{(-1)} + Z_{22,S}^{(0)} + \lambda Z_{22,S}^{(1)} + \cdots}\right).$$

As the leading order terms in each of the exponents are $cL\lambda^{-1}$ and $-cL\lambda^{-1}$, and we are considering the limit as $\lambda \to 0^+$, the first exponential will be of the form $\mathrm{e}^{cL\lambda^{-1}}$, so will dominate over the second exponential, which will be of the form $\mathrm{e}^{-cL\lambda^{-1}}$, which decays exponentially in the limit $\lambda \to 0^+$. The expansion of $\mathcal{G}_S(\lambda)$ is therefore simply:

$$\mathcal{G}_S(\lambda) = \lambda^{-1} Z_{11,S}^{(-1)} + Z_{11,S}^{(0)} + \lambda Z_{11,S}^{(1)} + \dots.$$

The first three conserved quantities appearing in this expansion can then be read from the Z-series:

$$\begin{aligned}
\mathcal{G}_S^{(-1)} &= cL, \\
\mathcal{G}_S^{(0)} &= \frac{1}{4c}\int_{-L}^{L} \frac{S_+S_-' - S_+'S_-}{c+S_z}\mathrm{d}x, \\
\mathcal{G}_S^{(1)} &= \frac{-1}{4c^3}\int_{-L}^{L} \big(S_+'S_-' + (S_z')^2\big)\mathrm{d}x,
\end{aligned} \tag{2.12}$$

the second and third of which can be recognised as the total momentum and Hamiltonian for the HM model, respectively (up to a factor of $-2c$) [9]:

$$P_S = -2c\mathcal{G}_S^{(0)}, \qquad\qquad H_S = -2c\mathcal{G}_S^{(1)}. \tag{2.13}$$

Each of the conserved quantities $\mathcal{G}_S^{(k)}$ generated through the expansion of $\mathcal{G}_S$ can be seen to describe the evolution of the system along a distinct time flow t_k, so that the equations of motion for each of these systems would be given by:

$$\partial_{t_k} S_\sigma = \{\mathcal{G}_S^{(k)}, S_\sigma\}_S. \tag{2.14}$$

Consequently, each of these systems should have some associated Lax pair. As we use the U-matrix to generate the conserved quantities we will be looking for a generator $\mathbb{V}$ that produces the V-matrices $V^{(k)}$ associated to each time flow t_k. We do so by first equating Hamilton's equation (as applied to U) and the zero-curvature condition:

$$\begin{aligned}
\mathbb{V}_b'(\lambda,\mu) - [U_b(\lambda), \mathbb{V}_b(\lambda,\mu)] = \partial_{\bar{t}} U_b(\lambda) &= \{\ln\left(\mathrm{tr}_a\left\{T_{S,a}(\mu)\right\}\right), U_b(\lambda)\}_S \\
&= t_S^{-1}(\mu)\,\mathrm{tr}_a\left\{\{T_{S,a}(\mu), U_b(\lambda)\}_S\right\}
\end{aligned}$$

where the $\bar{t}$ is used to denote some master time flow and the vector space subscripts are introduced to distinguish the space being traced over (the a vector space). Using the algebraic relations (2.3) and (2.6), we can extract from this the generator of the V-matrices associated to each time flow t_k, [20]:

$$\mathbb{V}_b(x;\lambda,\mu) = t_S^{-1}(\mu)\,\mathrm{tr}_a\left\{T_{S,a}(L,x;\mu) r_{ab}(\mu-\lambda) T_{S,a}(x,-L;\mu)\right\}, \tag{2.15}$$

such that the V-matrix associated to the t_k time flow appears as the coefficient of μ^k in the series expansion of this about μ. Using the diagonalisation of the monodromy matrix, the limit $\mu \to 0^+$ of the exponential of $Z_S(\mu)$, and the cyclic properties of the trace, this can be simplified to:

$$\mathbb{V}_b(x;\lambda,\mu) = \mathrm{tr}_a\left\{r_{ab}(\mu-\lambda)\big(\mathbb{I} + W_{S,a}(x;\mu)\big)e_{11,a}\big(\mathbb{I} + W_{S,a}(x;\mu)\big)^{-1}\right\},$$

where e_{ij} is the 2×2 matrix that obeys $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Finally, as the chosen r-matrix satisfies $r_{ab}M_a = M_b r_{ab}$ for any 2×2 matrix M, this can be simplified further to lie solely in the b vector space (so that we may drop the subscript b):

$$\mathbb{V}(x;\lambda,\mu) = \frac{1}{\mu-\lambda}\big(\mathbb{I} + W_S(x;\mu)\big)e_{11}\big(\mathbb{I} + W_S(x;\mu)\big)^{-1}. \tag{2.16}$$

If we expand this about powers of μ in the limit as $\mu \to 0^+$, the first three terms are:

$$\begin{aligned}
\mathbb{V}^{(0)} &= \frac{-1}{4\lambda}\mathbb{I} - \frac{1}{4c\lambda}S, \\
\mathbb{V}^{(1)} &= \frac{-1}{4\lambda^2}\mathbb{I} - \frac{1}{4c\lambda^2}S + \frac{1}{4c^3\lambda}S'S, \\
\mathbb{V}^{(2)} &= \frac{-1}{4\lambda^3}\mathbb{I} - \frac{1}{4c\lambda^3}S + \frac{1}{4c^3\lambda^2}S'S - \frac{1}{4c^3\lambda}S'' - \frac{3}{8c^5\lambda}(S')^2S.
\end{aligned} \tag{2.17}$$

After removing the overall commuting constant factors and scaling by $-2c$, the second of these can be identified as the V-matrix in the Lax pair (1.5):

$$V = -2c(\mathbb{V}^{(1)} + \frac{1}{4\lambda^2}\mathbb{I}).$$

It is the identification of U with $\mathbb{V}^{(0)}$ up to some constant factors, that suggests the introduction of a dual picture for this model, with the roles of time and space switched. Before we investigate this though, we briefly discuss how to adapt this construction to account for non-periodic boundary conditions.

2.3. Open Boundary Conditions

In order to study systems with open boundary conditions, we need to introduce some $K_\pm$-matrices that are associated to the $\pm L$ boundaries, and have a dependence on the spectral parameter and some additonal constants. In order for them to be used in generating conserved quantities, we require that they satisfy the classical analogue of the (non-dynamical) quantum reflection equation [13]:

$$0 = [r_{ab}(\lambda-\mu), K_{\pm,a}(\lambda)K_{\pm,b}(\mu)] + K_{\pm,a}(\lambda)r_{ab}(\lambda+\mu)K_{\pm,b}(\mu) - K_{\pm,b}(\mu)r_{ab}(\lambda+\mu)K_{\pm,a}(\lambda). \tag{2.18}$$

For the r-matrix (2.2), the most general choice of $K_\pm$-matrix (up to some rescaling and gauge transformations) is [23]:

$$K_\pm(\lambda) = \alpha_\pm\mathbb{I} + \lambda\begin{pmatrix} \delta_\pm & \beta_\pm \\ \gamma_\pm & -\delta_\pm \end{pmatrix}, \tag{2.19}$$

where $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and $\delta_\pm$ are some constants that describe the boundary conditions being considered[3]. If these are given a time dependence, then these would be dynamical boundary conditions. For this paper, however, we consider only the non-dynamical case where they have no time dependence (and when we move on to discuss time-like boundary conditions, we shall assume that the equivalent constants have no space dependence). These $K_\pm$-matrices are introduced into the transfer matrix $\mathfrak{t}_S$ as [12, 13]:

$$\bar{\mathfrak{t}}_S(\lambda) = \mathrm{tr}\left\{K_+(\lambda)T_S(L,-L;\lambda)K_-(\lambda)T_S^{-1}(L,-L;-\lambda)\right\}, \tag{2.20}$$

and from this definition it follows that:

$$\{\bar{\mathfrak{t}}_S(\lambda), \bar{\mathfrak{t}}_S(\mu)\}_S = 0.$$

Much as in the periodic case, we will consider the generator $\bar{\mathcal{G}}_S(\lambda) = \ln(\bar{\mathfrak{t}}_S(\lambda))$, as this will supply us with the known Hamiltonian. To diagonalise the T_S^{-1}, we use:

$$T_S^{-1}(x,y;-\lambda) = \big(\mathbb{I} + W_S(y;-\lambda)\big)\mathrm{e}^{-Z_S(x,y;-\lambda)}\big(\mathbb{I} + W_S(x;-\lambda)\big)^{-1},$$

in place of (2.9). Consequently, as the highest order term in Z_S is λ^{-1}, the effect of the $-$ sign outside of the Z_S and the change in sign of the λ will cancel out, so that the expansion of the exponential term in the limit $\lambda \to 0^+$ is:

$$\mathrm{e}^{-Z_S(x,y;-\lambda)} \to \mathrm{e}^{-Z_{11,S}(x,y;-\lambda)}e_{11} + \mathcal{O}(\mathrm{e}^{-\lambda^{-1}}). \tag{2.21}$$

Consequently, the expansion of the generator $\bar{\mathcal{G}}_S$ is:

$$\begin{aligned}\bar{\mathcal{G}}_S(\lambda) = Z_{11,S}(\lambda) - Z_{11,S}(-\lambda) &+ \ln\left(\Big[\big(\mathbb{I} + W_S(L;-\lambda)\big)^{-1}K_+(\lambda)\big(\mathbb{I} + W_S(L;\lambda)\big)\Big]_{11}\right) \\ &+ \ln\left(\Big[\big(\mathbb{I} + W_S(-L;\lambda)\big)^{-1}K_-(\lambda)\big(\mathbb{I} + W_S(-L;-\lambda)\big)\Big]_{11}\right),\end{aligned}$$

[3] The reflection equation satisfied by the K_+- and K_--matrices actually differ by a minus sign in the spectral parameter, but we absorb this factor into the β_+, γ_+, and δ_+ to keep the forms of the matrices the same.

where the $[...]_{ij}$ indicates that we are only considering the ijth component of the matrix inside the brackets. If we expand this expression, the order λ^0 coefficient is constant while the order λ^1 coefficient is:

$$\begin{aligned}\bar{\mathcal{G}}_S^{(1)} = \frac{-1}{2c^3}\int_{-L}^{L}\left(S'_+S'_- + (S'_z)^2\right)\mathrm{d}x + \frac{1}{2\alpha_+ c}\left[2\delta_+S_z + \beta_+S_+ + \gamma_+S_-\right]_{x=+L} \\ + \frac{1}{2\alpha_- c}\left[2\delta_-S_z + \beta_-S_+ + \gamma_-S_-\right]_{x=-L}.\end{aligned} \tag{2.22}$$

This can be recognised as $\mathcal{G}_S^{(1)}$ from (2.12), up to boundary contributions and an overall factor. As $\mathcal{G}_S^{(0)}$ was associated to the total momentum of the system, and $\bar{\mathcal{G}}_S^{(0)}$ is trivial, we can infer that the momentum is no longer conserved when boundary conditions are introduced.

By following an analogous derivation to that of (2.15), we can derive the generator of the V-matrices corresponding to the conserved quantities generated by $\bar{\mathcal{G}}_S$. There are three cases to consider in this setting [24], corresponding to the V-matrices in the bulk (labelled $\bar{\mathbb{V}}_{\mathrm{B}}$), and the V-matrices lying at each of the two boundaries (labelled $\bar{\mathbb{V}}_\pm$ for the $x = \pm L$ boundaries, respectively). The generator of the bulk V-matrices is:

$$\begin{aligned}\bar{\mathbb{V}}_{\mathrm{B},b}(x;\lambda,\mu) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a\Big\{K_{+,a}(\mu)T_{S,a}(L,x;\mu)r_{ab}(\mu-\lambda)T_{S,a}(x,-L;\mu)K_{-,a}(\mu)T_{S,a}^{-1}(-\mu) \\ + K_{+,a}(\mu)T_{S,a}(\mu)K_{-,a}(\mu)T_{S,a}^{-1}(x,-L;-\mu)r_{ab}(\mu+\lambda)T_{S,a}^{-1}(L,x;-\mu)\Big\},\end{aligned} \tag{2.23}$$

while the generator of the V-matrices at the positive boundary is:

$$\bar{\mathbb{V}}_{+,b}(\lambda,\mu) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a\Big\{K_{-,a}(\mu)T_{S,a}^{-1}(-\mu)\Big(K_{+,a}(\mu)r_{ab}(\mu-\lambda) + r_{ab}(\mu+\lambda)K_{+,a}(\mu)\Big)T_{S,a}(\mu)\Big\}, \tag{2.24}$$

and the generator of the V-matrices at the negative boundary is:

$$\bar{\mathbb{V}}_{-,b}(\lambda,\mu) = \bar{\mathfrak{t}}_S^{-1}(\mu)\mathrm{tr}_a\Big\{K_{+,a}(\mu)T_{S,a}(\mu)\Big(r_{ab}(\mu-\lambda)K_{-,a}(\mu) + K_{-,a}(\mu)r_{ab}(\mu+\lambda)\Big)T_{S,a}^{-1}(-\mu)\Big\}. \tag{2.25}$$

If we expand these three generators about as $\mu \to 0^+$, the order μ^0 contributions from each generator are trivial, corresponding to $\bar{\mathcal{G}}_S^{(0)}$ being constant. At order μ^1, they are:

$$\begin{aligned}\bar{\mathbb{V}}_{\mathrm{B}}^{(1)}(x;\lambda) &= \frac{-1}{2\lambda^2}\mathbb{I} - \frac{1}{2c\lambda^2}S + \frac{1}{2c^3\lambda}S'S, \\ \bar{\mathbb{V}}_{\pm}^{(1)}(\lambda) &= \frac{-1}{2\lambda^2}\mathbb{I} - \frac{1}{2c\lambda^2}S \pm \frac{1}{4\alpha_\pm c\lambda}\begin{pmatrix}\beta_\pm S_+ - \gamma_\pm S_- & 2(\delta_\pm S_- - \beta_\pm S_z) \\ 2(\gamma_\pm S_z - \delta_\pm S_+) & \gamma_\pm S_- - \beta_\pm S_+\end{pmatrix}.\end{aligned} \tag{2.26}$$

In order to extract the boundary conditions from the open Hamiltonian, we simply calculate the equations of motion as usual (through the Poisson brackets and Hamilton's equation), except gathering all of the boundary terms that arise (either from the integration of total derivatives in the bulk Hamiltonian, or from the Poisson bracket of the fields with the boundary Hamiltonians). We then impose the sewing conditions that the equations of motion away from the boundary smoothly transition to those at the boundary, i.e. that $\lim_{x\to\pm L}\dot{S}_\sigma(x) = \dot{S}_\sigma(\pm L)$.

Similarly, in order to extract the boundary conditions from the V-matrices, the condition that the equations of motion agree at the boundary manifests as the condition that $\lim_{x\to\pm L}\bar{\mathbb{V}}_{\mathrm{B},b} = \bar{\mathbb{V}}_{\pm,b}$. Performing either of these limits yields the same constraint on the boundary constants and the S_σ at the boundary [14]:

$$\begin{aligned}\alpha_\pm\left[S_+S'_- - S'_+S_-\right]_{x=\pm L} &= \pm c^2\left[\beta_\pm S_+ - \gamma_\pm S_-\right]_{x=\pm L}, \\ \alpha_\pm\left[S_+S'_z - S'_+S_z\right]_{x=\pm L} &= \pm c^2\left[\delta_\pm S_+ - \gamma_\pm S_z\right]_{x=\pm L}, \\ \alpha_\pm\left[S_-S'_z - S'_-S_z\right]_{x=\pm L} &= \pm c^2\left[\delta_\pm S_- - \beta_\pm S_z\right]_{x=\pm L}.\end{aligned} \tag{2.27}$$

3. The Dual Model

By considering the equal prominence of the space and time coordinates in the Lagrangian picture of a 1+1 dimensional system, a dual Hamiltonian formulation of the non-linear Schrödinger model was constructed in [2], which had equal-space Poisson brackets (in place of the equal-time Poisson brackets) and dual integrals of motion that are conserved with respect to space-evolution rather than time-evolution. In this paper we focus on the Lax pair construction rather than the Lagrangian picture emphasised in previous work.

In this Section, we build the dual construction of the isotropic Landau-Lifshitz model in the language of Lax pairs. It follows mostly in parallel with Section 2, with the only divergences being where we emphasise important differences between the two pictures, such as in the limiting procedure of the exponential in the case of open boundary conditions, and where we digress to give an example of how this dual picture can be used to find integrable systems depending non-trivially on additional fields.

The final subsection 3.4 considers the introduction of time-like boundary conditions. This idea was introduced in [10], where it was applied to the non-linear Schrödinger model.

3.1. Poisson Brackets

The first step in this dual construction is defining the equal-space Poisson brackets (3.5) through the use of the r-matrix and an analogue of the linear algebraic relation (2.3). However, as the hierarchy will now describe a series of commuting space flows, the S'_σ in the V-matrix (1.5) will all be derivatives with respect to a specific space-flow, namely the 0th order flow x_0 (as will be seen later). Consequently, to prevent later confusion, we define these as some new fields, Σ_σ. When we look at the 0th order Hamiltonian or V-matrix (that is, those that provide the original equations of motion (1.5)), we will find as part of the space-evolution equations the identification $\Sigma_\sigma = \partial_{x_0} S_\sigma$. Otherwise, these Σ_σ will be treated as entirely independent fields, as can be seen in Subsection 3.3.

With these new fields, the V-matrix that we consider is:

$$V = \frac{1}{2\lambda^2} S - \frac{1}{2c^2\lambda} \Sigma S, \tag{3.1}$$

with:

$$\Sigma = \begin{pmatrix} \Sigma_z & \Sigma_- \\ \Sigma_+ & -\Sigma_z \end{pmatrix}.$$

While the Poisson brackets were found from the U- and r-matrices via (2.3), we assume that a similar equation exists for the V-matrices, namely [1]:

$$\{V_a(t_1,\lambda), V_b(t_2,\mu)\}_T = [r_{ab}(\lambda-\mu), V_a(t_1,\lambda) + V_b(t_2,\mu)]\, \delta(t_1 - t_2). \tag{3.2}$$

Inserting both the V-matrix and the r-matrix into this expression, we find a collection of Poisson brackets between the various fields:

$$\begin{aligned}
&\{S_\pm(t_1), S_z(t_2)\}_T = \{S_+(t_1), S_-(t_2)\}_T = 0,\\
&\{S_\pm(t_1), \Sigma_z(t_2)\}_T = \{S_z(t_1), \Sigma_\pm(t_2)\}_T = S_\pm S_z\, \delta(t_1 - t_2),\\
&\{S_z(t_1), \Sigma_z(t_2)\}_T = -S_+ S_-\, \delta(t_1 - t_2),\\
&\{S_\pm(t_1), \Sigma_\pm(t_2)\}_T = S_\pm^2\, \delta(t_1 - t_2),\\
&\{S_\pm(t_1), \Sigma_\mp(t_2)\}_T = -(2S_z^2 + S_+ S_-)\, \delta(t_1 - t_2),\\
&\{\Sigma_\pm(t_1), \Sigma_z(t_2)\}_T = (S_\pm \Sigma_z - \Sigma_\pm S_z)\, \delta(t_1 - t_2),\\
&\{\Sigma_+(t_1), \Sigma_-(t_2)\}_T = (S_+ \Sigma_- - \Sigma_+ S_-)\, \delta(t_1 - t_2).
\end{aligned} \tag{3.3}$$

As well as sharing the Casimir element $c^2 = S_z^2 + S_+ S_-$ with the original model, these brackets have an additional commuting quantity:

$$\tilde{c} = 2S_z\Sigma_z + S_+\Sigma_- + S_-\Sigma_+, \tag{3.4}$$

where, in reference to when $\Sigma_\sigma = \partial_{x_0} S_\sigma$ in the HM model, we choose to set $\tilde{c} = 0$. Consequently, when the HM model is considered and we can write the Σ_σ directly as the derivatives of the S_σ, (3.4) becomes redundant as it is merely the derivative of the original Casimir, (2.4). At any other level of the hierarchy however, we cannot directly relate the Σ_σ and the S_σ, so the two Casimirs are distinct.

Introducing the fields Σ_x, Σ_y, and Σ_z in analogy to S_x, S_y, and S_z, these Poisson brackets can be written more compactly by using the indices $i, j \in \{x, y, z\}$:

$$\begin{aligned} \{S_i(t_1), S_j(t_2)\}_T &= 0, \\ \{S_i(t_1), \Sigma_j(t_2)\}_T &= (S_i S_j - c^2 \delta_{ij})\, \delta(t_1 - t_2), \\ \{\Sigma_i(t_1), \Sigma_j(t_2)\}_T &= (S_i \Sigma_j - S_j \Sigma_i)\, \delta(t_1 - t_2), \end{aligned} \tag{3.5}$$

where the two Casimir elements are now:

$$\begin{aligned} c^2 &= S_x^2 + S_y^2 + S_z^2, \\ 0 &= S_x \Sigma_x + S_y \Sigma_y + S_z \Sigma_z. \end{aligned} \tag{3.6}$$

By defining the quantities:

$$\psi_1 = S_x^2, \qquad \phi_1 = \frac{1}{2c^2}\left(\frac{\Sigma_z}{S_z} - \frac{\Sigma_x}{S_x}\right), \qquad \psi_2 = S_y^2, \qquad \phi_2 = \frac{1}{2c^2}\left(\frac{\Sigma_z}{S_z} - \frac{\Sigma_y}{S_y}\right), \tag{3.7}$$

the above Poisson brackets can be written as a canonical pair (where we use the 2 Casimir elements to discount two of the fields):

$$\{\psi_1(t_1), \psi_2(t_2)\}_T = \{\phi_1(t_1), \phi_2(t_2)\}_T = 0, \qquad \{\psi_i(t_1), \phi_j(t_2)\}_T = \delta_{ij}\delta(t_1 - t_2). \tag{3.8}$$

3.2. Periodic Boundary Conditions

In both this section and the next (where open boundary conditions are considered), we consider a system that lies on the interval $[-\tau, \tau]$, for some $\tau > 0$. The periodic boundary conditions in this setting are then $S_\sigma(\tau) = S_\sigma(-\tau)$ and $\Sigma_\sigma(\tau) = \Sigma_\sigma(-\tau)$.

The construction of the dual model follows in parallel with Section 2.2. The first object constructed is therefore the equal-space monodromy matrix, T_T, which is a solution to the temporal half of the auxiliary linear problem, (1.4), in place of Ψ. This is diagonalised (by analogy to the standard picture discussed in Section 2) through the use of a diagonal matrix Z_T and an anti-diagonal matrix W_T:

$$\begin{aligned} T_T(t_1, t_2; \lambda) &= \mathrm{P}\exp\int_{t_2}^{t_1} V(\xi)\mathrm{d}\xi \\ &= \big(\mathbb{I} + W_T(t_1;\lambda)\big)\mathrm{e}^{Z_T(t_1,t_2;\lambda)}\big(\mathbb{I} + W_T(t_2;\lambda)\big)^{-1}. \end{aligned} \tag{3.9}$$

Because we have chosen that the V-matrices satisfy a linear algebraic relation of the form (3.2), the full equal-space monodromy matrix $T_T(\lambda) = T_T(\tau, -\tau; \lambda)$ will satisfy a quadratic algebraic relation analogous to (2.6):

$$\{T_{T,a}(\lambda), T_{T,b}(\mu)\}_T = [r_{ab}(\lambda - \mu), T_{T,a}(\lambda) T_{T,b}(\mu)]. \tag{3.10}$$

Taking the trace of the equal-space monodromy matrix we get the equal-space transfer matrix, $\mathfrak{t}_T$:

$$\begin{aligned} \mathfrak{t}_T(\lambda) &= \mathrm{tr}\,\{T_T(\lambda)\} \\ &= \mathrm{e}^{Z_{11,T}(\lambda)} + \mathrm{e}^{Z_{22,T}(\lambda)}, \end{aligned} \tag{3.11}$$

which, by virtue of the equal-space monodromy matrix satisfying the quadratic relation (3.10), Poisson commute for different spectral parameters:

$$\{\mathfrak{t}_T(\lambda), \mathfrak{t}_T(\mu)\}_T = 0.$$

Finally, as these two series Poisson commute, so will each pair of the coefficients $\mathfrak{t}_T^{(k)}$. Therefore, if we take the logarithm of these, $\mathcal{G}_T(\lambda) = \ln(\mathfrak{t}_T(\lambda))$, we have that the coefficients in the series expansion of $\mathcal{G}_T(\lambda)$ Poisson commute with one another:

$$\{\mathcal{G}_T^{(k)}, \mathcal{G}_T^{(j)}\}_T = 0. \tag{3.12}$$

As in Section 2.2, in order to expand $\mathcal{G}_T$, we need to consider the leading order contribution in each of $Z_{11,T}$ and $Z_{22,T}$. Consequently, if we insert the diagonalisation of T_T into the temporal half of the auxiliary linear problem, (1.4), then we find relations for the W_T and Z_T:

$$\begin{aligned} 0 &= \dot{W}_T + [W_T, V_D] + W_T V_A W_T - V_A, \\ \dot{Z}_T &= V_D + V_A W_T, \end{aligned} \tag{3.13}$$

where now V_D and V_A are the diagonal and anti-diagonal components of the V-matrix, respectively. Expanding W_T and Z_T in powers of λ as[4]:

$$W_T(\lambda) = \sum_{k=0}^{\infty} \lambda^k W_T^{(k)}, \qquad Z_T(\lambda) = \sum_{k=-2}^{\infty} \lambda^k Z_T^{(k)},$$

then we can recursively solve (3.13). Solving the first few orders of these, we find the first three Z_T-matrices to be:

$$\begin{aligned} Z_T^{(-2)} &= c\tau \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Z_T^{(-1)} = 0, \\ Z_T^{(0)} &= \frac{1}{2c}\int_{-\tau}^{\tau} \left[\dot{S}_z \mathbb{I} + (c - S_z) \begin{pmatrix} \frac{\dot{S}_-}{S_-} & 0 \\ 0 & -\frac{\dot{S}_+}{S_+} \end{pmatrix} - \frac{1}{2c^2}(\Sigma_+\Sigma_- + \Sigma_z^2) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \mathrm{d}t. \end{aligned} \tag{3.14}$$

Then, due to the form of the highest order term, the $\mathrm{e}^{Z_{11,T}}$ dominate over the $\mathrm{e}^{Z_{22,T}}$ in (3.11), so that $\mathcal{G}_T = Z_{11,T} + \dots$. I.e. the first three conserved quantities generated this way will be:

$$\begin{aligned} \mathcal{G}_T^{(-2)} &= c\tau, \qquad \mathcal{G}_T^{(-1)} = 0, \\ \mathcal{G}_T^{(0)} &= \frac{1}{2c}\int_{-\tau}^{\tau} \left(\dot{S}_z + (c - S_z)\frac{\dot{S}_-}{S_-} - \frac{1}{2c^2}(\Sigma_+\Sigma_- + \Sigma_z^2) \right) \mathrm{d}t. \end{aligned} \tag{3.15}$$

Focussing on the third of these, if we use the periodic boundary conditions to remove any total derivatives and multiply by a factor of $-2c$, $\mathcal{G}_T^{(0)}$ reduces to:

$$H_T = \frac{1}{2}\int_{-L}^{L} \left(\frac{\dot{S}_+ S_- - S_+ \dot{S}_-}{c + S_z} + \frac{1}{c^2}(\Sigma_+\Sigma_- + \Sigma_z^2) \right) \mathrm{d}t. \tag{3.16}$$

This is the equal-space Hamiltonian for the HM model, i.e. the generator of the space-evolution along the space flow x_0, as can be seen by using H_T in Hamilton's equation to find the space-evolution equations:

$$S'_\sigma = \{H_T, S_\sigma\}_T, \qquad \Sigma'_\sigma = \{H_T, \Sigma_\sigma\}_T.$$

[4]Note that due to the underlying V-matrix having a dependence on λ^{-2} (as compared to the earlier construction where the underlying U-matrix depended only on λ^{-1}), the Z_T series needs to start at $k = -2$ instead of $k = -1$.

Doing so, the space-evolution equations for S_σ simply give the identification $S'_\sigma = \Sigma_\sigma$, which is similar to the sine-Gordon model (which has been studied in this description in [11]) and the dual construction of the NLS model [1], while the space-evolution equations for Σ_σ give:

$$\begin{aligned}\Sigma'_\pm &= \pm(S_\pm \dot{S}_z - \dot{S}_\pm S_z) - \frac{1}{c^2} S_\pm(\Sigma_+\Sigma_- + \Sigma_z^2),\\ \Sigma'_z &= \frac{1}{2}(\dot{S}_+ S_- - S_+\dot{S}_-) - \frac{1}{c^2} S_z(\Sigma_+\Sigma_- + \Sigma_z^2),\end{aligned} \tag{3.17}$$

which, after substituting in $S'_\sigma = \Sigma_\sigma$ can be compactly written as:

$$\vec{S}'' = \mathrm{i}\vec{S} \times \dot{\vec{S}} - \frac{1}{c^2}\vec{S}|\vec{S}'|^2, \tag{3.18}$$

and are equivalent to the original equations of motion, (1.3), after replacing S_x, S_y with $S_\pm = S_x \pm \mathrm{i}S_y$.

Using the equal space Poisson brackets and the tower of equal space conserved quantities, we can generate a whole hierarchy of space-evolution equations associated to distinct systems. Consequently, we will also be interested in generating Lax pairs for each of these systems. By following the derivation of (2.15) and (2.16), we can derive a generator $\mathbb{U}$ for the tower of U-matrices that partner with the underlying V-matrix, (3.1), which can be generally written as:

$$\mathbb{U}_b(t;\lambda,\mu) = \mathfrak{t}_T^{-1}(\mu)\mathrm{tr}_a\left\{T_{T,a}(\tau,t;\mu)r_{ab}(\mu-\lambda)T_{T,a}(t,-\tau;\mu)\right\}, \tag{3.19}$$

or by using the known results and properties for the r-matrix as well as the diagonalisation of T_T, this can be reduced to an expression that lies only in one vector space:

$$\mathbb{U}(t;\lambda,\mu) = \frac{1}{2(\mu-\lambda)}\big(\mathbb{I}+W_T(t;\mu)\big)e_{11}\big(\mathbb{I}+W_T(t;\mu)\big)^{-1}. \tag{3.20}$$

When we expand this generator about $\mu \to 0^+$, the first three terms are:

$$\begin{aligned}\mathbb{U}^{(0)} &= \frac{-1}{4\lambda}\mathbb{I} - \frac{1}{4c\lambda}S,\\ \mathbb{U}^{(1)} &= \frac{-1}{4\lambda^2}\mathbb{I} - \frac{1}{4c\lambda^2}S + \frac{1}{4c^3\lambda}\Sigma S,\\ \mathbb{U}^{(2)} &= \frac{-1}{4\lambda^3}\mathbb{I} - \frac{1}{4c\lambda^3}S + \frac{1}{4c^3\lambda^2}\Sigma S + \frac{1}{4c^3\lambda}\dot{S}S - \frac{1}{8c^5\lambda}\Sigma^2 S.\end{aligned} \tag{3.21}$$

If we remove the constant factor from the first of these and multiply by a factor of $-2c$, $\mathbb{U}^{(0)}$ can be identified with the spatial component of the original Lax pair (1.5):

$$U = -2c(\mathbb{U}^{(0)} + \frac{1}{4\lambda}\mathbb{I}).$$

This guarantees that the equations of motion for this model agree with the original equations, (1.3).

3.3. Higher Order Systems

The identification of the Σ_σ with the derivatives of the S_σ appears as part of the equations of motion for the system at order 0 in the hierarchy (the isotropic Landau-Lifshitz model). If we instead consider a different system, these will not necessarily be the same. To see this, we consider the system at order μ^2 in the hierarchy, which has Lax pair (U_2, V), where we define:

$$\begin{aligned}U_2 &= -2c(\mathbb{U}^{(2)} + \frac{1}{4\lambda^3}\mathbb{I})\\ &= \frac{1}{2\lambda^3}S - \frac{1}{2c^2\lambda^2}\Sigma S - \frac{1}{2c^2\lambda}\dot{S}S + \frac{1}{4c^4\lambda}\Sigma^2 S.\end{aligned} \tag{3.22}$$

Inserting this Lax pair into the zero-curvature condition, we find the space-evolution equations for this new system. The space-evolution of the three original fields, $S_\pm$ and S_z, are:

$$\begin{aligned} S'_+ &= \frac{1}{c^2}(S_+\dot\Sigma_z - S_z\dot\Sigma_+) + \frac{1}{2c^4}(\Sigma_z^2+\Sigma_+\Sigma_-)\Sigma_+,\\ S'_- &= \frac{1}{c^2}(S_z\dot\Sigma_- - S_-\dot\Sigma_z) + \frac{1}{2c^4}(\Sigma_z^2+\Sigma_+\Sigma_-)\Sigma_-,\\ S'_z &= \frac{1}{2c^2}(S_-\dot\Sigma_+ - S_+\dot\Sigma_-) + \frac{1}{2c^4}(\Sigma_z^2+\Sigma_+\Sigma_-)\Sigma_z, \end{aligned} \tag{3.23}$$

while the space-evolution of the three fields $\Sigma_\pm$ and Σ_z are:

$$\begin{aligned} \Sigma'_+ &= \frac{1}{c^2}(\Sigma_+\dot\Sigma_z - \dot\Sigma_+\Sigma_z) + \ddot S_+ + S_+\Big(\frac{1}{c^2}\big((\dot S_z)^2+\dot S_+\dot S_-\big) - \frac{1}{2c^6}\big(\Sigma_z^2+\Sigma_+\Sigma_-\big)^2\Big)\\ &\quad + \frac{1}{2c^4}\big(\Sigma_z^2(\dot S_+S_z - S_+\dot S_z) + \Sigma_+^2(\dot S_-S_z - S_-\dot S_z) + \Sigma_+\Sigma_z(\dot S_+S_- - S_+\dot S_-)\big),\\ \Sigma'_- &= \frac{1}{c^2}(\dot\Sigma_-\Sigma_z - \Sigma_-\dot\Sigma_z) + \ddot S_- + S_-\Big(\frac{1}{c^2}\big((\dot S_z)^2+\dot S_+\dot S_-\big) - \frac{1}{2c^6}\big(\Sigma_z^2+\Sigma_+\Sigma_-\big)^2\Big)\\ &\quad + \frac{1}{2c^4}\big(\Sigma_z^2(S_-\dot S_z - \dot S_-S_z) + \Sigma_-^2(S_+\dot S_z - \dot S_+S_z) + \Sigma_-\Sigma_z(\dot S_+S_- - S_+\dot S_-)\big),\\ \Sigma'_z &= \frac{1}{2c^2}(\dot\Sigma_+\Sigma_- - \Sigma_+\dot\Sigma_-) + \ddot S_z + S_z\Big(\frac{1}{c^2}\big((\dot S_z)^2+\dot S_+\dot S_-\big) - \frac{1}{2c^6}\big(\Sigma_z^2+\Sigma_+\Sigma_-\big)^2\Big)\\ &\quad + \frac{1}{2c^4}\big(\Sigma_-\Sigma_z(S_+\dot S_z - \dot S_+S_z) + \Sigma_+\Sigma_z(\dot S_-S_z - S_-\dot S_z) + \frac{1}{2}(\Sigma_z^2 - \Sigma_+\Sigma_-)(\dot S_+S_- - S_+\dot S_-)\big). \end{aligned} \tag{3.24}$$

These can be written more compactly in terms of the vectors $\vec S = (S_x, S_y, S_z)^T$ and $\vec\Sigma = (\Sigma_x, \Sigma_y, \Sigma_z)^T$ as:

$$\begin{aligned} \vec S' &= \frac{\mathrm{i}}{c^2}(\vec S\times\dot{\vec\Sigma}) + \frac{1}{2c^4}|\vec\Sigma|^2\vec\Sigma,\\ \vec\Sigma' &= \frac{\mathrm{i}}{c^2}(\vec\Sigma\times\dot{\vec\Sigma}) - \frac{\mathrm{i}}{2c^4}|\vec\Sigma|^2(\vec S\times\dot{\vec S}) + \ddot{\vec S} + \vec S\Big(\frac{1}{c^2}|\dot{\vec S}|^2 - \frac{1}{2c^6}|\vec\Sigma|^4\Big) + \frac{\mathrm{i}}{c^4}\vec\Sigma\big(\vec\Sigma\cdot(\vec S\times\dot{\vec S})\big). \end{aligned} \tag{3.25}$$

When deriving the above Lax pair and resulting equations of motion we started from a V-matrix at order μ^1 and found the corresponding U-matrix at order μ^2. We could instead, however, start by considering a U-matrix at order μ^2 and use that to find the corresponding V-matrix at order μ^1.

To find this order μ^2 U-matrix, we start from the base system (i.e. the Lax pair consisting of the U- and V-matrices appearing at order μ^0 (see (2.17) and (3.21)):

$$U = V = \frac{1}{2\lambda}S. \tag{3.26}$$

The equations of motion for this system are simply $\dot S_\sigma = S'_\sigma$. Then, the first three terms in the hierarchy of U-matrices constructed from the V-matrix are:

$$\begin{aligned} \mathbb{U}^{(0)} &= \frac{-1}{4\lambda}\mathbb{I} - \frac{1}{4c\lambda}S,\\ \mathbb{U}^{(1)} &= \frac{-1}{4\lambda^2}\mathbb{I} - \frac{1}{4c\lambda^2}S + \frac{1}{4c^3\lambda}\dot S S,\\ \mathbb{U}^{(2)} &= \frac{-1}{4\lambda^3}\mathbb{I} - \frac{1}{4c\lambda^3}S + \frac{1}{4c^3\lambda^2}\dot S S - \frac{1}{4c^3\lambda}\ddot S - \frac{1}{8c^5\lambda}(\dot S)^2 S, \end{aligned} \tag{3.27}$$

which should be compared with (2.17). Before we can construct the space-like (standard) hierarchy for the U-matrix found from $\mathbb{U}^{(2)}$ we need to define the fields $P_\sigma = \partial_{t_0}S_\sigma$ and $\mathbb{P}_\sigma = \partial_{t_0}^2 S_\sigma$ (in analogy to how we defined the field $\Sigma_\sigma = \partial_{x_0}S_\sigma$), so that the U-matrix is:

$$U = \frac{1}{2\lambda^3}S - \frac{1}{2c^2\lambda^2}PS + \frac{1}{2c^2\lambda}\mathbb{P} + \frac{3}{4c^4\lambda}P^2 S, \tag{3.28}$$

with:

$$P = \begin{pmatrix} P_z & P_- \\ P_+ & -P_z \end{pmatrix}, \qquad \mathbb{P} = \begin{pmatrix} \mathbb{P}_z & \mathbb{P}_- \\ \mathbb{P}_+ & -\mathbb{P}_z \end{pmatrix}.$$

This is the U-matrix appearing at order μ^2 that we consider in place of (3.22). Constructing the space-like hierarchy from this, the V-matrix appearing at order μ^1 is (after removing the constant factor and scaling by $-2c$):

$$V = \frac{1}{2\lambda^2} S - \frac{1}{2c^2\lambda} PS. \tag{3.29}$$

This Lax pair would appear to describe a system of equations different to (3.25), due to containing a total of nine fields, S_σ, P_σ, and $\mathbb{P}_\sigma$. When these matrices are inserted into the zero-curvature condition, however, one of these sets of fields is redundant and $\mathbb{P}$ can be written in terms of S and P as:

$$\mathbb{P} = S\dot{S} - \frac{1}{c^2} P^2 S.$$

The combination of this identification and the remaining equations of motion can then be recognised as the equations (3.25). Consequently, traversing the early ($n < 3$) part of these dual hierarchies is commutative for this model. It remains to be seen if any higher order parts of the dual hierarchies commute, however, there is no *a priori* justification for the commutativity and an investigation into this is left for future study.

3.4. Open Boundary Conditions

Finally, we consider the effect of introducing reflective boundary conditions to the time-axis. This idea was introduced in [10], where it was applied to the NLS model. Due to the r-matrix structure for the dual model, (3.2), being identical to the r-matrix structure of the original model, (2.3), we introduce boundary conditions in an identical manner. That is, we start by choosing a pair of matrices, $K_\pm$, that satisfy (2.18). Specifically, we use the same K-matrices as in the original picture, (2.19):

$$K_\pm(\lambda) = \alpha_\pm \mathbb{I} + \lambda \begin{pmatrix} \delta_\pm & \beta_\pm \\ \gamma_\pm & -\delta_\pm \end{pmatrix},$$

where the constants $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and $\delta_\pm$ could in general depend on the evolution parameter, x, but we choose them to be constant for simplicity. We introduce these K-matrices into the generator of the quantities conserved with respect to space as [12, 13, 10]:

$$\bar{\mathfrak{t}}_T(\lambda) = \operatorname{tr}\left\{K_+(\lambda) T_T(\tau, -\tau; \lambda) K_-(\lambda) T_T^{-1}(\tau, -\tau; -\lambda)\right\}, \tag{3.30}$$

from which we can use the quadratic relation (3.10) and the defining relation for the K-matrices, (2.18), to derive the time-like equivalent of (2.20), which tells us that the $\bar{\mathfrak{t}}_T$ Poisson commute for different spectral parameters. Again, we are actually interested in the coefficients in the expansion of $\bar{\mathcal{G}}_T(\lambda) = \ln(\bar{\mathfrak{t}}_T(\lambda))$, which will also Poisson commute with one another:

$$\{\bar{\mathcal{G}}_T^{(k)}, \bar{\mathcal{G}}_T^{(j)}\}_T = 0. \tag{3.31}$$

In order to evaluate the series expansion of $\bar{\mathcal{G}}_T(\lambda)$, as well as diagonalising T_T through (3.9), we need to also diagonalise T_T^{-1} through:

$$T_T^{-1}(t_1, t_2; -\lambda) = \big(\mathbb{I} + W_T(t_2; -\lambda)\big) \mathrm{e}^{-Z_T(t_1, t_2; -\lambda)} \big(\mathbb{I} + W_T(t_1; -\lambda)\big)^{-1}.$$

An important point here is that when we take the limit as $\lambda \to 0^+$ of the exponentiated term, due to the $-$ sign in front of the Z_T and the highest order term being $(-\lambda)^2 = \lambda^2$, the expansion of the exponential as $\lambda \to 0^+$ will instead be:

$$\mathrm{e}^{-Z_T(t_1, t_2; -\lambda)} \to \mathrm{e}^{-Z_{22,T}(t_1, t_2; -\lambda)} e_{22} + \mathcal{O}(\mathrm{e}^{-\lambda^{-2}}).$$

Consequently, when the diagonalisations are inserted into the generator $\bar{\mathcal{G}}_T$, we have (where we suppress the parameters by defining $\hat{f} = f(-\lambda)$ and $W_{\pm,T} = W_T(\pm\tau)$):

$$\bar{\mathcal{G}}_T(\lambda) = \ln\left(\mathrm{e}^{Z_{11,T}-\hat{Z}_{22,T}}\mathrm{tr}\left\{K_+\left(\mathbb{I}+W_{+,T}\right)e_{11}\left(\mathbb{I}+W_{-,T}\right)^{-1}K_-\left(\mathbb{I}+\hat{W}_{-,T}\right)e_{22}\left(\mathbb{I}+\hat{W}_{+,T}\right)^{-1}\right\}\right),$$

which can be separated into the bulk contribution and the two boundary contributions:

$$\bar{\mathcal{G}}_T(\lambda) = Z_{11,T}(\lambda) - Z_{22,T}(-\lambda) + \ln\left(\mathbb{W}_+(\lambda)\right) + \ln\left(\mathbb{W}_-(\lambda)\right), \tag{3.32}$$

where we define:

$$\begin{aligned}\mathbb{W}_+(\lambda) &= \left[\left(\mathbb{I}+W_T(\tau;-\lambda)\right)^{-1}K_+(\lambda)\left(\mathbb{I}+W_T(\tau;\lambda)\right)\right]_{21},\\ \mathbb{W}_-(\lambda) &= \left[\left(\mathbb{I}+W_T(-\tau;\lambda)\right)^{-1}K_-(\lambda)\left(\mathbb{I}+W_T(-\tau;-\lambda)\right)\right]_{12}.\end{aligned} \tag{3.33}$$

Due to the logarithmic dependence of $\bar{\mathcal{G}}_T$ on $\mathbb{W}_\pm$, the lowest order contribution of the boundary terms to the generator $\bar{\mathcal{G}}_T$ will appear at order λ^0. Specifically, this lowest order contribution will be:

$$\mathbb{W}^{(1)}_\pm = \frac{1}{2c}\left(\frac{\pm 2\alpha_\pm}{c}\left(\frac{S_\pm\Sigma_z}{S_z+c} - \Sigma_\pm\right) - 2\delta_\pm S_\mp - \beta_\pm\frac{S_\pm S_\pm}{S_z \pm c} - \gamma_\pm\frac{S_-S_\pm}{S_z\mp c}\right), \tag{3.34}$$

so that the first three terms in the expansion of $\bar{\mathcal{G}}_T$ are:

$$\begin{aligned}\bar{\mathcal{G}}_T^{(-2)} &= 2c\tau, \qquad\qquad \bar{\mathcal{G}}_T^{(-1)} = 0,\\ \bar{\mathcal{G}}_T^{(0)} &= \frac{1}{2c}\int_{-\tau}^{\tau}\left(\frac{S_+\dot{S}_- - \dot{S}_+S_-}{c+S_z} - \frac{1}{c^2}\left(\Sigma_+\Sigma_- + \Sigma_z^2\right)\right)\mathrm{d}t + \ln\left(\mathbb{W}^{(1)}_+\right) + \ln\left(\mathbb{W}^{(1)}_-\right).\end{aligned} \tag{3.35}$$

Multiplying $\bar{\mathcal{G}}_T^{(0)}$ by the factor $-c$ gives the Hamiltonian with open boundary conditions:

$$\bar{H}_T = \int_{-\tau}^{\tau}\left(\frac{1}{2c^2}\left(\Sigma_+\Sigma_- + \Sigma_z^2\right) + \frac{\dot{S}_+S_- - S_+\dot{S}_-}{2(c+S_z)}\right)\mathrm{d}t - c\ln\left(\mathbb{W}^{(1)}_+\right) - c\ln\left(\mathbb{W}^{(1)}_-\right). \tag{3.36}$$

Away from the boundaries, the Poisson brackets of $\bar{H}_T$ with each of the six fields returns the space-evolution equations, (3.17). At the boundaries, however, when the space-evolution is derived the condition that the fields at the boundary still satisfy the usual space-evolution equations imposes extra conditions on the fields S_σ and Σ_σ, as well as the $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and $\delta_\pm$. The requirement that $\lim_{t\to\pm\tau} S'_\sigma = S'_\sigma(\pm\tau)$ restricts us to the case $\alpha_\pm = 0$. If we combine this with the requirement that $\lim_{t\to\pm\tau}\Sigma'_\sigma = \Sigma'_\sigma(\pm\tau)$, then we find the time-like boundary conditions for the HM model:

$$0 = \alpha_\pm, \qquad\qquad 0 = \beta_\pm S_+ + \gamma_\pm S_- + 2\delta_\pm S_z. \tag{3.37}$$

We can also find a generator for the U-matrices both in the bulk and at the boundaries. The generator for the bulk U-matrices will be [10]:

$$\begin{aligned}\bar{\mathbb{U}}_{\mathrm{B},b}(t;\lambda,\mu) = \mathcal{T}^{-1}(\mu)\mathrm{tr}_a\Big\{&K_{+,a}(\mu)T_{T,a}(\tau,t;\mu)r_{ab}(\mu-\lambda)T_{T,a}(t,-\tau;\mu)K_{-,a}(\mu)T^{-1}_{T,a}(-\mu)\\ &+ K_{+,a}(\mu)T_{T,a}(\mu)K_{-,a}(\mu)T^{-1}_{T,a}(t,-\tau;-\mu)r_{ab}(\mu+\lambda)T^{-1}_{T,a}(\tau,t;-\mu)\Big\},\end{aligned} \tag{3.38}$$

and, being mindful of the different limit for the $T_T^{-1}(-\mu)$, this can be reduced to:

$$\bar{\mathbb{U}}_{\mathrm{B}}(t;\lambda,\mu) = \mathbb{U}(t;\lambda,\mu) + \frac{1}{2(\mu+\lambda)}\left(\mathbb{I}+W_T(t;-\mu)\right)e_{22}\left(\mathbb{I}+W_T(t;-\mu)\right)^{-1}, \tag{3.39}$$

where $\mathbb{U}(t;\lambda,\mu)$ is the generator of the U-matrices with periodic boundary conditions. Unlike in the original case, where the second term differed from the first only by the sign of the μ, here it differs both by the sign of the μ and in that the matrix e_{11} has become e_{22}. The lowest order term in the expansion of this appears as the coefficient of μ^0, and is:

$$\mathbb{U}_{\mathrm{B}}^{(0)} = \frac{-1}{2c\lambda}\begin{pmatrix} S_z & S_- \\ S_+ & -S_z \end{pmatrix} = 2\mathbb{U}^{(0)}, \tag{3.40}$$

where $\mathbb{U}^{(0)}$ is the U-matrix appearing at lowest order in the periodic case. The boundary U-matrices are found by considering the generators:

$$\begin{aligned} \bar{\mathbb{U}}_{+,b}(\lambda,\mu) &= \mathfrak{t}_T^{-1}(\mu)\mathrm{tr}_a\left\{K_{-,a}(\mu)T_{T,a}^{-1}(-\mu)\Big(K_{+,a}(\mu)r_{ab}(\mu-\lambda)+r_{ab}(\mu+\lambda)K_{+,a}(\mu)\Big)T_{T,a}(\mu)\right\}, \\ \bar{\mathbb{U}}_{-,b}(\lambda,\mu) &= \mathfrak{t}_T^{-1}(\mu)\mathrm{tr}_a\left\{K_{+,a}(\mu)T_{T,a}(\mu)\Big(r_{ab}(\mu-\lambda)K_{-,a}(\mu)+K_{-,a}(\mu)r_{ab}(\mu+\lambda)\Big)T_{T,a}^{-1}(-\mu)\right\}, \end{aligned} \tag{3.41}$$

which can be simplified to:

$$\begin{aligned} \bar{\mathbb{U}}_{+,b}(\lambda,\mu) = \frac{1}{2\mathbb{W}_+(\mu)}\bigg(&\frac{1}{\mu-\lambda}\big(\mathbb{I}+W_T(\tau;\mu)\big)e_{12}\big(\mathbb{I}+W_T(\tau;-\mu)\big)^{-1}K_+(\mu) \\ &+\frac{1}{\mu+\lambda}K_+(\mu)\big(\mathbb{I}+W_T(\tau;\mu)\big)e_{22}\big(\mathbb{I}+W_T(\tau;-\mu)\big)^{-1}\bigg), \end{aligned} \tag{3.42}$$

and:

$$\begin{aligned} \bar{\mathbb{U}}_{-,b}(\lambda,\mu) = \frac{1}{2\mathbb{W}_-(\mu)}\bigg(&\frac{1}{\mu-\lambda}K_-(\mu)\big(\mathbb{I}+W_T(-\tau;-\mu)\big)e_{21}\big(\mathbb{I}+W_T(-\tau;\mu))\big)^{-1} \\ &+\frac{1}{\mu+\lambda}\big(\mathbb{I}+W_T(-\tau;-\mu)\big)e_{21}\big(\mathbb{I}+W_T(-\tau;\mu)\big)^{-1}K_-(\mu)\bigg). \end{aligned} \tag{3.43}$$

The first non-trivial term in the expansion of each of these appears at order μ^0. For the $t=+\tau$ boundary, this is:

$$\begin{aligned} \mathbb{U}_+^{(0)} = \frac{1}{2c(c+S_z)\mathbb{W}_+^{(1)}}\Bigg[&\frac{\alpha_+}{\lambda^2}\begin{pmatrix} S_+(c+S_z) & -(c+S_z)^2 \\ S_+^2 & -S_+(c+S_z)\end{pmatrix} \\ &-\frac{1}{2\lambda}\begin{pmatrix} -\beta_+S_+^2-\gamma_+(c+S_z)^2 & 2(c+S_z)\big(\delta_+(c+S_z)+\beta_+S_+\big) \\ 2S_+\big(\delta_+S_+-\gamma_+(c+S_z)\big) & \beta_+S_+^2+\gamma_+(c+S_z)^2\end{pmatrix}\Bigg], \end{aligned} \tag{3.44}$$

while at the $t=-\tau$ boundary, the U-matrix is:

$$\begin{aligned} \mathbb{U}_-^{(0)} = \frac{1}{2c(c+S_z)\mathbb{W}_-^{(1)}}\Bigg[&\frac{\alpha_-}{\lambda^2}\begin{pmatrix} S_-(c+S_z) & S_-^2 \\ -(c+S_z)^2 & -S_-(c+S_z)\end{pmatrix} \\ &-\frac{1}{2\lambda}\begin{pmatrix} \beta_-(c+S_z)^2+\gamma_-S_-^2 & -2S_-\big(\delta_-S_--\beta_-(c+S_z)\big) \\ -2(c+S_z)\big(\delta_-(c+S_z)+\gamma_-S_-\big) & -\beta_-(c+S_z)^2-\gamma_-S_-^2\end{pmatrix}\Bigg]. \end{aligned} \tag{3.45}$$

Requiring that $\lim_{t\to\pm\tau}\mathbb{U}_{\mathrm{B}}^{(0)} = \mathbb{U}_{\pm}^{(0)}$ gives rise to both the condition that $\alpha_\pm = 0$ (from the order λ^{-2} terms) and that $\beta_\pm S_+ + \gamma_\pm S_- + 2\delta_\pm S_z = 0$, which agrees with the boundary conditions found from the Hamiltonian approach, (3.37).

By comparing the time-like boundary conditions, (3.37), with the space-like boundary conditions, (2.27), we can see that there is no evident connection between the two. This asymmetry is rooted in the fundamentally different dependence of the fields on the space and time coordinates, as can be seen by comparing the forms of the equations of motion in (1.1) and (3.18).

4. Summary

The main result of this paper, derived in Section 3, is the dual construction of the isotropic Landau-Lifshitz model, where space-evolution equations, spatially conserved quantities, and equal-space Poisson brackets are obtained. This was done by following the usual procedure for deriving Poisson brackets and conserved quantities for a system that is integrable via the existence of a Lax pair and r-matrix, except with the roles of the space and time variables switched. A consequence of this equal-space construction is the existence of a hierarchy of dual integrable systems, each of which has an infinite tower of conserved quantities, (3.15), and a Lax pair representation, (3.21). Then, through the combination of the usual equal-time hierarchy and this dual equal-space hierarchy, an infinite "lattice" of integrable models can be built (it is important to note here that this "lattice" is not commutative *a priori*, although it has been observed to commute for $n, m < 3$).

By considering a higher order system in the dual hierarchy of the isotropic Landau-Lifshitz model, (3.25), we have connected the 3-field HM model (with 1 Casimir element) with a novel 6-field model (which has 2 Casimir elements). As this system appears in the hierarchy of the HM model, it is likely to have a solitonic solution similar to that of the HM model, which would be discoverable through the use of the inverse scattering tools, or through a Darboux-Bäcklund/Dressing approach. The investigation of such a soliton could provide interesting insights into the dual construction, if not the original model itself, but we leave this for future consideration.

We have also studied the introduction of reflective boundary conditions to the time-axis in Section 3.4, in the vein of [10]. While seemingly unphysical, such boundary conditions could have applications as a particular type of initial condition for the system, where the time coordinate is considered on the half-line, $[0, \infty)$, instead. Thus, the boundary conditions discussed above would appear as a particular set of initial conditions that settle into (in the case of a soliton reflecting boundary) a 2-soliton solution. Potential applications and consequences of this however are left for later investigation.

Finally, we close by repeating that, due to the U- and V-matrices sharing the same r-matrix, the space and time coordinates in this construction are fully interchangeable. This means that all of the results described here will still hold when the space and time coordinates are switched, so that switching the space derivatives and time derivatives in (3.25) describes the time evolution of an integrable system:

$$\begin{aligned} \dot{\vec{S}} &= \frac{\mathrm{i}}{c^2}(\vec{S} \times \vec{\Sigma}') + \frac{1}{2c^4}|\vec{\Sigma}|^2\vec{\Sigma}, \\ \dot{\vec{\Sigma}} &= \frac{\mathrm{i}}{c^2}(\vec{\Sigma} \times \vec{\Sigma}') - \frac{\mathrm{i}}{2c^4}|\vec{\Sigma}|^2(\vec{S} \times \vec{S}') + \vec{S}'' + \vec{S}\Big(\frac{1}{c^2}|\vec{S}'|^2 - \frac{1}{2c^6}|\vec{\Sigma}|^4\Big) + \frac{\mathrm{i}}{c^4}\vec{\Sigma}\big(\vec{\Sigma} \cdot (\vec{S} \times \vec{S}')\big), \end{aligned} \tag{4.1}$$

and the results of Section 3.4 can be viewed instead as a description of (space-like) open boundary conditions for the time-evolution equations:

$$\ddot{\vec{S}} = \mathrm{i}(\vec{S} \times \vec{S}') - \frac{1}{c^2}\vec{S}|\dot{\vec{S}}|^2. \tag{4.2}$$

This dual construction has now been applied to the isotropic Landau-Lifshitz model, the non-linear Schrödinger model (originally in scalar [2] case and later extended to the vector [25] case) and its associated hierarchy (including, for example, the complex modified KdV equation) in [1], and the sine-Gordon model in [11]. All of these models can be found as special limits of the anisotropic Landau-Lifshitz model [8] and its hierarchy. Consequently, it would be expected that the fully anisotropic Landau-Lifshitz model also admits a space-time duality of this type, however, an investigation into this is left for future work.

Acknowledgements

The author would like to thank the EPSRC funding council for a PhD studentship, and his PhD supervisor Anastasia Doikou for feedback and encouragement. He would also like to thank Calum Ross and Lukas Müller for proofreading and comments, as well as the reviewer for useful feedback.

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- An equal-space Poisson structure for the isotropic Landau-Lifshitz model.
- A discussion on reflective boundary conditions along the time axis.
- The derivation of a novel six-field integrable model.