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# *Finite-Time Singularity Formation for Incompressible Euler Moving Interfaces in the Plane*

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**Abstract**

This paper provides a new general method for establishing a finite-time singularity formation for moving interface problems involving the incompressible Euler equations in the plane. This methodology is applied to two different problems. The first problem considered is the two-phase vortex sheet problem with surface tension, for which, under suitable assumptions of smallness of the initial height of the heaviest phase and velocity fields, is proved the finite-time singularity of the natural norm of the problem. This is in striking contrast with the case of finite-time splash and splat singularity formation for the one-phase Euler equations of [4] and [8], for which the natural norm (in the one-phase fluid) stays finite all the way until contact. The second problem considered involves the presence of a heavier rigid body moving in the inviscid fluid. For a very general set of geometries (essentially the contact zone being a graph) we first establish that the rigid body will hit the bottom of the fluid domain in finite time. Compared to the previous paper [20] for the rigid body case, the present paper allows for small square integrable vorticity and provides a characterization of acceleration at contact. A surface energy is shown to blow up and acceleration at contact is shown to oppose the motion: it is either strictly positive and finite if the contact zone is of non zero length, or infinite otherwise.

## 1. Introduction

Finite-time singularity formation in moving boundary problems has been an active field of research for at least the past 10 years. Historically the first cases studied were contact problems for a symmetric rigid body moving in a fluid (see [11–15,25] for the viscous fluid case and [16,20] for the inviscid case), which present the simplification at the level of the analysis of having a constant shape for the inclusion. More recently the case of one-phase and two-phase Euler interface problems have

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started to be considered. The present paper presents a new methodology addressing finite-time singularity formation for any type of problems when the fluid equations are the incompressible Euler equations and the physical law of the included phase provides spatial control of the position of the interface.

The first problem considered in this paper is the formation of finite-time singularity for the two-phase moving interface Euler equations with surface tension. This problem is known to be locally in time well-posed for a natural norm $N(t)$ encoding the Sobolev regularity of the velocity field in each phase and the regularity of the moving interface (see [2,3] for the irrotational case, and [5,22,23] for the case with vorticity).

The one-phase water waves problem is known to be locally in time well-posed in Sobolev spaces, as the pressure condition holding in this situation avoids any Rayleigh–Taylor instability ([1,18,27,28] for the case without vorticity and [6,7, 19,21,29] for the case with vorticity). The first type of singularity formation in finite time for this problem in Sobolev spaces was established by Castro et al. in [4] by introducing the notion of splash and splat singularity, which is the self-intersection of the moving free boundary while the curve remains smooth (but is no longer locally on one side of its boundary at contact). This result was generalised in 3-D and with vorticity by Coutand and Shkoller [8] by a very different approach. Our approach can be easily applied to many one-phase hyperbolic free boundary problems. It is to be noted that this type of splash singularity is purely restricted to a loss of injectivity, since the natural norm of the problem stays bounded until the time of contact.

A natural question that then arose was about what happens when we extend this type of self-contact along a smooth curve in the two-phase context (with surface tension to make the problem locally well-posed in Sobolev spaces). With different methods, Fefferman, Ionescu and Lie [10] and Coutand and Shkoller [9], established that the two-phase vortex sheet problem with surface tension does not have finite-time formation of a splash or splat singularity so long as the natural norm of the problem for the velocity field in one phase stays bounded. The results of [9] and [10] however do not exclude such a loss of injectivity; if it was to occur, it would involve blow-up of the natural norm of the problem in both phases.

The present paper introduces a new methodology, based upon studying the motion of the center of gravity of one of the two phases, which provides a differential inequality for a surface energy introduced in the present paper. We here establish that under some symmetry assumptions at time zero, and with gravity effects, there will either be a loss of injectivity or a natural norm of the problem for local in time existence will blow-up in finite time. In both cases of this alternative, we show a natural norm of the problem blows up. This result is in striking contrast with splash and splat singularity formation for the one-phase water-waves problem introduced in [4], and treated with different methods in a more general context in [8], where the natural norm $N(t)$ stays finite. This was essential in the analysis of these papers in order to establish the finite-time contact, as this ensures that the magnitude of the relative velocity between two parts of an almost self-intersecting curve coming towards each other will be in magnitude greater than some strictly positive quantity. Such an approach would be impossible here, as in the two-phase problem, any contact would involve the formation of a cusp, which would make impossible high order elliptic estimates.

We next turn our attention to the case of the rigid body. This is a simpler problem given that the shape of the interface stays constant for all time, which removes some considerable level of difficulty from the previous problem. The interest of this problem resides in allowing a more precise description of the behaviour at the time of singularity than for the case with deformable interface.

Recently, Glass and Sueur [17] proved that the motion of a rigid body in an inviscid fluid in a domain in the plane is globally in time well posed so long as no contact occurs between the moving rigid body and the boundary. The qualitative question of whether contact singularity formation in finite-time is possible in the natural case where $u \cdot n = 0$ on $\partial\Omega$ arises then naturally.

The first results for finite-time contact for the rigid body case with zero vorticity in the inviscid fluid were obtained by Houot and Munnier [16] for the case of the disk in the half plane, and generalized by Munnier and Ramdani [20], where they establish for the flat bottom case with the symmetric rigid body being a graph of the type $x_2 = C|x_1|^{1+\alpha}$ ($\alpha > 0$), that finite-time contact occurs, with a rigid body velocity which is shown to be either zero or non zero depending on $\alpha$. Other cases involving discussions on concavity of domains are also treated in [20]. It is to be noted that their methods, purely elliptic in nature in some rescaled infinite strip, require the zero vorticity assumption of their paper, as the rescaling of any non zero vorticity in this infinite strip would be problematic. By contrast, we never do any rescaling on some infinite strip in this paper, and allow for (small square integrable) vorticity.

For this problem of the rigid inclusion (where the shape of the inclusion does not change), our new methodology based on a differential inequality for a surface energy that we identify (which is a completely different approach from the methodology of [20], where no differential inequality appeared) allows us to consider small square integrable vorticity, and allows us to obtain a characterization of acceleration at the time of contact, which remarkably depends only on the size of the contact zone. We first establish here the question of finite-time contact at $T_{\max} > 0$ when gravity effects are taken into account (in particular the rigid body is assumed of higher density than the fluid phase). We then establish a set of blow-up properties satisfied by the fluid velocity and pressure fields and acceleration as $t \to T_{\max}$, which are new for this kind of problems:

- First, although the solid velocity stays bounded for all time of existence, the present paper establishes the fluid has a radically different behaviour, as the $L^2(\partial\Omega)$ norm of the fluid velocity approaches $\infty$ near contact. This happens in a neighborhood of the contact zone, whereas away from the contact zone, the fluid velocity stays bounded.
- Second, this work also establishes that the acceleration of the rigid body becomes infinite in the upward direction at the time of contact, except for the case where the contact zone contains a curve of non zero length, in which case the acceleration remains strictly positive and bounded close to the time of contact. This behaviour is strikingly different from the behaviour of a material point

falling in void (the basic question of elementary Newtonian mechanics), for which the motion has constant negative acceleration $-g$.

The plan of this paper is as follows: in Sections 2 and 3, we remind the vortex sheet problem with surface tension, precise notations, and our type of initial data. In the essential Section 4, we derive an equation linked to the motion of the center of mass of the included phase, using in particular the formulation (20) for the incompressible Euler equations (which allows us to replace the pressure on $\partial\Omega$ by an equivalent expression in terms of velocity and acceleration in the fluid, by integration of the tangential component of the Euler equations). This equation, where appears a signed surface energy (depending on the sign of the vertical component of the normal vector on the boundary), does not depend on the choice of constitutive relation in the included phase. In Section 5, we deduce from the equation obtained from the previous section a differential inequality for a surface energy. This differential inequality structure appears by some elliptic estimates away from the heavier phase (corresponding to parts of $\partial\Omega$ where the sign of the vertical component of the normal is opposite to where contact may occur). The elliptic estimates performed (based on conservation of rotational and on a priori control of the $L^2$ norm of velocity) allow to establish the velocity field is smooth (at least for the energy appearing in the differential inequality) away from the interface. The differential inequality obtained appears in a way quite natural to the problem of a moving Euler phase, and is also quite different from the pioneering works of Sideris [24] and Xin [26] for compressible Euler and Navier–Stokes equations. We then use this differential inequality to establish the first theorem on finite-time singularity formation for the Euler vortex sheet problem with surface tension and gravity effects:

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $H^{\frac{9}{2}}$, which is symmetric with respect to $x_1 = 0$ and satisfies the assumptions of Section 3 (in particular the bottom of $\partial\Omega$ is a general graph, see Fig. 1 in Section 3), and let $\overline{\Omega^+} \subset \Omega$ be a domain of same regularity whose center of gravity is at altitude h at time zero, and which is symmetric with respect to $x_1 = 0$. Let $\Omega^- = \Omega \cap \overline{\Omega^+}^c$. With $u_0^\pm$ being our initial velocity field in each phase and $\omega_0^- = curl\ u_0^-$, let us assume that*

$$\|u_0^\pm\|_{L^2(\Omega^\pm)} + \|\omega_0^-\|_{L^2(\Omega^-)} + h + |\partial\Omega^+|$$

*is small enough (satisfying (22) and (80) stated later), and that $\rho^+ > \rho^-$. We also assume our initial velocity field satisfying $u_0^\pm \in H^3(\Omega^\pm)$ with $(u_0^+ - u_0^-)\cdot n = 0$ on $\partial\Omega^+$ and $u_0^- \cdot n = 0$ on $\partial\Omega$ (where n is the exterior unit normal to $\partial\Omega^-$). Then, for some $T_{\max} \in (0, \infty)$, any solution of (4) satisfies that:*

*1) either $\lim_{t\to T_{\max}^-} N(t) = \infty$, where*

$$N(t) = \|u^-\|_{H^3(\Omega^-(t))} + \|u^+\|_{H^3(\Omega^+(t))} + \|\eta^-\|_{H^4(\partial\Omega^+)},$$

*where $\eta^-$ denote the Lagrangian flow map associated to $u^-$;*

*2) or there is either a self-intersection of the interface $\partial\Omega^+(T_{\max})$ with itself, or contact of $\partial\Omega^+(T_{\max})$ with $\partial\Omega$ at $T_{\max}$;*

*3) or $\int_0^{T_{\max}} \int_{\partial\Omega} |u^-|^2\, \mathrm{d}l\, \mathrm{d}t = \infty$.*

**Remark 1.** We can substitute in this Theorem any norm $N(t)$ for which the problem has a local in time existence. This particular norm was chosen due to our result of local in time existence in [5].

**Remark 2.** If we were to assume the density of the material initially inside $\Omega^+$ to be strictly smaller than the density of the material initially in $\Omega^-$, similar theorems would hold, assuming the top of $\partial\Omega$ to be under the form of a graph.

The cases 1) and 3) obviously involve a blow-up of $N$. We now show in Section 6.1 that the second case (corresponding to a loss of injectivity) leads to the blow-up of the following norm:

**Theorem 2.** *If the case 2) of Theorem 1 is satisfied then*

$$\lim_{t\to T_{\max}^-} [\|\nabla_\tau \tau\|_{L^\infty(\partial\Omega^+(t))} + \sum_{\pm} \int_0^t \|\nabla u^\pm\|_{L^\infty(\Omega^\pm(t))}] = \infty, \tag{1}$$

*where $\tau$ denotes the unit tangent on $\partial\Omega^+(t)$.*

**Remark 3.** If we assume our initial data to have a smoother curl in both phases: $\text{curl} u_0^\pm \in H^{\frac{7}{2}}(\Omega^\pm)$, local in time existence can be carried in a similar way as in [5], this time in the norm (of the same type as our earlier work [7])

$$\tilde{N}(t) = N(t) + \|\eta^+\|_{H^{\frac{9}{2}}(\Omega^+)} + \|\eta^-\|_{H^{\frac{9}{2}}(\Omega^-)},$$

for which Theorem 1 applies similarly. We prove in Section 6.2 that if we have the blow-up (1) of Theorem 2, then

$$\limsup_{t\to T_{\max}^-} \tilde{N}(t) = \infty. \tag{2}$$

We therefore have proved that with this higher regularity for the initial curl of velocity in each phase, the natural norm $\tilde{N}(t)$ blows up in finite time for all situations of Theorem 1.

We next consider the case of the rigid body in an inviscid fluid, governed by (92). For this problem we show a small curl guarantees a monotone fall simply by conservation of energy, whereas in the vortex sheet problem with surface tension, there is no guarantee that the fall even occurs (locally in time when the solution is smooth, it can be guaranteed, but not as the singularity forms). In Section 7, the problem is reminded, and in Section 8 is precised our initial data (which is more general than for the vortex sheet problem). In particular, since for this problem the contact area is precisely known in advance (vertical fall of a rigid body), we just require this time each part of $\partial\Omega$ where contact occurs to be under the form of a graph (in particular at the same horizontal coordinate, there could be different connected components of the contact zone). In Section 9, are provided elliptic estimates showing the velocity remains smooth away from contact (similarly as the one proved in Section 5 for the vortex sheet problem away from contact) and the velocity is shown to be strictly non zero before contact. In Section 10, finite-time contact is established (this is a generalization of the results of [20] with a very different approach), again by using our fundamental relation (34) established in Lemma 1.

**Theorem 3.** *Let $\Omega$ and $\Omega^s$ be $C^2$ bounded domains satisfying the assumptions of Section 8 (essentially each connected component of $\partial\Omega$ where contact potentially occurs satisfies $n_2 < 0$ ). Let us assume that $v^s(0) = (0, v_2^s)$, with $v_2^s < 0$, and that $\rho^f < \rho^s$. Let us furthermore assume that the odd (with respect to $x_1$) vorticity satisfies*

$$\|\omega_0\|^2_{L^2(\Omega^f)} < \min\left(\frac{m_s}{\rho^f D_\Omega}|v_2^s(0)|^2, \frac{(m_s - \rho^f|\Omega^s|)g}{2\rho^f(CD_\Omega + C)}\right), \tag{3}$$

*with $C_\Omega$ being the standard Poincaré constant in $H_0^1(\Omega)$, given by (107), $D_\Omega$ being given by (112) and $C$ being given by (100).*

*Then there exists $T_{\max} \in (0, \infty)$ such that the rigid body will touch $\partial\Omega$ at time $T_{\max}$ with a finite velocity $v_2^s(T_{\max}) \leqq 0$.*

Unlike in the case of the vortex sheet problem, there is no restriction on how far from the boundary the rigid body has to be initially. This is due to our use of the velocity of the rigid body being of constant sign in our set up of a differential inequality (which is done differently than in the vortex sheet problem) from our fundamental relation (34).

The next sections are for $\omega = 0$, and establish a characterization of the acceleration of the rigid body at contact. In Section 11, is established an essential comparison of various norms of the velocity in the fluid by elliptic techniques proper to this problem. In Section 12, we provide the essential and simpler (than (34)) formula for acceleration (128) in Lemma 5. This formula shows straightforwardly that if the $L^2(\partial\Omega)$ norm of $u^f$ blows up at the time of contact, and the velocity of the rigid body at the time of contact is nonzero, then the acceleration becomes infinite upwards (implying in particular a blow-up of the pressure, due to the definition (92e) of acceleration of the rigid body). In Section 13 is established the blow-up of the $L^2(\partial\Omega)$ norm (which is entirely new for this particular problem), which is an essential step in characterizing acceleration at contact. In Section 14 is proved the positive or infinite character of acceleration at contact depending on the size of the contact zone. Note that for the fall of a rigid body in the Navier–Stokes context, a different type of blow-up property is established in [15], with very different methods. The characterization of acceleration at contact presented hereafter is entirely new for this type of contact problem.

**Theorem 4.** *Let us assume furthermore that $\omega_0 = 0$. Then, with $T_{\max}$ obtained in Theorem 3, we have the following properties for the solution of (92):*

*1)* $\lim_{T_{\max}^-} \|u^f\|_{L^2(\partial\Omega)} = \infty$;

*2)*

$$\lim_{T_{\max}^-} \left|\int_{\partial\Omega^s(t)} pn\, \mathrm{d}l\right| = \lim_{T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t} = \infty,$$

*except for the case where the contact zone between $\partial\Omega$ and $\partial\Omega^s(T_{\max})$ contains a curve of nonzero length, in which case we have*

$$0 < \liminf_{T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t} \leqq \limsup_{T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t} < \infty.$$

**Remark 4.** Point 2) shows a drastic difference between the problem of the rigid body in an inviscid fluid and the basic problem of Newtonian mechanics in void (without fluid), since in the case with void, the acceleration remains constant ($= -g$) for all time even at contact. It shows that the rigid body does feel the imminence of contact by the presence of the fluid and tries to avoid it by an upward acceleration (finite or infinite according to the size of the contact zone) opposing the fall. We also note that point 2) establishes indirectly that the $L^1(\partial\Omega^s(t))$ norm of pressure at contact becomes infinite for the case where the contact zone is of zero length, although the present paper only keeps the fluid velocity and acceleration as its variables (by integrating the tangential component of the Euler equations along the boundary).

**Remark 5.** Any physical model for $\Omega^+(t)$ such that $\partial\Omega^+(t)$ can be shown to stay away from the part of $\partial\Omega$ where $n_2 \geqq 0$ (typically the lateral sides and the top of $\partial\Omega$) would be suitable for this theory (for Theorem 1). Standard models of nonlinear elastodynamics (such as the quasilinear Saint-Venant Kirchhoff model) for the included phase $\Omega^+$ would be suitable.

## 2. Preliminaries on the Vortex Sheet Problem with Surface Tension

### *2.1. Formulation of the Vortex Sheet Problem with Surface Tension*

The vortex sheet problem with surface tension is a moving interface problem locally in time well-posed from [2,3] for the irrotational case, and [5,22,23] for the case with vorticity.

Here, $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain of class $H^{\frac{9}{2}}$, and $\Omega^+ \subset \Omega$ is also a smooth bounded domain of class $H^{\frac{9}{2}}$, and such that $\overline{\Omega^+} \subset \Omega$. We consider the incompressible Euler equations for the motion of two fluids of densities $\rho^-$ and $\rho^+$ that are at time zero in $\Omega \cap \overline{\Omega^+}^c = \Omega^-$ and $\Omega^+$, with surface tension and gravity effects:

$$\rho^\pm(u_t^\pm + u^\pm \cdot \nabla u^\pm) + \nabla p^\pm = -\rho^\pm g e_2, \text{ in } \Omega^\pm(t) \tag{4a}$$

$$\operatorname{div} u^\pm = 0, \text{ in } \Omega^\pm(t), \tag{4b}$$

$$(p^- - p^+)\, n = -\sigma \nabla_\tau(\tau), \text{ on } \partial\Omega^+(t), \tag{4c}$$

$$u^- \cdot n = u^+ \cdot n, \text{ on } \partial\Omega^+(t), \tag{4d}$$

$$u^- \cdot n = 0, \text{ on } \partial\Omega, \tag{4e}$$

$$\Omega^\pm(0) = \Omega^\pm, \tag{4f}$$

$$u^\pm(x,0) = u_0^\pm, \text{ in } \Omega^\pm, \tag{4g}$$

where the material interface $\partial\Omega^+(t)$ moves with speed $u^-\cdot n = u^+\cdot n$, where $n$ is the outward unit normal to $\Omega^-(t)$, $\tau$ is the unit tangent, and $e_2$ is the unit vertical vector pointing upward. Also the surface tension coefficient $\sigma$ is classically assumed strictly positive.

If $\eta^\pm$ denote the Lagrangian flow map associated to $u^\pm$, defined by

$$\eta_t^\pm(x,t) = u^\pm(\eta^\pm(x,t),t),\ \forall x\in\Omega^\pm, t\geqq 0,$$
$$\eta^\pm(x,0) = x,$$

we showed in [5] that the problem has a local in time solution defined in the norm:

$$N(t) = \|\eta^-\|_{H^4(\partial\Omega^+)} + \|u^-\|_{H^3(\Omega^-(t))} + \|u^+\|_{H^3(\Omega^+(t))}. \tag{5}$$

We also define the Lagrangian velocity $v^\pm(x,t) = u^\pm(\eta^\pm(x,t),t)$.

We will show this problem has a finite-time singularity formation provided some assumptions are made on the initial domain and data and that

$$\rho^+ > \rho^-.$$

To this end, we will establish that if $N(t)$ stays finite for all time, then either a finite in time contact occurs (either self intersection of $\partial\Omega^+(t)$ or between $\partial\Omega^+(t)$ and $\partial\Omega$) or a surface energy blows up. In case of contact, we will show that this leads to the blow-up (1).

### 2.2. *Global Vector Field in $\overline{\Omega}$ Extending the Normal*

We will need later on a smooth vector field extending the normal to $\partial\Omega$ into $\Omega$.

We denote by $n$ the outward unit normal to $\Omega$, and by $\tilde{n}$ the smooth solution of the elliptic problem:

$$\triangle\tilde{n} = 0,\ \text{in } \Omega, \tag{6a}$$
$$\tilde{n} = n,\ \text{on } \partial\Omega. \tag{6b}$$

By the maximum and minimum principles we have that for each component of $\tilde{n}$,

$$|\tilde{n}_i| \leqq 1. \tag{7}$$

Given the regularity of $\partial\Omega$, we have by elliptic regularity that $\tilde{n}\in H^3(\Omega)\subset C^1(\overline{\Omega})$. Therefore,

$$\|\nabla\tilde{n}\|_{L^\infty(\Omega)} \leqq \beta_\Omega < \infty. \tag{8}$$

We then define the vector field $\tilde{\tau} = (\tilde{n}_2, -\tilde{n}_1)\in H^3(\Omega)$, which extends the tangent to $\partial\Omega$ inside $\Omega$.

### 2.3. Notations

We have $n = (n_1, n_2)$ denote the outer unit normal to $\Omega^-(t)$, and $\tau = (\tau_1, \tau_2) = (n_2, -n_1)$ denote the unit tangent vector field.

The euclidean norm of a vector will be denoted as $|\cdot|$.

For a smooth domain $A \subset \mathbb{R}^2$ we denote by $|A|$ its area and by $|\partial A|$ the length of its boundary.

Due to incompressibility, we have for all time of existence $|\Omega^\pm(t)| = |\Omega^\pm|$.

We also use the Einstein convention of summation with respect to repeated indices or exponents.

For a given vector $a \in \mathbb{R}^2$, we denote $\nabla_a u = a_i \dfrac{\partial u}{\partial x_i}$. Of particular interest will be the case when either $a = \tau(x)$, or $a = n(x)$. In that case the divergence of a vector field $u$ written in the $(\tau(x), n(x))$ basis instead of the $(e_1, e_2)$ basis reads as

$$\operatorname{div} u = (\nabla_{\tau(x)} u) \cdot \tau(x) + (\nabla_{n(x)} u) \cdot n(x), \tag{9}$$

while the curl reads as

$$\omega = \operatorname{curl} u = (\nabla_{\tau(x)} u) \cdot n(x) - (\nabla_{n(x)} u) \cdot \tau(x). \tag{10}$$

Another context in which these derivatives will be encountered is integration along closed curves. For instance, if $\theta$ is a smooth $1-$ periodic parameterization of a closed curve $\gamma$, we have the following properties that will be used extensively:

$$\tau(\theta(s)) = \frac{\frac{\partial \theta}{\partial s}}{\left|\frac{\partial \theta}{\partial s}\right|}(s), \tag{11}$$

$$\frac{\partial (u \circ \theta)}{\partial s} = \frac{\partial \theta_i}{\partial s} \frac{\partial u}{\partial x_i} \circ \theta = \left|\frac{\partial \theta}{\partial s}\right| (\nabla_\tau u)(\theta(s)), \tag{12}$$

$$\int_\gamma \nabla_\tau u \, \mathrm{d}l = \int_0^1 \frac{1}{\left|\frac{\partial \theta}{\partial s}\right|} \frac{\partial (u \circ \theta)}{\partial s} \underbrace{\left|\frac{\partial \theta}{\partial s}\right| \mathrm{d}s}_{=\mathrm{d}l} = [u \circ \theta]_0^1 = 0. \tag{13}$$

### 2.4. Conservation of Energy

For all time of existence it is classical that the quantity

$$\frac{1}{2} \sum_\pm \rho^\pm \int_{\Omega^\pm(t)} |u^\pm(x,t)|^2 \, \mathrm{d}x + \sum_\pm \rho^\pm g \int_{\Omega^\pm} \eta_2^\pm \, \mathrm{d}x + \sigma |\partial \Omega^+(t)|,$$

is independent of time. Now given that

$$\int_{\Omega^\pm} \eta_2^\pm \, \mathrm{d}x = \int_{\Omega^\pm(t)} x_2 \, \mathrm{d}x, \quad \int_{\Omega^-(t)} x_2 + \int_{\Omega^+(t)} x_2 \, \mathrm{d}x = \int_\Omega x_2 \, \mathrm{d}x,$$

we then infer from this conservation that the total energy,

$$E(t) = \sum_{\pm} \frac{\rho^{\pm}}{2} \int_{\Omega^{\pm}(t)} |u^{\pm}|^2 \, \mathrm{d}x + (\rho^+ - \rho^-) g \int_{\Omega^+(t)} x_2 \, \mathrm{d}x + \sigma |\partial \Omega^+(t)|$$
$$= \sum_{\pm} \frac{\rho^{\pm}}{2} \int_{\Omega^{\pm}} |u^{\pm}(x,t)|^2 \, \mathrm{d}x + \underbrace{(\rho^+ - \rho^-)}_{\geqq 0} g \underbrace{x_2^+(t)}_{\geqq 0} |\Omega^+| + \sigma |\partial \Omega^+(t)|, \tag{14}$$

is constant in time for all time of existence of a smooth solution (namely so long as no eventual collision with the boundary occurs, or that no self-intersection of $\partial\Omega^+(t)$ occurs, and so long as the norm (5) stays finite) and where we defined

$$x^+(t) = \frac{1}{|\Omega^+|} \int_{\Omega^+(t)} x \, \mathrm{d}x = \frac{1}{|\Omega^+|} \int_{\Omega^+} \eta^+ \, \mathrm{d}x \tag{15}$$

as the center of gravity of $\Omega^+(t)$. Tracking the motion of this center of gravity will prove a powerful tool in establishing our finite in time singularity formation result (since any pointwise estimate would be hopeless in a two-phase problem as a cusp forms in $\Omega^-(t)$ at the time of contact).

We then have for the velocity of the center of mass that

$$v^+(t) = \frac{1}{|\Omega^+|} \int_{\Omega^+} v^+ \, \mathrm{d}x = \frac{1}{|\Omega^+|} \int_{\Omega^+(t)} u^+ \, \mathrm{d}x, \tag{16}$$

and for the acceleration that

$$a^+(t) = \frac{1}{|\Omega^+|} \int_{\Omega^+} \frac{\mathrm{d}v^+}{\mathrm{d}t} \, \mathrm{d}x = \frac{1}{|\Omega^+|} \int_{\Omega^+(t)} u_t^+ + u^+ \cdot \nabla u^+ \, \mathrm{d}x. \tag{17}$$

Due to (14) and our definition (16), we have by Cauchy–Schwarz that

$$|v^+(t)|^2 \leqq \frac{1}{|\Omega^+|} \int_{\Omega^+(t)} |u^+|^2 \, \mathrm{d}x \leqq \frac{2E(0)}{m_+}, \tag{18}$$

where $m_+ = \rho^+ |\Omega^+|$, which establishes the uniform in time control of this velocity.

### 2.5. *Conservation of Curl in Each Phase*

We have in each phase $\omega^{\pm} + u^{\pm} \cdot \nabla \omega^{\pm} = 0$, which implies, with $\eta^{\pm}$ being the flow map associated to $u^{\pm}$, that we have similarly as for a problem on a fixed domain

$$\omega^{\pm}(\eta^{\pm})(x,t) = \omega^{\pm}(x,0) = \omega_0^{\pm}. \tag{19}$$

This implies conservation of the $L^2$ norm of the curl of $u^-$ in the $\Omega^-(t)$ phase, which will be useful for elliptic estimates in Sections 5 and 9.

We will use in a crucial way the following equivalent formulation of the incompressible Euler equations.

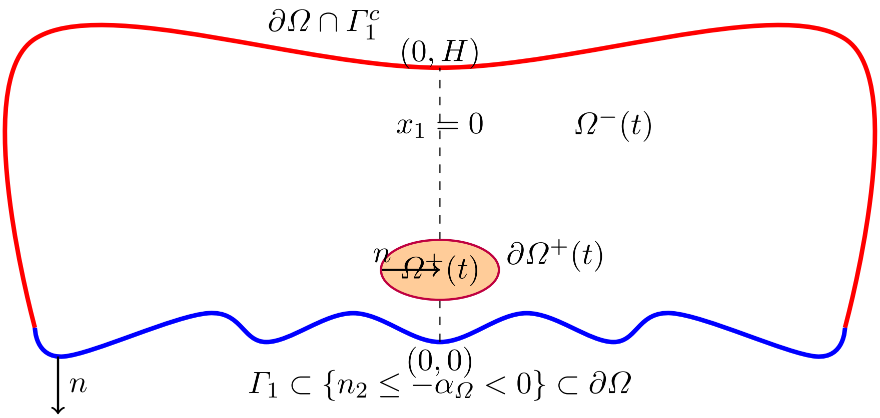


**Fig. 1.** In our convention, $n$ is always exterior to $\Omega^-(t)$

### *2.6. An Equivalent Formulation of the Problem*

First, using the definition of the curl, we see that

$$u_1^- \frac{\partial u_1^-}{\partial x_1} + u_2^- \frac{\partial u_1^-}{\partial x_2} = u_1^- \frac{\partial u_1^-}{\partial x_1} + u_2^- \frac{\partial u_2^-}{\partial x_1} - \omega^- u_2^- = \frac{1}{2} \frac{\partial |u^-|^2}{\partial x_1} - \omega^- u_2^- .$$

Similarly,

$$u_1^- \frac{\partial u_2^-}{\partial x_1} + u_2^- \frac{\partial u_2^-}{\partial x_2} = u_1^- \frac{\partial u_1^-}{\partial x_2} + u_2^- \frac{\partial u_2^-}{\partial x_2} + \omega^- u_1^- = \frac{1}{2} \frac{\partial |u^-|^2}{\partial x_2} + \omega^- u_1^- .$$

Therefore, the Euler equations in $\Omega^-(t)$ can be written as

$$\rho^- u_t^- + \nabla \left( \frac{\rho^- |u^-|^2}{2} + p^- \right) = -\rho^- g e_2 - \omega^- (-u_2^-, u_1^-). \tag{20}$$

## 3. Choice of Initial Data

We denote by $\Omega$ a bounded domain of class $H^{\frac{9}{2}}$, which is symmetric with respect to the vertical axis $x_1 = 0$ and whose boundary $\partial\Omega$ is connected. This domain is of height $H > 0$ along the vertical axis $x_1 = 0$, with the bottom point on the vertical axis being $(0, 0)$. We also assume that $\Omega$ has a part of its boundary $\Gamma_1$, centered at the origin, with $-L \leqq x_1 \leqq L$ $(L > 0)$, under the form of a graph $x_2 = f(x_1)$ and thus satisfying

$$n_2 \leqq -\alpha_\Omega < 0, \text{ on } \Gamma_1. \tag{21}$$

We then choose $\Omega^+$ such that $\overline{\Omega^+} \subset \Omega$ to be an equally symmetric domain with respect to the vertical axis $x_1 = 0$, which is of the same regularity class as $\Omega$.

We then define the initial fluid domain $\Omega^- = \Omega \cap (\overline{\Omega^+})^c$.

We choose $u^+(0) \in H^3(\Omega^+)$ and $u^-(0) \in H^3(\Omega^-)$ divergence free velocity fields such that their horizontal component is odd whereas their vertical one is even and satisfying at time 0, (4d) and (4e). At time 0, the center of gravity of $\Omega^+$ is located at $x^+(0) = (0, h)$. Given the symmetry of the initial data with respect to the $x_1 = 0$ axis, we have that for all time of existence $u_1^\pm(-x_1, x_2, \cdot) = -u_1^\pm(x_1, x_2, \cdot)$, $u_2^\pm(-x_1, x_2, \cdot) = u_2^\pm(x_1, x_2, \cdot)$. This implies for the center of gravity of $\Omega^+(t)$ that $x_1^+(t) = 0 = v_1^+(t)$. This can be seen by setting the fixed-point approach of [5] in a symmetric setting.

We moreover assume

$$\frac{E(0)}{(\rho^+ - \rho^-)g|\Omega^+|} + \frac{E(0)}{\sigma} < \min\left(\frac{L}{4}, \frac{H}{8}\right), \tag{22}$$

and

$$\left\{(x_1, x_2);\ x_1 \in \left[-\frac{L}{2}, \frac{L}{2}\right];\ f(x_1) \leqq x_2 \leqq \frac{H}{2}\right\} \subset \Omega. \tag{23}$$

We also assume

$$\Gamma_1^c \cap \partial\Omega \subset (\partial\Omega \cap \{|x_1| \geqq L\}) \cup \left(\partial\Omega \cap \left\{x_2 \geqq \frac{H}{2}\right\}\right), \tag{24}$$

where $\Gamma_1$ was defined earlier in this Section. The first condition can be satisfied by taking the dimensions of the container domain $\Omega$ large relative to $\Omega^+$ and the initial $x^+(0)$, and small square integrable velocities, whereas the second and third ones are conditions on the shape of $\partial\Omega$ (if $\Omega$ is for instance of essentially rectangular shape, with four smoothed corners, all these conditions are satisfied).

The conservation of (14) states that

$$\sum_\pm \frac{\rho^\pm}{2} \int_{\Omega^\pm(t)} |u^\pm|^2\, \mathrm{d}x + (\rho^+ - \rho^-)g x_2^+(t)|\Omega^+| + \sigma|\partial\Omega^+(t)| = E(0). \tag{25}$$

First, (25) shows that

$$x_2^+(t) \leqq \frac{E(0)}{(\rho^+ - \rho^-)g|\Omega^+|} < \frac{H}{8}, \tag{26}$$

where we used (22) to obtain the second inequality. This shows that the center of gravity of $\Omega^+(t)$ stays away from the top of $\partial\Omega$. Also (25) shows that

$$\sum_\pm \frac{\rho^\pm}{2} \int_{\Omega^\pm(t)} |u^\pm|^2\, \mathrm{d}x \leqq E(0). \tag{27}$$

Now, we prove $\Omega^+(t)$ stays away from the top of $\partial\Omega$ and from the lateral sides of $\partial\Omega$. Using (25) again, we have, since $x_2^+(t) \geqq 0$, that

$$|\partial\Omega^+(t)| \leqq \frac{E(0)}{\sigma} \leqq \min\left(\frac{H}{8}, \frac{L}{4}\right), \tag{28}$$

by using our assumption (22). Now, let $x^l(t)$ be a point of lowest altitude of $\Omega^+(t)$ and $x^h(t)$ be a point of highest altitude of $\Omega^+(t)$. Then, since the straight line from these two points is shorter than any of the two paths along $\partial\Omega^+(t)$ between them, we have

$$x_2^h(t) - x_2^l(t) \leqq |x^h(t) - x^l(t)| \leqq \frac{|\partial\Omega^+(t)|}{2} \leqq \frac{H}{16}, \tag{29}$$

where we used (28). Thus,

$$x_2^h(t) \leqq \frac{H}{16} + x_2^l(t) \leqq \frac{H}{16} + x_2^+(t),$$

which, with (26), provides

$$x_2^h(t) \leqq \frac{H}{16} + \frac{H}{8} < \frac{H}{4}. \tag{30}$$

By introducing at most on the left point $x^L(t)$ and at most on the right point $x^R(t)$ of $\Omega^+(t)$, we have, similarly, that

$$2x_1^R(t) = x_1^R(t) - x_1^L(t) \leqq \frac{|\partial\Omega^+(t)|}{2} \leqq \frac{L}{8},$$

where we used (28). Thus,

$$0 \leqq x_1^R(t) \leqq \frac{L}{16}, \tag{31a}$$

$$-\frac{L}{16} \leqq x_1^L(t) \leqq 0. \tag{31b}$$

Propositions (30), (31a) and (31b) then show that for all the time of existence,

$$\Omega^+(t) \subset \left\{(x_1, x_2);\ x_1 \in \left[-\frac{L}{16}, \frac{L}{16}\right];\ f(x_1) \leqq x_2 \leqq \frac{H}{4}\right\} \subset \Omega, \tag{32}$$

with our assumption (23). Therefore, due to our assumption (24), for all the time of existence,

$$d(\Omega^+(t), \partial\Omega \cap \Gamma_1^c) \geqq D = \min\left(\frac{H}{4}, \frac{15L}{16}\right) > 0, \tag{33}$$

where $\Gamma_1 \subset \partial\Omega$ was defined earlier in this Section as the bottom part of $\partial\Omega$ under the form of a graph.

## 4. Evolution of the Center of Gravity of the Moving Fluid Bubble $\Omega^+(t)$

The present section is crucial in obtaining an ODE linked to the motion of the center of mass of the inclusion. This ODE will be shown later to lead to finite-time blow-up for a differential inequality. These calculations are quite general and only require $\Omega^-(t)$ to be governed by the incompressible Euler equations. In particular they are true for both problems considered in this paper. Our aim is to establish

**Lemma 1.** *The vertical motion of the center of mass of $\Omega^+(t)$ satisfies the relation*

$$m_+ \frac{\mathrm{d}v_2^+}{\mathrm{d}t} = \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n_2 \,\mathrm{d}l - \rho^- \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^-(t)} u_2^- \,\mathrm{d}x - (m_+ - \rho^- |\Omega^+|) g \\ - \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \rho^- x_1 \, u^- \cdot \tau \,\mathrm{d}l. \qquad (34)$$

**Proof.** We have the fundamental equation for the center of mass

$$\begin{aligned} m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t}(t) &= \rho^+ \int_{\Omega^+} \frac{\mathrm{d}v^+}{\mathrm{d}t}(x,t) \,\mathrm{d}x \\ &= \rho^+ \int_{\Omega^+(t)} u_t^+ + u^+ \cdot \nabla u^+ \,\mathrm{d}x \\ &= - \int_{\Omega^+(t)} \nabla p^+ + \rho^+ g e_2 \,\mathrm{d}x \\ &= \int_{\partial\Omega^+(t)} p^+ \, n \,\mathrm{d}l(t) - m_+ g e_2, \end{aligned} \qquad (35)$$

where we remind $n$ is the outer unit normal to $\Omega^-(t)$, pointing inside $\Omega^+(t)$, which explains the sign in the boundary integral in (35). Using our boundary condition (4c), this provides

$$\begin{aligned} m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t} &= \int_{\partial\Omega^+(t)} (p^- \, n + \sigma \nabla_\tau(\tau)) \,\mathrm{d}l(t) - m_+ g e_2 \\ &= \int_{\partial\Omega^+(t)} p^- \, n \,\mathrm{d}l(t) - m_+ g e_2, \end{aligned} \qquad (36)$$

where we used

$$\int_{\partial\Omega^+(t)} \nabla_\tau(\tau) \,\mathrm{d}l = 0$$

for any closed smooth curve such as $\partial\Omega^+(t)$ (so long as the smooth solution exists).

By integrating by parts in $\Omega^-(t)$, we have

$$\int_{\Omega^-(t)} \nabla p^- \,\mathrm{d}x = \int_{\partial\Omega^+(t)} p^- \, n \,\mathrm{d}l(t) + \int_{\partial\Omega} p^- \, n \,\mathrm{d}l.$$

This provides, by substitution in (36)

$$m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t} = - \int_{\partial\Omega} p^- \, n \,\mathrm{d}l + \int_{\Omega^-(t)} \nabla p^- \,\mathrm{d}x - m_+ g e_2,$$

which with the Euler equations provides

$$\begin{aligned} m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t} &= - \int_{\partial\Omega} p^- \, n \,\mathrm{d}l - \rho^- \int_{\Omega^-(t)} u_t^- + u^- \cdot \nabla u^- + g e_2 \,\mathrm{d}x - m_+ g e_2, \\ &= - \int_{\partial\Omega} p^- \, n \,\mathrm{d}l - \rho^- \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^-(t)} u^- \,\mathrm{d}x - (m_+ + \rho^- |\Omega^-|) g e_2. \end{aligned} \qquad (37)$$

Next, we notice that on $\partial\Omega$, thanks to (20), we have

$$\rho^- u_t^- \cdot \tau + \nabla_\tau \left( p^- + \rho^- \frac{|u^-|^2}{2} \right) = -\rho^- g e_2 \cdot \tau + \omega^- \underbrace{u^- \cdot n}_{=0 \text{ on } \partial\Omega} = -\rho^- g e_2 \cdot \tau. \tag{38}$$

We now denote by $\theta : [0,1] \to \partial\Omega$ a 1-periodic smooth parameterization of $\partial\Omega$ with $\theta(0) = (0, H)$. We integrate (38) along $\partial\Omega$ between $\theta(0)$ and $\theta(s)$ to get

$$\left[ \left( p^- + \rho^- \frac{|u^-|^2}{2} \right) (\theta(\cdot), t) \right]_0^s = -\int_0^s (\rho^- g e_2 \cdot \tau + \rho^- u_t^- \cdot \tau)(\theta(\alpha), t) \underbrace{|\theta'(\alpha)| \, \mathrm{d}\alpha}_{=\mathrm{d}l},$$

which implies, by integrating (in the $s$ variable) along $\partial\Omega$, that

$$\begin{aligned} -\int_{\partial\Omega} p^- n \, \mathrm{d}l = \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n \, \mathrm{d}l - \left( p^- + \rho^- \frac{|u^-|^2}{2} \right) (\theta(0), t) \int_{\partial\Omega} n \, \mathrm{d}l \\ + \int_0^1 \int_0^s \rho^- (g e_2 + u_t^-) \cdot \tau(\theta(\alpha), t) |\theta'(\alpha)| \, \mathrm{d}\alpha \, n(\theta(s)) \, |\theta'(s)| \, \mathrm{d}s. \end{aligned}$$

Since $\partial\Omega$ is a closed curve, $\int_{\partial\Omega} n \, \mathrm{d}l = 0$, and thus the previous relation becomes

$$\begin{aligned} -\int_{\partial\Omega} p^- n \, \mathrm{d}l = \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n \, \mathrm{d}l \\ + \int_0^1 \int_0^s \rho^- (g e_2 + u_t^-) \cdot \tau(\theta(\alpha), t) |\theta'(\alpha)| \, \mathrm{d}\alpha \, n(\theta(s)) \, |\theta'(s)| \, \mathrm{d}s. \end{aligned} \tag{39}$$

We now substitute (39) into (37), leading to

$$\begin{aligned} m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t} = -\rho^- \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^-(t)} u^- \, \mathrm{d}x - (m_+ + \rho^- |\Omega^-|) g e_2 + \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n \, \mathrm{d}l \\ + \int_0^1 \int_0^s (\rho^- g e_2 \cdot \tau + \rho^- u_t^- \cdot \tau)(\theta(\alpha), t) |\theta'(\alpha)| \, \mathrm{d}\alpha \, n(\theta(s)) \, |\theta'(s)| \, \mathrm{d}s. \end{aligned} \tag{40}$$

We now write in a much simpler way the fourth term on the right-hand side of this equation. In order to do so, we define $f(x) = x_2$, so that $\nabla f = e_2$ and $\nabla_\tau f = e_2 \cdot \tau$. Therefore,

$$f(\theta(s)) = f(\theta(0)) + \int_0^s \underbrace{e_2 \cdot \tau(\theta(\alpha))}_{\nabla_\tau f(\theta(\alpha))} \underbrace{|\theta'(\alpha)| \, \mathrm{d}\alpha}_{\mathrm{d}l}. \tag{41}$$

Next, since

$$\int_\Omega e_2 \, \mathrm{d}x = \int_\Omega \nabla f \, \mathrm{d}x = \int_{\partial\Omega} f \, n \, \mathrm{d}l, \tag{42}$$

substituting (41) into (42) provides, (using $f(\theta(0))\int_{\partial\Omega} n\,\mathrm{d}l = 0$), that

$$\int_\Omega e_2\,\mathrm{d}x = \int_0^1 \int_0^s e_2 \cdot \tau(\theta(\alpha))|\theta'(\alpha)|\,\mathrm{d}\alpha\; n(\theta(s))\;|\theta'(s)|\,\mathrm{d}s. \tag{43}$$

Using (43) in (40) then yields

$$\begin{aligned} m_+ \frac{\mathrm{d}v^+}{\mathrm{d}t} &\\ &= \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n\,\mathrm{d}l - \rho^- \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^-(t)} u^-\,\mathrm{d}x - (m_+ + \rho^-|\Omega^-| - \rho^-|\Omega|)ge_2 \\ &\quad + \int_0^1 \int_0^s \rho^- u_t^- \cdot \tau(\theta(\alpha))|\theta'(\alpha)|\,\mathrm{d}\alpha\; n(\theta(s))\;|\theta'(s)|\,\mathrm{d}s \\ &= \rho^- \int_{\partial\Omega} \frac{|u^-|^2}{2} n\,\mathrm{d}l - \rho^- \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega^-(t)} u^-\,\mathrm{d}x - (m_+ - \rho^-|\Omega^+|)ge_2 \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \int_0^s \rho^- u^- \cdot \tau(\theta(\alpha))|\theta'(\alpha)|\,\mathrm{d}\alpha\; n(\theta(s))\;|\theta'(s)|\,\mathrm{d}s. \end{aligned} \tag{44}$$

Defining

$$F(t) = \int_0^1 \int_0^s \rho^- u^- \cdot \tau(\theta(\alpha))|\theta'(\alpha)|\,\mathrm{d}\alpha\; n(\theta(s))\;|\theta'(s)|\,\mathrm{d}s, \tag{45}$$

we next rewrite its vertical component $F_2$ in a simpler way.

First, since $n_2 = \tau_1$, we have

$$F_2(t) = \int_0^1 \int_0^s \rho^- u^- \cdot \tau(\theta(\alpha))|\theta'(\alpha)|\,\mathrm{d}\alpha\; \tau_1(\theta(s))\;|\theta'(s)|\,\mathrm{d}s,$$

which by integration by parts provides

$$F_2(t) = -\int_0^1 \rho^- u^- \cdot \tau(\theta(s))|\theta'(s)| \int_0^s \tau_1(\theta(\alpha))\;|\theta'(\alpha)|\,\mathrm{d}\alpha\,\mathrm{d}s. \tag{46}$$

Note here that we used the fact that $\int_{\partial\Omega} \tau_1\,\mathrm{d}l = 0$.

Moreover, in the same way as we obtained (41), this time for $f(x) = x_1$, so that $\nabla f = e_1$ and therefore $\nabla_\tau f = e_1 \cdot \tau$, we have

$$f(\theta(s)) = f(\theta(0)) + \int_0^s \underbrace{e_1 \cdot \tau(\theta(\alpha))}_{\nabla_\tau f(\theta(\alpha))} \underbrace{|\theta'(\alpha)|\,\mathrm{d}\alpha}_{\mathrm{d}l}, \tag{47}$$

which by substitution in (46) provides

$$F_2(t) = -\int_0^1 \rho^- u^- \cdot \tau(\theta(s))|\theta'(s)|\;(\theta_1(s) - \theta_1(0))\,\mathrm{d}s.$$

Using $\theta_1(0) = 0$, this yields

$$
\begin{aligned}
F_2(t) &= -\int_0^1 \rho^- u^- \cdot \tau(\theta(s))|\theta'(s)|\,\theta_1(s)\,\mathrm{d}s \\
&= -\int_{\partial\Omega} \rho^- x_1\, u^- \cdot \tau\,\mathrm{d}l. \qquad (48)
\end{aligned}
$$

Substituting (48) in the vertical component of (44) proves (34). □

We have by integrating (34) in time the corollary

$$
\begin{aligned}
&-\rho^- \int_{\partial\Omega} x_1 u^-(\cdot,t)\cdot\tau\,\mathrm{d}l \\
&= m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^-\,\mathrm{d}x - \frac{\rho^-}{2}\int_0^t \int_{\partial\Omega} |u^-|^2 n_2\,\mathrm{d}l\,\mathrm{d}t \\
&\quad + \underbrace{(\rho^+ - \rho^-)}_{>0}|\Omega^+|gt \\
&\quad \underbrace{-\rho^- \int_{\partial\Omega} x_1 (u_0^-)_2 \cdot \tau\,\mathrm{d}l - m_+ v_2^+(0) - \rho^- \int_{\Omega^-} (u_0^-)_2\,\mathrm{d}x}_{C_0}\,. \qquad (49)
\end{aligned}
$$

**Remark 6.** We notice the computations leading to (34) and (49) came purely from using the incompressible Euler equations with gravity in $\Omega^-(t)$ in the relation (36), and are valid for any law governing the phase $\Omega^+(t)$, including the case of the rigid body considered later in this paper.

## 5. Finite-Time Singularity Formation for the Vortex Sheet Problem with Surface Tension

We note that from our energy conservation (25) and (18), the first and second terms on the right-hand side of (49) are controlled for all the time of existence by a constant independent of time, while the fourth term is linear in time. We now address the question of the third term, which is not sign definite across $\partial\Omega$, due to the presence of $n_2$.

We remind that our assumptions from Section 3 imply that we can split $\partial\Omega$ into the graph $\Gamma_1$, centered on the vertical axis $x_1 = 0$, below the (potentially) falling moving body in the fluid, and where $n_2 \leqq -\alpha_\Omega < 0$ and its complementary, where we will show the integral is small relative to the fourth term of (49).

From (49) and (48) we infer that

$$
\begin{aligned}
F_2(t) \geqq\ & m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^-\,\mathrm{d}x - \rho^- \int_0^t \int_{\Gamma_1^c \cap \partial\Omega} \frac{|u^-|^2}{2} n_2\,\mathrm{d}l\,\mathrm{d}t \\
& + \rho^- \alpha_\Omega \int_0^t \int_{\Gamma_1} \frac{|u^-|^2}{2}\,\mathrm{d}l\,\mathrm{d}t + \underbrace{(\rho^+ - \rho^-)}_{>0}|\Omega^+|gt + C_0
\end{aligned}
$$

$$\geqq m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^- \, \mathrm{d}x \, \mathrm{d}x + C_0$$
$$- \rho^- \int_0^t \int_{\Gamma_1^c \cap \partial\Omega} \frac{|u^-|^2}{2} n_2 \, \mathrm{d}l \, \mathrm{d}t + \frac{1}{4}(\rho^+ - \rho^-)|\Omega^+| g t$$
$$+ \rho^- \alpha_\Omega \int_0^t \int_{\Gamma_1} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t + \frac{3}{4}(\rho^+ - \rho^-)|\Omega^+| g t. \tag{50}$$

We will prove later on that for initial height $h$ and initial velocities satisfying (80) stated later, that

$$(1 + \alpha_\Omega) \left| \rho^- \int_0^t \int_{\Gamma_1^c \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \, \mathrm{d}t \right| \leqq \frac{1}{4}(\rho^+ - \rho^-)|\Omega^+| g t. \tag{51}$$

Using the property (51), we have by $\partial\Omega = \Gamma_1 \cup (\Gamma_1^c \cap \partial\Omega)$ that

$$\rho^- \alpha_\Omega \int_0^t \int_{\Gamma_1} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t + \frac{1}{8}(\rho^+ - \rho^-)|\Omega^+| g t \geqq \rho^- \alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t,$$

and thus

$$\rho^- \alpha_\Omega \int_0^t \int_{\Gamma_1} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t + \frac{1}{4}(\rho^+ - \rho^-)|\Omega^+| g t \geqq \rho^- \alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t. \tag{52}$$

Using (51) again, we also have

$$-\rho^- \int_0^t \int_{\Gamma_1^c \cap \partial\Omega} \frac{|u^-|^2}{2} n_2 \, \mathrm{d}l \, \mathrm{d}t + \frac{1}{4}(\rho^+ - \rho^-)|\Omega^+| g t \geqq 0. \tag{53}$$

Using (52) and (53) in (50), we infer that

$$F_2(t) \geqq m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^- \, \mathrm{d}x + \rho^- \alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t$$
$$+ \frac{\rho^+ - \rho^-}{2} |\Omega^+| g t + C_0. \tag{54}$$

On the other hand, given (48) for $F_2$, we have the existence of $\tilde{C}_\Omega > 0$ (depending on $\Omega$) such that

$$|F_2(t)| \leqq \tilde{C}_\Omega \rho^- \int_{\partial\Omega} |u^-|(\cdot, t) \, \mathrm{d}l. \tag{55}$$

Using (55) in (54), we obtain

$$\tilde{C}_\Omega \rho^- \int_{\partial\Omega} |u^-|(\cdot, t) \, \mathrm{d}l$$
$$\geqq m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^- \, \mathrm{d}x + \rho^- \alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^-|^2}{2} \, \mathrm{d}l \, \mathrm{d}t$$
$$+ \frac{\rho^+ - \rho^-}{2} |\Omega^+| g t - |C_0|. \tag{56}$$

By Cauchy–Schwarz, this implies

$$\begin{aligned}\tilde{C}_\Omega \rho^- \int_{\partial\Omega} |u^-|(\cdot,t)\,\mathrm{d}l &\geqq m_+ v_2^+(t) + \rho^- \int_{\Omega^-(t)} u_2^-\,\mathrm{d}x - |C_0| \\ &+ \frac{\rho^- \alpha_\Omega}{2t|\partial\Omega|}\left(\int_0^t \int_{\partial\Omega} |u^-|\,\mathrm{d}l\,\mathrm{d}t\right)^2 + \frac{1}{2}(\rho^+ - \rho^-)|\Omega^+|gt. \end{aligned} \tag{57}$$

Using our energy bounds (27) and (18) in (57), we have

$$\begin{aligned}\tilde{C}_\Omega \rho^- \int_{\partial\Omega} |u^-|(\cdot,t)\,\mathrm{d}l \geqq -m_+\sqrt{\frac{2E(0)}{m_+}} - \rho^-\sqrt{\frac{2E(0)}{\rho^-}}\sqrt{|\Omega^-|} - |C_0| \\ + \frac{\rho^- \alpha_\Omega}{2t|\partial\Omega|}\left(\int_0^t \int_{\partial\Omega} |u^-|\,\mathrm{d}l\,\mathrm{d}t\right)^2 + \frac{1}{2}(\rho^+ - \rho^-)|\Omega^+|gt. \end{aligned} \tag{58}$$

Let

$$f(t) = \int_0^t \int_{\partial\Omega} |u^-|\,\mathrm{d}l\,\mathrm{d}t. \tag{59}$$

From (58), we have that with

$$t_0 = \frac{4}{g|\Omega^+|(\rho^+ - \rho^-)}\left(\sqrt{2E(0)m_+} + \sqrt{2E(0)\rho^-|\Omega^-|} + |C_0|\right) \tag{60}$$

for all $t \geqq t_0$, (58) implies

$$\tilde{C}_\Omega \rho^- f'(t) \geqq \frac{\rho^- \alpha_\Omega}{2t|\partial\Omega|} f^2 + \frac{1}{4}(\rho^+ - \rho^-)|\Omega^+|gt > 0.$$

Therefore, for all $t \geqq t_0$, $f(t) > 0$ and

$$\frac{f'}{f^2} \geqq \frac{\alpha_\Omega}{2|\partial\Omega|\tilde{C}_\Omega t}, \tag{61}$$

which by integration from $t_0$ to $t \geqq t_0$ provides

$$-\frac{1}{f(t)} + \frac{1}{f(t_0)} \geqq \frac{\alpha_\Omega}{2|\partial\Omega|\tilde{C}_\Omega} \ln\left(\frac{t}{t_0}\right).$$

Therefore,

$$0 < \frac{1}{f(t)} \leqq \frac{1}{f(t_0)} - \frac{\alpha_\Omega}{2|\partial\Omega|\tilde{C}_\Omega} \ln\left(\frac{t}{t_0}\right),$$

which shows that for $t \geqq t_0\, e^{\frac{2|\partial\Omega|\tilde{C}_\Omega}{\alpha_\Omega f(t_0)}}$, we have

$$0 < \frac{1}{f(t)} \leqq 0,$$

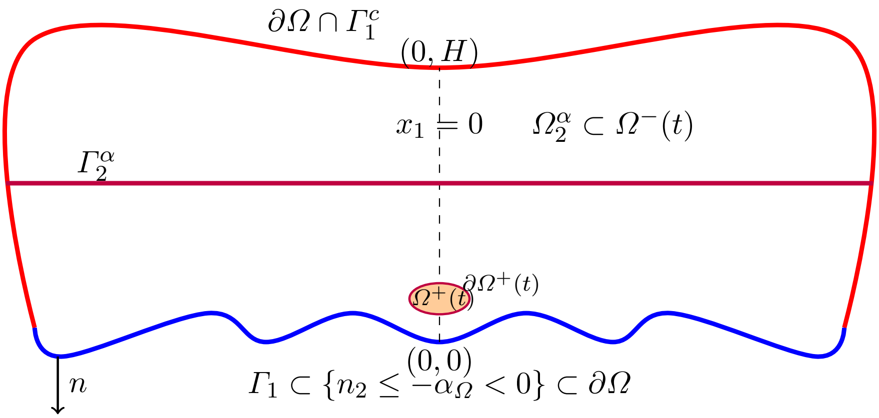


**Fig. 2.** $\Omega^+(t)$ stays below $\Gamma_2^\alpha$ for $\alpha > \frac{H}{4}$

which is an obvious impossibility. Therefore, the maximal time of existence of a smooth solution $T_{\max} > 0$ satisfies

$$T_{\max} \leqq t_0 \, e^{\frac{2|\partial\Omega|\tilde{C}_\Omega}{\alpha_\Omega f(t_0)}}. \tag{62}$$

We now have to turn back to proving our missing estimate (51) (provided our initial data satisfy (80)), which controls the velocity on $\Gamma_1^c$ (that we are sure the moving bubble $\Omega^+(t)$ stays away from, given (33)). Here the difficulty is to get the precise bound given by (51) and not just a generic constant, or a constant greater than the majorant of (51), and it calls for subtle observations of elliptic and geometric natures.

Our starting point is the fact that $u^-$ being divergence free

$$u^- = \nabla^\perp \phi, \text{ in } \Omega^-(t), \tag{63}$$

with

$$\phi = 0, \text{ on } \partial\Omega, \tag{64}$$

(we will not need the condition $\nabla^\perp\phi \cdot n = u^+ \cdot n$ on $\partial\Omega^+(t)$) and

$$\Delta\phi = \omega^-, \text{ in } \Omega^-(t). \tag{65}$$

We will also need the fact that

$$\|\nabla\phi\|^2_{L^2(\Omega^-(t))} = \|u^-\|^2_{L^2(\Omega^-(t))} \leqq 2\frac{E(0)}{\rho^-}, \tag{66}$$

due to(27). We now define, for any $\alpha > \frac{H}{4}$,

$$\Omega_2^\alpha = \{x \in \Omega;\ x_2 \geqq \alpha\}, \tag{67}$$

and

$$\Gamma_2^\alpha = \{x \in \Omega;\ x_2 = \alpha\}. \tag{68}$$

From our relation (32), we know that for $\alpha > \frac{H}{4}$, $\Omega_2^\alpha$ does not intersect $\Omega^+(t)$ for all time of existence, and we will work with such values of $\alpha$ in what follows.

Taking $\tilde{n}$ as being our global vector field extending $n$ into $\Omega$ defined in Section 2, we now take $|\tilde{n}|^2\nabla_{\tilde{n}}\phi$ as test function for (65) and integrate the relation in $\Omega_2^\alpha$.

We first notice that for any unit vector $a$, if $b = a^\perp$, we have

$$\omega^- = \Delta\phi = a_i a_j \frac{\partial^2\phi}{\partial x_i \partial x_j} + b_i b_j \frac{\partial^2\phi}{\partial x_i \partial x_j}.$$

Therefore, for any vector $a$, if $b = a^\perp$,

$$|a|^2\omega^- = |a|^2\Delta\phi = a_i a_j \frac{\partial^2\phi}{\partial x_i \partial x_j} + b_i b_j \frac{\partial^2\phi}{\partial x_i \partial x_j}.$$

We now simply use this expansion of $\Delta$ in the orthogonal $(\tilde{\tau}(x), \tilde{n}(x))$ system at each point $x \in \Omega^-(t)$, to get

$$|\tilde{n}(x)|^2\omega^- = |\tilde{n}(x)|^2\Delta\phi = \tilde{\tau}_i(x)\tilde{\tau}_j(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x,t) + \tilde{n}_i(x)\tilde{n}_j(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x,t). \tag{69}$$

Integration by parts in $\Omega_2^\alpha \subset \Omega^-(t)$ (and remembering that the normal exterior vector to $\Gamma_2^\alpha$ is $-e_2$ and to $\partial\Omega$ is $n = \tilde{n}$) provides us with

$$\begin{aligned}
\int_{\Omega_2^\alpha} |\tilde{n}|^2\omega^-\nabla_{\tilde{n}}\phi\,\mathrm{d}x &= \int_{\Omega_2^\alpha}\left(\tilde{\tau}_i\tilde{\tau}_j\frac{\partial^2\phi}{\partial x_i\partial x_j} + \tilde{n}_i\tilde{n}_j\frac{\partial^2\phi}{\partial x_i\partial x_j}\right)\tilde{n}_k\frac{\partial\phi}{\partial x_k}\,\mathrm{d}x\\
&= \underbrace{-\int_{\Omega_2^\alpha}\frac{\partial\phi}{\partial x_j}\frac{\partial(\tilde{\tau}_i\tilde{\tau}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k})}{\partial x_i}\,\mathrm{d}x}_{I_1} \underbrace{-\int_{\Omega_2^\alpha}\frac{\partial\phi}{\partial x_j}\frac{\partial(\tilde{n}_i\tilde{n}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k})}{\partial x_i}\,\mathrm{d}x}_{I_2}\\
&\quad - \int_{\Gamma_2^\alpha}\frac{\partial\phi}{\partial x_j}\tilde{\tau}_2\tilde{\tau}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k} + \tilde{n}_2\tilde{n}_j\tilde{n}_k\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_k}\,\mathrm{d}x_1\\
&\quad + \int_{\partial\Omega_2^\alpha\cap\partial\Omega}\frac{\partial\phi}{\partial x_j}\underbrace{\tilde{\tau}_i\tilde{n}_i}_{=0}\tilde{\tau}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k} + \frac{\partial\phi}{\partial x_j}\underbrace{\tilde{n}_i n_i}_{=1 \text{ on } \partial\Omega}\tilde{n}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k}\,\mathrm{d}l.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 + I_2 = &-\int_{\Gamma_2^\alpha}\frac{\partial\phi}{\partial x_j}\tilde{\tau}_2\tilde{\tau}_j\tilde{n}_k\frac{\partial\phi}{\partial x_k} + \tilde{n}_2\tilde{n}_j\tilde{n}_k\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_k}\,\mathrm{d}x_1\\
&+ \int_{\partial\Omega_2^\alpha\cap\partial\Omega}|\nabla_n\phi|^2\,\mathrm{d}l - \int_{\Omega_2^\alpha}|\tilde{n}|^2\omega^-\nabla_{\tilde{n}}\phi\,\mathrm{d}x.
\end{aligned} \tag{70}$$

We next rewrite $I_1$ and $I_2$. From the definition,

$$
\begin{aligned}
I_2 &= \int_{\Omega_2^\alpha} \tilde{n}_i \tilde{n}_j \frac{\partial \phi}{\partial x_j} \frac{\partial (\tilde{n}_k \frac{\partial \phi}{\partial x_k})}{\partial x_i} \,\mathrm{d}x + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{n}_i \tilde{n}_j)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&= \frac{1}{2} \int_{\Omega_2^\alpha} \tilde{n}_i \frac{\partial |\nabla_{\tilde{n}} \phi|^2}{\partial x_i} \,\mathrm{d}x + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{n}_i \tilde{n}_j)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&= -\int_{\Omega_2^\alpha} \frac{\partial \tilde{n}_i}{\partial x_i} \frac{|\nabla_{\tilde{n}} \phi|^2}{2} \,\mathrm{d}x + \int_{\partial \Omega_2^\alpha \cap \partial \Omega} \underbrace{|\tilde{n}|^2}_{=1 \text{ on } \partial \Omega} \frac{|\nabla_{\tilde{n}} \phi|^2}{2} \,\mathrm{d}l - \int_{\Gamma_2^\alpha} \tilde{n}_2 \frac{|\nabla_{\tilde{n}} \phi|^2}{2} \,\mathrm{d}x_1 \\
&\quad + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{n}_i \tilde{n}_j)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x. \qquad (71)
\end{aligned}
$$

We next move to $I_1$:

$$
\begin{aligned}
I_1 &= \int_{\Omega_2^\alpha} \tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_k \partial x_i} \,\mathrm{d}x + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&= \int_{\Omega_2^\alpha} \tilde{\tau}_j \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial (\tilde{\tau}_i \frac{\partial \phi}{\partial x_i})}{\partial x_k} \,\mathrm{d}x - \int_{\Omega_2^\alpha} \tilde{\tau}_j \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \tilde{\tau}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x \\
&\quad + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&= \int_{\Omega_2^\alpha} \frac{\tilde{n}_k}{2} \frac{\partial |\nabla_{\tilde{\tau}} \phi|^2}{\partial x_k} \,\mathrm{d}x - \int_{\Omega_2^\alpha} \tilde{\tau}_j \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \tilde{\tau}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x \\
&\quad + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&= -\int_{\Omega_2^\alpha} \frac{\partial \tilde{n}_k}{\partial x_k} \frac{|\nabla_{\tilde{\tau}} \phi|^2}{2} \,\mathrm{d}x + \int_{\partial \Omega_2^\alpha \cap \partial \Omega} \frac{|\tilde{n}|^2}{2} \underbrace{|\nabla_{\tilde{\tau}} \phi|^2}_{=0 \text{ on } \partial \Omega} \,\mathrm{d}l - \int_{\Gamma_2^\alpha} \frac{\tilde{n}_2}{2} |\nabla_{\tilde{\tau}} \phi|^2 \,\mathrm{d}x_1 \\
&\quad - \int_{\Omega_2^\alpha} \tilde{\tau}_j \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \tilde{\tau}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x. \qquad (72)
\end{aligned}
$$

By gathering (70), (71) and (72), we obtain

$$
\begin{aligned}
\int_{\partial \Omega_2^\alpha \cap \partial \Omega} \frac{|\nabla_n \phi|^2}{2} \,\mathrm{d}l &= -\frac{1}{2} \int_{\Omega_2^\alpha} \frac{\partial \tilde{n}_k}{\partial x_k} |\nabla_{\tilde{\tau}} \phi|^2 \,\mathrm{d}x - \frac{1}{2} \int_{\Gamma_2^\alpha} \tilde{n}_2 |\nabla_{\tilde{\tau}} \phi|^2 \,\mathrm{d}x_1 \\
&\quad - \int_{\Omega_2^\alpha} \tilde{\tau}_j \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \tilde{\tau}_i}{\partial x_k} \frac{\partial \phi}{\partial x_i} \,\mathrm{d}x + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{\tau}_i \tilde{\tau}_j \tilde{n}_k)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x \\
&\quad - \frac{1}{2} \int_{\Omega_2^\alpha} \frac{\partial \tilde{n}_i}{\partial x_i} |\nabla_{\tilde{n}} \phi|^2 - \frac{1}{2} \int_{\Gamma_2^\alpha} \tilde{n}_2 |\nabla_{\tilde{n}} \phi|^2 \,\mathrm{d}x_1 \\
&\quad + \int_{\Omega_2^\alpha} \frac{\partial (\tilde{n}_i \tilde{n}_j)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x + \int_{\Omega_2^\alpha} |\tilde{n}|^2 \omega^- \nabla_{\tilde{n}} \phi \,\mathrm{d}x \\
&\quad + \int_{\Gamma_2^\alpha} \frac{\partial \phi}{\partial x_j} \tilde{\tau}_2 \tilde{\tau}_j \tilde{n}_k \frac{\partial \phi}{\partial x_k} + \tilde{n}_2 \tilde{n}_j \tilde{n}_k \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \,\mathrm{d}x_1. \qquad (73)
\end{aligned}
$$

In what follows, $C_i$ is a generic constant which does not depend on our initial velocity and height $h$. Due to (7), (8) and (27), we have that

$$\left| \int_{\Omega_2^\alpha} \frac{\partial(\tilde{n}_i \tilde{n}_j)}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tilde{n}_k \frac{\partial \phi}{\partial x_k} \, \mathrm{d}x \right| \leqq C_1 \, \|u^-\|^2_{L^2(\Omega^-(t))} \leqq C_2 E(0), \tag{74}$$

with similar estimates for each of the integrals on $\Omega_2^\alpha$ appearing on the right-hand side of (73). With (73) and (74) and (19), we then obtain

$$\int_{\{x_2 \geqq \alpha\} \cap \partial\Omega} |\nabla_{\tilde{n}} \phi|^2 \, \mathrm{d}l \leqq C_3 (E(0) + \|\omega_0^-\|^2_{L^2(\Omega^-)}) + C_4 \int_{\Gamma_2^\alpha} |u^-|^2 \, \mathrm{d}x_1.$$

Remembering that $u \cdot n = \nabla_\tau \phi = 0$ on $\partial\Omega$, we deduce that for any $\alpha \geqq \frac{H}{4}$,

$$\int_{\{x_2 \geqq \alpha\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \leqq C_3 (\underbrace{E(0) + \|\omega_0^-\|^2_{L^2(\Omega^-)}}_{M_0}) + C_4 \int_{\Gamma_2^\alpha} |u^-|^2 \, \mathrm{d}x_1. \tag{75}$$

Remembering that from (32) we can take for $\alpha$ any value between $\frac{H}{4}$ to $\frac{H}{2}$, we get, by integrating (75) for $x_2$ between $\frac{H}{4}$ and $\frac{H}{2}$ (keeping in mind that

$$\forall \alpha \in \left[\frac{H}{4}, \frac{H}{2}\right], \int_{\{x_2 \geqq \alpha\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \geqq \int_{\{x_2 \geqq \frac{H}{2}\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \;),$$

$$\begin{aligned} \frac{H}{4} \int_{\{x_2 \geqq \frac{H}{2}\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l &\leqq \frac{C_3 H}{4} M_0 + C_4 \int_{\Omega^-(t) \cap \{\frac{H}{4} \leqq x_2 \leqq \frac{H}{2}\}} |u^-|^2 \, \mathrm{d}x \\ &\leqq \frac{C_3 H}{4} M_0 + C_4 \|u^-\|^2_{L^2(\Omega^-(t))} \leqq C_5 M_0. \end{aligned}$$

Therefore,

$$\int_{\{x_2 \geqq \frac{H}{2}\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \leqq C_6 M_0. \tag{76}$$

We now define

$$\Omega_1^\alpha = \{x \in \Omega; \; x_1 \geqq \alpha\}, \tag{77}$$

and

$$\Gamma_1^\alpha = \{x \in \Omega; \; x_1 = \alpha\}. \tag{78}$$

From our relation (32), we know that for $\alpha > \frac{L}{4}$, $\Omega_1^\alpha$ does not intersect $\Omega^+(t)$ for all time of existence, and we will work with such values of $\alpha$ in what follows.

By proceeding as for $\Omega_2^\alpha$ we obtain in a verbatim way that

$$\int_{\{x_1 \geqq \frac{L}{2}\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l \leqq C_7 M_0. \tag{79}$$

Due to our symmetry in $x_1$, the same estimate holds for $\int_{\{x_1 \leqq -\frac{L}{2}\} \cap \partial\Omega} |u^-|^2 \, \mathrm{d}l$.

Using now our assumption (24) on $\Omega$, we infer from (79) and (76) that

$$\int_{\Gamma_1^c} |u^-|^2 \, \mathrm{d}l \leqq C_9 M_0.$$

Taking $h$ and the $L^2$ norm of velocities as well as the $L^2$ norm of the initial vorticity small enough so that

$$C_9\left(E(0)+\|\omega_0^-\|^2_{L^2(\Omega^-)}\right) \leqq \frac{1}{4}\frac{\rho^+-\rho^-}{\rho^-(1+\alpha_\Omega)}g|\Omega^+| \tag{80}$$

then provides the desired estimate (51), which concludes our proof of finite in time singularity formation.

Therefore for $T_{\max}$ estimated by (62), we have established that so long as a smooth non self-intersecting and non contacting with $\partial\Omega$ solution exists, blow-up of $f$ will occur at $T_{\max}$, namely we proved Theorem 1. □

## 6. Blow-Up of Norms if Finite-Time Self-Contact or Contact with $\partial\Omega$

We first establish that the finite-time contact cases 2) of Theorem 1 lead to blow-up of a lower norm.

### *6.1. Blow-Up of a Lower Norm if Finite-Time Self-Contact or Contact with $\partial\Omega$*

**Proof.** We just provide the proof of the more difficult case of self-contact of $\partial\Omega^+(t)$ with itself at $T_{\max}$, the other case having a similar proof. Assume that $\partial\Omega^+(t)$ self-intersects at $T_{\max}$ and that there exists $C_0>0$ finite so that

$$\forall t\in[0,T_{\max}),\ \|\nabla_\tau\tau\|_{L^\infty(\partial\Omega^+(t))}+\sum_\pm\int_0^t\|\nabla u^\pm\|_{L^\infty(\Omega^\pm(t))}\leqq C_0. \tag{81}$$

From the fact that the length of the interface and the $L^2$ norm of the velocities $u^\pm$ are bounded, it is not difficult to infer from (81) that there exists $C_1>0$ such that

$$\forall t\in[0,T_{\max}),\ \int_0^t\|u^\pm\|_{L^\infty(\Omega^\pm(t))}\leqq C_1. \tag{82}$$

From (82) we can define by continuity in time as $t\to T_{\max}$

$$\eta^\pm(x,T_{\max})=x+\int_0^{T_{\max}}v^\pm(x,t)\,\mathrm{d}t,$$

since $v^\pm$ has the same $L^\infty$ norm as $u^\pm$. The self intersection assumption 2) simply means that there exists $x_0\neq x_1$ points of $\partial\Omega^+$ such that

$$\eta^-(x_0,T_{\max})=\eta^-(x_1,T_{\max}). \tag{83}$$

In [9] we proved there can only be a finite number of additional points $x_i\in\partial\Omega^+$ such that $\eta^-(x_i,T_{\max})=\eta^-(x_0,T_{\max})$. Note that although the assumptions about regularity in [9] are stronger than the ones involved here, in order to prove this statement (and the other statements we will make after), it is only the fact that the length of $\Gamma(t)$ stays bounded, as well as the uniform bounds (81) and (82), which are needed.

**Remark 7.** The present work does not exclude the possibility that the velocity field in one phase would remain smooth all the way until contact. This exclusion was done in [9]. In order to exclude this situation, which corresponds to the case of a splash singularity (in order to have an analogous of the one-phase problem, all relevant norms in one of the phases are assumed bounded), the extra regularity in the framework of [9] are needed.

From (81) we see that the tangent vector at $\eta(x_0, t)$ is a continuous function of space (due to the control of $\nabla_\tau \tau$) and time (due to the control of $\nabla u^-$). Given the fact the curve first self-intersect at time $T_{\max}$ we have that the tangent vector on $\partial\Omega^+(T_{\max})$ at each $\eta(x_i, T_{\max})$ is the same, and we call it $e_1$ (it is not necessarily horizontal).

By proceeding in a way similar to Section 6 of [9] (assuming contact occurs with $\Omega^-(t)$ being pinched), we have by the fundamental theorem of calculus applied in a path orthogonal to $e_1$ that $(\eta^-(x_0, t), \eta^-(z(t), t)) \subset \Omega^-(t)$, and a path alongside the interface between $\eta^-(z(t), t)$ and $\eta^-(x_1, t)$ (figure 4 of [9]) that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}(\eta^-(x_0, t) - \eta^-(x_1, t))\right| &= |u^-(\eta^-(x_0, t), t)) - u^-(\eta^-(x_1, t), t)| \\ &\leqq C_2 \|\nabla u^-\|_{L^\infty(\Omega^-(t))} |\eta^-(x_0, t) - \eta^-(x_1, t)|.\end{aligned}$$

**Remark 8.** The key to adapt Section 6 of [9] is simply to notice that the Claim 1 (before (6.20) of [9]) becomes in the present context $\eta_2^-(x_1, t) - \eta_2^-(z(t), t) = o(1)(\eta_1^-(x_1, t) - \eta_1^-(x_0, t))$, with $\lim_{T_{\max}^-} o(1) = 0$ (if 1 denote the coordinate along $e_1$ and 2 the coordinate along the direction orthogonal to $e_1$). This shows the length of each path involved is less than $2|\eta^-(x_0, t) - \eta^-(x_1, t)|$, for $t$ close enough to $T_{\max}$, and $C_2$ can be chosen as 2 if $t$ is close enough to $T_{\max}$.

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\eta^-(x_0, t) - \eta^-(x_1, t)|^2 \geqq -2C_2 \|\nabla u^-\|_{L^\infty(\Omega^-(t))} |\eta^-(x_0, t) - \eta^-(x_1, t)|^2,$$

which provides, by integration, that

$$0 = \frac{|\eta^-(x_0, T_{\max}) - \eta^-(x_1, T_{\max})|^2}{|\eta^-(x_0, 0) - \eta^-(x_1, 0)|^2} \geqq e^{-2C_2 \int_0^{T_{\max}} \|\nabla u^-\|_{L^\infty(\Omega^-(t))}\, \mathrm{d}t},$$

and thus

$$\int_0^{T_{\max}} \|\nabla u^-\|_{L^\infty(\Omega^-(t))} dt = \infty,$$

which is in contradiction with (81). If contact had occured with $\Omega^+(t)$ being pinched, we would have had the same identity with $\nabla u^+$ and $\Omega^+(t)$. This establishes Theorem 2. □

*6.2. Blow-Up of a Natural Norm of Local in Time Existence if Finite-Time Self-Intersection or Contact with the Boundary*

We now assume our initial data to have a smoother curl in both phases: $\text{curl}u_0^\pm \in H^{\frac{7}{2}}(\Omega^\pm)$. Local in time existence can be carried in a similar way as in [5] in the norm $\tilde{N}(t) = N(t) + \|\eta^+\|_{H^{\frac{9}{2}}(\Omega^+)} + \|\eta^-\|_{H^{\frac{9}{2}}(\Omega^-)}$, for which Theorem 1 applies similarly. We now prove that if we have the blow-up (1) of Theorem 2, then (2) holds.

**Proof.** For the sake of contradiction, let us assume (1) holds and that there is constant $C > 0$ such that

$$\forall t \in (0, T_{\max}), \quad \tilde{N}(t) \leqq C. \tag{84}$$

We have as a consequence of this bound (and the Sobolev embeddings in the initial smooth domains $\Omega^+$ and $\Omega^-$) that

$$\sum_{+,-} \|\eta^\pm\|_{C^3(\overline{\Omega^\pm})} \leqq C, \tag{85}$$

where we take the convention $C$ is a generic positive constant independent of $t$ approaching $T_{\max}$.

Moreover due to the Sobolev embeddings in the initial smooth domains $\Omega^+$ and $\Omega^-$, we have that for the Lagrangian velocities

$$\|v^\pm\|_{C^1(\overline{\Omega^\pm})} \leqq C\|v^\pm\|_{H^3(\Omega^\pm)}. \tag{86}$$

Due to $v = u \circ \eta$ and (84) and (85) we have

$$\|v^\pm\|_{H^3(\Omega^\pm)} \leqq C. \tag{87}$$

Therefore, (87) and (86) imply

$$\|v^\pm\|_{C^1(\overline{\Omega^\pm})} \leqq C. \tag{88}$$

Next we have from $u = v \circ \eta^{-1}$ (in each phase, interface included) that

$$\nabla u = \nabla v(\eta^{-1})\ \nabla(\eta^{-1}) = \nabla v(\eta^{-1})\ (\nabla\eta)^{-1}(\eta^{-1}) = \nabla v(\eta^{-1})\ (\text{Cof}\nabla\eta)^T(\eta^{-1}),$$

where we used $\det(\nabla\eta) = 1$ in the last equality. Therefore, from (85) and (88) we have

$$\|\nabla u^\pm\|_{L^\infty(\Omega^\pm(t))} \leqq C. \tag{89}$$

We now look at curvature on the interface and show (84) implies it stays finite as well. If $\theta$ denote a smooth parameterization from $[0, 1]$ of the initial $\Gamma$ (in particular $|\frac{\partial\theta}{\partial s}|$ stays by some $\alpha_1 > 0$ away from 0), we have

$$|\kappa| = \left| \frac{1}{|\frac{\partial(\eta\circ\theta)}{\partial s}|} | \frac{\partial^2(\eta\circ\theta)}{\partial s^2} \cdot n(\eta\circ\theta) \right|. \tag{90}$$

From this expression and (85), we just need to ensure that $|\frac{\partial(\eta\circ\theta)}{\partial s}|$ stays bounded from below. Let us show this now. From

$$|\frac{\partial(\eta\circ\theta)}{\partial s}| = |(\nabla_\tau\eta)(\theta)|\,|\frac{\partial\theta}{\partial s}| \geqq \alpha_1|(\nabla_\tau\eta)(\theta)|,$$
$$1 = |\det(\nabla_\tau\eta, \nabla_n\eta)| \leqq |\nabla_\tau\eta||\nabla_n\eta| \leqq C|\nabla_\tau\eta|$$

(where we used (85)), we infer $\left|\frac{\partial(\eta\circ\theta)}{\partial s}\right| \geqq \frac{\alpha_1}{C}$. Therefore, curvature stays bounded independently of time:

$$\|\kappa\|_{L^\infty(\Gamma(t))} \leqq C, \tag{91}$$

where we remind readers that $C > 0$ is a generic constant independent of time. From (91) and (89) we have a contradiction with (1), which shows that (2) holds. □

## 7. Equations of the Rigid Body Moving Inside an Inviscid Fluid, Stream Function and Conservation of Energy

We now consider a rigid body moving in the inviscid fluid. The rigid body dynamics is described by the following unknowns:

- The position of the center of the rigid body at time $t$: $x^s(t)$.
- The angular velocity of rigid body at time $t$: $r(t)$.
- The velocity field in the rigid body $\Omega^s(t) = \Omega^s(0) + (x^s(t) - x^s(0))$:

$$u^s(x,t) = v^s(t) + r(t)(x - x^s(t))^\perp, \quad (x_1, x_2)^\perp = (-x_2, x_1).$$

- The fluid phase is described by the incompressible Euler equations in $\Omega^f(t) = \Omega \cap \overline{\Omega^s(t)}^c$, with unknown velocity field $u^f(x,t)$ and pressure field $p(x,t)$. This classical interacting fluid-rigid solid system is written as

$$\rho_f(u^f_t + u^f\cdot\nabla u^f) + \nabla p = -\rho_f\, g\, e_2 \quad \text{in } \Omega^f(t), \tag{92a}$$
$$\operatorname{div} u = 0 \quad \text{in } \Omega^f(t), \tag{92b}$$
$$u^f\cdot n = u^s\cdot n \quad \text{on } \partial\Omega^s(t), \tag{92c}$$
$$u^f\cdot n = 0 \quad \text{on } \partial\Omega, \tag{92d}$$
$$m_s\frac{\mathrm{d}v^s}{\mathrm{d}t} = \int_{\partial\Omega^s(t)} p\, n\, \mathrm{d}l - m_s g\, e_2, \tag{92e}$$
$$I_s\frac{\mathrm{d}r}{\mathrm{d}t} = \int_{\partial\Omega^s(t)} p\,(x - x^s(t))^\perp\cdot n\,\mathrm{d}l, \tag{92f}$$
$$u^f(0) = u_0 \quad \text{in } \Omega^f, \tag{92g}$$
$$x^s(0) = x^s_0, \quad v^s(0) = v^s_0, \quad r(0) = r_0, \tag{92h}$$

where $n$ is the exterior unit normal to $\Omega^f(t)$, pointing inside $\Omega^s(t)$, and $e_2$ is the unit vertical vector pointing upwards. Also, $m_s = \rho^s|\Omega^s|$ is the mass of the rigid body, and $I_s$ the inertial moment. In this paper we assume that

$$\rho_s > \rho_f. \tag{93}$$

The existence and uniqueness of this system (if the initial data satisfies $u_0^f \cdot n = (v_0^s + r_0(x - x_0^s)^\perp) \cdot n$ on $\partial\Omega^s$ and $u_0^f \cdot n = 0$ on $\partial\Omega$) was established by Glass and Sueur in [17], which shows the existence and uniqueness of a solution to this problem so long as $\partial\Omega^s(t)$ does not touch $\partial\Omega$.

We next establish the boundary condition satisfied on the boundary of the solid body by the stream function and remind the conservation of energy.

### 7.1. Stream Function

Since $u^f$ is divergence free we have $u^f = \nabla^\perp\phi = (-\frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_1})$, with $\phi$ the solution of the elliptic system

$$\Delta\phi(\cdot, t) = \omega(\cdot, t) = \mathrm{curl}\, u^f(\cdot, t), \text{ in } \Omega^f(t), \tag{94a}$$

$$\phi(\cdot, t) = 0, \text{ on } \partial\Omega, \tag{94b}$$

$$\phi(x, t) = v_2^s(t)x_1, \text{ on } \partial\Omega^s(t). \tag{94c}$$

Since $\nabla_\tau\phi = u^f \cdot n$ we then have $\nabla_\tau\phi = 0$ on $\partial\Omega$ which ensures we can choose $\phi = 0$ on the connected $\partial\Omega$. On the other hand, we have on $\partial\Omega_s(t)$ that

$$\nabla_\tau\phi = v_2^s(t)n_2 = v_2^s(t)\tau_1 = v_2^s(t)\nabla_\tau x_1,$$

which provides $\phi(x, t) = v_2^s(t)x_1 + c(t)$. Next, by the fundamental theorem of calculus, if we denote by $r_2$ the distance from the centre of gravity to the lowest point on $x_1 = 0$ (which is not necessarily the lowest point of the rigid body, just the lowest on the vertical axis of symmetry), then

$$\begin{aligned}\phi(0, x_2(t) - r_2, \cdot) &= \phi(0, 0, \cdot) + \int_0^{x_2(t)-r_2} \frac{\partial\phi}{\partial x_2}(0, x_2, \cdot)\, \mathrm{d}x_2 \\ &= -\int_0^{x_2(t)-r_2} u_1^f(0, x_2, \cdot)\, \mathrm{d}x_2 = 0,\end{aligned}$$

due to the fact $u_1^f$ is odd. This in turn provides us with $c(t) = 0$ and (94c).

### 7.2. Energy Conservation

For all the time of existence it is classical that the quantity

$$\frac{1}{2}m_s|v^s|^2(t) + \frac{1}{2}\rho^f \int_{\Omega^f(t)} |u^f(x, t)|^2\, \mathrm{d}x + m_s g x_2^s(t) + \rho^f g \int_{\Omega^f} \eta_2^f\, \mathrm{d}x \tag{95}$$

is independent of time. Similarly as to establishing (14), this shows that the total energy

$$E(t) = \frac{1}{2} m_s |v^s|^2(t) + \frac{1}{2}\rho^f \int_{\Omega^f(t)} |u^f(x,t)|^2 \,\mathrm{d}x + m_s g x_2^s(t) - \rho^f g x_2^s(t)|\Omega^s|$$
$$= \frac{1}{2} m_s |v^s|^2(t) + \frac{1}{2}\rho^f \int_{\Omega^f(t)} |u^f(x,t)|^2 \,\mathrm{d}x + \underbrace{(\rho^s - \rho^f)}_{>0} g x_2^s(t)|\Omega^s| \quad (96)$$

is constant in time for all the time of existence of a smooth solution (namely from [17] so long as no eventual collision with the boundary occurs).

Thus,

$$\frac{1}{2} m_s |v^s|^2(t) + \frac{1}{2}\rho^f \int_{\Omega^f(t)} |u^f(x,t)|^2 \,\mathrm{d}x + (\rho^s - \rho^f) g x_2^s(t)|\Omega^s| = E(0). \quad (97)$$

Moreover, since $x_2^s > 0$, we also have from (97) the control of the kinetic energy:

$$\frac{1}{2} m_s |v^s|^2(t) + \frac{1}{2}\rho^f \int_{\Omega^f(t)} |u^f(x,t)|^2 \,\mathrm{d}x \leqq E(0). \quad (98)$$

## 8. Choice of Initial Data

We denote by $\Omega$ a bounded domain of class $C^2$, which is symmetric with respect to the axis $x_1 = 0$.

We then choose $\Omega^s$ such that $\overline{\Omega^s} \subset \Omega$ to be an equally symmetric connected domain with respect to the vertical axis $x_1 = 0$, which is of the same regularity class as $\Omega$.

We then define the initial fluid domain $\Omega^f = \Omega \cap (\overline{\Omega^s})^c$.

We choose

$$v^s(0) = (0, v_2^s(0)), \text{ with } v_2^s(0) < 0, \quad (99a)$$
$$u^f(0) \text{ divergence free with } u_1^f(0) \text{ odd, } u_2^f(0) \text{ even, and,} \quad (99b)$$
$$v_2^s(0) n_2 = u^f(0) \cdot n \text{ on } \partial\Omega^s, \quad (99c)$$
$$r(0) = 0, \quad (99d)$$
$$x^s(0) = (0, h), \quad (99e)$$

with $h > 0$ such that $\overline{\Omega^s} \subset \Omega$.

Given the symmetry of $\Omega$ and $\Omega^s$ with respect to the $x_1 = 0$ axis, as well as the symmetry of the initial data with respect to this axis, we then have that for all time of existence $u^f$ and $u^s$ are symmetric with respect to the vertical axis $x_1 = 0$: $u_1^f(-x_1, x_2) = -u_1^f(x_1, x_2)$ and $u_2^f(-x_1, x_2) = u_2^f(x_1, x_2)$ and $v_1^s(t) = 0$, $r(t) = 0$ for all time of existence. Therefore, the rigid solid falls in a vertical translation (at a speed dependent of time) and there is no rotation. The argument is simply to use the construction of solutions of [17] pp. 937–942, set up with $r = 0$ in the functional framework.

As we will see later on, the assumption (99a) together with a small square integrable vorticity ensures that if the rigid body falls from its initial position, then the rigid body keeps falling for all time of existence.

Here, since we know in advance where contact would occur (vertical fall of a body keeping its shape), we just need contact to occur on a strict subset of

$$\Gamma_1 = \{n_2 \leqq -\alpha_\Omega < 0\},$$

(for some $\alpha_\Omega \in (0, 1)$) with $\Gamma_1$ being not necessarily required to be connected (unlike in the case of the deformable interface). The vertically falling rigid body then stays away from $\partial\Omega \cap \Gamma_1^c$ by a strictly positive distance.

## 9. Elliptic Estimate Away from the Contact Zone and Non zero Velocity for the Vertically Falling Rigid Body

Our starting point is (34) which is valid for this problem as well, since it was established from (36) which is satisfied for this problem as well.

Due to the nature of the vertical fall of a rigid body not rotating, any point of $\partial\Omega \cap \Gamma_1^c$ will stay away from $\Omega^s(t)$ by a positive distance $D > 0$ for all time in $[0, T_{\max})$, $T_{\max}$ being the maximal time of existence of a smooth solution (that we do not assume finite or not here). In a manner similar to that in which we proved the boundary estimate (75) for the vortex sheet problem (for which we used the conservation of curl (19) in $\Omega^-(t)$, which holds in $\Omega^f(t)$ for the problem with a rigid body), we have by using $\xi^2(x)\nabla_{n(x_0)}\phi(x, t)$ (where $\xi$ is a cut-off function in a neighborhood of $x_0 \in \Gamma_1^c$) as a test function in the same elliptic system (94)

**Lemma 2.** *For all the time of existence of a smooth solution,*

$$\|u^f\|^2_{L^2(\partial\Omega\cap\Gamma_1^c)} \leqq C \left(\|u^f\|^2_{L^2(\Omega^f(t))} + \|\omega_0\|^2_{L^2(\Omega^f)}\right), \tag{100}$$

*where $C > 0$ is independent of time.*

**Remark 9.** Of course the energy estimate implies that $\|u^f\|_{L^2(\Omega^f(t))}$ is bounded uniformly in time, implying that the right-hand side of (100) can be replaced by just a constant $C$ independent of time. Although having just $C$ is enough for most of our purposes, it turns out that the more precise form (100) is used in Section 10 in a crucial way.

In a similar way we also have, for $\Gamma_1^s(t)$ being the vertical projection of $\Gamma_1$ on $\partial\Omega^s(t)$,

**Lemma 3.** *For all the time of existence of a smooth solution,*

$$\|u^f\|_{L^2((\Gamma_1^s(t))^c\cap\partial\Omega^s(t))} \leqq C, \tag{101}$$

*where $C > 0$ is independent of time.*

We next establish that the rigid body keeps falling for all time of existence of a smooth solution with our small square integrable curl assumption.

**Lemma 4.** *With our choice of initial data in Section 8, for all time $t > 0$ of existence of a smooth solution, we have $v_2^s(t) < 0$.*

**Proof.** Since $v_2^s(0) < 0$, we know that for some time $T > 0$ we will have $v_2^s(t) < 0$ for all $t < T$. Now let us assume that there exists a first value of $t_0 > 0$ such that

$$v_2^s(t_0) = 0, \tag{102}$$

while there is no contact with $\partial\Omega$ at $t_0$, with

$$\forall t \in [0, t_0),\ \ v_2^s(t) < 0 \tag{103}$$

(namely the rigid body is with zero speed at $t_0$, and does not touch $\partial\Omega$, and was before that time falling at a negative vertical speed). From the start of this Section, we have $u^f = \nabla^\perp\phi$ satisfying (94).

From the elliptic system (94) we immediately have by Green's theorem that

$$\begin{aligned}\int_{\Omega^f(t)} |u^f|^2\, \mathrm{d}x &= -\int_{\Omega^f(t)} \omega\phi\, \mathrm{d}x + v_2^s(t)\int_{\partial\Omega^s(t)} \nabla_n\phi\ x_1\, \mathrm{d}l \\ &= -\int_{\Omega^f(t)} \omega\phi\, \mathrm{d}x - v_2^s(t)\int_{\partial\Omega^s(t)} u^f\cdot\tau\ x_1\, \mathrm{d}l. \end{aligned}\tag{104}$$

Therefore,

$$\|u^f\|^2_{L^2(\Omega^f(t))} \leqq \|\phi\|_{L^2(\Omega^f(t))}\|\omega_0\|_{L^2(\Omega^f)} + |v_2^s(t)|\left|\int_{\partial\Omega^s(t)} u^f\cdot\tau\ x_1\, \mathrm{d}l\right|. \tag{105}$$

We now need to establish a Poincaré inequality for $\phi$ (independent of how close to contact we are), in order to control $\|\phi\|_{L^2(\Omega^f(t))}$. To do so we simply notice that if we define $\bar\phi$ as

$$\bar\phi(x,t) = 1_{\Omega^f(t)}(x)\phi(x) + 1_{\overline{\Omega^s(t)}}(x)v_2^s(t)x_1, \tag{106}$$

we have, due to the continuity (94c), that $\bar\phi \in H^1(\Omega)$, and due to (94b), that $\bar\phi \in H^1_0(\Omega)$. Note here that this is done for any $t$ such that $\partial\Omega^s(t)$ and $\partial\Omega$ do not intersect.

By the standard Poincaré inequality for $\bar\phi$ in $\Omega$, we then have (independently of any $t$ such that $\partial\Omega^s(t)$ and $\partial\Omega$ do not intersect)

$$\int_\Omega \bar\phi^2\, \mathrm{d}x \leqq C_\Omega\int_\Omega |\nabla\bar\phi|^2\, \mathrm{d}x = C_\Omega\left(\int_{\Omega^f(t)} |\nabla\phi|^2\, \mathrm{d}x + v_2^s(t)^2|\Omega^s(t)|\right). \tag{107}$$

From (105) and (107) we infer successively, that

$$\|u^f\|^2_{L^2(\Omega^f(t))} \leqq \frac{\|\phi\|^2_{L^2(\Omega^f(t))}}{2C_\Omega} + \frac{C_\Omega}{2}\|\omega_0\|^2_{L^2(\Omega^f)} + |v_2^s(t)|\left|\int_{\partial\Omega^s(t)} u^f\cdot\tau x_1\, \mathrm{d}l\right|$$

$$\leqq \frac{1}{2}\|\nabla\phi\|^2_{L^2(\Omega^f(t))} + \frac{1}{2}v_2^s(t)^2|\Omega^s| + \frac{C_\Omega}{2}\|\omega_0\|^2_{L^2(\Omega^f)} + |v_2^s(t)|\left|\int_{\partial\Omega^s(t)} u^f\cdot\tau\, x_1\,\mathrm{d}l\right|.$$

Therefore,

$$\|u^f\|^2_{L^2(\Omega^f(t))} \leqq v_2^s(t)^2|\Omega^s| + C_\Omega\|\omega_0\|^2_{L^2(\Omega^f)} + 2|v_2^s(t)|\left|\int_{\partial\Omega^s(t)} u^f\cdot\tau\, x_1\,\mathrm{d}l\right|. \tag{108}$$

We will need later on to replace the integral set on $\partial\Omega^s(t)$ by an integral set on $\partial\Omega$. This is done in the following way:

$$\int_{\partial\Omega^s(t)} x_1\, u^f\cdot\tau\,\mathrm{d}l = \int_{\partial\Omega^s(t)} x_1\,(u_1^f n_2 - u_2^f n_1)\,\mathrm{d}l. \tag{109}$$

Now, by integration by parts in $\Omega^f(t)$ for the right-hand side of (109), we have

$$\begin{aligned}
\int_{\partial\Omega^s(t)} x_1 u^f\cdot\tau\,\mathrm{d}l &= -\int_{\partial\Omega} x_1(u_1^f n_2 - u_2^f n_1)\,\mathrm{d}l + \int_{\Omega^f(t)} \frac{\partial(x_1u_1^f)}{\partial x_2} - \frac{\partial(x_1u_2^f)}{\partial x_1}\,\mathrm{d}x\\
&= -\int_{\partial\Omega} x_1u^f\cdot\tau\,\mathrm{d}l - \int_{\Omega^f(t)} x_1\omega + u_2^f\,\mathrm{d}x\\
&= -\int_{\partial\Omega} x_1u^f\cdot\tau\,\mathrm{d}l - \int_{\Omega^f(t)} x_1\omega + \frac{\partial\phi}{\partial x_1}\,\mathrm{d}x\\
&= -\int_{\partial\Omega} x_1u^f\cdot\tau\,\mathrm{d}l - \int_{\Omega^f(t)} x_1\omega\,\mathrm{d}x - v_2^s(t)\int_{\partial\Omega^s(t)} x_1n_1\,\mathrm{d}l,
\end{aligned} \tag{110}$$

where we used (94b), (94c) and integration by parts to obtain the last term above. Therefore using (110) in (108) we obtain

$$\begin{aligned}
\|u^f\|^2_{L^2(\Omega^f(t))} \leqq\ & v_2^s(t)^2(|\Omega^s| + 2\mathrm{diam}(\Omega^s)|\partial\Omega^s|) + C_\Omega\|\omega_0\|^2_{L^2(\Omega^f)}\\
&+ 2|v_2^s(t)|\left|\int_{\partial\Omega} u^f\cdot\tau\, x_1\,\mathrm{d}l\right| + 2|v_2^s(t)|\left|\int_{\Omega^f(t)} x_1\omega\,\mathrm{d}x\right|.
\end{aligned}$$

Using Young's inequality for the last term of the right-hand side, we obtain

$$\begin{aligned}
\|u^f\|^2_{L^2(\Omega^f(t))} &\leqq v_2^s(t)^2(|\Omega^s| + 2\mathrm{diam}(\Omega^s)|\partial\Omega^s| + \mathrm{diam}(\Omega)^2)\\
&\quad + (C_\Omega + |\Omega^f|)\|\omega_0\|^2_{L^2(\Omega^f)} + 2|v_2^s(t)|\left|\int_{\partial\Omega} u^f\cdot\tau\, x_1\,\mathrm{d}l\right|\\
&\leqq D_\Omega\left(v_2^s(t)^2 + |v_2^s(t)|\left|\int_{\partial\Omega} u^f\cdot\tau\, x_1\,\mathrm{d}l\right| + \|\omega_0\|^2_{L^2(\Omega^f)}\right),
\end{aligned} \tag{111}$$

with

$$D_\Omega = \max(|\Omega^s| + 2\mathrm{diam}(\Omega^s)|\partial\Omega^s| + \mathrm{diam}(\Omega)^2, 2, C_\Omega + |\Omega^f|). \tag{112}$$

Thus, if $v_2^s(t_0) = 0$, we infer from (111) that

$$\|u^f\|^2_{L^2(\Omega^f(t_0))} \leqq D_\Omega \|\omega_0\|^2_{L^2(\Omega^f)}. \tag{113}$$

Therefore, the total energy satisfies

$$E(t_0) \leqq \frac{D_\Omega}{2}\rho^f \|\omega_0\|^2_{L^2(\Omega^f)} + (\rho^s - \rho^f) g x_2^s(t_0)|\Omega^s|. \tag{114}$$

Now, from (103) we infer that

$$\forall t \in [0, t_0),\ x_2^s(t) > x_2^s(t_0). \tag{115}$$

With our assumption (3) of smallness of $\omega_0$ relative to $v_2^s(0)$, (114) and (115) lead to

$$E(t_0) < \frac{m_s}{2}|v_2^s(0)|^2 + (\rho^s - \rho^f) g x_2^s(0) = E(0),$$

which is in contradiction with the conservation of energy. Therefore, for all time $t$ such that $\Omega^s(t)$ does not intersect $\partial\Omega$, we have $v_2^s(t) < 0$, which proves the lemma. □

We can now prove our general finite-time contact Theorem 3.

## 10. Finite-Time Contact for the Rigid Body Vertically Falling Over a Contact Zone Locally Under the Form of a Graph

**Proof.** We now assume that the rigid body does not touch $\partial\Omega$ at any finite $t > 0$. From [17], we then know the maximal time of existence of a smooth solution satisfies

$$T_{\max} = \infty. \tag{116}$$

From Lemma 4, we infer

$$\forall t \geqq 0,\ x_2^s(t) = h - \int_0^t |v_2^s(s)|\, \mathrm{d}s,$$

which shows that

$$\int_0^\infty |v_2^s(s)|\, \mathrm{d}s \leqq h. \tag{117}$$

We now integrate (34) from 0 to $t$:

$$m_s\ (v_2^s(t) - v_2^s(0)) = \rho^f \int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2\, \mathrm{d}l\, \mathrm{d}t - (m_s - \rho^f|\Omega^s|) g t$$

$$
\begin{aligned}
&- \rho^f \int_{\partial\Omega} x_1 u^f(\cdot,t)\cdot\tau \,\mathrm{d}l + \rho^f \int_{\partial\Omega} x_1 u^f(\cdot,0)\cdot\tau\,\mathrm{d}l \\
&- \rho^f \int_{\Omega^f(t)} u_2^f(\cdot,t)\,\mathrm{d}x + \rho^f \int_{\Omega^f} u_2^f(\cdot,0)\,\mathrm{d}x. \qquad (118)
\end{aligned}
$$

We now write

$$
\int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \,\mathrm{d}l = \int_{\Gamma_1} \frac{|u^f|^2}{2} n_2\,\mathrm{d}l + \int_{\Gamma_1^c\cap\partial\Omega} \frac{|u^f|^2}{2}(-\alpha_\Omega + (n_2+\alpha_\Omega))\,\mathrm{d}l, \qquad (119)
$$

which, thanks to $\Gamma_1 \subset \{n_2 \leqq -\alpha_\Omega < 0\}$, $0 < \alpha_\Omega \leqq 1$, provides us with

$$
\int_{\partial\Omega} \frac{|u^f|^2}{2} n_2\,\mathrm{d}l \leqq -\alpha_\Omega \int_{\partial\Omega} \frac{|u^f|^2}{2}\,\mathrm{d}l + \int_{\Gamma_1^c\cap\partial\Omega} |u^f|^2\,\mathrm{d}l. \qquad (120)
$$

Using our elliptic estimate (100) away from the contact zone, together with the other elliptic estimate (111), we infer that

$$
\begin{aligned}
\int_{\partial\Omega} \frac{|u^f|^2}{2} n_2\,\mathrm{d}l \leqq & - \alpha_\Omega \int_{\partial\Omega} \frac{|u^f|^2}{2}\,\mathrm{d}l + C D_\Omega (\|\omega_0\|^2_{L^2(\Omega^f)} + v_2^s(t)^2 \\
& + |v_2^s(t)| \left| \int_{\partial\Omega} u^f\cdot\tau\, x_1\,\mathrm{d}l \right|) + C\|\omega_0\|^2_{L^2(\Omega^f)} \\
\leqq & - \alpha_\Omega \int_{\partial\Omega} \frac{|u^f|^2}{2}\,\mathrm{d}l + (C D_\Omega + C)|\|\omega_0\|^2_{L^2(\Omega^f)} \\
& + C D_\Omega \left( v_2^s(t)^2 + |v_2^s(t)| \mathrm{diam}(\Omega)\sqrt{|\partial\Omega|}\sqrt{\int_{\partial\Omega} |u^f\cdot\tau|^2\,\mathrm{d}l} \right).
\end{aligned}
$$

Integrating this in time and using Young for $\varepsilon > 0$ for the last term, we get

$$
\begin{aligned}
\int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2\,\mathrm{d}l\,\mathrm{d}t \leqq & - \alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{2}\,\mathrm{d}l\,\mathrm{d}t + (C D_\Omega + C)\|\omega_0\|^2_{L^2(\Omega^f)}\, t \\
& + C D_\Omega \|v_2^s\|_{L^\infty(0,t)} \int_0^t |v_2^s(t)|\,\mathrm{d}t \\
& + \frac{C D_\Omega \mathrm{diam}(\Omega)^2}{4\varepsilon} \int_0^t v_2^s(t)^2\,\mathrm{d}t \\
& + C D_\Omega \varepsilon |\partial\Omega| \int_0^t \int_{\partial\Omega} |u^f|^2\,\mathrm{d}l\,\mathrm{d}t. \qquad (121)
\end{aligned}
$$

Noticing that due to (117),

$$
\int_0^t (v_2^s)^2\,\mathrm{d}t \leqq \|v_2^s\|_{L^\infty(0,t)} \int_0^t |v_2^s(t)|\,\mathrm{d}t \leqq h\|v_2^s\|_{L^\infty(0,t)} \leqq \sqrt{\frac{2E(0)}{m_s}}h,
$$

we infer from (121) that for $\varepsilon = \dfrac{\alpha_\Omega}{4CD_\Omega|\partial\Omega|}$,

$$\int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \, \mathrm{d}l \, \mathrm{d}t \leqq -\alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{4} \, \mathrm{d}l \, \mathrm{d}t + (CD_\Omega + C)\|\omega_0\|^2_{L^2(\Omega^f)} t + CD_\Omega\Big(1 + \frac{CD_\Omega|\partial\Omega|\mathrm{diam}(\Omega)^2}{\alpha_\Omega}\Big)\sqrt{\frac{2E(0)}{m_s}} h. \quad (122)$$

Reporting (122) in (118) we obtain

$$\rho^f \int_{\partial\Omega} x_1 u^f(\cdot,t)\cdot\tau \, \mathrm{d}l \leqq -\rho^f\alpha_\Omega \int_0^t \int_{\partial\Omega} \frac{|u^f|^2}{4} \, \mathrm{d}l \, \mathrm{d}t + \rho^f C(D_\Omega+1)t\|\omega_0\|^2_{L^2(\Omega^f)} - (\rho^s-\rho^f)|\Omega^s|gt + C_1, \quad (123)$$

where we also used the conservation of energy to have an estimate uniform in time for the terms not explicitly reported in (123) and controlled by some $C_1 > 0$ independent of time. Therefore, remembering our small curl assumption (3),

$$\rho^f \|x_1\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u^f|(\cdot,t) \, \mathrm{d}l \geqq \frac{\alpha_\Omega\rho^f}{4} \int_0^t \int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l \, \mathrm{d}t + \frac{(\rho^s-\rho^f)|\Omega^s|g}{2} t - C_1.$$

Proceeding in an identical manner as to the one in which we obtain (62) from (58) we infer that the maximal time of existence $T_{\max}$ of a smooth solution is finite, which is in contradiction with our assumption that it was infinite. Therefore,

$$T_{\max} < \infty. \quad (124)$$

From [17], the rigid body will then touch $\partial\Omega$ at $T_{\max}$. □

From now on we assume that $\omega_0 = 0$.

## 11. Equivalence of Norms for the Velocity Field when $\omega = 0$

We have by integration by parts that

$$0 = \int_{\Omega^f(t)} \triangle u^f \cdot u^f \, \mathrm{d}x = -\int_{\Omega^f(t)} |\nabla u^f|^2 \, \mathrm{d}x + \int_{\partial\Omega^f(t)} \nabla_n u^f \cdot u^f \, \mathrm{d}l.$$

Therefore, expanding in the $(\tau, n)$ basis,

$$0 = -\int_{\Omega^f(t)} |\nabla u^f|^2 \, \mathrm{d}x + \int_{\partial\Omega^f(t)} \nabla_n u^f \cdot n \, u^f \cdot n + \nabla_n u^f \cdot \tau \, u^f \cdot \tau \, \mathrm{d}l,$$

and using the divergence and curl relations on the boundary integral,

$$0 = -\int_{\Omega^f(t)} |\nabla u^f|^2 \, dx + \int_{\partial\Omega^f(t)} -\nabla_\tau u^f \cdot \tau \, u^f \cdot n + (\nabla_\tau u^f \cdot n) \, u^f \cdot \tau \, dl.$$

Rearranging the last term of the boundary integral, this identity yields

$$\begin{aligned} 0 = &-\int_{\Omega^f(t)} |\nabla u^f|^2 \, dx - \int_{\partial\Omega^f(t)} \nabla_\tau u^f \cdot \tau \, u^f \cdot n \, dl \\ &+ \int_{\partial\Omega^f(t)} (\nabla_\tau (u^f \cdot n) - u^f \cdot \nabla_\tau n) \, u^f \cdot \tau \, dl. \end{aligned} \tag{125}$$

By integrating by parts the first integral set on $\partial\Omega^f(t)$, (125) becomes

$$\begin{aligned} \int_{\Omega^f(t)} |\nabla u^f|^2 \, dx &= \int_{\partial\Omega^f(t)} u^f \cdot \nabla_\tau \tau \, u^f \cdot n + u^f \cdot \tau \, \nabla_\tau \, (u^f \cdot n) \, dl \\ &\quad + \int_{\partial\Omega^f(t)} \nabla_\tau (u^f \cdot n) \, u^f \cdot \tau \; - u^f \cdot \nabla_\tau n \, u^f \cdot \tau \, dl \\ &= \int_{\partial\Omega^f(t)} \kappa u^f \cdot n \, u^f \cdot n + u^f \cdot \tau \nabla_\tau \, (u^f \cdot n) \, dl \\ &\quad + \int_{\partial\Omega^f(t)} \nabla_\tau (u^f \cdot n) u^f \cdot \tau \; + \kappa u^f \cdot \tau u^f \cdot \tau \, dl, \end{aligned} \tag{126}$$

where we used $\nabla_\tau \, \tau = \kappa \, n$ and $\nabla_\tau \, n = -\kappa\tau$ on $\partial\Omega^f(t)$ (we remind $n$ points outside $\Omega^f(t)$). Using the boundary conditions $u^f \cdot n = v_s(t) \cdot n$ on $\partial\Omega^s(t)$ and $u^f \cdot n = 0$ on $\partial\Omega$, we obtain from (126) that

$$\begin{aligned} \int_{\Omega^f(t)} |\nabla u^f|^2 \, dx = &\int_{\partial\Omega_s(t)} \kappa \, (v_s \cdot n)^2 + 2u^f \cdot \tau \, v_s \cdot \nabla_\tau n + \kappa \, (u^f \cdot \tau)^2 \, dl \\ &+ \int_{\partial\Omega} \kappa \, (u^f \cdot \tau)^2 \, dl. \end{aligned} \tag{127}$$

## 12. A Formula for Acceleration at Time of Contact for the Case Without Vorticity

Our aim is to establish the following formula for the case without vorticity, which will be crucial to characterize the acceleration at contact later:

**Lemma 5.**

$$\underbrace{2\frac{E(0) + (\rho^f - \rho^s) g x_2^s |\Omega^s|}{(v_2^s)^2}}_{>0 \; by \; (135) \; seen \; after} \frac{dv_2^s}{dt} = -\rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} \, n_2 \, dl + (\rho^f - \rho^s) g |\Omega^s|. \tag{128}$$

**Proof.** Using (104) we obtain that

$$\int_{\Omega^f(t)} |u^f|^2 \, \mathrm{d}x = -v_2^s(t) \int_{\partial\Omega^s(t)} u^f \cdot \tau \; x_1 \, \mathrm{d}l.$$

Using $u^f = \nabla\psi$ for some potential $\psi$ (since in this Section $u^f$ is curl free), this provides

$$\begin{aligned}\int_{\Omega^f(t)} |u^f|^2 \, \mathrm{d}x &= -v_2^s(t) \int_{\partial\Omega^s(t)} \nabla_\tau \psi \; x_1 \, \mathrm{d}l \\ &= v_2^s(t) \int_{\partial\Omega^s(t)} \psi \; \nabla_\tau x_1 \, \mathrm{d}l \\ &= v_2^s(t) \int_{\partial\Omega^s(t)} \psi \; \tau_1 \, \mathrm{d}l \\ &= v_2^s(t) \int_{\partial\Omega^s(t)} \psi \; n_2 \, \mathrm{d}l. \end{aligned} \tag{129}$$

We will also need the following simple identity:

$$\begin{aligned}\int_{\Omega^f(t)} u_2^f \, \mathrm{d}x &= \int_{\Omega^f(t)} \frac{\partial\psi}{\partial x_2} \, \mathrm{d}x \\ &= \int_{\partial\Omega} \psi \; n_2 \, \mathrm{d}l + \int_{\partial\Omega_s(t)} \psi \; n_2 \, \mathrm{d}l. \end{aligned} \tag{130}$$

From (34), we obtain that

$$\begin{aligned} m_s \frac{\mathrm{d}v_2^s}{\mathrm{d}t} &= \rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \, \mathrm{d}l - \rho^f \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega^f(t)} u_2^f \, \mathrm{d}x - (m_s - \rho^f |\Omega_s|) g \\ &\quad + \rho^f \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega} \psi \; n_2 \, \mathrm{d}l, \end{aligned} \tag{131}$$

where we used $\int_{\partial\Omega} u^f \cdot \tau x_1 \, \mathrm{d}l = -\int_{\partial\Omega} \psi n_2 \, \mathrm{d}l$ which is established similarly as in the proof of (129). Using (130) in (131) yields

$$m_s \frac{\mathrm{d}v_2^s}{\mathrm{d}t} = \rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \, \mathrm{d}l - (m_s - \rho^f |\Omega_s|) g - \rho^f \frac{\mathrm{d}}{\mathrm{d}t} \int_{\partial\Omega^s(t)} \psi \; n_2 \, \mathrm{d}l. \tag{132}$$

Using (129) in (132) yields

$$m_s \frac{\mathrm{d}v_2^s}{\mathrm{d}t} = \rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \, \mathrm{d}l - (m_s - \rho^f |\Omega_s|) g - \rho^f \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\|u^f\|^2_{L^2(\Omega^f(t))}}{v_2^s} \right). \tag{133}$$

From our conservation of energy, (133) becomes

$$m_s \frac{\mathrm{d}v_2^s}{\mathrm{d}t} = \rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \,\mathrm{d}l \; - (\rho_s - \rho^f)|\Omega_s| g - \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{2E(0) - m_s (v_2^s)^2 + 2(\rho^f - \rho^s) g x_2^s |\Omega^s|}{v_2^s} \right),$$

and thus by noticing that the second term in the time derivative on the right hand side of this relation equals $m_s \frac{\mathrm{d}v_2^s}{\mathrm{d}t}$, we obtain

$$0 = \rho^f \int_{\partial\Omega} \frac{|u^f|^2}{2} n_2 \,\mathrm{d}l - (\rho^s - \rho^f)|\Omega_s| g - \frac{\mathrm{d}}{\mathrm{d}t}\left( \frac{2E(0) + 2(\rho^f - \rho^s) g x_2^s |\Omega^s|}{v_2^s} \right). \tag{134}$$

The quotient rule for the last derivative of the right hand side of (134) then yields the desired (128). □

As a corollary of Lemma 5 we notice the following to define velocity at time of contact: since $n_2 \leqq -\alpha_\Omega$ on $\Gamma_1$, we have that $\int_0^{T_{\max}} \int_{\Gamma_1} |u^f|^2 n_2 \,\mathrm{d}l dt$ is well defined in $[-\infty, 0]$. Since $u^f$ is bounded away from contact in $L^2(\Gamma_1^c \cap \partial\Omega)$ we also have that $\int_0^{T_{\max}} \int_{\Gamma_1^c \cap \partial\Omega} |u^f|^2 n_2 \,\mathrm{d}l dt$ is well defined in $\mathbb{R}$. Therefore (134) shows that

$$\lim_{t \to T_{\max}^-} \frac{2E(0) + 2(\rho^f - \rho^s) g x_2^s(t) |\Omega^s|}{v_2^s(t)} \in [-\infty, \infty).$$

Since the coefficient

$$2E(0) + 2(\rho^f - \rho^s) g x_2^s(t)|\Omega^s| \geqq m_s |v_s(0)|^2 + 2 \underbrace{(\rho^s - \rho^f)}_{>0} g \underbrace{(h - x_2^s(t))}_{>0} |\Omega^s|$$
$$\geqq C_0 > 0 \tag{135}$$

is positive this provides that the following limit is well-defined:

$$v_2^s(T_{\max}) = \lim_{t \to T_{\max}^-} v_2^s(t) \in (-\infty, 0],$$

which allows us to speak of a velocity at contact.

## 13. Blow-Up of the $L^2(\partial\Omega^f(t))$ Norm of the Velocity Field in the Fluid in the Case Without Vorticity

### *13.1. Blow-Up of the $L^2(\partial\Omega^f(t))$ Norm of $u^f$ as $t \to T_{\max}^-$ for the Case $v_2^s(T_{\max}) = 0$*

Integrating (134) from 0 to $T_{\max}$ and using (135) we infer

$$\int_0^{T_{\max}} \int_{\partial\Omega} |u^f|^2 n_2 \,\mathrm{d}l = -\infty.$$

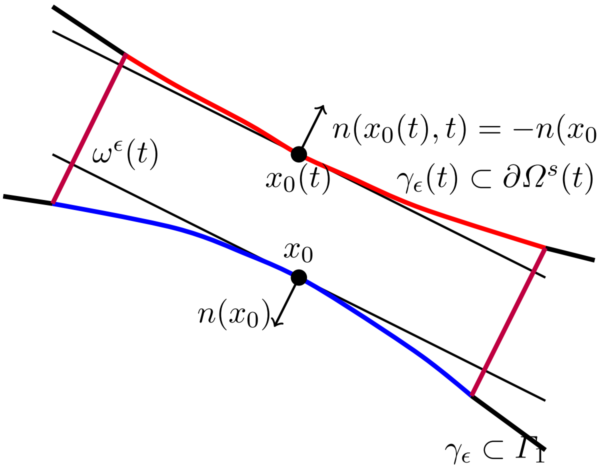


**Fig. 3.** $\omega^\varepsilon(t)$ = fluid region between blue, red and purple curves

*13.2. Blow-Up of the $L^2(\partial\Omega^f(t))$ Norm of $u^f$ as $t \to T_{\max}^-$ for the Case $v_2^s(T_{\max}) < 0$*

**Proof.** In this case we have the existence of $\alpha > 0$ such that

$$\forall t \in [0, T_{\max}), \ -\sqrt{\frac{2E(0)}{m_s}} \leqq v_2^s(t) < -\alpha < 0.$$

Let us now assume that we have the existence of $\beta > 0$ such that for a sequence of points $t_n$ converging to $T_{\max}$, we have

$$\int_{\partial\Omega^f(t_n)} |u^f(t_n)|^2 \, \mathrm{d}l \leqq \beta. \tag{136}$$

In what follows we work exclusively with this sequence of points, that we denote $t$.

Let us denote by $x_0 \in \partial\Omega$ a point where intersection occurs at $T_{\max}$. By assumption our normal vector satisfies

$$n_2(x_0) \leqq -\alpha_\Omega < 0.$$

We also know that at the intersection, the direction of the normal vector to $\partial\Omega^s(T_{\max})$ at $x_0$ will be the same as $n(x_0)$. We now for $\varepsilon > 0$ small consider the curve $\gamma_\varepsilon \subset \partial\Omega$ centered at $x_0$, and with length $\varepsilon$. For $t$ close to $T_{\max}$, we then call $\gamma_\varepsilon(t)$ the projection of $\gamma_\varepsilon$ on $\partial\Omega^s(t)$ parallel to $n(x_0)$. Namely, for $t$ close to $T_{\max}$ and $\varepsilon > 0$ small, these two curves are almost like segments of length $\varepsilon$ which are orthogonal to $n(x_0)$.

**Remark 10.** $\gamma^\varepsilon(t)$ and $\gamma_\varepsilon$ do not need to be locally on one side of the tangent at $x_0(t)$ and $x_0$, respectively.

Since our curves are of class $C^2$ we have the existence of $C > 0$ such that the area $A_1$ between $\gamma_\varepsilon$ and the tangent line passing through $x_0$ satisfies

$$|A_1| \leqq C\varepsilon^3. \tag{137}$$

Next we remember that since the fall of the rigid body is purely vertical, the vertical projection of $x_0$ onto $\partial\Omega^s(t)$, that we call $x_0(t)$ satisfies

$$n(x_0(t)) = -n(x_0), \tag{138}$$

as well as $x_0(t) \in \gamma_\varepsilon(t)$ if $t$ is close enough to $T_{\max}$. Then, similarly, by increasing $C$ if necessary, the area $A_2(t)$ between the tangent line passing though $x_0(t)$ (which is perpendicular to $n(x_0)$) and $\gamma_\varepsilon(t)$ satisfies

$$|A_2(t)| \leqq C\varepsilon^3. \tag{139}$$

Next the distance between between $x_0$ and $x_0(t)$ satisfies (since $x_0(T_{\max}) = x_0$)

$$|x_0 - x_0(t)| \leqq \underbrace{\sqrt{\frac{2E(0)}{m_s}}}_{C_0}(T_{\max} - t). \tag{140}$$

If we denote by $\omega_\varepsilon(t)$ the region comprised between $\gamma_\varepsilon$, $\gamma_\varepsilon(t)$, and the two segments parallel to $n(x_0)$ and starting at an extremity point of $\gamma_\varepsilon$, we have

$$|\omega_\varepsilon(t)| \leqq |A_1(t)| + |A_2(t)| + \varepsilon|x_0 - x_0(t)| \leqq 2C\varepsilon^3 + C_0(T_{\max} - t)\varepsilon. \tag{141}$$

Now, by integration by parts,

$$\int_{\omega_\varepsilon(t)} \nabla_{n(x_0)} u^f \cdot n(x_0)\,\mathrm{d}x = \int_{\gamma_\varepsilon \cup \gamma_\varepsilon(t)} u^f \cdot n(x_0)\, n(x_0) \cdot n\,\mathrm{d}l. \tag{142}$$

Thus, with (127) and our assumption (136) we obtain by Cauchy–Schwarz that

$$\left|\int_{\gamma_\varepsilon \cup \gamma_\varepsilon(t)} u^f \cdot n(x_0)\, n(x_0) \cdot n\,\mathrm{d}l\right| \leqq C\sqrt{|\omega^\varepsilon(t)|} \tag{143}$$

for some $C > 0$ independent of $t$ and $\varepsilon$. We next have on $\gamma_\varepsilon$ that

$$|n(x_0) - n| \leqq \max|\kappa||\gamma_\varepsilon| \leqq C\varepsilon \tag{144}$$

for some $C > 0$ independent of $t$ and $\varepsilon$. Taking $\varepsilon > 0$ and $T_{\max} - t > 0$ small enough, we have

$$|\gamma_\varepsilon(t)| \leqq 2\varepsilon. \tag{145}$$

Due to $x_0(t) \in \gamma_\varepsilon(t)$ for $t$ close enough to $T_{\max}$ and (145),

$$\forall x \in \gamma_\varepsilon(t), \quad |x - x_0(t)| \leqq 2\varepsilon.$$

Therefore the distance on $\gamma_\varepsilon(t)$ satisfies (for $\varepsilon > 0$ small enough) that

$$\forall x \in \gamma_\varepsilon(t), \;\; \mathrm{d}_{\gamma_\varepsilon(t)}(x, x_0(t)) \leqq 3\varepsilon.$$

Therefore,

$$\forall x \in \gamma_\varepsilon(t),\ \ |n(x_0(t),t) - n(x,t)| \leqq \max|\kappa| max_{\gamma_\varepsilon(t)} d_{\gamma_\varepsilon(t)}(x, \tilde{x}_0(t)) \leqq C\varepsilon \tag{146}$$

for some $C > 0$ independent of $t$ and $\varepsilon$. We now write for $\gamma_\varepsilon$ that

$$u^f \cdot n(x_0)n(x_0)\cdot n = u^f \cdot (n(x_0) - n)n(x_0)\cdot n + u^f \cdot n(n(x_0) - n)\cdot n + u^f \cdot n \underbrace{n\cdot n}_{=1},$$

while for $\gamma_\varepsilon(t)$,

$$u^f \cdot n(x_0)n(x_0)\cdot n = u^f \cdot (n(x_0) + n)n(x_0)\cdot n - u^f \cdot n(n(x_0) + n)\cdot n + u^f \cdot n \underbrace{n\cdot n}_{=1}.$$

Using these two equations in (143), (144) and (146), we infer that for some $C > 0$ independent of $t$ and $\varepsilon$,

$$\begin{aligned}\left|\int_{\gamma_\varepsilon(t)} u^f \cdot n \,\mathrm{d}l\right| &\leqq C\sqrt{|\omega^\varepsilon(t)|} + C\varepsilon \int_{\gamma_\varepsilon(t)\cup\gamma_\varepsilon} |u^f|\,\mathrm{d}l \\ &\leqq C\sqrt{|\omega^\varepsilon(t)|} + C\varepsilon(\sqrt{|\gamma_\varepsilon|} + \sqrt{|\gamma_\varepsilon(t)|})\|u^f\|_{L^2(\partial\Omega^f(t))}. \end{aligned} \tag{147}$$

Using the boundary condition $u^f \cdot n = v_2^s n_2$ on $\partial\Omega^s(t)$ and our estimate (141) as well as our crucial assumption (136) we then obtain from (147) that for some $C > 0$ independent of $t$ and $\varepsilon$,

$$\left|v_2^s(t)\int_{\gamma_\varepsilon(t)} n_2 \,\mathrm{d}l\right| \leqq C\sqrt{\varepsilon^3 + \varepsilon(T_{\max} - t)} + C\varepsilon\sqrt{\varepsilon}. \tag{148}$$

Therefore, since for $\varepsilon > 0$ small enough and $t$ close enough to $T_{\max}$, $n$ on $\gamma_\varepsilon(t)$ is close to $-n(x_0)$, which satisfies $n_2(x_0) < -\alpha_\Omega$; we then infer from (148) that

$$|v_2^s(t)|\alpha_\Omega\varepsilon \leqq 2C\sqrt{\varepsilon^3 + \varepsilon(T_{\max} - t)} + 2C\varepsilon\sqrt{\varepsilon}.$$

Letting (with $\varepsilon > 0$ fixed) $t$ converge to $T_{\max}$ we then have

$$|v_2^s(T_{\max})|\alpha_\Omega \leqq 2C\sqrt{\varepsilon} + 2C\sqrt{\varepsilon}.$$

This identity being true for any $\varepsilon > 0$ small enough leads us to obtain that $v_2^s(T_{\max}) = 0$, which is in contradiction with our assumption that $v_2^s(T_{\max}) < 0$. Therefore our assumption (136) has to be rejected, which means we proved 1) of Theorem 4:

$$\lim_{t\to T_{\max}^-} \|u^f\|_{L^2(\partial\Omega^f(t))} \to \infty. \tag{149}$$

□

**Remark 11.** Away from the contact points at time $T_{\max}$, the velocity field in the fluid stays smooth by elliptic regularity (by (100) and (101)), so it is indeed at the contact points that the blow-up is localized.

We now establish that contact occurs with an infinite upward acceleration for the solid, except for the case where the contact zone at $T_{\max}$ contains a curve of non-zero length, in which case the acceleration becomes strictly positive as contact nears, while staying bounded.

The blow-up (149) shows that the first term in the expression of acceleration in (34), established in Lemma 1, will tend to $-\infty$ (since $n_2 < 0$ in the contact zone). This term indeed contributes oppositely to the announced result. It is the detailed treatment of the last term of (34) for the case without vorticity which allows us to rewrite this term and obtain (128) in Lemma 5. In turn, (128) shows in the next section that the acceleration of the rigid body is either positive finite or infinite at contact, establishing a repelling effect of the boundary at contact.

## 14. Positive or Infinite Upward Solid Acceleration at Time of Contact for the Case Without Vorticity

From (128) in Lemma 5 we immediately have

$$2\frac{E(0)+(\rho^f-\rho^s)gx_2^s|\Omega^s|}{(v_2^s)^2}\frac{\mathrm{d}v_2^s}{\mathrm{d}t} \geqq \rho^f\alpha_\Omega\int_{\partial\Omega}\frac{|u^f|^2}{2}\,\mathrm{d}l+(\rho^f-\rho^s)g|\Omega^s| \\ -\rho^f\underbrace{\int_{\partial\Omega\cap\Gamma_1^c}\frac{|u^f|^2}{2}(n_2+\alpha_\Omega)\mathrm{d}l}_{\text{bounded by (100)}}, \tag{150}$$

where we used in (150) the fact that $n_2 < -\alpha_\Omega$ on $\Gamma_1$.

We can now conclude on our acceleration. We will have to distinguish three cases. The first case is when the velocity at contact is nonzero. The next case is when the velocity at contact is zero, and contact occurs on a set of zero length, which has the same conclusion with an infinite upward acceleration. The final case considered is when contact occurs on a set containing a connected component with non zero length, for which we establish that the velocity at contact is zero, and the acceleration remains finite and stays away from zero.

**Case 1.** $v_2^s(T_{\max}) < 0$

From (135), we infer from (150), the fact that the velocity stays away from zero, and our blow-up (149) that

$$\lim_{t\to T_{\max}^-}\frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t)=\infty. \tag{151}$$

From (151) and (92e), we immediately have

$$\lim_{t\to T_{\max}^-}\int_{\partial\Omega^s(t)} pn\,\mathrm{d}l=\infty, \tag{152}$$

which establishes the blow-up of the normalized (to zero on top of $\partial\Omega$ on $x_1 = 0$) pressure on $\partial\Omega^s(t)$ as we approach contact.

We now get back to (150). From the fact that the velocity stays away from zero as $t \to T_{\max}^-$ (from our assumption $v_2^s(T_{\max}) < 0$) we then infer that $\frac{1}{v_2^s(t)}$ stays bounded as $t \to T_{\max}^-$. Therefore, from (128) and $\frac{\mathrm{d}(v_2^s)^{-1}}{\mathrm{d}t} = -\frac{1}{(v_2^s)^2}\frac{\mathrm{d}v_2^s}{\mathrm{d}t}$, this implies that

$$\int_0^{T_{\max}} \|u^f\|^2_{L^2(\partial\Omega)} \,\mathrm{d}t < \infty.$$

**Case 2.** $v_2^s(T_{\max}) = 0$. The calculations in this part are for any contact zone covered in Theorem 4, with $v_2^s(T_{\max}) = 0$, until (161) included. After this relation, they are for a contact zone of zero length. Here

$$\lim_{t\to T_{\max}^-} v_2^s(t) = 0 \tag{153}$$

simply translates into

$$\lim_{t\to T_{\max}^-} (v_2^s(t))^{-1} = -\infty. \tag{154}$$

By integrating (128) from 0 to $T_{\max}$ we then obtain

$$\int_0^{T_{\max}} \int_{\partial\Omega} |u^f|^2 n_2 \,\mathrm{d}l \,\mathrm{d}t = -\infty. \tag{155}$$

By conservation of total energy, we have by (135) that

$$\rho^f \int_{\Omega^f(t)} |u^f|^2 \,\mathrm{d}x \to 2(E(0) - (\rho^s - \rho^f)g|\Omega^s|x_2(T_{\max})) > 0, \text{ as } t \to T_{\max}^-. \tag{156}$$

Using (129), we obtain

$$\int_{\partial\Omega^s(t)} \phi n_2 \,\mathrm{d}x\; v_2^s(t) \to \frac{2}{\rho^f}(E(0) - (\rho^s - \rho^f)g|\Omega^s|x_2(T_{\max})), \text{ as } t \to T_{\max}^-. \tag{157}$$

Using (130) in (157) then yields

$$\int_{\partial\Omega} \phi n_2 \,\mathrm{d}x\; v_2^s(t) \to -\frac{2}{\rho^f}(E(0) - (\rho^s - \rho^f)g|\Omega^s|x_2(T_{\max})), \text{ as } t \to T_{\max}^-, \tag{158}$$

where we also used the fact that $u^f$ is bounded in $L^2(\Omega^f(t))$. By reasoning similar to that used to obtain (129), this is equivalent to

$$\int_{\partial\Omega} u^f \cdot \tau x_1 \,\mathrm{d}l\; v_2^s(t) \to \underbrace{\frac{2}{\rho^f}(E(0) - (\rho^s - \rho^f)g|\Omega^s|x_2(T_{\max}))}_{C_0>0}, \text{ as } t \to T_{\max}^-. \tag{159}$$

Using Cauchy–Schwarz in the integral on the left hand side of (159), this then provides some $C_1 > 0$ such that for $t$ close enough to $T_{\max}$,

$$\int_{\partial\Omega} |u^f|^2 \,\mathrm{d}l \geqq \frac{C_1}{v_2^s(t)^2}. \tag{160}$$

Reporting (160) in (150) yields

$$\frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t) \geqq \frac{\rho^f C_1 \alpha_\Omega}{5E(0)} = a_0 > 0 \tag{161}$$

for any $t$ close enough to $T_{\max}$, which again shows a positive upward acceleration for the rigid body as contact nears, opposing the fall.

We now prove that for the case when the part of $\partial\Omega$ intersecting $\partial\Omega^s(T_{\max})$ is of zero measure and $v_2^s(T_{\max}) = 0$, we have an infinite upward acceleration for the solid at the time of contact.

From now on $C$ denotes a generic positive constant independent of $t < T_{\max}$.

Let us now fix $\varepsilon > 0$.

Using our assumption that the intersecting part of $\partial\Omega$ is of zero length, we write

$$\partial\Omega = \Gamma_\varepsilon \cup (\Gamma_\varepsilon^c \cap \partial\Omega), \tag{162}$$

where $\Gamma_\varepsilon$ is a union of curves containing the contact points at $T_{\max}$ and whose total length is less than $\varepsilon$. By Cauchy–Schwarz applied on $\Gamma_\varepsilon$ and that which is complementary to it, we have

$$\left|\int_{\partial\Omega} u^f \cdot \tau x_1 \,\mathrm{d}l\right| \leqq C\sqrt{\varepsilon}\sqrt{-\int_{\Gamma_\varepsilon} n_2 |u^f \cdot \tau|^2 \,\mathrm{d}l} + C\sqrt{\int_{\Gamma_\varepsilon^c \cap \partial\Omega} |u^f \cdot \tau|^2 \,\mathrm{d}l}, \tag{163}$$

since $n_2 \leqq -\alpha_\Omega < 0$ on the contact part of $\partial\Omega$.

Due to our control of $u^f$ away from the contact zone by (100), we have from (163) that

$$\left|\int_{\partial\Omega} u^f \cdot \tau x_1 \,\mathrm{d}l\right| \leqq C\sqrt{\varepsilon}\sqrt{-\int_{\Gamma_\varepsilon} n_2 |u^f \cdot \tau|^2 \,\mathrm{d}l} + C_\varepsilon, \tag{164}$$

where $C_\varepsilon$ is independent of $t$ (but blows up as $\varepsilon \to 0$). With (159), (164) provides for $t$ close enough to $T_{\max}$ (with $\varepsilon > 0$ fixed, and remembering that $\lim_{t\to T_{\max}^-} v_2^s(t) = 0$, and $C$ is generic) such that

$$\left|\frac{(E(0) - (\rho^s - \rho^f)g|\Omega^s| x_2^s(T_{\max}))}{\rho^f v_2^s(t)}\right| \leqq C\sqrt{\varepsilon}\sqrt{-\int_{\Gamma_\varepsilon} n_2 |u^f \cdot \tau|^2 \,\mathrm{d}l}. \tag{165}$$

Therefore,

$$\frac{(E(0) - (\rho^s - \rho^f)g|\Omega^s| x_2^s(T_{\max}))^2}{(\rho^f)^2 v_2^s(t)^2 C^2 \varepsilon} \leqq \int_{\partial\Omega} |u^f \cdot \tau|^2 \,\mathrm{d}l. \tag{166}$$

Using (166) in (150) then yields, for $t$ close, that enough to $T_{\max}$, that

$$\frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t) \geqq \frac{(E(0) - (\rho^s - \rho^f)g|\Omega^s| x_2^s(T_{\max}))}{5\rho^f C^2}\frac{\alpha_\Omega}{\varepsilon},$$

which, given the arbitrary nature of $\varepsilon > 0$, provides

$$\lim_{t\to T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t) = \infty. \tag{167}$$

**Case 3. When the contact zone is of non zero length.**

We now treat the remaining case where the contact zone contains a curve $\Gamma_c \subset \Gamma_1 \cap \partial\Omega^s(T_{\max})$ of non zero length. We will first prove the velocity of the rigid body at contact is zero (therefore allowing the inequality (161)), and that the acceleration remains bounded.

Since $n_2 < -\alpha_\Omega$ on $\Gamma_1$, we have the existence of $f$ smooth such that $\Gamma_1$ is the graph of a function $f$ for $x_1 \in \cup_{i\in I}[\alpha_i, \beta_i]$ for some $\alpha_i \leqq \beta_i$. In a neighborhood of the region of the rigid body which intersects $\Gamma_1$ at $T_{\max}$, we also have that $\partial\Omega^s(T_{\max})$ is the graph of a function $g$, which equals $f$ on the contact zone. We have that $f = g$ for $x_1 \in \cup_{i\in J}[a_i, b_i]$ ($J \subset I$ and $[a_i, b_i] \subset [\alpha_i, \beta_i]$) and for $\varepsilon > 0$ small, we have the existence of $c_\varepsilon > 0$ small such that

$$\forall x \in [a_i - c_\varepsilon, b_i + c_\varepsilon],\ \ f(x_1) \leqq g(x_1) \leqq f(x_1) + \varepsilon. \tag{168}$$

We now define for $t < T_{\max}$ the distance in the vertical direction between the two curves at time $t$:

$$\eta_2^s(t) = \int_{T_{\max}}^t v_2^s(s)\,\mathrm{d}s > 0. \tag{169}$$

We denote this by (dropping the $i$ index)

$$\begin{aligned}
S^-_{a,b,\varepsilon} &= \{(x_1, f(x_1));\, x_1 \in [a - c_\varepsilon, b + c_\varepsilon]\} \subset \Gamma_1,\\
S^+_{a',b'}(t) &= \{(x_1, f(x_1) + \eta_2^s(t));\, x_1 \in [a', b'] \subset [a, b]\},\\
\Omega^f_{a,b,\varepsilon}(t) &= \{(x_1, x_2);\, x_1 \in [a - c_\varepsilon, b + c_\varepsilon];\, x_2 \in (f(x_1), g(x_1) + \eta_2^s(t))\},\\
\Omega^f_{a',b'}(t) &= \{(x_1, x_2);\, x_1 \in [a', b'] \subset [a, b];\, x_2 \in (f(x_1), f(x_1) + \eta_2^s(t))\}.
\end{aligned}$$

Since the fall is vertical with velocity constant in space, we have that

$$\begin{aligned}
\Omega^f_{a,b}(t) \subset \Omega^f_{a,b,\varepsilon}(t) &\subset \Omega^f(t),\\
S^+_{a,b}(t) &\subset \partial\Omega^s(t).
\end{aligned}$$

For $\alpha \in [0, \frac{b-a}{4}]$ we now denote this by

$$\Omega_\alpha(t) = \Omega^f_{a+\alpha, a+3\frac{b-a}{4}+\alpha}(t) \subset \Omega^f(t). \tag{170}$$

From the divergence theorem, $\int_{\partial\Omega_\alpha(t)} u^f \cdot n\,\mathrm{d}l = 0$, which provides, if we denote $S_\alpha(t) = S^+_{a+\alpha, a+3\frac{b-a}{4}+\alpha}(t) \subset \partial\Omega^s(t)$,

$$v_2^s(t) \int_{S_\alpha(t)} n_2\,\mathrm{d}l = \int_{\partial\Omega_\alpha(t)\cap\{x_1 = a+\alpha\}} u_1^f\,\mathrm{d}x_2 - \int_{\partial\Omega_\alpha(t)\cap\{x_1 = a+3\frac{b-a}{4}+\alpha\}} u_1^f\,\mathrm{d}x_2. \tag{171}$$

Integrating (171) with respect to $\alpha$ (variable $x_1$) between 0 and $\frac{b-a}{4}$ yields

$$v_2^s(t) \int_0^{\frac{b-a}{4}} \int_{S_\alpha(t)} n_2\,\mathrm{d}l\,\mathrm{d}x_1 = \int_{\Omega^f_{a,a+\frac{b-a}{4}}(t)} u_1^f\,\mathrm{d}x - \int_{\Omega^f_{a+3\frac{b-a}{4},b}(t)} u_1^f\,\mathrm{d}x.$$

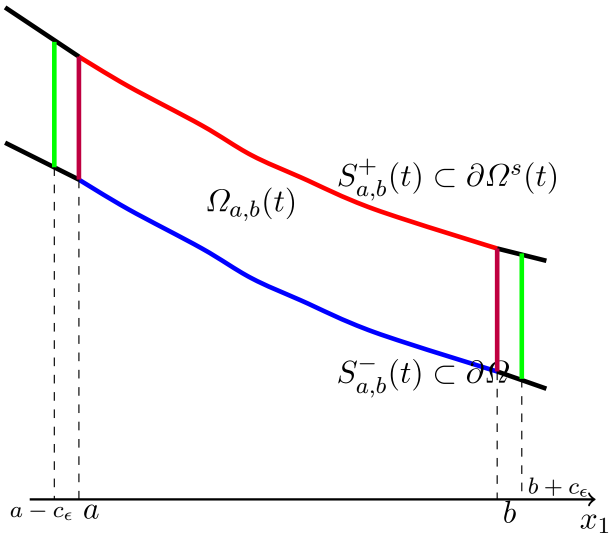


**Fig. 4.** Here, the blue and red curves are translated vertically from each other. Contact at time $T_{\max}$ does not occur along the black curves. $\Omega^f_{a,b,\varepsilon}(t)$ is defined as $\Omega^f_{a,b}(t)$ with the green vertical lines replacing the purple ones

Therefore, by Cauchy–Schwarz applied to the right-hand side of this identity,

$$|v_2^s(t)| \left| \int_0^{\frac{b-a}{4}} \int_{S_\alpha(t)} n_2 \,\mathrm{d}l \,\mathrm{d}x_1 \right| \leqq 2\sqrt{|\Omega^f_{a,b}(t)|} \|u^f\|_{L^2(\Omega^f_{a,b}(t))},$$

which provides us (since $u^f$ is bounded in $L^2(\Omega^f(t))$ and $|\Omega^f_{a,b}(t)| = \eta_2^s(t) \times (b-a)$ ) with the existence of $C > 0$ independent of $t < T_{\max}$ such that

$$|v_2^s(t)| \leqq C\sqrt{\eta_2^s(t)}. \tag{172}$$

**Remark 12.** This inequality uses in a crucial way the fact contact occurs on a zone containing a curve of non-zero length. It also establishes that the contact velocity is zero whenever a curve of non-zero length is part of the contact zone.

**Remark 13.** After completion of this work, the author was informed that in the Navier–Stokes context, Starovoitov in [25] previously defined similar geometric sets as the various sets appearing in the previous page. The present paper obtains the inequality before (172) in the same way as the original inequality (11) of [25]. The rest of the present proof (in particular to obtain (179), as well as the type of conclusions reached) and the analysis of [25] however differ significantly. For instance, [25] establishes the estimate (12) from (11) by means of a Sobolev inequality in $\Omega$ (domain without cusp), which is allowed since the velocity field for a viscous problem is in $H^1_0(\Omega)$ (unlike in the inviscid case where there is tangential discontinuity).

We define the vertical distance between the two graphs at time $t$ at $x_1$ as follows

$$d(x_1,t)=g(x_1)-f(x_1)+\eta_2^s(t),$$

and we define for any $x_1\in[a-c_\varepsilon,b+c_\varepsilon]$ the vertical average of $u_1^f$ as:

$$\bar{u}_1(x_1,t)=\frac{1}{\mathrm{d}(x_1,t)}\int_{f(x_1)}^{g(x_1)+\eta_2^s(t)}u_1^f(x_1,x_2,t)\,\mathrm{d}x_2.$$

Due to $u^f=\nabla^\perp\phi$ with $\phi=0$ on $\partial\Omega$ and $\phi=v_2^s(t)x_1$ on $\partial\Omega^s(t)$, we have

$$\bar{u}_1(x_1,t)=-\frac{v_2^s(t)x_1}{\mathrm{d}(x_1,t)}. \tag{173}$$

Next since $\bar{u_1}(x_1,t)$ is a value taken by $u_1^f$ on the vertical segment $\{x_1\}\times[f(x_1),g(x_1)+\eta_2^s(t)]$, we have the existence of $\alpha(x_1,t)\in[f(x_1),g(x_1)+\eta_2^s(t)]$ such that $\bar{u}_1(x_1,t)=u_1^f(x_1,\alpha(x_1,t),t)$, which leads to

$$u_1^f(x_1,f(x_1),t)-\bar{u}_1(x_1,t)=\int_{\alpha(x_1,t)}^{f(x_1)}\frac{\partial u_1^f}{\partial x_2}(x_1,x_2,t)\,\mathrm{d}x_2. \tag{174}$$

This implies by Cauchy–Schwarz that

$$(u_1^f(x_1,f(x_1),t)-\bar{u}_1(x_1,t))^2\leqq d(x_1,t)\int_{f(x_1)}^{g(x_1)+\eta_2^s(t)}\left|\frac{\partial u_1^f}{\partial x_2}(x_1,x_2,t)\right|^2\,\mathrm{d}x_2. \tag{175}$$

We now multiply (175) by the length element $\sqrt{1+f'^2(x_1)}$ on $\partial\Omega$ and integrate the resulting relation with respect to $x_1\in[a-c_\varepsilon,b+c_\varepsilon]$. Remembering that $c_\varepsilon$ was chosen so that (168) was satisfied, we then obtain

$$\int_{S^-_{a,b,\varepsilon}(t)}|u_1^f-\bar{u}_1|^2\,\mathrm{d}l\leqq C(\varepsilon+\eta_2^s(t))\int_{\Omega^f_{a,b,\varepsilon}(t)}\left|\frac{\partial u_1^f}{\partial x_2}(x_1,x_2,t)\right|^2\,\mathrm{d}x. \tag{176}$$

Using the triangular inequality we infer from (176) and (173) that

$$\int_{S^-_{a,b,\varepsilon}(t)}|u_1^f|^2\,\mathrm{d}l\leqq C(\varepsilon+\eta_2^s(t))\int_{\Omega^f_{a,b,\varepsilon}(t)}|\frac{\partial u_1^f}{\partial x_2}|^2\,\mathrm{d}x+C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2}, \tag{177}$$

where we remind the reader $C$ is a generic constant independent of time. Since by our assumption on $\Gamma_1$, $n_2\leqq-\alpha_\Omega<0$ , and since $u^f\cdot n=0$ on $S^-_{a,b,\varepsilon}$, we have that $|u_1^f|\geqq C|u^f|$ on $\Gamma_1$ for some $C>0$ independent of time. Therefore, we infer from (177) that (we remind the reader $C$ is generic)

$$\int_{S^-_{a,b,\varepsilon}(t)}|u^f|^2\,\mathrm{d}l\leqq C(\varepsilon+\eta_2^s(t))\int_{\Omega^f_{a,b,\varepsilon}(t)}|\frac{\partial u^f}{\partial x_2}|^2\,\mathrm{d}x+C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2}. \tag{178}$$

We now work with $t$ close enough to $T_{\max}$ so that $\eta_2^s(t) \leqq \varepsilon$. Therefore, with our generic constant $C$, (178) becomes

$$\int_{S^-_{a,b,\varepsilon}(t)} |u^f|^2 \, \mathrm{d}l \leqq C\varepsilon \int_{\Omega^f_{a,b,\varepsilon}(t)} |\frac{\partial u^f}{\partial x_2}|^2 \, \mathrm{d}x + C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2}. \tag{179}$$

Summing over all regions of the type $S^-_{a,b,\varepsilon}(t)$ in case contact occurs on a non–connected set, we then have from (179) that

$$\int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l \leqq C\varepsilon \int_{\Omega^f(t)} |\frac{\partial u^f}{\partial x_2}|^2 \, \mathrm{d}x + C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2} + C_\varepsilon, \tag{180}$$

with $C_\varepsilon$ being a constant independent of time (and becoming large as $\varepsilon$ is small). Due to (127), this inequality implies that

$$\int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l \leqq C\varepsilon \int_{\partial\Omega^f(t)} |u^f|^2 \, \mathrm{d}l + C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2} + C_\varepsilon.$$

By choosing $\varepsilon > 0$ small enough, this inequality implies (we remind readers $C > 0$ is generic and $\partial\Omega^f(t) = \partial\Omega^s(t) \cup \partial\Omega$)

$$\int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l \leqq C\varepsilon \int_{\partial\Omega^s(t)} |u^f|^2 \, \mathrm{d}l + C\frac{(v_2^s(t))^2}{\eta_2^s(t)^2} + C_\varepsilon. \tag{181}$$

Next we notice that since $n_2 \leqq -\alpha_\Omega$ on $\Gamma_1$, we have $n_2 \geqq \alpha_\Omega > 0$ in the region of $\partial\Omega^s(t)$ near $\partial\Omega$. Since $u^f$ is bounded away from the contact zone, we have

$$\int_{\partial\Omega^s(t)} |u^f|^2 \, \mathrm{d}l \leqq \frac{1}{\alpha_\Omega} \int_{\partial\Omega^s(t)} |u^f|^2 n_2 \, \mathrm{d}l + C. \tag{182}$$

By integration by parts in $\Omega^f(t)$,

$$\int_{\partial\Omega^s(t)} |u^f|^2 n_2 \, \mathrm{d}l = -\int_{\partial\Omega} |u^f|^2 n_2 \, \mathrm{d}l + 2\int_{\Omega^f(t)} \frac{\partial u^f}{\partial x_2} \cdot u^f \, \mathrm{d}x.$$

Thus, for $\delta > 0$ small (to be made precised later), we have

$$\begin{aligned}\int_{\partial\Omega^s(t)} |u^f|^2 n_2 \, \mathrm{d}l &\leqq \int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l + \delta \int_{\Omega^f(t)} |\nabla u^f|^2 \, \mathrm{d}x + \frac{1}{\delta}\int_{\Omega^f(t)} |u^f|^2 \, \mathrm{d}x \\ &\leqq \int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l + \delta \int_{\Omega^f(t)} |\nabla u^f|^2 \, \mathrm{d}x + \frac{C}{\delta} \\ &\leqq \int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l + C\delta \int_{\partial\Omega^f(t)} |u^f|^2 \, \mathrm{d}l + \frac{C}{\delta},\end{aligned} \tag{183}$$

where we used (127). By using (182) we then see that for $C\delta < \frac{\alpha_\Omega}{2}$ we have, from (183),

$$\int_{\partial\Omega^s(t)} |u^f|^2 n_2 \, \mathrm{d}l \leqq C \int_{\partial\Omega} |u^f|^2 \, \mathrm{d}l + C.$$

From (182), (and remembering that $C$ is generic)

$$\int_{\partial\Omega^s(t)} |u^f|^2 \,\mathrm{d}l \leqq C \int_{\partial\Omega} |u^f|^2 \,\mathrm{d}l + C. \tag{184}$$

Therefore, by picking $\varepsilon > 0$ small enough, (184) used in (181) implies

$$\int_{\partial\Omega} |u^f|^2 \,\mathrm{d}l \leqq C \frac{(v_2^s(t))^2}{\eta_2^s(t)^2} + C. \tag{185}$$

Using (185) in our formula for acceleration (128), we obtain

$$\frac{\mathrm{d}v_2^s}{\mathrm{d}t} \leqq C \frac{(v_2^s(t))^4}{\eta_2^s(t)^2} + C v_2^s(t)^2.$$

Using (172) in the previous inequality provides $\dfrac{\mathrm{d}v_2^s}{\mathrm{d}t} \leqq C$, and therefore with (161),

$$0 < a_0 \leqq \liminf_{t\to T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t) \leqq \limsup_{t\to T_{\max}^-} \frac{\mathrm{d}v_2^s}{\mathrm{d}t}(t) < \infty,$$

which finishes the proof of Theorem 4. □

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**Compliance with ethical standards**

**Ethical Responsibilities and Ethical Standard** The author satisfies all required ethical responsibilities and standards required by Archive for Rational Mechanics and Analysis.

**Conflict of interest** The author declares that he has no conflict of interest.



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