

ALF GRAVITATIONAL INSTANTONS AND COLLAPSING RICCI-FLAT METRICS ON THE $K3$ SURFACE

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Abstract

We construct large families of new collapsing hyperkähler metrics on the $K3$ surface. The limit space is a flat Riemannian 3-orbifold T^3/\mathbb{Z}_2 . Away from finitely many exceptional points the collapse occurs with bounded curvature. There are at most 24 exceptional points where the curvature concentrates, which always contains the 8 fixed points of the involution on T^3 . The geometry around these points is modelled by ALF gravitational instantons: of dihedral type (D_k) for the fixed points of the involution on T^3 and of cyclic type (A_k) otherwise.

The collapsing metrics are constructed by deforming approximately hyperkähler metrics obtained by gluing ALF gravitational instantons to a background (incomplete) S^1 -invariant hyperkähler metric arising from the Gibbons–Hawking ansatz over a punctured 3-torus.

As an immediate application to submanifold geometry, we exhibit hyperkähler metrics on the $K3$ surface that admit a strictly stable minimal sphere which cannot be holomorphic with respect to any complex structure compatible with the metric.

1. Introduction

Soon after Yau’s proof of the Calabi Conjecture showed that the (smooth 4-manifold underlying a complex) $K3$ surface carries Kähler Ricci-flat metrics, Gibbons and Pope [24] suggested the construction (further explored by Page in [43]) of explicit approximately Ricci-flat metrics on Kummer surfaces. They considered the quotient of a flat 4-torus by an involution and resolved the 16 orbifold singularities by gluing in copies of the Eguchi–Hanson metric. This *Kummer construction* was later made rigorous by LeBrun–Singer [33] and Topiwala [47] using twistor methods. Much more recently Donaldson [19] gave a different proof using analysis, closest to the approach taken in the current paper.

From a broader perspective the Kummer construction furnishes the prototypical example of the appearance of orbifold singularities in non-collapsing sequences of Einstein 4-manifolds. In [1, Theorem C] Anderson showed that a sequence of Einstein 4-manifolds (M_i, g_i) with a uniform lower bound on volume and upper bounds on diameter and Euler characteristic converges (up to subsequences) to an Einstein 4-orbifold M_∞ with finitely many singular points. The formation of orbifold singularities is modelled on complete Ricci-flat ALE spaces which appear as rescaled limits, or “bubbles”, of the sequence (M_i, g_i) around points that approach one of the singularities of the orbifold M_∞ .

In the Ricci-flat case collapsing can also occur. Anderson [2, Theorem II] showed that every sequence of Ricci-flat metrics (M, g_i) of unit volume but unbounded diameter collapses everywhere, *i.e.* $\text{inj}_{g_i}(x) \rightarrow 0$ for all $x \in M$. The collapse is in the sense of Cheeger–Gromov outside finitely many points x_1, \dots, x_n , *i.e.* $\text{inj}_{g_i}(x) \rightarrow 0$ and $\text{inj}_{g_i}(x)^2 |\text{Rm}_{g_i}|_{g_i}(x) \leq \epsilon_0$ for all $x \in M \setminus \{x_1, \dots, x_n\}$, for a universal constant $\epsilon_0 > 0$. In fact, Cheeger and Tian [10, Theorems 0.1 and 0.8] have shown that the collapse occurs with *bounded* curvature away from a definite number of points.

Contrary to the case of non-collapsed singularities, almost nothing is known about the structure of the singular points arising in collapsing sequences of Ricci-flat metrics [3, §6]. One would expect that the geometry around these points is modelled on complete Ricci-flat manifolds with non-maximal volume growth. A simple example of the expected phenomena was suggested by Page [44] in 1981. Consider the Kummer construction of Ricci-flat metrics on the K3 surface along a family of split tori $T^4 = T^3 \times S_\ell^1$ with a circle factor of length $\ell \rightarrow 0$. We can then think of the 2-spheres arising in the resolution of the 16 singularities of T^4/\mathbb{Z}_2 as coming in pairs aligned along the collapsing circle over each of the 8 singular points of T^3/\mathbb{Z}_2 . If we now rescale the sequence of Kähler Ricci-flat metrics on the K3 surface by ℓ^{-2} around one of these pairs, Page suggests, in the limit $\ell \rightarrow 0$ we should obtain a complete Ricci-flat (hyperkähler) metric on a noncompact space which at infinity looks like $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$.

In this paper we regard this example as a simple case of a more general construction of sequences of Ricci-flat metrics on the K3 surface that collapse to a 3-dimensional limit. In this more general construction, the 4-torus $T^3 \times S_\ell^1$ is replaced by a non-trivial circle bundle over a (punctured) 3-torus and the role of Page’s “periodic but nonstationary gravitational instanton” is played by other ALF gravitational instantons.

A gravitational instanton is a complete hyperkähler 4-manifold with decaying curvature at infinity. Since every hyperkähler manifold is in particular Ricci-flat, gravitational instantons have constrained volume

growth: the volume of a geodesic ball of radius r grows at most as r^4 . Gravitational instantons of maximal volume growth are the ALE spaces constructed and classified by Kronheimer [32] following earlier work of Eguchi–Hanson, Gibbons–Hawking and Hitchin. We have seen how ALE spaces arise as models for the formation of orbifold singularities of non-collapsed sequences of Einstein 4-manifolds. In [41] Minerbe showed that if $\text{Vol}(B_r(p)) = O(r^a)$ for some $3 \leq a < 4$ and all p , then we must have $a = 3$. Gravitational instantons of cubic volume growth are called ALF. By [41] the (unique) end of an ALF space looks like a circle fibration over the complement of a ball in \mathbb{R}^3 or $\mathbb{R}^3/\mathbb{Z}_2$ with fibres of asymptotically finite length. If the base of the circle fibration at infinity is \mathbb{R}^3 (respectively, $\mathbb{R}^3/\mathbb{Z}_2$) then we say that the ALF space is of cyclic (dihedral) type, since the boundary of large geodesic balls is diffeomorphic to S^3/Γ , where $\Gamma \subset SU(2)$ is a cyclic group in the first case and a binary dihedral group in the second.

The prototypical example of an ALF space of cyclic type is the Taub–NUT metric on \mathbb{R}^4 . This metric is explicit and the circle fibration at infinity is induced by the Hopf projection $S^3 \rightarrow S^2$. The first example of an ALF metric of dihedral type was found by Atiyah–Hitchin [5] by studying moduli spaces of magnetic monopoles on \mathbb{R}^3 , *i.e.* the solutions of the dimensional reduction of the Yang–Mills self-duality equations from 4 to 3 dimensions. The Atiyah–Hitchin manifold is diffeomorphic to the complement of a Veronese $\mathbb{R}P^2$ in S^4 and the metric is explicitly given in terms of elliptic integrals.

Recently ALF gravitational instantons have been the focus of intense research with the aim of constructing and classifying examples. Minerbe [42] classified ALF spaces of cyclic type. These are all explicitly given by the Gibbons–Hawking construction of hyperkähler 4-manifolds with a triholomorphic circle action [25]. Most (if not all) the known methods of construction of hyperkähler metrics have been applied to the dihedral ALF case: twistor methods [13, 29], hyperkähler quotient constructions [15, 16], gauge-theoretic constructions as in the case of the Atiyah–Hitchin manifold [13], Kummer-type constructions [8] and complex Monge–Ampère methods [6]. For example, Page’s “periodic but nonstationary” gravitational instantons of [44], more commonly known as D_2 ALF spaces, were first constructed rigorously by Hitchin [29] using twistor methods, then by Dancer [15] using a hyperkähler quotient construction and more recently by Biquard–Minerbe [8] using an extension of the Kummer construction to non-compact spaces. The D_2 ALF spaces can also be thought of as the moduli spaces of centred charge 2 $SO(3)$ monopoles on \mathbb{R}^3 with two singularities endowed with their natural L^2 -metric [14]. The fact that all these constructions yield equivalent families of ALF metrics was shown only recently by Chen–Chen [11].

Despite this rich theory of ALF gravitational instantons, until now it has remained unclear how they can appear as models for the formation of singularities in collapsing sequences of hyperkähler metrics on the K3 surface. The aim of this paper is to exploit singular perturbation methods to construct examples of Ricci-flat metrics on the K3 surface collapsing to a 3-dimensional limit and exhibit ALF gravitational instantons as the “bubbles” appearing in the process.

Theorem 1.1. *Every collection of 8 ALF spaces of dihedral type M_1, \dots, M_8 and $n \leq 16$ ALF spaces of cyclic type N_1, \dots, N_n satisfying*

$$\sum_{j=1}^8 \chi(M_j) + \sum_{i=1}^n \chi(N_i) = 24$$

arises as the collection of “bubbles” forming in a sequence of hyperkähler metrics on the K3 surface which collapse to T^3/\mathbb{Z}_2 with bounded curvature away from $n + 8$ points.

We refer to Theorem 6.17 for a more precise statement.

In [26] Gross–Wilson studied hyperkähler metrics on elliptic K3 surfaces with fibres of small size. They considered the generic case when all singular fibres (24 of them) are of Kodaira type I_1 (*i.e.* a pinched torus). The hyperkähler metric is approximated by a semi-flat metric on the locus of the smooth fibres and by a certain (incomplete) explicit hyperkähler metric, the Ooguri–Vafa metric, in the neighbourhood of each singular fibre. As the size of the fibres converges to zero, the K3 surface collapses to a metric on S^2 (the base of the elliptic fibration) with 24 singular points. To the knowledge of the author, besides Gross–Wilson’s work, Theorem 1.1 is the only study of collapsing sequences of hyperkähler metrics on the K3 surface.

Now, one way to make precise Page’s observations in [44] about the Kummer construction for a degenerating family of tori is to consider a gluing construction in which one glues 8 copies of the D_2 ALF space to the \mathbb{Z}_2 quotient of the trivial circle bundle $T^3 \times S^1$ over the flat 3-torus. The proof of Theorem 1.1 is also based on a gluing construction. In order to allow for more general ALF spaces to appear as rescaled limits, the main idea is to replace $T^3 \times S^1$ with an (incomplete) background hyperkähler metric on a *non-trivial* circle bundle over a punctured 3-torus. The tool to construct such a background metric is the Gibbons–Hawking construction of hyperkähler metrics with a triholomorphic S^1 symmetry, *i.e.* an isometric circle action that also preserves the 2-sphere of complex structures compatible with the metric. The S^1 -invariant hyperkähler metrics we seek are explicitly given in terms of a positive harmonic function h on T^3 with prescribed singularities at a finite number of points. For most configurations of punctures the harmonic function h becomes negative somewhere. However, by multiplying h by a small

number $\epsilon > 0$ (which geometrically corresponds to making the circle fibres have small length) it is possible to construct highly collapsed hyperkähler metrics g_ϵ^{gh} outside of an arbitrarily small neighbourhood of the punctures. Furthermore, the construction of this background metric can be made invariant under the action of an involution.

The key observation now is that the asymptotic model of any ALF metric (up to a double cover in the dihedral case) can be written in Gibbons–Hawking coordinates. By choosing the configuration of punctures appropriately it is then possible to glue in copies of ALF spaces to extend the Gibbons–Hawking metric g_ϵ^{gh} to an approximately hyperkähler metric g_ϵ : close to a fixed point of the \mathbb{Z}_2 -action on T^3 we glue in an ALF space of dihedral type (this explains why we need 8 of them in Theorem 1.1); close to a puncture which is not a fixed point of the \mathbb{Z}_2 -action we glue in an ALF space of cyclic type. The Euler characteristic constraint in the statement of Theorem 1.1 is necessary for the resulting 4-manifold to have the same Euler characteristic as the K3 surface, but it can also be reinterpreted as the necessary and sufficient condition for the existence of the harmonic function h in the first place.

The approximate solution g_ϵ is then deformed into an exact hyperkähler metric by means of the Implicit Function Theorem. Since some of the ALF spaces are not biholomorphic to their asymptotic model outside a compact set, it is necessary to set up the problem as a gluing problem for hyperkähler structures, rather than the most standard procedure (as in the classical Kummer construction) of first constructing a complex surface using complex geometry and then solving a complex Monge–Ampère equation on this given complex manifold.

REMARK. At least in some form this “Gibbons–Hawking approximation” of hyperkähler metrics on the K3 surface seems to be known to physicists in the context of the duality between M theory compactified on the K3 surface and Type IIA String theory compactified on T^3/\mathbb{Z}_2 . For example, in [45] Sen discusses the physical interpretation of dihedral ALF spaces thought of as a “superposition” of Taub–NUT spaces and the Atiyah–Hitchin manifold, *cf.* Remark 3.7.

REMARK. One can also wonder what happens when, instead of T^3/\mathbb{Z}_2 , we start from an arbitrary closed flat 3-orbifold with only isolated singularities modelled on $\mathbb{R}^3/\mathbb{Z}_2$. Any such orbifold is the quotient of a closed orientable flat 3-manifold by an orientation-reversing isometric involution with finitely many fixed points. Indeed, let X be a flat 3-orbifold with only m isolated singularities each modelled on $\mathbb{R}^3/\mathbb{Z}_2$. Then there are m disjoint open subsets $\mathcal{U}_j \subset X$ each isometric to a ball centred at the origin in $\mathbb{R}^3/\mathbb{Z}_2$, such that $X^* = X \setminus \bigcup_{j=1}^m \mathcal{U}_j$ is a flat 3-manifold with boundary. X^* is not orientable since each boundary component is a two-sided \mathbb{RP}^2 . By gluing m flat 3-balls to the oriented double-cover of X^* along the m disjoint round 2-spheres forming its boundary,

we define a closed flat orientable 3-manifold \tilde{X} with $X = \tilde{X}/\iota$ for an orientation-reversing isometric involution ι with m isolated fixed points, *cf.* [23, Lemma 1.7]. There are six closed orientable flat 3-manifolds: in the notation of [49, §3.5] they are $\mathcal{G}_1 = T^3$, $\mathcal{G}_i = T^3/\mathbb{Z}_i$ for $i = 2, 3, 4$, $\mathcal{G}_5 = T^3/\mathbb{Z}_6$ and $\mathcal{G}_6 = T^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Luft–Sjerve [36] have shown that only $\mathcal{G}_1, \mathcal{G}_2$ and \mathcal{G}_6 admit an involution with finitely many fixed points (8, 4 and 2 of them, respectively). By working on the 3-torus equivariantly with respect to a finite group action, the Gibbons–Hawking construction yields (incomplete) Ricci-flat metrics on circle bundles over any (punctured) flat 3-manifold \tilde{X} . These metrics are hyperkähler only when $\tilde{X} = T^3$, Kähler if $\tilde{X} = \mathcal{G}_i$ for $i = 2, 3, 4, 5$ and have generic holonomy when $\tilde{X} = \mathcal{G}_6$, since only \mathcal{G}_1 has $b_1 = 3$ and $b_1(\mathcal{G}_i) = 1$ in all other cases except for \mathcal{G}_6 , which has purely torsion first homology. By the result of Luft–Sjerve, only when $\tilde{X} = \mathcal{G}_1, \mathcal{G}_2$ or \mathcal{G}_6 we are able to construct background Ricci-flat metrics that can be extended to complete metrics by gluing in copies of ALF spaces of cyclic and dihedral type. On the other hand, Hitchin [28, Theorem 1] showed that the only Ricci-flat 4-manifolds covered by the K3 surface are the Enriques surfaces (quotients of a K3 surface by an involution without fixed points) with their Kähler Ricci-flat metrics and the quotient of an Enriques surface by an anti-holomorphic involution without fixed points. Carrying out our gluing construction equivariantly with respect to a finite group action then allows us to produce collapsing sequences of Ricci-flat metrics on an Enriques surface (the metrics are Kähler in this case) and its quotient by an anti-holomorphic involution: the collapsed limit is $\mathcal{G}_2/\mathbb{Z}_2$ and $\mathcal{G}_6/\mathbb{Z}_2$, respectively, and ALF gravitational instantons appear as “bubbles”.

We leave aside for future work the question of understanding the relation between the metric degenerations described in this paper and degenerations of a compatible complex structure on the K3 surface. Similarly, it would be very interesting to understand to what extent the collapsing behaviour exhibited in this paper is typical of an arbitrary sequence of Ricci-flat metrics on the K3 surface collapsing to a 3-dimensional limit.

We give instead an application of our gluing construction to the theory of minimal surfaces. It is well known that holomorphic submanifolds of a Kähler manifold minimise volume in their homology class. A classical problem in minimal surface theory is to understand to what extent area minimising surfaces (and more generally stable minimal surfaces) in Kähler manifolds must be (anti)holomorphic. For example, in 1993 Yau asked whether it is possible to classify all stable minimal 2-spheres in a simply connected Kähler Ricci-flat manifold [50, Question 64]. In [39] Micallef showed that every stable minimal surface in a flat 4-torus must be holomorphic for some complex structure compatible with the metric.

For some time there was hope to prove a similar result in the case of the K3 surface endowed with a hyperkähler metric. Eventually, Micallef–Wolfson [38] showed that this is not the case. A simple application of our gluing construction allows us to give an alternative (simpler) counterexample: there exist hyperkähler metrics on the K3 surface that admit a strictly stable minimal sphere which is not holomorphic with respect to any complex structure compatible with the metric, *cf.* Theorem 7.1. In [37, Proposition 5.5] Micallef–Wolfson show that the double cover of the Atiyah–Hitchin manifold, the rotationally symmetric D_1 ALF space, contains a strictly stable minimal 2-sphere Σ with self-intersection -4 . By the adjunction formula Σ cannot be holomorphic with respect to any complex structure compatible with the hyperkähler metric. Using the rotationally symmetric D_1 ALF space as one of the building blocks in our gluing construction allows us to construct (non-holomorphic) stable minimal spheres with self-intersection -4 in the K3 surface. These examples are optimal since Donaldson [17, Proposition 26] (*cf.* also [38, Theorem 5.1]) has shown that in a hyperkähler 4-manifold any compact minimal surface that is not holomorphic with respect to any complex structure and has self-intersection larger than -4 cannot be stable.

Plan of the paper. As we have already mentioned, in this paper we will need to glue hyperkähler structures rather than solving a complex Monge–Ampère equation on a given complex manifold. In Section 2, following Donaldson [18], we explain how to set up the problem of deforming approximately hyperkähler metrics based on the notion of definite triples.

Section 3 is a detailed summary of the theory of ALF spaces: we give precise definitions, describe detailed asymptotics for such metrics and recall the construction and classification of examples.

In Section 4 we use the Gibbons–Hawking ansatz to construct (incomplete) hyperkähler metrics on circle bundles over a punctured 3-torus. In Section 5 we use ALF spaces of cyclic and dihedral type together with the metrics constructed in Section 4 to produce families of approximately hyperkähler metrics. In Section 6 we use analysis to deform these approximate solutions into exact hyperkähler metrics. This is done by means of an Implicit Function Theorem in weighted Hölder spaces. As usual in gluing problems, most of the work goes into showing that the relevant linear operator has no small eigenvalues as $\epsilon \rightarrow 0$ and the geometry degenerates.

Finally, Section 7 contains the proof of Theorem 7.1 about the existence of non-holomorphic strictly stable minimal spheres.

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2. Definite triples and hyperkähler structures on 4-manifolds

The standard approach in the Kummer construction of Kähler Ricci-flat metrics on the K3 surface is to proceed in two steps. First one constructs a complex surface with vanishing first Chern class by blowing up the singularities of a flat orbifold T^4/\mathbb{Z}_2 . On this given complex manifold one then constructs a Kähler Ricci-flat metric by solving a complex Monge-Ampère equation. Some of the building blocks we are going to use in the gluing construction of this paper are not biholomorphic to their asymptotic model outside a compact set. This will force us to adopt a different strategy and glue hyperkähler structures all together. In this initial preliminary section we explain how Donaldson [18] suggested an approach to this problem, based on the notion of definite (or hypersymplectic) triples.

Recall that the space of 2-forms on an oriented 4-dimensional vector space carries a natural non-degenerate bilinear form of signature $(3, 3)$.

Definition 2.1. Let (M^4, μ_0) be an oriented 4-manifold with volume form μ_0 . A *definite triple* is a triple $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ of 2-forms on M such that $\text{Span}(\omega_1, \omega_2, \omega_3)$ is a 3-dimensional positive definite subspace of $\Lambda^2 T_x^* M$ at every point $x \in M$.

Given a triple $\underline{\omega}$ of 2-forms on (M, μ_0) we consider the matrix $Q \in \Gamma(M, \text{Sym}^2(\mathbb{R}^3))$ defined by

$$(2.2) \quad \frac{1}{2} \omega_i \wedge \omega_j = Q_{ij} \mu_0.$$

$\underline{\omega}$ is a definite triple if and only if Q is a positive definite matrix.

To every definite triple $\underline{\omega}$ we associate a volume form $\mu_{\underline{\omega}}$ by

$$(2.3) \quad \mu_{\underline{\omega}} = (\det Q)^{\frac{1}{3}} \mu_0$$

and the new matrix $Q_{\underline{\omega}} = (\det Q)^{-\frac{1}{3}} Q$ which satisfies (2.2) with $\mu_{\underline{\omega}}$ in place of μ_0 . Note that the volume form $\mu_{\underline{\omega}}$ and the matrix $Q_{\underline{\omega}}$ are independent of the choice of volume form μ_0 . We refer to $\mu_{\underline{\omega}}$ and $Q_{\underline{\omega}}$ as the *associated volume form* and *intersection matrix* of the definite triple $\underline{\omega}$.

Now, let (M^4, μ_0) be an oriented 4-dimensional manifold. It is well known that the choice of the 3-dimensional positive definite subspace $\Lambda^2_+ T_x^* M$ of $\Lambda^2 T_x^* M$ for all $x \in M$ is equivalent to the choice of a conformal class on M . Thus every definite triple defines a Riemannian metric $g_{\underline{\omega}}$ by requiring that $\text{Span}(\omega_1, \omega_2, \omega_3)|_x = \Lambda^2_+ T_x^* M$ for all $x \in M$ and $\text{dv}_{g_{\underline{\omega}}} = \mu_{\underline{\omega}}$.

Definition 2.4. A definite triple $\underline{\omega}$ is said

- (i) *closed* if $d\omega_i = 0$ for $i = 1, 2, 3$;
- (ii) an $SU(2)$ -*structure* if $Q_{\underline{\omega}} \equiv \text{id}$;
- (iii) *hyperkähler* if it is both closed and an $SU(2)$ -structure.

The metric $g_{\underline{\omega}}$ associated to a hyperkähler triple is hyperkähler, in the sense that it has holonomy contained in $Sp(1) \simeq SU(2)$.

2.1. The deformation problem. In Section 5 we will construct closed definite triples $\underline{\omega}$ which are approximately hyperkähler, in the sense that the intersection matrix $Q_{\underline{\omega}}$ is close to the identity. We now explain how to formulate the problem of deforming such a triple $\underline{\omega}$ into a hyperkähler structure.

Let $\underline{\omega}$ be a closed definite triple on a 4-manifold M and assume that $\|Q_{\underline{\omega}} - \text{id}\|_{C^0} < \sigma$ for some small $\sigma > 0$. We want to deform $\underline{\omega}$ into a hyperkähler triple, *i.e.* we look for a triple of closed 2-forms $\underline{\eta} = (\eta_1, \eta_2, \eta_3)$ on M such that

$$(2.5) \quad \frac{1}{2} (\omega_i + \eta_i) \wedge (\omega_j + \eta_j) = \delta_{ij} \mu_{\underline{\omega}}.$$

Decompose $\underline{\eta}$ into self-dual and anti-self dual parts $\underline{\eta} = \underline{\eta}^+ + \underline{\eta}^-$ with respect to $g_{\underline{\omega}}$. The self-dual part can be written in terms of a $M_{3 \times 3}(\mathbb{R})$ -valued function A by

$$\eta_i^+ = \sum_{j=1}^3 A_{ij} \omega_j.$$

Denote by $\underline{\eta}^- * \underline{\eta}^-$ the symmetric (3×3) -matrix with entries $(\frac{1}{2} \eta_i^- \wedge \eta_j^-) / \mu_{\underline{\omega}}$. Then we can rewrite (2.5) as

$$(2.6) \quad Q_{\underline{\omega}} + Q_{\underline{\omega}} A^T + A Q_{\underline{\omega}} + A Q_{\underline{\omega}} A^T + \underline{\eta}^- * \underline{\eta}^- = \text{id}.$$

Now, consider the map

$$M_{3 \times 3}(\mathbb{R}) \longrightarrow \text{Sym}^2(\mathbb{R}^3); \quad A \longmapsto Q_{\underline{\omega}} A^T + A Q_{\underline{\omega}} + A Q_{\underline{\omega}} A^T$$

and its differential $A \mapsto Q_{\underline{\omega}} A^T + A Q_{\underline{\omega}}$. Since $Q_{\underline{\omega}}$ is arbitrarily close to the identity, this linear map induces an isomorphism $\text{Sym}^2(\mathbb{R}^3) \rightarrow \text{Sym}^2(\mathbb{R}^3)$ for σ sufficiently small. We can therefore define a smooth function $\mathcal{F}: \text{Sym}^2(\mathbb{R}^3) \rightarrow \text{Sym}^2(\mathbb{R}^3)$ such that $Q_{\underline{\omega}} A^T + A Q_{\underline{\omega}} + A Q_{\underline{\omega}} A^T = S$ if and only if $A = \mathcal{F}(S)$.

REMARK. When $\underline{\omega}$ is hyperkähler (thus $Q_{\underline{\omega}} = \text{id}$) the kernel of $A \mapsto Q_{\underline{\omega}} A^T + A Q_{\underline{\omega}}$ corresponds to infinitesimal hyperkähler rotations.

Hence we reformulate (2.6) as

$$(2.7) \quad \underline{\eta}^+ = \mathcal{F} \left((\text{id} - Q_{\underline{\omega}}) - \underline{\eta}^- * \underline{\eta}^- \right).$$

Now, let $\mathcal{H}_{\underline{\omega}}^+$ be the space of self-dual harmonic 2-forms with respect to $g_{\underline{\omega}}$. If a solution of (2.6) exists on a compact manifold M then M must be either a 4-torus or a K3 surface with the standard orientation and therefore $\mathcal{H}_{\underline{\omega}}^+$ is 3-dimensional. Since $\omega_1, \omega_2, \omega_3$ are closed and self-dual (therefore harmonic) and linearly independent (since $\underline{\omega}$ is a definite triple) we deduce that $\mathcal{H}_{\underline{\omega}}^+$ consist of constant linear combinations of $\omega_1, \omega_2, \omega_3$.

By Hodge theory with respect to $g_{\underline{\omega}}$ we can finally rewrite (2.7) as the *elliptic* equation

$$(2.8) \quad d^+ \underline{a} + \underline{\zeta} = \mathcal{F} \left((\text{id} - Q_{\underline{\omega}}) - \underline{\eta}^- * \underline{\eta}^- \right), \quad d^* \underline{a} = 0,$$

for a triple \underline{a} of 1-forms on M and a triple $\underline{\zeta} \in \mathcal{H}_{\underline{\omega}}^+ \otimes \mathbb{R}^3$. Here $2d^+ a = da + *da$ is the self-dual part of da .

REMARK. Note that in general it is necessary to deform the cohomology classes of $\omega_1, \omega_2, \omega_3$ since every hyperkähler triple must satisfy

$$\frac{1}{2} \langle [\omega_i] \cup [\omega_j], [M] \rangle = \delta_{ij} \text{Vol}_{g_{\underline{\omega}}}(M).$$

The linearisation of (2.8) is the operator $(\Omega^1(M) \oplus \mathcal{H}_{\underline{\omega}}^+) \otimes \mathbb{R}^3 \rightarrow (\Omega^0(M) \oplus \Omega^+(M)) \otimes \mathbb{R}^3$ defined by

$$(2.9) \quad (\underline{a}, \underline{\zeta}) \longmapsto (d^* \underline{a}, d^+ \underline{a} + \underline{\zeta}).$$

Note that on a compact manifold this operator is always surjective with kernel consisting of harmonic 1-forms since the images of d and $*d$ and \mathcal{H}^+ are L^2 -orthogonal.

3. ALF gravitational instantons

In this section we collect known results about gravitational instantons of type ALF, with an emphasis on their asymptotic geometry. ALF gravitational instantons will appear as local models for the geometry of high curvature regions in sequences of hyperkähler metrics on K3 collapsing to a 3-dimensional limit.

A *gravitational instanton* is a complete hyperkähler 4-manifold (M, g) with decaying Riemannian curvature at infinity. The minimum requirement (automatically satisfied for rescaled limits of Einstein metrics on 4-manifolds with bounded Euler characteristic by the Chern–Gauss–Bonnet formula) is that (M, g) has finite *energy* $\|\text{Rm}\|_{L^2}$. In order to

say something about the structure of gravitational instantons it has often been necessary to strengthen this finite energy assumption to *faster than quadratic curvature decay* $|\text{Rm}| = O(r^{-2-\epsilon})$, $\epsilon > 0$ (or a slightly weaker finite weighted energy assumption). Note however that there are examples of gravitational instantons which do *not* satisfy this stronger decay assumption [27, Theorem 1.5].

Since hyperkähler manifolds are in particular Ricci-flat, gravitational instantons have only one end and constrained *volume growth*: the volume of a geodesic ball of radius r can grow at most as r^4 and at least linearly. An initial rough classification of gravitational instantons can be given in terms of their volume growth. The gravitational instantons of maximal volume growth are the ALE spaces classified by Kronheimer [32] following earlier work of Eguchi–Hanson, Gibbons–Hawking and Hitchin. Under the assumption of faster than quadratic curvature decay (or a slightly weaker finite weighted energy assumption) Minerbe [41, Theorem 0.1] has shown that if we assume $\text{Vol}(B_r(p)) = O(r^a)$ for some $3 \leq a < 4$ and all p , then $a = 3$. Minerbe also described the asymptotic geometry of gravitational instantons of cubic volume growth and faster than quadratic curvature decay: they are all ALF spaces, in the following sense.

Definition 3.1. A gravitational instanton (M, g) is called *ALF* if there exists a compact set $K \subset M$, $R > 0$ and a finite group $\Gamma < O(3)$ acting freely on \mathbb{S}^2 such that $M \setminus K$ is the total space of a circle fibration $\pi: M \setminus K \rightarrow (\mathbb{R}^3 \setminus B_R)/\Gamma$ and the metric is asymptotically a Riemannian submersion

$$(3.2) \quad g = \pi^* g_{\mathbb{R}^3/\Gamma} + \theta^2 + O(r^{-\tau})$$

for a connection θ on π and some $\tau > 0$. There are two possibilities for the finite group Γ : if $\Gamma = \text{id}$ we say that M is an ALF gravitational instanton of *cyclic type*; if $\Gamma = \mathbb{Z}_2$ we say that M is an ALF gravitational instanton of *dihedral type*.

We are interested in refining these asymptotics. In order to describe a more precise model for the end of an ALF gravitational instanton it is necessary to recall the explicit construction of 4-dimensional hyperkähler metrics with a triholomorphic circle action known as the Gibbons–Hawking ansatz. We will use this same ansatz later in the paper to construct (incomplete) hyperkähler metrics on circle bundles over a punctured 3-torus.

3.1. The Gibbons–Hawking ansatz. The Gibbons–Hawking ansatz describes 4-dimensional hyperkähler metrics with an isometric S^1 -action that also preserves the whole hyperkähler structure. Such an S^1 -action is therefore called *triholomorphic*.

Let U be an open set of \mathbb{R}^3 and $\pi: P \rightarrow U$ be a principal $U(1)$ -bundle. Suppose that there exists a positive harmonic function h on U

such that $*dh$ is the curvature $d\theta$ of a connection θ on P . Then

$$(3.3a) \quad g^{\text{gh}} = h \pi^* g_{\mathbb{R}^3} + h^{-1} \theta^2$$

is a hyperkähler metric. Indeed, we can exhibit an explicit hyperkähler triple $\underline{\omega}^{\text{gh}}$ that induces the metric g^{gh} . Fix coordinates (x_1, x_2, x_3) on $U \subset \mathbb{R}^3$ and define

$$(3.3b) \quad \omega_i^{\text{gh}} = dx_i \wedge \theta + h dx_j \wedge dx_k.$$

Here and in the rest of the paper we use the convention that for every $i = 1, 2, 3$ the indices j, k are chosen so that $\epsilon_{ijk} = 1$. One can check explicitly that $\underline{\omega}^{\text{gh}}$ defines an $SU(2)$ -structure and it induces the Riemannian metric g^{gh} . Moreover, the requirement that $\underline{\omega}^{\text{gh}}$ is also closed is equivalent to the abelian *monopole equation*

$$(3.4) \quad *dh = d\theta$$

The fibre-wise circle action on P preserves $\underline{\omega}^{\text{gh}}$ and π is nothing but a hyperkähler moment map for this action. Conversely, every 4-dimensional hyperkähler metric with a triholomorphic circle action is described by (3.3).

The basic example of the Gibbons–Hawking construction is given in terms of so-called Dirac monopoles on \mathbb{R}^3 . Fix a set of distinct points p_1, \dots, p_n in \mathbb{R}^3 and consider the harmonic function

$$h = \lambda + \sum_{j=1}^n \frac{k_j}{2|x - p_j|},$$

where $\lambda > 0$ and k_1, \dots, k_n are constants. Since $\mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ has non-trivial second homology, we must require $k_j \in \mathbb{Z}$ for all j in order to be able to solve (3.4). If these integrality constraints are satisfied then $*dh$ defines the curvature $d\theta$ of a connection θ (unique up to gauge transformations) on a principal $U(1)$ -bundle P over $\mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ which restricts to the principal $U(1)$ -bundle associated with the line bundle $\mathcal{O}(k_j) \rightarrow S^2$ on a small punctured neighbourhood of p_j . The pair (h, θ) is a solution of (3.4) which we call a *Dirac monopole* with singularities at p_1, \dots, p_n .

The Gibbons–Hawking ansatz (3.3) associates a hyperkähler metric g^{gh} to every Dirac monopole on the open set where $h > 0$. When $k_j > 0$ then g^{gh} is certainly defined on the restriction of P to a small punctured neighbourhood of p_j . By a change of variables one can check that g^{gh} can be extended to a smooth (orbifold) metric modelled on $\mathbb{C}^2/\mathbb{Z}_{k_j}$ by adding a single point. In particular g^{gh} is a complete metric whenever $\lambda \geq 0$ and $k_j = 1$ for all $j = 1, \dots, n$. One can check that g^{gh} is an ALE metric when $\lambda = 0$ and an ALF metric of cyclic type when $\lambda > 0$. Note also that when $\lambda > 0$ we can always rescale the metric so that $\lambda = 1$.

3.2. Families of ALF gravitational instantons. We are now going to use the Gibbons–Hawking ansatz to define a refined asymptotic model for ALF gravitational instantons.

Let H^k be the total space of the principal $U(1)$ –bundle associated with the line bundle $\mathcal{O}(k)$ over S^2 radially extended to $\mathbb{R}^3 \setminus B_R$ for any $R > 0$. θ_k will denote the $SO(3)$ –invariant connection on H^k . The Gibbons–Hawking ansatz (3.3) yields a hyperkähler metric

$$(3.5) \quad g_k = \left(1 + \frac{k}{2\rho}\right) (d\rho^2 + \rho^2 g_{S^2}) + \left(1 + \frac{k}{2\rho}\right)^{-1} \theta_k^2$$

on H^k for all $k \in \mathbb{Z}$. Here ρ is a radial function on \mathbb{R}^3 . We denote by $\underline{\omega}_k$ the associated hyperkähler triple defined by (3.3b). Note that we could replace the harmonic function $1 + \frac{k}{2\rho}$ with $\lambda + \frac{k}{2\rho}$ for any $\lambda > 0$ but we can always reduce to the case $\lambda = 1$ by scaling.

Finally, on H^{2k} we consider the \mathbb{Z}_2 –action which is defined as the simultaneous standard involutions on the base \mathbb{R}^3 and the fibre. Here the involution on the fibre $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the one induced by the standard involution on the universal cover \mathbb{R} . Throughout the paper we refer to this as the standard involution of S^1 .

Definition 3.6. Let (M^4, g) be an ALF gravitational instanton of cyclic type. By scaling assume that the length of the circle fibres at infinity is 1.

- (i) We say that M is of type A_k for some $k \geq -1$ if there exists a compact set $K \subset M$, $R > 0$ and a diffeomorphism $\phi: H^{k+1} \rightarrow M \setminus K$ such that

$$|\nabla_{g_{k+1}}^l (g_{k+1} - \phi^* g)|_{g_{k+1}} = O(r^{-3-l})$$

for every $l \geq 0$.

- (ii) We say that M is of type D_m for some $m \geq 0$ if there exists a compact set $K \subset M$, $R > 0$ and a double cover $\phi: H^{2m-4} \rightarrow M \setminus K$ such that the group \mathbb{Z}_2 of deck transformations is generated by the standard involution on H^{2m-4} and

$$|\nabla_{g_{2m-4}}^l (g_{2m-4} - \phi^* g)|_{g_{2m-4}} = O(r^{-3-l})$$

for every $l \geq 0$.

By [11, Theorem 1.1] every ALF gravitational instanton is either of type A_k for some $k \geq -1$ or D_m for some $m \geq 0$ (the constraints $k \geq -1$ and $m \geq 0$ follow from [40, Theorem 0.1] and [8, Corollary 3.2], respectively).

REMARK. In the cyclic case Chen–Chen [11, Theorem 1.1] have a worse decay $O(r^{-2})$ of an ALF metric of type A_k to g_{k+1} . However, from the explicit description of cyclic ALF gravitational instantons as multi-Taub–NUT spaces, as we will recall below, it is clear that one can

always change coordinates by a translation on \mathbb{R}^3 so that every A_k ALF space satisfies the stronger decay stated in Definition 3.6 (i).

3.2.1. ALF spaces of cyclic type. We saw that gravitational instantons of type A_k can be constructed from Dirac monopoles on \mathbb{R}^3 with $k + 1$ singularities via the Gibbons–Hawking ansatz. These are usually called *multi-Taub–NUT* metrics. The case $k = 0$ is the Taub–NUT metric on \mathbb{R}^4 and $k = -1$ is $\mathbb{R}^3 \times \mathbb{S}^1$ with its flat metric. Minerbe [42, Theorem 0.2] has shown that every ALF space of cyclic type must be isometric to a multi-Taub–NUT metric.

From their explicit description one can easily compute basic information about cyclic ALF spaces: the fundamental group $\pi_1(M)$, the second Betti number $b_2(M)$, the Euler characteristic and the dimension of the moduli space \mathcal{M} of A_k metrics:

k	$\pi_1(M)$	$b_2(M)$	$\chi(M)$	$\dim(\mathcal{M})$
-1	\mathbb{Z}	0	0	0
$k > -1$	1	k	$k + 1$	$3k$

Here we assume that the asymptotic length of the circle fibre is normalised to be 1 so that $\dim(\mathcal{M})$ does not include rescalings.

3.2.2. ALF spaces of dihedral type. ALF metrics of dihedral type are not globally given by the Gibbons–Hawking construction and in most cases are not explicit. A number of different constructions have appeared over the past 30 years, but only recently Chen–Chen [11, Theorem 1.2] have shown that all these constructions yield equivalent families of ALF metrics. We distinguish the cases $m = 0, 1, 2$ and $m \geq 3$.

$m = 0$: The D_0 ALF manifold is the moduli space of centred charge 2 monopoles on \mathbb{R}^3 with its natural L^2 -metric, known as the *Atiyah–Hitchin manifold*. The metric admits a cohomogeneity one isometric action of $SU(2)$ and is explicitly given in terms of elliptic integrals [5, Chapter 11]. The Atiyah–Hitchin manifold is diffeomorphic to the complement of a Veronese \mathbb{RP}^2 in \mathbb{S}^4 and therefore it retracts to \mathbb{RP}^2 . The Atiyah–Hitchin metric does not admit deformations as a D_0 ALF metric except for scaling.

$m = 1$: The double cover of the Atiyah–Hitchin manifold is a D_1 ALF space. As a smooth manifold it is diffeomorphic to the complement of \mathbb{RP}^2 in \mathbb{CP}^2 , or equivalently to the total space of $\mathcal{O}(-4)$ over S^2 . Exploiting the rotational invariance of the metric it can be shown [37, Proposition 5.5] that the 2-sphere in the interior is a strictly stable minimal sphere which is not holomorphic with respect to any complex structure compatible with the metric, a fact that will play a crucial role in the proof of Theorem 7.1. This rotationally invariant D_1 ALF metric admits a 3-dimensional family of D_1 ALF deformations, sometimes referred to as the *Dancer metrics*.

REMARK. The fact that the double cover of the Atiyah–Hitchin manifold admits a 3-parameter family of D_1 ALF deformations can also

be shown using methods similar to the ones developed in this paper. Indeed, it is known [30, §5.4] that the rotationally invariant D_1 ALF metric admits a unique L^2 -integrable (in fact, exponentially decaying) anti-self-dual harmonic form η . This form yields a 3-dimensional space of infinitesimal hyperkähler deformations and an extension of the analysis needed for the proof of Theorem 6.17 could be used to integrate these infinitesimal deformations to genuine D_1 ALF metrics. In fact Dancer [16] has constructed a 3-parameter family of hyperkähler deformations of the rotationally invariant D_1 ALF metric using Nahm's equations and hyperkähler quotient techniques: there exists a hyperkähler 8-manifold \mathcal{N} constructed as a moduli space of solutions to Nahm's equations which admits a triholomorphic $U(1)$ -action. Denote by $\mu: \mathcal{N} \rightarrow \mathbb{R}^3$ the corresponding hyperkähler moment map. Dancer identifies the rotationally symmetric D_1 ALF metric with the hyperkähler quotient $\mu^{-1}(0)/U(1)$. By varying the level set of the moment map he then obtains a 3-parameter family of hyperkähler deformations of the Atiyah–Hitchin metric. By a general formula for the infinitesimal deformation of the symplectic form of a symplectic quotient corresponding to varying the level set of the moment map [21], the infinitesimal deformations of the Atiyah–Hitchin metric corresponding to Dancer's metrics coincide with those determined by the L^2 harmonic form η , which is interpreted in this context as the curvature of the natural hyperholomorphic connection on the $U(1)$ -bundle $\mu^{-1}(0) \rightarrow \mu^{-1}(0)/U(1)$ induced by the Levi–Civita connection of \mathcal{N} .

$m = 2$: D_2 ALF metrics were constructed by Hitchin [29, §7] using twistor methods, by Dancer [15, pp. 88-89] using a hyperkähler quotient construction and by Biquard–Minerbe [8, Theorem 2.4] using a non-compact version of the Kummer construction: one considers the quotient of $\mathbb{R}^3 \times \mathbb{S}^1$ by an involution and resolves the two singularities gluing in copies of the Eguchi–Hanson metric.

REMARK. Biquard–Minerbe [8, Theorem 2.4] use singular perturbation methods to solve a complex Monge–Ampère equation on the minimal resolution of $(\mathbb{R}^3 \times \mathbb{S}^1)/\mathbb{Z}_2$. Using the more general approach adopted in this paper to glue hyperkähler structures one could extend their construction to recover a 6-dimensional family of D_2 ALF metrics.

$m \geq 3$: D_m ALF metrics (for all $m \geq 1$) appeared in the work of Dancer [15, 16] on finite dimensional hyperkähler quotients of moduli spaces of solutions to Nahm's equations and in the work of Cherkis–Kapustin [14] on moduli spaces of singular monopoles on \mathbb{R}^3 . A construction based on twistor methods and the generalised Legendre transform was given by Cherkis–Hitchin [13]. In

the case $m \geq 3$ a more transparent construction due to Biquard–Minerbe [8, Theorem 2.5] yields D_m ALF metrics by desingularising the quotient of the Taub–NUT metric by the binary dihedral group \mathcal{D}_m of order $4(m-2)$ using ALE dihedral spaces. Using complex Monge–Ampère methods Auvray [6] has then constructed $3m$ -dimensional families of D_m ALF metrics on the smooth 4-manifold underlying the minimal resolution of $\mathbb{C}^2/\mathcal{D}_m$.

REMARK 3.7. The gluing construction presented in this paper could be extended to the non-compact setting to yield yet another construction of dihedral ALF metrics. Indeed, as suggested by Sen [45], one considers a Gibbons–Hawking metric obtained from the harmonic function

$$h = \lambda - \frac{2}{|x|} + \sum_{i=1}^m \frac{1}{2|x-x_i|} + \frac{1}{2|x+x_i|}$$

for m distinct points $x_1, \dots, x_m \in \mathbb{R}^3 \setminus \{0\}$. Observe that for $\lambda > 0$ sufficiently large $h > 0$ outside an arbitrarily small neighbourhood of the origin. Since the configuration of punctures is invariant under the standard involution of \mathbb{R}^3 , this (incomplete) metric descends to a hyperkähler metric on a \mathbb{Z}_2 quotient. For λ sufficiently large one can then complete this metric by gluing in a copy of the D_0 ALF space close to the origin. This approximate solution could then be deformed to an exact hyperkähler metric in a way similar to the proof of Theorem 6.17.

We summarise some of the properties of a D_m ALF gravitational instanton M in the following table:

m	$\pi_1(M)$	$b_2(M)$	$\chi(M)$	$\dim(\mathcal{M})$
0	\mathbb{Z}_2	0	1	0
$m > 0$	1	m	$m+1$	$3m$

As in the cyclic case, \mathcal{M} is the moduli space of D_m ALF metrics modulo scaling.

4. Gibbons–Hawking ansatz on a punctured 3-torus

In this section we use the Gibbons–Hawking ansatz (3.3) to construct families $\{g_\epsilon^{\text{gh}}\}_{\epsilon>0}$ of (incomplete) hyperkähler metrics on circle bundles over a punctured 3-torus modulo an involution. The parameter $\epsilon > 0$ essentially determines the length of the circle fibres. As $\epsilon \rightarrow 0$ the metric g_ϵ^{gh} collapses to the flat orbifold T^3/\mathbb{Z}_2 with bounded curvature away from the punctures. The family $\{g_\epsilon^{\text{gh}}\}_{\epsilon>0}$ will serve as the model for a family of hyperkähler metrics on the K3 surface collapsing to a 3-dimensional limit in the region where the collapsing occurs with bounded curvature. In the next section we will use ALF gravitational instantons as models for high curvature regions to extend the metric g_ϵ^{gh} to a complete almost hyperkähler metric.

4.1. Dirac monopoles on a punctured torus. Let $\mathbb{T} = \mathbb{R}^3/\Lambda$ be a 3-torus for some lattice $\Lambda \simeq \mathbb{Z}^3$. Endow \mathbb{T} with a flat metric $g_{\mathbb{T}}$.

Let $\tau: \mathbb{T} \rightarrow \mathbb{T}$ be the standard involution $x \mapsto -x$ on \mathbb{T} and denote by q_1, \dots, q_8 its fixed points. For each $j = 1, \dots, 8$ choose a non-negative integer m_j . Fix a τ -symmetric configuration of a further $2n$ distinct points $p_1, \tau(p_1), \dots, p_n, \tau(p_n)$. Sometimes we will use the notation $-p_i$ for $\tau(p_i)$. Denote by \mathbb{T}^* the punctured torus

$$\mathbb{T}^* = \mathbb{T} \setminus \{q_1, \dots, q_8, p_1, \tau(p_1), \dots, p_n, \tau(p_n)\}.$$

Finally choose integer weights $k_1, \dots, k_n > 0$ and assume the following *balancing condition* holds:

$$(4.1) \quad \sum_{j=1}^8 m_j + \sum_{i=1}^n k_i = 16.$$

In particular, $n \leq \sum_{i=1}^n k_i \leq 16$.

For each $j = 1, \dots, 8$ let ρ_j denote the distance function from the point q_j with respect to $g_{\mathbb{T}}$. Similarly, by abuse of notation we let ρ_i denote the distance function from $\pm p_i$ in \mathbb{T}/τ . By restricting the branched double cover $\mathbb{T} \rightarrow \mathbb{T}/\tau$ to a sufficiently small ball centred at $\pm p_i$ in \mathbb{T}/τ we will also regard ρ_i as the distance function on \mathbb{T} from the point p_i or $\tau(p_i)$.

We look for a Dirac monopole (h, θ) on \mathbb{T}^* with the following singular behaviour: h is a harmonic function on \mathbb{T}^* with prescribed singularities at the punctures

$$(4.2) \quad h \sim \frac{2m_j - 4}{2\rho_j} \text{ as } \rho_j \rightarrow 0, \quad h \sim \frac{k_i}{2\rho_i} \text{ as } \rho_i \rightarrow 0.$$

Proposition 4.3. *Assume the balancing condition (4.1) is satisfied.*

- (i) *There exists a harmonic function h on \mathbb{T}^* with prescribed singular behaviour (4.2) such that $*_{g_{\mathbb{T}}} dh$ is the curvature $d\theta$ of a connection θ on some principal $U(1)$ -bundle $P \rightarrow \mathbb{T}^*$.*
- (ii) *The moduli space of Dirac monopoles (h, θ) on P is isomorphic to $\mathbb{R} \times \hat{\mathbb{T}}$, where $\hat{\mathbb{T}}$ is the dual torus parametrising flat $U(1)$ -connections on \mathbb{T} .*
- (iii) *The involution τ lifts to the involution $\tilde{\tau}$ of the $U(1)$ -bundle P which acts simultaneously as τ on \mathbb{T}^* and as the standard involution on the circle fibres.*

Proof. The necessary and sufficient condition for the existence of the harmonic function h is

$$(4.4) \quad \sum_{j=1}^8 (2m_j - 4) + 2 \sum_{i=1}^n k_i = \frac{1}{2\pi} \lim_{\sigma \rightarrow 0} \int_{\partial \mathbb{T}_{\sigma}} *dh = 0,$$

where \mathbb{T}_{σ} denotes the complement of the union of small balls of radius σ centred at the punctures. Thus if (4.1) is satisfied, a harmonic function

h with the singular behaviour (4.2) does indeed exist and is unique up to the addition of a constant.

By Lefschetz–Poincaré duality $H_2(\mathbb{T}^*) \simeq H_c^1(\mathbb{T}^*)$. The latter group sits in a long exact sequence

$$0 \rightarrow H^0(\mathbb{T}) \rightarrow \mathbb{Z}^{2n+8} \rightarrow H_c^1(\mathbb{T}^*) \rightarrow H^1(\mathbb{T}) \rightarrow 0,$$

where \mathbb{Z}^{2n+8} is generated by the $2n + 8$ punctures and we have used the isomorphism $H^1(\mathbb{T}^*) \simeq H^1(\mathbb{T})$ given by the reduced Mayer–Vietoris sequence. Thus $H_2(\mathbb{T}^*)$ is $(2n + 10)$ –dimensional and maps onto $H_2(\mathbb{T})$ with kernel spanned by the classes of $2n + 8$ spheres centred at the punctures. Note that the sum of these $2n + 8$ homology classes vanishes.

Because of (4.2), $2n + 7$ of the $2n + 10$ integrality constraints on $\frac{i}{2\pi} *dh$ to represent the first Chern class of a line bundle are automatically satisfied since we chose $2m_j - 4, k_i \in \mathbb{Z}$. The remaining 3 constraints can be reinterpreted in terms of the position of the punctures following the arguments in the proof of [9, Proposition 3.5]:

$$\sum_{j=1}^8 (2m_j - 4) q_j + \sum_{i=1}^n k_i (p_i + \tau(p_i)) \in \Lambda.$$

Since the points q_j belong to the half-lattice $\frac{1}{2}\Lambda$ this condition is automatically satisfied.

We have therefore proved the existence of a principal $U(1)$ bundle $P \rightarrow \mathbb{T}^*$ endowed with a connection θ with curvature $*dh$. Since by the reduced Mayer–Vietoris sequence $H^1(\mathbb{T}^*) \simeq H^1(\mathbb{T}) \neq 0$, θ is uniquely determined up to a flat connection, *i.e.* a point of the dual torus $\hat{\mathbb{T}}$. This concludes the proof of (i) and (ii).

By uniqueness up to the addition of a constant the harmonic function h is τ –invariant and therefore can be thought of as defined on \mathbb{T}^*/τ . Since $\tau^>(*dh) = -*dh$, we can lift τ (uniquely up to gauge transformations) to an involution $\tilde{\tau}$ of the circle bundle P by requiring that $\tilde{\tau}$ acts simultaneously as τ on \mathbb{T}^* and as the standard involution on the circle fibres. q.e.d.

4.2. Collapsed S^1 –invariant hyperkähler metrics. Fix once and for all a Dirac monopole (h, θ) amongst the ones produced by Proposition 4.3. Via the Gibbons–Hawking ansatz (3.3) we now use (h, θ) to construct (incomplete) hyperkähler metrics with a triholomorphic circle action with orbits of small length.

Fix a (small) positive number $\epsilon > 0$ and define

$$(4.5) \quad h_\epsilon = 1 + \epsilon h.$$

The Gibbons–Hawking ansatz (3.3) with potential $\epsilon^{-1}h_\epsilon$ yields a hyperkähler structure on $P|_{\mathcal{U}_\epsilon}$, where \mathcal{U}_ϵ is the open set of \mathbb{T}^* where $h_\epsilon > 0$.

After multiplying the metric by ϵ to normalise the diameter, the hyperkähler triple $\underline{\omega}_\epsilon^{\text{gh}}$ and the induced metric g_ϵ^{gh} are

$$(4.6) \quad \omega_{\epsilon,i}^{\text{gh}} = \epsilon \theta_i \wedge \theta + h_\epsilon \theta_j \wedge \theta_k, \quad g_\epsilon^{\text{gh}} = h_\epsilon \pi^* g_{\mathbb{T}} + \epsilon^2 h_\epsilon^{-1} \theta^2.$$

Here $(\theta_1, \theta_2, \theta_3)$ is a triple of closed 1-forms on \mathbb{T} such that $g_{\mathbb{T}} = \theta_1^2 + \theta_2^2 + \theta_3^2$. Note that the hyperkähler structure $\underline{\omega}_\epsilon^{\text{gh}}$ is $\tilde{\tau}$ -invariant and therefore defines an induced hyperkähler structure on the quotient $M_\epsilon^{\text{gh}} = (P|_{\mathcal{U}_\epsilon})/\tilde{\tau}$. For ease of notation, we will denote the induced hyperkähler triple and metric on M_ϵ^{gh} with the same symbols.

In the rest of the section we study the properties of the hyperkähler manifold M_ϵ^{gh} . We aim to (i) study the local structure of the metric g_ϵ^{gh} close to the punctures, (ii) determine the set where $h_\epsilon > 0$, and (iii) understand the limit of g_ϵ^{gh} as $\epsilon \rightarrow 0$.

The following asymptotic expansions for the harmonic function h_ϵ close to the punctures are standard. In the case of a fixed point of the involution τ the improved decay follows from the fact that linear harmonic functions on \mathbb{R}^3 are not \mathbb{Z}_2 -invariant.

Lemma 4.7. *There exists $0 < \rho_0 < \frac{1}{4} \text{inj} g_{\mathbb{T}}$ such that the balls $B_{2\rho_0}(q_j)$, $j = 1, \dots, 8$, and $B_{2\rho_0}(\pm p_i)$, $i = 1, \dots, n$, in \mathbb{T} are all disjoint and such that the following holds.*

(i) *For $j = 1, \dots, 8$ there exists $\lambda_j \in \mathbb{R}$ such that in $B_{2\rho_0}(q_j)$*

$$h_\epsilon = (1 + \epsilon \lambda_j) + \frac{\epsilon(m_j - 2)}{\rho_j} + O(\epsilon \rho_j^2).$$

(ii) *For each $i = 1, \dots, n$ there exists $\lambda_i \in \mathbb{R}$ and a linear function ℓ_i on \mathbb{R}^3 with $|\ell_i| \leq C\rho$ such that in $B_{2\rho_0}(\pm p_i)$*

$$h_\epsilon = (1 + \epsilon \lambda_i) + \frac{\epsilon k_i}{2\rho_i} + \epsilon \ell_i + O(\epsilon \rho_i^2).$$

Moreover, $\rho_0, \lambda_j, \lambda_i, \ell_i$ depend continuously on the position of the punctures p_1, \dots, p_n and on the flat metric $g_{\mathbb{T}}$ and $f = O(\epsilon \rho^2)$ means that there exists a constant C depending continuously on these data such that $|\nabla^k f| \leq C\epsilon \rho^{2-k}$ for $k = 0, 1, 2$.

Since we chose $k_i > 0$ for $i = 1, \dots, n$ certainly a punctured neighbourhood of $\pm p_i$ is contained in the set \mathcal{U}_ϵ where $h_\epsilon > 0$. As already mentioned, the Gibbons–Hawking metric can be extended by adding a single point to a smooth orbifold metric modelled on $\mathbb{C}^2/\mathbb{Z}_{k_i}$. The obvious way to smooth out such an orbifold singularity is to replace the “multiplicity” k_i point p_i with k_i points each with weight 1. Then the Gibbons–Hawking ansatz yields a smooth metric that is modelled on a rescaled Taub–NUT space in a neighbourhood of p_i . However we prefer to leave the freedom to choose $k_i > 1$ so that we can consider configurations of punctures that “degenerate” as $\epsilon \rightarrow 0$ and see an A_{k_i-1} ALF space appearing as a rescaled limit.

REMARK. One could also consider (but we will not pursue this in the paper) more general degenerating families of punctures with various clusters of points coalescing at different rates as $\epsilon \rightarrow 0$. One would expect “bubble trees” of ALF and ALE spaces appearing as rescaled limits in this case, cf. [2, Remark 5.2].

Next, we consider the structure of g_ϵ^{gh} in a neighbourhood of q_j . By Lemma 4.7 (i) certainly h_ϵ is positive in a punctured neighbourhood of q_j whenever $m_j > 2$. In this case the Gibbons–Hawking metric on M_ϵ^{gh} can be extended to a smooth orbifold metric with a singularity of the form $\mathbb{C}^2/\mathcal{D}_{m_j}$, where \mathcal{D}_{m_j} is the binary dihedral group of order $4(m_j - 2)$. In contrast with the previous case, there is no explicit way to remove this singularity, but for *fixed* $\epsilon > 0$ one can imagine using the methods of [8, §2.4] to resolve this singularity by gluing in a rescaled ALE dihedral space. However, we are interested in the limit $\epsilon \rightarrow 0$ and in the next section we will directly glue in a D_{m_j} ALF space to resolve this singularity.

Similarly, when $m_j = 2$ one can choose ϵ sufficiently small so that $1 + \epsilon\lambda_j > 0$. Note that in this case h_ϵ and the $U(1)$ -bundle P are well defined at q_j . After quotienting by $\tilde{\tau}$, the Gibbons–Hawking metric g_ϵ^{gh} becomes an orbifold metric modelled on $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$. As before, the two orbifold singularities could be resolved by (i) fixing $\epsilon > 0$ and gluing in two copies of the Eguchi–Hanson metric, or (ii) letting $\epsilon \rightarrow 0$ and gluing in a single copy of a D_2 ALF metric. We will follow the second approach.

REMARK 4.8. Note that if $m_j = 2$ for all $j = 1, \dots, 8$ then $n = 0$ by (4.1) and the bundle P extends over every puncture: in fact P is a 4-torus and our construction reduces to the usual Kummer construction along a family of 4-tori collapsing to a 3-dimensional torus. This is the case considered by Page in [44].

It remains to study the case when $m_j = 0, 1$ for some j . Assume this is the case for $j = 1, \dots, k$ for some $1 \leq k \leq 8$. Then $k = 0$ if and only if $m_j \geq 2$ for all $j = 1, \dots, 8$. Since $\sum_{j=1}^8 m_j = 16 - \sum_{i=1}^n k_i \leq 16 - n$, we must have $k \geq 1$ as soon as $n \geq 1$ or $n = 0$ and $m_j > 2$ for some j , *i.e.* in every case except for the usual Kummer construction. The case $m_j = 0, 1$ is “bad” in the sense that $h_\epsilon \rightarrow -\infty$ as $\rho_j \rightarrow 0$.

Lemma 4.9. *There exists $\epsilon_0 > 0$ depending continuously on p_1, \dots, p_n and $g_{\mathbb{T}}$ such that for every $\epsilon < \epsilon_0$ we have $h_\epsilon > \frac{1}{2}$ on the complement of $\bigcup_{j=1}^k B_{8\epsilon}(q_j)$.*

Proof. Restrict attention to the ball $B_{2\rho_0}(q_j)$. First note that $1 + \frac{\epsilon(m_j-2)}{\rho} \geq 1 - \frac{2\epsilon}{\rho} = \frac{3}{4}$ for $\rho = 8\epsilon$. Now choose $\epsilon_0 > 0$ so that $\epsilon(\lambda_j + C\frac{\epsilon^2}{64}) \leq \frac{1}{4}$ for all $\epsilon \leq \epsilon_0$. Here λ_j, C are the constants of Lemma 4.7 (i). We conclude that $h_\epsilon > \frac{1}{2}$ on $\partial B_{8\epsilon}(q_j)$ for $\epsilon < \epsilon_0$. Since h_ϵ blows up to

$+\infty$ at the punctures $q_{k+1}, \dots, q_8, \pm p_1, \dots, \pm p_n$ the maximum principle completes the proof. q.e.d.

In other words, by choosing ϵ small enough we can assume that $h_\epsilon > 0$ outside an arbitrarily small neighbourhood of the points q_1, \dots, q_k where $m_j = 0, 1$.

Finally, we consider the limit $\epsilon \rightarrow 0$.

Lemma 4.10. *As $\epsilon \rightarrow 0$ the harmonic function h_ϵ converges to the constant function 1. More precisely, given $k \geq 0$ and $\alpha \in (0, 1)$ fix $\beta > 0$ with $\beta^{-1} > k + 1 + \alpha$. Then $\|h_\epsilon - 1\|_{C^{k, \alpha}} \rightarrow 0$ as $\epsilon \rightarrow 0$ on the complement of the union of balls of radius ϵ^β around the punctures. In particular, M_ϵ^{gh} collapses to the flat orbifold \mathbb{T}/τ with bounded curvature away from the punctures.*

Proof. The first statement is a simple application of Lemma 4.7, since close to each puncture we have

$$\rho^k |\nabla^k (h_\epsilon - 1)| \leq C\epsilon\rho^{-1},$$

where ρ is the distance from the puncture. It follows that away from the punctures g_ϵ^{gh} is $C_{loc}^{k, \alpha}$ -close to the \mathbb{Z}_2 -quotient of $g_{\mathbb{T}} + \epsilon^2\theta^2$ for any $k \geq 0$ and ϵ sufficiently small. q.e.d.

5. Approximate hyperkähler metrics

In this section we patch together ALF gravitational instantons and the incomplete hyperkähler structure $\underline{\omega}_\epsilon^{\text{gh}}$ of the previous section to construct a closed definite triple $\underline{\omega}_\epsilon$ which is approximately hyperkähler. In the next section we will use analysis to deform $\underline{\omega}_\epsilon$ into a genuine hyperkähler structure for each $\epsilon > 0$ sufficiently small.

5.1. The 4-manifold M_ϵ . Let $\underline{\omega}_\epsilon^{\text{gh}}$ be the hyperkähler triple defined in (4.6). By Lemma 4.9 for $\epsilon > 0$ small enough we think of $\underline{\omega}_\epsilon^{\text{gh}}$ as defined on M_ϵ^{gh} , a smooth manifold with boundary obtained by restricting the line bundle P to the complement of (arbitrarily) small balls centred at the punctures and then taking the quotient by the involution $\tilde{\tau}$. The boundary of M_ϵ^{gh} has $n + 8$ components, each of which has a collar neighbourhood diffeomorphic either to H^{2m_j-4}/\mathbb{Z}_2 for $j = 1, \dots, 8$, or H^{k_i} for $i = 1, \dots, n$.

For each $j = 1, \dots, 8$ let M_j be the smooth 4-manifold underlying a D_{m_j} ALF space and for each $i = 1, \dots, n$ let N_i be the smooth manifold underlying an A_{k_i-2} ALF space. We construct a smooth 4-manifold M_ϵ by cutting the ends of M_j and N_i and gluing the resulting manifolds with boundary to M_ϵ^{gh} in a neighbourhood of q_j or $\pm p_i$, respectively.

We will construct an approximate hyperkähler structure on M_ϵ in the next subsection. Here we pause for a moment to determine the Betti numbers of M_ϵ . While we will not use this result in an essential way in

the rest of the paper, it is interesting to note how the balancing condition (4.1) appears naturally in the calculation of the Euler characteristic of M_ϵ .

Proposition 5.1. *The Betti numbers of the compact orientable 4-manifold M_ϵ are*

$$b_1(M_\epsilon) = 0, \quad b_2^+(M_\epsilon) = 3, \quad b_2^-(M_\epsilon) = 19.$$

Proof. Decompose M_ϵ into the union of a piece $P/\tilde{\tau}$, an A_{k_i-1} ALF space for each $i = 1, \dots, n$ and a D_{m_j} ALF space for each $j = 1, \dots, 8$. These pieces are identified along their common boundaries, which are homology spheres. Note that all the pieces have vanishing first Betti number. Indeed, all ALF spaces except $\mathbb{R}^3 \times \mathbb{S}^1$ have finite fundamental group. On the other hand, the Gysin sequence for the circle fibration $P \rightarrow \mathbb{T}^*$ yields $0 \rightarrow H^1(\mathbb{T}^*) \rightarrow H^1(P) \rightarrow H^0(\mathbb{T}^*)$, where the last arrow is the zero map unless $P = \mathbb{T}^* \times S^1$. The exact sequence is equivariant with respect to the \mathbb{Z}_2 -actions induced by τ and $\tilde{\tau}$ on $H^1(\mathbb{T}^*) \simeq H^1(\mathbb{T})$ and $H^1(P)$, respectively, and the \mathbb{Z}_2 -action generated by -1 on $H^0(\mathbb{T}^*) \simeq \mathbb{R}$. Since there are no fixed points for the \mathbb{Z}_2 -actions on $H^1(\mathbb{T}^*)$ and $H^0(\mathbb{T}^*)$, we deduce that $H^1(P/\tilde{\tau}) \simeq H^1(P)^{\mathbb{Z}_2}$ is trivial.

Now, since all components have vanishing first Betti number, the reduced Mayer–Vietoris sequence yields $b_1(M_\epsilon) = 0$. The Euler characteristic is also easily calculated:

$$\chi(M_\epsilon) = \chi(P/\mathbb{Z}_2) + \sum_{i=1}^n \chi(A_{k_i-1}) + \sum_{j=1}^8 \chi(D_{m_j}) = \sum_{i=1}^n k_i + \sum_{j=1}^8 m_j + 8 = 24$$

by the balancing condition (4.1).

It remains to calculate the signature $\tau(M_\epsilon)$. Below we will construct a definite triple $\underline{\omega}_\epsilon$ on M_ϵ which is close to defining a hyperkähler structure. By changing the basis of $\Lambda^+ T^* M_\epsilon$ one can always deform this triple to a genuine $SU(2)$ -structure (without requiring any differential constraint). In particular, M_ϵ can be endowed with an almost complex structure J with $c_1(M_\epsilon, J) = 0$. Since $c_2(M_\epsilon, J) = \chi(M_\epsilon) = 24$, Hirzebruch’s Signature Theorem and the equality of characteristic classes $p_1 = c_1^2 - 2c_2$ yield $\tau(M_\epsilon) = -16$. q.e.d.

REMARK. In Remark 4.8 we noted that the case where $n = 0$ and $m_j = 2$ for all $j = 1, \dots, 8$ reduces to the usual Kummer construction. Hence we know that M_ϵ is *diffeomorphic* to the K3 surface in this special case. It seems likely one can prove that the diffeomorphism type of M_ϵ does not depend on the configuration of punctures satisfying the balancing condition (4.1). Since we are going to construct a hyperkähler metric on M_ϵ , the calculation of the Betti numbers will anyway imply that M_ϵ is always diffeomorphic to the K3 surface.

5.2. The definite triple ω_ϵ . We are now going to define an approximately hyperkähler triple ω_ϵ on the 4-manifold M_ϵ .

For each $j = 1, \dots, 8$ denote by $\underline{\omega}_{q_j, \epsilon}$ the hyperkähler triple obtained by first applying the Gibbons–Hawking ansatz (3.3) using the harmonic function $\epsilon^{-1}h_{q_j}$, where

$$(5.2) \quad h_{q_j} = (1 + \epsilon\lambda_j) + \frac{\epsilon(m_j - 2)}{\rho},$$

and then multiplying the metric by ϵ . By abuse of notation we think of $\underline{\omega}_{q_j, \epsilon}$ as defined both on the circle bundle $H^{2m_j-4} \rightarrow \mathbb{R}^3 \setminus B_{8\epsilon}(0)$ as well as on its quotient by the involution that acts as the simultaneous standard involution on \mathbb{R}^3 and the circle fibres.

Similarly, for each $i = 1, \dots, n$ let $\underline{\omega}_{p_i, \epsilon}$ be the hyperkähler triple obtained up to a factor of ϵ from the Gibbons–Hawking ansatz using the harmonic function $\epsilon^{-1}h_{p_i}$, where

$$(5.3) \quad h_{p_i} = (1 + \epsilon\lambda_i) + \frac{\epsilon k_i}{2\rho} + \epsilon \ell_i.$$

Here λ_j, λ_i and ℓ_i are the constants and linear functions appearing in Lemma 4.7. We will assume that ϵ is small enough to guarantee that $\frac{1}{2} < 1 + \epsilon\lambda_i, 1 + \epsilon\lambda_j < \frac{3}{2}$.

For all $j = 1, \dots, 8$ let $(M_j, \underline{\omega}_{M_j})$ be a complete D_{m_j} ALF space with asymptotic length of the circle fibre $2\pi(1 + \epsilon\lambda_j)$. By Definition 3.6 there exists a compact set $K \subset M_j$, $R_0 > 0$ and a diffeomorphism $M_j \setminus K \simeq H^{2m_j-4}/\mathbb{Z}_2$ such that, after the rescaling $x \mapsto \epsilon x$ on the base $\mathbb{R}^3/\mathbb{Z}_2$ of the asymptotic circle fibration,

$$\epsilon^2 \underline{\omega}_{M_j} = \underline{\omega}_{q_j, \epsilon} + \underline{\eta}_{q_j, \epsilon}$$

for $\rho > \epsilon R_0$ with $\underline{\eta}_{q_j, \epsilon} = O(\epsilon^3 \rho^{-3})$ and similar estimates on the derivatives.

For each $i = 1, \dots, n$ let $(N_i, \underline{\omega}_{N_i})$ be a complete A_{k_i-1} ALF space. By the classification of ALF spaces of cyclic type [42] the hyperkähler structure on N_i is explicitly given via the Gibbons–Hawking ansatz starting from a harmonic function on \mathbb{R}^3 with k_i singularities. We can add to this function the smooth harmonic function $\epsilon^2 \ell_i$. Over a ball in \mathbb{R}^3 of radius much smaller than ϵ^{-2} we can regard the resulting hyperkähler structure as a small perturbation of the ALF A_{k_i-1} hyperkähler structure $\underline{\omega}_{N_i}$. By abuse of notation we denote this perturbed hyperkähler structure with the same symbol $\underline{\omega}_{N_i}$. The advantage of this modification is that now $\epsilon^2 \underline{\omega}_{N_i}$ approaches $\underline{\omega}_{p_i, \epsilon}$ with a smaller error: by Definition 3.6 there exists a compact set $K \subset N_i$, $R_0 > 0$ and a diffeomorphism $N_i \setminus K \simeq H^{k_i}|_{\mathbb{R}^3 \setminus B_{R_0}}$ such that, after the rescaling $x \mapsto \epsilon x$ on the base \mathbb{R}^3 of the circle fibration,

$$\epsilon^2 \underline{\omega}_{N_i} = \underline{\omega}_{p_i, \epsilon} + \underline{\eta}_{p_i, \epsilon}$$

with (by scaling) $\underline{\eta}_{p_i, \epsilon} = O(\epsilon^3 \rho^{-3})$ and similar estimates on the derivatives.

REMARK. While not strictly necessary, the choice of perturbing N_i by a linear function makes the exposition more uniform. In particular, the closed definite triple we will construct below fails to be hyperkähler by the *same* amount in a neighbourhood of q_j and $\pm p_i$.

We will assume that ϵ is chosen so small as to make sure that $\epsilon R_0 \ll \rho_0$, where $\rho_0 > 0$ was fixed in Lemma 4.7. By choosing R_0 larger if necessary we can assume that the harmonic functions h_{q_j} and h_{p_i} in (5.2) and (5.3) are as close to constant functions as we please for $\epsilon R_0 \leq \rho \leq \rho_0$. Then the hyperkähler triples $\underline{\omega}_{q_j, \epsilon}$ and $\underline{\omega}_{p_i, \epsilon}$ define metrics $g_{q_j, \epsilon}$ and $g_{p_i, \epsilon}$ which are uniformly equivalent to $g_{\mathbb{T}} + \epsilon^2 \theta^2$ in the regions $\epsilon R_0 \leq \rho_j, \rho_i \leq 2\rho_0$. In the rest of the section all norms and covariant derivatives will be computed with respect to this metric.

A crucial observation is that we can take $\underline{\eta}_{p_i, \epsilon}$ and $\underline{\eta}_{q_j, \epsilon}$ to be exact.

Lemma 5.4. (i) *For all $i = 1, \dots, n$ there exists a triple $\underline{a}_{p_i, \epsilon}$ of 1-forms on H^{k_i} such that*

$$|\nabla^k \underline{a}_{p_i, \epsilon}| \leq C \epsilon^3 \left(\frac{1}{\rho} \right)^{2+k}$$

and $\epsilon^2 \underline{\omega}_{N_i} = \underline{\omega}_{p_i, \epsilon} + d\underline{a}_{p_i, \epsilon}$.

(ii) *For all $j = 1, \dots, 8$ there exists a triple $\underline{a}_{q_j, \epsilon}$ of \mathbb{Z}_2 -invariant 1-forms on H^{2m_j-4} such that*

$$|\nabla^k \underline{a}_{q_j, \epsilon}| \leq C \epsilon^3 \left(\frac{1}{\rho} \right)^{2+k}$$

and $\epsilon^2 \underline{\omega}_{M_j} = \underline{\omega}_{q_j, \epsilon} + d\underline{a}_{q_j, \epsilon}$.

Proof. The proof is identical in the two cases and therefore we work on H^m for some $m \geq 0$, possibly restricting to \mathbb{Z}_2 -invariant forms when m is even. By scaling we can also assume that $\epsilon = 1$.

It is enough to prove that every closed 2-form η with $\eta = O(\rho^{-3})$ can be written as $\eta = da$ with $|\nabla^k a| = O(\rho^{-2-k})$.

Since the restriction of H^m to an exterior domain in \mathbb{R}^3 is diffeomorphic to $(R, \infty) \times \Sigma$ with Σ an homology sphere, we can write $\eta = d\rho \wedge \alpha + \beta$ for some ρ -dependent 1-form α and 2-form β on Σ with $|\alpha| + |\beta| = O(\rho^{-3})$.

The condition $d\eta = 0$ implies $\partial_\rho \beta - d_\Sigma \alpha = 0$. We then define $a = -\int_\rho^\infty \alpha$. The Lemma follows. q.e.d.

By a similar radial integration, Lemma 4.7 (i) implies that in the regions $\epsilon R_0 \leq \rho_j \leq 2\rho_0$ and $\epsilon R_0 \leq \rho_i \leq 2\rho_0$, respectively, we can write

$$(5.5a) \quad \underline{\omega}_\epsilon^{\text{gh}} = \underline{\omega}_{q_j, \epsilon} + d\underline{a}_{q_j, \epsilon}^{\text{gh}}, \quad \underline{\omega}_\epsilon^{\text{gh}} = \underline{\omega}_{p_i, \epsilon} + d\underline{a}_{p_i, \epsilon}^{\text{gh}}$$

for triples $\underline{\mathbf{a}}_{q_j, \epsilon}^{\text{gh}}$ and $\underline{\mathbf{a}}_{p_i, \epsilon}^{\text{gh}}$ of 1-forms such that

$$(5.5b) \quad |\nabla^k \underline{\mathbf{a}}_{q_j, \epsilon}^{\text{gh}}| \leq C\epsilon \rho_j^{3-k}, \quad |\nabla^k \underline{\mathbf{a}}_{p_i, \epsilon}^{\text{gh}}| \leq C\epsilon \rho_i^{3-k}$$

for $k = 0, 1, 2, 3$.

Now, let χ_{q_j} and χ_{p_i} be cut-off functions with the following properties:

$$(5.6) \quad \begin{aligned} \chi_{q_j} &\equiv 1 \text{ for } \rho_j \leq \epsilon^{\frac{2}{5}}, & \chi_{q_j} &\equiv 0 \text{ for } \rho_j \geq 2\epsilon^{\frac{2}{5}}, & |\nabla \chi_{q_j}| &\leq C\rho_j^{-1}, \\ \chi_{p_i} &\equiv 1 \text{ for } \rho_i \leq \epsilon^{\frac{2}{5}}, & \chi_{p_i} &\equiv 0 \text{ for } \rho_i \geq 2\epsilon^{\frac{2}{5}}, & |\nabla \chi_{p_i}| &\leq C\rho_i^{-1}. \end{aligned}$$

We finally define a triple of *closed* 2-forms on M_ϵ by

$$(5.7) \quad \underline{\omega}_\epsilon = \begin{cases} \epsilon^2 \underline{\omega}_{M_j} & \text{if } \rho_j \leq \epsilon^{\frac{2}{5}}, \\ \underline{\omega}_{q_j, \epsilon} + d\left(\chi_{q_j} \underline{\mathbf{a}}_{q_j, \epsilon} + (1 - \chi_{q_j}) \underline{\mathbf{a}}_{q_j, \epsilon}^{\text{gh}}\right) & \text{if } \epsilon^{\frac{2}{5}} \leq \rho_j \leq 2\epsilon^{\frac{2}{5}}, \\ \epsilon^2 \underline{\omega}_{N_i} & \text{if } \rho_i \leq \epsilon^{\frac{2}{5}}, \\ \underline{\omega}_{p_i, \epsilon} + d\left(\chi_{p_i} \underline{\mathbf{a}}_{p_i, \epsilon} + (1 - \chi_{p_i}) \underline{\mathbf{a}}_{p_i, \epsilon}^{\text{gh}}\right) & \text{if } \epsilon^{\frac{2}{5}} \leq \rho_i \leq 2\epsilon^{\frac{2}{5}}, \\ \underline{\omega}_\epsilon^{\text{gh}} & \text{if } \rho_j \geq 2\epsilon^{\frac{2}{5}} \text{ and } \rho_i \geq 2\epsilon^{\frac{2}{5}} \text{ for all } i, j. \end{cases}$$

REMARK. When $k_i = 1$ there is no need to glue in an A_0 ALF space (\mathbb{R}^4 endowed with the Taub–NUT metric), since $\underline{\omega}_\epsilon^{\text{gh}}$ already extends smoothly over $\pm p_i$. It is however useful (we will use this in setting up the analysis for the deformation problem) to think of a rescaled Taub–NUT space localised around $\pm p_i$.

5.2.1. The error. We conclude this section by quantifying the failure of $\underline{\omega}_\epsilon = (\omega_\epsilon^1, \omega_\epsilon^2, \omega_\epsilon^3)$ to define a hyperkähler structure. Since by construction $d\omega_\epsilon^i = 0$ for all $i = 1, 2, 3$, we only have to check that $\underline{\omega}_\epsilon$ is a definite triple and estimate the difference between the associated intersection matrix and the identity.

In the regions $\rho_j \leq \epsilon^{\frac{2}{5}}$, $\rho_i \leq \epsilon^{\frac{2}{5}}$ and when $\rho_i, \rho_j \geq 2\epsilon^{\frac{2}{5}}$ for all i, j the triple $\underline{\omega}_\epsilon$ defines a genuine hyperkähler structure. In the *transition regions* $\epsilon^{\frac{2}{5}} \leq \rho_j \leq 2\epsilon^{\frac{2}{5}}$ and $\epsilon^{\frac{2}{5}} \leq \rho_i \leq 2\epsilon^{\frac{2}{5}}$ we have, respectively,

$$\underline{\omega}_\epsilon - \underline{\omega}_{q_j, \epsilon} = O(\epsilon^{2-\frac{1}{5}}), \quad \underline{\omega}_\epsilon - \underline{\omega}_{p_i, \epsilon} = O(\epsilon^{2-\frac{1}{5}}).$$

Since $\underline{\omega}_{q_j, \epsilon}$ and $\underline{\omega}_{p_i, \epsilon}$ are hyperkähler triples, we conclude that $\underline{\omega}_\epsilon$ is a definite triple for ϵ sufficiently small.

Let μ_ϵ , g_ϵ and Q_ϵ be the volume form, metric and intersection matrix associated to the definite triple $\underline{\omega}_\epsilon$ as in Section 2. Using Lemma 5.4, (5.5), (5.6) and the definition of $\underline{\omega}_\epsilon$ we calculate that

$$(5.8) \quad |Q_\epsilon - \text{id}| \leq C\epsilon^{2-\frac{1}{5}}$$

in every transition region $\epsilon^{\frac{2}{5}} \leq \rho_j \leq 2\epsilon^{\frac{2}{5}}$, $j = 1, \dots, 8$, and $\epsilon^{\frac{2}{5}} \leq \rho_i \leq 2\epsilon^{\frac{2}{5}}$, $i = 1, \dots, n$. Outside the transition regions $Q_\epsilon \equiv \text{id}$.

6. Perturbation to hyperkähler metrics

In the previous section we have constructed a 4-manifold M_ϵ together with a closed definite triple $\underline{\omega}_\epsilon$ which is approximately hyperkähler in the sense that the intersection matrix Q_ϵ associated with $\underline{\omega}_\epsilon$ differs from the identity by arbitrarily small terms as $\epsilon \rightarrow 0$. We would like to deform $\underline{\omega}_\epsilon$ into a genuine hyperkähler triple using analysis. Since the geometry of M_ϵ degenerates as $\epsilon \rightarrow 0$ we need to take some care in applying the Implicit Function Theorem.

As explained in Section 2 we can reformulate the problem in terms of an elliptic PDE. Let g_ϵ be the Riemannian metric on M_ϵ defined by $\underline{\omega}_\epsilon$. Denote by \mathcal{H}_ϵ^+ the space of self-dual harmonic forms with respect to g_ϵ . By Proposition 5.1 \mathcal{H}_ϵ^+ is 3-dimensional spanned by ω_ϵ^i , $i = 1, 2, 3$. The equation we want to solve is (2.8), *i.e.*

$$(6.1) \quad d^+ \underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}} = \mathcal{F}((\text{id} - Q_\epsilon) - d^- \underline{\mathbf{a}}^- * d^- \underline{\mathbf{a}}), \quad d^* \underline{\mathbf{a}} = 0,$$

for a triple $\underline{\mathbf{a}}$ of 1-forms on M_ϵ and a triple $\underline{\boldsymbol{\zeta}} \in \mathcal{H}_\epsilon^+ \otimes \mathbb{R}^3$.

The linearisation of (6.1) is an isomorphism since $b_1(M_\epsilon) = 0$ by Proposition 5.1. Our main task is to show that its inverse has bounded norm as $\epsilon \rightarrow 0$ and to control the non-linearities in (6.1) by introducing appropriate Banach spaces.

6.1. The linear operator $d^* + d^+$ for collapsing Gibbons–Hawking metrics. Before introducing weighted Hölder spaces and proving the main estimates, it is helpful to look more closely at the linearisation of (6.1). It involves the operator $D = d^* + d^+ : \Omega^1(M_\epsilon) \rightarrow \Omega^0(M_\epsilon) \oplus \Omega^+(M_\epsilon)$, where adjoints and projections are computed using the metric g_ϵ .

We are interested in understanding the behaviour of the operator D (in particular, the presence of small eigenvalues) for a sequence of collapsing metrics in the Gibbons–Hawking form (3.3a).

Consider then the metric

$$g_\epsilon^{\text{gh}} = h_\epsilon g_{\mathbb{T}} + \epsilon^2 h_\epsilon^{-1} \theta^2$$

of (4.6) over the circle bundle $\pi: P \rightarrow \mathcal{U}_\epsilon \subset \mathbb{T}^*$. By Lemma 4.9 the open sets \mathcal{U}_ϵ form an exhaustion of \mathbb{T}^* as $\epsilon \rightarrow 0$.

We first consider the geometry of g_ϵ^{gh} and in particular calculate its Levi–Civita connection.

Let ξ be the vertical vector field normalised so that $\theta(\xi) = 1$. As before let $\theta_1, \theta_2, \theta_3$ denote closed 1-forms on \mathbb{T} such that $g_{\mathbb{T}} = \theta_1^2 + \theta_2^2 + \theta_3^2$. Let ξ_1, ξ_2, ξ_3 be the dual vector fields with respect to $g_{\mathbb{T}}$. We will not distinguish between a vector tangent to \mathbb{T} and its horizontal lift to P with respect to the connection θ . In particular, $[\xi_i, \xi_j] = -d\theta(\xi_i, \xi_j) \xi$ as vector fields on P .

Since $\theta([\xi, \xi_i]) = -d\theta(\xi, \xi_i) = 0$ and $\pi_*[\xi, \xi_i] = [\pi_*\xi, \xi_i] = 0$, we have $[\xi, \xi_i] = 0$. The Koszul formula

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle \\ &\quad + X \cdot \langle Y, Z \rangle + Y \cdot \langle X, Z \rangle - Z \cdot \langle X, Y \rangle \end{aligned}$$

then allows to calculate the Levi-Civita connection ∇ of the metric g_ϵ^{gh} :

$$\begin{aligned} \nabla_\xi \xi &= -\frac{1}{4}\epsilon^2 \nabla^\mathbb{T} h_\epsilon^{-2}, & \nabla_{\xi_i} \xi &= \nabla_\xi \xi_i = -\frac{1}{2}h_\epsilon^{-1}(\xi_i \cdot h_\epsilon) \xi + \frac{1}{2}\epsilon^2 h_\epsilon^{-2}(\xi_i \lrcorner d\theta)^{\sharp\mathbb{T}}, \\ \nabla_{\xi_i} \xi_j &= -\frac{1}{2}d\theta(\xi_i, \xi_j) \xi + \frac{1}{2}h_\epsilon^{-1} \left((\xi_i \cdot h_\epsilon) \xi_j - (\xi_j \cdot h_\epsilon) \xi_i - \delta_{ij} \nabla^\mathbb{T} h_\epsilon \right). \end{aligned}$$

Here $\nabla^\mathbb{T}$ and $\sharp\mathbb{T}$ denote gradient and musical isomorphism with respect to the flat metric $g_\mathbb{T}$ and we used the fact that h_ϵ is S^1 -invariant. Since ∇ is a metric connection, we calculate the covariant derivatives of the 1-forms θ, θ_i by duality. Since we will need this later, we write out formulas for $\nabla_\xi \theta$ and $\nabla_\xi \theta_i$:

$$(6.2) \quad \nabla_\xi \theta = \frac{1}{2}h_\epsilon^{-1}dh_\epsilon, \quad \nabla_\xi \theta_i = -\frac{1}{2}\epsilon^2 h_\epsilon^{-3}(\xi_i \cdot h_\epsilon) \theta + \frac{1}{2}\epsilon^2 h_\epsilon^{-2}(\xi_i \lrcorner d\theta).$$

Now, the cotangent bundle of M_ϵ^{gh} is trivial as it is spanned by $\theta, \theta_1, \theta_2, \theta_3$. Thus we can write every 1-form a as

$$(6.3) \quad a = \epsilon a_0 \theta + a_1 \theta_1 + a_2 \theta_2 + a_3 \theta_3$$

for functions a_0, a_1, a_2, a_3 . Note that

$$|a|_{g_\epsilon^{\text{gh}}}^2 = h_\epsilon |a_0|^2 + h_\epsilon^{-1} (|a_1|^2 + |a_2|^2 + |a_3|^2).$$

A direct computation using the fact that θ_i is closed for $i = 1, 2, 3$ and $(\epsilon^{-1}h_\epsilon, d\theta)$ is a solution of the monopole equation (3.4) with respect to the flat metric $g_\mathbb{T}$ shows that

(6.4)

$$d^*a = -h_\epsilon^{-1} \left(\sum_{i=1}^3 \xi_i \cdot a_i + \frac{1}{\epsilon} h_\epsilon^2 \xi \cdot a_0 \right),$$

$$2d^+a = \sum_{i=1}^3 \left(\xi_i \cdot a_0 + h_\epsilon^{-1}(\xi_j \cdot a_k - \xi_k \cdot a_j) + h_\epsilon^{-1}(\xi_i \cdot h_\epsilon) a_0 - \frac{1}{\epsilon}(\xi \cdot a_i) \right) \omega_{\epsilon, i}^{\text{gh}},$$

where $\omega_{\epsilon, i}^{\text{gh}}$, $i = 1, 2, 3$, is the hyperkähler triple $\omega_\epsilon^{\text{gh}}$ of (4.6).

By Fourier analysis along the circle fibres we define projections Π_0 and Π_\perp onto S^1 -invariant and oscillatory components of functions. Via the trivialisation (6.3) Π_0 and Π_\perp extend to 1-forms. Since h_ϵ is S^1 -invariant we see from (6.4) that the operator D respects this decomposition.

For any fixed $\tau \in (0, 1)$ restrict attention to the region in M_ϵ^{gh} where $\rho_j, \rho_i \geq c\epsilon^{\frac{1-\tau}{2}}$ for all $i = 1, \dots, n$ and $j = 1, \dots, 8$. Then

$$(6.5) \quad \|h_\epsilon - 1\|_{C^0} \leq C\epsilon^{\frac{1+\tau}{2}}, \quad \|\nabla^\mathbb{T} h_\epsilon\|_{C^0} \leq C\epsilon^\tau$$

by Lemma 4.10. We conclude that the operator D of (6.4) acting on S^1 -invariant 1-forms approaches the Dirac operator

$$(6.6) \quad D_0: \Omega^0(\mathbb{T}) \oplus \Omega^1(\mathbb{T}) \rightarrow \Omega^0(\mathbb{T}) \oplus \Omega^1(\mathbb{T}), \quad (f, \gamma) \mapsto (d^*\gamma, df + *d\gamma)$$

of the flat torus \mathbb{T} .

Moreover, using the expressions (6.2) for $\nabla_\xi \theta$ and $\nabla_\xi \theta_i$ we find

$$(6.7) \quad \begin{aligned} \epsilon^2 h_\epsilon^{-1} |\nabla a|_{g_\epsilon^{\text{gh}}}^2 &\geq |\nabla_\xi a|_{g_\epsilon^{\text{gh}}}^2 \geq h_\epsilon |\xi \cdot a_0|^2 + h_\epsilon^{-1} \sum_{i=1}^3 |\xi \cdot a_i|^2 \\ &\quad - Ch_\epsilon^{-4} |\nabla^\mathbb{T} h_\epsilon|_{g_\mathbb{T}}^2 \left(h_\epsilon |a_0|^2 + h_\epsilon^{-1} \sum_{i=1}^3 |a_i|^2 \right). \end{aligned}$$

Thus in the region where $\rho_j, \rho_i \geq c\epsilon^{\frac{1-\tau}{2}}$ for some $\tau \in (0, 1)$ we have

$$(6.8) \quad \|\Pi_\perp a\|_{C^0(S^1)} \leq C\epsilon \|\nabla a\|_{C^0(S^1)}$$

on each circle fibre for all ϵ sufficiently small. Indeed, on one side $|\Pi_\perp a|^2 \leq \sum_{i=1}^3 |\xi \cdot a_i|^2$. On the other side, for ϵ small enough (6.5) implies that

$$h_\epsilon |a_0|^2 + h_\epsilon^{-1} \sum_{i=1}^3 |a_i|^2$$

is uniformly equivalent to $\sum_{i=0}^3 |a_i|^2$ and $Ch_\epsilon^{-4} |\nabla^\mathbb{T} h_\epsilon|_{g_\mathbb{T}}^2$ can be made as small as needed to absorb the last term in (6.7).

These two observations—the convergence of the operator D to the Dirac operator D_0 of the flat 3-torus as $\epsilon \rightarrow 0$ and the strong control of the oscillatory part of 1-forms—will be crucial in the rest of the section. We will exploit the same remarks when considering blow-downs of ALF gravitational instantons. Let (M, g_M) be a complete ALF space. Given a sequence $R_i \rightarrow \infty$ consider the blow down $R_i^{-2} g_M$. Since by Definition 3.6 $R_i^{-2} g_M$ is asymptotic (up to a double cover in the dihedral case) to the rescaled Gibbons–Hawking metric

$$\left(1 + R_i^{-1} \frac{k}{2\rho}\right) g_{\mathbb{R}^3} + R_i^{-2} \left(1 + R_i^{-1} \frac{k}{2\rho}\right)^{-1} \theta^2,$$

as $i \rightarrow \infty$ the behaviour of the operator D with respect to the metric $R_i^{-2} g_M$ is the same as the one observed for the metric g_ϵ^{gh} as $\epsilon \rightarrow 0$ with the flat \mathbb{R}^3 in place of the flat 3-torus \mathbb{T} . Namely, on the region $\rho \geq cR_i^{-\frac{1-\tau}{2}}$ (6.8) holds with $\epsilon = R_i^{-1}$ and the operator D converges to the Dirac operator D_0 of flat space \mathbb{R}^3 .

REMARK. The behaviour of natural differential operators (the Laplacian acting on p -forms, the Dirac operator) associated with Riemannian metrics collapsing with bounded curvature and diameter have been studied by many authors, *cf.* for example [34, 35]. The concrete situation we

are interested in is a simple case of this more general theory and it seemed more appropriate to exploit the explicit nature of the Gibbons–Hawking metric rather than appealing to these more general results.

In order to control the growth of differential forms close to the punctures on \mathbb{T}^* and on the end of an ALF space we will now introduce weighted Hölder spaces.

6.2. Weighted Hölder spaces. We work on the Riemannian 4-manifold (M_ϵ, g_ϵ) constructed in Section 5. The aim of this subsection is to introduce weighted Hölder spaces and prove a weighted Schauder estimate for the operator $D = d^* + 2d^+$ associated with the metric g_ϵ .

For R_0 sufficiently large and $\epsilon = \epsilon(R_0)$ sufficiently small we define a *weight function* ρ_ϵ as follows: we set

$$(6.9) \quad \rho_\epsilon = \begin{cases} \epsilon & \text{if } \rho_j \leq R_0\epsilon, \\ \rho_j & \text{if } 2R_0\epsilon \leq \rho_j \leq \rho_0, \\ \epsilon & \text{if } \rho_i \leq R_0\epsilon, \\ \rho_i & \text{if } 2R_0\epsilon \leq \rho_i \leq \rho_0, \\ 1 & \text{if } \rho_j, \rho_i \geq 2\rho_0 \text{ for all } j = 1, \dots, 8, i = 1, \dots, n, \end{cases}$$

and let ρ_ϵ interpolate smoothly and monotonically between the various regions. By abuse of notation we think of ρ_ϵ as defined both on M_ϵ and on M_ϵ^{gh} or its double cover $P|_{\mathcal{U}_\epsilon}$.

Definition 6.10. For each $\delta \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$ and $\alpha \in (0, 1)$ define the weighted Hölder norm $C_\delta^{k, \alpha}$ by

$$\begin{aligned} \|a\|_{C_\delta^{k, \alpha}} &= \sum_{j=1}^k \|\rho_\epsilon^{-\delta+j} \nabla^j a\|_{C^0} \\ &+ \sup_{|x-y| < \text{inj}_{g_\epsilon}} \min \left\{ \rho_\epsilon(x)^{-\delta+k+\alpha}, \rho_\epsilon(y)^{-\delta+k+\alpha} \right\} \frac{|\nabla^k a(x) - \nabla^k a(y)|}{|x-y|^\alpha}. \end{aligned}$$

Here all norms and covariant derivatives are computed with respect to the metric g_ϵ and $\nabla^k a(x)$ and $\nabla^k a(y)$ are compared using parallel transport along the unique geodesic connecting x and y . Similarly set $\|a\|_{C_\delta^0} = \|\rho_\epsilon^{-\delta} a\|_{C^0}$.

The following simple estimate for products in $C_{\delta-1}^{0, \alpha}$ will be used to control the non-linearities.

Lemma 6.11. *For every $\delta < 1$ there exists a constant $C > 0$ independent of ϵ such that*

$$\|uv\|_{C_{\delta-1}^{0, \alpha}} \leq C\epsilon^{\delta-1} \|u\|_{C_{\delta-1}^{0, \alpha}} \|v\|_{C_{\delta-1}^{0, \alpha}}.$$

Proof. From the definition of the $C_{\delta-1}^{0,\alpha}$ -norm it is immediate to check that

$$\|uv\|_{C_{\delta-1}^{0,\alpha}} \leq C \|\rho_\epsilon^{\delta-1}\|_{C^0} \|u\|_{C_{\delta-1}^{0,\alpha}} \|v\|_{C_{\delta-1}^{0,\alpha}}.$$

Since $\rho_\epsilon \geq c\epsilon$ and $\delta - 1 < 0$ the result follows. q.e.d.

We now consider the operator $D = d^* + d^+$ acting on 1-forms of class $C_\delta^{1,\alpha}$. We prove the following weighted Schauder estimate.

Proposition 6.12. *For every $\delta \in \mathbb{R}$ there exists a constant $C > 0$ independent of ϵ such that*

$$\|a\|_{C_\delta^{1,\alpha}} \leq C \left(\|Da\|_{C_{\delta-1}^{0,\alpha}} + \|a\|_{C_\delta^0} \right).$$

Proof. In order to prove this estimate it is convenient to cut the manifold M_ϵ in various pieces and analyse the geometry separately in each of them. The global estimate follows by combining the “local” estimates obtained in each of these pieces.

Consider first the region $\rho_j \leq 2R_0\epsilon$ for some $j = 1, \dots, 8$. The rescaled metric $\epsilon^{-2}g_\epsilon$ is isometric to a compact region in the D_{m_j} ALF space (M_j, g_{M_j}) .

Now, given a 1-form a on M_ϵ , restrict a to the region $\rho_j \leq 2R_0\epsilon$ and define $\tilde{a} = \epsilon^{-1-\delta}a$. The standard Schauder estimate for the elliptic operator D associated with the metric g_{M_j} is

$$\|\tilde{a}\|_{C^{1,\alpha}} \leq C (\|D\tilde{a}\|_{C^{0,\alpha}} + \|\tilde{a}\|_{C^0}).$$

Since $|\tilde{a}|_{\epsilon^{-2}g_\epsilon} = \epsilon^{-\delta}|a|_{g_\epsilon}$ and the norms $\nabla\tilde{a}$ and $D\tilde{a}$ are related in a similar way to those of ∇a and Da , the weighted Schauder estimate follows immediately.

The same argument can be applied in the region $\rho_i \leq 2R_0\epsilon$: the role of M_j is now played by a small perturbation (*cf.* the beginning of Section 5.2) of the A_{k_i-1} ALF space N_i .

Consider now the transition region $R_0\epsilon \leq \rho_j \leq \rho_0$ for some $j = 1, \dots, 8$. We can work on the double cover H^{2m_j-4} and restrict to \mathbb{Z}_2 -invariant forms. Scaling by ϵ as above we reduce to consider the restriction of H^{2m_j-4} to the region $R_0 \leq \rho \leq \frac{\rho_0}{\epsilon}$ in \mathbb{R}^3 endowed with a metric

$$g = \left(1 + \epsilon\lambda_j + \frac{m_j - 2}{\rho}\right) g_{\mathbb{R}^3} + \left(1 + \epsilon\lambda_j + \frac{m_j - 2}{\rho}\right)^{-1} \theta^2 + O(\rho^{-3}) + O(\epsilon^3\rho^2).$$

Moreover, after rescaling the weight function ρ_ϵ coincides with the radial function ρ on \mathbb{R}^3 .

Fix a number $\sigma \in (0, 1)$. For each point x let $\pi(x)$ be its image in \mathbb{R}^3 and set $R = \sigma\rho(x)$. Up to changing R_0 and ρ_0 into $(1 - \sigma)R_0$ and $(1 + \sigma)\rho_0$ we can assume that $B_R(\pi(x))$ is contained in the annulus $R_0 \leq \rho \leq \frac{\rho_0}{\epsilon}$ and that the restriction of H^{2m_j-4} to this ball is trivial. We can then work on a “square” $B_R \times [-R, R]$ in the universal cover of

$H^{2m_j-4}|_{B_R}$. Rescaling the metric by R^{-2} , applying standard Schauder estimates, rescaling back and multiplying by $R^{-\delta}$ we obtain

$$\|a\|_{C_\delta^{1,\alpha}(B_R)} \leq C \left(\|Da\|_{C_{\delta-1}^{0,\alpha}(B_R)} + \|a\|_{C_\delta^0(B_R)} \right).$$

The case of the region $R_0\epsilon \leq \rho_i \leq \rho_0$ is completely analogous.

Finally, in the region where $\rho_j, \rho_i \geq \frac{1}{2}\rho_0$ the weight function ρ_ϵ is uniformly equivalent to the constant 1 and therefore weighted spaces coincide with standard Hölder spaces. Moreover the harmonic function h_ϵ is C^∞ -close to the constant 1. The metric g_ϵ is therefore C^∞ -close to the metric $g_\infty = g_{\mathbb{T}} + \epsilon^2\theta^2$. The Schauder estimate for forms supported in this region is immediate since we can restrict to small balls in the torus \mathbb{T} on which the circle bundle P is trivial and then work on the universal cover, which has bounded geometry. q.e.d.

6.3. The linear estimate. We can now prove the main result about the linearisation of (6.1): the operator D has uniformly bounded inverse.

Proposition 6.13. *For ϵ sufficiently small and $\delta \in (-2, 0)$ there exists C independent of ϵ such that*

$$\|a\|_{C_\delta^{1,\alpha}} \leq C \|Da\|_{C_{\delta-1}^{0,\alpha}}.$$

Proof. By contradiction assume that there exists a sequence $\epsilon_i \rightarrow 0$ and 1-forms a_i on M_{ϵ_i} such that $\|a_i\|_{C_\delta^{1,\alpha}} = 1$ but $\|Da_i\|_{C_{\delta-1}^{0,\alpha}} \rightarrow 0$.

First of all we show that for every compact set K in M_ϵ^{gh} contained in the complement of balls of radius of order $O(\epsilon_i^{\frac{2}{5}})$ around the punctures we must have $\|a_i\|_{C_\delta^{1,\alpha}(K)} \rightarrow 0$.

Over K we can work on the double cover of M_ϵ^{gh} and regard a_i as \mathbb{Z}_2 -invariant forms. Write $a_i = \epsilon_i f_i \theta + \gamma_i$, for a function f_i and a 1-form γ_i such that $\xi \lrcorner \gamma_i = 0$ (recall that ξ is the vector field dual to θ). Over K we can also decompose $a_i = \Pi_0 a_i + \Pi_\perp a_i$. Observe that for any $0 < \tau < 1$ (6.8) implies that

$$\epsilon_i^{\frac{1-\tau}{2}} \|\Pi_\perp a_i\|_{C_\delta^0} \leq \|\Pi_\perp a_i\|_{C_{\delta-1}^0} \leq C \epsilon_i \|a\|_{C_{\delta-1}^{1,\alpha}}$$

provided $\rho_j, \rho_i \geq c \epsilon_i^{\frac{1-\tau}{2}}$. Since the gluing regions occur for $\rho_j \sim \epsilon_i^{\frac{2}{5}}$ and $\rho_i \sim \epsilon_i^{\frac{2}{5}}$ we can choose $\tau > 0$ sufficiently small so that these assumptions are satisfied on M_ϵ^{gh} . Thus $\|\Pi_\perp a_i\|_{C_\delta^0(K)} \rightarrow 0$.

By the Arzelá–Ascoli Theorem we can therefore assume that (f_i, γ_i) converges to $(f_0, \gamma) \in \Omega^0(\mathbb{T}) \oplus \Omega^1(\mathbb{T})$. By (6.4) (f_0, γ) satisfies

$$(6.14) \quad *d\gamma + df_0 = 0 = d^*\gamma$$

on \mathbb{T} . The control on $\|a_i\|_{C_\delta^{1,\alpha}}$ guarantees that $|f_0| + |\gamma| \leq C\rho^\delta$ close to the punctures.

We want to conclude that f_0 is constant and γ is a smooth harmonic 1-form on \mathbb{T} . By trivialising the cotangent bundle of the 3-torus \mathbb{T} by harmonic 1-forms $\theta_1, \theta_2, \theta_3$ we can write $\gamma = f_1\theta_1 + f_2\theta_2 + f_3\theta_3$. Then f_0, f_1, f_2, f_3 are harmonic functions on the punctured 3-torus with controlled blow-up rate at the punctures. Since $\delta > -2$, close to each puncture we must have $f_i = \lambda_i + c_i\rho^{-1} + O(\rho)$ for some constants λ_i, c_i . However, c_i must vanish for all $i = 0, 1, 2, 3$ if (f_0, f_i) is a solution of the first order system (6.14) and not only of the second order PDE this implies. Hence the functions f_i are bounded harmonic functions on \mathbb{T} and must be constant.

However, by \mathbb{Z}_2 -invariance of the 1-forms a_i the functions f_i must be odd with respect to the involution τ on \mathbb{T} and must therefore vanish. Thus $(f_0, \gamma) = 0$ and the Schauder estimate of Proposition 6.12 implies that $\|a_i\|_{C_\delta^{1,\alpha}(K)} \rightarrow 0$.

Next we look at what happens close to one of the punctures. By what we have just proved and the assumption $\|a_i\|_{C_\delta^{1,\alpha}(M_\epsilon)} = 1$, there exists at least a $j = 1, \dots, 8$ or $i = 1, \dots, n$ such that $\|a_i\|_{C_\delta^{1,\alpha}} = 1$ in the region $\rho_j \leq \epsilon^{\frac{2}{5}}$ or $\rho_i \leq \epsilon^{\frac{2}{5}}$. We fix attention to such a region. Rescaling the metric by ϵ_i^{-2} and replacing a_i with $\tilde{a}_i = \epsilon_i^{-\delta-1}a_i$, from now on we will work on a D_m or an A_{k-1} ALF space. Denote either of these non-compact manifolds by M . Because of the behaviour of the weighted Hölder norm in Definition 6.10 under rescaling, we have $\|\tilde{a}_i\|_{C_\delta^{1,\alpha}} = \|a_i\|_{C_\delta^{1,\alpha}} = 1$. For ease of notation we replace \tilde{a}_i with a_i until the end of the proof.

By the Arzelá–Ascoli Theorem over every compact set of M we can extract a subsequence of $\{a_i\}$ that converges to a solution a of $Da = 0$ on M . Moreover, $|a| \leq C\rho^\delta$. Since $\delta < 0$ we conclude that $a = 0$. Indeed, the equation $Da = 0$ in particular implies that $\Delta a = 0$. Since M is Ricci-flat, the Weitzenböck formula yields $|a|\Delta|a| \leq 0$ and therefore $|a| = 0$ by the maximum principle.

Now, if $\|a_i\|_{C_\delta^0(M)} \rightarrow 0$ then the Schauder estimate of Proposition 6.12 would yield a contradiction to the assumption $\|a_i\|_{C_\delta^{1,\alpha}} = 1$. Assume therefore that there exists some $\nu > 0$ such that $\|a_i\|_{C_\delta^0} \geq \nu$. Since $a_i \rightarrow 0$ in $C_\delta^{1,\alpha}(K)$ for every compact set $K \subset M$ there must exist a sequence of points $x_i \in M$ going off to infinity (in particular $R_i := \rho(x_i) \rightarrow \infty$) such that $|a_i(x_i)| \geq \nu\rho(x_i)^\delta$.

Now rescale the metric on M by R_i^{-2} and replace a_i by $R_i^{-\delta-1}a_i$. Then $(M, R_i^{-2}g_M)$ is converging to the tangent cone C at infinity of M , *i.e.* either $C = \mathbb{R}^3$ or $C = \mathbb{R}^3/\mathbb{Z}_2$ depending on whether M is of cyclic or dihedral type.

As in the first step of the proof, we can use (6.4) and (6.8) to conclude that $R_i^{-\delta-1}a_i$ sub-converges over compact subsets of $C \setminus \{0\}$ to a pair

$(f_0, \gamma) \in \Omega^0(C) \oplus \Omega^1(C)$ such that

$$*d\gamma + df_0 = 0 = d^*\gamma,$$

$|f_0| + |\gamma| \leq C\rho^\delta$ and $(|f_0| + |\gamma|)(x_0) = \nu > 0$ for some $x_0 \in C \setminus \{0\}$. As before, the fact that $\delta > -2$ implies that f_0, γ are bounded close to the origin in C . The fact that $\delta < 0$ then forces (f_0, γ) to vanish. However this contradicts the fact that $(|f_0| + |\gamma|)(x_0) > 0$. q.e.d.

6.4. The non-linear problem. We are now ready to deform the triple $\underline{\omega}_\epsilon$ into a genuine hyperkähler triple by using the following Implicit Function Theorem.

Lemma 6.15. *Let $\Phi: E \rightarrow F$ be the smooth function between Banach spaces and write $\Phi(x) = \Phi(0) + L(x) + N(x)$, where L is linear and N contains the non-linearities. Assume that there exists constants r, C, q such that*

- (i) L is invertible with $\|L^{-1}\| \leq C$;
- (ii) $\|N(x) - N(y)\|_F \leq q\|x + y\|_E\|x - y\|_E$ for all $x, y \in B_r(0) \subset E$;
- (iii) $\|\Phi(0)\|_F < \min\left\{\frac{r}{2C}, \frac{1}{4qC^2}\right\}$.

Then there exists a unique $x \in E$ with $\|x\|_E \leq 2C\|\Phi(0)\|_F$ such that $\Phi(x) = 0$.

In our situation we set

$$E := \left(C_\delta^{1,\alpha}(T^*M_\epsilon) \oplus \mathcal{H}_\epsilon^+\right) \otimes \mathbb{R}^3,$$

where \mathcal{H}_ϵ^+ denotes the space of self-dual harmonic forms with respect to g_ϵ , i.e. constant linear combinations of $\omega_\epsilon^1, \omega_\epsilon^2, \omega_\epsilon^3$. We endow E with the product of the $C_\delta^{1,\alpha}$ -norm and the norm on the finite dimensional vector space $\mathcal{H}_\epsilon^+ \otimes \mathbb{R}^3 \simeq \mathbb{R}^9$ induced by the L^2 -norm. Similarly we set

$$F := C_{\delta-1}^{0,\alpha}(\mathbb{R} \oplus \Lambda^+ T^*M_\epsilon) \otimes \mathbb{R}^3$$

endowed with the $C_{\delta-1}^{0,\alpha}$ -norm.

The operator Φ is the one defined by (6.1). Thus $\Phi(0) = -\mathcal{F}(\text{id} - Q_\epsilon)$, $L(\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}}) = D\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}}$ and the non-linear term is

$$N(\underline{\mathbf{a}} + \underline{\boldsymbol{\zeta}}) = \mathcal{F}(\text{id} - Q_\epsilon) - \mathcal{F}(\text{id} - Q_\epsilon - d^-\underline{\mathbf{a}} * d^-\underline{\boldsymbol{\zeta}}).$$

We need to check that the hypotheses of Lemma 6.15 are satisfied.

We use Proposition 6.13 to show that L has uniformly bounded inverse for $\delta \in (-\frac{1}{2}, 0)$.

Lemma 6.16. *For $\delta \in (-\frac{1}{2}, 0)$ and ϵ sufficiently small there exists a constant $C > 0$ independent of ϵ such that for every triple of self-dual 2-forms $\underline{\boldsymbol{\xi}} \in C_{\delta-1}^{0,\alpha}$ there exists a unique $(\underline{\mathbf{a}}, \underline{\boldsymbol{\zeta}}) \in E$ with*

$$\|\underline{\mathbf{a}}\|_{C_\delta^{1,\alpha}} + \|\underline{\boldsymbol{\zeta}}\| \leq C\|\underline{\boldsymbol{\xi}}\|_{C_{\delta-1}^{0,\alpha}}.$$

and $L(\underline{\mathbf{a}}, \underline{\boldsymbol{\zeta}}) = \underline{\boldsymbol{\xi}}$.

Proof. First of all, note that the 2-forms ω_ϵ^i have uniformly bounded $C_{\delta-1}^{0,\alpha}$ -norm. Indeed, outside the gluing regions $\underline{\omega}_\epsilon$ is a hyperkähler triple and thus ω_ϵ^i is parallel and bounded. On the gluing regions, $\underline{\omega}_\epsilon$ differs from the hyperkähler triple $\underline{\omega}_{q_j,\epsilon}$ or $\underline{\omega}_{p_i,\epsilon}$ by terms of order $O(\epsilon\rho^2 + \epsilon^3\rho^{-3})$ (with similar estimates on their derivatives). Finally, $\rho_\epsilon^{-\delta+1}$ is bounded above since $\delta < 0$.

Now, let $\tilde{\omega}$ be an L^2 -orthonormal triple of harmonic self-dual forms with respect to g_ϵ . Consider the positive definite matrix B defined by

$$B_{ij} = \int_{M_\epsilon} \omega_\epsilon^i \wedge \omega_\epsilon^j = 2 \int_{M_\epsilon} (Q_\epsilon)_{ij} \, dv_{g_\epsilon}.$$

Since Q_ϵ is close to the identity and $\text{Vol}_{g_\epsilon}(M_\epsilon) = O(\epsilon)$ we have $B = \epsilon(\text{id} + T)$ for some matrix T with $\|T\| \rightarrow 0$ as $\epsilon \rightarrow 0$. Then since $\tilde{\omega} = A\underline{\omega}$ for a matrix A with $AA^\top = B^{-1}$ we have

$$\|\tilde{\omega}\|_{C_{\delta-1}^{0,\alpha}} \leq C\epsilon^{-\frac{1}{2}} \|\underline{\omega}_\epsilon\|_{C_{\delta-1}^{0,\alpha}} \leq C\epsilon^{-\frac{1}{2}}.$$

Finally, observe that for every $u \in C_{\delta-1}^{0,\alpha}$ we have

$$\|u\|_{L^2} \leq \|\rho_\epsilon^{\delta-1}\|_{L^2} \|u\|_{C_{\delta-1}^0} \leq C(\epsilon^{\frac{1}{2}} + \epsilon^{\delta+1}) \|u\|_{C_{\delta-1}^0}.$$

Indeed, using the definition (6.9) of ρ_ϵ and the construction of $\underline{\omega}_\epsilon$ it is not difficult to estimate $\|\rho_\epsilon^{\delta-1}\|_{L^2} \leq C(\epsilon^{\frac{1}{2}} + \epsilon^{\delta+1})$.

Now let $\pi: C_{\delta-1}^{0,\alpha}(\Lambda^+ T^* M_\epsilon) \rightarrow \mathcal{H}_\epsilon^+$ be the L^2 -orthogonal projection

$$\pi(\xi) = \sum_{i=1}^3 \lambda_i \tilde{\omega}_i, \quad \lambda_i = \int \xi \wedge \tilde{\omega}_i,$$

and regard $\text{id} - \pi$ as a map $C_{\delta-1}^{0,\alpha}(\Lambda^+ T^* M_\epsilon) \rightarrow C_{\delta-1}^{0,\alpha}(\Lambda^+ T^* M_\epsilon)$. By the remarks above we have

$$|\lambda_i| \leq C(\epsilon^{\frac{1}{2}} + \epsilon^{\delta+1}) \|\xi\|_{C_{\delta-1}^{0,\alpha}}, \quad \|\pi(\xi)\|_{C_{\delta-1}^{0,\alpha}} \leq C(1 + \epsilon^{\delta+\frac{1}{2}}) \|\xi\|_{C_{\delta-1}^{0,\alpha}}.$$

Thus if $\delta \geq -\frac{1}{2}$ the projections π and $\text{id} - \pi$ are uniformly bounded. Proposition 6.13 and the surjectivity of L then yield the result. \square

Next, we consider the non-linear term N . Note that this does not involve the harmonic part $\underline{\zeta}$. The function \mathcal{F} is pointwise smooth with uniformly controlled norm for ϵ sufficiently small. Using the Taylor expansion of \mathcal{F} at $\text{id} - Q_\epsilon$ and Lemma 6.11 to control products we can therefore find uniform constants $r, C > 0$ independent of ϵ such that assumption (ii) in Lemma 6.15 is satisfied with $q = C\epsilon^{\delta-1}$.

Finally,

$$\|\mathcal{F}(\text{id} - Q_\epsilon)\|_{C_{\delta-1}^{0,\alpha}} \leq C \|\text{id} - Q_\epsilon\|_{C_{\delta-1}^{0,\alpha}} \leq C\epsilon^{2+\frac{1}{5}-\frac{2}{5}\delta}.$$

Indeed, setting $\rho = \rho_j$ for $j = 1, \dots, 8$ or $\rho = \rho_i$ for some $i = 1, \dots, n$, in the region $\epsilon^{\frac{2}{5}} \leq \rho \leq 2\epsilon^{\frac{2}{5}}$ we have $|\text{id} - Q_\epsilon| = O(\epsilon\rho^2 + \epsilon^3\rho^{-3})$ by (5.8) and $\rho_\epsilon = \rho$ by (6.9).

Thus assumption (iii) in Lemma 6.15 is therefore satisfied as soon as $\epsilon^{2+\frac{1}{5}-\frac{2}{5}\delta} \ll \epsilon^{1-\delta}$, i.e.

$$\epsilon^{\frac{3}{5}(\delta+2)} \ll 1.$$

If $\delta > -2$ this condition is satisfied for $\epsilon > 0$ sufficiently small.

Theorem 6.17. *Let $(\mathbb{T}, g_{\mathbb{T}})$ be a flat 3-torus with standard involution $\tau: \mathbb{T} \rightarrow \mathbb{T}$. Let q_1, \dots, q_8 be the fixed points of τ and let $p_1, \tau(p_1), \dots, p_n, \tau(p_n)$ be further $2n$ distinct points. Denote by \mathbb{T}^* the punctured torus $\mathbb{T} \setminus \{q_1, \dots, q_8, p_1, \dots, \tau(p_n)\}$.*

Let $m_1, \dots, m_8 \in \mathbb{Z}_{\geq 0}$ and $k_1, \dots, k_n \in \mathbb{Z}_{\geq 1}$ satisfy

$$\sum_{j=1}^8 m_j + \sum_{i=1}^n k_i = 16.$$

For each $j = 1, \dots, 8$ fix a D_{m_j} ALF space M_j and for each $i = 1, \dots, n$ an A_{k_i-1} ALF space N_i .

Then there exists a 1-parameter family of hyperkähler metrics $\{g_\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ on the K3 surface with the following properties. We can decompose the K3 surface into the union of open sets $K^\epsilon \cup \bigcup_{j=1}^8 M_j^\epsilon \cup \bigcup_{i=1}^n N_i^\epsilon$ such that

- (i) (K^ϵ, g_ϵ) collapses to the flat orbifold $\mathbb{T}^*/\mathbb{Z}_2$ with bounded curvature away from the punctures;
- (ii) for each $j = 1, \dots, 8$ and $k \geq 0$, $(M_j^\epsilon, \epsilon^{-2}g_\epsilon)$ converges in $C_{loc}^{k, \alpha}$ to the D_{m_j} ALF space M_j ;
- (iii) for each $i = 1, \dots, n$ and $k \geq 0$, $(N_i^\epsilon, \epsilon^{-2}g_\epsilon)$ converges in $C_{loc}^{k, \alpha}$ to the A_{k_i-1} ALF space N_i .

Proof. Given data as in the statement we constructed a 4-manifold M_ϵ and a 1-parameter family of closed definite triples $\underline{\omega}_\epsilon$ which are approximately hyperkähler. For ϵ sufficiently small we can apply Lemma 6.15 to find unique $\underline{\mathbf{a}}_\epsilon \in C_\delta^{1, \alpha}(T^*M_\epsilon)$ for $\delta \in (-\frac{1}{2}, 0)$ and $\underline{\zeta}_\epsilon \in \mathcal{H}_\epsilon^+$ such that $\|\underline{\mathbf{a}}_\epsilon\|_{C_\delta^{1, \alpha}} + \|\underline{\zeta}_\epsilon\| \leq C\epsilon^{\frac{11-2\delta}{5}}$ and $\underline{\omega}_\epsilon + d\underline{\mathbf{a}}_\epsilon + \underline{\zeta}_\epsilon$ is a hyperkähler structure on M_ϵ . In particular, since $b_1(M_\epsilon) = 0$ by Proposition 5.1, M_ϵ must be diffeomorphic to the K3 surface.

Away from the gluing regions $\underline{\mathbf{a}}_\epsilon$ satisfies $d^+\underline{\mathbf{a}}_\epsilon = \mathcal{F}(d^-\underline{\mathbf{a}}_\epsilon * d^-\underline{\mathbf{a}}_\epsilon) - \underline{\zeta}_\epsilon$, $d^*\underline{\mathbf{a}}_\epsilon = 0$. Note that we can write $\mathcal{F}(d^-\underline{\mathbf{a}}_\epsilon * d^-\underline{\mathbf{a}}_\epsilon)$ as a smooth function $Q(x, y, z)$ of $x \in \Omega$ and $y, z \in \Lambda^{-1}T_x^*\Omega \otimes \mathbb{R}^3$ evaluated at $y = z = d^-\underline{\mathbf{a}}_\epsilon(x)$. By the chain rule we deduce that derivatives of order k of $\underline{\mathbf{a}}_\epsilon$ solve a linear equation of the form $(D + A_k)u = f_k$, where the operator norm of $A_k: C^{1, \alpha} \rightarrow C^{0, \alpha}$ is controlled by $\|\underline{\mathbf{a}}_\epsilon\|_{C^{k, \alpha}}$ and $\|f_k\|_{C^{0, \alpha}}$ is controlled by $\|\underline{\mathbf{a}}_\epsilon\|_{C^{k, \alpha}}$ and $\|\underline{\zeta}_\epsilon\|_{C^{k, \alpha}}$.

For domains $\Omega \subset\subset \Omega'$ contained either in compact sets of M_ϵ^{gh} or in regions where $\rho_i \leq R\epsilon$ or $\rho_j \leq R\epsilon$ for some $R > 0$ and ϵ sufficiently small, the proof of Proposition 6.12 shows that we have a uniform interior elliptic estimate

$$\|u\|_{C^{1,\alpha}(\Omega)} \leq C \left(\|Du\|_{C^{0,\alpha}(\Omega')} + \|u\|_{C^{0,\alpha}(\Omega')} \right).$$

If $\|\underline{\mathbf{a}}_\epsilon\|_{C^{k,\alpha}(\Omega')}$ is sufficiently small then we can replace D with $D + A_k$ in this estimate and obtain

$$(6.18) \quad \begin{aligned} \|\underline{\mathbf{a}}_\epsilon\|_{C^{k+1,\alpha}(\Omega)} &\leq C \left(\|\underline{\zeta}_\epsilon\|_{C^{k,\alpha}(\Omega')} + \|\underline{\mathbf{a}}_\epsilon\|_{C^{k,\alpha}(\Omega')} \right) \\ &\leq C \left(\epsilon^{-\frac{1}{2}} \|\underline{\zeta}_\epsilon\| + \|\underline{\mathbf{a}}_\epsilon\|_{C^{k,\alpha}(\Omega')} \right), \end{aligned}$$

since, as in the proof of Lemma 6.16, we have

$$\|\underline{\zeta}_\epsilon\|_{C^{k,\alpha}(\Omega')} \leq C\epsilon^{-\frac{1}{2}} \|\underline{\zeta}_\epsilon\|.$$

Now, taking into account that $\rho_\epsilon \geq c\epsilon$ and $\delta < 0$ we have

$$\|a_\epsilon\|_{C^{1,\alpha}} \leq C\epsilon^{\delta-1} \|\underline{\mathbf{a}}_\epsilon\|_{C_\delta^{1,\alpha}} = O(\epsilon^{\frac{3}{5}(\delta+2)}), \quad \epsilon^{-\frac{1}{2}} \|\underline{\zeta}_\epsilon\| = O(\epsilon^{\frac{17-4\delta}{10}}).$$

Since $\delta + 2 > 0$ we can use (6.18) inductively to prove that in regions Ω as above $\|\underline{\mathbf{a}}_\epsilon\|_{C^{k+1,\alpha}(\Omega)} = O(\epsilon^{\frac{3}{5}(\delta+2)})$ for all $k \geq 0$. Taking Ω contained in M_ϵ^{gh} implies that on compact sets of M_ϵ^{gh} the hyperkähler metric induced by $\underline{\omega}_\epsilon + d\underline{\mathbf{a}}_\epsilon + \underline{\zeta}_\epsilon$ is arbitrarily $C^{k,\alpha}$ -close to $g_{\mathbb{T}} + \epsilon^2\theta^2$ as $\epsilon \rightarrow 0$. Similarly, by taking Ω contained in regions where either $\rho_i \leq R_0\epsilon$ or $\rho_j \leq R_0\epsilon$ and observing that $\|\epsilon^{-2}\sigma\|_{C^{k,\alpha}(\epsilon^{-2}g_\epsilon)} = \|\sigma\|_{C^{k,\alpha}(g_\epsilon)}$ for every 2-form σ , we deduce that the rescaled metric $\epsilon^{-2}g_\epsilon$ is arbitrarily $C^{k,\alpha}$ -close to g_{N_i} or g_{M_j} , respectively, as $\epsilon \rightarrow 0$. q.e.d.

By varying all parameters involved in the construction we can in fact realise a whole open set in the moduli space of hyperkähler metrics on the K3 surface. Indeed,

- (i) the moduli space of flat tori is 6-dimensional;
- (ii) the choice of punctures p_1, \dots, p_n yields additional $3n$ parameters;
- (iii) once the punctured torus and weights are fixed, the moduli space of abelian Dirac monopoles with prescribed singularities is 4-dimensional (one has to choose ϵ and the 3 moduli of a flat connection);
- (iv) each D_{m_j} ALF space contributes $3m_j$ parameters and every A_{k_i-1} ALF space contributes $3(k_i - 1)$ parameters.

Hence the total number of parameters in the construction is

$$6 + 3n + 4 + 3 \sum_{j=1}^8 m_j + 3 \sum_{i=1}^n k_i - 3n = 10 + 3 \times 16 = 58,$$

which is exactly the dimension of the moduli space of Ricci-flat metrics on the K3 surface (without any normalisation on volume).

7. Stable minimal surfaces

In this final section we exploit our gluing construction of hyperkähler metrics on the K3 surface to deduce some information about their submanifold geometry.

It is well known that holomorphic submanifolds of a Kähler manifold are volume minimising. It is a classical problem in the theory of minimal submanifolds in Kähler manifolds to understand to what extent volume minimising submanifolds must be holomorphic or anti-holomorphic.

In [39] Micallef showed that every stable minimal surface in a flat 4-torus must be holomorphic with respect to a complex structure compatible with the metric. (Note however that this is no longer the case for higher dimensional tori [4].) In view of Micallef's result it was thought for some time that a similar result could hold for the K3 surface endowed with a hyperkähler metric. Partial results in this direction were established by Micallef–Wolfson [37, Theorem 5.3] and motivation for the conjecture came from the fact that, given an arbitrary hyperkähler metric on the K3 surface, every homology class can be represented by the sum of surfaces each of which is holomorphic with respect to some complex structure compatible with the metric. However, Micallef–Wolfson [38] have eventually shown that no analogue of the result for 4-tori holds for the K3 surface. Indeed they found a class $\alpha \in H_2(K3, \mathbb{Z})$ and a hyperkähler metric g on the K3 surface such that the volume minimiser in α decomposes into a sum of branched minimal surfaces $\Sigma_1 \cup \dots \cup \Sigma_k$ not all of which can be holomorphic with respect to some complex structure compatible with g .

We can use our gluing construction to construct further (simpler) examples of strictly stable minimal spheres with respect to some hyperkähler metric on the K3 surface which cannot be holomorphic for any complex structure compatible with the metric.

Theorem 7.1. *There exist hyperkähler metrics on the K3 surface that contain a strictly stable minimal sphere which is not holomorphic with respect to any complex structure compatible with the metric.*

Proof. In [37, Proposition 5.5] Micallef–Wolfson show that the double cover of the Atiyah–Hitchin manifold, the rotationally symmetric D_1 ALF space, contains a strictly stable minimal 2-sphere Σ with $[\Sigma] \cdot [\Sigma] = -4$. Since every holomorphic curve Σ of genus γ in a hyperkähler 4-manifold must have $[\Sigma] \cdot [\Sigma] = 2\gamma - 2$ by the adjunction formula, this minimal 2-sphere cannot be holomorphic with respect to any complex structure. One can also use the isometric action of $SU(2)$ on the Atiyah–Hitchin metric to prove this fact: the $SU(2)$ action preserves the metric but rotates the complex structures (equivalently, the hyperkähler triple) and the minimal 2-sphere is an $SU(2)$ -orbit. Hence the periods $\int_{\Sigma} \omega_i$ are forced to vanish.

Now, consider an approximate hyperkähler metric g_ϵ obtained in Section 5 by using the rotationally symmetric D_1 ALF space as one of the building blocks. Thus g_ϵ contains a strictly stable minimal sphere Σ with $[\Sigma] \cdot [\Sigma] = -4$.

Because of strict stability, Σ has no Jacobi fields. Then we can invoke White's Implicit Function Theorem for minimal immersions with respect to variations of the ambient metric [48, Theorem 2.1] to deform Σ into a minimal immersion with respect to the hyperkähler metric produced by Theorem 6.17 starting from g_ϵ . As before, this minimal 2-sphere cannot be holomorphic with respect to any complex structure because of its self-intersection number. It is strictly stable by continuity of the spectrum of the Jacobi operator. q.e.d.

REMARK 7.2. By work of Chen–Li [12, Corollary 3.7] each of these minimal spheres is the image of a smooth harmonic map u from S^2 into the K3 surface satisfying the first-order equation

$$(7.3) \quad du \circ J_{S^2} = -(x_1 J_1 \circ du + x_2 J_2 \circ du + x_3 J_3 \circ du).$$

Here (x_1, x_2, x_3) defines the standard embedding of S^2 as the unit sphere in \mathbb{R}^3 and (J_1, J_2, J_3) is the triple of parallel complex structures associated to a hyperkähler structure on the K3 surface. By taking the radial extension f of u to \mathbb{R}^3 and further extending f by translation invariance to a map F from $\mathbb{H} = \mathbb{R}^3 \times \mathbb{R}$ into the K3 surface we obtain the first known examples of (homogeneous) Fueter and triholomorphic maps with codimension-3 singular set and compact target. A triholomorphic map is a harmonic map between hyperkähler manifolds satisfying a quaternionic del-bar equation. Triholomorphic maps (and their dimensional reduction to Fueter maps from 3-manifolds into hyperkähler manifolds) arise in different contexts related to gauge theory in higher dimensions, *cf.* for example [20, §6] and [22], the hyperkähler Floer theory of [31] and the definition of enumerative invariants of hyperkähler manifolds [46]. For these geometric applications it is essential to understand the regularity theory of limits of smooth triholomorphic maps. The importance of the existence or non-existence of homogeneous triholomorphic maps with a singular set of codimension 3 was stressed in the recent work of Bellettini–Tian [7, Remark 1.2]. We also note that since for the minimal spheres Σ of Theorem 7.1 the integrals $\int_\Sigma \omega_i$ all vanish, one can further show that the solutions to (7.3) that we produce are also balanced, *i.e.* $\int_{S^2} x_i |\nabla u|^2 = 0$ for $i = 1, 2, 3$, and therefore the Fueter and triholomorphic tangent maps f and F obtained from Theorem 7.1 are stationary harmonic maps, *i.e.* critical points of the Dirichlet energy with respect to variations both of the domain and the target.

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